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Disturbance Rejection in Multivariable Systems

Salem A.K. Al-Assadi

A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University
Montréal, Québec, Canada

November, 1990

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ABSTRACT

Disturbance Rejection in Multivariable Systems

Salem A.K. Al-Assadi Ph. D.,
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This thesis is concerned with developing computational algorithms for disturbance rejection in multivariable systems. The theoretical basis for the algorithms is a factorization procedure for the transfer function matrix between the outputs and the disturbances. This enable us to use the concept of a minimal order inverse to determine the position of "disturbance blocking zeros" which play an important part in disturbance rejection. It has been shown that using the transmission properties of the disturbance blocking zeros, it is possible to choose closed-loop positions for these zeros in order to eliminate the steady-state effect of a class of disturbances at the outputs of the system. The algorithms presented in this thesis can be used to assign as many disturbance blocking zeros as required by any multivariable system described by 4-tuples $\sum_d \left[A, B, C, E\right]$ or 6-tuples $\sum_d \left[A, B, C, D, E, F\right]$. These algorithms use state feedback controllers (constant or dynamic feedback) to position these zeros at desired locations in the complex plane, such that certain measurable or unmeasurable disturbances are rejected in the steady state. In addition, the resulting closed-loop system is stabilized and/or meets some transient performance. This is achieved by designing an output feedback controllers to assign all the poles at desired locations in the complex plane. Moreover, a new approach for designing robust controllers by means of dynamic output feedback is also developed. This provides a way of solving the general servomechanism problem. The numerical performance of the algorithms proposed in this thesis is illustrated by applying them to several practical examples.

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LIST OF SYMBOLS

$\mathbb{R}$	Real n-dimensional vector space
$\mathbb{R}^{n \times m}$	The set of real $n \times m$ matrices
C^+	The right half of the complex plane
$c-c$	Complex-conjugate
O	A zero matrix of appropriate dimension
I_n	An $n \times n$ identity matrix
A^T	Transpose of A
A^{-1}	Inverse of A
A^+	Pseudo inverse of A
$\sigma(A)$	The set of eigenvalues of A
$\sigma(A, B)$	The set of eigenvalues of the square matrix A which are common to those of the square matrix $(A + BL)$ for all real L of appropriate dimension
rank (A)	Rank of A
adj (A)	Adjoint of A
det (A)	Determinant of A
UHM	Upper Hessenberg matrix
UHF	A single-input system$(A, \mathbf{b}, C)$ in upper Hessenberg form
BUHM	Block upper Hessenberg matrix
BUHF	A multivariable system (A, B, C) in block upper Hessenberg form
RSF	A multivariable system (A, B, C) in real Schur form
SVD	Singular Value Decomposition

EVA	Eigenvalue Assignment
D.B.Z.'s	Disturbance blocking zeros
D.Z.'s	Disturbance zeros
Z_{id}	Input-decoupling zeros
Z_{od}	Output-decoupling zeros
Z_{iod}	Input-output decoupling zeros
Z_t	Transmission zeros
Z_i	Invariant zeros
Z_b	Blocking zeros
$P(s)$	System matrix of the 4-tuple system $\left[A, B, C, D\right]$
$W_u(s)$	Transfer function matrix between the control inputs and the outputs
$W_d(s)$	Transfer function matrix between the disturbances and the outputs
$W_d^T(s)$	Transpose of $W_d(s)$
$W_d^R(s)$	Right inverse of $W_d(s)$
$W_d^L(s)$	Left inverse of $W_d(s)$
$W_d^{R_m}(s)$	Minimal right inverse of $W_d(s)$
$W_d^{L_m}(s)$	Minimal left inverse of $W_d(s)$
$P_d(s)$	The "numerator" rational function matrix of $W_d(s)$
$Q_d(s)$	The "denominator" rational function matrix of $W_d(s)$
$P_d^R(s)$	Right inverse of $P_d(s)$
$P_d^{R_m}(s)$	Minimal right inverse of $P_d(s)$

NOTATIONS AND ABBREVIATIONS

Throughout the thesis, the notation $\Sigma\left[A, B, C, D\right]$ (or $\Sigma\left[A, B, C\right]$ when $D = 0$) will be used to denote the state-space equations of a multivariable system given by

$$\Sigma : \begin{array}{l} \dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) \\ \mathbf{y}(t) = C\ \mathbf{x}(t) + D\ \mathbf{u}(t) \end{array}$$

When we refer to a multivariable system with disturbances, we shall use the notation $\Sigma_d\left[A, B, C, D, E, F\right]$ (or $\Sigma_d\left[A, B, C, E\right]$ when $D = 0$ and $F = 0$) to describe the state-space equations given as

$$\Sigma_{\mathbf{d}} : \begin{array}{l} \dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) + E\ \mathbf{d}(t) \\ \mathbf{y}(t) = C\ \mathbf{x}(t) + D\ \mathbf{u}(t) + F\ \mathbf{d}(t) \end{array}$$

The notation $\Sigma_d\left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ will be used to denote a system $\Sigma_d\left[A, B, C, E\right]$ after performing column compressions on the output matrix. This notation is also used to denote a higher order system obtained by incorporating a dynamic output feedback compensator at the outputs of the system $\Sigma_d\left[A, B, C, D, E, F\right]$.

In the definitions for zeros of system $\Sigma_d\left[A, B, C, D, E, F\right]$, we will use the following abbreviations:

$Z_o^D\left\{\Sigma_d\left[A, B, C, D, E, F\right]\right\}$ for the set of open-loop disturbance zeros of system $\Sigma_d\left[A, B, C, D, E, F\right]$

$Z_o^B\left\{\sum_d \left[A, B, C, D, E, F\right]\right\}$ for the set of open-loop disturbance blocking zeros of system $\sum_d \left[A, B, C, D, E, F\right]$.

$Z_c^B\left\{\sum_d \left[A, B, C, E, D, E, F\right]\Big|_{K_2}\right\}$ for the set of closed-loop disturbance blocking zeros of system $\sum_d \left[A, B, C, D, E, F\right]$ which are affected by state feedback K_2.

All vectors are denoted by lower-case bold letters.

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CHAPTER I

INTRODUCTION

1.1 THE NATURE OF DISTURBANCES

Most realistic control systems operate in environments where persistent external disturbances are present, which may degrade their performance under certain operating conditions. These disturbances can roughly be classified as noise-type disturbances or waveform-structured disturbances. The former require a statistical description and are studied in stochastic control theory [1]. In some cases the disturbances may be treated by considering them to be equivalent to plant initial conditions of the system state vectors. In most practical situations, disturbances cannot be treated in this way, especially for the cases where the disturbances take the form of persistently acting fluctuating forces, torques, voltages, etc, which are no longer equivalent to state initial conditions of the system. Disturbance inputs can arise from a number of different sources. For example, in regulator-type control problems, the unknown external disturbances are: frictional forces, disturbance torques, disturbance voltages and the external loads on the system which may fluctuate from time to time giving rise to disturbance terms in the control equations. When considering the automatic guidance and control of ships, aircrafts, rockets, etc. one frequently encounters external disturbing forces acting on the system due to the action of waves, cross-winds, updrafts, wind gusts, gravity gradients, etc.

The *a priori* knowledge about the nature of disturbances and their characteristics differs from one situation to another. If a disturbance is an accurately known function of time, then it can be considered as a known time-varying part of the plant parameters. In other situations, the disturbance may be totally unknown. The situation for most practical control systems lies in between these two extreme cases. In particular, a designer usually has some information concerning the nature of the expected disturbances. This information may take the form of reliable

statistical data which has been collected from previous tests and experiments or it may appear in the form of some knowledge concerning the possible waveforms, amplitudes, duration, etc. of the disturbance function. In any case, the *a priori* information about the disturbance function is usually not complete enough to allow the design of an appropriate controller to act against this disturbance. Thus, the problem is how to design the most suitable controller using incomplete information about disturbances. From the given information, a mathematical model for the disturbances can be formulated. The design procedure for choosing a suitable control law then depends on this model [2-7].

1.2 AN EFFECTIVE MATHEMATICAL MODEL OF DISTURBANCES

Disturbances $\mathbf{d}(t) = \left[d_1(t), d_2(t), \ldots, d_r(t)\right]$ are by definition, plant inputs, which cannot be manipulated by the designer and are not completely known beforehand. Various schemes have been proposed for mathematically modeling partially unknown functions $\mathbf{d}(t)$ of this type. The traditional approach consists of using classical Fourier series methods on representative experimental recordings of $d_i(t)$ to estimate the discrete harmonic content of the disturbance. Another method consists of treating the disturbance as a random process with *a priori* information about its statistical properties. An excellent bibliography and summary of the status of this subject has been given by Wonham [8]. Practical applications of statistical methods in disturbance problems has so far been severely hampered by computational complexities as well as practical difficulties in obtaining reliable *a priori* statistical data about the expected disturbances. A less common, and conceptually different, way to model a partially unknown function $d_i(t)$ is to give a differential equation which $d_i(t)$ is known to satisfy. This approach, first used by Johnson [3,6] and called disturbance "state" modeling, was based on the idea of characterizing the possible waveform modes of disturbances. It was shown [5] that this approach is effective for the kind of disturbances which consist of any linear combinations of constants, ramps, polynomials in time, exponentials, sinusoids, decaying or growing sinusoids, pulses, etc. encountered in

practical regulator and servomechanism control problems. For instance, consider a scalar disturbances $d(t)$ and assume that it can be mathematically described as

$$d(t) = c_1 f_1(t) + c_2 f_2(t) + \ldots + c_\kappa f_\kappa(t) \tag{1.2.1}$$

where the c_i, $i = 1, 2, \ldots, \kappa$, are unknown constant weighting coefficients that may jump in value every once in a while in a completely unknown manner, and the $f_i(t)$ are completely known linearly independent functions given in the "state" description form:

$$\frac{d^{\eta_i} f_i}{dt^{\eta_i}} + \beta^i{}_{\eta_i} \frac{d^{\eta_i - 1} f_i}{dt^{\eta_i - 1}} + \ldots + \beta^i{}_2 \frac{df_i}{dt} + \beta^i{}_1 f_i = 0 \tag{1.2.2}$$

Let $F_i(s)$ denote the Laplace transform of $f_i(t)$ and assume that $F_i(s)$ is, (or can be closely approximated by) a rational algebraic function of s, i.e.

$$F_i(s) = \frac{p_i(s)}{q_i(s)} \qquad i \in \kappa \tag{1.2.3}$$

where $p_i(s)$, $q_i(s)$ are finite-degree real polynomials in s. Then, taking the Laplace transform of (1.2.1), and collecting terms, gives

$$d(s) = \frac{\hat{P}(s)}{\hat{Q}(s)} \tag{1.2.4}$$

where the numerator polynomial $\hat{P}(s)$ involves the coefficients c_i and $\hat{Q}(s)$ is the monic least common denominator of the set of denominator polynomials $\left[q_1, q_2, \ldots, q_\kappa\right]$ in eqn.(1.2.3). If the right hand side of eqn.(1.2.4) is now viewed as the 'transfer function' of a scalar linear stationary system with a unit impulse forcing function, then the polynomial $\hat{Q}(s)$ represents the characteristic polynomial of that linear dynamic system:

$$\begin{aligned} \hat{Q}(s) &= s^\nu + \alpha_\nu s^{\nu-1} + \ldots + \alpha_2 s + \alpha_1 \\ &= \prod_{j=1}^{\nu} (s - \lambda_j) \end{aligned} \tag{1.2.5}$$

where λ_j, $j = 1, 2, \ldots, \nu$ are the zeros of $\hat{Q}(s)$.

1.3 DISTURBANCES ACCOMMODATION IN CONTROL SYSTEMS

Three basic approaches regarding the disturbance-accommodation problem are possible in control systems: First, one can take the point of view that the effect of disturbances on the plant response is always undesirable. This leads to the rather common attitude that disturbances are ideally accommodated when the control completely counteracts (e.g., cancels out) the effect of the disturbance on the plant response. Suppose it turns out that one cannot achieve exact counteraction of the disturbance, owing to say, the inherent structural properties of the plant. Then one might attempt to design a controller which minimizes the effect of disturbances on the plant response [4] and [9-12]. This represents a second primary attitude to disturbance-accommodation and can take on many different forms, depending on the particular quantity one chooses to minimize. However, in practice, some of disturbance effects on the plant response are not necessarily completely undesirable. In fact some of the actions of disturbances may be useful in accomplishing the primary control task. It has been remarked that the idea of trying to make constructive use of disturbances is not new in science (e.g., the classic idea of harnessing the "free energy" of tides, storms, etc.), but this idea apparently has not been exploited in automatic control theory. This leads to a third primary attitude to disturbance-accommodation in which one choose the controller so as to achieve the primary control task, and, at the same time, make maximum utilization of the potentially useful effects of any disturbances that may be present. Needless to say, the latter requires some finesse in the choice of the controller using modern optimal control theory. This approach was investigated by Johnson [4] where he derives bounded optimal control laws.

1.4 DIFFERENT METHODOLOGIES FOR DISTURBANCE ACCOMMODATION

The disturbance-accommodation problem that has been studied extensively has been con-

cerned with designing a feedback control law which ensures that the effect of some or all disturbances acting on a linear system are completely rejected or reduced to an acceptable level in steady state. References [7, 9-20] provide different approaches to this disturbance-accommodation problem.

In some special kinds of applications, particularly in chemical process control, it turns out that one can directly measure, on-line the instantaneous values of disturbances that are acting on the system, for example the flow-rate variations in chemical reactors. For such cases, one can sometimes employ well-known classical methods based on the 'feedforward principle' to design a satisfactory controller [5]. In the majority of realistic applications, however, it is either not economically feasible or not physically possible to perform direct on-line measurements of the disturbances acting on a system. Our point of interest is to design a feedback controller for the situation when various external disturbances acting on the system are not accessible for direct on-line measurement.

Mathematical problems concerning disturbance rejection controllers have received somewhat more attention than those concerning disturbance minimization. In disturbance minimization problems, the design objective is to design a controller which will minimize, in some sense, the effect of unabsorbable external disturbances acting on the plant. Some techniques which have been previously proposed for disturbance minimization are (i) maximum partial absorption, (ii) norm-minimization, (iii) critical state variable and (iv) indirect disturbance absorption. These are explored in details in several papers by Johnson [2-7].

In the case where disturbances can be considered as being equivalent to plant initial conditions, conventional linear-quadratic regulator and servomechanism theories may be used [21]. An attempt to solve a modified version of the classical linear-quadratic regulator problem in which persistently acting disturbances have been added, leads to physically unrealizable control laws. Applications of the general linear-quadratic, time-varying regulator and servomechanism problem (with disturbances) can be found in the series of research papers by Johnson [2-7,22].

Various approaches based on statistical properties of disturbances have been developed by several researchers e.g. see [8], to accommodate disturbances. These approaches have high computational complexities as well as practical difficulties in obtaining reliable statistical data about the expected disturbances.

As early as 1970, a question was raised about the conditions under which there exists a set of appropriate *robust* control laws that will stabilize and control a multivariable system in a desired manner, in spite of allowable unknown external disturbances and changes in the system parameters. Several different versions of this problem have been formulated and examined using the "Robust Servomechanism Approach" by Davison and his co-worker [15,23,24]. Generally this approach leads to dynamical compensators of high order.

Anderson and Moore[25], Kwakernaak and Sivan [26], developed an approach for designing a proportional-plus-integral feedback controller to completely counter the effect of unmeasurable constant disturbances in the steady state. Then, Smith and Davison [13] derived the necessary and sufficient conditions for the existence of feedforward as well as integral-feedback controllers for multivariable systems with constant disturbances, which may be measurable [13] or unmeasurable [14]. This control was designed such that it possesses desirable properties: if the system matrices are arbitrarily perturbed, output regulation is retained, provided that the system remains stable. Pruess [17] presented a method by replacing the integral feedback by appropriately structured state feedback alone to increase the freedom in the multivariable feedback design procedure. In the geometric approach [18,27,28], a state feedback controller is used to ensure that the disturbances are completely decoupled from the outputs. However, for some systems, it may be impossible to reduce the effect of the disturbances below a certain threshold value. Hence, the disturbance decoupling problem would have no solution in this case. Peterson [19] solved this problem by designing a stabilizing state feedback control which reduces the effect of the disturbances to a prespecified level. The results obtained in [20] are also applicable to a class of H^∞ optimization problems, in which the effect of disturbances is minimized in some sense. The standard H^∞ optimization problem [29] is concerned with constructing a dynamic

feedback compensator to minimize the H^{∞} norm of the transfer function from the disturbance to the output of the system. Thus, the systematic method presented in [20], in constructing a control law is arbitrarily close to the H^{∞} optimum. However, results on the H^{∞} optimization problem [29] are based on solving several algebraic Riccati equations and lead to quite complicated design procedures.

Based on combined eigenvalue/eigenvector assignment, a procedure for synthesizing multivariable controllers is developed in Shah et al. [30] to achieve disturbance localization, i.e. complete "undisturbability" with respect to arbitrary disturbances. But the assumptions and consequences often impose limitations on the applicability of this method. Another technique that was used by Patel et al. [31] for linear multivariable systems described by state-space models $\sum_{d} \left[A, B, C, E\right]$ is based on the transmission properties of "disturbance zeros". This approach shows that using constant state feedback, it is possible to choose closed-loop positions for a certain number of these zeros in order to eliminate the effect of a single-disturbance in the steady state.

1.5 OUTLINE OF THE THESIS

The work described in this thesis is based on the approach of Patel et al. [31]. Algorithms are developed for assigning as many disturbance blocking zeros (d.b.z.'s) as required for multivariable systems denoted by $\sum_{d} \left[A, B, C, E\right]$ and $\sum_{d} \left[A, B, C, D, E, F\right]$. These Algorithms use state feedback controllers (constant or dynamic) to position these zeros at desired locations in the complex plane, such that the class of single as well as multiple exponential type, measurable or unmeasurable are rejected in the steady state. In addition, the resulting closed-loop system is stabilized and possibly meets some transient performance requirements. This is achieved by designing an output feedback controller (constant or dynamic) to assign all the poles at desired locations in the complex plane. Moreover, a new approach for designing "robust"

controllers by means of dynamic output feedback is also developed which stabilizes and controls a multivariable system in a desired manner in spite of allowable unknown external disturbances and changes in the system parameters.

The thesis is organized as follows: In Chapter II, we present different numerical algorithms for solving the eigenvalue assignment (EVA) (pole assignment) problem in multivariable systems by means of state and output feedback. Chapter III contains some preliminary results on factorization of transfer function matrices of multivariable systems described by triples (A, C, E) or 4-tuples (A, C, E, F) that can be used to determine the d.b.z.'s of the systems using the concept of minimal-order inverses. Chapters IV and V are mainly concerned with the use of state feedback laws to assign the required d.b.z.'s of the resulting closed-loop systems at desired locations, such that the specified disturbances are rejected in the steady state. It is also shown that when the system does not have any d.b.z.'s and/or the number of d.b.z.'s is not large enough to achieve steady-state rejection of all the disturbances, we can generalize the results using dynamic state feedback to introduce new d.b.z.'s in the system. In Chapter VI, the transient performance of the closed-loop system obtained after assigning d.b.z.'s, is improved by designing an output feedback controller to position all the system poles at desired locations in the complex plane. It is also shown that, a "robust" controller can be designed by using dynamic output feedback to achieve disturbance rejection in the presence of non-destabilizing perturbations in the system parameters. The numerical performance of all the algorithms presented in this thesis are illustrated by means of numerical examples. Finally some concluding remarks concerning the research described in this thesis and suggestions for future work are given in Chapter VII.

The specific contents of each chapter are outlined next.

Chapter II: *Eigenvalue Assignment in Multivariable Systems*

This chapter introduces the problem of eigenvalue assignment by means of state and output feedback. A brief survey of existing computational methods for solving the problem of EVA by state feedback is given and followed by an outline of a numerically reliable algorithm, that can be considered as the converse of the algebraic eigenvalue problem. The problem of EVA by

output feedback is then stated, and a survey of existing techniques for solving the problem is given. Two algorithms are considered to solve the problem. The first algorithm uses constant gain output feedback and the second use dynamic output feedback. Both algorithms are described as two-stage procedures. The underlying principle of these algorithms is the implicitly shift QR algorithm. However, for the case when the poles of a compensator are prespecified, we propose a new algorithm which is a modification of Seraji's method [32] for the design of dynamic output feedback compensators having low order.

Chapter III: *Multivariable Zeros and Their Properties*

This chapter is concerned with the definitions and the relationships of different types of zeros used in multivariable systems. Some important properties of these zeros are also determined. The factorizing technique developed by Patel [33] for the transfer function matrices of the triples (A, C, E) is extended to 4-tuples (A, C, E, F). Based on these results, the problem of computing the d.b.z.'s for different cases is discussed. This chapter also includes, an approach for choosing the positions of d.b.z.'s in multivariable systems, such that steady-state rejection of all exponential type disturbances is achieved.

Chapter IV: *Assignment of Disturbance Blocking Zeros : Single Disturbance Case*

This chapter introduces the problem of designing state feedback controllers to assign d.b.z.'s and/or introducing more d.b.z.'s for linear multivariable systems that contains a single disturbance. The algorithms presented in this chapter are used to achieve arbitrary d.b.z.'s placement, such that complete disturbance rejection is achieved in the steady state.

Chapter V: *Assignment of Disturbance Blocking Zeros : Multiple Disturbance Case*

This is an extension of the d.b.z.'s assignment problem presented in Chapter IV. The underlying principle of the algorithms developed is the reduction of the multi-disturbance problem to one or more single-disturbance problems. The procedure is sequential in nature, in that for each disturbance, we compute a local state feedback to assign the required number of d.b.z.'s without altering those which have been assigned in the preceding steps.

Chapter VI: *Dynamic Output Feedback in Multivariable Systems*

In this chapter, we show that two alternative approaches can be used for computing dynamic output feedback to stabilize and/or improve the transient performance of the closed-loop system. It is also shown that, dynamic output feedback introduces additional d.b.z.'s between the outputs and the disturbances at the poles of the compensator. This important feature is then used to develop a new approach for designing dynamic output feedback to solve the problem of disturbance rejection as well as pole assignment in multivariable systems. In addition, the resulting dynamic compensator is "robust" in the sense that asymptotic regulation takes place for some or all disturbances acting on the system independent of any non-destabilizing perturbations in the system parameters.

Chapter VII: *Conclusions and Future Work*

The main results and algorithms presented in this thesis are summarized and discussed, also suggestions are made for future work in this area.

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CHAPTER II

EIGENVALUE ASSIGNMENT IN MULTIVARIABLE SYSTEMS

This chapter presents numerical algorithms for solving the eigenvalue assignment (EVA) problem in multivariable systems by means of state and output feedback. These algorithms will be used to assign "disturbance blocking zeros" in Chapters IV and V, and system poles in Chapter VI.

The outline of the chapter is as follows: In Section 2.1, we briefly review some results from linear algebra which will be used extensively in the algorithms described in subsequent sections. The problem of EVA by state feedback in single-input system is stated in Section 2.2, and a survey of existing computational methods for solving the problem is given. Then a numerically reliable algorithm [1] which solves the EVA problem is described briefly. The approach used in this algorithm is based on the numerically stable QR algorithm for finding the eigenvalues of a matrix. Section 2.3 is concerned with solving the problem of the EVA by means of constant gain as well as dynamic output feedback. It contains a brief survey of existing techniques for solving the problem followed by various approaches described as two-step eigenvalues assignment problems. A condition under which the eigenvalue of a controllable and observable system can be assigned arbitrarily close to desired locations in the complex plane by means of constant gain output feedback is that the sum of the number of inputs (m) and the number of outputs (l) is greater than the number of states (n). Based on this condition, Misra and Patel [2] developed accurate and efficient numerical algorithms for EVA by means of constant gain and dynamic output feedback. The constant gain output feedback algorithm is based on the implicitly shifted QR algorithm. However, for the case when the condition ($m + l > n$) is not satisfied, two distinct approaches for the design of dynamic output feedback compensators are considered such that all the poles of the augmented closed-loop system consisting of the compensator and the plant can be positioned arbitrarily in the complex plane. These approaches compute the dynamic compen-

sator in two-steps, the first step to assign a subset of poles and the second to assign the remaining poles while preserving the previously assigned ones. The first approach which was reported in [2.3], is based on reformulation of the problem to one of EVA by constant gain output feedback. In the second (new) approach, we give a modification of Seraji's method [4] for the design of dynamic output feedback compensators with lower order than required by other existing methods. By this approach, the simplicity of the unity-rank compensator design is utilized in the design of a non-unity rank compensator. The proposed approach can be implemented as a two-step method: In the first step, we assign a number of poles by means of constant output feedback using a method based on the implicitly shifted QR algorithm for solving the algebraic eigenvalue problem, while in the second step, the assigned poles are preserved and a number of additional poles are placed using a unity-rank dynamic compensator computed entirely in the frequency domain, by solving a set of linear equations relating the parameters of the compensator to the desired closed-loop characteristic polynomial. The problem of assigning the poles of the compensator has also been considered in this approach to ensure a stable compensator. Finally, the numerical performance of the algorithms is illustrated in Section 2.4 by means of numerical examples.

2.1 REVIEW OF PRELIMINARY RESULTS

In this section, we shall review some relevant results from linear algebra that will be used in the algorithms for solving the EVA problem.

Definition 2.1 [5,6] : A matrix $A \in \mathbb{R}^{n\times n}$ is said to be an upper Hessenberg matrix if $a_{ij} = 0$, $i \geq j+2$, where a_{ij} denotes the element in the ith row and jth column of A. Furthermore, if $a_{i,i-1} \neq 0, i = 2, \ldots, n$, then A is said to be an unreduced upper Hessenberg matrix.

Definition 2.2 [2] : A matrix $A \in \mathbb{R}^{n\times n}$ is said to be in real Schur form (RSF) if it is a quasi-upper triangular matrix with only scalars and 2×2 blocks on the (block) diagonal. Each scalar corresponds to a real eigenvalue and each 2×2 block to a complex-conjugate (c-c) pair of

eigenvalues of A. Any matrix $A \in \mathbb{R}^{n \times n}$ can be reduced to an RSF by means of an orthogonal transformation.

Theorem 2.1 [7,8,1] : Given a controllable single-input, multi-output system

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \mathbf{b}\,u(t) \tag{2.1.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{2.1.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $\mathbf{y}(t) \in \mathbb{R}^l$, the triple $(A, \mathbf{b}, C)$ can be reduced to an upper Hessenberg form (UHF) by applying an orthogonal state coordinate transformation such that

$$T^T A T = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1,n-1} & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2,n-1} & f_{2n} \\ 0 & f_{32} & \cdots & f_{3,n-1} & f_{3n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & f_{n,n-1} \quad f_{nn} \end{bmatrix} \triangleq F \tag{2.1.2}$$

$$T^T \mathbf{b} = \begin{bmatrix} g_1 & 0 \ldots 0 \end{bmatrix}^T \triangleq \mathbf{g} \tag{2.1.3}$$

and

$$C\,T = \begin{bmatrix} \mathbf{h}_1 & \mathbf{h}_2 \ldots \mathbf{h}_n \end{bmatrix} \tag{2.1.4}$$

Comments: The matrix F is an upper Hessenberg matrix. Also, it can be easily shown that F is an unreduced upper Hessenberg matrix and $g_1 \neq 0$ if and only if $(A, \mathbf{b})$ is a controllable pair. It may be noted that if any of the subdiagonal elements $f_{i+1,i}$, $i = 1, 2, \ldots, n-1$, is equal to zero then F become a block upper triangular matrix i.e.

$$F = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \tag{2.1.5}$$

The structure of $\mathbf{g}$ then implies that the mode corresponding to the eigenvalues of F_{22} are uncontrollable. Note that F_{11} is an unreduced upper Hessenberg matrix, and that F_{11} together with the corresponding part of $\mathbf{g}$ forms a controllable pair. Note that, the algorithm for reducing a single-input, multi-output system $(A, \mathbf{b}, C)$ to UHF can be found in [3].

Theorem 2.2 [3,6] : A matrix $A \in \mathbb{R}^{n \times n}$ having linearly independent columns can be written uniquely in the form $A = QR$, where $Q \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $R \in \mathbb{R}^{n \times n}$ is an upper triangular matrix with positive diagonal elements.

2.2 ALGORITHM FOR EIGENVALUE ASSIGNMENT BY STATE FEEDBACK

Consider a linear, time-invariant, single-input system described by its state equation

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + \mathbf{b}\,u(t) \tag{2.2.1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}$. Assume that the pair $(A, \mathbf{b})$ is controllable. It is well known [9] that the eigenvalues of a system can be assigned at any desired locations in the complex plane, subject to complex-conjugate (c-c) pairing, by means of state feedback if and only if the system is controllable. Note that if the system is not controllable, the eigenvalues corresponding to the uncontrollable modes of the system cannot be altered.

The problem that we will consider in this section is to find a $1 \times n$ feedback vector $\mathbf{k}_x$ such that, under the feedback law

$$u(t) = v(t) - \mathbf{k}_x \mathbf{x}(t) \tag{2.2.2}$$

the resulting closed-loop state matrix

$$A_{cl} = A - \mathbf{b}\,\mathbf{k}_x \tag{2.2.3}$$

has n eigenvalues at desired location in the complex plane (symmetric about the real axis). This

problem has been investigated by several researchers and many algorithms already exist for solving the problem e.g., see [1,7] and [9-20]. Some of the conventional techniques require a reduction of the system to a canonical form e.g.[9-12], where the state matrix is in "companion" form. The feedback vector is then determined by comparing the coefficients of the characteristic polynomials of the open-loop and the desired closed-loop systems. But, because of the sensitivity of the roots of a polynomial to perturbations in its coefficients and the numerical ill-conditioning associated with the reduction of a system to its companion canonical form, this approach is numerically unreliable. Some other algorithms such as in [13] require the system to be in transfer function form and use polynomial arithmetic which can cause numerical difficulties. The algorithms presented in [1,7] and [15-20] have attempted to address the numerical issues involved in the EVA problem. In [7], the algorithm is based on the well-known QR algorithm and uses only numerically stable orthogonal transformations. The algorithm in [15,17] reduces the system to a block upper Hessenberg form by means of orthogonal state coordinate transformations. However, the method in [15,17] are not straightforward extensions of the method in [7] and in fact it can be shown that they can lead to floating point overflows or underflows. The algorithms in [19,20] are based on the reduction of the system state matrix to RSF by means of orthogonal state coordinate transformations. If the eigenvalue problem of the state matrix of an open-loop system is ill-conditioned [5,6], the RSF (and hence the computed eigenvalues) obtained can be inaccurate. If the feedback gains are computed using inaccurate values of the open-loop eigenvalues, then on applying feedback, the closed-loop poles can be far from the desired ones. This is, therefore a weak point of this and other algorithms that require knowledge of the open-loop eigenvalues.

Numerically, the EVA problem can be solved using an accurate and efficient approach developed by Patel and Misra [1]. This algorithm can be regarded as the converse of the implicitly QR algorithm for eigenvalue determination. A double step implicit shift is applied to assign two real eigenvalues or a c-c pair of eigenvalues using only real arithmetic.

In computing the eigenvalues of a matrix using the QR algorithm, the shifts converge to the true eigenvalue while in the EVA problem, the shifts are known, being the desired closed-loop

eigenvalues which are denoted by $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. The basic idea of Patel and Misra's algorithm is to use constant state feedback to modify the given state matrix so that it has eigenvalues corresponding to a specified shift.

For sake of completeness, we shall briefly describe this algorithm (denoted by Algorithm 2.1) for assigning the poles of a controllable single-input system $(A, \mathbf{b})$:

Algorithm 2.1: (EVA by Constant State Feedback)

(i) Reduce the pair $(A, \mathbf{b})$ to UHF $(F, \mathbf{g}) = (T^T A T, T^T \mathbf{b})$, by applying an orthogonal state coordinate transformation T; F and $\mathbf{g}$ have the structure shown in (2.1.2) and (2.1.3) respectively, and the matrix F is an unreduced upper Hessenberg matrix.

Comment: Note that, from the structure of F and $\mathbf{g}$ it is clear that the pair $(F, \mathbf{g})$ is controllable if and only if $(A, \mathbf{b})$ is a controllable pair. Also, the eigenvalues of F are the same as the eigenvalues of A. We can therefore assign the eigenvalues of A by carrying out EVA for the pair $(F, \mathbf{g})$.

(ii) Determine a constant state feedback $\bar{\mathbf{k}}_x \in \mathbb{R}^n$ by using the implicitly shifted algorithm of Patel and Misra [1] such that the unreduced upper Hessenberg matrix

$$F_{cl} = F - \mathbf{g}\bar{\mathbf{k}}_x$$

has all its eigenvalues at desired locations in the complex plane. The desired locations in the complex plane, λ_i, $i = 1, 2, \ldots, n$ are assumed to be arranged such that the two terms of every c-c pair of eigenvalues appear consecutively.

Comment: Since $\bar{\mathbf{k}}_x$ is the state feedback vector in the coordinate system $(F, \mathbf{g})$, we can write

$$\mathbf{k}_x = \bar{\mathbf{k}}_x T^T$$

Also, it can be easily shown that if F is an unreduced upper Hessenberg matrix, then

$$F - \mathbf{g}\bar{\mathbf{k}}_x = T^T (A - \mathbf{b}\mathbf{k}_x) T$$

For more details concerning the algorithm and its implementation, the interested reader is referred to [1,3,8]. The algorithm will be illustrated by a numerical example in Section 2.4.

2.3 ALGORITHMS FOR EIGENVALUE ASSIGNMENT BY OUTPUT FEEDBACK

In this section, we will present algorithms for eigenvalue assignment by constant as well as dynamic output feedback. In computing the constant output feedback, use is made of the fact that the closed-loop eigenvalues can "almost" always be assigned arbitrarily close to desired locations in the complex plane, provided the system satisfies the condition $m + l > n$, where m, l and n are respectively, the number of inputs, outputs, and states of the system. We then extend the results to systems which do not meet this condition for arbitrary EVA and, therefore, require dynamic output feedback. This can be done by converting the dynamic output feedback problem to a constant gain output feedback problem for the augmented system. This approach is numerically reliable but it may result in an unstable dynamic compensator which is clearly undesirable. In order to overcome this drawback, we design dynamic output feedback with prespecified poles by developing a new algorithm which is an improvement over Seraji's method [4].

2.3.1 Algorithm for EVA by Constant Gain Output Feedback

Consider a linear time-invariant multivariable system described by its state-space equations

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \tag{2.3.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{2.3.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ and $\mathbf{y}(t) \in \mathbb{R}^l$. We assume that (A, B, C) is a controllable and observable system. It is desired to compute a constant gain output feedback $K_p \in \mathbb{R}^{m \times l}$ defined by the feedback law

$$\mathbf{u}(t) = \mathbf{v}(t) - K_p\,\mathbf{y}(t) \tag{2.3.2}$$

such that the resulting closed-loop state matrix

$$A_{cl} = A - BK_p C \tag{2.3.3}$$

has a desired set of eigenvalues. We shall assume that the desired locations are such that they can be achieved by finite output feedback matrix. In eqn.(2.3.3), if the matrix C is an identity matrix of order n, then the problem is reduced to that of EVA by means of state feedback for multi-input systems. Note that, Algorithm 2.1 for EVA in single-input systems considered in the previous section has also been extended in [1,3] to treat the multi-input case. Eigenvalue assignment by constant output feedback may, therefore be regarded as a general case.

In a controllable and observable single-input system with l outputs, we can always assign l closed-loop eigenvalues at almost any desired locations in the complex plane by means of finite constant gain output feedback, e.g., see [10] and [12].

For EVA by means of constant output feedback, considerable theoretical work has been done and several numerical algorithms are available in the literature. However, many of them are not based on sound numerical analysis principles and therefore their numerical reliability is questionable. The algorithms presented in [9,10,12,21] require the system to be in companion or other canonical form. Such algorithms will invariably incur numerical difficulties because the reduction of system (A, B, C) to a canonical form is a numerically ill-conditioned problem. Algorithms that use the transfer function matrix [9,10,12,21] of the given system will be sensitive to perturbations in the coefficients of the numerator and denominator polynomials of the transfer function matrix.

In order to overcome the drawbacks of the above mentioned methods, the algorithm described in [2,3] was developed. It avoids the use of potentially unstable transformation. This algorithm, which we shall call Algorithm 2.2, uses only orthogonal (numerically stable) state coordinate transformations together with constant output feedback to assign the closed-loop eigenvalues arbitrarily close to desired locations in the complex plane. The problem of EVA using constant output feedback considered in this algorithm (as that using state feedback) is treated as the converse of the algebraic eigenvalue problem, the underlying principle being the QR decomposition of a matrix and the use of implicit shifts.

In order to achieve EVA for a set of desired locations which we denote by $\Lambda = \left\{\lambda_1, \lambda_2, \ldots, \lambda_n\right\}$, we reduce the problem to a two single-input EVA problems: the first to assign $l-1$ eigenvalues and the second to assign the remaining $n-l+1$ eigenvalues while preserving the $l-1$ previously assigned ones. This algorithm is similar to the two step procedure in [22]. We assume that the given system is controllable from any one input, and that if the system is not controllable from one of the inputs, then, we can use a randomly generated matrix $K_r \in \mathbb{R}^{m \times l}$ and column vector $\mathbf{d}_r \in \mathbb{R}^m$ to get a controllable single-input system $(A - BK_r C, B\mathbf{d}_r)$. It can be shown that this single-input system will "almost always" be controllable. The effect of K_r is to make the resulting closed-loop state matrix cyclic, so that if A is already cyclic, then K_r can be chosen as the null matrix. For a controllable pair (A, B) with A cyclic, it can be shown that almost any linear combination of the inputs (via the vector $\mathbf{d}_r$) will result in a controllable pair $(A, B\mathbf{d}_r)$. Having done that, we can easily reduced the EVA problem to two single-input multi-output EVA problems. The first EVA problem is to assign $l-1$ eigenvalues and the second to assign the remaining $n-l+1$ eigenvalues while preserving the $l-1$ previously assigned eigenvalues. The design philosophy of the algorithm for EVA in a single-input, l-output system is based on using only orthogonal similarity transformations together with output feedback gains to assign the desired eigenvalues. Therefore the algorithm has good numerical properties.

Now, for the sake of completeness, we will describe this algorithm considering a controllable and observable triple (A, B, C) defining the state equation of a general multi-input multi-output system. We assume that $m+l>n$ so that arbitrary EVA can almost always be achieved using constant output feedback.

Algorithm 2.2: (EVA by Constant Output Feedback)

Step I: *(Assign the first $l-1$ eigenvalues)*

(i) Obtain a controllable single-input system $(A, \mathbf{b}, C)$, where $\mathbf{b} = B\mathbf{d}_1$.

Comment: The vector $\mathbf{d}_1$ can be generated randomly or chosen as described above. If the matrix is not cyclic, then a randomly generated output feedback K_r should be applied to make the resulting state matrix cyclic.

(ii) Reduce the single-input system $(A, \mathbf{b}, C)$ to UHF and compute the constant gain output feedback using implicit shifts, to assign $l-1$ eigenvalues to desired locations in the complex plane. In terms of the system (A, B, C), the constant output feedback after this step is $K_1 = \mathbf{d}_1\mathbf{k}_1$, where $\mathbf{k}_1$ is the output feedback row vector required to assign the desired $l-1$ eigenvalues for the single-input system $(A, \mathbf{b}, C)$. The compensated system $(A_{l-1}, B_{l-1}, C_{l-1})$, will then be in block upper triangular form, where $A_{l-1} = (A - BK_1C)$ is the state matrix of the system.

Step II: *(Assign the Remaining Eigenvalues)*

(i) Form the dual system, i.e. set $F = (A_{l-1})^T$, $G = (C_{l-1})^T$, and $H = (B_{l-1})^T$. Then partition F, G, and H as

$$F \triangleq \begin{bmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix}, \quad G \triangleq \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H \triangleq = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$$

(ii) Determine $\mathbf{d}_2 \in \mathbb{R}^l$ such that $G_1\mathbf{d}_2 = 0$ and $G_2\mathbf{d}_2 \neq 0$. If (F_{22}, G_2) is a controllable pair go to Step II-(iii), ELSE change Λ and go to Step I.

(iii) Reduce the system $(F_{22}, G_2\mathbf{d}_2, H_2)$ to UHF and compute the constant output feedback using implicit shifts to assign $n-l+1$ eigenvalues for the single-input system $(F_{22}, G_2\mathbf{d}_2, H_2)$. At the end of this step, we get a constant gain output feedback $K_2 = \mathbf{d}_2\mathbf{k}_2$ which preserves the $l-1$ eigenvalues assigned in the first step and assigns the remaining $n-l+1$ eigenvalues, where $\mathbf{k}_2$ is the constant output feedback vector of the system $(F_{22}, G_2\mathbf{d}_2, H_2)$.

(iv) The constant gain output feedback required to position all n eigenvalues for the given system (A, B, C) is then given by

$$K_p = K_r + K_1 + K_2^T$$

where K_r can be set equal to the $m \times l$ null matrix if the state matrix of the given system is cyclic. Note that K_2 is transposed because we used the dual system in the second step.

For more details about the implementation of this algorithm and the development of various shift strategies, see [2,3]. Algorithm 2.2 is illustrated by a numerical example in Section 2.4.

2.3.2 Algorithms for EVA by Dynamic Output Feedback

Consider a system described by eqns.(2.3.1), and the case when the condition $m + l > n$ is not satisfied. It is required to compute a dynamic output compensator such that the poles of the closed-loop system can be positioned arbitrarily close to desired locations in the complex plane.

The problem of designing a dynamic output feedback compensator for pole placement in linear multivariable systems has been considered by several researchers [23-28]. Brasch and Pearson [24] developed a state-space method to solve the problem by using a dynamic compensator of order $r = \min\left[(r_c - 1), (r_o - 1)\right]$, where r_c and r_o are the controllability and observability indices of the system. This represents only an upper bound on the compensator order required for placement of all closed-loop poles. Ahmari and Vacroux [23] have generalized this result to the case of fixed-order compensators. They obtain a lower bound on the number of poles that can be placed arbitrarily for a given order of the compensator.

Other methods developed by Chen and Hsu [25], Patel [27] and Seraji [28] used frequency domain solutions to this problem. The structure of the dynamic compensator considered by these authors is restrictive in that a unity-rank constraint is imposed on the compensator transfer function matrix at the outset. This, in effect, transforms the multi-input, multi-output system to a single-input or a single-output system and thus considerably simplifies the design problem. On the other hand, the unity-rank structure reduces the number of adjustable parameters in the compensator, and consequently results in a compensator of higher order than would otherwise be necessary.

Recently, Misra and Patel [2], develop an accurate and efficient algorithm for computing non-unity rank dynamic compensators to achieve eigenvalue assignment in multivariable systems. Analytically, this algorithm is based on reducing the problem of dynamic output feedback to that of EVA by constant output feedback for an augmented system. We consider a controllable and observable linear multivariable system described by eqns.(2.3.1a,b) with a cyclic state feedback matrix A. We define the dynamic output feedback compensator for EVA by

$$\mathbf{u}(t) = \mathbf{v}(t) - H\,\mathbf{z}(t) - J\,\mathbf{y}(t) \tag{2.3.5a}$$

$$\dot{\mathbf{z}}(t) = F\,\mathbf{z}(t) + G\,\mathbf{y}(t) \tag{2.3.5b}$$

where $\mathbf{z}(t) \in \mathbb{R}^r$ and $\mathbf{v}(t) \in \mathbb{R}^m$ and r is the order of the dynamic compensator that assigns all the eigenvalues of the resulting closed-loop system arbitrarily close to $n + r$ desired locations.

We now find the state equation of the compensated closed-loop system consisting of the system in eqns.(2.3.1a,b) and the compensator in eqns.(2.3.5a,b) as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}(t) \end{bmatrix} = \begin{bmatrix} A - BJC & -BH \\ GC & F \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{v}(t) \tag{2.3.6a}$$

$$\mathbf{y}(t) = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} \tag{2.3.6b}$$

The closed-loop state matrix is given by

$$A_{cl} = \begin{bmatrix} A - BJC & -BH \\ GC & F \end{bmatrix}$$

and can be written as

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & Y \end{bmatrix} - \begin{bmatrix} B & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} J & H \\ G & F-Y \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \tag{2.3.7}$$

In (2.3.7) $Y \in \mathbb{R}^{r \times r}$ is selected such that all its eigenvalues are distinct and different from those

of A and the transmission zeros of $(A, B\mathbf{d}_1, C)$ where $\mathbf{d}_1$ is chosen as discussed in section 2.3.1. From (2.3.7), we note that the same closed-loop state matrix would result if we were to apply constant gain output feedback.

$$\mu = \nu - \begin{bmatrix} J & H \\ G & F-Y \end{bmatrix} \xi$$

to the system $(\hat{A}, \hat{B}, \hat{C})$, where

$$\hat{A} = \begin{bmatrix} A & 0 \\ 0 & Y \end{bmatrix}, \hat{B} = \begin{bmatrix} B & 0 \\ 0 & -I \end{bmatrix}, \hat{C} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}$$

It should be noted that the matrices H, J, G, Y, and F define the required dynamic output feedback completely. The order of the dynamic compensator (r) can be determined such that $m+l+2r > n+r$, i.e., $r > n-m-l$.

From the above result, it is clear that the dynamic compensator design problem is thus reduced to an equivalent constant output feedback design problem and Algorithm 2.2 considered in Section 2.2.1 can now be applied to a higher order system. The resulting algorithm which we shall call Algorithm 2.3, can be summarized as follows:

Algorithm 2.3: (EVA using Dynamic Output Feedback)

(i) For a controllable and observable system (A, B, C) with a cyclic state matrix A, to design a dynamic compensator (F, G, H, J), we first form the auxiliary system $(\hat{A}, \hat{B}, \hat{C})$ by selecting Y such that its eigenvalues are distinct and different from those of A.

Comment: For the case when A is not cyclic, so that the given system is not controllable from one input or a linear combinations of inputs, we can use the procedure discussed in Section 2.2.1, to make $(A - BK_rC, B\mathbf{d}_r, C)$ controllable with cyclic state matrix $(A - BK_rC)$.

(ii) Select the vector $\hat{\mathbf{d}} \in \mathbb{R}^m$ such that $(\hat{A}, \hat{B}\hat{\mathbf{d}})$ is a controllable pair. The vector $\hat{\mathbf{d}}$ has the

form $[\hat{\mathbf{d}}_1^T, \hat{\mathbf{d}}_2^T]^T$ where $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ are chosen such that $(A, B\hat{\mathbf{d}}_1)$ and $(Y, -\hat{\mathbf{d}}_2)$ are controllable pairs.

Comments: From the structure of $\hat{\mathbf{d}}$ and Y, we can easily find the vector $\hat{\mathbf{d}}$ such that $(\hat{A}, \hat{B}\hat{\mathbf{d}})$ is a controllable pair. Since the matrix Y can be selected such that its eigenvalues are distinct and different from those of A, then $\hat{A}$ will be cyclic if A is cyclic. Then the pair $(\hat{A}, \hat{B}\hat{\mathbf{d}})$ will be controllable if $(A, B\hat{\mathbf{d}}_1)$ is controllable. The structure of $\hat{C}$ ensures that $(\hat{A}, \hat{C})$ is an observable pair if (A, C) is an observable pair.

(iii) Apply Algorithm 2.2 to accomplish the EVA for $(\hat{A}, \hat{B}, \hat{C})$ and get the matrices H, J, G and F that define the required dynamic output feedback.

This Algorithm is numerically reliable but it may result in an unstable dynamic compensator which is clearly undesirable in practical systems.

Another method, presented by Seraji [4], uses a transfer function approach for the design of non-unity rank dynamic compensators to achieve eigenvalue assignment in multivariable systems. The compensator constructed by this method is a sum of a constant output feedback and a unity-rank dynamic output feedback i.e.

$$G_r(s) = K_c + \mathbf{k}\,\mathbf{g}(s) \tag{2.3.8}$$

The constant output feedback $K_c \in \mathbb{R}^{m\times l}$ is applied initially to assign a number of poles at desired locations. The unity-rank dynamic compensator $\mathbf{k}\,\mathbf{g}(s)$, where $\mathbf{k}$ and $\mathbf{g}(s)$ are $m\times 1$ and $1\times l$ vectors, is applied subsequently such that the assigned poles are preserved and a number of additional poles are placed at desired positions. By this method, the resulting dynamic compensator required for arbitrary placement of all closed-loop eigenvalues has lower order given by $r_{\min} = \left\lceil (n-m-l+1)/\max(l,m) \right\rceil$. This technique reduces the state matrix of the system to a "companion" form and then determines the required output feedback (constant and unity-rank compensator) by comparing the coefficients of the characteristic polynomials of the open and the augmented closed-loop systems. However, for the case when the roots of the open-loop characteristic polynomial are sensitive to perturbations in its coefficients, this approach is numerically

unreliable and may lead to unsatisfactory performance. Also the transfer function approach can lead to numerical difficulties, especially when the given system description is in state-space form, and therefore has to be converted to its transfer function form. In this method, all parameters of the compensator including the coefficients of its characteristic polynomial which determine its stability are calculated so as to position the remaining $n + r - l + 1$ eigenvalues of the augmented system at desired locations. This design procedure has no restrictions imposed on it to ensure the stability of the compensator. It is therefore possible that a compensator which achieves arbitrary pole placement in the closed-loop system is unstable, although the resulting system is stable.

In order to case the above mentioned difficulties, we will present a new algorithm for the more accurate calculation of the dynamic output feedback compensator $G_r(s)$ parameters with prespecified poles having the same structure as that given by Seraji [4], such that the resulted closed-loop poles can be positioned arbitrarily close to desired locations in the complex plane. The proposed algorithm is carried out in two step methods: In the first step, we improve the numerical performance of Seraji's method [4] for computing the constant output feedback by applying Algorithm 2.2 to a single-input system and assigning $l-1$ eigenvalues. In the second step, the assigned $l-1$ poles are preserved by reducing the multivariable system to a partially uncontrollable single-input system, and a number of additional poles $(n+r-l+1)$ poles are placed by designing unity-rank dynamic output feedback having prespecified poles. The preservation of the poles is achieved by using the fact that in a multivariable system with distinct poles, the numerator of the transfer function matrix has a unity-rank at the system poles. The approach used for computing a unity-rank dynamic compensator with prespecified poles is carried out entirely in the frequency domain using the approach developed by Patel [27]. The key step in the design is the formation of a set of linear equations relating the parameters of the compensator to the desired closed-loop characteristic polynomial. By this method, the designer has complete freedom in the choice of the compensator poles and uses only the numerator parameters of the compensator transfer function to assign the remaining poles of the augmented system. Thus, by using this approach for assigning all the closed-loop poles at desired locations, we can ensure

that the required dynamic output feedback compensator is stable.

We now describe the algorithm to solve the EVA problem by means of a non-unity rank dynamic output feedback compensator with prespecified poles.

Algorithm 2.4: (EVA using Dynamic Output Feedback Compensator with Prespecified Poles)

Step I: *(Assign the first $l-1$ eigenvalues)*

In this step, the unity-rank $m \times l$ constant output feedback matrix $K_c = \mathbf{p}\,\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are $m \times 1$ and $1 \times l$ vectors respectively, is determined so as to place $l-1$ poles of the system (A, B, C) at distinct specified locations denoted by $\lambda_1, \lambda_2, \ldots, \lambda_{l-1}$. The matrix A is either cyclic or is made cyclic by an initial application of an arbitrary output feedback matrix as discussed before in Section 2.2, and the matrices B and C have full rank m and l respectively. The vector $\mathbf{p}$ is specified arbitrarily such that the resulting single-input system $(A, B\mathbf{p}, C)$ is controllable and observable. We shall assume without any loss of generality that the triple $(A, \mathbf{b}, C)$ is in UHF, where $\mathbf{b} = B\mathbf{p}$. Moreover, since the pair $(A, \mathbf{b})$ is controllable, the state matrix A is an unreduced upper Hessenberg matrix. The problem of EVA by means of constant output feedback is to determine a vector $\mathbf{q} \in \mathbb{R}^l$ such that the closed-loop state matrix

$$A_1 = A - \mathbf{b}\,\mathbf{q}\,C \tag{2.3.9}$$

has $l-1$ eigenvalues at desired locations in the complex plane. The vector $\mathbf{q}$ can be computed by applying Algorithm 2.2. At the end of this step, the resulting closed-loop system is (A_1, B, C), where A_1 has $l-1$ poles at $\lambda_1, \lambda_2, \ldots, \lambda_{l-1}$.

Comment: Note that the feedback matrix K_c is not unique since $\mathbf{p}$ is specified arbitrarily such that the pair $(A, B\mathbf{p})$ is controllable.

Step II: *(Assign the Remaining Eigenvalues)*

In this step, the unity rank $m \times l$ dynamic compensator $G_c(s) = \mathbf{k}\,\mathbf{g}(s)$ is applied to the system (A_1, B, C), where $\mathbf{k}$ is $m \times 1$ constant vector and

$$\mathbf{g}(s) = \frac{\beta(s)}{\alpha(s)} = \frac{\beta_r s^r + \beta_{r-1} s^{r-1} + \ldots + \beta_0}{s^r + \alpha_{r-1} s^{r-1} + \ldots + \alpha_0} \tag{2.3.10}$$

is a $1 \times l$ rational function vector of degree r. This structure for $G_c(s)$ effectively reduces the multivariable problem to the design of the compensator $\mathbf{g}(s)$ for the single-input system $(A_1, B\mathbf{k}, C)$, and results in the closed-loop characteristic polynomial given by [4]

$$H(s) = D(s)\alpha(s) + \beta(s)\mathbf{w}(s) \tag{2.3.11}$$

where,

$$D(s) = \det\left[sI - A_1\right] = s^n + d_{n-1}s^{n-1} + \ldots + d_0$$

$$\mathbf{w}(s) = W(s)\mathbf{k} = \mathbf{w}_{n-1}s^{n-1} + \mathbf{w}_{n-2}s^{n-2} + \ldots + \mathbf{w}_0$$

and

$$W(s) = C\ adj\left[sI - A_1\right]B$$

The vector $\mathbf{k}$ is used to preserve the $l-1$ poles of (A_1, B, C) at $\lambda_1, \lambda_2, \ldots, \lambda_{l-1}$ in the closed-loop system. In order to preserve the poles at $\lambda_1, \lambda_2, \ldots, \lambda_{l-1}$ irrespective of $\mathbf{g}(s)$, from eqn.(2.3.11), we require [4]

$$W(\lambda_i)\mathbf{k} = 0 \qquad i = 1, 2, \ldots, l-1 \tag{2.3.12}$$

since the matrices $adj(\lambda_i I - A_1)$, for $i = 1, 2, \ldots, l-1$, have rank one [4], each $W(\lambda_i)$ contains only one independent row, denoted by ω_i. Thus the vector $\mathbf{k}$ is found from the $l-1$ linear equations

$$\omega_i \mathbf{k} = 0 \qquad i = 1, 2, \ldots, l-1 \tag{2.3.13}$$

It is noted that the required $\mathbf{k}$ makes the single-input system $(A_1, B\mathbf{k}, C)$ partially uncontrollable through pole-zero cancellations at $s = \lambda_1, \lambda_2, \ldots, \lambda_{l-1}$ in the transfer function vector $W(s)\mathbf{k}/D(s)$, and subsequently the uncontrollable poles remain invariant under the compensator $\mathbf{g}(s)$. Once $\mathbf{k}$ is found, the number of closed-loop poles in the single-input system $(A_1, B\mathbf{k}, C)$ that can be placed arbitrarily by the rth order compensator $\mathbf{g}(s)$ is given by [4,27]

$$\nu_1 = r + rank \left\{ R_c^{\,T} \left[C^T, A_1 C^T, \ldots, (A_1^{\,T})^r C^T \right] \right\} \tag{2.3.14}$$

where $R_c = rank \left[B\mathbf{k}, A_1 B\mathbf{k}, \ldots, A_1^{\,n-1} B\mathbf{k} \right]$ is the controllability matrix of the partially uncontrollable single-input system $(A_1, B\mathbf{k}, C)$ and has rank = $(n-l+1)$ [27].

If we select r such that

$$rank \left[C^T, A_1 C^T, \ldots, (A_1^{\,T})^r C^T \right] = n-l+1$$

Then, the closed-loop poles that can be assigned by an rth order compensator is

$$\nu_1 = r + n - l + 1$$

From eqn.(2.3.11), the closed-loop characteristic polynomial can be factored as

$$H(s) = \left[\prod_{i=1}^{l-1} (s - \lambda_i) \right] \hat{H}(s)$$

$$= \left[\prod_{i=1}^{l-1} (s - \lambda_i) \right] \left[\hat{D}(s)\alpha(s) + \beta(s)\hat{\mathbf{w}}(s) \right] \tag{2.3.15}$$

with

$$\hat{D}(s) = \frac{D(s)}{\prod_{i=1}^{l-1} (s - \lambda_i)}$$

and

$$\hat{\mathbf{w}}(s) = \frac{\mathbf{w}(s)}{\prod_{i=1}^{l-1} (s - \lambda_i)}$$

where $\hat{H}(s)$, $\hat{D}(s)$ and $\hat{\mathbf{w}}(s)$ are polynomials because $\lambda_1, \lambda_2, \ldots, \lambda_{l-1}$ are roots of $H(s)$. Thus, the part of the closed-loop characteristic polynomial which is affected by the dynamic unity-rank

compensator is

$$\hat{H}(s) = \hat{D}(s)\,\alpha(s) + \beta(s)\,\hat{w}(s) \tag{2.3.16}$$

and has a degree $\nu_1 = \left[n+r-l+1\right]$.

Now, in order to place ν_1 additional closed-loop poles at the desired locations $\lambda_l, \ldots, \lambda_{\nu_1+l-1}$ using only the numerator parameters of the compensator transfer function $\mathbf{g}(s)$ we need to solve eqn.(2.3.16) for $\beta(t)$. The solution of eqn.(2.3.16) can be obtained by following the procedure developed by Patel [27]:

Eqn.(2.3.16) can be written as

$$\hat{H}(s) = \alpha(s)\,\hat{D}(s) + \beta(s)\,\hat{w}(s) \tag{2.3.17}$$

Note that, $\hat{H}(s)$ is a polynomial of degree $\hat{n}+r$, where $\hat{n} = n-l+1$, and can be written as

$$\hat{H}(s) = h_{\hat{n}+r}\, s^{\hat{n}+r} + h_{\hat{n}+r-1}\, s^{\hat{n}+r-1} + \ldots + h_0 \tag{2.3.18}$$

Similarly, we write $\hat{D}(s)$ and $\hat{w}(s)$ as

$$\hat{D}(s) = s^{\hat{n}} + d_{\hat{n}-1} s^{\hat{n}-1} + \ldots + d_0$$

$$\hat{w}(s) = \hat{w}_{\hat{n}-1} s^{\hat{n}-1} + \hat{w}_{\hat{n}-2} s^{\hat{n}-2} + \ldots + \hat{w}_0$$

To find the solution of (2.3.17), we first express it as a set of linear equations in terms of the coefficients of the numerator and denominator polynomials of $\mathbf{g}(s)$. Equating coefficients of like powers of s on both sides of (2.3.17) yields the set of linear equations

$$\Phi\,\xi = \hat{\mathbf{h}} \tag{2.3.19}$$

where Φ is a matrix constructed from the coefficients of $\hat{D}(s)$ and $\hat{w}(s)$, the vector ξ contains the unknown coefficients of $\alpha(s)$ and $\beta(s)$ defining $\mathbf{g}(s)$, and the vector $\hat{\mathbf{h}}$ contains the coefficients of the polynomial vector $\hat{H}(s)$ together with the coefficients of the difference of the polynomials $\hat{H}(s)$ and $\hat{D}(s)$. The elements of Φ, ξ and $\hat{\mathbf{h}}$ are shown explicitly in (2.3.20) for the case where a dynamic feedback compensator of degree r is used.

$$
\begin{array}{c}
\hat{n}\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\
r\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right.
\end{array}
\underbrace{\left[\begin{array}{cccc}
d_0 & 0 & \cdots & 0 \\
d_1 & d_0 & \cdots & 0 \\
\vdots & \vdots & & d_0 \\
d_{\hat{n}-1} & d_{\hat{n}-2} & \cdots & d_{\hat{n}-r} \\
\hline
1 & d_{\hat{n}-1} & \cdots & d_{\hat{n}-r+1} \\
0 & 1 & & \vdots \\
\vdots & \vdots & & d_{\hat{n}-1} \\
0 & 0 & \cdots & 1
\end{array}\right.}_{r}
\underbrace{\left.\begin{array}{c|c|c|c|c}
 & 0 & & 0 & 0 \\
 & & & \vdots & \vdots \\
L & & \ddots & & \\
 & L & & & \\
\hline
0 & & & L & \\
 & 0 & & & L \\
\vdots & \vdots & & & \\
0 & 0 & & 0 &
\end{array}\right]}_{l(r+1)}
\left[\begin{array}{c}
\alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{r-1} \\ \hline \beta_0^t \\ \hline \beta_1^t \\ \hline \vdots \\ \hline \beta_{r-1}^t \\ \hline \beta_r^t
\end{array}\right]
=
\left[\begin{array}{c}
h_0 \\ h_1 \\ \vdots \\ h_{r-1} \\ h_r - d_0 \\ h_{r+1} - d_1 \\ \vdots \\ h_{r+\hat{n}-1} - d_{\hat{n}-1}
\end{array}\right]
\tag{2.3.20}
$$

$$\phi\xi = \mathfrak{h}$$

In (2.3.20), 0 is $1\times l$ null vector and L is the $\hat{n}\times l$ matrix formed from the coefficients of the polynomial vector $\hat{\mathbf{w}}(s)$ as

$$\mathbf{L} = \begin{bmatrix} \hat{\mathbf{w}}_0^{\,T} \\ \hat{\mathbf{w}}_1^{\,T} \\ \cdot \\ \cdot \\ \hat{\mathbf{w}}_{\hat{n}-1}^{\,T} \end{bmatrix}$$

Note that the matrix Φ in eqn.(2.3.19) is an $\left[(\hat{n}+r)\times\left[r+l(r+1)\right]\right]$ matrix. Hence for (2.3.19) to have a solution for ξ for any value of the vector $\hat{\mathbf{h}}$ we require that $rank(\Phi) = \hat{n}+r$ which also implies that $\left[r+l(r+1)\right]\geq\hat{n}+r$.

If the matrix Φ is square and of full rank, (2.3.19) has a unique solution for ξ given by

$$\xi = \Phi^{-1}\hat{\mathbf{h}} \tag{2.3.21}$$

However, in general Φ is a rectangular matrix with more columns than rows and if it has full rank, then eqn.(2.3.19) has an infinite number of exact solutions for ξ. This implies that there exists an infinite number of unity-rank compensators with the same order, which achieve arbitrary EVA.

Using the concept of a matrix pseudo-inverse, the general "exact" solution of (2.3.20) is given by [29]

$$\xi = \Phi^{+}\hat{\mathbf{h}} + (I-\Phi^{+}\Phi)\,\mathbf{z} \tag{2.3.22}$$

where the $\left[r+l(r+1)\times\hat{n}+r\right]$ matrix Φ^{+} is a pseudo inverse of Φ and $\mathbf{z}$ is an arbitrary $\left[r+l(r+1)\right]$ column vector. Since Φ has a full rank, Φ^{+} is given by $\Phi^{+} = \Phi^{T}(\Phi\Phi^{T})^{-1}$.

Solving (2.3.20) for any desired $\hat{H}(s)$ results in a dynamic compensator $\mathbf{g}(s)$ i.e. both $\alpha(s)$ and $\beta(s)$ are found to achieve arbitrary eigenvalue assignment.

In order to ensure that the required dynamic compensator is stable and its poles are at prespecified locations, we use only the numerator polynomial vector of the compensator to assign the poles of the augmented system. Eqn.(2.3.22) can be written in the form

$$\xi = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \hat{\mathbf{h}} + \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \mathbf{z} \tag{2.3.23}$$

i.e

$$\hat{\alpha} = T_1 \hat{\mathbf{h}} + S_1 \mathbf{z} \tag{2.3.24}$$

and

$$\hat{\beta} = T_2 \hat{\mathbf{h}} + S_2 \mathbf{z} \tag{2.3.25}$$

where

$$\hat{\alpha} = \left[\alpha_0 \, \alpha_1 \ldots \alpha_{r-1}\right]^T, \hat{\beta} = \left[\beta_0 \, \beta_1 \ldots \beta_r\right]^T, \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \Phi^+$$

and

$$\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \left[I - \Phi^+ \Phi \right]$$

The poles of the compensator can be specified through the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{r-1}$ of the denominator polynomial $\alpha(s)$ of the compensator. Therefore, from eqn.(2.3.24), we can obtain a set of r equations in the elements of the vector $\mathbf{z}$ as

$$S_1 \mathbf{z} = (\hat{\alpha} - T_1 \hat{\mathbf{h}}) \tag{2.3.26}$$

A necessary and sufficient condition for (2.3.26) to be consistent [29] is that

$$S_1 S_1^+ (\hat{\alpha} - T_1 \hat{\mathbf{h}}) = \hat{\alpha} - T_1 \hat{\mathbf{h}} \tag{2.3.27}$$

where S_1^+ is a pseudo inverse of S_1. When the condition in (2.3.27) is satisfied, the solution of

(2.3.26) is given by [29]

$$\mathbf{z}_0 = S_1^{+}(\hat{\boldsymbol{\alpha}} - T_1\hat{\mathbf{h}}) + (I - S_1^{+}S_1)\boldsymbol{\eta} \tag{2.3.28}$$

where $\boldsymbol{\eta}$ is an arbitrary $\left[r + l(r+1)\right]$ column vector. All the solutions given by (2.3.28) achieve the desired poles for the compensator. For simplicity, we choose the solution corresponding to $\boldsymbol{\eta} = 0$. This gives

$$\mathbf{z}_0 = S_1^{+}(\hat{\boldsymbol{\alpha}} - T_1\hat{\mathbf{h}}) \tag{2.3.29}$$

Then the numerator dynamics of the resulting compensator are given by

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= T_2\hat{\mathbf{h}} + S_2\mathbf{z}_0 \\ &= T_2\hat{\mathbf{h}} + S_2S_1^{+}(\hat{\boldsymbol{\alpha}} - T_1\hat{\mathbf{h}})\end{aligned} \tag{2.3.30}$$

i.e

$$\hat{\boldsymbol{\beta}} = (T_2 - S_2S_1^{+}T_1)\hat{\mathbf{h}} + S_2S_1^{+}\hat{\boldsymbol{\alpha}}$$

Therefore, all parameters of the compensator including the coefficients of its characteristic polynomial $\hat{\boldsymbol{\alpha}}$ which determine its stability are determined so as to position ν_1 poles of the augmented system at desired locations.

The compensator required for eigenvalue assignment in the multivariable system (A, B, C) is then given by

$$\begin{aligned}G_r(s) &= K_c + \mathbf{k}\,\mathbf{g}(s) \\ &= \frac{(K_c + \mathbf{k}\boldsymbol{\beta}_r)s^r + (\alpha_r K_c + \mathbf{k}\boldsymbol{\beta}_{r-1})s^{r-1} + \ldots + (\alpha_0 K_c + \mathbf{k}\boldsymbol{\beta}_0)}{s^r + \alpha_{r-1}s^{r-1} + \ldots + \alpha_0}\end{aligned} \tag{2.3.31}$$

and the matrix $G_c(s)$ has rank >1. The total number of closed-loop poles that can be assigned arbitrarily by means of $G_c(s)$ is [4]

$$\rho = (l-1) + r + rank\left\{R_c^T\left[C^T, A_1^TC^T, \ldots, (A_1^T)^rC^T\right]\right\} \tag{2.3.32}$$

We note that, from [9,30] that

$$rank\ \left[C^T, A_1^T C^T, \ldots, (A_1^T)^r C^T\right] = rank\ \left[C^T, A^T C^T, \ldots, (A^T)^r C^T\right]$$

Then from (2.3.32) by using Sylvester's inequality we obtain [4]

$$\Delta + r \le \rho \le \gamma - 1 + r \tag{2.3.33}$$

where

$$\gamma = l + \min(\Delta, n-l+1)$$

and

$$\Delta = rank\left[C^T, A^T C^T, \ldots, (A^T)^r C^T\right]$$

Eqn.(2.3.33) gives bounds on the number of poles that can be assigned by an rth order compensator using this method.

Finally, the lowest order of the compensator for arbitrary placement of all $n+r$ closed-loop poles by this method is given [4] by

$$r_{\min} = \min r \mid rank\ \left[R_c^T\ \left[C^T, A_1^T C^T, \ldots, (A_1^T)^r C^T\right]\right] = n-l+1 \tag{2.3.34}$$

From (2.3.34), a lower bound r_l and an upper bound r_u on $r_{\min}$ are given by

$$r_l = \min r \mid rank\left[C^T, A^T C^T, \ldots, (A^T)^r C^T\right] = n-l+1$$

$$r_u = \min r \mid rank\left[C^T, A^T C^T, \ldots, (A^T)^r C^T\right] = n$$

$$= (r_o - 1)$$

where r_o is the observability index of the system (A, B, C). Hence the minimum order of the compensator is bounded by

$$r_l \le r_{\min} \le (r_o - 1)$$

From (2.3.34), it follow that $r_l \ge (n-m-l+1)/l$.

A few remarks are now required in order to clarify certain points regarding the implementation and properties of the proposed algorithm.

Remark 2.1: When $m \geq l$, by considering the dual system (A_1^T, C^T, B^T), we can show that the upper bound on the minimum degree of compensator required to achieve arbitrary pole placement in the single-input system $(A_1^T, C^T\mathrm{k}, B^T)$ is equal to $r_c - 1$ where r_c is the controllability index of (A_1, B) and is defined as the smallest integer r such that the rank $\left[B, A_1B, \ldots, A_1^rB\right] = n$. Hence we get the result obtained by Brasch and Pearson [24] that for a controllable and observable multivariable system (A_1, B, C) a compensator of degree $r = \min\left[(r_o - 1), (r_c - 1)\right]$ is sufficient to achieve arbitrary pole placement. In general, the value of $r_{\min}$ obtained using this method is smaller than $\min\left[(r_o - 1), (r_c - 1)\right]$ and is bound by

$$r_l \leq r_{\min} \leq \min\left[(r_o - 1), (r_c - 1)\right]$$

A lower bound r_l is given by $(r_l \geq (n - m - l + 1)/m)$.

Remark 2.2: The results of step two depends on the fact that the condition of eqn.(2.3.27) must be satisfied in order for eqn.(2.3.26) to have a solution for $\mathbf{z}$. It may be observed that when the matrix S_1 has full (row) rank, say $\bar{r}$ $(\geq r_l)$, the condition of eqn.(2.3.27) is satisfied and therefore, in this particular case, we can specify all the poles of the compensator arbitrarily in addition to the poles of the closed-loop system using a compensator of order $\bar{r}$. If for some chosen compensator order $\bar{r}$, the matrix S_1 does not have full rank, then the condition of eqn.(2.3.27) may not be satisfied, in which case it would be necessary to use a compensator of order $> \bar{r}$. This may then provide the extra degree of freedom required to assign all the compensator poles in addition to those of the closed-loop system. It is useful to note that the only constraint on r for arbitrary closed-loop pole assignment is that the matrix Φ in (2.3.19) has full rank $\hat{n}+r$, and this constraint is satisfied for all $r \geq r_l$.

Remark 2.3: The method for computing the unity-rank dynamic compensator considered in the

second step uses the transfer function matrix of the system (A_1, B, C). Computation of a transfer function matrix from a state-space description is a potential source of numerical rounding errors. There are a number of good numerical methods available for computing transfer function matrices from state-space description [31]. The computation of the dynamic compensator via the characteristic polynomials approach may cause more serious numerical difficulties resulting in unsatisfactory performance of the algorithm, especially for high order systems or systems with badly conditioned data. However, we had to resort to this approach to assign the poles of the compensator in addition to the overall closed-loop poles. To our knowledge, solving this problem in a numerically reliable way remains an unsolved problem..

2.4 NUMERICAL EXAMPLES

In this section, we illustrate the performance of the algorithms described in this chapter by means of some numerical examples. The computations were performed on a VAX 11/780 . The desired closed-loop eigenvalues have been selected for the purpose of illustration and performance evaluation of the algorithms and not to meet any specific design criteria.

Example 2.1: This example illustrates the use of Algorithm 2.1 for EVA by means of state feedback. The system considered is a $5th$ -order model given in [32]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.1094 & 0.0628 & 0.0 & 0.0 & 0.0 \\ 1.306 & -2.132 & 0.9807 & 0.0 & 0.0 \\ 0.0 & 1.595 & -3.149 & 1.547 & 0.0 \\ 0.0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0.0 & 0.00227 & 0.0 & 0.1636 & -0.6125 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 & 0.0 \\ 0.0632 & 0.0 \\ 0.0838 & -0.1396 \\ 0.1004 & -0.206 \\ 0.063 & -0.0128 \end{bmatrix} \mathbf{u}(t) \tag{2.4.1}$$

It is required to assign the eigenvalues by using either of the two inputs, such that the closed-loop poles are positioned at -0.2, -0.5 , -1.0 and $-1.0 \pm j$. Note that, the system in eqn.(2.4.1) is

controllable with respect to both inputs separately, and the state matrix A is cyclic.

In solving the EVA problem with respect to each input by applying Algorithm 2.1, we need to find the constant state feedback vector $\mathbf{k}_x^{u_i}$, $i = 1,2$ to modify the given state matrix, so that $A_{cl} = A - \mathbf{b}_i \mathbf{k}_x^{u_i}$ has the desired eigenvalues. The corresponding state feedback gains obtained by Algorithm 2.1 are as follows

$$\mathbf{k}_x^{u_1} = \begin{bmatrix} 74.5894 & -200.9530 & 282.6692 & -177.2991 & 40.234 \end{bmatrix}$$

and

$$\mathbf{k}_x^{u_2} = \begin{bmatrix} -23.0594 & 76.601 & -170.8379 & 150.7487 & -50.4210 \end{bmatrix}$$

Example 2.2 : This example illustrates the use of Algorithm 2.2 for EVA by constant gain output feedback. The example is a linearized model of the $5th$-order double-effect evaporator model given in [32]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0.0 & -0.00156 & 0.0 & 0.0 & 0.0 \\ 0.0 & -0.1419 & 0.1711 & -2.0 & -1.0 \\ 0.0 & -0.00875 & -1.102 & 0.0 & 0.0 \\ 0.0 & -0.00128 & -0.1489 & 0.0 & 0.00013 \\ 0.0 & 0.0605 & 0.1489 & 0.0 & -0.059 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 & -0.143 & 0.0 \\ 0.0 & 0.0 & 0.0 \\ 0.392 & 0.0 & 0.0 \\ 0.0 & 0.108 & -0.592 \\ 0.0 & -0.0486 & 0.0 \end{bmatrix} \mathbf{u}(t) \tag{2.4.2a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \tag{2.4.2b}$$

It is required to assign the closed-loop eigenvalues at $-50 \pm j10$, -9.0, -8.0 and -7.0. Note that, the system in eqns.(2.4.2a,b) is controllable and observable with A a cyclic matrix.

By assigning the first $l-1$ (=2) poles at -50 ± j10, using Step I in Algorithm 2.2, we obtained a constant gain output feedback matrix K_1 as given in Table 2.1a.

In order to preserve these poles and further assign the remaining poles at -9.0, -8.0 and -7.0, we require a constant gain output feedback matrix K_2^T, to be computed using Step II. This was obtained as in Table 2.1b. Thus, the constant gain output feedback $K_p = K_1 + K_2^T$ required for EVA is given in Table 2.1c, and the closed-loop poles are obtained as desired.

Example 2.3 : We have selected this example to illustrate Algorithms 2.3 and 2.4 for EVA by means of dynamic output feedback. The system being considered is an unstable $5th$-order model given by the following state space equations [4]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{u}(t) \tag{2.4.3a}$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x} \tag{2.4.3b}$$

It is required to design a dynamic compensator of minimum order to place the closed-loop poles arbitrarily, for example at -1 , -1 ,-2 and -1 ± j .

In order to solve the EVA problem by applying Algorithm 2.3, we need to design a dynamic output feedback of order two. This compensator can be computed by the augmented system $(\hat{A}, \hat{B}, \hat{C})$ and by selecting a matrix $Y \in \mathbb{R}^{2\times 2}$ with its eigenvalues located at -4.0 and -5.0 , respectively . Then, we applied Algorithm 2.2 to accomplish EVA in the augmented system $(\hat{A}, \hat{B}, \hat{C})$, and get the constant gain output feedback matrix $\hat{K}_p$. The constant matrix

$\hat{K}_p = \begin{bmatrix} J & H \\ G & F \end{bmatrix}$ defines the parameters of the dynamic compensator and is given in Table 2.2. This assigns all the closed-loop poles at the desired values as shown in Table 2.3.

To solve the EVA problem by Algorithm 2.4, so that the resulting dynamic output feedback compensator is stable and to have prespecified poles , we need a dynamic output feedback of the minimum order bounded by $1 \le r_{\min} \le 2$. In fact, if we use Algorithm 2.4, we require at least a second-order compensator to assign all the poles of the closed-loop system at the desired values in addition to the poles of the compensator e.g. at -1, -1. This compensator can be computed in two steps as follows: In the first step, we assign one pole at -1, by computing a constant gain output feedback

$$K_c = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

In the second step, we assign the remaining closed-loop poles and preserve the the one at -1, by computing a unity-rank dynamic compensator of degree two with prespecified poles located at (-1, -1). This gives

$$\mathbf{k}\,\mathbf{g}(s) = \frac{1}{(s+1.0)^2} \begin{bmatrix} -5.0\,s^2 - 25.0\,s + 4.0 & 59.0\,s^2 + 10.0\,s + 47.0 \\ 5.0\,s^2 + 25.0\,s - 4.0 & -59.0\,s^2 - 10.0\,s - 47.0 \end{bmatrix}$$

Thus, the required dynamic output feedback compensator for the given system is

$$G_r(s) = \frac{1}{(s+1.0)^2} \begin{bmatrix} -4.0\,s^2 - 23.0\,s + 5.0 & 59.0\,s^2 + 10.0\,s + 47.0 \\ 6.0\,s^2 + 27.0\,s - 3.0 & -59.0\,s^2 - 10.0\,s - 47.0 \end{bmatrix}$$

and the closed-loop poles are positioned at desired locations.

2.5 CONCLUDING REMARKS

In this chapter, we have described four numerical algorithms for solving the EVA (pole assignment) problem in linear multivariable systems. Algorithm 2.1 was used to solve the EVA

problem for single-input, multi-output systems by means of state feedback. This algorithm has good numerical properties which are similar to those of the implicitly shift QR algorithm for eigenvalue determination. In the eigenvalue computation problem, the shifts converge to the true eigenvalues while in the eigenvalue assignment problem, the shifts are known a priori, being the desired closed-loop eigenvalues. The basic idea is to use constant state feedback to modify the given state matrix so that it has an eigenvalue corresponding to the specified shift. The algorithm uses only orthogonal transformations together with state feedback to assign the desired eigenvalues and is therefore numerically robust.

Algorithms 2.2 and 2.3 were used to solve the EVA problem by means of constant and dynamic output feedback. The problem was treated by generalization of the state feedback EVA problem. We have also proposed a new approach for EVA using dynamic output feedback compensators (with lower order) which have prespecified poles. The design procedure of this approach uses Algorithm 2.4 to compute a dynamic compensator which has the same structure as that used by Seraji [22]. This algorithm is carried out into two step. In the first step, we assign $l-1$ eigenvalues by means of constant output feedback using a numerical reliable algorithm which is based on the QR decomposition of a matrix and implicit shifts. In the second step, we preserve the previously assigned eigenvalues and assign additional $(n+r-l+1)$ eigenvalues using unity-rank dynamic output feedback computed entirely in the frequency domain. The minimum order ($r_{\min}$) of the resulting compensator required for arbitrary placement of the all closed-loop poles is bounded by:

$$(n-m-l+1)/\max(l,m) \leq r_{\min} \leq \min\left[(r_o-1),(r_c-1)\right]$$

It was shown that the price paid for the prespecification of the dynamic compensator poles is a reduction in the number of adjustable parameters of the unity-rank compensator. This may result in higher order for the dynamic compensator. Also the use of a transfer function approach may cause numerical difficulties due to the inherent ill-conditioning associated with working with polynomials and transfer functions.

$$\begin{bmatrix} 2.542312254408E+04 & -5.2933984621732E+06 & 5.9580875988278E+06 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

(a) Constant Output Feedback Matrix K_1

$$\begin{bmatrix} -3.2256195788E+11 & -6.7540483777E+08 & 7.7631373778E+08 \\ -8.5872336792E+02 & -1.7980605053E+00 & 2.0666984222E+00 \\ 4.1745699796E+03 & 8.7410331282E+00 & -1.0046987703E+01 \end{bmatrix}$$

(b) Constant Output Feedback matrix K_2^T

$$\begin{bmatrix} -3.2256193246E+11 & -6.8069823622E+08 & 7.8227182538E+08 \\ -8.5872336792E+02 & -1.7980605053E+00 & 2.0666984222E+00 \\ 4.1745699796E+03 & 8.7410331282E+00 & -1.0046987703E+01 \end{bmatrix}$$

(c) Constant Output Feedback matrix K_p

Table 2.1 Constant output feedback matrices for Example 2.2.

$$F = \begin{bmatrix} -3.1875E-01 & -4.8617E+00 \\ 1.0776E+00 & -7.2430E+00 \end{bmatrix}$$

$$J = \begin{bmatrix} 1.7766E+01 & 5.8371E+00 \\ 4.3821E-01 & -1.1410E+00 \end{bmatrix}$$

$$G = \begin{bmatrix} -3.4827E+01 & -2.4889E+01 \\ -2.9643E+01 & -2.3826E+01 \end{bmatrix}$$

$$H = \begin{bmatrix} 2.1806E+00 & -2.4111E+00 \\ -3.7194E-01 & -1.4118E-01 \end{bmatrix}$$

Table 2.2. Parameters of the dynamic output feedback compensator for Example 2.3.

Desired c-l e.v.'s	Computed c-l e.v.'s
-1.0 + j	-1.0000 + 1.0000 j
-1.0 - j	-1.0000 - 1.0000 j
-1.0	-1.0003 + 2.5858 E-7 j
-1.0	-1.0003 - 2.5858 E-7 j
-1.0	-1.0000 + 3.0449 E-4 j
-1.0	-1.0000 - 3.0449 E-4 j
-2.0	-2.0000

Table 2.3 Desired and computed closed-loop eigenvalues for Example 2.3.

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CHAPTER III

MULTIVARIABLE ZEROS AND THEIR PROPERTIES

This chapter provides a comprehensive treatment of the concepts of zeros in a linear multivariable system represented by a state-space model $\sum\left[A, B, C, D\right]$ (or $\sum\left[A, B, C\right]$, when $D=0$). Various definitions and properties of zeros - decoupling zeros, transmission zeros, invariant zeros, system zeros and blocking zeros are given, and the relationships between them are discussed. Then, the zeros of the "disturbance transfer function matrix" i.e. that relating the outputs to the disturbances (called 'disturbance zeros') of invertible system represented by a state-space model $\sum_d\left[A, B, C, D, E, F\right]$ (or $\sum_d\left[A, B, C, E\right]$) are defined. From these definitions, a systematic procedure is developed for computing the 'disturbance blocking zeros' of a large class systems via the concept of minimal order system inverses. In Chapters IV and V, these results are used to assign these disturbance blocking zeros at suitable locations in the complex plane, such that the effect of a class of disturbances at the outputs is eliminated in the steady state.

The layout of the chapter is as follows. In Section 3.1 several types of zeros in multivariable system as discussed by various researchers, are summarized. A technique for factorizing the disturbance transfer function matrices of state-space systems is described, and some of the properties of the factorization are presented in Section 3.2, while Section 3.3 shows the relationship between disturbance blocking zero positions and rejection of disturbances at the outputs of the system. Numerical examples to illustrate the main results of this chapter are given in Section 3.4, and finally in Section 3.5, we discuss the results presented in this chapter.

3.1 ZEROS OF MULTIVARIABLE SYSTEMS

The concept of zeros of a linear, time-invariant multivariable system, which are not necessarily zeros of individual elements of the transfer function matrix, has received considerable attention during the past two decades [1-9]. The significance of these zeros has been shown by the important role they play in several aspects of control system theory and design, e.g. optimal control [10], system responses [6,7], model matching [11-12], decoupling theory [3], convolution coding [13], regulator synthesis [14] and disturbance rejection [15]. The occurrence of these zeros can be thought of as a consequence of the structural nature of multi-input, multi-output systems. These zeros have been defined in several, mostly equivalent ways, by different researchers [1-9].

3.1.1 Definition of Zeros

First, we recall the definition of the zeros of a system described by the state-space equations

$$\dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) \tag{3.1.1a}$$

$$\mathbf{y}(t) = C\ \mathbf{x}(t) + D\ \mathbf{u}(t) \tag{3.1.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the vector of state variables, $\mathbf{u}(t) \in \mathbb{R}^m$ is the vector of control inputs $(m<n)$, $\mathbf{y}(t) \in \mathbb{R}^l$ is the vector of outputs $(l<n)$, and A, B, C and D are matrices of appropriate dimensions with $rank(C) = l$, $rank(B) = m$ and $rank(D) = \min(l, m)$.

The transfer function matrix relating the outputs to the control inputs is given by

$$W_u{}^o(s) = C\ \left[sI_n - A\right]^{-1} B + D \tag{3.1.2}$$

where I_n is $n \times n$ identity matrix.

The matrix

$$\mathrm{P}(s) = \begin{bmatrix} sI_n - A & B \\ -C & D \end{bmatrix} \tag{3.1.3}$$

plays a key role in the study of the zeros associated with linear dynamical systems and is called the system matrix [5].

The difference and the relationships between the two matrix-valued functions $W_u^{\,o}(s)$ and $P(s)$ can be summarized as follows: The matrix $W_u^{\,o}(s)$ gives a description of the way in which the system appears to its environment, and can be thought of as an external description of the system. The matrix $P(s)$ exhibits the structure associated with the state-space model, and can be thought of as internal description of the system. The matrix $P(s)$ is the one used most in studying the way in which frequency response and state-space methods are inter-related. The matrix $W_u^{\,o}(s)$ is a matrix-valued rational function of s whereas $P(s)$ is a matrix-valued polynomial function of s; this introduces important technical differences in the way in which the respective matrices are handled. The matrix $P(s)$ conveys more information about the system than $W_u^{\,o}(s)$, which represents only the controllable and observable subsystem [16] associated with the system defined by eqn.(3.1.2). The difference between the two matrices results in a difference in the sets of zeros defined via them. In general a larger set of zeros is defined via $P(s)$ than via $W_u^{\,o}(s)$; if the system is controllable and observable, then both sets of zeros coincide.

There are in the current literature five important types of zeros of multivariable system. They are defined as:

(i) **Decoupling Zeros** (Z_{id} , Z_{od} , Z_{iod}) : For a system described by a system matrix $P(s)$, the input-decoupling zeros (Z_{id}) are the values of s for which the matrix $\left[sI_n - A, B\right]$ is rank deficient. The output-decoupling zeros (Z_{od}) are the values of s for which the matrix $\left[sI_n - A^T, C^T\right]$ is rank deficient. The input output-decoupling zeros (Z_{iod}) are defined as those values of s for which both $\left[sI_n - A, B\right]$ and $\left[sI_n - A^T, C^T\right]$ are rank deficient. Input-decoupling zeros and output-decoupling zeros correspond to the uncontrollable and unobservable modes respectively in the state-space description. They do not appear in the transfer function matrix of the system. These zeros can be computed efficiently by reducing the system to block

upper and lower Hessenberg forms [39].

(ii) **Transmission Zeros** (Z_t) : These are defined in terms of Smith-McMillan form of the transfer function matrix of the system [5,19-21]. A system described by an $l \times m$ transfer function matrix $W_u{}^o(s)$ can be transformed by unimodular transformations to its Smith-McMillan form [5,19-21] $M(s)$ i.e.

$$M(s) = U(s)\, W_u{}^o(s)\, V(s)$$

$$= \begin{bmatrix} diag\left\{\alpha_i(s)/\beta_i(s)\right\} \\ O_{l-m,m} \end{bmatrix} \qquad , l>m$$

or

$$= diag\left[\alpha_i(s)/\beta_i(s)\right] \qquad , l = m$$

or

$$= \left[diag\left\{\alpha_i(s)/\beta_i(s)\right\} \; O_{l,m-l}\right] \quad , l < m$$

where $\alpha_i(s)$ and $\beta_i(s), i = 1, 2, \ldots, \min(l,m)$ are relatively prime and $\alpha_i(s)$ divides $\alpha_{i+1}(s)$ and $\beta_{i+1}(s)$ divides $\beta_i(s)$. In all cases, the transmission zeros of the system or of $W_u{}^o(s)$ are the roots of all the numerator polynomials $\alpha_i(s)$. The roots of all the denominator polynomials $\beta_i(s)$ are the poles of the system. This method can be quite involved as far as determination of transmission zeros is concerned, and a more direct approach via the minors of the transfer function matrix is proposed as an alternative by MacFarlane and Karcanias[16]. These transfer function based approaches are not recommended for computing transmission zeros.

(iii) **Invariant Zeros** (Z_i) : For a system described by a system matrix P(s), the invariant zeros are defined as those (complex) values of s for which $rank\left[\mathrm{P}(s)\right] < n + min(l, m)$. The invariant zeros of general non-square systems comprise the transmission zeros and some of the

decoupling zeros.

(iv) **System Zeros** (Z_s) : This set consists of all the transmission and decoupling zeros of the system. They are defined via a suitable set of minors of $P(s)$, that includes the set used to define the invariant zeros.

(v) **Blocking Zeros** (Z_b) : A scalar $\lambda \in C$ where $\lambda \notin \sigma(A)$, is a "blocking zeros" [23] of the system $\sum\left[A, B, C, D\right]$, with A a cyclic matrix if

$$C\left[\lambda I_n - A\right]^{-1} B + D = 0$$

A scaler $\lambda \in C$ where $\lambda \in \sigma(A)$ is a blocking zero of $\sum\left[A, B, C, D\right]$ with A a cyclic matrix if

$$C\ adj\left[\lambda I_n - A\right] B = 0$$

where $adj(\bullet)$ denotes the adjoint of the matrix($\bullet$).

Some authors define zeros without the need to form the corresponding transfer function matrix or calculate the McMillan form. Kouvaritakis and MacFarlane [24], use a geometric state-space approach for square systems, Bengtsson [2] gives a more general formulation by constructing the characteristic polynomial of minimal order inverse using a geometric approach while Desoer and Schulman[6], Wolovich [7] and Davison and Wang [9] define the zeros in terms of frequencies at which $W_u^{\ o}(s)$ and $P(s)$ lose rank. The definitions of zeros presented by Desoer and Schulman [6] and Wolovich [7] are given in terms of a coprime matrix fraction description of a matrix $W_u^{\ o}(s)$, into the product of a polynomial matrix and the inverse of another polynomial matrix.

Several researchers have developed methods for computing zeros - most notably transmission zeros. The algorithm developed by Davison and Wang [9] for determining transmission

zeros is applicable for general linear, time-invariant, multivariable systems represented by state-space equations. This algorithm has the advantage that it can deal with most types of systems. However, it does not give invariant zeros which are not transmission zeros and, in its final stage, the method involves the computation of eigenvalues of a matrix of dimension greater than the order of the system. Sinswat et al [25] developed an algorithm in the state-space for computing invariant zeros and transmission zeros of invertible systems with any number of inputs and outputs that has advantages over the Davison and Wang method [9]. In this algorithm the zeros are obtained from the eigenvalues of a matrix, derived from the system matrix, of order not larger than that of the system. In [26-29], the method of computing transmission zeros is equivalent to solving a "generalized eigenvalue problem" [30-33], for which the numerically reliable QZ algorithm can be used [34-36]. This approach is conceptually simple and is considerably superior numerically to computing transmission zeros via the Smith-McMillan form. Finally, Emami and Dooren [27] developed a complete computer package for computing various types of zeros of an arbitrary state-space systems. This package is also applicable to non-square and/or degenerate systems. The algorithm used by these authors is a modified version of Silverman's structure algorithm.

3.1.2 Relationship Between Various Zeros

The relationship between the different types of zeros in multivariable systems which we have denoted by $Z_t, Z_i, Z_s, Z_{id}, Z_{od}$ and Z_b can be summarized as follows. Since the set of decoupling zeros are cancelled by the poles associated with the uncontrollable and/or unobservable modes, they do not appear as zeros in the transfer function matrix $W_u^{\,o}(s)$. The set of invariant zeros, defined via $P(s)$, may differ from the set of transmission zeros, defined via $W_u^{\,o}(s)$, by the presence of a sub-set of decoupling zeros. The set of system zeros differs from the set of transmission zeros in that the former includes all the decoupling zeros of the system. This corresponds to the situation when the given system is not controllable and/or observable. When a system has more inputs than outputs, $P(s)$ loses rank at an input-decoupling zero. In

such circumstances the input-decoupling zero involved is also an invariant zero. If the system has more outputs than inputs, then P(s) loses rank at an output-decoupling zero. In such cases the output decoupling zero involved is also an invariant zero. When a system is controllable and observable, the set of transmission zeros, define via $W_u^{\,o}(s)$, and the set of the invariant zeros, defined via P(s), coincide.

The relationships between system zeros, invariant zeros and transmission zeros can be summarized as:

$$Z_t \subseteq Z_i \subseteq Z_s$$

The relationship between blocking zeros and various zeros in a given state-space description of a multivariable system is given by Patel [23] as follows:

Let $\lambda \in Z_b$, the set of blocking zeros of $\sum\left[A, B, C, D\right]$. Then

(a) $\lambda \in Z_t$ if $\lambda \notin \sigma(A)$

(b) $\lambda \in Z_i$ *but* $\lambda \notin Z_t$ if $\lambda \in Z_{id}$ and $l \leq m$

or $\lambda \in Z_{od}$ and $l \geq m$

(c) $\lambda \in Z_s$ *but* $\lambda \notin Z_i$ if $\lambda \in Z_{id}$ and $l > m$

or $\lambda \in Z_{od}$ and $l < m$

Note that (a) implies that every blocking zero of $\sum\left[A, B, C, D\right]$ which is not cancelled by an eigenvalue of A is also a transmission zero of the system.

3.1.3 Properties of Transmission Zeros

3.1.3.1 Structural properties

(i) Transmission zeros are unaffected by feedback, by non-singular cascade controllers (Davison and Wang [9]) and under duality.

(ii) For a square system, if the elements of an output feedback matrix are allowed to approach infinite values, a number of closed-loop poles equal to the number of transmission zeros will approach the locations of these zeros while the rest of the closed-loop poles will tend to infinity (Kouvaritakis [4]).

3.1.3.2 Some properties in relation to the dynamic behaviour

(i) A system having one of its transmission zeros in the right half of the s-plane is difficult to control [2,5]. Such a system is said to be non-minimum phase in the multivariable sense.

(ii) If α is a transmission zero of the transfer function matrix $W_u^{\,o}(s)$, then there exists an $m \times 1$ vector β such that $W_u^{\,o}(\alpha)\,\beta = 0$ [6,16,38]. This means that if a system has a transmission zeros at $s=\alpha$, there exists a (complex) vector $\beta \neq 0$, such that the system blocks transmission of an input of the form $\beta \exp(\alpha t)$ in the steady state.

3.1.4 Disturbance Zeros

Let us now consider the system denoted by $\sum_d \left[A, B, C, D, E, F\right]$ and having the form

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + E\,\mathbf{d}(t) \tag{3.1.4a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) + F\,\mathbf{d}(t) \tag{3.1.4b}$$

where $\mathbf{d}(t) \in \mathbb{R}^r$ is the vector of some arbitrary disturbances acting on the system which may or may not be measurable, and E and F are constant matrices of appropriate dimensions, with $rank\left[E\right] = r$ and $rank\left[F\right] = \min(l, r)$. If $D = 0$ and $F = 0$, the system in eqns.(3.1.4a,b) is denoted by $\sum_d \left[A, B, C, E\right]$.

The transfer matrix relating the outputs to the disturbances is given by

$$W_d{}^o(s) = C\left[sI_n - A\right]^{-1} E + F \tag{3.1.5}$$

where I_n is the $n \times n$ identity matrix.

We can define the zeros of $W_d{}^o(s)$ in an analogous way to the zeros of $W_u{}^o(s)$. However, the "transmission" zeros of $W_d{}^o(s)$ (which we shall call 'disturbance zeros' (d.z.'s)) are not invariant under state feedback to control inputs. Thus, we can use this property to assign a subset of these disturbance zeros - the 'disturbance blocking zeros' (d.b.z.'s) - to appropriate positions in the complex plane, such that the effect of disturbances of the form $\beta \exp(\alpha t)$ for all $r \times 1$ vectors β and scalars α, is rejected at the outputs in steady state. The choice of suitable positions for these d.b.z.'s follows from their transmission properties which are analogous to transmission properties of transmission zeros [25]. This will be discussed further in Section 3.3.

In the next two sections, we will present a systematic procedure for computing the sets of d.b.z.'s for the systems described by $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$ using the concept of minimal order right or left inverses (whichever exist).

3.2 FACTORIZATION OF THE TRANSFER FUNCTION MATRIX

In this section, we present a factorization procedure for the transfer function matrix $W_d{}^o(s)$. This will enable us to compute the d.b.z.'s for both $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$.

3.2.1 Disturbance Blocking Zeros of the System $\sum_d \left[A, B, C, E\right]$

Consider a linear time-invariant, multivariable system described by eqn.(3.1.4a-b) with D and F equal to zero. Assume that $\sum_d \left[A, B, C, E\right]$ is non-degenerate i.e it has a finite number of transmission zeros and disturbance zeros. We also assume that the $l \times n$ matrix C has full rank l. Then, by means of a state coordinate transformation, the columns of the matrix C can be "compressed" as

$$C \rightarrow \left[\hat{C}_1 \;\; 0\right]$$

where $\hat{C}_1 \in \mathbb{R}^{l \times l}$.

This can be done by using the singular value decomposition (SVD) or the QR factorization (via Householder transformations). If we use the SVD, there exists $l \times l$ and $n \times n$ orthogonal matrices M and T such that

$$MCT^T = \left[\Sigma_l \;\; 0\right]$$

where $\Sigma_l = diag\,(\sigma_1, \sigma_2, \ldots, \sigma_l)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l > 0$, σ_i, $i = 1, 2, \ldots, l$ being the non-zero singular values of C. The SVD of a matrix has been studied extensively in the numerical analysis literature e.g. see [33]. The important advantages of using orthogonal transformations based on SVD has been discussed by Patel [39] and its use leads to efficient, numerically stable and accurate column compression. Therefore, the coordinate transformation on the state vector is defined as

$$\hat{\mathbf{x}}(t) = T\mathbf{x}(t)$$

Thus, on performing column compressions on C, the system $\sum_d \left[A, B, C, E\right]$ is transformed

to

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\,\hat{\mathbf{x}}(t) + \hat{B}\,\mathbf{u}(t) + \hat{E}\,\mathbf{d}(t) \tag{3.2.1a}$$

$$\mathbf{y}(t) = \hat{C}\,\hat{\mathbf{x}}(t) \tag{3.2.1b}$$

where

$$\hat{A} = TAT^T, \hat{B} = TB, \hat{E} = TE \text{ and } \hat{C} = CT^T = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$$

with $\hat{C}_1 = M^T\,\Sigma_l$. It must be pointed out that, since T is nonsingular, the transfer function matrix $W_d{}^o(s)$ is invariant under coordinate transformations with T [40].

Partitioning $\hat{A}$ and $\hat{E}$ in the form

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \quad \text{and} \quad \hat{E} = \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix},$$

the transfer function matrix $W_d{}^o(s)$ can be written as

$$W_d{}^o(s) = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \begin{bmatrix} sI_l - \hat{A}_{11} & -\hat{A}_{12} \\ -\hat{A}_{21} & sI_{n-l} - \hat{A}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} \tag{3.2.2}$$

From the relation

$$\begin{bmatrix} sI_l - \hat{A}_{11} & -\hat{A}_{12} \\ -\hat{A}_{21} & sI_{n-l} - \hat{A}_{22} \end{bmatrix} \begin{bmatrix} I_l & 0 \\ \left[sI_{n-l} - \hat{A}_{22}\right]^{-1} \hat{A}_{21} & I_{n-l} \end{bmatrix}$$

$$= \begin{bmatrix} sI_l - \hat{A}_{11} - \hat{A}_{12}\left[sI_{n-l} - \hat{A}_{22}\right]^{-1} \hat{A}_{21} & -\hat{A}_{12} \\ 0 & sI_{n-l} - \hat{A}_{22} \end{bmatrix},$$

we have

$$\begin{bmatrix} sI_l - \hat{A}_{11} & -\hat{A}_{12} \\ -\hat{A}_{21} & sI_{n-l} - \hat{A}_{22} \end{bmatrix}^{-1} = \begin{bmatrix} I_l & 0 \\ \left[sI_{n-l} - \hat{A}_{22}\right]^{-1} \hat{A}_{21} & I_{n-l} \end{bmatrix} \begin{bmatrix} Q(s) & -\hat{A}_{12} \\ 0 & sI_{n-l} - \hat{A}_{22} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} I_l & 0 \\ \left[sI_{n-l}-\hat{A}_{22}\right]^{-1}\hat{A}_{21} & I_{n-l} \end{bmatrix} \begin{bmatrix} Q^{-1}(s) & Q^{-1}(s)\hat{A}_{12}\left[sI_{n-l}-\hat{A}_{22}\right]^{-1} \\ 0 & \left[sI_{n-l}-\hat{A}_{22}\right] \end{bmatrix} \tag{3.2.3}$$

where we have substituted $Q(s)$ for $\left[sI_l-\hat{A}_{11}-\hat{A}_{12}\left[sI_{n-l}-\hat{A}_{22}\right]^{-1}\hat{A}_{21}\right]$.

Using (3.2.3) in (3.2.2) and simplifying yields the transfer function matrix $W_d{}^o(s)$ in the form

$$W_d{}^o(s) = \left[Q_d{}^o(s)\right]^{-1} P_d{}^o(s) \tag{3.2.4}$$

where $Q_d{}^o(s)$ and $P_d{}^o(s)$ are $l \times l$ and $l \times r$ rational function matrices respectively, and are given by

$$Q_d{}^o(s) = \left[\left[sI_l-\hat{A}_{11}\right]-\hat{A}_{12}\left[sI_{n-l}-\hat{A}_{22}\right]^{-1}\hat{A}_{21}\right]\hat{C}_1^{-1} \tag{3.2.5}$$

$$P_d{}^o(s) = \left[\hat{E}_1+\hat{A}_{12}\left[sI_{n-l}-\hat{A}_{22}\right]^{-1}\hat{E}_2\right] \tag{3.2.6}$$

From eqn.(3.2.6), it can be seen that $P_d{}^o(s)$ is the transfer function matrix of the $(n-l)$th-order, r-input, l-output system described by the 4-tuple $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ i.e. of the system

$$\dot{\xi} = \hat{A}_{22}\xi + \hat{E}_2\mu \tag{3.2.7a}$$
$$\upsilon = \hat{A}_{12}\xi + \hat{E}_1\mu \tag{3.2.7b}$$

Now, we discuss some properties of the system (3.2.7a-b) which may not necessarily be of minimal order, and present some results concerning the d.z.'s of the system $\sum_d\left[A, B, C, E\right]$.

To do this we first make the following assumptions:

Assumptions

In all that follows, we shall assume that the transfer function matrix $W_d{}^o(s)$ has full *rank* $= \min(l, r)$, where the rank of $W_d{}^o(s)$ is defined as the order of its largest minor which is

not identically zero. This assumption implies that the system $\sum_d \left[A, B, C, E\right]$ is invertible. Consequently from eqn.(3.2.6), it follows that $P_d^{\,o}(s)$ has full rank (= $\min(l,r)$), which in turn implies that the system in eqns.(3.2.7a-b) is invertible.

We also assume that the $l \times r$ matrix $\hat{E}_1$ has full rank equal to $\min(l, r)$. Since

$$CE = CT^T TE$$

$$= \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} = \hat{C}_1 \hat{E}_1$$

and $rank\left[\hat{C}_1\right] = l$, this assumption is equivalent to having $rank\left[CE\right] = \min(l, r)$.

Theorem 3.1: If the system $\sum_d \left[A, B, C, E\right]$ is observable, then the pair $\left[\hat{A}_{12}, \hat{A}_{22}\right]$ is observable.

Proof: If the pair (C, A) is observable, then the pair $(\hat{C}, \hat{A})$ is observable. Now, the pair $(\hat{C}, \hat{A})$ is observable if and only if the rank of the matrix

$$\Gamma\left[\lambda\right] = \begin{bmatrix} \hat{A} - \lambda I_n \\ \hat{C} \end{bmatrix}$$

has full rank $= n$ for all complex values of λ [5].

The above matrix Γ can also be written as

$$\Gamma\left[\lambda\right] = \begin{bmatrix} \hat{A}_{11} - \lambda I_l & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} - \lambda I_{n-l} \\ \hat{C}_1 & 0 \end{bmatrix} \tag{3.2.8}$$

which implies that

$$rank\ \Gamma\left[\lambda\right] = rank\left[\hat{C}_1\right] + rank\begin{bmatrix}\hat{A}_{12}\\ \hat{A}_{22}-\lambda I_{n-l}\end{bmatrix}$$

Since the rank $\left[\hat{C}_1\right] = l$. Then the above condition holds if and only if

$$rank\begin{bmatrix}\hat{A}_{12}\\ \hat{A}_{22}-\lambda I_{n-l}\end{bmatrix} = n-l$$

for all λ, which is the condition for the pair $\left[\hat{A}_{12}, \hat{A}_{22}\right]$ to be observable, completing the proof.

The system $\sum_d \left[A, B, C, E\right]$ with $rank\left[W_d{}^o(s)\right] = \min(l, r)$ is right invertible for $l<r$ and left invertible for $l>r$, for $l=r$ right and left inverses are the same as the ordinary inverse. A right inverse of $W_d{}^o(s)$ satisfies the relationship

$$W_d{}^o(s)\, W_d{}^R(s) = I_l \tag{3.2.9}$$

i.e.

$$W_d{}^R(s) = \left[W_d{}^o(s)\right]^T \left[W_d{}^o(s)\left[W_d{}^o(s)\right]^T\right]^{-1} \tag{3.2.10}$$

and a left inverse satisfies the relationship

$$W_d{}^L(s) W_d{}^o(s) = I_r \tag{3.2.11}$$

i.e.

$$W_d{}^L(s) = \left[\left[W_d{}^o(s)\right]^T W_d{}^o(s)\right]^{-1}\left[W_d{}^o(s)\right]^T \tag{3.2.12}$$

Note that in general $W_d{}^R(s)$ and $W_d{}^L(s)$ satisfying eqns.(3.2.9) and (3.2.11) respectively are not unique [20].

Next, we define a minimal right (left) inverse of $W_d^{\,o}(s)$ as that $W_d^{\,R}(s)\left[W_d^{\,L}(s)\right]$ which has a characteristic polynomial of lowest degree. A minimal order inverse is also not necessarily unique [20]. We shall denote a minimal order right (left) inverse of $W_d^{\,o}(s)$ by $W_d^{\,R_m}(s)\left[W_d^{\,L_m}(s)\right]$. The d.z.'s of $W_d^{\,o}(s)$ can then be defined as follows:

Definition 3.1: The d.z.'s of $W_d^{\,o}(s)$ are the poles of $W_d^{\,R_m}(s)$ for $l \le r$ and of $W_d^{\,L_m}(s)$ for $l \ge r$.

Remark 3.1: It can be shown [2,40] that the characteristic polynomial of a minimal order right (left) inverse divides the characteristic polynomials of all other right (left) inverses. In other words if a complex number κ is a pole of a minimal order right (left) inverse then it is also a pole of any other right (left) inverse.

Remark 3.2: Taking the transpose of both sides of eqns.(3.2.11) and (3.2.12), it can be seen that a left inverse of $W_d^{\,o}(s)$ is a right inverse of $\left[W_d^{\,o}(s)\right]^T$. Therefore, in the remainder of this section, unless stated otherwise, we shall assume that $l \le r$ and consider only right and ordinary inverses of $W_d^{\,o}(s)$. The case $r < l$ can be treated in a similar way by considering the dual of the system $\sum_d \left[A, B, C, E\right]$.

Since $l \le r$, $W_d^{\,o}(s)$ has full rank l and therefore from the factorization in (3.2.4), the $l \times r$ rational function $P_d^{\,o}(s)$ has full rank l. Consequently we can express $W_d^{\,R}(s)$ as

$$W_d^{\,R}(s) = P_d^{\,R}(s)\, Q_d^{\,o}(s)$$

where the $r \times l$ rational function matrix $P_d^{\,R}(s)$ is a right inverse of $P_d^{\,o}(s)$, and is defined in a similar manner to $W_d^{\,R}(s)$. Denoting the corresponding minimal order inverse by $P_d^{\,R_m}(s)$, the d.z.'s of $P_d^{\,o}(s)$ can be defined in an analogous way to the d.z.'s of $W_d^{\,o}(s)$.

Definition 3.2: The d.z.'s of $P_d^{\,o}(s)$ are the poles of $P_d^{\,R_m}(s)$.

Note that $P_d^{R_m}(s)$ is the transfer function matrix of a minimal order right inverse of the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$. Consequently if the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ is a minimal order realization of $P_d^{o}(s)$, then its d.z.'s are the same as those of $P_d^{o}(s)$ defined above. However, if the realization is not of minimal order, then it has some cancellation of poles and d.z.'s which will not appear in a minimal order realization of the corresponding minimal order inverse. Now, since the pair $\left[\hat{A}_{12}, \hat{A}_{22}\right]$ is observable (Theorem 3.1), the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ not having minimal order implies uncontrollability only for the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$. Using this information, we can construct a controllable and observable state-space representation (and therefore one of minimal order) $\sum\left[\bar{A}_{22}, \bar{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ which has the same set of d.z.'s as $P_d^{o}(s)$.

Theorem 3.2: If the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ is a non-minimal realization of $P_d^{o}(s)$, then a minimal order system $\sum\left[\bar{A}_{22}, \bar{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ can always be constructed, such that its transmission zeros are the same as the d.z.'s of $P_d^{o}(s)$.

Proof: Consider the system

$$\dot{\xi}^* = \hat{A}_{22}^T \xi^* + \hat{A}_{12}^T \mu^* \tag{3.2.13a}$$

$$\upsilon^* = \hat{E}_2^T \xi^* + \hat{E}_1^T \mu^* \tag{3.2.13b}$$

which is the dual of the system (3.2.7a-b). Since the pair $\left[\hat{A}_{12}, \hat{A}_{22}\right]$ is observable (Theorem 3.1), the pair $\left[\hat{A}^T{}_{22}, \hat{A}^T{}_{12}\right]$ is controllable. Therefore we can always find a state feedback

matrix L^T define by

$$\mu^* = \hat{\mu}^* - L^T \xi^*$$

such that the pair $\left[\hat{E}_2^T - \hat{E}_1^T L^T, \hat{A}_{22}^T - \hat{A}_{12}^T L^T\right]$ is observable. It is well known [40] that, state feedback does not affect the controllability of a system, i.e. the pair $\left[\hat{A}_{22}^T - \hat{A}_{12}^T L^T, \hat{A}_{12}^T\right]$ is controllable. Hence, the system $\sum\left[\bar{A}_{22}, \bar{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ has minimal order, where $\bar{A}_{22}^{\ T} = \left[\hat{A}_{22}^T - \hat{A}_{12}^T L^T\right]$ and $\bar{E}_2^{\ T} = \left[\hat{E}_2^T - \hat{E}_1^T L^T\right]$. Next, since transmission zeros are invariant under state feedback, we note that the matrix L^T does not change the transmission zeros of (3.2.13a,b) which we have defined as the d.z.'s of $P_d^{\ o}(s)$. To complete the proof, we use the fact that a system and its dual have the same set of transmission zeros. Then it follows that $\sum\left[\bar{A}_{22}, \bar{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ is of minimal order and its transmission zeros are the same as the d.z.'s of $P_d^{\ o}(s)$.

As a consequence of Theorem 3.2, in the rest of this section we shall assume without loss of generality that the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ has minimal order.

Theorem 3.3: The d.z.'s of the system $\sum_d\left[A, B, C, E\right]$ are the same as the transmission zeros of the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$.

Proof: The d.z.'s of the system $\sum_d\left[A, B, C, E\right]$ are defined as those complex numbers λ for which

$$rank\ \Gamma\left[\lambda\right] = rank \begin{bmatrix} A - \lambda I_n & E \\ C & 0 \end{bmatrix}$$

$$= rank \begin{bmatrix} \hat{A}-\lambda I_n & \hat{E} \\ \hat{C} & 0 \end{bmatrix} < n+\min(l,r)$$

The above condition can also be written as

$$rank\ \Gamma\left[\lambda\right] = rank \begin{bmatrix} \hat{A}_{11}-\lambda I_l & \hat{A}_{12} & \hat{E}_1 \\ \hat{A}_{21} & \hat{A}_{22}-\lambda I_{n-l} & \hat{E}_2 \\ \hat{C}_1 & 0 & 0 \end{bmatrix}$$

which implies that

$$rank\ \Gamma\left[\lambda\right] = rank\left[\hat{C}_1\right] + rank \begin{bmatrix} \hat{A}_{12} & \hat{E}_1 \\ \hat{A}_{22}-\lambda I_{n-l} & \hat{E}_2 \end{bmatrix} \tag{3.2.14}$$

Since $rank\ \left[\hat{C}_1\right]$ is equal to l, it follows that rank $\Gamma < n+min(l,r)$ if and only if

$$rank \begin{bmatrix} \hat{A}_{12} & \hat{E}_1 \\ \hat{A}_{22}-\lambda I_{n-l} & \hat{E}_2 \end{bmatrix} < (n-l) + \min(l, r)$$

which is the condition for λ to be a transmission zeros of the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$,

thus completing the proof.

Theorem 3.4: The system $\sum_d \left[A, B, C, E\right]$ with $l=r$ and $rank\left[CE\right] = l$ has exactly $(n-l)$ d.z.'s.

Proof: From the coordinate transformation $\hat{\mathbf{x}}(t) = T\mathbf{x}(t)$, it is easy to see that $CE = \left[\hat{C}_1\hat{E}_1\right]$.

Therefore $rank\left[CE\right] = l$ with $rank\left[\hat{C}_1\right] = l$, it implies that the $rank\left[\hat{E}_1\right] = l$. From Theorem 3.3, the d.z.'s of the system $\sum_d \left[A, B, C, E\right]$ are the same as the transmission zeros of $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$. Since $\hat{E}_1$ is an $l \times l$ nonsingular matrix, the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ has a unique inverse given by

$$\dot{\xi} = \left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^{-1}\hat{A}_{12}\right]\xi + \left[\hat{E}_2\hat{E}_1^{-1}\right]\upsilon \tag{3.2.15a}$$

$$\mu = \left[-\hat{E}_1^{-1}\hat{A}_{12}\right]\xi + \left[\hat{E}_1^{-1}\right]\upsilon \tag{3.2.15b}$$

Assuming the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ is a minimal order realization of $P_d{}^o(s)$. Therefore, the unique inverse in eqns.(3.2.15a,b) has minimal order. Then from Theorem 3.3, it follows that the d.z.'s of $\sum_d \left[A, B, C, E\right]$ are the eigenvalues of the $(n-l)\times(n-l)$ matrix $\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^{-1}\hat{A}_{12}\right]$, completing the proof.

Based on Theorem 3.3, the problem of computing the set of d.z.'s of the system $\sum_d \left[A, B, C, E\right]$ is reduced to that of computing the set of transmission zeros of the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$. Therefore, in the remainder of this section, we will compute the set of d.z.'s using the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ with $rank\ (\hat{E}_1) = \min(l, r)$, that has either a right inverse ($l < r$), or a left inverse ($l > r$); for $l = r$, the right and left inverses are the same.

Case 1 $\left[l<r\right]$:

The general solution of eqn.(3.2.7b) for μ is given by

$$\mu = \hat{E}_1^+\left[\upsilon - \hat{A}_{12}\xi\right] + \left[I_r - \hat{E}_1^+\hat{E}_1\right]\omega \tag{3.2.16}$$

where $\hat{E}_1^+ = \hat{E}_1^T\left[\hat{E}_1\hat{E}_1^T\right]^{-1}$ is a right inverse of $\hat{E}_1$, and ω is an arbitrary r-vector. Since we are primarily interested in the eigenvalues of the state matrix of a right inverse of $\Sigma\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ or more precisely in the eigenvalues of the state matrix of its minimal order right inverse. We can write ω, without any loss of generality, as

$$\omega = \hat{M}\,\xi$$

where $\hat{M}$ is an arbitrary $\left[r\times(n-l)\right]$ matrix. Then substituting for μ from (3.2.16) into (3.2.7a) yields a general representation of a right inverse of $\Sigma\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$

$$\dot{\xi} = \left[\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^+\hat{A}_{12}\right] + \left[\hat{E}_2(I_r - \hat{E}_1^+\hat{E}_1)\right]\hat{M}\right]\xi + \left[\hat{E}_2\hat{E}_1^+\right]\upsilon \tag{3.2.17a}$$

$$\mu = \left[\left[I_r - \hat{E}_1^+\hat{E}_1\right]\hat{M} - \hat{E}_1^+\hat{A}_{12}\right]\xi + \left[\hat{E}_1^+\right]\upsilon \tag{3.2.17b}$$

Case 2 $\left[l=r\right]$:

Solving eqn.(3.2.7b) for μ, the solution is uniquely given by

$$\mu = \hat{E}_1^{-1}\left[\upsilon - \hat{A}_{12}\xi\right] \tag{3.2.18a}$$

on substituting this value of μ into (3.2.7a), we obtain

$$\dot{\xi} = \left[\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^{-1}\hat{A}_{12}\right]\right]\xi + \left[\hat{E}_2\hat{E}_1^{-1}\right]\upsilon \tag{3.2.18b}$$

Case 3 $\left[l > r\right]$:

In this case the $l \times r$ matrix $\hat{E}_1$ has more rows than columns and we can find a left inverse. However we cannot, in general solve (3.2.7b) exactly for μ. To get a left inverse of (3.2.7b), we consider the dual system

$$\dot{\xi}^* = \hat{A}_{22}^T \xi^* + \hat{A}_{12}^T \mu^* \tag{3.2.19a}$$

$$\upsilon^* = \hat{E}_2^T \xi^* + \hat{E}_1^T \mu^* \tag{3.2.19b}$$

The system in eqns.(3.2.19a-b) is an $(n-l)$th-order, l-input, r-output system where $l > r$. Hence the procedure of Case 1 above can be used to obtain a general right inverse which can be easily shown to be a general left inverse of the system in eqns.(3.2.7a-b). This general inverse is given by

$$\dot{\xi}^* = \left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T (\hat{E}_1^T)^+ \hat{E}_2^T\right] + \left[\hat{A}_{12}^T (I_l - (\hat{E}_1^T)^+ \hat{E}_1^T)\right]\overline{M}\right]\xi^* + \left[\hat{A}_{12}^T (\hat{E}_1^T)^+\right]\upsilon^* \tag{3.2.20a}$$

$$\mu^* = \left[\left[I_l - (\hat{E}_1^T)^+ \hat{E}_1^T\right]\overline{M} - (\hat{E}_1^T)^+ \hat{E}_2^T\right]\xi^* + \left[(\hat{E}_1^T)^+\right]\upsilon^* \tag{3.2.20b}$$

where $\overline{M}$ is an arbitrary $\left[l \times (n-r)\right]$ matrix.

Theorem 3.5: The system $\sum_d \left[A, B, C, E\right]$ with $l > r$ has at most $n-l$ d.z.'s.

Proof: For an invertible system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$, we can always construct an inverse given by eqns.(3.2.20a,b) with a characteristic polynomial of degree less or equal to $n-l$. The result then follows using Theorem 3.3.

From the above results, we can now give an analogous definitions for the sets of d.z.'s (Z_o^D) and the d.b.z.'s (Z_o^B) of the system $\sum_d \left[A, B, C, E\right]$.

Definition 3.3 $\left[l<r\right]$:

$$Z_o^D\left[\sum_d\left[A,B,C,E\right]\right]=\sigma\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right],\left[\hat{E}_2(I_r-\hat{E}_1^{+}\hat{E}_1)\right]\right]$$

$$Z_o^B\left[\sum_d\left[A,B,C,E\right]\right]=\left\{\lambda\in C\mid\lambda\in Z_o^D\left[\sum_d\left[A,B,C,E\right]\right],\lambda\notin\sigma\left[\hat{A}_{22}\right]\right.$$

$$\left.\text{and } P_d^{\,o}(\lambda)=0\right\}$$

Definition 3.4 $\left[l=r\right]$:

$$Z_o^D\left[\sum_d\left[A,B,C,E\right]\right]=\sigma\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{-1}\hat{A}_{12}\right]\right]$$

$$Z_o^B\left[\sum_d\left[A,B,C,E\right]\right]=\left\{\lambda\in C\mid\lambda\in Z_o^D\left[\sum_d\left[A,B,C,E\right]\right],\lambda\notin\sigma\left[\hat{A}_{22}\right]\right.$$

$$\left.\text{and } P_d^{\,o}(\lambda)=0\right\}$$

Definition 3.5 $\left[l>r\right]$:

$$Z_o^D\left[\sum_d\left[A,B,C,E\right]\right]=\sigma\left[\left[\hat{A}_{22}^T-\hat{A}_{12}^T(\hat{E}_1^T)^{+}\hat{E}_2^T\right],\left[\hat{A}_{12}^T(I_l-(\hat{E}_1^T)^{+}\hat{E}_1^T)\right]\right]$$

$$Z_o^B\left[\sum_d\left[A,B,C,E\right]\right]=\left\{\lambda\in C\mid\lambda\in Z_o^D\left[\sum_d\left[A,B,C,E\right]\right],\lambda\notin\sigma\left[\hat{A}_{22}^T\right]\right.$$

$$\left.\text{and } P_d^{\,o}(\lambda)=0\right\}$$

In order to justify these definitions, we first review two important results obtained by Bengtsson [2], which are relevant to the following discussion. One is that the poles of a minimal order left or right inverse (whichever exists), of a system $\sum\left[A,B,C,D\right]$ are the invariant

zeros of the $\sum\left[A,B,C,D\right]$. The other result states that all minimal order inverses of $\sum\left[A,B,C,D\right]$, have the same characteristic polynomial and that it divides the characteristic polynomial of all other inverses of $\sum\left[A,B,C,D\right]$. Applying these results to the system (3.2.7a,b), we see that the d.z.'s of the system are the common poles of all inverses of (3.2.7a,b). The definitions given above for d.z.'s follow from results as poles of minimal order inverses given in [20].

To justify Definitions 3.3, 3.4 and 3.5, we recall (Theorem 3.3), that d.z.'s of the system $\sum_d\left[A,B,C,E\right]$ in eqns.(3.2.1a,b) are identical to the transmission zeros of the system $\sum\left[\hat{A}_{22},\hat{E}_2,\hat{A}_{12},\hat{E}_1\right]$ with the transfer function matrix $P_d{}^o(s)$ as given in eqns.(3.2.7a,b).

Consider Definition 3.3, since the characteristic polynomial of a minimal order inverse divides the characteristic polynomial of all other inverses, it follows that those eigenvalues of the matrix $\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]+\left[\hat{E}_2(I_r-\hat{E}_1^+\hat{E}_1)\right]\hat{M}\right]$ which remain invariant when $\hat{M}$ is varied are the poles of a minimal order inverse. To determine the poles of a minimal order inverse, we consider the eigenvalues of the matrix $\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]+\left[\hat{E}_2(I_r-\hat{E}_1^+\hat{E}_1)\right]\hat{M}\right]$. This matrix can be represent as the closed-loop state matrix of the system

$$\dot{\Psi}=\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]\Psi+\left[\hat{E}_2(I_r-\hat{E}_1^+\hat{E}_1)\right]\Phi \tag{3.2.21a}$$

subject to the state feedback law

$$\Phi=\hat{M}\,\Psi \tag{3.2.21b}$$

Hence if the rank of the controllability matrix of the pair

$\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right],\left[\hat{E}_2(I_r-\hat{E}_1^{+}\hat{E}_1)\right]\right]$ is n_c , then $\left[(n-l)-n_c\right]$ eigenvalues of the state matrix of (3.2.21a) will remain invariant under the feedback law (3.2.21b). Thus, it follows that the order of the minimal order inverse of $\sum\left[\hat{A}_{22},\hat{E}_2,\hat{A}_{12},\hat{E}_1\right]$ given in eqns.(3.2.17a-b) is $\left[=(n-l)-n_c\right]$. In order to determine the poles of this minimal order inverse , the pair $\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right],\left[\hat{E}_2(I_r-\hat{E}_1^{+}\hat{E}_1)\right]\right]$ can be reduced to its BUHF, by applying an orthogonal state coordinate transformation matrix U i.e. $F = U^T\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]U$ and $G = U^T\left[\hat{E}_2(I_r-\hat{E}_1^{+}\hat{E}_1)\right]$, and the $\left[(n-l)-n_c\right]$ eigenvalues of the sub-matrix F_{22} which are uncontrollable are the required poles. Thus, the term

$$\sigma\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right],\left[\hat{E}_2(I_r-\hat{E}_1^{+}\hat{E}_1)\right]\right]$$

represents the set of common poles of all inverses of the system $\sum\left[\hat{A}_{22},\hat{E}_2,\hat{A}_{12},\hat{E}_1\right]$ in eqns.(3.2.7a-b) and hence they are the d.z.'s of the system.

Now let $\lambda \in C$, is the set of d.z.'s of the system $\sum_d\left[A,B,C,E\right]$. If λ are not an eigenvalues of $\hat{A}_{22}$ and the rational function $P_d^{\,o}(\lambda) = 0$, then all λ are the d.b.z.'s of the system.

In th case of Definition 3.4, we note that the inverse is uniquely obtained and thus the definition can be easily verified.

Definition 3.5 can be verified in a similar manner as Definition 3.3, using duality.

It should be noted that, an alternative way to define the sets of d.z.'s and d.b.z.'s is by representing the state-space equations (3.2.1a,b) for a multivariable system $\sum_d\left[A,B,C,E\right]$ by:

$$\dot{\mathbf{x}}(t) = \hat{A}\,\hat{\mathbf{x}}(t) + \hat{B}\mathbf{u}(t) + \sum_{i=1}^{r}\hat{E}_i d_i(t) \tag{3.2.22a}$$

$$y(t) = \hat{C}\,\hat{x}(t) \tag{3.2.22b}$$

with

$$\hat{E}_i = \begin{bmatrix} \hat{E}_{1,i} \\ \hat{E}_{2,i} \end{bmatrix}, \qquad i = 1, 2, \ldots, r$$

And then Definition 3.3 can be written as

$$Z_o^D\left\{\sum_d \left[A, B, C, E\right]\right\} = \bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}$$
$$= \bigcap_{i=1}^{r}\left\{\sigma\left[\left[\hat{A}_{22} - \hat{E}_{2,i}\hat{E}_{1,i}^{+}\hat{A}_{12}\right], \left[\hat{E}_{2,i}(I - \hat{E}_{1,i}^{+}\hat{E}_{1,i})\right]\right]\right\}$$

$$Z_o^B\left\{\sum_d \left[A, B, C, E\right]\right\} = \bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}$$
$$- \left[\left\{\bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}\right\} \cap \sigma\left[\hat{A}_{22}\right]\right]$$

Similarly Definitions 3.4 and 3.5 can be written respectively as

$$Z_o^D\left\{\sum_d \left[A, B, C, E\right]\right\} = \bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}$$
$$= \bigcap_{i=1}^{r}\left\{\sigma\left[\left[\hat{A}_{22} - \hat{E}_{2,i}\hat{E}_{1,i}^{-1}\hat{A}_{12}\right]\right]\right\}$$

$$Z_o^B\left\{\sum_d \left[A, B, C, E\right]\right\} = \bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}$$
$$- \left[\left\{\bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}\right\} \cap \sigma\left[\hat{A}_{22}\right]\right]$$

and

$$Z_o^D\left\{\sum_d \left[A, B, C, E\right]\right\} = \bigcap_{i=1}^{r} Z_o^{D_i}\left\{\sum_d \left[A, B, C, E_i\right]\right\}$$
$$= \bigcap_{i=1}^{r}\left\{\sigma\left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T(\hat{E}_{1,i}^T)^{+}\hat{E}_{2,i}^T\right], \left[\hat{A}_{12}^T(I_l - (\hat{E}_{1,i}^T)^{+}\hat{E}_{1,i}^T)\right]\right]\right\}$$

$$Z_o^B\left\{\sum_d\left[A,B,C,E\right]\right\}=\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}$$
$$-\left[\left[\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}\right]\bigcap\sigma\left[\hat{A}_{22}^T\right]\right]$$

The term $\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}$ represents the set of common d.z.'s of all transfer function matrices of the system $\sum_d\left[A,B,C,E_i\right]$,$i = 1, 2, \ldots, r$ between the outputs and the disturbances. The term

$$\left[\left[\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}\right]\bigcap\sigma\left[\hat{A}_{22}\right]\right]$$

represents the subset of common d.z.'s which cancel with the eigenvalues of $\hat{A}_{22}$ and hence the term

$$\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}-\left[\left[\bigcap_{i=1}^{r}Z_o^{D_i}\left\{\sum_d\left[A,B,C,E_i\right]\right\}\right]\bigcap\sigma\left[\hat{A}_{22}\right]\right]$$

represents the subset of common d.z.'s which do not cancel with the eigenvalues of $\hat{A}_{22}$, and thus it is the set of d.b.z.'s of the system $\sum_d\left[A,B,C,E\right]$.

Remark 3.3: In order to simplify the mathematics, we assume that $W_d^{\,o}(s)$ has min $\left[(n-l),(n-r)\right]$ d.z.'s. This assumption is not necessary for achieving disturbance rejection, and implies that $n_c = 0$, which in turn implies that for $r>l$, the matrix $\left[\hat{E}_2(I_r-\hat{E}_1^{\,+}\hat{E}_1)\right] = 0$ in eqns.(3.2.17a,b) and for $r<l$, the matrix $\left[\hat{A}_{12}^T(I_l-(\hat{E}_1^T)^+\hat{E}_1^T)\right] = 0$ in eqns.(3.2.20a,b). Therefore, in illustrating the main results of this section, the first numerical example considered in Section 3.4 uses this assumption to construct a left inverse of the system.

Remark 3.4: If $rank\left[CE\right] < \min(l, r)$, we can apply the structure algorithm of Silverman [41]

to the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ to get an invertible system $\sum\left[\hat{A}_{22}, \hat{E}_2, \bar{A}_{12}, \bar{E}_1\right]$ with $rank\left[\bar{E}_1\right] = l$. Then the above procedure for constructing a general right or left inverse (whichever exists) can be applied. However, $\sum\left[\hat{A}_{22}, \hat{E}_2, \bar{A}_{12}, \bar{E}_1\right]$ will have some additional transmission zeros at the origin which are not the transmission zeros of $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ [20].

Remark 3.5: The factorization of $W_d{}^o(s)$ given above is a left factorization in the sense that nonsingular matrix factors common to $Q_d{}^o(s)$ and $P_d{}^o(s)$ can be factored out on the left without affecting $W_d{}^o(s)$. We can obtain a right factorization by applying the above procedure to get a left factorization of the dual system represented by the triple (A^T, C^T, E^T), which is then a right factorization of the system $\sum_d\left[A, B, C, E\right]$.

3.2.2 Disturbance Blocking Zeros of the System $\sum_d\left[A, B, C, D, E, F\right]$

Consider an nth-order, m-input, l-output, r-disturbance, linear time-invariant system $\sum_d\left[A, B, C, D, E, F\right]$ described by eqns.(3.1.4a-b). Assume that $rank\left[F\right] = \min(l, r)$, a the system $\sum_d\left[A, B, C, D, E, F\right]$ is non-degenerate (i.e. it has a finite number of transmission zeros and disturbance zeros).

It is possible to obtain a higher order system denoted by $\sum_d\left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ from the given system $\sum_d\left[A, B, C, D, E, F\right]$, such that the factorization procedure and the concept of

minimal order inverses can be applied to compute the set of d.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$.

To create the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$, we consider a dynamic compensator of order p $\left[\geq l\right]$ defined by

$$\dot{\mathbf{z}}(t) = \Omega\, \mathbf{z}(t) + \Theta\, \mathbf{y}(t) \tag{3.2.23a}$$

$$\hat{\mathbf{y}}(t) = \mathbf{z}(t) \tag{3.2.23b}$$

where $\mathbf{z}(t) \in \mathbb{R}^p$ is the vector of state variables of the compensator, $\hat{\mathbf{y}}(t) \in \mathbb{R}^p$ is the output vector of the compensator, Ω and Θ are matrices of compatible sizes which should be chosen to be a controllable pair. It should be noted that, this compensator is not unique and may be designed in a number of ways e.g. using observer theory or pole assignment [20].

By incorporating of the above compensator at the output of the system of eqns.(3.1.4a,b), the following augmented system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is obtained:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\, \hat{\mathbf{x}}(t) + \hat{B}\, \mathbf{u}(t) + \hat{E}\, \mathbf{d}(t) \tag{3.2.24a}$$

$$\hat{\mathbf{y}}(t) = \hat{C}\, \hat{\mathbf{x}}(t) \tag{3.2.24b}$$

with

$$\mathbf{y}(t) = \begin{bmatrix} 0 & C \end{bmatrix} \hat{\mathbf{x}}(t) + D\, \mathbf{u}(t) + F\, \mathbf{d}(t) \tag{3.2.24c}$$

where

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{x}(t) \end{bmatrix}$$

and

$$\hat{A} = \begin{bmatrix} \Omega & \Theta C \\ 0 & A \end{bmatrix}, \hat{B} = \begin{bmatrix} \Theta D \\ B \end{bmatrix}, \hat{E} = \begin{bmatrix} \Theta F \\ E \end{bmatrix}, \hat{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

Remark 3.6: The output $\mathbf{y}(t)$ and the output $\hat{\mathbf{y}}(t)$ are related by

$$\hat{\mathbf{y}}(s) = G_c(s)\,\mathbf{y}(s)$$

where $\mathbf{y}(s)$ and $\hat{\mathbf{y}}(s)$ denotes the Laplace transform of $\mathbf{y}(t)$ and $\hat{\mathbf{y}}(t)$ respectively, and $G_c(s) = \left[sI_p - \Omega\right]^{-1}\Theta$ is the transfer function matrix of the dynamic compensator. This relationship can be obtained by first computing the output $\mathbf{y}(s)$ to the inputs $\mathbf{u}(s)$ and $\mathbf{d}(s)$ of the system $\sum_d \left[A, B, C, D, E, F\right]$ i.e.

$$\mathbf{y}(s) = \left[P_u{}^o(s)\ P_d{}^o(s)\right] \begin{bmatrix} \mathbf{u}(s) \\ \mathbf{d}(s) \end{bmatrix}$$

where $P_u{}^o(s)$ and $P_d{}^o(s)$ are rational function matrices given by

$$P_u{}^o(s) = C\left[sI_n - A\right]^{-1} B + D$$
$$P_d{}^o(s) = C\left[sI_n - A\right]^{-1} E + F$$

Then, we compute the output $\hat{\mathbf{y}}(s)$ to the inputs $\mathbf{u}(s)$ and $\mathbf{d}(s)$ of the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ i.e.

$$\hat{\mathbf{y}}(s) = \hat{Q}^{-1}(s)\left[\hat{P}_u{}^o(s)\ \hat{P}_d{}^o(s)\right] \begin{bmatrix} \mathbf{u}(s) \\ \mathbf{d}(s) \end{bmatrix}$$

where $\hat{Q}^{-1}(s)$, $\hat{P}_u{}^o(s)$ and $\hat{P}_d{}^o(s)$ are $p \times p$, $p \times m$ and $p \times r$ rational function matrices given by

$$\hat{Q}^{-1}(s) = \left[sI_p - \Omega\right]^{-1}$$
$$\hat{P}_u{}^o(s) = \Theta P_u{}^o(s)$$
$$\hat{P}_d{}^o(s) = \Theta P_d{}^o(s)$$

Thus, it follows that

$$\hat{\mathbf{y}}(s) = \hat{Q}^{-1}(s)\,\Theta\,\mathbf{y}(s)$$

$$= G_c(s)\mathbf{y}(s)$$

Now, in order to relate the d.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$ and $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ and to use the augmented system in assigning d.b.z.'s, we need the following results.

Theorem 3.6:

(i) The augmented system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is stabilizable if the pair (A , B) is stabilizable and Ω is stable

(ii) The system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is detectable if and only if the pair (ΘC ,A) is detectable.

Proof:

(i) Stabilizability:

The system (3.2.24a-b) is stabilizable, if and only if, the pair

$$\left\{ \begin{bmatrix} \Omega & \Theta C \\ 0 & A \end{bmatrix}, \begin{bmatrix} \Theta D \\ B \end{bmatrix} \right\} \tag{3.2.25}$$

is stabilizable, i.e. if and only if the matrix

$$\Gamma\left(\lambda\right) = \begin{bmatrix} \Omega - \lambda I_p & \Theta C & \Theta D \\ 0 & A - \lambda I_n & B \end{bmatrix} \tag{3.2.26}$$

has full rank $n+p$ for all λ equal to the unstable eigenvalues of $\hat{A}$. Then the matrix Γ has full rank, if

$$rank\left[A - \lambda I_n \;\; B\right] = n \tag{3.2.27a}$$

and

$$rank\left[\Omega - \lambda I_p\right] = p \tag{3.2.27b}$$

Now (3.2.27a) holds for $\lambda \in C^+$ since Ω is a stable matrix. If (A, B) is a stabilizable pair, then (3.2.27a) holds for all unstable eigenvalues of A which are also the unstable eigenvalues of $\hat{A}$. Therefore rank $(\Gamma) = n + p$ for all λ which are unstable eigenvalues of $\hat{A}$ implying that $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is stabilizable.

(ii) Detectability:

The system (3.2.24a-b) is detectable, if and only if, the pair

$$\left\{ \begin{bmatrix} I_p & 0 \end{bmatrix}, \begin{bmatrix} \Omega & \Theta C \\ 0 & A \end{bmatrix} \right\} \tag{3.2.28}$$

is detectable.

Let us form the matrix

$$\Lambda\left[\lambda\right] = \begin{bmatrix} \Omega - \lambda I_p & \Theta C \\ 0 & A - \lambda I_n \\ I_p & 0 \end{bmatrix} \tag{3.2.29}$$

Then $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is detectable if and only if the matrix Λ has full rank $(n+p)$ for all λ equal to the unstable eigenvalues of $\hat{A}$. Now

$$rank\ \left[\Lambda\left[\lambda\right]\right] = p + rank \begin{bmatrix} \Theta C \\ A - \lambda I_n \end{bmatrix} \tag{3.2.30}$$

Since Ω is a stable matrix, the unstable eigenvalues of $\hat{A}$ are the same as the unstable eigenvalues of A. Therefore

$$rank \begin{bmatrix} \Theta C \\ A - \lambda I_n \end{bmatrix} = n$$

for all λ equal to the unstable eigenvalues of $\hat{A}$ if and only if the pair $(\Theta C, A)$ is detectable completing the proof.

If the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is stabilizable and detectable, then we can always stabilize the system by state or output feedback. However, if it is required to achieve arbitrary pole assignment in the overall closed-loop system using state (or output feedback), then the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ should be controllable (and observable). In that case in Theorem 3.6, we replace the stabilizability and detectability conditions by controllability and observability conditions respectively with the additional requirement that the pair (Ω, Θ) should be controllable.

To show the mechanism by which the additional d.z.'s are introduced in the augmented system (3.2.24a,b) due to the dynamic compensator (3.2.23a,b), we assume without loss of generality that the pairs (A, E) and (C, A) are controllable and observable respectively. Note that, this assumption is made to simplify the presentation and implies that the system $\sum_d \left[A, B, C, D, E, F\right]$ has at most n d.z.'s.

Remark 3.7: If the pairs (A, E) and/or (C, A) are uncontrollable and/or unobservable, then the system $\sum_d \left[A, B, C, D, E, F\right]$ has less than n d.z.'s. This is due to cancellation of zeros and poles which correspond to uncontrollable and/or unobservable modes.

Theorem 3.7: Let $l \geq r$. Then the set of d.z.'s of the transfer function matrix relating the outputs $\mathbf{y}(t)$ to the disturbances $\mathbf{d}(t)$ of the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ consists of:

(i) the d.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$ together with,

(ii) the poles of the dynamic compensator.

Proof: The d.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$ between the outputs $\mathbf{y}(t)$ and the disturbances $\mathbf{d}(t)$ are defined as those values of λ for which

$$rank \begin{bmatrix} A-\lambda I_n & E \\ C & F \end{bmatrix} < n + r$$

while the d.z.'s of the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ between the outputs $\mathbf{y}(t)$ and the disturbances $\mathbf{d}(t)$ are defined as those values of λ for which

$$rank\ \Gamma\left[\lambda\right] = rank \begin{bmatrix} \Omega-\lambda I_p & \Theta C & \Theta F \\ 0 & A-\lambda I_n & E \\ 0 & C & F \end{bmatrix} < n+p+r$$

from which it follows that the values of λ for which

$$rank\ \Gamma\left[\lambda\right] < n+p+r$$

are those values for which

$$rank \begin{bmatrix} A-\lambda I_n & E \\ C & F \end{bmatrix} < n+r$$

or

$$rank\ \left[\Omega-\lambda I_p\right] < p$$

and the result of the theorem follows.

Now, to compute the sets of d.z.'s and d.b.z.'s of the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$, we first find the transfer function matrix $W_d{}^o(s)$ between the outputs $\hat{\mathbf{y}}(t)$ and the disturbances $\mathbf{d}(t)$ of the sys-

tem $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$. This is given by

$$W_d{}^o(s) = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} sI_p - \Omega & -\Theta C \\ 0 & sI_n - A \end{bmatrix}^{-1} \begin{bmatrix} \Theta F \\ E \end{bmatrix}$$

By following the factorization procedure as discussed in Section 3.2, the transfer function matrix $W_d{}^o(s)$ can be written as

$$W_d{}^o(s) = \left[Q_d{}^o(s)\right]^{-1} P_d{}^o(s)$$

where $Q_d{}^o(s)$ and $P_d{}^o(s)$ are $p \times p$ and $p \times r$ rational function matrices respectively, and are given by

$$Q_d{}^o(s) = \left[sI_p - \Omega\right] \tag{3.2.31a}$$

$$P_d{}^o(s) = \Theta\left[C\left[sI_n - A\right]^{-1} E + F\right] \tag{3.2.31b}$$

From eqn.(3.2.31b), it can be seen that $P_d{}^o(s)$ is the product of the matrix Θ and the transfer matrix of an nth-order, r-input and l-output system described by the 4-tuple $\sum\left[A, E, C, F\right]$

i.e. of the system

$$\dot{\xi} = A\,\xi + E\,\mu \tag{3.2.32a}$$
$$\upsilon = C\xi + F\,\mu \tag{3.2.32b}$$

Assumptions:

From now on we shall assume that the $l \times r$ matrix F has full rank equal to $\min(l, r)$. We also assume that the transfer matrix $W_d{}^o(s)$ has full rank $\left[= \min(l, r)\right]$. The latter assumption implies that $W_d{}^o(s)$ is invertible. Consequently, it follows that $P_d{}^o(s)$ is invertible, which in turn implies that the 4-tuple system $\sum\left[A, E, C, F\right]$ is invertible. Therefore we can always con-

struct controllable and observable right or left inverses (whichever exist) for the nth-order, r-input, l-output system $\sum\left[A, E, C, F\right]$.

Remark 3.8: From the above discussions, it follows that, the system $\sum\left[A, E, C, F\right]$ in eqns.(3.2.32a-b) can be treated in the same manner as the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ in eqns.(3.2.7a-b). Thus, whatever properties are true for the system $\sum\left[\hat{A}_{22}, \hat{E}_2, \hat{A}_{12}, \hat{E}_1\right]$ are also true for the system $\sum\left[A, E, C, F\right]$.

Therefore the problem of computing the d.z.'s of the system $\sum_d\left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is reduced to that of computing the transmission zeros of the system $\sum\left[A, E, C, F\right]$.

For $l < r$, a general right inverse of the system $\sum\left[A, E, C, F\right]$ is given by

$$\dot{\xi} = \left[\left[A - EF^{+}C\right] + \left[E(I_r - F^{+}F)\right]\hat{M}\right]\xi + \left[EF^{+}\right]\upsilon \tag{3.2.33a}$$

$$\mu = \left[\left[I_r - F^{+}F\right]\hat{M} - F^{+}C\right]\xi + \left[F^{+}\right]\upsilon \tag{3.2.33b}$$

where $F^{+} = F^{T}\left[FF^{T}\right]^{-1}$ is a right inverse of F and $\hat{M}$ is an arbitrary $r \times n$ matrix.

For $l = r$, the inverse of $\sum\left[A, E, C, F\right]$ is unique and is given by

$$\dot{\xi} = \left[\left[A - EF^{-1}C\right]\right]\xi + \left[EF^{-1}\right]\upsilon \tag{3.2.34a}$$

$$\mu = \left[-F^{-1}C\right]\xi + \left[F^{-1}\right]\upsilon \tag{3.2.34b}$$

For $l > r$, the system $\sum\left[A, E, C, F\right]$ has a left inverse which is the transpose of the right inverse of the dual system given by

$$\dot{\xi}^* = A^T \xi^* + C^T \mu^* \tag{3.2.35a}$$

$$\upsilon^* = E^T \xi^* + F^T \mu^* \tag{3.2.35b}$$

A right inverse of the system (3.2.35a-b) is given by

$$\dot{\xi}^* = \left[\left[A^T - C^T(F^T)^+E^T\right] + \left[C^T(I_l - (F^T)^+F^T)\right]\overline{M}\right]\xi^* + \left[C^T(F^T)^+\right]\upsilon^* \tag{3.2.36a}$$

$$\mu^* = \left[\left[I_l - (F^T)^+F^T\right]\overline{M} - (F^T)^+E^T\right]\xi^* + \left[(F^T)^+\right]\upsilon^* \tag{3.2.36b}$$

where $\overline{M}$ is an arbitrary $l \times n$ matrix.

We shall now give an analogous definition for the sets of transmission zeros and blocking zeros of the system $\sum\left[A, E, C, F\right]$, which are consequently equivalent to those of d.z.'s and d.b.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$.

Definition 3.6 $\left[l < r\right]$

$$Z_o^D\left\{\sum_d \left[A, B, C, D, E, F\right]\right\} = \sigma\left[\left[A - EF^+C\right], \left[E(I_r - F^+F)\right]\right]$$

$$Z_o^B\left\{\sum_d \left[A, B, C, D, E, F\right]\right\} = \left\{\lambda \in C \mid \lambda \in Z_o^D\left\{\sum_d \left[A, B, C, D, E, F\right]\right\}\right.$$

$$\left., \lambda \notin \sigma\left[A\right] \text{ and } P_d^{\,o}(\lambda) = 0\right\}$$

Definition 3.7: $\left[l=r\right]$

$$Z_o^D\left\{\sum_d \left[A,B,C,D,E,F\right]\right\} = \sigma\left[\left[A-EF^{-1}C\right]\right]$$

$$Z_o^B\left\{\sum_d \left[A,B,C,D,E,F\right]\right\} = \left\{\lambda \in C \mid \lambda \in Z_o^D\left\{\sum_d \left[A,B,C,D,E,F\right]\right\}\right.$$

$$\left.,\lambda \notin \sigma\left[A\right] \text{ and } P_d{}^o(\lambda) = 0\right\}$$

Definition 3.8: $\left[l>r\right]$

$$Z_o^D\left\{\sum_d \left[A,B,C,D,E,F\right]\right\} = \sigma\left[\left[A^T-C^T(F^T)^+E^T\right],\ \left[C^T(I_l-(F^T)^+F^T)\right]\right]$$

$$Z_o^B\left\{\sum_d \left[A,B,C,D,E,F\right]\right\} = \left\{\lambda \in C \mid \lambda \in Z_o^D\left\{\sum_d \left[A,B,C,D,E,F\right]\right\}\right.$$

$$\left.,\lambda \notin \sigma\left[A^T\right] \text{ and } P_d{}^o(\lambda) = 0\right\}$$

Remark 3.9: As a consequence of Remark 3.3, in all that follows we shall assume that the transfer matrix $W_d{}^o(s)$ has n d.z.'s. This assumption is made to simplify the mathematics and is not necessary for achieving disturbance rejection. The assumption implies that either (i) $l = r$ or (ii) for $l<r$ $\left[E(I_r-F^+F)\right] = 0$ for $l>r$ $\left[C^T(I_l-(F^T)^+F^T)\right] = 0$.

3.3 CHOICES OF DISTURBANCE BLOCKING ZERO POSITIONS

In general, the positions of zeros in multivariable systems play an important role in regulation and asymptotic tracking problems [6]. Desoer and Schulman [6] have shown that a transmission zero at $s = \alpha$ of the transfer matrix $W_u{}^o(s)$ has the property that for appropriate initial conditions, it completely blocks the transmission of some input proportional to $\exp(\alpha t)$, in the

steady state. Therefore, if α is a transmission zero of an $l \times m$ transfer function matrix, then there exists an $m \times 1$ vector $\beta \neq 0$, such that if the input of the system is $\mathbf{u}(t) = \beta \exp(\alpha t)$, the effect of mode $\exp(\alpha t)$ will not appear at the outputs in the steady state i.e. the condition [38,44]

$$W_u^{\,o}(s)\big|_{s=\alpha}\,\beta = 0 \tag{3.3.1}$$

is satisfied. Note that (3.3.1) is satisfied only for some specific vector β. For single-input system β is a scaler and the result is therefore true for any β. In order for (3.3.1) to hold for any vector β, we must have $W_u^{\,o}(s)\big|_{s=\alpha} = 0$. This condition is satisfied when α is a blocking zero of $W_u^{\,o}(s)$. As we have noted earlier, in the single-input case (and by duality, in the single-output case), transmission zeros are also blocking zeros so that (3.3.1) holds for all β. Based on this result, Patel et al. [15] developed an algorithm for single-disturbance state-space system $\sum_d \left[A, B, C, E\right]$ for assigning d.b.z.'s at appropriate locations e.g. at $s = \alpha_i$, to completely block the effect of exponential disturbances $\beta_i \exp(\alpha_i t)$, for all complex β_i and α_i, in the steady state.

A generalization of this algorithm to block the effect of multiple disturbances on systems described by $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$ will be developed later on in this thesis.

3.3.1 Multiple Disturbances in Multivariable Systems

In Section 3.2, the system described by the state-space model $\sum_d \left[A, B, C, E\right]$ or $\sum_d \left[A, B, C, D, E, F\right]$ has been represented by the $\hat{n}th$-order, m-input, r- disturbance, $\hat{l}$-output system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ defined by

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\,\hat{\mathbf{x}}(t) + \hat{B}\,\mathbf{u}(t) + \hat{E}\,\mathbf{d}(t) \tag{3.3.2a}$$

$$\hat{\mathbf{y}}(t) = \hat{C}\,\hat{\mathbf{x}}(t) \tag{3.3.2b}$$

where $\hat{A}, \hat{B}, \hat{C}$ and $\hat{E}$ are matrices of appropriate dimensions, $\mathbf{d}(t) \in \mathbb{R}^r$ is the vector of disturbances which may or may not be measurable and $\hat{n}=n$ and $\hat{l}=l$ for $\sum_d \left[A, B, C, E\right]$ and $\hat{n}=n+p$ and $\hat{l}=p$ for $\sum_d \left[A, B, C, D, E, F\right]$. Assume that, each element of the disturbance vector is described by

$$d_i(t) = \beta_i \exp(\alpha_i t) \quad i = 1, 2, \ldots, r \tag{3.3.3}$$

where β_i and α_i are arbitrary complex scalars with real$(\alpha_i) \geq 0$.

Partitioning the matrix $\hat{E}$ as

$$\hat{E} = \left[\hat{E}_1\ \hat{E}_2\ \ldots\ \hat{E}_r\right] \tag{3.3.4}$$

The disturbance transfer function matrix of the system in eqns.(3.3.2a-b) i.e. relating the outputs to the disturbance is given by

$$W_d^{\,o}(s) = \left[\mathbf{w}_{d_1}^{o}(s)\ \mathbf{w}_{d_2}^{o}(s)\ \ldots\ \mathbf{w}_{d_r}^{o}(s)\right] \tag{3.3.5}$$

where $\mathbf{w}_{d_i}^{o}(s)$, represents the transfer function vector relating the outputs and i^{th} disturbance and is given by

$$\mathbf{w}_{d_i}^{o}(s) = \hat{C}\left[sI_{\hat{n}} - \hat{A}\right]^{-1}\hat{E}_i \quad i = 1, 2, \ldots, r \tag{3.3.6}$$

The response at the outputs due to the $i-th$ disturbance is

$$\hat{\mathbf{y}}_{d_i}(s) = \left[\hat{C}\left[sI_{\hat{n}} - \hat{A}\right]^{-1}\hat{E}_i\right]d_i(s) \quad i = 1, 2, \ldots, r \tag{3.3.7}$$

The condition under which each exponential disturbance leads to an identically zero output in the steady state is given by the following theorem:

Theorem 3.8: The condition under which complete rejection in the steady state of all

exponential disturbances $\beta_i \exp(\alpha_i t)$ affecting a system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is

$$\mathbf{w}_{d_i}^{o}(s)\big|_{s=\alpha_i} = 0 \qquad i = 1, 2, \ldots, r \tag{3.3.8}$$

We shall show later how these conditions can be satisfied by assigning one or more d.b.z.'s between the outputs and each disturbance at $s = \alpha_i$ by means of state feedback.

3.3.2 Effect of State Feedback on the Disturbance Transfer Function Matrix

The results to be derived in this section are concerned with the effect of constant state feedback defined by

$$\mathbf{u}(t) = \mathbf{v}(t) - \hat{K}\,\hat{\mathbf{x}}(t)$$

on the disturbance transfer function matrix relating the outputs to the disturbances.

On implementing the feedback law in eqn.(3.3.2a,b), the resulting closed-loop system is

$$\dot{\hat{\mathbf{x}}}(t) = \left[\hat{A} - \hat{B}\hat{K}\right]\hat{\mathbf{x}}(t) + \hat{B}\,\mathbf{v}(t) + \sum_{i=1}^{r} \hat{E}_i d_i(t) \tag{3.3.9a}$$

$$\hat{\mathbf{y}}(t) = \hat{C}\,\hat{\mathbf{x}}(t) \tag{3.3.9b}$$

The closed-loop transfer matrix $\mathbf{w}_{d_i}^{c}(s)$ relating the outputs to each disturbance is given by

$$\mathbf{w}_{d_i}^{c}(s) = \hat{C}\left[sI_{\hat{n}} - \left[\hat{A} - \hat{B}K\right]\right]^{-1}\hat{E}_i \qquad i = 1, 2, \ldots, r \tag{3.3.10}$$

Partitioning $\hat{B}$, $\hat{E}_i$ and K as

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \hat{E}_i = \begin{bmatrix} \hat{E}_{1,i} \\ \hat{E}_{2,i} \end{bmatrix}, \hat{K} = \begin{bmatrix} \hat{K}_1 & \hat{K}_2 \end{bmatrix} \tag{3.3.11}$$

and following the factorization procedure described in Section 3.2, $\mathbf{w}_{d_i}^{c}(s)$ can be expressed in the form

$$\mathbf{w}_{d_i}^{c}(s) = \left[Q_{d_i}{}^{c}(s)\right]^{-1} P_{d_i}{}^{c}(s) \quad i \in r \tag{3.3.12}$$

where

$$Q_{d_i}{}^{c}(s) = \left\{sI_{\hat{l}} - \left[\hat{A}_{11} - \hat{B}_1\hat{K}_1\right] - \left[\hat{A}_{12} - \hat{B}_1\hat{K}_2\right]\left[sI_{\hat{n}-\hat{l}} - \hat{A}_{22} - \hat{B}_2\hat{K}_2\right]^{-1}\left[\hat{A}_{21} - \hat{B}_2\hat{K}_2\right]\right\}\hat{C}_1^{-1} \tag{3.3.13a}$$

$$P_{d_i}{}^{c}(s) = \hat{E}_{1,i} + \left[\hat{A}_{12} - \hat{B}_1\hat{K}_2\right]\left[sI_{\hat{n}-\hat{l}} - \left[\hat{A}_{22} - \hat{B}_2\hat{K}_2\right]\right]^{-1}\hat{E}_{2,i} \tag{3.3.13b}$$

From eqn.(3.3.13b), it can be seen that $P_{d_i}{}^{c}(s)$ for $i = 1,2,\ldots,r$ is the transfer function vector of the $(\hat{n}-\hat{l})$th-order, single-input, $\hat{l}$-output system given by

$$\xi_i = \left[\hat{A}_{22} - \hat{B}_2\hat{K}_2\right]\xi_i + \hat{E}_{2,i}\mu_i \tag{3.3.14a}$$

$$\upsilon_i = \left[\hat{A}_{12} - \hat{B}_1\hat{K}_2\right]\xi_i + \hat{E}_{1,i}\mu_i \tag{3.3.14b}$$

From Definitions 3.4 and 3.5, it follows that the set of d.b.z.'s of the system in eqns.(3.3.9a,b) for $i = 1,2,\ldots,r$ is equivalent to the set of blocking zeros of the system in eqns.(3.3.14a,b). Therefore, it follows that only the submatrix $\hat{K}_2$ affects the position of the closed-loop d.b.z.'s between the outputs and each disturbance, and that $\hat{K}_1$ has no effect on these zeros. Consequently a state feedback matrix $\begin{bmatrix}\hat{K}_1 & 0\end{bmatrix}$ has no effect on the d.b.z.'s and is used to assign the system poles. It should be noted that after the coordinate transformation, the outputs of the system in eqns.(3.3.9a,b) are simply the first $\hat{l}$ elements of the state vector. Therefore it is clear that d.z.'s and d.b.z.'s between the outputs and the disturbances are invariant under output feedback. The algorithms for assigning the d.b.z.'s and system poles for the systems $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$ by means of state and output feedback will be described in Chapters IV, V and VI.

3.4 NUMERICAL EXAMPLES

To illustrate the main results of this chapter, we consider the following examples:

Example 3.1: Consider a 4th-order system $\sum_d \left[A, B, C, E\right]$ which represents the linearized model of a nuclear rocket engine [44,45]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -65 & 65 & -19.5 & 19.5 \\ 0.1 & -0.1 & 0 & 0 \\ 1.0 & 0 & -0.5 & -1.0 \\ 0 & 0 & 0.4 & -0.4 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 65 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.4 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} 50 \\ 0 \\ -0.1 \\ 0 \end{bmatrix} d(t) \tag{3.4.1a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \tag{3.4.1b}$$

where, $\mathbf{x}(t) = \left[T, p, N, c\right]^T$, $\mathbf{u}(t) = \left[\delta_d, V\right]^T$, $\mathbf{y}(t) = \left[T, p\right]^T$, $d(t) = \delta_c$ and

T : Thrust chamber temperature,

p : Thrust chamber pressure,

N : Nuclear power,

c : Delayed neutrons,

δ_d : Control drum reactivity,

δ_c : Change in flow of delayed neutrons,

V : Turbine power control valve.

It is required to determine the sets of d.z.'s and d.b.z.'s for the following pairs:

(a) $d(t)$ and $\mathbf{y}(t)$

(b) $d(t)$ and $y_1(t)$.

In order to implement the factorization procedure on the system given in eqns.(3.4.1a,b), we need to compress the columns of C to get $\left[\hat{C}_1 \;\; 0\right]$, where $\hat{C}_1$ is 2×2 nonsingular matrix. In this example this can be accomplished by a rearrangement of the state of the system $\sum_d \left[A, B, C, E\right]$. This results in the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$, given by

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} -0.5 & -1.0 & 1.0 & 0.0 \\ 0.4 & -0.4 & 0.0 & 0.0 \\ -19.5 & 19.5 & -65.0 & 65.0 \\ 0.0 & 0.0 & 0.1 & -0.1 \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.4 \\ 65.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} -0.1 \\ 0.0 \\ 50.0 \\ 0.0 \end{bmatrix} d(t) \tag{3.4.2a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{3.4.2b}$$

Note that for $l > r$ and the condition $\left[\hat{A}_{12}^T (I_l - (\hat{E}_1^T)^+ \hat{E}_1^T)\right] = 0$ is satisfied. Therefore the transfer function vector relating the outputs to the disturbance has $(n-l)$ d.z.'s.

(a) Between $d(t)$ and $\mathbf{y}(t)$

$$\sigma\left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T (\hat{E}_1^T)^+ \hat{E}_2^T\right]\right] = \left\{-0.115\,,\, 435.015\right\}$$

$$\sigma\left[\hat{A}_{22}^T\right] = \left\{0.0\,,\, -65.1\right\}$$

Therefore, from Definition 3.5, there are two d.z.'s and (since this is a single-disturbance system) two d.b.z.'s located at -0.115 , 435.015.

(b) Between $d(t)$ and $y_1(t)$

$$\sigma\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{-1}\hat{A}_{12}\right]\right] = \left\{-0.115\,,-0.40\,,435.015\right\}$$

$$\sigma\left[\hat{A}_{22}\right] = \left\{0.0\,,-0.40\,,-65.1\right\}$$

Therefore, from Definition 3.4, there are three d.z.'s at -0.115 , -0.40 , 435.015 and two d.b.z.'s at -0.115 , 435.015.

Example 3.2: Consider a 3rd-order system $\sum_d \left[A\,,B\,,C\,,D\,,E\,,F\right]$ given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -6 & 2 & -3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ 0 & 5 \end{bmatrix}\mathbf{u}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{d}(t) \tag{3.4.3a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}\mathbf{u}(t) + \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{d}(t) \tag{3.4.3b}$$

It is required to determine the sets of d.z.'s and d.b.z.'s for the following pairs:

(a) $d_1(t)$ and $\mathbf{y}(t)$

(b) $d_1(t)$ and $y_2(t)$.

In order to compute the sets of d.z.'s and d.b.z.'s for this example, we note that these zeros are the same as the transmission zeros and blocking zeros of the 4-tuple system $\sum\left[A\,,E\,,C\,,F\right]$

given by eqns.(3.2.32a,b).

(a) Between $d_1(t)$ and $y(t)$

Since $l > r$ and the condition $\left[C^T(I_l-(F^T)^+F^T)\right] \neq 0$, system $\sum\left[A, E, C, F\right]$ has at most n transmission zeros. Therefore, we choose several arbitrary values of $\overline{M}$ and find that the eigenvalue of

$$\sigma\left[\left[A^T-C^T(F^T)^+E^T\right] + \left[C^T(I_l-(F^T)^+F^T)\right]\overline{M}\right]$$

at 0.0 is invariant under $\overline{M}$. Since

$$\sigma\left[A\right] = \left\{-1.0, -1.0, -3.0\right\},$$

it follows that the system (3.4.3a,b) has one d.z. and (since we have a single-disturbance case) one d.b.z. located at 0.0 .

(b) Between $d_1(t)$ and $y_2(t)$

$$\sigma\left[\left[A-EF^{-1}C\right]\right] = \left\{0.0, -1.0, -5.0\right\}$$

$$\sigma\left[A\right] = \left\{-1.0, -1.0, -3.0\right\}$$

Therefore from Definition 3.7, there are three d.z.'s at 0.0, -1.0, -5.0 and two d.b.z.'s at 0.0, -5.0.

3.5 CONCLUDING REMARKS

In this chapter the zeros of invertible linear, time-invariant, multivariable systems have been defined and relationships between them have been discussed. Based on the concepts of

invariant zeros and blocking zeros, definitions were given for the disturbance zeros (d.z.'s) and the disturbance blocking zeros (d.b.z.'s) of systems described by state-space models $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$. The relationship between d.b.z.'s locations and the steady-state of rejection of a class of disturbances at the outputs of the system was also discussed. An orthogonal state coordinate transformation on the state-space system $\sum_d \left[A, B, C, E\right]$ was used to compress the columns of the output matrix into the form $\left[\hat{C}_1 \;\; 0\right]$, where $\hat{C}_1$ is an $l \times l$ nonsingular matrix. Following this transformation, a factorization procedure was described which enabled the sets of d.z.'s and d.b.z.'s of the system to be computed using the concept of minimal order system inverses. It was shown that the d.b.z.'s were only affected by feedback of that part of the state vector which was not contained in the output and were invariant under output feedback. Two numerical examples were given to illustrate the methods of determining the sets of d.z.'s and d.b.z.'s for the systems $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$.

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CHAPTER IV

ASSIGNMENT OF DISTURBANCE BLOCKING ZEROS : SINGLE DISTURBANCE CASE

In this chapter some new results concerning the assignment of "disturbance blocking zeros"(d.b.z.'s) for a linear, time-invariant, multivariable system with a single disturbance are presented. The chapter is organized as follows: In Section 4.1, we present algorithms for solving the problem of assignment of the d.b.z.'s by means of state feedback using a constant gain as well as a dynamic compensator for the system described by $\sum_d \left[A, B, C, E\right]$. The theoretical basis of the algorithms is the factorization procedure of the disturbance transfer function matrix between the outputs and the disturbances. This enables us to use the concept of a minimal order inverse to determine the positions of d.b.z.'s which can be assigned by the state feedback laws. In Section 4.2, the results are then extended to the system described by $\sum_d \left[A, B, C, D, E, F\right]$, where it is shown that the assignment of d.b.z.'s can be achieved by increasing the order of the system, such that constant gain or dynamic state feedback can be applied to position the d.b.z.'s to appropriate positions in the complex plane, so as to eliminate in the steady state the effect of disturbances at the outputs. Numerical examples to illustrate the proposed algorithms are given in Section 4.3, followed by a brief discussion of the main results presented in this chapter.

4.1 ASSIGNMENT OF D.B.Z.'S OF THE SYSTEM $\sum_d \left[A, B, C, E\right]$

Consider a linear, time-invariant multivariable system described by its state-space equations

$$\dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) + E\ d(t) \tag{4.1.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{4.1.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^l$, $d(t) \in \mathbb{R}$ and A, B, C, E are of appropriate dimensions. In order to apply the factorization procedure considered in the previous chapter, we need to compress the columns of output matrix C i.e. $CT^T = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$ by using the SVD algorithm to get the system

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\,\hat{\mathbf{x}}(t) + \hat{B}\,\mathbf{u}(t) + \hat{E}\,d(t) \tag{4.1.2a}$$

$$\mathbf{y}(t) = \hat{C}\,\hat{\mathbf{x}}(t) \tag{4.1.2b}$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \hat{E} = \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix}, \hat{C} = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}$$

with $\hat{C}_1$ a nonsingular $l \times l$ matrix.

It was shown in Section 3.3, that the d.b.z.'s of the 'disturbance transfer matrix' $\mathbf{w}_d^c(s)$, (i.e. relating the outputs to the disturbances) are not invariant under state feedback applied to (4.1.2a,b) of the form $\begin{bmatrix} 0 & \hat{K}_2 \end{bmatrix}$, where $\hat{K}_2 \in \mathbb{R}^{m \times (n-l)}$. Therefore, this property can be used to assign the d.b.z.'s of the system in eqns.(4.1.1a,b) at desired locations, in order to eliminate the steady state effect of a class of disturbances at the outputs.

4.1.1 Assignment of D.B.Z.'s by Constant Gain State Feedback

The problem that we consider now is to show the effect of an $m \times n$ constant gain matrix $\begin{bmatrix} 0 & \hat{K}_2 \end{bmatrix}$ defined by the feedback law

$$\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & \hat{K}_2 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{4.1.3}$$

on the d.b.z.'s of the system (4.1.2a,b). Note that the state feedback matrix $\begin{bmatrix} 0 & \hat{K}_2 \end{bmatrix}$ in (4.1.3) corresponds to state feedback matrix K_2 for (4.1.1a,b), where $K_2 T^T = \begin{bmatrix} 0 & \hat{K}_2 \end{bmatrix}$. On

implementing the feedback law in eqn.(4.1.3), the system in eqns.(4.1.2a,b) becomes

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12}-\hat{B}_1\hat{K}_2 \\ \hat{A}_{21} & \hat{A}_{22}-\hat{B}_2\hat{K}_2 \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} d(t) \tag{4.1.4a}$$

$$\mathbf{y}(t) = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix} \tag{4.1.4b}$$

Then by using the factorization procedure, it was shown in Section 3.2 that the d.z.'s of the system in eqns.(4.1.4a,b) are equivalent to the d.z.'s of the $(n-l)$th-order, single-input, l-output system given by

$$\dot{\xi} = \left[\hat{A}_{22}-\hat{B}_2\hat{K}_2\right] \xi + \hat{E}_2 \mu \tag{4.1.5a}$$

$$\upsilon = \left[\hat{A}_{12}-\hat{B}_1\hat{K}_2\right] \xi + \hat{E}_1 \mu \tag{4.1.5b}$$

Now, by using the concept of a minimal order inverse (assuming that the system in (4.1.5a,b) is invertible), we can examine the problem of assignment of the d.b.z.'s of the system (4.1.5a,b). We also assume that, the open-loop system in eqns.(4.1.2a,b) has $(n-l)$ d.z.'s. This assumption is made to simplify the mathematical fomulation (see Remark 3.3 in Chapter III), and implies that

$$\hat{A}_{12}^T \left[I_l-(\hat{E}_1^T)^+\hat{E}_1^T\right] = 0 \tag{4.1.6}$$

We also assume further that, the closed-loop system in eqns.(4.1.4a,b) has $(n-l)$ d.z.'s. This assumption is not necessary for achieving disturbance rejection and implies that a condition of the form

$$\hat{K}_2^{\,T} \left[\hat{B}_1^{\,T} \left[I_l-(\hat{E}_1^T)^+\hat{E}_1^T\right]\right] = 0 \tag{4.1.7}$$

must be satisfied. This condition can be satisfied by implementing a unity-rank $m\times(n-l)$ constant matrix $\hat{K}_2 = \mathbf{q}_2\,\mathbf{p}_2$, where $\mathbf{q}_2$ and $\mathbf{p}_2$ are $m\times 1$ and $1\times(n-l)$ vectors respectively. The vector $\mathbf{q}_2$ can be specified arbitrarily such that

$$\mathbf{q}_2^T \left[\hat{B}_1^T \left[I_l - (\hat{E}_1^T)^+ \hat{E}_1^T\right]\right] = 0 \tag{4.1.8}$$

By restricting the constant gain matrix $\hat{K}_2$ to have unity-rank, the system in eqns.(4.1.5a,b) can be written as

$$\dot{\xi} = \left[\hat{A}_{22} - \mathbf{b}_2 \mathbf{p}_2\right]\xi + \hat{E}_2 \mu \tag{4.1.9a}$$

$$\upsilon = \left[\hat{A}_{12} - \mathbf{b}_1 \mathbf{p}_2\right]\xi + \hat{E}_1 \mu \tag{4.1.9b}$$

where $\mathbf{b}_1 = \hat{B}_1 \mathbf{q}_2$ and $\mathbf{b}_2 = \hat{B}_2 \mathbf{q}_2$, respectively.

For $\left[l \geq 1\right]$, the system in eqns.(4.1.9a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}^* = \left[\hat{A}_{22}^T - \mathbf{p}_2^T \mathbf{b}_2^T\right]\xi^* + \left[\hat{A}_{12}^T - \mathbf{p}_2^T \mathbf{b}_1^T\right]\mu^* \tag{4.1.10a}$$

$$\upsilon^* = \left[\hat{E}_2^T\right]\xi^* + \left[\hat{E}_1^T\right]\mu^* \tag{4.1.10b}$$

A right inverse of the system in eqns.(4.1.10a,b) is given by [1],

$$\dot{\xi}^* = \left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T (\hat{E}_1^T)^+ \hat{E}_2^T\right] - \mathbf{p}_2^T \left[\mathbf{b}_2^T - \mathbf{b}_1^T (\hat{E}_1^T)^+ \hat{E}_2^T\right]\right]\xi^*$$

$$- \left[\left[\hat{A}_{12}^T - \mathbf{p}_2^T \mathbf{b}_1^T\right](\hat{E}_1^T)^+\right]\upsilon^* \tag{4.1.11a}$$

$$\mu^* = \left[-(\hat{E}_1^T)^+ \hat{E}_2^T\right]\xi^* + \left[(\hat{E}_1^T)^+\right]\upsilon^* \tag{4.1.11b}$$

Now, we can define the sets of the transmission zeros and blocking zeros of the system (4.1.9a,b) which are equivalent to closed-loop d.z.'s (Z_c^D) and d.b.z.'s (Z_c^B) of the system $\sum_d \left[A, B, C, E\right]$ and which are affected by the constant state feedback K_2.

Definition 4.1:

$$Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_s}\right\} = \sigma\left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T(\hat{E}_1^T)^+\hat{E}_2^T\right] - \mathbf{p}_2^T\left[\mathbf{b}_2^T - \mathbf{b}_1^T(\hat{E}_1^T)^+\hat{E}_2^T\right]\right]$$

$$Z_c^B\left\{\sum_d \left[A,B,C,E\right]|_{K_s}\right\} = Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_s}\right\}$$
$$- \left[Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_s}\right\} \cap \sigma\left[\hat{A}_{22}^T - \mathbf{p}_2^T\mathbf{b}_2^T\right]\right]$$

From the above definition, it follows that the closed-loop d.b.z.'s are those eigenvalues of $\left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T(\hat{E}_1^T)^+\hat{E}_2^T\right] - \mathbf{p}_2^T\left[\mathbf{b}_2^T - \mathbf{b}_1^T(\hat{E}_1^T)^+\hat{E}_2^T\right]\right]$ which are not also the eigenvalues of $\left[\hat{A}_{22}^T - \mathbf{p}_2^T\mathbf{b}_2^T\right]$. Now the matrix

$$\left[\left[\hat{A}_{22}^T - \hat{A}_{12}^T(\hat{E}_1^T)^+\hat{E}_2^T\right] - \mathbf{p}_2^T\left[\mathbf{b}_2^T - \mathbf{b}_1^T(\hat{E}_1^T)^+\hat{E}_2^T\right]\right]$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi} = \left[\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]\right]\Psi + \left[\left[\mathbf{b}_2 - \hat{E}_2\hat{E}_1^+\mathbf{b}_1\right]\right]\Phi \tag{4.1.12}$$

subject to the state feedback law

$$\Phi = -\mathbf{p}_2\Psi \tag{4.1.13}$$

Similarly the matrix $\left[\hat{A}_{22}^T - \mathbf{p}_2^T\mathbf{b}_2^T\right]$ is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi} = \hat{A}_{22}\Psi + \mathbf{b}_2\Phi \tag{4.1.14}$$

subject to the state feedback law in eqn.(4.1.13).

It is well known [2] that if the system in eqn.(4.1.12) is controllable, then all the eigenvalues of the matrix $\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]$ can be arbitrarily assigned by state feedback, otherwise only those eigenvalues of $\left[\hat{A}_{22} - \hat{E}_2\hat{E}_1^+\hat{A}_{12}\right]$ can be assigned arbitrarily by state feedback

which correspond to the controllable modes of the system.

From the above results, we can now outline an algorithm for the placement of the d.b.z.'s of the system $\sum_d \left[A, B, C, E\right]$ using constant state feedback:

Algorithm 4.1: *(Assignment of the d.b.z.'s of the system* $\sum_d \left[A, B, C, E\right]$ *using constant state feedback)*

(i) Apply an orthogonal state coordinate transformation on the system $\sum_d \left[A, B, C, E\right]$ to compress the output matrix C to $\left[\hat{C}_1 \;\; 0\right]$ where $\hat{C}_1$ is an $l \times l$ nonsingular matrix, and obtain a system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ given by eqns.(4.1.2a,b).

(ii) Specify $\mathbf{q}_2$ arbitrarily such that the condition in eqn.(4.1.8) is satisfied.

(iii) Calculate the eigenvalues of the matrices $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right]$ and $\left[\hat{A}_{22}\right]$. Then, the open-loop d.b.z.'s of the system (4.1.2a,b), are those eigenvalues of $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right]$ which are not also the eigenvalues of $\left[\hat{A}_{22}\right]$.

(iv) Determine if the system in eqn.(4.1.12) is controllable; if it is not controllable, determine which eigenvalues of $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right]$ correspond to the controllable mode of the system.

(v) Calculate the state feedback $\mathbf{p}_2$ in eqn.(4.1.13) using Algorithm 2.1, such that those eigenvalue of $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right]$ which are the d.b.z.'s and which correspond to controllable modes of the system in eqn.(4.1.12) are positioned at desired locations in the complex plane. These assigned eigenvalues are the required d.b.z.'s and their locations are chosen such that disturbance rejection is achieved in the steady state.

(vi) The unity-rank state feedback matrix is then obtained as $\hat{K}_2 = \mathbf{q}_2 \mathbf{p}_2$ for the system

$\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$, so that for the system $\sum_d \left[A, B, C, E\right]$, $K_2 = \left[0 \ \hat{K}_2\right]T$, where T is the orthogonal coordinate transformation matrix.

(vii) Implement the feedback law

$$\mathbf{u}(t) = \mathbf{v}(t) - K_2\,\mathbf{x}(t) \tag{4.1.15}$$

on the system in eqns.(4.1.1a,b) to get the closed-loop system

$$\dot{\mathbf{x}}(t) = \tilde{A}\,\mathbf{x}(t) + B\,\mathbf{v}(t) + E\,d(t) \tag{4.1.16a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{4.1.16b}$$

where

$$\tilde{A} = \left[A - BK_2\right]$$

which has all its d.b.z.'s assigned at the desired values.

4.1.2 Assignment of D.B.Z.'s by Dynamic State Feedback

For the system described by eqns.(4.1.2a,b), the problem of assigning the d.b.z.'s that we investigate is to determine a dynamic state feedback matrix $\hat{K}_2(s) = K_2(s)T^T$, which is the transfer function matrix of the system

$$\dot{\mathbf{z}}_2(t) = F_2\,\mathbf{z}_2(t) + \left[0 \ \hat{G}_2\right]\hat{\mathbf{x}}(t) \tag{4.1.17a}$$

$$\mathbf{u}_2(t) = H_2\,\mathbf{z}_2(t) + \left[0 \ \hat{J}_2\right]\hat{\mathbf{x}}(t) \tag{4.1.17b}$$

with dynamic state feedback defined by

$$\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t) \tag{4.1.17c}$$

In (4.1.17), $\mathbf{z}_2(t) \in \mathbb{R}^q$ is the state vector of the compensator, $\mathbf{u}_2(t) \in \mathbb{R}^m$ is the output of the dynamic compensator and $F_2, \hat{G}_2, H_2, \hat{J}_2$ are matrices of appropriate dimensions. This type of control law introduces additional d.b.z.'s and at the same time assigns d.b.z.'s of the resulting closed-loop system at appropriate locations in the complex plane. These locations can be chosen,

such that complete disturbance rejection is achieved in the steady state. On implementing the feedback law in eqn.(4.1.17c), the system in eqns.(4.1.2a,b) becomes

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12}-\hat{B}_1\hat{J}_2 \\ \hat{A}_{21} & \hat{A}_{22}-\hat{B}_2\hat{J}_2 \end{bmatrix}\hat{\mathbf{x}}(t) - \begin{bmatrix} \hat{B}_1 H_2 \\ \hat{B}_2 H_2 \end{bmatrix}\mathbf{z}_2(t) + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}\mathbf{v}(t) + \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \end{bmatrix} d(t) \tag{4.1.18a}$$

$$\mathbf{y}(t) = \begin{bmatrix} \hat{C}_1 & 0 \end{bmatrix}\hat{\mathbf{x}}(t) \tag{4.1.18b}$$

Combining eqns.(4.1.18a,b) and eqns.(4.1.17a,b), the augmented closed-loop feedback system is described by

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{z}}_2(t) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12}-\hat{B}_1\hat{J}_2 & -\hat{B}_1 H_2 \\ \hat{A}_{21} & \hat{A}_{22}-\hat{B}_2\hat{J}_2 & -\hat{B}_2 H_2 \\ 0 & \hat{G}_2 & F_2 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \end{bmatrix}\mathbf{v}(t) + \begin{bmatrix} \hat{E}_1 \\ \hat{E}_2 \\ 0 \end{bmatrix} d(t) \tag{4.1.19a}$$

$$\mathbf{y}(t) = \begin{bmatrix} \hat{C}_1 & 0 & 0 \end{bmatrix}\begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{z}_2(t) \end{bmatrix} \tag{4.1.19b}$$

Then, by following the factorization procedure, the closed-loop disturbance transfer function matrix $\mathbf{w}_d^c(s)$ can be written as

$$\mathbf{w}_d^c(s) = \left[Q_d{}^c(s)\right]^{-1} P_d{}^c(s) \tag{4.1.20}$$

where,

$$Q_d{}^c(s) = \left\{\left[sI_l-\hat{A}_{11}\right] - \left[(\hat{A}_{12}-\hat{B}_1\hat{J}_2)\ -\hat{B}_1 H_2\right]\left[Q(s)\right]^{-1}\begin{bmatrix} \hat{A}_{21} \\ 0 \end{bmatrix}\right\}\hat{C}_1^{-1} \tag{4.1.21a}$$

$$P_d{}^c(s) = \left\{\hat{E}_1 + \left[(\hat{A}_{12}-\hat{B}_1\hat{J}_2)\ -\hat{B}_1 H_2\right]\left[Q(s)\right]^{-1}\begin{bmatrix} \hat{E}_2 \\ 0 \end{bmatrix}\right\} \tag{4.1.21b}$$

and

$$Q(s) = \begin{bmatrix} \left[sI_{n-l}-(\hat{A}_{22}-\hat{B}_2\hat{J}_2)\right] & \hat{B}_2H_2 \\ -\hat{G}_2 & \left[sI_q-F_2\right] \end{bmatrix}$$

The matrix $P_d{}^c(s)$ is the transfer matrix of the $(n-l+q)$th, single-input, l-output system given by

$$\dot{\xi} = \begin{bmatrix} \hat{A}_{22}-\hat{B}_2\hat{J}_2 & -\hat{B}_2H_2 \\ \hat{G}_2 & F_2 \end{bmatrix}\xi + \begin{bmatrix} \hat{E}_2 \\ 0 \end{bmatrix}\mu \tag{4.1.22a}$$

$$\upsilon = \left[\hat{A}_{12}-\hat{B}_1\hat{J}_2 \quad -\hat{B}_1H_2\right]\xi + \hat{E}_1\mu \tag{4.1.22b}$$

It was shown in Chapter III (Theorem 3.3) that, the d.z.'s of the system in eqns.(4.1.19a,b) are equivalent to the transmission zeros of the system in eqns.(4.1.22a,b) and that there are at most $(n-l+q)$ such zeros. Also, the problem of computing the d.b.z.'s for the system in eqns.(4.1.19a,b) is reduced to that of computing the blocking zeros of the system in eqns.(4.1.22a,b), which are the same as its transmission zeros when there are no pole-zero cancellation, since it is a single-input system.

In order to compute the d.z.'s and d.b.z.'s of the system in eqns.(4.1.22a,b) using the concept of minimal order inverses, we assume that the transfer matrix $\mathbf{w}_d^c(s)$ is invertible. Then, it follows that $P_d{}^c(s)$ is invertible, which in turn implies that the system in eqns.(4.1.22a,b) is invertible. We also assume without loss of generality that, the open-loop system in eqns.(4.1.2a,b) has $(n-l)$ d.z.'s. This assumption implies that

$$\hat{A}_{12}^T\left[I_l-(\hat{E}_1^T)^+\hat{E}_1^T\right] = 0 \tag{4.1.23}$$

We assume further that, the closed-loop transfer matrix $\mathbf{w}_d^c(s)$ has $(n-l+q)$ d.z.'s. This assump-

tion is made to simplify the analysis and is not necessary for achieving disturbance rejection. It implies that a condition of the form

$$\begin{bmatrix} \hat{J}_2^T \hat{B}_1^T \\ H_2^T \hat{B}_1^T \end{bmatrix} \left[\left(I_l - (\hat{E}_1^T)^+ \hat{E}_1^T \right) \right] = 0 \tag{4.1.24}$$

is satisfied. This condition can be satisfied by implementing unity-rank constant matrices $\hat{J}_2$ and H_2 given by

$$\hat{J}_2 = \mathbf{q}_2 \mathbf{p}_2$$
$$H_2 = \mathbf{q}_2 \mathbf{h}_2$$

where $\mathbf{q}_2$, $\mathbf{p}_2$ and $\mathbf{h}_2$ are $m \times 1$, $1 \times q$ and $1 \times (n-l)$ respectively.

The vector $\mathbf{q}_2$ can be specified arbitrarily such that a condition of the form

$$\mathbf{q}_2^T \left[\hat{B}_1^T \left(I_l - (\hat{E}_1^T)^+ \hat{E}_1^T \right) \right] = 0 \tag{4.1.25}$$

is satisfied. Therefore, on implementing the unity-rank constant matrices $\hat{J}_2$ and H_2, the system in eqns.(4.1.22a,b) becomes

$$\dot{\xi} = \begin{bmatrix} \hat{A}_{22} - \mathbf{b}_2 \mathbf{p}_2 & -\mathbf{b}_2 \mathbf{h}_2 \\ \hat{G}_2 & F_2 \end{bmatrix} \xi + \begin{bmatrix} \hat{E}_2 \\ 0 \end{bmatrix} \mu \tag{4.1.26a}$$

$$\upsilon = \left[\hat{A}_{12} - \mathbf{b}_1 \mathbf{p}_2 \quad -\mathbf{b}_1 \mathbf{h}_2 \right] \xi + \hat{E}_1 \mu \tag{4.1.26b}$$

where $\mathbf{b}_1 = \hat{B}_1 \mathbf{q}_2$ and $\mathbf{b}_2 = \hat{B}_2 \mathbf{q}_2$, respectively.

For $\left[l \geq 1 \right]$, the system in eqn.(4.1.26a,b) has a left inverse which is the transpose of a right inverse of the dual system [1]

$$\dot{\xi}^* = \begin{bmatrix} \left(\hat{A}_{22}^T - \mathbf{p}_2^T \mathbf{b}_2^T \right) & \hat{G}_2^T \\ -\mathbf{h}_2^T \mathbf{b}_2^T & F_2^T \end{bmatrix} \xi^* + \begin{bmatrix} \left(\hat{A}_{12}^T - \mathbf{p}_2^T \mathbf{b}_1^T \right) \\ -\mathbf{h}_2^T \mathbf{b}_1^T \end{bmatrix} \mu^* \tag{4.1.27a}$$

$$\upsilon^* = \left[\hat{E}_2^T \; 0\right]\xi^* + \left[\hat{E}_1^T\right]\mu^* \tag{4.1.27b}$$

A right inverse of the system in eqns.(4.1.27a,b) is given by

$$\dot{\xi}^* = \begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T\mathbf{b}_2{}^T\right] - \left[\hat{A}_{12}^T - \mathbf{p}_2{}^T\mathbf{b}_1{}^T\right](\hat{E}_1^T)^+\hat{E}_2^T & \hat{G}_2^T \\ \left[-\mathbf{h}_2{}^T\mathbf{b}_2{}^T + \mathbf{h}_2{}^T\mathbf{b}_1{}^T(\hat{E}_1^T)^+\hat{E}_2^T\right] & F_2^T \end{bmatrix}\xi^*$$

$$+ \begin{bmatrix} \left[\hat{A}_{12}^T - \mathbf{p}_2{}^T\mathbf{b}_1{}^T\right] \\ -\mathbf{h}_2{}^T\mathbf{b}_1{}^T \end{bmatrix}(\hat{E}_1^T)^+\upsilon^* \tag{4.1.28a}$$

$$\mu^* = -(\hat{E}_1^T)^+\left[\hat{E}_2^T \; 0\right]\xi^* + \left[(\hat{E}_1^T)^+\right]\upsilon^* \tag{4.1.28b}$$

Now, we can define the sets of transmission zeros and blocking zeros of the system (4.1.26a,b), which are equivalent to those closed-loop d.z.'s (Z_c^D) and d.b.z.'s (Z_c^B) of the system $\sum_d \left[A, B, C, E\right]$ which are affected by the dynamic state feedback compensator $\left[K_2(s) = \hat{K}_2(s)T\right]$:

Definition 4.2:

$$Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_2(s)}\right\} = \sigma\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T\mathbf{b}_2{}^T\right] - \left[\hat{A}_{12}^T - \mathbf{p}_2{}^T\mathbf{b}_1{}^T\right](\hat{E}_1^T)^+\hat{E}_2^T & \hat{G}_2^T \\ \left[-\mathbf{h}_2{}^T\mathbf{b}_2{}^T + \mathbf{h}_2{}^T\mathbf{b}_1{}^T(\hat{E}_1^T)^+\hat{E}_2^T\right] & F_2^T \end{bmatrix}$$

$$Z_c^B\left\{\sum_d \left[A,B,C,E\right]|_{K_2(s)}\right\} = Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_2(s)}\right\}$$

$$- \left[Z_c^D\left\{\sum_d \left[A,B,C,E\right]|_{K_2(s)}\right\} \cap \sigma\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T\mathbf{b}_2{}^T\right] & \hat{G}_2^T \\ -\mathbf{h}_2{}^T\mathbf{b}_2{}^T & F_2^T \end{bmatrix}\right]$$

From above definition, it follows that the closed-loop d.b.z.'s are those eigenvalues of the matrix

$$\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T \mathbf{b}_2{}^T\right] - \left[\hat{A}_{12}^T - \mathbf{p}_2{}^T \mathbf{b}_1{}^T\right](\hat{E}_1^T)^+ \hat{E}_2^T & \hat{G}_2^T \\ \left[-\mathbf{h}_2{}^T \mathbf{b}_2{}^T + \mathbf{h}_2{}^T \mathbf{b}_1{}^T (\hat{E}_1^T)^+ \hat{E}_2^T\right] & F_2^T \end{bmatrix}$$

which are not also the eigenvalues of the matrix

$$\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T \mathbf{b}_2{}^T\right] & \hat{G}_2^T \\ -\mathbf{h}_2{}^T \mathbf{b}_2{}^T & F_2^T \end{bmatrix}$$

Now the matrix

$$\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T \mathbf{b}_2{}^T\right] - \left[\hat{A}_{12}^T - \mathbf{p}_2{}^T \mathbf{b}_1{}^T\right](\hat{E}_1^T)^+ \hat{E}_2^T & \hat{G}_2^T \\ \left[-\mathbf{h}_2{}^T \mathbf{b}_2{}^T + \mathbf{h}_2{}^T \mathbf{b}_1{}^T (\hat{E}_1^T)^+ \hat{E}_2^T\right] & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop state matrix of a system given by:

$$\dot{\Psi} = \begin{bmatrix} \left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^+ \hat{A}_{12}\right] & 0 \\ \hat{G}_2 & F_2 \end{bmatrix} \Psi + \begin{bmatrix} \left[\mathbf{b}_2 - \hat{E}_2 \hat{E}_1^+ \mathbf{b}_1\right] \\ 0 \end{bmatrix} \Phi \tag{4.1.29}$$

subject to the state feedback law

$$\Phi = -\left[\mathbf{p}_2 \;\; \mathbf{h}_2\right] \Psi \tag{4.1.30}$$

Similarly, the matrix

$$\begin{bmatrix} \left[\hat{A}_{22}^T - \mathbf{p}_2{}^T \mathbf{b}_2{}^T\right] & \hat{G}_2^T \\ -\mathbf{h}_2{}^T \mathbf{b}_2{}^T & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi} = \begin{bmatrix} \hat{A}_{22} & 0 \\ \hat{G}_2 & F_2 \end{bmatrix} \Psi + \begin{bmatrix} \mathbf{b}_2 \\ 0 \end{bmatrix} \Phi \tag{4.1.31}$$

subject to the state feedback law in eqn.(4.1.30).

Therefore, if the system in eqn.(4.1.29) is controllable, then all the eigenvalues of the matrix

$$\begin{bmatrix} \left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right] & 0 \\ \hat{G}_2 & F_2 \end{bmatrix}$$

can be arbitrarily assigned by state feedback (4.1.30), otherwise only those eigenvalues of the above matrix can be assigned by state feedback which correspond to the controllable modes of the system. We now examine the controllability of (4.1.29).

Theorem 4.1: The system in (4.1.29) is controllable if the matrix $\hat{G}_2$ is selected such that

$$rank \left[-\hat{G}_2 \left[\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right] - \lambda I_{n-l} \right]^{-1} \left[\mathbf{b}_2 - \hat{E}_2 \hat{E}_1^{+} \mathbf{b}_1\right] \quad \left[F_2 - \lambda I_q\right] \right] = q \tag{4.1.32}$$

for all complex values of $\lambda \in \sigma(F_2)$.

Proof: The system (4.1.29) is controllable if, and only if the matrix

$$\Gamma\left[\lambda\right] = \begin{bmatrix} \left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^{+} \hat{A}_{12}\right] - \lambda I_{n-l} & 0 & \left[\mathbf{b}_2 - \hat{E}_2 \hat{E}_1^{+} \mathbf{b}_1\right] \\ \hat{G}_2 & (F_2 - \lambda I_q) & 0 \end{bmatrix} \tag{4.1.33}$$

has full rank $\left[= n - l + q\right]$ for all complex values of λ.

Postmultiplying Γ by

$$\begin{bmatrix} I_{n-l} & 0 & 0 \\ 0 & 0 & I_q \\ 0 & I_m & 0 \end{bmatrix}$$

we obtain the matrix

$$\Lambda\left[\lambda\right] = \begin{bmatrix} \left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l} & \left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & 0 \\ \hat{G}_2 & 0 & \left[F_2-\lambda I_q\right] \end{bmatrix} \tag{4.1.34}$$

Assuming that $\lambda \notin \sigma\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]$, we can factor the matrix Λ as

$$\Lambda\left[\lambda\right] = \begin{bmatrix} \left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l} & 0 \\ 0 & I_q \end{bmatrix}$$

$$\begin{bmatrix} I_{n-l} & \left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & 0 \\ \hat{G}_2 & 0 & (F_2-\lambda I_q) \end{bmatrix}$$

$$\equiv \begin{bmatrix} \left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l} & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_{n-l} & 0 \\ \hat{G}_2 & I_q \end{bmatrix}$$

$$\begin{bmatrix} I_{n-l} & \left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & 0 \\ 0 & -\hat{G}_2\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & (F_2-\lambda I_q) \end{bmatrix} \tag{4.1.35}$$

which implies that

$$rank\ \Lambda\left[\lambda\right] = rank \begin{bmatrix} I_{n-l} & \left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & 0 \\ 0 & -\hat{G}_2\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] & \left[F_2-\lambda I_q\right] \end{bmatrix}$$

$$= n-l+rank\left[-\hat{G}_2\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right]\ (F_2-\lambda I_q)\right]$$

Therefore the above matrix Λ has full row rank $\left[=n-l+q\right]$ if and only if

$$rank\left[-\hat{G}_2\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right]\ (F_2-\lambda I_q)\right] = q$$

Note that we need to check the above rank condition only for $\lambda \in \sigma\left[F_2\right]$ since the rank condition is always satisfied for all $\lambda \notin \sigma\left[F_2\right]$. This completes the proof of the Theorem.

Using the result of the above Theorem, we can find the state feedback $\left[\mathbf{p}_2\ \mathbf{h}_2\right]$ given in eqn.(4.1.30), such that all the d.b.z.'s consisting of:

(a) the eigenvalues of the matrix $\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]$ which do not cancel out with the eigenvalues of $\left[\hat{A}_{22}\right]$, and

(b) the eigenvalues of $\left[F_2\right]$,

can be assigned arbitrarily at any locations in the complex plane, in order to asymptotically reject the effect of a class of disturbances at the outputs.

Remark 4.1: The matrix F_2 can be selected such that its eigenvalues are stable and different from the eigenvalues of $\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]$; the matrix $\hat{G}_2$ can be chosen to ensure that $\lambda \in \sigma\left[F_2\right]$ is not a transmission zero of $\Sigma\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right], \left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right], \hat{G}_2\right]$ i.e.

$$\hat{G}_2\left[\left[\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right]-\lambda I_{n-l}\right]^{-1}\left[\mathbf{b}_2-\hat{E}_2\hat{E}_1^{+}\mathbf{b}_1\right] \neq 0$$

This is necessary to satisfy Theorem 4.1.

We now outline an algorithm for the placement of d.b.z.'s of systems described by $\sum_d \left[A, B, C, E\right]$ using dynamic state feedback:

Algorithm 4.2: *(Assignment of the d.b.z.'s of the system* $\sum_d \left[A, B, C, E\right]$ *by means of dynamic state feedback)*

(i) Apply orthogonal state coordinate transformation on the system $\sum_d \left[A, B, C, E\right]$ to compress the output matrix C to $\left[\hat{C}_1 \;\; 0\right]$ and obtain a system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ given by eqns.(4.1.2a,b)

(ii) Specify $\mathbf{q}_2$ such that the condition in eqn.(4.1.25) is satisfied.

(iii) Calculate the eigenvalues of matrix $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^+ \hat{A}_{12}\right]$ and $\left[\hat{A}_{22}\right]$, and select a matrix F_2 such that its eigenvalues are different from those of $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^+ \hat{A}_{12}\right]$. The d.b.z.'s which are affected by the state feedback law in eqn.(4.1.30) are those eigenvalues of the matrix $\left[\hat{A}_{22} - \hat{E}_2 \hat{E}_1^+ \hat{A}_{12}\right]$ which are not also the eigenvalues of $\left[\hat{A}_{22}\right]$, together with the eigenvalues of the matrix $\left[F_2\right]$.

(iv) Ensure that the system in eqn.(4.1.29) is controllable using Theorem 4.1, otherwise only those eigenvalues of the system can be assigned by state feedback which correspond to the controllable modes of the system.

(v) Calculate the state feedback $\left[\mathbf{p}_2 \;\; \mathbf{h}_2\right]$ given in eqn.(4.1.30) using Algorithm 2.1 to achieve desired values for those eigenvalues of

$$\begin{bmatrix} \left(\hat{A}_{22}-\hat{E}_2\hat{E}_1^{+}\hat{A}_{12}\right) & 0 \\ \hat{G}_2 & F_2 \end{bmatrix}$$

which are the d.b.z.'s.

(vi) The dynamic state feedback is then given by eqns.(4.1.17a,b) for the system $\sum_d \left(\hat{A},\hat{B},\hat{C},\hat{E}\right)$ so that for the system $\sum_d \left(A,B,C,E\right)$, the dynamic state feedback is given by

$$\dot{\mathbf{z}}_2(t) = F_2\mathbf{z}_2(t)+G_2\mathbf{x}(t) \tag{4.1.36a}$$

$$\mathbf{u}_2(t) = H_2\mathbf{z}_2(t)+J_2\mathbf{x}(t) \tag{4.1.36b}$$

where $G_2 = \left[0 \;\; \hat{G}_2\right]T$ and $J_2 = \left[0 \;\; \hat{J}_2\right]T$.

(vii) Implement the feedback law

$$\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t)$$

on the system in eqns.(4.1.1a,b) to get the closed-loop system

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{A}\,\tilde{\mathbf{x}}(t) + \tilde{B}\,\mathbf{v}(t) + \tilde{E}d(t) \tag{4.1.37a}$$

$$\mathbf{y}(t) = \tilde{C}\,\tilde{\mathbf{x}}(t) \tag{4.1.37b}$$

where

$$\tilde{A} = \begin{bmatrix} A-BJ_2 & -BH_2 \\ G_2 & F_2 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

with

$$\tilde{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_2(t) \end{bmatrix}$$

This system will have d.b.z.'s assigned at the desired locations in the complex plane.

4.2 ASSIGNMENT OF D.B.Z.'S OF THE SYSTEM $\sum_d \left[A, B, C, D, E, F\right]$

In this section, the results of the preceding section are extended to solve the problem of assignment of d.b.z.'s by means of constant gain as well as dynamic state feedback for systems described by the state-space model $\sum_d \left[A, B, C, D, E, F\right]$, i.e.

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + E\,d(t) \tag{4.2.1a}$$
$$\mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) + F\,d(t) \tag{4.2.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^l$, $d(t) \in \mathbb{R}$ and the matrices A, B, C, D, E, F have appropriate dimensions.

To handle this problem (see Section 3.2), we created a higher order system denoted by $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ obtained by incorporating at the outputs of the system in eqns.(4.2.1a,b) a dynamic output compensator of order p. The resulting system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ is described by the following equations

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\,\hat{\mathbf{x}}(t) + \hat{B}\,\mathbf{u}(t) + \hat{E}\,d(t) \tag{4.2.2a}$$
$$\hat{\mathbf{y}}(t) = \hat{C}\,\hat{\mathbf{x}}(t) \tag{4.2.2b}$$

and

$$\mathbf{y}(t) = \begin{bmatrix} 0 & \hat{C}_1 \end{bmatrix} \hat{\mathbf{x}}(t) + D\,\mathbf{u}(t) + F\,d(t) \tag{4.2.2c}$$

where

$$\hat{A} = \begin{bmatrix} \Omega & \Theta C \\ 0 & A \end{bmatrix}, \hat{B} = \begin{bmatrix} \Theta D \\ B \end{bmatrix}, \hat{E} = \begin{bmatrix} \Theta F \\ E \end{bmatrix}, \hat{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

with

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{x}(t) \end{bmatrix}$$

Note that, the above system enables us to use the factorization procedure and the concept of minimal order inverses to determine the positions of the d.b.z.'s which can be assigned by state feedback laws.

4.2.1 Assignment of D.B.Z.'s by Constant Gain State Feedback

Let us define the constant state feedback by

$$\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & K_2 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{4.2.3}$$

Implementing the feedback law in eqn.(4.2.3) on the system in eqns.(4.2.2a-c), we get

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \Omega & \Theta\left[C-DK_2\right] \\ 0 & \left[A-BK_2\right] \end{bmatrix} \hat{\mathbf{x}}(t) + \begin{bmatrix} \Theta D \\ B \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} \Theta F \\ E \end{bmatrix} d(t) \tag{4.2.4a}$$

$$\hat{\mathbf{y}}(t) = \begin{bmatrix} I_p & 0 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{4.2.4b}$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & \left[C-DK_2\right] \end{bmatrix} \hat{\mathbf{x}}(t) + D\,\mathbf{v}(t) + Fd(t) \tag{4.2.4c}$$

Then, by using the factorization procedure, the transfer function matrix between the outputs $\hat{\mathbf{y}}(t)$ and the disturbances $d(t)$, can be expressed as:

$$\mathbf{w}_d^c(s) = \left[Q_d{}^c(s)\right]^{-1} P_d{}^c(s) \tag{4.2.5}$$

where $Q_d{}^c(s)$ and $P_d{}^c(s)$ are $p \times p$ and $p \times 1$ rational function matrices, and are given by

$$Q_d{}^c(s) = \left[sI_p - \Omega\right] \tag{4.2.6a}$$

$$P_d{}^c(s) = \Theta\left\{F + \left[C-DK_2\right]\left[sI_n-(A-BK_2)\right]^{-1} E\right\} \tag{4.2.6b}$$

From eqn.(4.2.6b) it can be seen that the rational function matrix $P_d{}^c(s)$ is the product of the

matrix Θ and the transfer matrix of the n-th order, single-input, l-output system given by the following state space model

$$\dot{\xi} = \left[A - BK_2\right]\xi + E\,\mu \tag{4.2.7a}$$

$$\upsilon = \left[C - DK_2\right]\xi + F\,\mu \tag{4.2.7b}$$

Based on Theorem 3.7 in Section 3.3, the problem of computing the d.z.'s of the system in eqns.(4.2.5a,b) is reduced to that of computing the d.z.'s of the system represented by eqns.(4.2.7a,b). The system in eqns.(4.2.7a,b) is the same as the system in eqns.(4.1.4a,b) with the only difference being that A, B, C, D, E, F and K_2 are replaced by $\hat{A}_{22}, \hat{B}_2, \hat{A}_{21}, \hat{B}_1, \hat{E}_1, \hat{E}_2$ and $\hat{K}_2$, respectively. Therefore, we can compute the d.z.'s of the system in eqns.(4.2.7a,b) in the same manner as those of the system in eqns.(4.1.4a,b) using the concept of a minimal order inverse.

We first assume that the system in eqns.(4.2.1a,b) has n d.z.'s. This assumption implies that a condition of the form

$$C^T\left[I_l - (F^T)^+ F^T\right] = 0$$

is satisfied. We also assume that the system in eqns.(4.2.7a,b) is invertible and has n d.z.'s. This assumption is made to simplify the mathematical analysis presented in the rest of this section. It implies that

$$K_2^T\left[D^T\left[I_l - (F^T)^+ F^T\right]\right] = 0$$

and is not necessary for achieving disturbance rejection. The above condition can be achieved by means of a constant matrix K_2 having unity-rank i.e. $K_2 = \mathbf{q}_2\mathbf{p}_2$, where $\mathbf{q}_2$ and $\mathbf{p}_2$ are $m\times 1$ and $1\times n$ vectors respectively. We can specify the vector $\mathbf{q}_2$ arbitrarily such that a condition of the form

$$\mathbf{q}_2^T\left[D^T\left[I_l - (F^T)^+ F^T\right]\right] = 0 \tag{4.2.8}$$

is satisfied. Therefore, on implementing the unity-rank constant feedback, the system in eqns.(4.2.7a,b) becomes

$$\dot{\xi} = \left[A - \mathbf{b}\,\mathbf{p}_2^{\,T}\right]\xi + E\,\mu \tag{4.2.9a}$$

$$\upsilon = \left[C - \mathbf{d}\,\mathbf{p}_2^{\,T}\right]\xi + F\,\mu \tag{4.2.9b}$$

where $\mathbf{b} = B\,\mathbf{q}_2$ and $\mathbf{d} = D\,\mathbf{q}_2$, respectively.

For $\left[l \geq 1\right]$, the system in eqns.(4.2.9a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}^* = \left[\left[A^T - \mathbf{p}_2^{\,T}\mathbf{b}^T\right]\right]\xi^* + \left[C^T - \mathbf{p}_2^{\,T}\mathbf{d}^T\right]\mu^* \tag{4.2.10a}$$

$$\upsilon^* = \left[E^T\right]\xi^* + \left[F^T\right]\mu^* \tag{4.2.10b}$$

A right inverse of the system in eqns.(4.2.10a,b) is

$$\dot{\xi}^* = \left[\left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2^{\,T}\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right]\right]\xi^* + \left[\left[C^T - \mathbf{p}_2^{\,T}\mathbf{d}^T\right](F^T)^+\right]\upsilon^* \tag{4.2.11a}$$

$$\mu^* = \left[-(F^T)^+E^T\right]\xi^* + \left[(F^T)^+\right]\upsilon^* \tag{4.2.11b}$$

Now, we can define the sets of transmission zeros and blocking zeros of the system in eqns.(4.2.9a,b) which are equivalent to the closed-loop d.z.'s and d.b.z.'s respectively of the system $\sum_d \left[A, B, C, D, E, F\right]$ and which are affected by constant state feedback K_2.

Definition 4.3:

$$Z_c^D\left\{\sum_d \left[A, B, C, D, E, F\right]\Big|_{K_2}\right\} = \sigma\left[\left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2^{\,T}\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right]\right]$$

$$Z_c^B\left\{\sum_d \left[A,B,C,D,E,F\right]|_{K_2}\right\} = Z_c^D\left\{\sum_d \left[A,B,C,D,E,F\right]|_{K_2}\right\}$$

$$-\left[Z_c^D\left\{\sum_d \left[A,B,C,D,E,F\right]|_{K_2}\right\} \cap \sigma\left[A^T - \mathbf{p}_2{}^T\mathbf{b}^T\right]\right]$$

From the above definition, it follows that the closed-loop d.b.z.'s are those eigenvalues of $\left[\left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2{}^T\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right]\right]$ which are not also the eigenvalues of $\left[A^T - \mathbf{p}_2{}^T\mathbf{b}^T\right]$. Now the matrix

$$\left[\left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2{}^T\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right]\right]$$

can be represented as the transpose of the closed-loop state matrix of the system

$$\dot{\Psi} = \left[\left[A - EF^+C\right]\right]\Psi + \left[\left[\mathbf{b} - EF^+\mathbf{d}\right]\right]\Phi \tag{4.2.12}$$

subject to the state feedback law

$$\Phi = -\mathbf{p}_2\Psi \tag{4.2.13}$$

Similarly the matrix $\left[A^T - \mathbf{p}_2{}^T\mathbf{b}^T\right]$ is the transpose of closed-loop state matrix of the system

$$\dot{\Psi} = A\ \Psi + \mathbf{b}\ \Phi \tag{4.2.14}$$

subject to the state feedback law in eqn.(4.2.13). Therefore, if the system in eqn.(4.2.12) is controllable, then all the eigenvalues of the matrix $\left[A - EF^+C\right]$ can be arbitrarily assigned by the state feedback in eqn.(4.2.13), otherwise only those eigenvalues of $\left[A - EF^+C\right]$ can be assigned arbitrarily by state feedback which correspond to the controllable modes of the system.

From the above results, we can now outline the following algorithm for the placement of d.b.z.'s of the system $\sum_d \left[A,B,C,D,E,F\right]$ by means of constant state feedback:

Algorithm 4.3: *(Assignment of the d.b.z.'s of the system* $\sum_d \left[A, B, C, D, E, F\right]$ *by means of constant state feedback)*

(i) Create a higher order system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ so that the factorization procedure for the transfer function vector between the outputs and the disturbance can be used.

(ii) Specify the vector $\mathbf{q}_2$ such that the condition in eqn.(4.2.8) is satisfied.

(iii) Calculate the eigenvalues of the matrix $\left[A - EF^{+}C\right]$ and A. The d.b.z.'s which are affected by unity-rank constant state feedback i.e. $K_2 = \mathbf{q}_2\mathbf{p}_2$ are those eigenvalues of the matrix $\left[A - EF^{+}C\right]$ which are not also the eigenvalues of A.

(iv) Determine if the system in eqns.(4.2.12) is controllable, if it is not controllable, determine which eigenvalues of $\left[A - EF^{+}C\right]$ correspond to the controllable modes of the system.

(v) Calculate the state feedback vector $\mathbf{p}_2$ in eqn.(4.2.13) using Algorithm 2.1 so as to achieve desired values for those eigenvalues of $\left[A - EF^{+}C\right]$ which are the d.b.z.'s and which correspond to the controllable modes of the system in eqns.(4.2.12). These assigned eigenvalues are the required d.b.z.'s which achieve disturbance rejection in the steady state.

(vi) Implement the feedback law in eqn.(4.2.3) on the system in eqns.(4.2.1a,b) to get the closed-loop system

$$\dot{\mathbf{x}}(t) = \bar{A}\,\mathbf{x}(t) + B\,\mathbf{v}(t) + E\,d(t)$$
$$\mathbf{y}(t) = \bar{C}\,\mathbf{x}(t) + D\,\mathbf{v}(t) + F\,d(t)$$

where

$$\bar{A} = A - BK_2$$

and

$$\bar{C} = C - DK_2$$

This system will have d.b.z.'s assigned at the desired locations in the complex plane.

4.2.2 Assignment of D.B.Z.'s by Dynamic State Feedback

In this section, we will discuss the cases when the system $\sum_d \left[A, B, C, D, E, F\right]$ has no d.b.z.'s or the d.b.z.'s assigned by a constant gain state feedback are not sufficient for the steady state rejection of all the disturbances acting on the system.

Based on the results of section 4.1.2, a dynamic state feedback compensator of the form

$$\dot{\mathbf{z}}_2(t) = F_2 \mathbf{z}_2(t) + \begin{bmatrix} 0 & G_2 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{4.2.15a}$$

$$\mathbf{u}_2(t) = H_2 \mathbf{z}_2(t) + \begin{bmatrix} 0 & J_2 \end{bmatrix} \hat{\mathbf{x}}(t) \tag{4.2.15b}$$

is needed to introduce additional d.b.z.'s as well as to assign all the closed-loop d.b.z.'s at desired locations in the complex plane such that the effects of all the disturbances at the outputs are eliminated in the steady state.

The dynamic state feedback is implemented on the system (4.2.2) using the feedback law

$$\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t) \tag{4.2.16}$$

The closed-loop system is therefore described by

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t) \\ \dot{\mathbf{z}}_2(t) \end{bmatrix} = \begin{bmatrix} \Omega & \Theta\left[C - DJ_2\right] & -\Theta D H_2 \\ 0 & \left[A - BJ_2\right] & -BH_2 \\ 0 & G_2 & F_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} \Theta D \\ B \\ 0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} \Theta F \\ E \\ 0 \end{bmatrix} d(t) \tag{4.2.17a}$$

$$\hat{\mathbf{y}}(t) = \begin{bmatrix} I_p & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{z}_2(t) \end{bmatrix} \tag{4.2.17b}$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & \left(C - DJ_2\right) & -DH_2 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \mathbf{z}_2(t) \end{bmatrix} + D\mathbf{v}(t) + Fd(t) \tag{4.2.17c}$$

where

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{x}(t) \end{bmatrix}$$

The disturbance transfer matrix $\mathbf{w}_d^c(s)$ i.e. between the outputs $\hat{\mathbf{y}}(t)$ and the disturbance $d(t)$ can be written as

$$\mathbf{w}_d^c(s) = \left[Q_d{}^c(s)\right]^{-1} P_d{}^c(s) \tag{4.2.18}$$

where

$$Q_d{}^c(s) = \left[sI_q - \Omega\right] \tag{4.2.19a}$$

and

$$P_d{}^c(s) = \Theta\left\{F + \left[\left(C - DJ_2\right) \quad -DH_2\right]\left[Q(s)\right]^{-1}\begin{bmatrix} E \\ 0 \end{bmatrix}\right\} \tag{4.2.19b}$$

where

$$Q(s) = \begin{bmatrix} sI_n - \left(A - BJ_2\right) & BH_2 \\ -G_2 & \left(sI_q - F_2\right) \end{bmatrix}$$

From eqn.(4.2.19b), the $p \times 1$ rational function vector $P_d{}^c(s)$ is the product of the matrix Θ and the transfer function vector of an $\left(n+q\right)$th-order, single-input and l-output system represented by the following state-space equations

$$\dot{\xi} = \begin{bmatrix} \left(A - BJ_2\right) & -BH_2 \\ G_2 & F_2 \end{bmatrix} \xi + \begin{bmatrix} E \\ 0 \end{bmatrix} \mu \tag{4.2.20a}$$

$$\upsilon = \left[\left[C - DJ_2 \right] \; -DH_2 \right] \xi + \left[F \right] \mu \tag{4.2.20b}$$

From Theorem 3.3 and the results of Section 4.1.2, it follows that the transmission zeros of the system in eqns.(4.2.20a,b) are the closed-loop d.z.'s of the system in eqns.(4.2.1a,b) together with those introduced by the dynamic state feedback in eqns.(4.2.15a,b), and there are at most $n+q$ of them.

Since the system in eqns.(4.2.20a,b) can be treated in the same way as the system in eqns.(4.1.24a,b), the procedure given in section 4.1.2 can be repeated now to compute and assign d.b.z.'s. We start by assuming without loss of generality that, the system in eqns.(4.2.20a,b) is invertible and has $(n+q)$ d.z.'s. This assumption is made to simplify the mathematical analysis and is not necessary for achieving disturbance rejection. It implies that the following conditions are satisfied:

$$C^T \left[I_l - (F^T)^+ F^T \right] = 0 \tag{4.2.21}$$

$$\begin{bmatrix} J_2^T D^T \\ H_2^T D^T \end{bmatrix} \left[I_l - (F^T)^+ F^T \right] = 0 \tag{4.2.22}$$

The condition in eqn.(4.2.21) is satisfied by assuming that the open-loop system in eqns.(4.2.1a,b) has n d.z.'s; while the condition in eqn.(4.2.22) can be achieved by using unity-rank constant matrices J_2 and H_2 given by

$$J_2 = \mathbf{q}_2 \mathbf{p}_2$$

$$H_2 = \mathbf{q}_2 \mathbf{h}_2$$

where $\mathbf{q}_2$, $\mathbf{p}_2$ and $\mathbf{h}_2$ are $m \times 1$, $1 \times q$ and $1 \times n$ respectively. The vector $\mathbf{q}_2$ can be specified such that the condition in eqn.(4.2.22) is reduced to the form

$$\mathbf{q}_2^T \left[D^T \left[I_l - (F^T)^+ F^T \right] \right] = 0 \tag{4.2.23}$$

Therefore, on implementing the unity-rank matrices J_2 and H_2, the system in eqns.(4.2.20a,b) becomes

$$\dot{\xi} = \begin{bmatrix} \left(A - \mathbf{b}\,\mathbf{p}_2\right) & -\mathbf{b}\,\mathbf{h}_2 \\ G_2 & F_2 \end{bmatrix} \xi + \begin{bmatrix} E \\ 0 \end{bmatrix} \mu \tag{4.2.24a}$$

$$\upsilon = \left[\left(C - \mathbf{d}\,\mathbf{p}_2\right) \quad -\mathbf{d}\,\mathbf{h}_2 \right] \xi + \left[F \right] \mu \tag{4.2.24b}$$

where, $\mathbf{b} = B\,\mathbf{q}_2$ and $\mathbf{d} = D\,\mathbf{q}_2$ respectively.

For $\left(l \geq 1\right)$, the system in eqns.(4.2.24a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}^* = \begin{bmatrix} \left(A^T - \mathbf{p}_2^T \mathbf{b}^T\right) & G_2^T \\ -\mathbf{h}_2^T \mathbf{b}^T & F_2^T \end{bmatrix} \xi^* + \begin{bmatrix} \left(C^T - \mathbf{p}_2^T \mathbf{d}^T\right) \\ -\mathbf{h}_2^T \mathbf{d}^T \end{bmatrix} \mu^* \tag{4.2.25a}$$

$$\upsilon^* = \left[E^T \quad 0 \right] \xi^* + \left[F^T \right] \mu^* \tag{4.2.25b}$$

A right inverse of the system in eqns.(4.2.25a,b) is given by

$$\dot{\xi}^* = \begin{bmatrix} \left(A^T - C^T (F^T)^+ E^T\right) - \mathbf{p}_2^T \left(\mathbf{b}^T - \mathbf{d}^T (F^T)^+ E^T\right) & G_2^T \\ \left(-\mathbf{h}_2^T \mathbf{b}^T + \mathbf{h}_2^T \mathbf{d}^T (F^T)^+ E^T\right) & F_2^T \end{bmatrix} \xi^*$$

$$+ \begin{bmatrix} \left(C^T - \mathbf{p}_2^T \mathbf{d}^T\right) \\ -\mathbf{h}_2^T \mathbf{d}^T \end{bmatrix} (F^T)^+ \upsilon^* \tag{4.2.26a}$$

$$\mu^* = -(F^T)^+ \left[E^T \quad 0 \right] \xi^* + \left[(F^T)^+ \right] \upsilon^* \tag{4.2.26b}$$

Now, we can define the sets of transmission zeros and blocking zeros of the system (4.2.24a,b) which are equivalent to the closed-loop d.z.'s and d.b.z.'s respectively of the system $\sum_d \left[A, B, C, D, E, F\right]$ and which are affected by the dynamic state feedback $K_2(s)$.

Definition 4.4:

$$Z_c^D\left\{\sum_d \left[A, B, C, D, E, F\right]|_{K_2(s)}\right\} =$$

$$\sigma\begin{bmatrix} \left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2{}^T\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right] & G_2^T \\ \left[-\mathbf{h}_2{}^T\mathbf{b}^T + \mathbf{h}_2{}^T\mathbf{d}^T(F^T)^+E^T\right] & F_2^T \end{bmatrix}$$

$$Z_c^B\left\{\sum_d \left[A, B, C, D, E, F\right]|_{K_2(s)}\right\} = Z_c^D\left\{\sum_d \left[A, B, C, D, E, F\right]|_{K_2(s)}\right\}$$

$$- \left[Z_c^D\left\{\sum_d \left[A, B, C, D, E, F\right]|_{K_2(s)}\right\} \cap \sigma\begin{bmatrix} \left[A^T - \mathbf{p}_2{}^T\mathbf{b}^T\right] & G_2^T \\ -\mathbf{h}_2{}^T\mathbf{b}^T & F_2^T \end{bmatrix}\right]$$

From the above definition, it follows that the closed-loop d.b.z.'s are those eigenvalues of the matrix

$$\begin{bmatrix} \left[A^T - C^T(F^T)^+E^T\right] - \mathbf{p}_2{}^T\left[\mathbf{b}^T - \mathbf{d}^T(F^T)^+E^T\right] & G_2^T \\ \left[-\mathbf{h}_2{}^T\mathbf{b}^T + \mathbf{h}_2{}^T\mathbf{d}^T(F^T)^+E^T\right] & F_2^T \end{bmatrix}$$

which are not also the eigenvalues of the matrix

$$\begin{bmatrix} \left(A^T - \mathbf{p}_2{}^T \mathbf{b}^T\right) & G_2^T \\ -\mathbf{h}_2{}^T \mathbf{b}^T & F_2^T \end{bmatrix}$$

Now the matrix

$$\begin{bmatrix} \left(A^T - C^T (F^T)^+ E^T\right) - \mathbf{p}_2{}^T \left(\mathbf{b}^T - \mathbf{d}^T (F^T)^+ E^T\right) & G_2^T \\ \left(-\mathbf{h}_2{}^T \mathbf{b}^T + \mathbf{h}_2{}^T \mathbf{d}^T (F^T)^+ E^T\right) & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop state matrix of a system

$$\dot{\Psi} = \begin{bmatrix} \left(A - EF^+ C\right) & 0 \\ G_2 & F_2 \end{bmatrix} \Psi + \begin{bmatrix} \left(\mathbf{b} - EF^+ \mathbf{d}\right) \\ 0 \end{bmatrix} \Phi \tag{4.2.27}$$

subject to the state feedback

$$\Phi = -\left[\mathbf{p}_2 \ \mathbf{h}_2\right] \Psi \tag{4.2.28}$$

Similarly, the matrix

$$\begin{bmatrix} \left(A^T - \mathbf{p}_2{}^T \mathbf{b}^T\right) & G_2^T \\ -\mathbf{h}_2{}^T \mathbf{b}^T & F_2^T \end{bmatrix} \tag{4.2.29}$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi} = \begin{bmatrix} A & 0 \\ G_2 & F_2 \end{bmatrix} \Psi + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \Phi \tag{4.2.30}$$

subject to the state feedback law in eqn.(4.2.28).

Therefore, if the system in eqn.(4.2.27) is controllable, then all the eigenvalues of the matrix

$$\begin{bmatrix} \left(A-EF^{+}C\right) & 0 \\ G_2 & F_2 \end{bmatrix}$$

can be arbitrarily assigned by the state feedback (4.2.28); otherwise only those eigenvalues of the above matrix can be assigned by the state feedback which correspond to the controllable modes of the system.

Remark 4.2: By Theorem 4.1, the system in eqn.(4.2.27) is controllable if the matrix G_2 is selected such that

$$rank \left[G_2 \left[\left(A-EF^{+}C\right)-\lambda I_n \right]^{-1} \left(\mathbf{b}-EF^{+}\mathbf{d}\right) \quad \left[F_2-\lambda I_q\right] \right] = q$$

for all complex values of $\lambda \in \sigma \left[F_2\right]$.

Using the above results, we will now outline an algorithm for assigning d.b.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$ using a dynamic state feedback control law.

Algorithm 4.4: *(Assignment of the d.b.z.'s of the system* $\sum_d \left[A, B, C, D, E, F\right]$ *by means of dynamic state feedback)*

(i) Create the higher order system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ so that the factorization procedure for the transfer function vector between the outputs and the disturbances can be applied.

(ii) Specify a value of $\mathbf{q}_2$ such that the condition in eqn.(4.2.23) is satisfied.

(iii) Calculate the eigenvalues of the matrices $\left[A-EF^{+}C\right]$ and A, and select a matrix F_2 such that its eigenvalues are different from the eigenvalues of the matrix $\left[A-EF^{+}C\right]$. The d.b.z.'s which are affected by the state feedback law in eqn.(4.2.28) are therefore equivalent to those

eigenvalues of the matrix $\left[A-EF^{+}C\right]$ which are not also the eigenvalues of A together with the eigenvalues of $\left[F_2\right]$.

(iv) Ensure that the system in eqn.(4.2.27) is controllable using the extension of Theorem 4.1 described in Remark 4.2, otherwise only those eigenvalues of the system can be assigned by state feedback which correspond to the controllable modes of the system.

(v) Use Algorithm 2.1 to calculate the state feedback $\left[\mathbf{p}_2 \ \mathbf{h}_2\right]$ which achieves desired values for those eigenvalues of the matrix

$$\begin{bmatrix} \left[A-EF^{+}C\right] & 0 \\ G_2 & F_2 \end{bmatrix}$$

which are the d.b.z.'s.

(vi) Implement the feedback law in eqn.(4.2.15a,b) on the system in eqns.(4.2.1a,b) to get the closed-loop system

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{A}\,\tilde{\mathbf{x}}(t) + \tilde{B}\,\mathbf{v}(t) + \tilde{E}\,d(t) \tag{4.2.31a}$$

$$\mathbf{y}(t) = \tilde{C}\,\tilde{\mathbf{x}}(t) + D\,\mathbf{v}(t) + F\,d(t) \tag{4.2.31b}$$

where

$$\tilde{A} = \begin{bmatrix} \left[A-BJ_2\right] & -BH_2 \\ G_2 & F_2 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tilde{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$$

and

$$\tilde{C} = \left[\left[C-DJ_2\right] \ -DH_2\right]$$

with

$$\tilde{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_2(t) \end{bmatrix}$$

This will have d.b.z.'s at the desired locations in the complex plane.

4.3 NUMERICAL EXAMPLES

In this section, we illustrate the performance of the algorithms described in this chapter by means of some numerical examples.

Example 4.1: This example illustrates the use of Algorithms 4.1 and 4.2 considered in Section 4.1 for the system $\sum_d \left[A, B, C, E\right]$. The example is a 4th-order linearized model [3-5], whose parameters were given in the previous chapter (eqns.(3.1.2a,b). For the purpose of illustration, we require the rejection of the following class of disturbances:

(a) $d(t) = \left[\beta_1 \, exp(t)\right]$.

(b) $d(t) = \left[\beta_1 \, exp(t) + \beta_2 + \beta_3 \, t\right]$.

where β_1, β_2 and β_3 are unknown constant values.

Note that eqns.(3.1.2a,b) are already in the form of eqns.(4.1.2a,b). We also note that this system has two disturbance zeros at -0.115 and 435.01 , and hence the condition in eqn.(4.1.5) is satisfied.

(a) *Disturbances of the form* $\beta_1 exp(t)$:

For this case, it is required to assign at least one d.b.z at 1.0, to achieve complete steady-state rejection of all disturbances of the form $\beta_1 exp(t)$. Since the system has d.b.z.'s at - 0.115 and 435.01, a constant state feedback matrix $\left[0 \;\; K_2\right]$ is sufficient to assign at least one of the d.b.z.'s at 1.0, while the other can be placed at any location e.g. -0.1. To compute the constant gain K_2, we need to apply Algorithm 4.1.

We specify the vector $\mathbf{q}_2$ as

$$q_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

such that the condition in eqn.(4.1.7) is satisfied. Now to assign the d.b.z.'s, we write the closed-loop state matrix given in eqn.(4.1.12) for the system

$$\dot{\Psi} = \begin{bmatrix} 435.0 & 65.0 \\ 0.1 & -0.1 \end{bmatrix} \Psi + \begin{bmatrix} 65.0 \\ 0.0 \end{bmatrix} \Phi \tag{4.3.1}$$

The state feedback law is

$$\Phi = -\mathbf{p}_2 \Psi \tag{4.3.2}$$

The system in eqn.(4.3.1) is controllable, and hence a state feedback $\mathbf{p}_2$ can be found to position the two eigenvalues of the state matrix at 1.0 and -0.1. Since one of these eigenvalues at 1.0 is the desired d.b.z, then complete steady-state disturbance rejection will be achieved.

In order to compute $\mathbf{p}_2$, we use Algorithm 2.1 for state feedback pole assignment to position the eigenvalues of the system in eqn.(4.3.1) at 1.0 and -0.1. The vector $\mathbf{p}_2$ was found to be

$$\mathbf{p}_2 = \begin{bmatrix} 6.676923 & 1.0 \end{bmatrix}$$

Hence the required constant gain matrix K_2 is

$$K_2 = \begin{bmatrix} 6.676923 & 1.0 \\ 0.0 & 0.0 \end{bmatrix}$$

By implementing the constant matrix K_2 using the state feedback control law $\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & K_2 \end{bmatrix} \mathbf{x}(t)$ in eqns.(3.1.2a,b), the following closed-loop system is obtained

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.5 & -1.0 & 1.0 & 0.0 \\ 0.4 & -0.4 & 0.0 & 0.0 \\ -19.5 & 19.50 & -499.0 & 0.0 \\ 0.0 & 0.0 & 0.1 & -0.1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.4 \\ 65.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} -0.1 \\ 0.0 \\ 50.0 \\ 0.0 \end{bmatrix} d(t) \tag{4.3.3a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t) \tag{4.3.3b}$$

The above system is stable and has two d.b.z.'s located at 1.0 and -0.1. When $\beta_1 = 1.0$, the response at the outputs $y_1(t)$ and $y_2(t)$ to exponential disturbances are shown in Figs.(4.1a) and (4.1b) for the open-loop system and in Figs.(4.2a) and (4.2b) for the closed-loop system. It can be readily seen that in the closed-loop system the exponential disturbances are rejected completely in the steady state. An attempt was then made to position the poles of the above system at desired values [3,4], i.e. -0.1, -50 ± j 10.0 , -500.0, by means of constant output feedback matrix K_1. The corresponding constant output feedback matrix K_1 was found by implementing Algorithm 2.2:

$$K_1 = \begin{bmatrix} 32.78354069 & 5.09197423E+4 \\ 0.20036029 & 2.50249999E+2 \end{bmatrix}$$

Therefore, the overall state feedback $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is given by

$$K = \begin{bmatrix} 32.78354069 & 5.09197423E+4 & 6.6769230769 & 1.0 \\ 0.20036029 & 2.50249999E+2 & 0.0 & 0.0 \end{bmatrix}$$

By implementing the constant output feedback matrix K_1 on the system in eqns.(4.3.3a,b), the resulting closed-loop system is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.5 & -1.0 & 1.0 & 0.0 \\ 0.3198558 & -1.005E+2 & 0.0 & 0.0 \\ -2.15043E+3 & -3.30976375E+6 & -4.9900E+2 & 0.0 \\ 0.0 & 0.0 & 1.0 & -0.1 \end{bmatrix} \mathbf{x}(t)$$

$$+\begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.4 \\ 65.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} -0.1 \\ 0.0 \\ 50.0 \\ 0.0 \end{bmatrix} d(t) \tag{4.3.4a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{x}(t) \tag{4.3.4b}$$

which has two d.b.z.'s 1.0 and -0.1; and the closed-loop poles are assigned at the values specified above. It can be observed that pole-zero cancellation occurs at -0.1 , and thus the output response of the closed-loop system will be extremely fast compared with the open-loop system. The responses at the outputs $y_1(t)$ and $y_2(t)$ to the exponential disturbance are shown in Figs.(4.3a) and (4.3b) for the above closed-loop system. It can be seen that all the disturbances are rejected and the system has a good transient response.

(b) *Disturbances of the form* $\beta_1 exp\,(t)+\beta_2+\beta_3 t$:

For this case it is required to assign three d.b.z.'s at 0.0, 0.0 and 1.0 to achieve complete steady-state rejection of all disturbances with combination of step, ramp and $exp\,(t)$ functions. Since, the system has d.b.z.'s at -0.115 and 43:'.01, we need to introduce an additional d.b.z in the system by means of dynamic state feedback of order one. Hence, Algorithm 4.2 can be applied to compute the parameters of the dynamic state feedback compensator.

We first specify the vector $\mathbf{q}_2$ as

$$\mathbf{q}_2 = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}$$

such that the condition in eqn.(4.1.23) is achieved. Then, we select the eigenvalue of the dynamic compensator F_2 (e.g. at -5) which is different from the eigenvalues of the matrix

$$\left[A_{22}-E_2E_1^{+}A_{21}\right].$$

The vector G_2 is chosen as

$$G_2 = \left[1.0 \;\; 0.0\right]$$

such that the condition in eqn.(4.1.32) is satisfied.

In order to assign the d.b.z.'s, we write eqn.(4.1.29) for this case

$$\dot{\Psi} = \begin{bmatrix} 435.0 & 65.0 & 0.0 \\ 0.1 & -0.1 & 0.0 \\ 1.0 & 0.0 & -5.0 \end{bmatrix} \Psi + \begin{bmatrix} 65.0 \\ 0.0 \\ 0.0 \end{bmatrix} \Phi \tag{4.3.5}$$

subject to the state feedback law

$$\Phi = -\left[\mathbf{p}_2 \;\; \mathbf{h}_2\right]\Psi \tag{4.3.6}$$

The state matrix of the system in eqns.(4.3.5) has three eigenvalues at -0.115, 435.01 and -5.0. Since the system in eqn.(4.3.5) is controllable, the state feedback in eqn.(4.3.6) can be used to position the eigenvalues at 0.0, 0.0, and 1.0. It can be easily verified that these three eigenvalues correspond to the closed-loop d.b.z.'s of the system $\sum_d \left[A, B, C, E\right]$ subject to the dynamic state feedback compensator. In order to compute the constant state feedback in eqn.(4.3.6), we use Algorithm 2.1. The result was found to be

$$\left[\mathbf{p}_2 \;\; \mathbf{h}_2\right] = \left[6.5984615384 \;\; 0.9965463108 \;\; 0.4709576138\right]$$

Hence the required dynamic state feedback can be written as

$$\dot{z}_2(t) = \left[-5\right]z_2(t) + \left[0.0 \;\; 0.0 \;\; 1.0 \;\; 0.0\right]\mathbf{x}(t) \tag{4.3.7a}$$

$$\mathbf{u}_2(t) = \begin{bmatrix} 0.4709576138 \\ 0.0 \end{bmatrix} z_2(t)$$

$$+ \begin{bmatrix} 0.0 & 0.0 & 6.5984615384 & 0.9965463108 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{x}(t) \tag{4.3.7b}$$

On implementing the dynamic state feedback using the control law

$$\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t)$$

the following closed-loop system was obtained:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -0.5 & -1.0 & 1.0 & 0.0 & 0.0 \\ 0.4 & -0.4 & 0.0 & 0.0 & 0.0 \\ -19.5 & 19.5 & -493.9 & 0.022449 & -30.612245 \\ 0.0 & 0.0 & 0.1 & -0.1 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & -5 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 0.4 \\ 65.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} -0.1 \\ 0.0 \\ 50.0 \\ 0.0 \\ 0.0 \end{bmatrix} d(t) \tag{4.3.8a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_2(t) \end{bmatrix} \tag{4.3.8b}$$

The above system is stable. It can be easily verified that the resulting closed-loop system has d.b.z.'s at 0.0, 0.0 and 1.0.

Example 4.2 : To illustrate Algorithms 4.3 and 4.4 for system $\sum_d \left[A, B, C, D, E, F\right]$, we consider a linearized model of a d.c. motor described by the following state-space equations [6]

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.08 & 5.2 \\ -205 & -199.0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 \\ 188.0 \end{bmatrix} u(t) + \begin{bmatrix} -4.7 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \qquad (4.3.9a)$$

$$y(t) = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 \end{bmatrix} u(t) + \begin{bmatrix} -1.0 & 1.0 \end{bmatrix} \mathbf{d}(t) \qquad (4.3.9b)$$

where

$x_1(t)$: The motor speed, rad./sec.,

$x_2(t)$: The armature current, Amp.,

$u(t)$: The armature voltage, Volt,

$d_1(t)$: The load torque, Nm,

$d_2(t)$: The speed set point, rad./sec.,

$y(t)$: The speed error, rad./sec.

In order to illustrate the results for a single-disturbance case we shall only consider disturbance $d_1(t)$. It is required to control the speed of the motor by designing a state feedback controller, to reject in steady-state the following classes of disturbances for $d_1(t)$:

(a) $d_1(t) = \left[\beta_1 + \beta_2 \, exp(t)\right]$.

(b) $d_1(t) = \left[\beta_1 + \beta_2 \, sin(t)\right]$.

where β_1 and β_2 are unknown constant values. The system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ for this example is given by

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} -1.0 & 1.0 & 0.0 \\ 0.0 & -0.08 & 5.2 \\ 0.0 & -205 & -199.0 \end{bmatrix} \begin{bmatrix} z(t) \\ \mathbf{x}(t) \end{bmatrix} + \begin{bmatrix} 0.0 \\ 0.0 \\ 188.0 \end{bmatrix} u(t) + \begin{bmatrix} -1.0 & 1.0 \\ -4.7 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \tag{4.3.10a}$$

$$\hat{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} z(t) \\ \mathbf{x}(t) \end{bmatrix} \tag{4.3.10b}$$

$$y(t) = \begin{bmatrix} 0.0 & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} z(t) \\ \mathbf{x}(t) \end{bmatrix} + \begin{bmatrix} 0.0 \end{bmatrix} u(t) + \begin{bmatrix} -1.0 & 1.0 \end{bmatrix} \mathbf{d}(t) \tag{4.3.10c}$$

with $\left[\Omega, \Theta\right] = \left[-1, 1\right]$ has been chosen to be controllable pair (see Section 3.2.2). The condition of eqn.(4.2.8) is satisfied and there are two d.z.'s for the system(4.3.9) located at -10.43319 and -193.3468.

(a) *Disturbances of the form* $\beta_1 + \beta_2 exp\,(t)$ *:*

Since the system(4.3.9) has d.b.z.'s at -10.43319 and -193.346, it follows that constant state feedback is sufficient to assign these at 0.0 and 1.0, in order to achieve steady state rejection of all disturbances with step and $exp\,(t)$ functions. Then, following the procedure of Algorithm 4.3, the constant unity-rank $\mathbf{k}_2$ which assigns the d.b.z.'s at 0.0 and 1.0 was found to be

$$\mathbf{k}_2 = \begin{bmatrix} -1.062164 & -1.089255 \end{bmatrix}$$

By implementing the constant gain $\mathbf{k}_2$ (in the state feedback law $u(t) = v(t) - \mathbf{k}_2 \mathbf{x}(t)$) on the system (4.3.9a,b), the following closed-loop system is obtained

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.08 & 5.2 \\ -5.31315 & 5.778 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 \\ 188.0 \end{bmatrix} v(t) + \begin{bmatrix} -4.7 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \tag{4.3.11a}$$

$$y(t) = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0.0 \end{bmatrix} v(t) + \begin{bmatrix} -1.0 & 1.0 \end{bmatrix} \mathbf{d}(t) \tag{4.3.11b}$$

The above system is unstable and has d.b.z.'s located at 0.0 and 1.0.

By implementing dynamic output feedback (Algorithm 2.3) defined by

$$\begin{bmatrix} \dot{\mathbf{z}}_1(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -50.023642 & -29.415894 & -36.558466 \\ 53.465260 & 26.323642 & 44.009066 \\ 0.0 & 0.0 & -1.0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} y(t) \qquad (4.3.12a)$$

$$v(t) = \bar{v}(t) - \begin{bmatrix} 1.224944 & 0.138980 & 1.433032 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ z(t) \end{bmatrix} \qquad (4.3.12b)$$

we can stabilize the above system and assign all the poles of the augmented closed-loop system consisting the system (4.3.11a,b) together with dynamic output feedback in (4.3.12a,b) at desired values, e.g. at -2, -3, -4, $-5 \pm j5$.

(b) *Disturbances of the form* $\beta_1 + \beta_2 sin(t)$ *:*

For this case it is required to assign the d.b.z.'s at 0.0, j and -j in order to achieve complete steady-state rejection of this class of disturbances. Since the system(4.3.9) has two d.b.z.'s, we need to implement dynamic state feedback of order one, which will introduce another d.b.z and assign all the resulting d.b.z.'s at desired locations i.e. 0.0, j and -j. Using Algorithm 4.4, it was found that the required dynamic state feedback is given by

$$\dot{z}_2(t) = \begin{bmatrix} -1 \end{bmatrix} z_2(t) + \begin{bmatrix} 1.0 & 1.0 \end{bmatrix} \mathbf{x}(t) \qquad (4.3.13a)$$

$$u(t) = v(t) - \begin{bmatrix} -1.18466568E-3 \end{bmatrix} z_2(t) - \begin{bmatrix} -1.05989042 & 1.08925531 \end{bmatrix} \mathbf{x}(t) \qquad (4.3.13b)$$

The resulting closed-loop system is

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} -0.08 & 5.2 & 0.0 \\ -5.7406 & 5.78 & 0.22717 \\ 1.0 & 1.0 & -1.0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 188 \\ 0 \end{bmatrix} v(t) + \begin{bmatrix} -4.7 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \qquad (4.3.14a)$$

$$y(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0.0 \end{bmatrix} v(t) + \begin{bmatrix} -1 & 1.0 \end{bmatrix} \mathbf{d}(t) \tag{4.3.14b}$$

The above system is unstable and has d.b.z.'s located at 0.0, j and -j.

By using one of the approaches discussed in Chapter II for computing the dynamic output feedback, we can stabilize and assign all the closed-loop poles at the desired values, e.g. at -1, -2, -3, -4, -5 ± j 5. The dynamic output feedback that achieves our requirement was found by implementing Algorithm 2.3 to be:

$$\begin{bmatrix} \dot{\mathbf{z}}_1(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -83.485535 & -56.495068 & -58.930760 \\ 92.842644 & 59.785535 & 69.390866 \\ 0.0 & 0.0 & -1.0 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} y(t) \tag{4.3.15a}$$

$$v(t) = \bar{v}(t) - \begin{bmatrix} 1.453465 & 0.690141 & 1.373860 \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ z(t) \end{bmatrix} \tag{4.3.15b}$$

4.4 CONCLUDING REMARKS

In this chapter, we have presented Algorithms for assigning d.b.z.'s of a linear time-invariant system by means of state feedback, such that the effect of a class of disturbance signals at the outputs are eliminated in the steady state. In the proposed algorithms, the factorization procedure described in Chapter III and the concept of minimal order inverses are used to determine the positions of the d.b.z.'s for systems $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$.

In Algorithm 4.1, it was shown how a unity-rank constant matrix $\begin{bmatrix} 0 & K_2 \end{bmatrix}$ can be computed for the system $\sum_d \left[A, B, C, E\right]$ such that its d.b.z.'s can be assigned at desired locations

in the complex plane. For the case where the system has no d.b.z.'s or the number of d.b.z.'s is not large enough to achieve the rejection of all disturbances in the steady state, Algorithm 4.2 can be used. It was shown how dynamic state feedback can be employed to introduce additional d.b.z.'s and to assign all the d.b.z.'s at desired locations in the complex plane. In Section 4.2, these results were extended to systems $\sum_{d} \left[A, B, C, D, E, F\right]$ and Algorithms 4.3 and 4.4 were proposed.

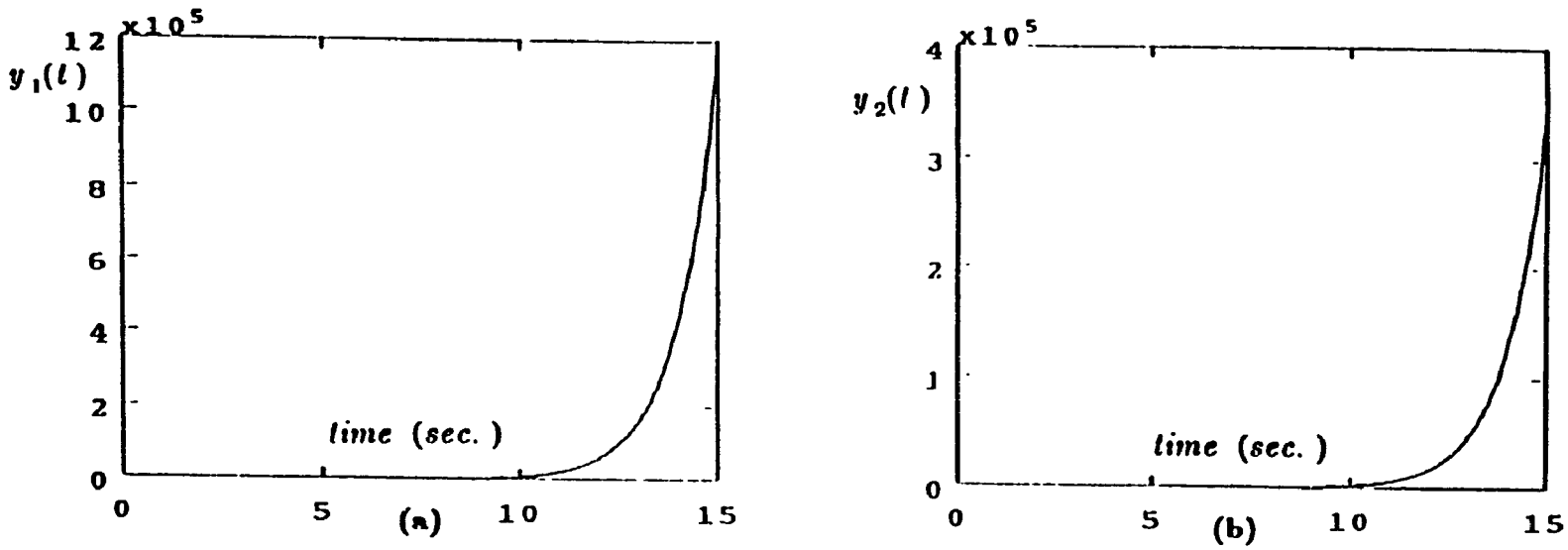


Fig.(4.1) Output responses of the open-loop system in Example 4.1-a to an exponential disturbance.

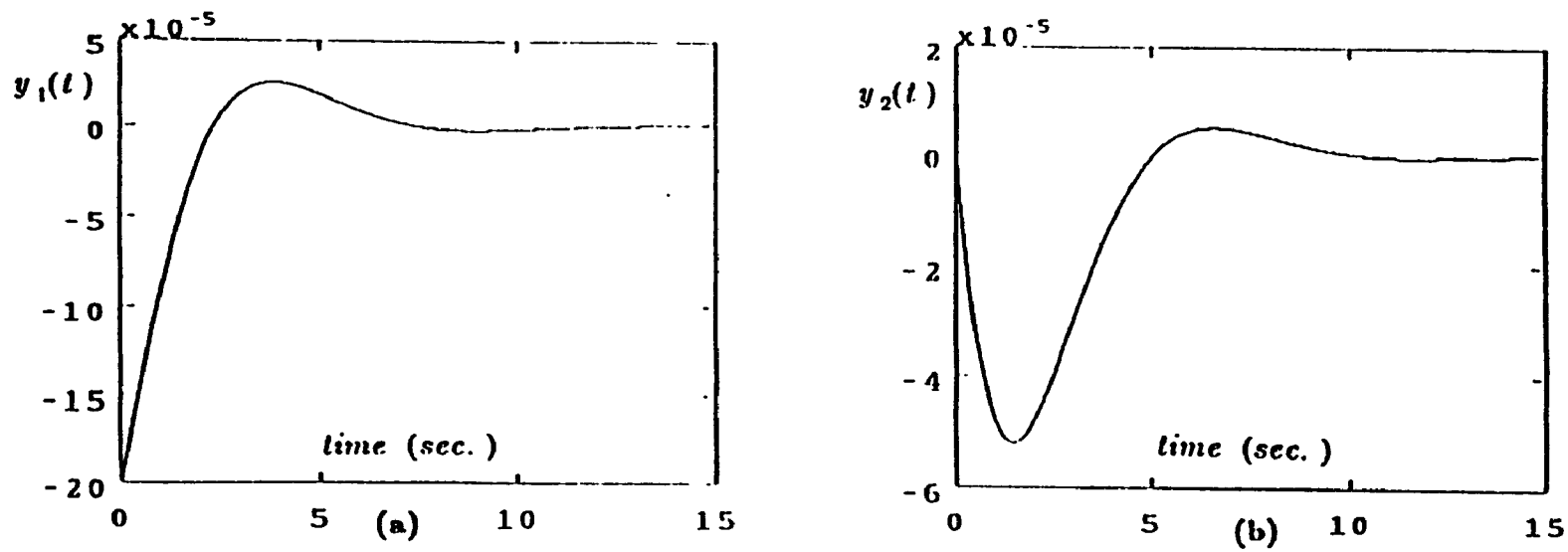
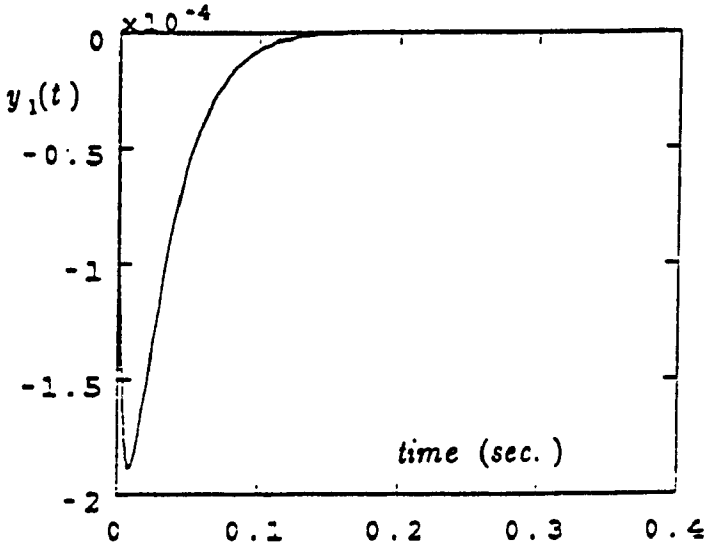
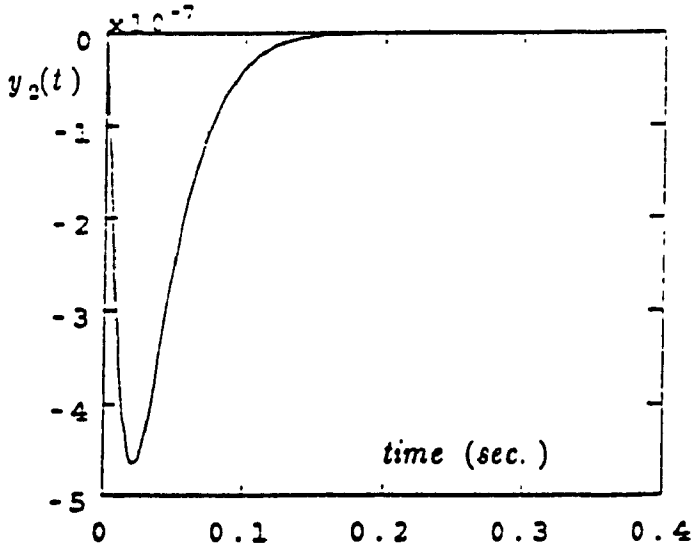


Fig.(4.2) Output responses of the closed-loop system in Example 4.1-a to an exponential disturbance.



(a)



(b)

Fig.(4.3) Output responses of the closed-loop system in Example 4.1-a to an exponential disturbance.

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CHAPTER V

ASSIGNMENT OF DISTURBANCE BLOCKING ZEROS: MULTIPLE DISTURBANCE CASE

In this chapter, the results of the previous chapter are extended to solve the problem of steady-state rejection of measurable and unmeasurable multiple disturbances by means of state feedback. The problem of multi-disturbance rejection is first reduced to one or more single-disturbance rejection problems, and then the required state feedback compensator is constructed as a sum of dyads. The dyadic design is carried out by implementing (with slight modifications) the algorithms proposed in the preceding chapter, to assign the required number of d.b.z.'s at specified locations in the complex plane so as to achieve complete disturbance rejection in steady state.

This chapter is organized as follows. In Section 5.2, we show how disturbance rejection can be carried out for a multivariable system described by $\sum_d \left[A, B, C, E\right]$ having multiple disturbances by achieving disturbance rejection for single-disturbance systems $\sum_d \left[A, B, C, E_i\right]$, for $i = 1,2,\ldots,r$; where r is the number of disturbances. A numerical Algorithm is developed to assign max $(n-l+m-1)$ d.b.z.'s arbitrarily close to desired locations in the complex plane by means of constant gain state feedback, such that a class of exponential disturbances can be rejected in steady state, where n, m and l are the state, input and output dimensions of the system, respectively. It is also shown that when the system does not have any d.b.z.'s between the outputs and each disturbance and/or that there are not enough d.b.z.'s to reject completely in steady state all the disturbances, we can introduce more d.b.z.'s in the system by using dynamic

state feedback. In Section 5.3, we generalize these results to systems $\sum_d \left[A, B, C, D, E, F\right]$. General remarks concerning the implementation and advantages of the proposed algorithms are given in Section 5.4. Numerical examples are provided in Section 5.5 to illustrate the use of the proposed algorithms, and finally in Section 5.6, we discuss the results presented in this chapter.

5.1 STATEMENT OF THE PROBLEM

In general, in multivariable systems with multiple disturbances, state feedback affects all the d.b.z.'s between the outputs and each disturbance. In order to treat this problem, we first reduce the multivariable system with measurable or unmeasurable multiple disturbances to one or more single-disturbance systems depending on the class of disturbances affecting the system. And then, under certain conditions, we can use the Algorithms developed in the previous chapter, to solve the single-disturbance rejection problem by means of state feedback. The problem of designing a state feedback compensator for the single-disturbance case considered in this chapter consist of two subproblems: First, for a given disturbance, it is required to assign at most $m-1$ d.b.z.'s at desired locations by means of state feedback without altering the number of d.b.z.'s corresponding to any other disturbance; and second, it is necessary to ensure that the state feedback used to assign d.b.z.'s for any subsequent disturbances preserves those assigned from the preceding ones. The design procedure is therefore sequential in nature. For each single-disturbance system, we assign the required number of d.b.z.'s at desired locations in the complex plane by means of state feedback to eliminate the steady-state effect of that disturbance, without altering those which have been assigned from the preceding disturbances.

The algorithms described in subsequent sections use the dyadic design procedure in order to compute constant or dynamic state feedback to assign the required number of d.b.z.'s at specified locations in the complex plane. In developing the algorithms, we need to specify some conditions, under which the required state feedback law exists.

5.2 MULI-DISTURBANCE REJECTION FOR THE SYSTEM $\sum_d \left[A, B, C, E\right]$

For simplicity of presentation, we assume that we have a multivariable system with two disturbances (i.e. $r = 2$) described by

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + E_1\,d_1(t) + E_2\,d_2(t) \tag{5.2.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{5.2.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^l$, $d_i(t) \in \mathbb{R}$ for $i = 1, 2$ and A, B, C, E_1 and E_2 have appropriate dimensions with $rank(B) = m$ and $rank(C) = l$. The disturbances $d_i(t)$ may or may not be measurable. Also, we assume that $m \geq r$. We assume further, without loss of generality that $\left[A, B, C, E_1, E_2\right]$ is non-degenerate (i.e. the system in eqns.(5.2.1a,b) has a finite number of transmission zeros and disturbance zeros), and the matrices $\left[A, B, C\right]$ with E_1 and E_2 can be partitioned as follows:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, E_1 = \begin{bmatrix} E_{11} \\ E_{21} \end{bmatrix}, E_2 = \begin{bmatrix} E_{12} \\ E_{22} \end{bmatrix},$$

and

$$C = \begin{bmatrix} C_1 & 0 \end{bmatrix}$$

with $rank\left[C_1\right] = l$. Note that C can be brought into this form by means of a nonsingular (orthogonal) coordinate transformation (see Chapter III). We also assume that, the transfer matrices of the open-loop system and the closed-loop system (with state feedback), relating the outputs to the disturbances have $(n-l)$ d.z.'s. This assumption implies that for the open-loop system, the following equations hold (see Section 4.1):

$$A_{12}^T\left[I_l - (E_{11}^T)^+ E_{11}^T\right] = 0 \tag{5.2.2a}$$

$$A_{12}^T\left[I_l - (E_{12}^T)^+ A_{12}^T\right] = 0 \tag{5.2.2b}$$

while for the closed-loop system, we require that

$$B_1^T\left[I_l-(E_{11}^T)^+E_{11}^T\right]=0 \tag{5.2.3a}$$

$$B_1^T\left[I_l-(E_{12}^T)^+E_{12}^T\right]=0 \tag{5.2.3b}$$

This assumption is made to simplify the mathematics and the presentation in the rest of this section. Furthermore we assume that the number of d.b.z.'s between the outputs and the first and second disturbances are $n_1\left[\le m-1\right]$ and $n_2\left[\le n-l\right]$, respectively.

5.2.1 Assignment of d.b.z.'s by Constant Gain State Feedback

In this section, a numerical algorithm is developed for determining an $m\times n$ constant state feedback matrix of the form $\left[0\ K_2\right]$, given by the feedback law $\mathbf{u}(t)=\mathbf{v}(t)-\left[0\ K_2\right]\mathbf{x}(t)$, where $\mathbf{v}(t)$ is an $m\times n$ external input vector, such that the n_1 and n_2 d.b.z.'s between the outputs and each disturbances are placed at any specified locations in the complex plane (subject to complex-conjugate pairing). The constant state matrix K_2 is constructed as a sum of two unity-rank matrices $K_2^{(1)}$ and $K_2^{(2)}$. The first assigns n_1 d.b.z.'s at specified locations to achieve asymptotic rejection of the first disturbance; the second, while preserving the n_1 d.b.z.'s which have been assigned in the preceding step, assigns an additional n_2 d.b.z.'s at desired positions to achieve asymptotic rejection of the second disturbance. Thus, max $n_1+n_2\le\left[m-1+n-l\right]$ d.b.z.'s of the system $\sum_d\left[A,B,C,E\right]$ are placed arbitrarily by a simple calculation of two unity-rank matrices. The design procedure is sequential in nature and can be described by the following steps:

Step I: *(Assign n_1 d.b.z.'s)*

In this step, the $m\times(n-l)$ unity-rank constant matrix $K_2^{(1)}=\mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$, where $\mathbf{q}_2^{(1)}$ and $\mathbf{p}_2^{(1)}$ are $m\times 1$ and $1\times(n-1)$ vectors respectively, is determined so as to place n_1 d.b.z.'s at specified locations to eliminate the effect of the first disturbance at the outputs in steady state. It

will be shown later that the design mechanism can be carried out by using Algorithm 4.1 (with a small modification).

Let us define the state feedback law as

$$\mathbf{u}(t) = \mathbf{v}_1(t) - \left[0 \;\; K_2^{(1)}\right]\mathbf{x}(t) \tag{5.2.4}$$

On implementing the feedback law (5.2.4), the system in eqns.(5.2.1a,b) becomes

$$\dot{\mathbf{x}}(t) = A_1\,\mathbf{x}(t) + B\;\mathbf{v}_1(t) + E_1\,d_1(t) + E_2\,d_2(t) \tag{5.2.5a}$$

$$\mathbf{y}(t) = C\;\mathbf{x}(t) \tag{5.2.5b}$$

where

$$A_1 = \begin{bmatrix} A_{11} & A_{12}-B_1K_2^{(1)} \\ A_{21} & A_{22}-B_2K_2^{(1)} \end{bmatrix}$$

is the closed-loop state matrix. By following the factorization procedure in Section 4.1.1, the sets of d.b.z.'s between the outputs $\mathbf{y}(t)$ and each disturbance $d_1(t)$ and $d_2(t)$ can be computed by using the following $(n-l)$th-order, single-input, l-output reduced systems:

$$\dot{\xi}_1 = \left[A_{22}-B_2K_2^{(1)}\right]\xi_1 + E_{21}\,\mu_1 \tag{5.2.6a}$$

$$\upsilon_1 = \left[A_{12}-B_1K_2^{(1)}\right]\xi_1 + E_{11}\,\mu_1 \tag{5.2.6b}$$

and

$$\dot{\xi}_2 = \left[A_{22}-B_2K_2^{(1)}\right]\xi_2 + E_{22}\,\mu_2 \tag{5.2.7a}$$

$$\upsilon_2 = \left[A_{12}-B_1K_2^{(1)}\right]\xi_2 + E_{12}\,\mu_2 \tag{5.2.7b}$$

Now, by assuming that the systems in eqns.(5.2.6a,b) and (5.2.7a,b) are invertible and using the concept of a minimal order system inverse, we can examine the problem of assigning of the n_1 d.b.z.'s of the system in eqns.(5.2.6a,b).

For $\left[l \geq 1\right]$, the system in eqns.(5.2.6a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}_1^* = \left[A_{22}^T - K_2^{(1)T} B_2^T\right] \xi_1^* + \left[A_{12}^T - K_2^{(1)T} B_1^T\right] \mu_1^* \tag{5.2.8a}$$

$$\upsilon_1^* = E_{21}^T \xi_1^* + E_{11}^T \mu_1^* \tag{5.2.8b}$$

A right inverse of the system in eqns.(5.2.8a,b) is given by [1]

$$\dot{\xi}_1^* = \left[\left[A_{22}^T - A_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(1)T} \left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right] \xi_1^*$$

$$+ \left[\left[A_{12}^T - K_2^{(1)T} B_1^T\right] (E_{11}^T)^+\right] \upsilon_1^* \tag{5.2.9a}$$

$$\mu_1^* = \left[-(E_{11}^T)^+ E_{21}^T\right] \xi_1^* + \left[(E_{11}^T)^+\right] \upsilon_1^* \tag{5.2.9b}$$

From Definition 4.3, the closed-loop d.b.z.'s of the above system are defined as those eigenvalues of $\left[\left[A_{22}^T - A_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(1)T} \left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right]$ which are not also the eigenvalues of $\left[\left[A_{22}^T - K_2^{(1)T} B_2^T\right]\right]$. Now the matrix

$$\left[\left[A_{22}^T - A_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(1)T} \left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right]$$

is the transpose of the closed-loop state matrix of the system [2]

$$\dot{\Psi}_1 = \left[A_{22} - E_{21} E_{11}^+ A_{12}\right] \Psi_1 + \left[B_2 - E_{21} E_{11}^+ B_1\right] \Phi_1 \tag{5.2.10}$$

subject to the state feedback law

$$\Phi_1 = \hat{\Phi}_1 - K_2^{(1)} \Psi_1 \tag{5.2.11}$$

where $\hat{\Phi}_1$ is the external input.

Similarly the matrix $\left[A_{22}^T - K_2^{(1)} B_2^T\right]$ is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_1 = A_{22}\Psi_1 + B_2\Phi_1 \tag{5.2.12}$$

subject to the state feedback law in eqn.(5.2.11). It is well known [3] that if the system in eqn.(5.2.10) is completely controllable, then all the eigenvalues of the matrix $\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right]$ can be arbitrarily assigned by state variable feedback, otherwise only those eigenvalues of $\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right]$ can be assigned arbitrarily by state variable feedback which correspond to the controllable modes of the system.

We restrict the constant gain matrix $K_2^{(1)}$ to have unity-rank i.e. $K_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$. The vector $\mathbf{q}_2^{(1)}$ is specified such that the controllability matrix of the single-input system $\left\{\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right], \left[B_2-E_{21}E_{11}^{+}B_1\right]\mathbf{q}_2^{(1)}\right\}$ has rank at least $n-l \leq \left[n_1\right]$. We can use the generic results in [4-6] to generate such a controllable pair. This involves using a randomly generated matrix K_2^r and vector $\mathbf{q}_2^{(1)}$ to get a controllable single-input system

$$\left\{\left[\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right]-\left[B_2-E_{21}E_{11}^{+}B_1\right]K_2^r\right], \left[B_2-E_{21}E_{11}^{+}B_1\right]\mathbf{q}_2^{(1)}\right\}$$

It can be shown that a single-input system generated in this way will "almost always" be controllable. The effect of K_2^r is to make the resulting closed-loop state matrix cyclic, so that if $\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right]$ is already cyclic, K_2^r can be chosen as the null matrix. Having done this, we reduce the design of the constant state feedback matrix $K_2^{(1)}$ for the multi-input system (5.2.10) to that of a constant state feedback vector $\mathbf{p}_2^{(1)}$ which achieves arbitrary assignment of the eigenvalues in the single-input system described by the following equations

$$\dot{\Psi}_1 = \left[A_{22}-E_{21}E_{11}^{+}A_{12}\right]\Psi_1 + \left[\mathbf{b}_2-E_{21}E_{11}^{+}\mathbf{b}_1\right]\Phi_1 \tag{5.2.13a}$$

with feedback defined by

$$\Phi_1 = \hat{\Phi}_1 - \mathbf{p}_2^{(1)}\Psi_1 \tag{5.2.13b}$$

where $\hat{\Phi}_1$ is the external input and $\mathbf{b}_1 = B_1\mathbf{q}_2^{(1)}, \mathbf{b}_2 = B_2\mathbf{q}_2^{(1)}$.

From the above results, we can now determine the vector $\mathbf{p}_2^{(1)}$ which places the n_1 d.b.z.'s of the system (5.2.13a,b) at any specified locations. The vector $\mathbf{p}_2^{(1)}$ can be computed by applying Algorithm 2.1 to the system in eqns.(5.2.13a,b). Note that the resulting state feedback matrix $K_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$ obtained is not unique since $\mathbf{q}_2^{(1)}$ is not unique.

At the end of this step, the closed-loop system is $\sum_d \left[A_1, B, C, E\right]$ which has n_1 d.b.z.'s between $d_1(t)$ and $\mathbf{y}(t)$ assigned at specified locations.

Step II : *(Assign n_2 d.b.z.'s)*

In this step, we determine the $m\times(n-l)$ unity-rank state feedback matrix $K_2^{(2)} = \mathbf{q}_2^{(2)}\mathbf{p}_2^{(2)}$ for the system $\sum_d \left[A_1, B, C, E\right]$ which preserves the n_1 d.b.z.'s assigned in the first step and assigns additional n_2 d.b.z.'s at specified locations, such that the effect of the second disturbance at the outputs is eliminated in steady state. The preservation of the n_1 d.b.z.'s is achieved by a suitable choice of the vector $\mathbf{q}_2^{(2)}$, while the placement of the n_2 d.b.z.'s at desired locations is accomplished by an appropriate choice of the vector $\mathbf{p}_2^{(2)}$.

On applying the state feedback matrix $K_2^{(2)}$ (in the state feedback law $\mathbf{v}_1(t) = \mathbf{v}_2(t) - \left[0 \;\; K_2^{(2)}\right]\mathbf{x}(t)$) to the system $\sum_d \left[A_1, B, C, E\right]$, the resulting closed-loop system is given by

$$\dot{\mathbf{x}}(t) = A_2\,\mathbf{x}(t) + B\,\mathbf{v}_2(t) + E_1\,d_1(t) + E_2\,d_2(t) \tag{5.2.14a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{5.2.14b}$$

where

$$A_2 = \begin{bmatrix} A_{11} & \tilde{A}_{12} - B_1 K_2^{(2)} \\ A_{21} & \tilde{A}_{22} - B_2 K_2^{(2)} \end{bmatrix}$$

with

$$\bar{A}_{12} = \left[A_{12} - B_1 K_2^{(1)}\right]$$

and

$$\bar{A}_{22} = \left[A_{22} - B_2 K_2^{(1)}\right]$$

By applying the factorization procedure to the above system, the sets of d.b.z.'s between the outputs and the two disturbances can now be computed using the following systems

$$\dot{\xi}_1 = \left[\bar{A}_{22} - B_2 K_2^{(2)}\right] \xi_1 + E_{21}\,\mu_1 \tag{5.2.15a}$$

$$\upsilon_1 = \left[\bar{A}_{12} - B_1 K_2^{(2)}\right] \xi_1 + E_{11}\,\mu_1 \tag{5.2.15b}$$

and

$$\dot{\xi}_2 = \left[\bar{A}_{22} - B_2 K_2^{(2)}\right] \xi_2 + E_{22}\,\mu_2 \tag{5.2.16a}$$

$$\upsilon_2 = \left[\bar{A}_{12} - B_1 K_2^{(2)}\right] \xi_2 + E_{12}\,\mu_2 \tag{5.2.16b}$$

Assuming that the system in eqns.(5.2.15a,b) and (5.2.16a,b) are invertible and using the concept of a minimal order system inverse, we can examine the problem of assigning an additional n_2 d.b.z.'s while preserving the n_1 d.b.z.'s which have been assigned in the first step.

For $\left[l \geq 1\right]$, the system in eqns.(5.2.15a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}_1^{\,*} = \left[\bar{A}_{22}^{\,T} - K_2^{(2)T} B_2^T\right] \xi_1^{\,*} + \left[\bar{A}_{12}^{\,T} - K_2^{(2)T} B_1^T\right] \mu_1^{\,*} \tag{5.2.17a}$$

$$\upsilon_1^{\,*} = E_{21}^T\, \xi_1^{\,*} + E_{11}^T\, \mu_1^{\,*} \tag{5.2.17b}$$

Similarly the dual of the system in eqns.(5.2.16a,b) is

$$\dot{\xi}_2^{\,*} = \left[\bar{A}_{22}^{\,T} - K_2^{(2)T} B_2^T\right] \xi_2^{\,*} + \left[\bar{A}_{12}^{\,T} - K_2^{(2)T} B_1^T\right] \mu_2^{\,*} \tag{5.2.18a}$$

$$\upsilon_2^* = E_{22}^T \xi_2^* + E_{12}^T \mu_2^* \tag{5.2.18b}$$

The system in eqns.(5.2.17a,b) has a right inverse given by [1]

$$\dot{\xi}_1^* = \left[\left[\tilde{A}_{22}^T - \tilde{A}_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(2)T}\left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right] \xi_1^*$$

$$+ \left[\left[\tilde{A}_{12}^T - K_2^{(2)T} B_1^T\right](E_{11}^T)^+\right] \upsilon_1^* \tag{5.2.19a}$$

$$\mu_1^* = \left[-(E_{11}^T)^+ E_{21}^T\right] \xi_1^* + \left[(E_{11}^T)^+\right] \upsilon_1^* \tag{5.2.19b}$$

The closed-loop d.b.z.'s of the system in eqns.(5.2.19a,b) are defined as those eigenvalues of $\left[\left[\tilde{A}_{22}^T - \tilde{A}_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(2)T}\left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right]$ which are not also the eigenvalues of $\left[\tilde{A}_{22}^T - K_2^{(2)T} B_2^T\right]$. Now the matrix

$$\left[\left[\tilde{A}_{22}^T - \tilde{A}_{12}^T (E_{11}^T)^+ E_{21}^T\right] - K_2^{(2)T}\left[B_2^T - B_1^T (E_{11}^T)^+ E_{21}^T\right]\right]$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_1 = \left[\tilde{A}_{22} - E_{21} E_{11}^+ \tilde{A}_{12}\right] \Psi_1 + \left[B_2 - E_{21} E_{11}^+ B_1\right] \hat{\Phi}_1 \tag{5.2.20}$$

subject to the state feedback law

$$\hat{\Phi}_1 = \bar{\Phi}_1 - K_2^{(2)} \Psi_1 \tag{5.2.21}$$

where $\bar{\Phi}_1$ is the external input.

Since the constant state feedback is restricted to have unity-rank and can be written as $K_2^{(2)} = \mathbf{q}_2^{(2)} \mathbf{p}_2^{(2)}$, the closed-loop characteristic polynomial of the system in eqns.(5.2.20) and (5.2.21) i.e.

$$H_{d_1}^c(s) = \det\left\{sI_{n-l} - \left[\left[\tilde{A}_{22} - E_{21} E_{11}^+ \tilde{A}_{12}\right] - \left[B_2 - E_{21} E_{11}^+ B_1\right] K_2^{(2)}\right]\right\}$$

can be expressed as [7,8]

$$H_{d_1}^{c}(s) = H_{d_1}^{o}(s) + \mathbf{p}_2^{(2)} V_{d_1}^{o}(s)\, \mathbf{q}_2^{(2)} \tag{5.2.22}$$

where, $V_{d_1}^{o}(s) = adj\left\{sI_{n-l} - \left[\left[A_{22}-B_2K_2^{(1)}\right]-E_{21}E_{11}^{+}\left[A_{12}-B_1K_2^{(1)}\right]\right]\right\}$ and

$H_{d_1}^{o}(s) = \det\left\{sI_{n-l} - \left[\left[A_{22}-B_2K_2^{(1)}\right]-E_{21}E_{11}^{+}\left[A_{12}-B_1K_2^{(1)}\right]\right]\right\}$ are the numerator polynomial matrix of the closed-loop transfer function matrix and the characteristic polynomial respectively at the end of the first step.

In order to preserve the n_1 d.b.z.'s at $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ which were assigned in Step I, we need $H_{d_1}^{c}(\lambda_i) = 0$ for $i = 1, 2, \ldots, n_1$. Since $n_1 \leq m-1$ and $H_{d_1}^{o}(\lambda_i) = 0$ for $i = 1, 2, \ldots, n_1$, from eqn.(5.2.22), we require that

$$\mathbf{p}_2^{(2)} V_{d_1}^{o}(\lambda_i)\, \mathbf{q}_2^{(2)} = 0 \quad ,i = 1, 2, \ldots, n_1 \tag{5.2.23}$$

For eqn.(5.2.23) to hold irrespective of the value of $\mathbf{p}_2^{(2)}$, the vector $\mathbf{q}_2^{(2)}$ must be chosen to satisfy the equations

$$V_{d_1}^{o}(\lambda_i)\, \mathbf{q}_2^{(2)} = 0 \quad ,i = 1, 2, \ldots, n_1 \tag{5.2.24}$$

It can be shown [9] that each of the matrices $V_{d_1}^{o}(\lambda_i), i = 1, 2, \ldots, n_1$ contains only one independent row. Let us denote these rows by $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_{n_1}$. Thus eqn.(5.2.24) will be satisfied for $i = 1, 2, \ldots, n_1$ if the following $n_1 \left[\leq m-1\right]$ linear equations are satisfied:

$$\bar{\mathbf{v}}_i\, \mathbf{q}_2^{(2)} = 0 \quad i = 1, 2, \ldots, n_1 \tag{5.2.25}$$

Equation (5.2.25) implies that the vector $\mathbf{q}_2^{(2)}$ can be chosen to be any $m \times 1$ vector which is orthogonal to the vectors $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_{n_1}$.

It should be noted that, the acquired $\mathbf{q}_2^{(2)}$ makes the system in eqns.(5.2.19a,b) partially uncontrollable through pole-zero cancellations at $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ in the transfer function vector

$V_{d_1}^o(s)\,\mathbf{q}_2^{(2)}/H_{d_1}^o(s)$, and subsequently the uncontrollable d.b.z.'s remain invariant under the constant vector $\mathbf{p}_2^{(2)}$.

Once $\mathbf{q}_2^{(2)}$ is found, the multi-input system in eqns.(5.2.16a,b) becomes

$$\dot{\xi}_2 = \left[\bar{A}_{22} - \mathbf{b}_2\mathbf{p}_2^{(2)}\right]\xi_2 + E_{22}\,\mu_2 \tag{5.2.26a}$$

$$\upsilon_2 = \left[\bar{A}_{12} - \mathbf{b}_1\mathbf{p}_2^{(2)}\right]\xi_2 + E_{12}\,\mu_2 \tag{5.2.26b}$$

where $\mathbf{b}_1 = B_1\mathbf{q}_2^{(2)}, \mathbf{b}_2 = B_2\mathbf{q}_2^{(2)}$. A general left inverse of the above system is given by

$$\dot{\xi}_2^* = \left[\left[\bar{A}_{22}^T - \bar{A}_{12}^T(E_{12}^T)^+E_{22}^T\right] - \mathbf{p}_2^{(2)T}\left[\mathbf{b}_2^T - \mathbf{b}_1^T(E_{12}^T)^+E_{22}^T\right]\right]\xi_2^*$$

$$+ \left[\left[\bar{A}_{12}^T - \mathbf{p}_2^{(2)T}\mathbf{b}_1^T\right](E_{12}^T)^+\right]\upsilon_2^* \tag{5.2.27a}$$

$$\mu_2^* = \left[-(E_{12}^T)^+E_{22}^T\right]\xi_2^* + \left[(E_{12}^T)^+\right]\upsilon_2^* \tag{5.2.27b}$$

The closed-loop d.b.z.'s of the system in eqns.(5.2.27a,b) are those eigenvalues of the matrix $\left[\left[\bar{A}_{22}^T - \bar{A}_{12}^T(E_{12}^T)^+E_{22}^T\right] - \mathbf{p}_2^{(2)T}\left[\mathbf{b}_2^T - \mathbf{b}_1^T(E_{12}^T)^+E_{22}^T\right]\right]$ which are not also the eigenvalues of $\left[\bar{A}_{22}^T - \mathbf{p}_2^{(2)T}\mathbf{b}_2^T\right]$. Now the matrix

$$\left[\left[\bar{A}_{22}^T - \bar{A}_{12}^T(E_{12}^T)^+E_{22}^T\right] - \mathbf{p}_2^{(2)T}\left[\mathbf{b}_2^T - \mathbf{b}_1^T(E_{12}^T)^+E_{22}^T\right]\right]$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_2 = \left[\bar{A}_{22} - E_{22}E_{12}^+\bar{A}_{12}\right]\Psi_2 + \left[\mathbf{b}_2 - E_{22}E_{12}^+\mathbf{b}_1\right]\hat{\Phi}_2 \tag{5.2.28}$$

subject to the state feedback law

$$\hat{\Phi}_2 = \bar{\Phi}_2 - \mathbf{p}_2^{(2)}\,\Psi_2 \tag{5.2.29}$$

Similarly the matrix $\left[\bar{A}_{22}^{T} - \mathbf{p}_2^{(2)T}\mathbf{b}_2^{T}\right]$ is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_2 = \bar{A}_{22}\Psi_2 + \mathbf{b}_2\hat{\Phi}_2 \tag{5.2.30}$$

subject to the state feedback law in eqn.(5.2.29). If the system in eqn.(5.2.28) is completely controllable, then all the $n-l$ eigenvalues of the matrix $\left[\bar{A}_{22} - E_{22}E_{12}^{+}\bar{A}_{12}\right]$ can be arbitrarily assigned by state variable feedback, otherwise only those eigenvalues of $\left[\bar{A}_{22} - E_{22}E_{12}^{+}\bar{A}_{12}\right]$ which correspond to controllable modes of the system can be assigned arbitrarily by state feedback.

From the above results, we can now use Algorithm 2.1 to determine the vector $\mathbf{p}_2^{(2)}$ which assigns $n_2 \leq (n-l)$ d.b.z.'s for the system in eqns.(5.2.28) and (5.2.29). At the end of this step, the closed-loop system is $\sum_d \left[A_2, B, C, E\right]$ and has n_1 (for $d_1(t)$) and n_2 (for $d_2(t)$) d.b.z.'s assigned at specified locations. Thus the constant state feedback matrix $K_2 = K_2^{(1)} + K_2^{(2)}$ assigns $n_1 + n_2 \leq m-1+n-l$ d.b.z.'s of the original system $\sum_d \left[A, B, C, E\right]$ at specified locations to achieve complete disturbance rejection in the steady state.

It should be noted that, for the case when we have r disturbances, we need to carry out r steps in the above procedure to compute the constant state feedback matrix K_2, where K_2 is constructed as a sum of dyads, i.e.

$$\begin{aligned} K_2 &= \sum_{i=1}^{r} K_2^{(i)} \\ &= \sum_{i=1}^{r} \mathbf{q}_2^{(i)}\,\mathbf{p}_2^{(i)} \end{aligned} \tag{5.2.31}$$

From the above results, we can now outline an Algorithm for assigning the d.b.z.'s of the

system $\sum_d \left[A, B, C, E\right]$ using constant state feedback.

Algorithm 5.1 : (Multi-disturbance rejection for the system $\sum_d \left[A, B, C, E\right]$ by means of constant state feedback)

Step I: *(Assignment of n_1 d.b.z.'s using unity-rank feedback matrix $K_2^{(1)}$)*

(i) Set $i = 1$ to select the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$.

(ii) Let $n_1 \left[\leq m-1\right]$ be the number of d.b.z.'s for the system $\sum_d \left[A, B, C, E_1\right]$.

(iii) Specify the vector $\mathbf{q}_2^{(1)}$ arbitrarily, such that the controllability matrix of the single-input system $\left\{\left[A_{22}-E_{21}E_{11}^{+}A_{12}\right], \left[B_2-E_{21}E_{11}^{+}B_1\right]\mathbf{q}_2^{(1)}\right\}$ has rank $\geq n_1$.

(iv) Apply Algorithm 2.1 to the single-input system in eqns.(5.2.13a,b), to determine the vector $\mathbf{p}_2^{(1)}$ which places the n_1 d.b.z.'s at the specified locations.

(v) The unity-rank state feedback matrix is then obtained as $K_2 = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$ and the resulting closed-loop system is $\sum_d \left[A_1, B, C, E\right]$ with $A_1 = \left[A-B\left[0 \;\; K_2^{(1)}\right]\right]$.

Step II: *(Assignment of n_2 d.b.z.'s using unity-rank feedback matrix $K_2^{(2)}$)*

(i) Set $i = 2$ to select the second single-disturbance system $\sum_d \left[A_1, B, C, E_2\right]$.

(ii) Let $n_2 \left[\leq n-l\right]$ be the number of d.b.z.'s for the system $\sum_d \left[A_1, B, C, E_2\right]$.

(iii) Determine the vector $\mathbf{q}_2^{(2)}$ to preserve the n_1 d.b.z.'s which were assigned in the first step.

(iv) Apply Algorithm 2.1 to the single-input system in eqn.(5.2.28), to determine the vector $\mathbf{p}_2^{(2)}$

to place additional n_2 d.b.z.'s at the specified locations.

(v) The unity-rank state feedback matrix is then given by $K_2^{(2)} = \mathbf{q}_2^{(2)} \mathbf{p}_2^{(2)}$, and the resulting closed-loop system is $\sum_d \left[A_2, B, C, E\right]$ with $A_2 = \left[A_1 - B\left[0 \;\; K_2^{(2)}\right]\right]$

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Step r: *(Assignment of n_r ($\leq n-l$) d.b.z.'s using unity-rank feedback matrix $K_2^{(r)}$, while preserving the $n_1, \ldots, n_{r-1}$ $\left[\max(m-1)\right]$ d.b.z.'s assigned in Steps $1-(r-1)$).*

The above procedure allows us to assign at the desired locations a maximum of $\left[m-1+n-l\right]$ d.b.z.'s between the outputs and the r disturbances in r steps. We note that at each step, those d.b.z.'s which are not preserved by choice of $q_i^{(i)}$, $i \geq 2$ and which are not being moved to desired locations will move in an arbitrary manner. However, after r steps, all the sets $\left\{n_i\right\}$; $i = 1, \ldots, r$ of the required d.b.z.'s would have been moved to the desired locations *provided* enough freedom exists in the feedback matrix K_2 to accomplish this.

5.2.2 Assignment of d.b.z.'s by Dynamic State Feedback

In this section, we discuss the cases when one or more single-disturbance systems $\sum_d \left[A, B, C, E_i\right]$ for $i = 1, 2, \ldots, r$ does not have any d.b.z.'s and/or there are not enough d.b.z.'s to completely eliminate all the effects of the disturbances at the outputs in steady state.

To solve this problem, we assume that the system (5.2.1a,b) with disturbances $d_1(t)$ and $d_2(t)$ has $n_1 \leq m-1$ and $n_2 \leq n-l$ d.b.z.'s, respectively. It will be shown later that the design procedure used to solve this problem is an extension of Algorithm 5.1 using a dynamic state feedback compensator defined by the following equations

$$\dot{\mathbf{z}}_2(t) = F_2 \mathbf{z}_2(t) + \begin{bmatrix} 0 & G_2 \end{bmatrix} \mathbf{x}(t) \tag{5.2.32a}$$

$$\mathbf{u}_2(t) = H_2 \mathbf{z}(t) + \begin{bmatrix} 0 & J_2 \end{bmatrix} \mathbf{x}(t) \tag{5.2.32b}$$

$$\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t) \tag{5.2.32c}$$

where $\mathbf{z}_2(t) \in \mathbb{R}^q$ is the state vector of the compensator, $\mathbf{u}_2(t) \in \mathbb{R}^m$ is the output of the compensator, $\mathbf{v}(t) \in \mathbb{R}^m$ is the external input vector, and F_2, G_2, H_2 and J_2 are constant matrices with H_2 and J_2 obtained as sums of dyads i.e $H_2 = H_2^{(1)} + H_2^{(2)}$ and $J_2 = J_2^{(1)} + J_2^{(2)}$. It was shown in Chapter IV, that dynamic state feedback of this form introduces q additional d.b.z.'s between the outputs and each disturbance.

We now consider the assignment of the required number of d.b.z.'s for the two single-disturbance systems $\sum_d \left[A, B, C, E_i\right]$, $i = 1, 2$, by means of a dynamic state feedback law. The following three cases can arise:

Case I :

In this case, we consider that the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$ does not have any d.b.z.'s and/or the number n_1 of the d.b.z.'s is not large enough to achieve complete steady-state rejection of the first disturbance, while the second single-disturbance system $\sum_d \left[A, B, C, E_2\right]$ has the required number n_2 of the d.b.z.'s, such that if we assign them at desired locations, the second disturbance will be rejected completely in steady state.

The above situation can be resolved in two steps: In the first step, the additional number of the d.b.z.'s (q) needed for the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$ are introduced by means of dynamic state feedback of order q. Note that, the dynamic state feedback introduces q d.b.z.'s (equivalent to the poles of the compensator) as well as assigns all the d.b.z.'s

(consisting of the d.b.z.'s of the open-loop system together with those introduced by the dynamic compensator) at desired locations in the complex plane. Second, while preserving the $n_1+q \leq m-1$ closed-loop d.b.z.'s which have been assigned in the preceding step , we compute a constant state feedback law to assign additional n_2 d.b.z.'s for the second single-disturbance system at specified locations in order to reject completely the effect of the second disturbance in steady state.

From the above discussion, the design procedure can be described by the following two steps:

Step I : *(Reject the effect of the first disturbance by means of dynamic state feedback)*

In this step, let us assume that the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$ has n_1 d.b.z.'s which are not enough to eliminate completely the effect of the first disturbance in steady state. It is required to introduce an additional q d.b.z.'s by using dynamic state feedback defined by the following equations

$$\dot{\mathbf{z}}_2(t) = F_2\,\mathbf{z}_2(t) + \begin{bmatrix} 0 & G_2 \end{bmatrix} \mathbf{x}(t) \tag{5.2.33a}$$

$$\mathbf{u}_2(t) = H_2^{(1)}\,\mathbf{z}_2(t) + \begin{bmatrix} 0 & J_2^{(1)} \end{bmatrix} \mathbf{x}(t) \tag{5.2.33b}$$

$$\mathbf{u}(t) = \mathbf{v}_1(t) - \mathbf{u}_2(t) \tag{5.2.33c}$$

where $H_2^{(1)}$ and $J_2^{(1)}$ are $m \times q$ and $m \times (n-l)$ unity-rank matrices, respectively and are given by

$$H_2^{(1)} = \mathbf{q}_2^{(1)}\,\mathbf{h}_2^{(1)}$$

$$J_2^{(1)} = \mathbf{q}_2^{(1)}\,\mathbf{p}_2^{(1)}$$

with $\mathbf{q}_2^{(1)}$, $\mathbf{h}_2^{(1)}$ and $\mathbf{p}_2^{(1)}$ being $m \times 1$, $1 \times q$ and $1 \times (n-l)$ vectors respectively.

On applying the state feedback law defined by eqns.(5.2.33a – c) to the system in eqns.(5.2.1a,b), we obtain

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{A}_1\,\tilde{\mathbf{x}}(t) + \tilde{B}\,\mathbf{v}_1(t) + \tilde{E}_1\,d_1(t) + \tilde{E}_2\,d_2(t) \tag{5.2.34a}$$

$$\mathbf{y}(t) = \tilde{C}\,\tilde{\mathbf{x}}(t) \tag{5.2.34b}$$

with

$$\tilde{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_2(t) \end{bmatrix}$$

and $\tilde{A}_1, \tilde{B}, \tilde{E}_1, \tilde{E}_2$ and $\tilde{C}$ are given by

$$\tilde{A}_1 = \begin{bmatrix} A_{11} & A_{12}-B_1J_2^{(1)} & -B_1H_2^{(1)} \\ A_{21} & A_{22}-B_2J_2^{(1)} & -B_2H_2^{(1)} \\ 0 & G_2 & F_2 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \tilde{E}_1 = \begin{bmatrix} E_{11} \\ E_{21} \\ 0 \end{bmatrix}, \tilde{E}_2 = \begin{bmatrix} E_{12} \\ E_{22} \\ 0 \end{bmatrix},$$

and

$$\tilde{C} = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix}$$

Then, by using the factorization procedure on the above system, the sets of d.b.z.'s between the outputs and the disturbances can be calculated using the following systems

$$\dot{\xi}_1 = \begin{bmatrix} A_{22}-B_2J_2^{(1)} & -B_2H_2^{(1)} \\ G_2 & F_2 \end{bmatrix} \xi_1 + \begin{bmatrix} E_{21} \\ 0 \end{bmatrix} \mu_1 \tag{5.2.35a}$$

$$\upsilon_1 = \begin{bmatrix} A_{12}-B_1J_2^{(1)} & -B_1H_2^{(1)} \end{bmatrix} \xi_1 + \begin{bmatrix} E_{11} \end{bmatrix} \mu_1 \tag{5.2.35b}$$

and

$$\dot{\xi}_2 = \begin{bmatrix} A_{22}-B_2J_2^{(1)} & -B_2H_2^{(1)} \\ G_2 & F_2 \end{bmatrix} \xi_2 + \begin{bmatrix} E_{22} \\ 0 \end{bmatrix} \mu_2 \tag{5.2.36a}$$

$$\upsilon_2 = \left[A_{12} - B_1 J_2^{(1)} \;\; -B_1 H_2^{(1)}\right] \xi_2 + \left[E_{12}\right] \mu_2 \tag{5.2.36b}$$

Now, assuming that the systems in eqns.(5.2.35a,b) and (5.2.36a,b) are invertible and using the concept of a minimal order system inverse, we can examine the problem of assignment of n_1+q closed-loop d.b.z.'s of the system in eqns.(5.2.35a,b) at specified locations in the complex plane.

For $\left[l \geq 1\right]$, the system in eqns.(5.2.35a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}_1^{\,*} = \begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] & G_2^T \\ -H_2^{(1)T} B_2^T & F_2^T \end{bmatrix} \xi_1^{\,*} + \begin{bmatrix} \left[A_{12}^T - J_2^{(1)T} B_1^T\right] \\ -H_2^{(1)T} B_1^T \end{bmatrix} \mu_1^{\,*} \tag{5.2.37a}$$

$$\upsilon_1^{\,*} = \left[E_{21}^T \;\; 0\right] \xi_1^{\,*} + \left[E_{11}^T\right] \mu_1^{\,*} \tag{5.2.37b}$$

A right inverse of the system in eqns.(5.2.37a,b) is given by

$$\dot{\xi}_1^{\,*} = \begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] - \left[A_{12}^T - J_2^{(1)T} B_1^T\right] (E_{11}^T)^+ E_{21}^T & G_2^T \\ \left[-H_2^{(1)T} B_2^T + H_2^{(1)T} B_1^T (E_{11}^T)^+ E_{21}^T\right] & F_2^T \end{bmatrix} \xi_1^{\,*}$$

$$+ \begin{bmatrix} \left[A_{12}^T - J_2^{(1)T} B_1^T\right] \\ -H_2^{(1)T} B_1^T \end{bmatrix} (E_{11}^T)^+ \upsilon_1^{\,*} \tag{5.2.38a}$$

$$\mu_1^{\,*} = -(E_{11}^T)^+ \left[E_{21}^T \;\; 0\right] \xi_1^{\,*} + \left[(E_{11}^T)^+\right] \upsilon_1^{\,*} \tag{5.2.38b}$$

The set of n_1+q closed-loop d.b.z.'s of the system (5.2.34a,b) between $\mathbf{y}(t)$ and $d_1(t)$ are those eigenvalues of the matrix

$$\begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] - \left[A_{12}^T - J_2^{(1)T} B_1^T\right](E_{11}^T)^+ E_{21}^T & G_2^T \\ \left[-H_2^{(1)T} B_2^T + H_2^{(1)T} B_1^T (E_{11}^T)^+ E_{21}^T\right] & F_2^T \end{bmatrix}$$

which are not also the eigenvalues of the matrix

$$\begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] & G_2^T \\ -H_2^{(1)T} B_2^T & F_2^T \end{bmatrix}$$

Now, the matrix

$$\begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] - \left[A_{12}^T - J_2^{(1)T} B_1^T\right](E_{11}^T)^+ E_{21}^T & G_2^T \\ \left[-H_2^{(1)T} B_2^T + H_2^{(1)T} B_1^T (E_{11}^T)^+ E_{21}^T\right] & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop matrix of a system given by

$$\dot{\Psi}_1 = \begin{bmatrix} \left[A_{22} - E_{21} E_{11}^+ A_{12}\right] & 0 \\ G_2 & F_2 \end{bmatrix} \Psi_1 + \begin{bmatrix} \left[B_2 - E_{21} E_{11}^+ B_1\right] \\ 0 \end{bmatrix} \Phi_1 \tag{5.2.39}$$

subject to the state feedback law

$$\Phi_1 = \hat{\Phi}_1 - \left[J_2^{(1)} \; H_2^{(1)}\right] \Psi_1 \tag{5.2.40}$$

where $\hat{\Phi}_1$ is the external input.

Similarly the matrix

$$\begin{bmatrix} \left[A_{22}^T - J_2^{(1)T} B_2^T\right] & G_2^T \\ -H_2^{(1)T} B_2^T & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_1 = \begin{bmatrix} A_{22} & 0 \\ G_2 & F_2 \end{bmatrix} \Psi_1 + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} \Phi_1 \tag{5.2.41}$$

subject to the state feedback law in eqn.(5.2.40).

If the condition in Theorem 4.1 is satisfied for system (5.2.39), then all the eigenvalues of the matrix

$$\begin{bmatrix} \left(A_{22}-E_{21}E_{11}^{+}A_{12}\right) & 0 \\ G_2 & F_2 \end{bmatrix}$$

can be arbitrarily assigned by state feedback, otherwise only those eigenvalues of the above matrix can be assigned arbitrarily by state feedback which correspond to the controllable modes of the system.

By restricting the constant state matrices $J_2^{(1)}$ and $H_2^{(1)}$ to have unity-rank i.e. $J_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$ and $H_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{h}_2^{(1)}$; the vector $\mathbf{q}_2^{(1)}$ can be specified such that the controllability matrix of the single-input system

$$\left\{ \begin{bmatrix} \left(A_{22}-E_{21}E_{11}^{+}A_{12}\right) & 0 \\ G_2 & F_2 \end{bmatrix}, \begin{bmatrix} \left(B_2-E_{21}E_{11}^{+}B_1\right) \\ 0 \end{bmatrix} \mathbf{q}_2^{(1)} \right\}$$

has rank $\geq n_1+q$. The system (5.2.39) with feedback (5.2.40) is then equivalent to the single-input system

$$\dot{\Psi}_1 = \begin{bmatrix} \left(A_{22}-E_{21}E_{11}^{+}A_{12}\right) & 0 \\ G_2 & F_2 \end{bmatrix} \Psi_1 + \begin{bmatrix} \mathbf{b}_2-E_{21}E_{11}^{+}\mathbf{b}_1 \\ 0 \end{bmatrix} \Phi_1 \tag{5.2.42}$$

subject to the state feedback law

$$\Phi_1 = \hat{\Phi}_1 - \left[\mathbf{p}_2^{(1)} \;\; \mathbf{h}_2^{(1)}\right] \Psi_1 \tag{5.2.43}$$

where $\mathbf{b}_1 = B_1\mathbf{q}_2^{(1)}$, $\mathbf{b}_2 = B_2\mathbf{q}_2^{(1)}$.

The state feedback vector $\left[\mathbf{p}_2^{(1)} \;\; \mathbf{h}_2^{(1)}\right]$ can then be computed using Algorithm 2.1. At the end of this step, the closed-loop system is $\sum_d \left[\tilde{A}_1, \tilde{B}, \tilde{C}, \tilde{E}\right]$. This has n_1+q d.b.z.'s placed at specified positions, such that the effect of the first disturbance at the outputs is eliminated in steady state.

Step II : *(Reject the effect of the second disturbance by means of constant state feedback)*

In this step, we determine the $q\times m$ and $m\times(n-l)$ unity-rank matrices $J_2^{(2)} = \mathbf{q}_2^{(2)}\mathbf{p}_2^{(2)}$ and $H_2^{(2)} = \mathbf{q}_2^{(2)}\mathbf{h}_2^{(2)}$, respectively defined via the feedback law

$$\mathbf{v}_1(t) = \mathbf{v}(t) - H_2^{(2)}\mathbf{z}_2(t) - \left[0 \;\; J_2^{(2)}\right]\mathbf{x}(t) \tag{5.2.44}$$

for the system $\sum_d \left[\tilde{A}_1, \tilde{B}, \tilde{C}, \tilde{E}\right]$ which preserves the $n_1+q\le m-1$ d.b.z.'s obtained in the first step and assigns additional $n_2+q\le n-l+q$ d.b.z.'s at specified locations in the complex plane. The preservation of n_1+q d.b.z.'s is achieved by suitable choice of $\mathbf{q}_2^{(2)}$, while the assignment of n_2+q d.b.z.'s at desired locations is accomplished by appropriate choice of the vectors $\mathbf{h}_2^{(2)}$ and $\mathbf{p}_2^{(2)}$.

On applying the control law defined by eqn.(5.2.44) to the system $\sum_d \left[\tilde{A}_1, \tilde{B}, \tilde{C}, \tilde{E}\right]$, the following closed-loop system is obtained

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{A}_2\tilde{\mathbf{x}}(t) + \tilde{B}\,\mathbf{v}(t) + \tilde{E}_1 d_1(t) + \tilde{E}_2 d_2(t) \tag{5.2.45a}$$

$$\mathbf{y}(t) = \tilde{C}\,\tilde{\mathbf{x}}(t) \tag{5.2.45b}$$

where $\tilde{A}_2$ is given by

$$\bar{A}_2 = \begin{bmatrix} A_{11} & \bar{A}_{12}-B_1J_2^{(2)} & -\bar{B}_1-B_1H_2^{(2)} \\ A_{21} & \bar{A}_{22}-B_2J_2^{(2)} & -\bar{B}_2-B_2H_2^{(2)} \\ 0 & G_2 & F_2 \end{bmatrix}$$

with

$$\bar{A}_{12} = \left[A_{12}-B_1J_2^{(1)}\right]$$

$$\bar{A}_{22} = \left[A_{22}-B_2J_2^{(1)}\right]$$

and

$$\bar{B}_1 = B_1H_2^{(1)}$$
$$\bar{B}_2 = B_2H_2^{(1)}$$

Then, by applying the factorization procedure to the above system, the sets of n_1+q and n_2+q closed-loop d.b.z.'s between the outputs and the disturbance $d_1(t)$ and $d_2(t)$ respectively can be calculated from the following reduced-order systems

$$\dot{\xi}_1 = \begin{bmatrix} \bar{A}_{22}-B_2J_2^{(2)} & -\bar{B}_2-B_2H_2^{(2)} \\ G_2 & F_2 \end{bmatrix} \xi_1 + \begin{bmatrix} E_{21} \\ 0 \end{bmatrix} \mu_1 \tag{5.2.46a}$$

$$\upsilon_1 = \left[\left(\bar{A}_{12}-B_1J_2^{(2)}\right) \quad -\bar{B}_1-B_1H_2^{(2)}\right] \xi_1 + \left[E_{11}\right] \mu_1 \tag{5.2.46b}$$

and

$$\dot{\xi}_2 = \begin{bmatrix} \bar{A}_{22}-B_2J_2^{(2)} & -\bar{B}_2-B_2H_2^{(2)} \\ G_2 & F_2 \end{bmatrix} \xi_2 + \begin{bmatrix} E_{22} \\ 0 \end{bmatrix} \mu_2 \tag{5.2.47a}$$

$$\upsilon_2 = \left[\left(\bar{A}_{12}-B_1J_2^{(2)}\right) \quad -\bar{B}_1-B_1H_2^{(2)}\right] \xi_2 + \left[E_{12}\right] \mu_2 \tag{5.2.47b}$$

Now assuming that, the systems in eqns.(5.2.46a,b) and (5.2.47a,b) are invertible and using the concept of a minimal order inverse, we can preserve the n_1+q closed-loop d.b.z.'s which have been assigned in Step I, and assign additional n_2+q of d.b.z.'s at specified locations in the complex plane.

For $\left[l \geq 1\right]$, the system in eqns.(5.2.46a,b) has a left inverse which is the transpose of a right inverse of the dual system

$$\dot{\xi}_1^* = \begin{bmatrix} \left[\bar{A}_{22}^T - J_2^{(2)T} B_2^T\right] & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \bar{B}_2^T\right] & F_2^T \end{bmatrix} \xi_1^* + \begin{bmatrix} \left[\bar{A}_{12}^T - J_2^{(2)T} B_1^T\right] \\ \left[-H_2^{(2)T} B_1^T - \bar{B}_1^T\right] \end{bmatrix} \mu_1^* \tag{5.2.48}$$

$$\upsilon_1^* = \left[E_{21}^T \;\; 0\right] \xi_1^* + \left[E_{11}^T\right] \mu_1^* \tag{5.2.48b}$$

A right inverse of the system in eqns.(5.2.48a,b) is given by

$$\dot{\xi}_1^* = \begin{bmatrix} \left[\bar{A}_{22}^T - J_2^{(2)T} B_2^T\right] - \left[\bar{A}_{12}^T - J_2^{(2)T} B_1^T\right](E_{11}^T)^+ E_{21}^T & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \bar{B}_2^T\right] + \left[H_2^{(2)T} B_1^T + \bar{B}_1^T\right](E_{11}^T)^+ E_{21}^T & F_2^T \end{bmatrix} \xi_1^*$$

$$+ \begin{bmatrix} \left[\bar{A}_{12}^T - J_2^{(2)T} B_1^T\right] \\ \left[-H_2^{(2)T} B_1^T - \bar{B}_1^T\right] \end{bmatrix} (E_{11}^T)^+ \upsilon_1^* \tag{5.2.49a}$$

$$\mu_1^* = -(E_{11}^T)^+ \left[E_{21}^T \;\; 0\right] \xi_1^* + \left[(E_{11}^T)^+\right] \upsilon_1^* \tag{5.2.49b}$$

Similarly the dual of the system in eqns.(5.2.47a,b) is

$$\dot{\xi}_2^* = \begin{bmatrix} \left[\tilde{A}_{22}^T - J_2^{(2)T} B_2^T\right] & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \tilde{B}_2^T\right] & F_2^T \end{bmatrix} \xi_2^* + \begin{bmatrix} \left[\tilde{A}_{12}^T - J_2^{(2)T} B_1^T\right] \\ \left[-H_2^{(2)T} B_1^T - \tilde{B}_1^T\right] \end{bmatrix} \mu_2^* \tag{5.2.50a}$$

$$\upsilon_2^* = \left[E_{22}^T \; 0\right] \xi_2^* + \left[E_{12}^T\right] \mu_2^* \tag{5.2.50b}$$

which has a right inverse given by

$$\dot{\xi}_2^* = \begin{bmatrix} \left[\tilde{A}_{22}^T - J_2^{(2)T} B_2^T\right] - \left[\tilde{A}_{12}^T - J_2^{(2)T} B_1^T\right](E_{12}^T)^+ E_{22}^T & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \tilde{B}_2^T\right] + \left[H_2^{(2)T} B_1^T + \tilde{B}_1^T\right](E_{11}^T)^+ E_{21}^T & F_2^T \end{bmatrix} \xi_2^*$$

$$+ \begin{bmatrix} \left[\tilde{A}_{12}^T - J_2^{(2)T} B_1^T\right] \\ \left[-H_2^{(2)T} B_1^T - \tilde{B}_1^T\right] \end{bmatrix} (E_{12}^T)^+ \upsilon_2^* \tag{5.2.51a}$$

$$\mu_2^* = -(E_{12}^T)^+ \left[E_{22}^T \; 0\right] \xi_2^* + \left[(E_{12}^T)^+\right] \upsilon_2^* \tag{5.2.51b}$$

The set of the closed-loop d.b.z.'s between $\mathbf{y}(t)$ and $d_1(t)$ for the system in eqns.(5.2.45a,b) are those eigenvalues of

$$\bar{A}_c = \begin{bmatrix} \left[\tilde{A}_{22}^T - J_2^{(2)T} B_2^T\right] - \left[\tilde{A}_{12}^T - J_2^{(2)T} B_1^T\right](E_{11}^T)^+ E_{21}^T & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \tilde{B}_2^T\right] + \left[H_2^{(2)T} B_1^T + \tilde{B}_1^T\right](E_{11}^T)^+ E_{21}^T & F_2^T \end{bmatrix}$$

which are not also the eigenvalues of

$$\begin{bmatrix} \left[\tilde{A}_{22}^T - J_2^{(2)T} B_2^T\right] & G_2^T \\ \left[-H_2^{(2)T} B_2^T - \tilde{B}_2^T\right] & F_2^T \end{bmatrix}$$

Now, the matrix $\bar{A}_c$ is the transpose of the closed-loop matrix of the system

$$\dot{\boldsymbol{\Psi}}_1 = \begin{bmatrix} \left(\bar{A}_{22}-E_{21}E_{11}^{+}\bar{A}_{12}\right) & \left(-\bar{B}_2+E_{21}E_{11}^{+}\bar{B}_1\right) \\ G_2 & F_2 \end{bmatrix} \boldsymbol{\Psi}_1 + \begin{bmatrix} \left(B_2-E_{21}E_{11}^{+}B_1\right) \\ 0 \end{bmatrix} \hat{\boldsymbol{\Phi}}_1 \tag{5.2.52}$$

subject to the state feedback law

$$\hat{\boldsymbol{\Phi}}_1 = \bar{\boldsymbol{\Phi}}_1 - \left[J_2^{(2)} \;\; H_2^{(2)}\right] \boldsymbol{\Psi}_1 \tag{5.2.53}$$

Since the constant state feedback $J_2^{(2)}$ and $H_2^{(2)}$ are restricted to have unity-rank, i.e. $J_2^{(2)} = \mathbf{q}_2^{(2)}\mathbf{p}_2^{(2)}$ and $H_2^{(2)} = \mathbf{q}_2^{(2)}\mathbf{h}_2^{(2)}$, then the closed-loop characteristic polynomial

$$H_{d_1}^{c}(s) = \det\left\{ sI_{n-l+q} - \left[\begin{bmatrix} \left(\bar{A}_{22}-E_{21}E_{11}^{+}\bar{A}_{12}\right) & \left(-\bar{B}_2+E_{21}E_{11}^{+}\bar{B}_1\right) \\ G_2 & F_2 \end{bmatrix} - \begin{bmatrix} \left(B_2-E_{21}E_{11}^{+}B_1\right) \\ 0 \end{bmatrix} \left[J_2^{(2)} \;\; H_2^{(2)}\right] \right] \right\}$$

can be expressed as [9]:

$$H_{d_1}^{c}(s) = H_{d_1}^{o}(s) + \left[\mathbf{p}_2^{(2)} \;\; \mathbf{h}_2^{(2)}\right] V_{d_1}^{o}(s)\, \mathbf{q}_2^{(2)} \tag{5.2.54}$$

where

$$V_{d_1}^{o}(s) = adj\left\{ sI_{n-l+q} - \left[\begin{bmatrix} \left(\bar{A}_{22}-E_{21}E_{11}^{+}\bar{A}_{12}\right) & \left(-\bar{B}_2+E_{21}E_{11}^{+}\bar{B}_1\right) \\ G_2 & F_2 \end{bmatrix} \right] \right\}$$

and

$$H_{d_1}^{o}(s) = \det\left\{ sI_{n-l+q} - \left[\begin{bmatrix} \left(\bar{A}_{22} - E_{21}E_{11}^{+}\bar{A}_{12}\right) & \left(-\bar{B}_2 + E_{21}E_{11}^{+}\bar{B}_1\right) \\ G_2 & F_2 \end{bmatrix} \right] \right\}$$

are the polynomial matrix of the closed-loop transfer function matrix and characteristic polynomial, respectively of the system obtained after the first step.

The vector $\mathbf{q}_2^{(2)}$ is used to preserve the set $n_1+q \leq m-1$ of the d.b.z.'s obtained in Step I at $\lambda_1, \lambda_2, \ldots, \lambda_{n_1+q}$ in the closed-loop system. In order to preserve the d.b.z.'s at $\lambda_1, \lambda_2, \ldots, \lambda_{n_1+q}$ irrespective of $\left[\mathbf{p}_2^{(2)} \;\; \mathbf{h}_2^{(2)}\right]$, from eqn.(5.2.54) we require

$$V_{d_1}^{o}(\lambda_i)\,\mathbf{q}_2^{(2)} = 0 \qquad i = 1, 2, \ldots, n_1+q \tag{5.2.55}$$

Since the matrices $V_{d_1}^{o}(\lambda_i)$, $i = 1, 2, \ldots, n_1+q$, have rank one [9], each matrix $V_{d_1}^{o}(\lambda_i)$ contains only one independent row. Let us denote this row by $\bar{\mathbf{v}}_i$. Thus the vector $\mathbf{q}_2^{(2)}$ is found from the $n_1+q\left(\leq m-1\right)$ linear equations:

$$\bar{\mathbf{v}}_i\,\mathbf{q}_2^{(2)} = 0 \qquad i = 1, 2, \ldots, n_1+q \tag{5.2.56}$$

It is noted that the acquired $\mathbf{q}_2^{(2)}$ makes the single-input system

$$\left(\begin{bmatrix} \left(\bar{A}_{22} - E_{21}E_{11}^{+}\bar{A}_{12}\right) & \left(-\bar{B}_2 + E_{21}E_{11}^{+}\bar{B}_1\right) \\ G_2 & F_2 \end{bmatrix}, \begin{bmatrix} \left(B_2 - E_{21}E_{11}^{+}B_1\right) \\ 0 \end{bmatrix} \mathbf{q}_2^{(2)} \right)$$

partially uncontrollable through pole-zero cancellation at $\lambda_1, \lambda_2, \ldots, \lambda_{n_1+q}$ in the transfer function vector $V_{d_1}^{o}(s)\,\mathbf{q}_2^{(2)}/H_{d_1}^{o}(s)$, and consequently the uncontrollable d.b.z.'s remain invariant under the control law given by eqn.(5.2.53).

Once $\mathbf{q}_2^{(2)}$ is found, the multi-input system in eqns.(5.2.51a,b) becomes

$$\dot{\xi}_2^* = \begin{bmatrix} \left[\bar{A}_{22}-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_2^T\right]-\left[\bar{A}_{12}^T-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_1^T\right](E_{12}^T)^T E_{22}^T & G_2^T \\ \left[-\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_2^T-\bar{B}_2^T\right]+\left[\bar{B}_1^T+\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_1^T\right](E_{12}^T)^+ E_{22}^T & F_2^T \end{bmatrix} \xi_2^*$$

$$+ \begin{bmatrix} \left[\bar{A}_{12}^T-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_1^T\right] \\ \left[-\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_1^T-\bar{B}_1^T\right] \end{bmatrix} (E_{12}^T)^+ \upsilon_1^* \tag{5.2.57a}$$

$$\mu_2^* = -(E_{12}^T)^+ \begin{bmatrix} E_{22}^T & 0 \end{bmatrix} \xi_2^* + \left[(E_{12}^T)^+\right] \upsilon_2^* \tag{5.2.57b}$$

where $\bar{\mathbf{b}}_1 = B_1\mathbf{q}_2^{(2)}$ and $\bar{\mathbf{b}}_2 = B_2\mathbf{q}_2^{(2)}$, respectively.

The set of the closed-loop d.b.z.'s of the system in eqns.(5.2.57a,b) are those eigenvalues of

$$\bar{A}_c = \begin{bmatrix} \left[\bar{A}_{22}-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_2^T\right]-\left[\bar{A}_{12}^T-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_1^T\right](E_{12}^T)^+ E_{22}^T & G_2^T \\ \left[-\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_2^T-\bar{B}_2^T\right]-\left[\bar{B}_1^T+\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_1^T\right](E_{12}^T)^+ E_{22}^T & F_2^T \end{bmatrix}$$

which are not also the eigenvalues of

$$\begin{bmatrix} \left[\bar{A}_{22}-\mathbf{p}_2^{(2)T}\bar{\mathbf{b}}_2^T\right] & G_2^T \\ \left[-\mathbf{h}_2^{(2)T}\bar{\mathbf{b}}_2^T-\bar{B}_2^T\right] & F_2^T \end{bmatrix}$$

Now, the matrix $\bar{A}_c$ is the transpose of the closed-loop state matrix of the single input system

$$\dot{\Psi}_2 = \begin{bmatrix} \left[\bar{A}_{22}-E_{22}E_{12}^+\bar{A}_{12}\right] & \left[-\bar{B}_2-E_{22}E_{12}^+\bar{B}_1\right] \\ G_2 & F_2 \end{bmatrix} \Psi_2 + \begin{bmatrix} \left[\bar{\mathbf{b}}_2-E_{22}E_{12}^+\bar{\mathbf{b}}_1\right] \\ 0 \end{bmatrix} \hat{\Phi}_2 \tag{5.2.58}$$

subject to the state feedback law

$$\hat{\Phi}_2 = \bar{\Phi}_2 - \left[\mathbf{p}_2^{(2)} \;\; \mathbf{h}_2^{(2)}\right] \Psi_2 \tag{5.2.59}$$

Similarly, the matrix

$$\begin{bmatrix} \left(\tilde{A}_{22} - \mathbf{p}_2^{(2)T} \tilde{\mathbf{b}}_2^{\,T}\right) & G_2^T \\ \left(-\mathbf{h}_2^{(2)T} \tilde{\mathbf{b}}_2^{\,T} - \tilde{B}_2^T\right) & F_2^T \end{bmatrix}$$

is the transpose of the closed-loop state matrix of the system

$$\dot{\Psi}_2 = \begin{bmatrix} \tilde{A}_{22} & -\hat{B}_2 \\ G_2 & F_2 \end{bmatrix} \Psi_2 + \begin{bmatrix} \tilde{\mathbf{b}}_2 \\ 0 \end{bmatrix} \hat{\Phi}_2 \tag{5.2.60}$$

subject to the state feedback in eqn.(5.2.59). If the system in eqn.(5.2.58) is controllable, than all the $n-l+q$ eigenvalues of the matrix

$$\begin{bmatrix} \left[\tilde{A}_{22} - E_{22} E_{12}^{+} \tilde{A}_{12}\right] & \left(-\tilde{B}_2 - E_{22} E_{12}^{+} \tilde{B}_1\right) \\ G_2 & F_2 \end{bmatrix}$$

can be arbitrarily assigned by state feedback; otherwise only those eigenvalues of the above matrix can be assigned arbitrarily by state feedback which correspond to the controllable modes of the system.

The final closed-loop system $\sum_d \left(\tilde{A}_2, \tilde{B}, \tilde{C}, \tilde{E}\right)$ will then have the n_1+q (for $d_1(t)$) and n_2 (for $d_2(t)$) d.b.z.'s assigned at specified locations in the complex plane. The required dynamic state feedback is

$$\dot{\mathbf{z}}_2(t) = F_2 \mathbf{z}_2(t) + \left[0 \;\; G_2\right] \mathbf{x}(t) \tag{5.2.61a}$$

$$\mathbf{u}_2(t) = H_2 \mathbf{z}_2(t) + \left[0 \;\; J_2\right] \mathbf{x}(t) \tag{5.2.61b}$$

It should be noted that the matrices $H_2 = H_2^{(1)} + H_2^{(2)}$ and $J_2 = J_2^{(1)} + J_2^{(2)}$ are not in general

unique because of the freedom in the choice of $\mathbf{q}_2^{(1)}$ in the first step.

Case II:

In this case, we consider the complement of the above case for which the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$ has the required number n_1 of the d.b.z.'s; while the second single-disturbance system $\sum_d \left[A, B, C, E_2\right]$ does not have any d.b.z.'s and/or the number n_2 of the d.b.z.'s is not large enough to reject completely the effect of the second disturbance in steady state. To solve this problem, we first use a unity-rank matrix $J_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$ in the state feedback law:

$$\mathbf{u}_2(t) = \mathbf{v}_1(t) - \left[0 \;\; J_2^{(1)}\right] \mathbf{x}(t) \tag{5.2.62}$$

to assign n_1 d.b.z.'s for the system $\sum_d \left[A, B, C, E_1\right]$ at specified locations, such that the effect of the first disturbance is eliminated in steady state. The unity-rank matrix $J_2^{(1)} = \mathbf{q}_2^{(1)}\mathbf{p}_2^{(1)}$ can be calculated by carrying out Step I in Algorithm 5.1. The closed-loop system at the end of this step is $\sum_d \left[A_1, B, C, E\right]$, where $A_1 = \left[A - B\left[0 \;\; J_2^{(1)}\right]\right]$.

In the second step, while preserving these $n_1 \leq m-1$ closed-loop d.b.z.'s, an additional q d.b.z.'s are introduced by means of dynamic state feedback defined by

$$\dot{\mathbf{z}}_2(t) = F_2\mathbf{z}_2(t) + \left[0 \;\; G_2\right] \mathbf{x}(t) \tag{5.2.63a}$$

$$\mathbf{u}_2(t) = H_2\mathbf{z}_2(t) + \left[0 \;\; J_2^{(2)}\right] \mathbf{x}(t) \tag{5.2.63b}$$

$$\mathbf{v}_1(t) = \mathbf{v}(t) - \mathbf{u}_2(t) \tag{5.2.63c}$$

such that the resulting n_2+q closed-loop d.b.z.'s are placed at specified locations in the complex plane to achieve complete rejection of the second disturbance in steady state. Note that this

compensator can be determined by carrying out the procedure in Step I of Case I. Therefore, the resulting dynamic state feedback compensator is of order q and is given by

$$\dot{\mathbf{z}}_2(t) = F_2 \mathbf{z}_2(t) + \begin{bmatrix} 0 & G_2 \end{bmatrix} \mathbf{x}(t) \tag{5.2.64a}$$

$$\mathbf{u}_2(t) = H_2 \mathbf{z}_2(t) + \begin{bmatrix} 0 & J_2 \end{bmatrix} \mathbf{x}(t) \tag{5.2.64b}$$

where the matrix $H_2 = \mathbf{q}_2^{(2)} \mathbf{h}_2$ has unity-rank and the matrix $J_2 = J_2^{(1)} + J_2^{(2)}$ with $J_2^{(2)} = \mathbf{q}_2^{(2)} \mathbf{p}_2^{(2)}$. The final closed-loop system $\sum_d \left[\tilde{A}_2, \tilde{B}, \tilde{C}, \tilde{E}\right]$ has the n_1 (for $d_1(t)$) and n_2+q (for $d_2(t)$) closed-loop d.b.z.'s assigned at specified locations where

$$\tilde{A}_2 = \begin{bmatrix} A_{11} & A_{12}-B_1 J_2 & -B_1 H_2 \\ A_{21} & A_{22}-B_2 J_2 & -B_2 H_2 \\ 0 & G_2 & F_2 \end{bmatrix}$$

Case III:

This is the general case, for which the two single-disturbance systems do not have any d.b.z.'s and/or the numbers n_1 and n_2 of the d.b.z.'s are not large enough to achieve complete disturbance rejection in steady state. The design procedure can be carried out in two steps using dynamic state feedback $K_2(s)$: In the first step (Step I in Case I), we compute a dynamic state feedback $K_2^{(1)}(s)$ of order $q^{(1)}$ depending on the required number of d.b.z.'s for the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$. Then the resulting closed-loop system $\sum_d \left[\tilde{A}_1, \tilde{B}, \tilde{C}, \tilde{E}\right]$ has $n_1+q^{(1)}$ d.b.z.'s assigned at desired locations to achieve complete rejection of the first disturbance in steady state. In the second step, the assigned $n_1+q^{(1)} \left[\leq m-1\right]$

d.b.z.'s are preserved and a set of upto $n_2+q^{(1)}+q^{(2)}$ (or $n_2+q^{(1)}$) additional d.b.z.'s are assigned by using a unity-rank dynamic state feedback compensator $K_2^{(2)}(s)$ of order $q^{(2)}$ (or constant state feedback matrix $\left[J_2^{(2)}\ H_2^{(2)}\right]$) depending on the number of additional d.b.z.'s to be assigned for the second single-disturbance system $\sum_d \left[\bar{A}_1, \bar{B}, \bar{C}, \bar{E}_2\right]$. Thus the resulting dynamic state feedback $K_2(s)$ is constructed as a sum of two unity-rank compensators $K_2(s) = K_2^{(1)}(s)+K_2^{(2)}(s)$ (or a sum of a unity-rank compensator $K_2^{(1)}(s)$ and a unity-rank constant matrix $\left[J_2^{(2)}\ H_2^{(2)}\right]$) and has order $q = q^{(1)}+q^{(2)}$ (or $q = q^{(1)}$). For the case when dynamic compensation is used in both steps, the overall compensator is given by:

$$\dot{z}_2(t) = F_2 z_2(t) + \left[0\ G_2\right] x(t) \tag{5.2.65a}$$

$$u_2(t) = H_2 z_2(t) + \left[0\ J_2\right] x(t) \tag{5.2.65b}$$

where

$$F_2 = \begin{bmatrix} F_2^{(1)} & 0 \\ 0 & F_2^{(2)} \end{bmatrix}, G_2 = \begin{bmatrix} G_2^{(1)} \\ G_2^{(2)} \end{bmatrix}$$

$$H_2 = \left[H_2^{(1)}\ H_2^{(2)}\right], J_2 = \left[J_2^{(1)} + J_2^{(2)}\right]$$

Now, for the sake of completeness we will outline the Algorithm used to reject in steady state all disturbances affecting a system $\sum_d \left[A, B, C, E\right]$ by means of dynamic state feedback.

Algorithm 5.2: (Multi-disturbance rejection for the system $\sum_d \left[A, B, C, E\right]$ by means of dynamic state feedback)

Step I: *(Reject the effect of the first disturbance)*

(i) Set $i = 1$, to select the first single-disturbance system $\sum_d \left[A, B, C, E_1\right]$.

(ii) let $n_1 \left[\leq m-1\right]$ be the number of d.b.z.'s for the system $\sum_d \left[A, B, C, E_1\right]$.

(iii) Suppose that $\bar{n}_1 \left[\leq m-1\right]$ is the number of the d.b.z.'s required to achieve complete rejection of the first disturbance in steady state.

(iv) If $\bar{n}_1 \leq n_1$ go to Step (v); else go to Step (vi).

(v) Assign $\bar{n}_1$ d.b.z.'s by means of unity-rank constant state feedback matrix $J_2^{(1)} = \mathbf{q}_2^{(1)} \mathbf{p}_2^{(1)}$ given by the feedback law in eqn.(5.2.62):

(a) Specify $\mathbf{q}_2^{(1)}$ such that the controllability matrix of the single-input system

$$\left\{ \left[A_{22} - E_{21} E_{11}^{+} A_{12}\right], \left[B_2 - E_{21} E_{11}^{+} B_1\right] \mathbf{q}_2^{(1)} \right\}$$

has a rank at least $\bar{n}_1$.

(b) Apply Algorithm 2.1 to the single-input system given in eqn.(5.2.13) by using the state feedback law in eqn(5.2.14) and determine the vector $\mathbf{p}_2^{(1)}$ which assigns the set of $\bar{n}_1$ closed-loop d.b.z.'s at specified locations.

(c) The resulting closed-loop system is $\sum_d \left[A_1, B, C, E\right]$, where $A_1 = \left[A - B\left[0 \;\; J_2^{(1)}\right]\right]$.

(d) Go to Step II.

(vi) Assign $\bar{n}_1$ d.b.z.'s by means of a unity-rank dynamic compensator $K_2^{(1)}(s)$ of order $q^{(1)} = (\bar{n}_1 - n_1)$, defined by the feedback law given in eqns.(5.2.33a-c).

(a) Specify $\mathbf{q}_2^{(1)}$ arbitrarily, such that the controllability matrix of the single-input system

$$\left\{ \begin{bmatrix} \left[A_{22}-E_{21}E_{11}^{+}A_{12}\right] & 0 \\ G_2 & F_2 \end{bmatrix}, \begin{bmatrix} \left[B_2-E_{21}E_{11}^{+}B_1\right] \\ 0 \end{bmatrix} \mathbf{q}_2^{(1)} \right\}$$

has rank at least $\bar{n}_1$.

(b) Apply Algorithm 2.1 to the single-input system given in eqn.(5.2.42) by using the state feedback law in eqn.(5.2.43) and determine the vector $\left[\mathbf{p}_2^{(1)} \;\; \mathbf{h}_2^{(1)}\right]$ which assigns $\bar{n}_1$ closed-loop d.b.z.'s at specified locations.

(c) The resulting closed-loop system is $\sum_d \left[\bar{A}_1, \bar{B}, \bar{C}, \bar{E}\right]$ and is given by eqns.(5.2.34a,b).

Step II: *(Reject the effect of the second disturbance)*

(i) Set $i = 2$, to select the second disturbance system $\sum_d \left[A_1, B, C, E_2\right]$ (or $\sum_d \left[\bar{A}_1, \bar{B}, \bar{C}, \bar{E}_2\right]$).

(ii) Let n_2 (or $n_2+q^{(1)}$) $\left[\leq n-l \;(or\; n-l+q^{(1)})\right]$ be the number of d.b.z.'s for the system $\sum_d \left[A_1, B, C, E_2\right]$ (or $\sum_d \left[\bar{A}_1, \bar{B}, \bar{C}, \bar{E}_2\right]$).

(iii) Suppose that $\bar{n}_2 \left[\leq n-l \;(or\; n-l+q^{(1)})\right]$ is the number of d.b.z.'s required to achieve complete rejection of the second disturbance in steady state.

(iv) If $\bar{n}_2 \leq n_2$ ($or\; n_2+q^{(1)}$) go to Step (v); else go to Step (vi).

Comment: Since the case $\bar{n}_2 \leq n_2$ is considered in Step II of Algorithm 5.1, here we will present only the case when $\bar{n}_2 \leq n_2+q^{(1)}$.

(v) Assign the set $\bar{n}_2$ d.b.z.'s by means of a unity-rank matrix $\left[J_2^{(2)} \; H_2^{(2)}\right]$ defined by the state feedback law in eqn.(5.2.44).

(a) Calculate the vector $\mathbf{q}_2^{(2)}$ which preserves the set $\bar{n}_1 \leq m-1$ assigned in Step I.

(b) Apply Algorithm 2.1 to the single-input system given in eqn.(5.2.58) using the state feedback law in eqn.(5.2.59), and determine the vector $\left[\mathbf{p}_2^{(2)} \; \mathbf{h}_2^2\right]$ to assign $\bar{n}_2$ closed-loop d.b.z.'s at specified locations such that the effect of the second disturbance is eliminated in steady state.

(c) The resulting closed-loop system $\sum_d \left[\bar{A}_2, \bar{B}, \bar{C}, \bar{E}\right]$ is given by eqns.(5.2.45a,b).

(d) Go to Step III.

(vi) Assign $\bar{n}_2$ d.b.z.'s by means of a unity-rank dynamic feedback compensator $K_2^{(2)}(s)$ of order $q^{(2)}$.

Comment: In all that follows, for the purpose of illustration and simplicity, we will consider the system $\sum_d \left[A_1, B, C, E\right]$ with dynamic compensator of order $q^{(2)} = \bar{n}_2 - n_2$ given by the feedback law in eqns.(5.2.63). Note that the system $\sum_d \left[\bar{A}_1, \bar{B}, \bar{C}, \bar{E}\right]$ with dynamic compensator of order $q^{(2)}$ can be treated in a similar way to that given in Case II.

(a) Calculate the vector $\mathbf{q}_2^{(2)}$ which preserves the $\bar{n}_1$ closed-loop d.b.z.'s assigned in Step I.

(b) Apply Algorithm 2.1 to determine the vector $\left[\mathbf{p}_2^{(2)} \; \mathbf{h}_2\right]$ which assigns $n_2 + q^{(2)}$ additional d.b.z.'s at specified locations.

(c) The resulting closed-loop system is $\sum_d \left[\bar{A}_2, \bar{B}, \bar{C}, \bar{E}\right]$ and has $\bar{n}_1$ and $\bar{n}_2$ d.b.z.'s between the outputs and the first and second disturbances respectively, which are assigned at

specified locations by means of the dynamic state feedback compensator given in eqns.(5.2.64a,b).

•
•
•

Step r: *(Reject the rth disturbance).*

Continuing in the above manner, it can be seen that this procedure allows us to compute a dynamic compensator of order q and assign a maximum of $(n-l+m-1+q)$ d.b.z.'s between the outputs and the r disturbances at desired locations in r steps.

5.3 MULTI-DISTURBANCE REJECTION FOR THE SYSTEM $\sum_d \left[A, B, C, D, E, F\right]$

In this section, the results of the preceding section are extended to solve the problem of assigning sets of d.b.z.'s by means of constant gain as well as dynamic state feedback for the system described by the state-space model $\sum_d \left[A, B, C, D, E, F\right]$, i.e.

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \sum_{i=1}^{r} E_i\, d_i(t) \tag{5.3.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) + \sum_{i=1}^{r} F_i\, d_i(t) \tag{5.3.1b}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector, $\mathbf{y}(t) \in \mathbb{R}^l$ is the output vector and $d_i(t) \in \mathbb{R}$ for $i = 1, 2, \ldots, r$ are the disturbances which may or may not be measurable, with $m \geq r$. It is assumed that the systems $\sum_d \left[A, B, C, D, E_i, F_i\right]$, $i = 1, 2, \ldots, r$ are

non-degenerate and that $rank(B) = m$ and $rank(C) = l$.

We also assume that, the transfer function matrices relating the outputs to the disturbances for the open-loop systems and the closed-loop systems with state feedback have n disturbance zeros. This assumption implies that for the open-loop system, the following equations hold:

$$A^T \left[I_n - (E_i^T)^+ E_i^T \right] = 0 \qquad i = 1, 2, \ldots, r \tag{5.3.2}$$

while for the closed-loop system, we require that

$$B^T \left[I_n - (E_i^T)^+ F_i^T \right] = 0 \qquad i = 1, 2, \ldots, r \tag{5.3.3}$$

Furthermore we assume that the number of d.b.z.'s to be assigned between the outputs and $d_i(t)$ are $n_i, i = 1, 2, \ldots, r$, with $\sum_{i=1}^{r-1} n_i \leq m-1$ and $n_r \leq n$.

We have shown in Sections 3.3 and 4.1 that the system in eqns.(5.3.1a,b) can be transformed into a higher order system denoted by $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E} \right]$ described by the following equations:

$$\dot{\hat{\mathbf{x}}}(t) = \hat{A}\, \hat{\mathbf{x}}(t) + \hat{B}\, \mathbf{u}(t) + \sum_{i=1}^{r} \hat{E}_i\, d_i(t) \tag{5.3.4a}$$

$$\hat{\mathbf{y}}(t) = \hat{C}\, \hat{\mathbf{x}}(t) \tag{5.3.4b}$$

$$\mathbf{y}(t) = \begin{bmatrix} 0 & C \end{bmatrix} \hat{\mathbf{x}}(t) + D\, \mathbf{u}(t) + \sum_{i=1}^{r} F_i\, d_i(t) \tag{5.3.4c}$$

where,

$$\hat{A} = \begin{bmatrix} \Omega & \Theta C \\ 0 & A \end{bmatrix}, \hat{B} = \begin{bmatrix} \Theta D \\ B \end{bmatrix}, \hat{E}_i = \begin{bmatrix} \Theta F_i \\ E_i \end{bmatrix}, \hat{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}$$

with

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{z}(t) \\ \mathbf{x}(t) \end{bmatrix}$$

and Ω, Θ and I_p are constant matrices defining the dynamic output feedback of order p.

The system (5.3.4a-c) enables us to use the factorization procedure of Section 3.2 and the concept of a minimal order inverse to determine the numbers and locations of the d.b.z.'s which can be assigned by state feedback. Hence, the procedures described in Algorithms 5.1 and 5.2, using either constant or dynamic state feedback, can be applied to the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$ and used to assign the sets of n_i, $i = 1, 2, \ldots, r$ d.b.z.'s at specified locations. From the property of the factorization, it then follows that the d.b.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$ will also be assigned by the constant or dynamic state feedback together with the dynamic output feedback used to get the system $\sum_d \left[\hat{A}, \hat{B}, \hat{C}, \hat{E}\right]$.

5.4 GENERAL REMARKS

The following remarks are required to clarify certain points regarding the implementation and advantages of the proposed algorithms to reject multiple disturbances affecting the multivariable systems described by $\sum_d \left[A, B, C, E\right]$ or $\sum_d \left[A, B, C, D, E, F\right]$:

Remark 5.1: For measurable disturbances the computations and the structure of the controllers in the proposed algorithms can be considerably simplified. To see this, consider the system $\sum_d \left[A, B, C, E\right]$ with two disturbances given by

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \sum_{i=1}^{2} E_i\, d_i(t) \tag{5.4.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{5.4.1b}$$

Assume that the disturbances $d_1(t)$ and $d_2(t)$ are measurable. Consider two cases:

(a) $d_1(t) = \beta_1$, and $d_2(t) = \beta_2$

(b) $d_1(t) = \beta_1$ and $d_2(t) = \beta_2 + \beta_3\, exp\,(t)$

where β_1, β_2 and β_3 are known constant values.

(a) In this case, since the disturbance $d_1(t)$ and $d_2(t)$ are of identical type (i.e step disturbances) but with different amplitudes, the problem of multi-disturbance rejection can be reduced to that of single-disturbance rejection for the system $\sum_d \left[A, B, C, \bar{E}\right]$, where $\bar{E} = E_1\beta_1 + E_2\beta_2$. We can then apply Algorithm 4.1 or 4.2 to assign at least one d.b.z at 0.0 to achieve complete rejection of the step disturbances in steady state.

Note that, for unmeasurable disturbances, (i.e. when β_1 and β_2 are unknown) this reduction cannot be used, and the problem should be solved by applying Algorithms 5.1 or 5.2.

(b) In order to reject this type of disturbances, we note that from the design point of view, it is desirable to split the problem into that for two single-disturbance systems : The first denoted $\sum_d \left[A, B, C, \bar{E}_1\right]$, is affected by a step disturbance and the second, $\sum_d \left[A, B, C, \bar{E}_2\right]$, is affected by an exponential disturbance, where the disturbance vectors $\bar{E}_1$ and $\bar{E}_2$ are given respectively by

$$\bar{E}_1 = E_1\beta_1 + E_2\beta_2$$

and

$$\bar{E}_2 = E_2\beta_3$$

Then we can apply the two step procedure of Algorithm 5.1 or 5.2 to find the state feedback controller which assigns at least one d.b.z each for the system $\sum_d \left[A, B, C, \bar{E}_1\right]$ at 0.0 and for the system $\sum_d \left[A, B, C, \bar{E}_2\right]$ at 1.0. Note that in this way the construction of the state feedback

controllers could be considerably simplified and may also have simpler structure because of the possibility of lower crder.

Remark 5.2: In some cases, it may not be possible to position d.b.z.'s at the same location, say $\hat{\lambda}$, for two or more disturbance inputs using the approach described above. These situations may arise where (i) the 4-tuple (A, B, C, D) is degenerate i.e. the system $\sum\left[A, B, C, D\right]$ is not invertible or (ii) the system $\sum\left[A, B, C, D\right]$ has a blocking zero at $\hat{\lambda}$. This is a consequence of the way in which a d.b.z.'s assigned at $\hat{\lambda}$ for the ith disturbance is preserved while assigning d.b.z.'s for subsequent disturbances.

Remark 5.3: In the case that the conditions in equations (5.2.2a,b) or (5.2.3a,b) are not satisfied for system $\sum_d \left[A, B, C, E\right]$ or system $\sum_d \left[A, B, C, D, E, F\right]$, we can carry out Step I using either Algorithm 4.1 or 4.2 to assign the required number of d.b.z.'s to achieve complete steady-state rejection of the first disturbance. Then in Step II, we can apply dynamic output feedback (or use a servocompensator) to reject the other disturbances. It will be shown later (see Chapter VI), that dynamic output feedback introduces additional d.b.z.'s (corresponding to the poles of the compensator).

5.5 NUMERICAL EXAMPLES

In this section, we illustrate the use of the algorithms described in the preceding sections by means of some numerical examples:

Example 5.1: Consider the following 4th-order system [2] whose parameters were given in Chapter III (eqns.(3.4.2a,b)). To demonstrate the steady- state rejection of multiple disturbances, the following input and disturbance matrices were used:

$$B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 50.0 \\ 20.0 & 2.0 \end{bmatrix}, E = \begin{bmatrix} -0.1 & -0.4 \\ 0.0 & 0.0 \\ 50.0 & 250.0 \\ 0.0 & 0.0 \end{bmatrix}$$

Hence, we have a system $\sum_d \left[A, B, C, E\right]$ described by

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \left[E_1 \;\; E_2\right]\mathbf{d}(t) \tag{5.5.1a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \tag{5.5.1b}$$

For the purpose of illustration, suppose that it is required to reject the following measurable disturbances:

(i) $\mathbf{d}(t) = \begin{bmatrix} 5.0 \\ 3.0 \end{bmatrix}$, and

(ii) $\mathbf{d}(t) = \left\{ \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} \exp(t) \right\}$.

Note that the given system with these disturbances satisfies the conditions given in eqns.(5.2.2a,b) and (5.2.3a,b).

(i) From Remark 5.1a, the system $\sum_d \left[A, B, C, E\right]$ can be reduced to a single-disturbance system denoted by $\sum_d \left[A, B, C, \bar{E}\right]$ which has two disturbance zeros between the output and the disturbance. Hence the condition in eqn.(5.2.2a,b) is satisfied. The disturbance vector $\bar{E} = 5.0\,E_1 + 3.0\,E_2$, where

$$\tilde{E} = \begin{bmatrix} -1.7 & 0.0 & 1000.0 & 0.0 \end{bmatrix}^T$$

In order to reject the effect of the unity-step disturbances $\tilde{d}(t)$ on the system $\sum_d \left[A, B, C, \tilde{E}\right]$, it is required to assign at least one of the d.b.z.'s at 0.0. Since the system $\sum_d \left[A, B, C, \tilde{E}\right]$ has two d.b.z.'s (at 523.2477 and - 0.11242), it follows that a constant state matrix K_2 is sufficient to assign the d.b.z.'s at 0.0 and as some other value, say - 0.1. To compute the constant feedback matrix K_2, we apply Algorithm 4.1, and obtain

$$K_2 = \begin{bmatrix} 0.0 & 0.0 \\ 10.412974 & 1.29329 \end{bmatrix}$$

By implementing K_2 using the state feedback law $\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & K_2 \end{bmatrix} \mathbf{x}(t)$ to the system in eqns.(5.5.1a,b), we obtain the closed-loop system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.5 & -1.0 & 1.0 & 0.0 \\ 0.4 & -0.4 & 0.0 & 0.0 \\ -19.5 & 19.5 & -585.6487 & 0.335285 \\ 0.0 & 0.0 & -20.7259 & -2.686588 \end{bmatrix} \mathbf{x}(t)$$

$$+ \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 50.0 \\ 20.0 & 2.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} -1.7 \\ 0.0 \\ 1000.0 \\ 0.0 \end{bmatrix} \tilde{d}(t) \tag{5.5.2a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{x}(t) \tag{5.5.2b}$$

This system is stable and the responses at the outputs $y_1(t)$ and $y_2(t)$ to the step disturbances are

shown in Figs.(5.1a) and (5.1b) for the open-loop system and in Figs.(5.2a) and (5.2b) for the closed-loop system. It can be readily seen that in the closed-loop system, the step disturbances are rejected completely in steady state.

(ii) From Remark 5.1b, system $\sum_d \left[A, B, C, E\right]$ with multiple disturbances can be reduced to two single-disturbance systems, such that the conditions in eqns.(5.2.2a,b) and (5.2.3a,b) are satisfied.

The first is denoted by $\sum_d \left[A, B, C, \bar{E}_1\right]$ and is affected by a step disturbance ($\bar{E}_1 = E_1$), while the second system $\sum_d \left[A, B, C, \bar{E}_2\right]$ is affected by exponential disturbance ($\bar{E}_2 = E_2$).

Note that, the two single-disturbance systems $\sum_d \left[A, B, C, \bar{E}_1\right]$ and $\sum_d \left[A, B, C, \bar{E}_2\right]$ satisfy conditions (5.2.2a,b) and (5.2.3a,b). The transfer matrices of systems $\sum_d \left[A, B, C, \bar{E}_1\right]$ and $\sum_d \left[A, B, C, \bar{E}_2\right]$ have two d.b.z.'s located at (523.2477, -0.11242) and (560.0116, -0.11160), respectively.

In order to reject the disturbances, we need to assign one d.b.z. for system $\sum_d \left[A, B, C, \bar{E}_1\right]$ at 0.0 and one d.b.z. for system $\sum_d \left[A, B, C, \bar{E}_2\right]$ at 1.0. Therefore we use Algorithm 5.1 and compute the constant state feedback matrix K_2 in two steps: In Step I, the unity-rank matrix $K_2^{(1)}$ given by

$$K_2^{(1)} = \begin{bmatrix} 0.0 & 0.0 \\ 10.41297 & 1.293294 \end{bmatrix}$$

is obtained to assign the d.b.z.'s of the system $\sum_d \left[A, B, C, \bar{E}_1\right]$ at 0.0 and -0.1. In Step II, while preserving the d.b.z at 0.0 for the first disturbance, we assign d.b.z.'s at 1.0 and -0.1 by using a unity-rank feedback matrix $K_2^{(2)}$, where

$$K_2^{(2)} = \begin{bmatrix} 0.9607769 & -0.0059635 \\ -11.294833 & 0.070107100 \end{bmatrix}$$

Thus, the total constant feedback matrix $K_2 = K_2^{(1)} + K_2^{(2)}$ to assign d.b.z.'s for both disturbances is given by

$$K_2 = \begin{bmatrix} 0.9607769 & -0.0059635 \\ -0.881859 & 1.3634014 \end{bmatrix} \tag{5.5.3}$$

By implementing the constant matrix K_2 (in the state feedback law $\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & K_2 \end{bmatrix} \mathbf{x}(t)$) on the system in eqns.(5.5.1a,b), the following closed-loop system is obtained:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -0.5 & -1.0 & 0.0392233 & 0.0059635 \\ 0.4 & -0.4 & 0.0 & 0.0 \\ -19.5 & 19.5 & -20.907037 & -3.170070 \\ 0.0 & 0.0 & -17.3518152 & -2.70753186 \end{bmatrix} \mathbf{x}(t)$$

$$+ \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 0.0 \\ 0.0 & 50.0 \\ 20.0 & 2.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} -0.1 & -0.4 \\ 0.0 & 0.0 \\ 50.0 & 250.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \tag{5.5.4a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{x}(t) \tag{5.5.4b}$$

The system is stable and the responses for the open-loop and closed-loop systems to a unit step disturbance are shown in Figs.(5.3a,b) and (5.4a,b) and for the exponential disturbance are shown in Figs.(5.5a,b) and (5.6a,b). Figs.(5.7a,b) and (5.8a,b) show the output responses for the open-loop and closed-loop systems when both disturbances affect the system at the same time.

Example 5.2: We have selected this example to illustrate Algorithms 5.1 and 5.2 for rejecting three unmeasurable disturbances which affect a 5th-order system [10], whose parameters are given in Chapter II (Example 2.2). The output and disturbance matrices are assumed to be:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -0.74 & 0.143 & 0.5 \\ -0.036 & 1.0 & -0.1 \\ 0.0 & 1.0 & -0.35 \\ 0.0 & 0.0 & 0.2 \end{bmatrix}$$

For the purpose of illustration, we also assume that it is required to reject the following disturbances:

$$\text{(i)}\ \mathbf{d}(t) = \begin{bmatrix} \beta_1 exp(t) \\ \beta_2 exp(2t) \\ \beta_3 + \beta_4 sin(t) \end{bmatrix}$$

$$\text{(ii) } \mathbf{d}(t) = \begin{bmatrix} \beta_1 exp(t) \\ \beta_2 exp(2t) \\ \beta_3 + \beta_4 sin(t) + \beta_5 t \end{bmatrix}$$

where β_1, β_2, β_3, β_4 and β_5 are unknown constant values.

Note that, the given system with these disturbances satisfies the conditions (5.2.2a,b) and (5.2.3a,b). The transfer function vectors between the outputs and disturbances $d_1(t)$, $d_2(t)$ and $d_3(t)$ have three d.b.z.'s each located respectively at (-1.104, -0.0155, -0.0497), (12.809, -1.12605, -0.054637) and (-1.02163±j0.0581, -0.0836).

(i) In order to reject these disturbances, it is required to assign the d.b.z.'s at (1.0), (2.0) and (0.0, -j , +j), i.e. $n_1 = 1, n_2 = 1, n_3 = 3$. This can be done by applying Algorithm 5.1 in three steps: In Step I, it is found that a unity-rank matrix $K_2^{(1)}$, where

$$K_2^{(1)} = \begin{bmatrix} -2.572554 & 2.3478649 & 6.842427 \\ 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 \end{bmatrix}$$

assigns d.b.z.'s for the first disturbance at (0.0, 0.5, 1.0). In Step II, the following unity-rank matrix

$$K_2^{(2)} = \begin{bmatrix} 0.29375363 & 2.2577222 & -0.2567452 \\ -0.13560431 & -1.042223240 & 0.118520295 \\ -1.5607582 & -11.9956254 & 1.364127205 \end{bmatrix}$$

preserves one d.b.z for the first disturbance at 1.0 and assigns an additional d.b.z for the second disturbance at 2.0. Finally, in Step III, while preserving the d.b.z.'s at 1.0 and 2.0 which have been assigned in Steps I and II respectively, additional d.b.z.'s for the third disturbance are assigned at 0.0, -j and +j using a unity-rank matrix $K_2^{(3)}$, where

$$K_2^{(3)} = \begin{bmatrix} -5.1952017 & 21.7477965 & 43.458829 \\ -2.189447 & 9.16531570 & 18.3151378 \\ -13.8006 & 57.771336 & 115.445012 \end{bmatrix}$$

Thus, the final constant state feedback matrix K_2 required to assign all the d.b.z.'s is $K_2 = K_2^{(1)} + K_2^{(2)} + K_2^{(3)}$ and is given by:

$$K_2 = \begin{bmatrix} -2.27880097 & 4.60558718 & 6.585681 \\ -0.135604 & -1.042223 & 0.11852029 \\ -1.5607582 & -11.9956254 & 1.3641272 \end{bmatrix}$$

By applying the constant feedback matrix K_2 using the state feedback law $\mathbf{u}(t) = \mathbf{v}(t) - \begin{bmatrix} 0 & K_2 \end{bmatrix} \mathbf{x}(t)$, we obtain the closed-loop system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0.0 & -0.00156 & 0.0 & 0.0 & 0.0 \\ 0.0 & -0.1419 & -0.897682 & 1.7685338 & 6.156365 \\ 0.0 & -0.00875 & 12.35288 & -44.64362 & -95.81752 \\ 0.0 & -0.00128 & -8.551914 & 24.659209 & 64.66801 \\ 0.0 & 0.0605 & 4.65160 & -14.772331 & -35.607789 \end{bmatrix} \mathbf{x}(t)$$

$$+\begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -0.143 & 0.0 & 0.0 \\ 1.0 & 0.392 & 0.330 \\ 0.108 & -0.05 & -0.592 \\ -0.0486 & 1.3 & 0.120 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -0.74 & 0.143 & 0.5 \\ -0.036 & 1.0 & -0.1 \\ 0.0 & 1.0 & -0.35 \\ 0.0 & 0.0 & 0.2 \end{bmatrix} \mathbf{d}(t) \tag{5.5.5a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{x}(t) \tag{5.5.5b}$$

which has closed-loop d.b.z.'s located at (1.0), (2.0) and (0.0 , -j , +j) for the three disturbances. The above system is unstable. We can stabilize it and improve its transient performance by performing pole assignment using dynamic output feedback. This will be discussed in Chapter VI.

(ii) For these disturbances, we need to assign the d.b.z.'s for the three disturbances at (1.0), (2.0) and (0.0, 0.0, -j, +j) to achieve complete rejection of all three disturbances in steady state. This can be done using Algorithm 5.2 in three steps. Note that in Steps I and II, the unity rank matrices $J_2^{(1)}$ and $J_2^{(2)}$ have the same values as those values of $K_2^{(1)}$ and $K_2^{(2)}$ obtained by applying Algorithm 5.1. In Step III, in order to assign additional d.b.z.'s at (0.0, 0.0, -j ,+j) , while preserving the d.b z.'s at (1.0) and (2.0) assigned in Steps I and II respectively, we need a dynamic state feedback compensator $K_2^{(3)}(s)$ of order one. This compensator is found to be

$$\dot{z}_2(t) = \begin{bmatrix} -5.0 \end{bmatrix} z_2(t) + \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) \tag{5.5.6a}$$

$$\mathbf{u}_2(t) = \begin{bmatrix} -21.80046932 \\ -9.1875139 \\ -57.9112569 \end{bmatrix} z_2(t) + \begin{bmatrix} -0.2781697 & 13.3517006 & 11.2237299 \\ -0.1172309 & 5.6268942 & 4.7300897 \\ -0.7389364 & 35.4677579 & 29.8149688 \end{bmatrix} \mathbf{x}(t)$$

(5.5.6b)

Thus, the dynamic state feedback $K_2(s)$ required to assign all three sets of d.b.z.'s is given by

$$\dot{z}_2(t) = \begin{bmatrix} -5.0 \end{bmatrix} z_2(t) + \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{x}(t) \tag{5.5.7a}$$

$$\mathbf{u}_2(t) = \begin{bmatrix} -21.80046932 \\ -9.1875139 \\ -57.9112569 \end{bmatrix} z_2(t) + \begin{bmatrix} -2.5569707 & 17.9572878 & 17.8094118 \\ -0.2528352 & 4.5846710 & 4.8486100 \\ -2.2996947 & 23.4721324 & 31.1790960 \end{bmatrix} \mathbf{x}(t) \tag{5.5.7b}$$

By applying the above dynamic state feedback using the feedback law $\mathbf{u}(t) = \mathbf{v}(t) - \mathbf{u}_2(t)$, the following closed-loop system is obtained

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & -0.001560 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0 & -0.14190 & -0.1945468 & 0.56789216 & 1.5467459 & -3.1174671 \\ 0 & -0.00875 & 2.312981 & -27.5002826 & -29.9991686 & 44.5126895 \\ 0 & 0.00128 & -1.246808 & 12.1853489 & 16.7771688 & -32.3883891 \\ 0 & 0.06050 & 0.629280 & -7.9040040 & -9.2382471 & 17.8336161 \\ 0 & 0.0 & 0.0 & 1.0 & 1.0 & -5.0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -0.143 & 0.0 & 0.0 \\ 1.0 & 0.392 & 0.330 \\ 0.108 & -0.05 & -0.592 \\ -0.0486 & 1.3 & 0.120 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ -0.74 & 0.143 & 0.5 \\ -0.036 & 1.0 & -0.1 \\ 0.0 & 1.0 & -0.35 \\ 0.0 & 0.0 & 0.2 \\ 0.0 & 0.0 & 0.0 \end{bmatrix} \mathbf{d}(t) \tag{5.5.8a}$$

$$y(t) = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} x(t) \\ z_2(t) \end{bmatrix} \tag{5.5.8b}$$

This has closed-loop d.b.z.'s located at (1.0), (2.0) and (0.0, 0.0, -j, +j) for the first, second and third disturbances respectively. The above system is unstable. So in this case also, we need to stabilize the system and shape its transient performance by assigning the closed-loop poles at desired locations in the complex plane by means of (dynamic) output feedback. This will be discussed in Chapter VI.

5.6 CONCLUDING REMARKS

In this chapter, we have presented algorithms for assigning d.b.z.'s of a linear time-invariant multivariable system by means of state feedback, such that the effects of a class of measurable or unmeasurable disturbances acting on the system are eliminated in the steady state. In the proposed algorithms, the dyadic feedback design mechanism was used to compute constant and dynamic state feedback compensators for arbitrarily assigning the required number of d.b.z.'s at specified locations in the complex plane. The design procedure is sequential in nature in that at each step we assign additional d.b.z.'s while preserving those which have been assigned in the preceding steps.

In Algorithm 5.1, it was shown how a constant feedback matrix K_2 can be computed as a sum of dyads for the system $\sum_d \left[A, B, C, E\right]$, such that the required d.b.z.'s can be assigned at desired locations in the complex plane. For the case where the system has no d.b.z.'s or the number of d.b.z.'s is not large enough to achieve the rejection of all disturbances in the steady state, Algorithm 5.2 was proposed. It was shown that dynamic state feedback of appropriate order can be used to introduce additional d.b.z.'s as well as to assign all the closed-loop d.b.z.'s at desired locations in the complex plane. In Section 5.3, we extended these results to the multivariable system $\sum_d \left[A, B, C, D, E, F\right]$ and showed how Algorithms 5.1 and 5.2 can be applied

by creating a higher order system $\sum_{d} \left\{\hat{A}, \hat{B}, \hat{C}, \hat{E}\right\}$.

It should be noted that Algorithms 5.1 and 5.2 can be applied to reject both measurable as well as unmeasurable multiple disturbances in steady state. However, the problem of measurable multiple disturbances can be reduced to one or more single-disturbance problems.

Finally, it should be pointed out that, our approach does not have the robustness property of the robust servomechanism approach [11,12]. However, the state feedback controllers constructed by the proposed algorithms have simpler structure than the servocompensators used in the robust servomechanism problem. Also, the proposed algorithms allow us to selectively reject some parts of disturbance signals in steady-state, e.g. only the step part from a disturbance $d_1(t) = \beta_1 + \beta_2 sin(t)$ and the exponential part from a disturbance $d_2(t) = \beta_3 + \beta_4 exp(t)$. Such a situation arises in designing what is called a "disturbance utilizing controller" [13], where some parts of disturbance signals may have not need to be rejected and may sometimes produce effects that are beneficial to the system.

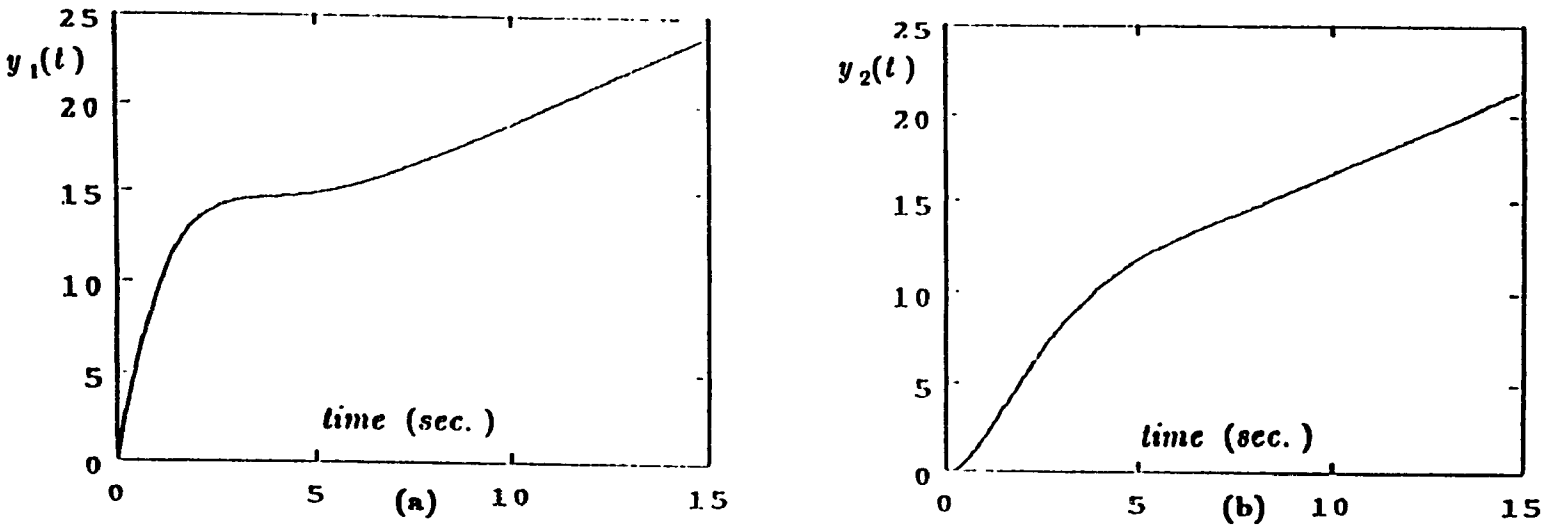

Fig.(5.1) Output responses of the open-loop system in Example 5.1-i to step disturbances.

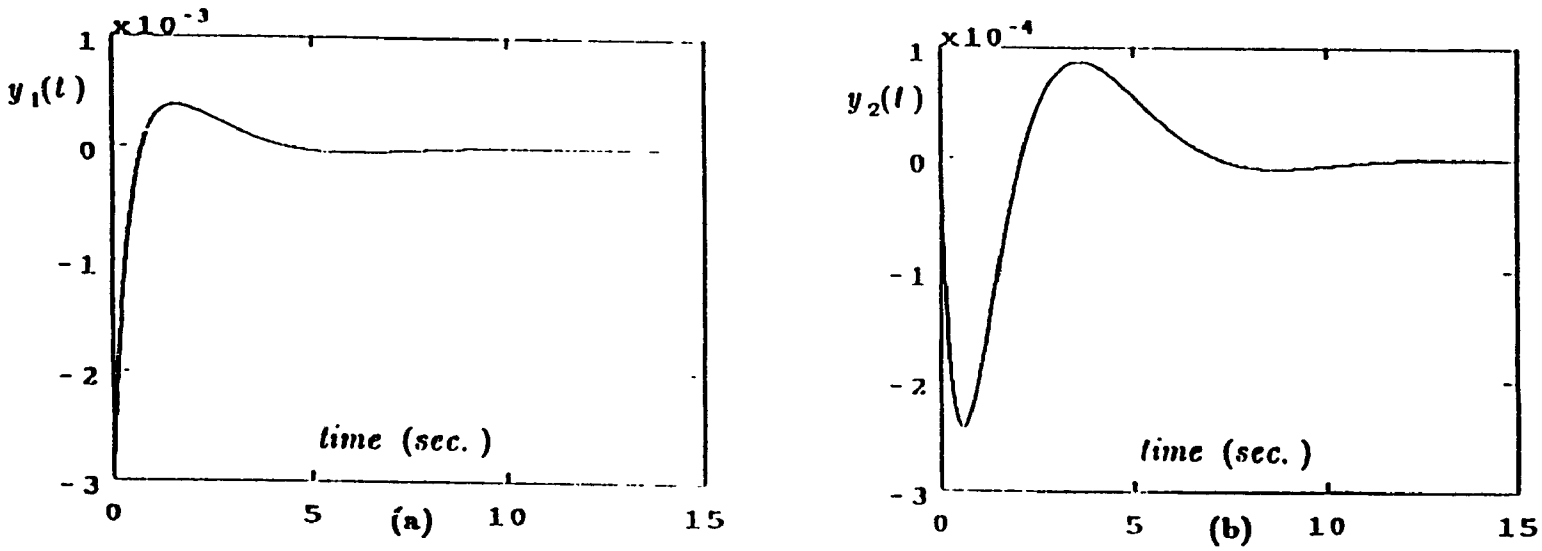

Fig.(5.2) Output responses of the closed-loop system in Example 5.1-i to step disturbances.

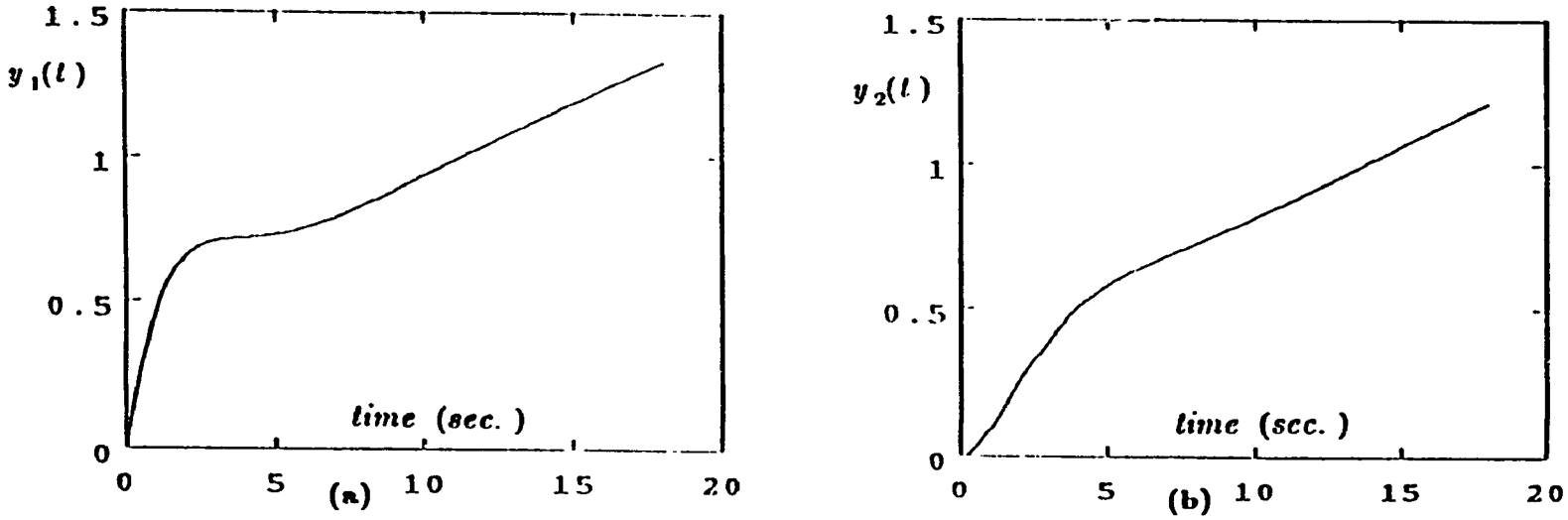

Fig.(5.3) Output responses of the open-loop system in Example 5.1-ii to step disturbance.

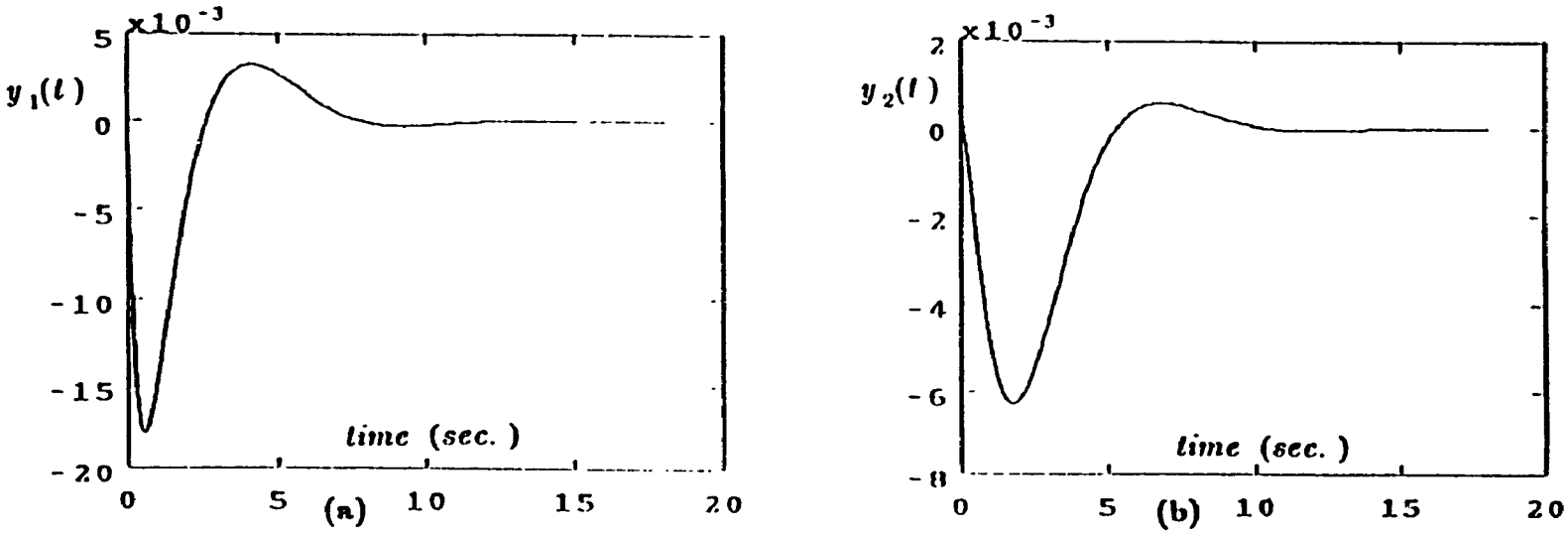

Fig.(5.4) Output responses of the closed-loop system in Example 5.1-ii to step disturbance.

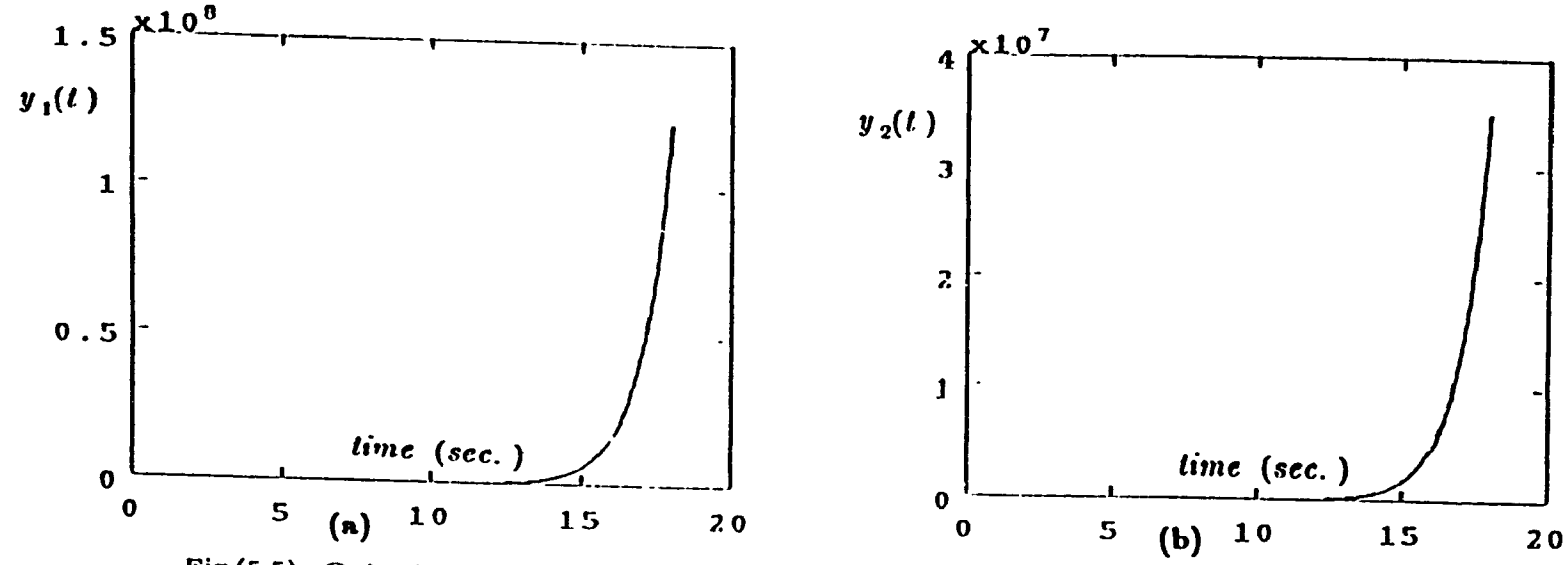

Fig.(5.5) Output responses of the open-loop system in Example 5.1-ii to an exponential disturbance.

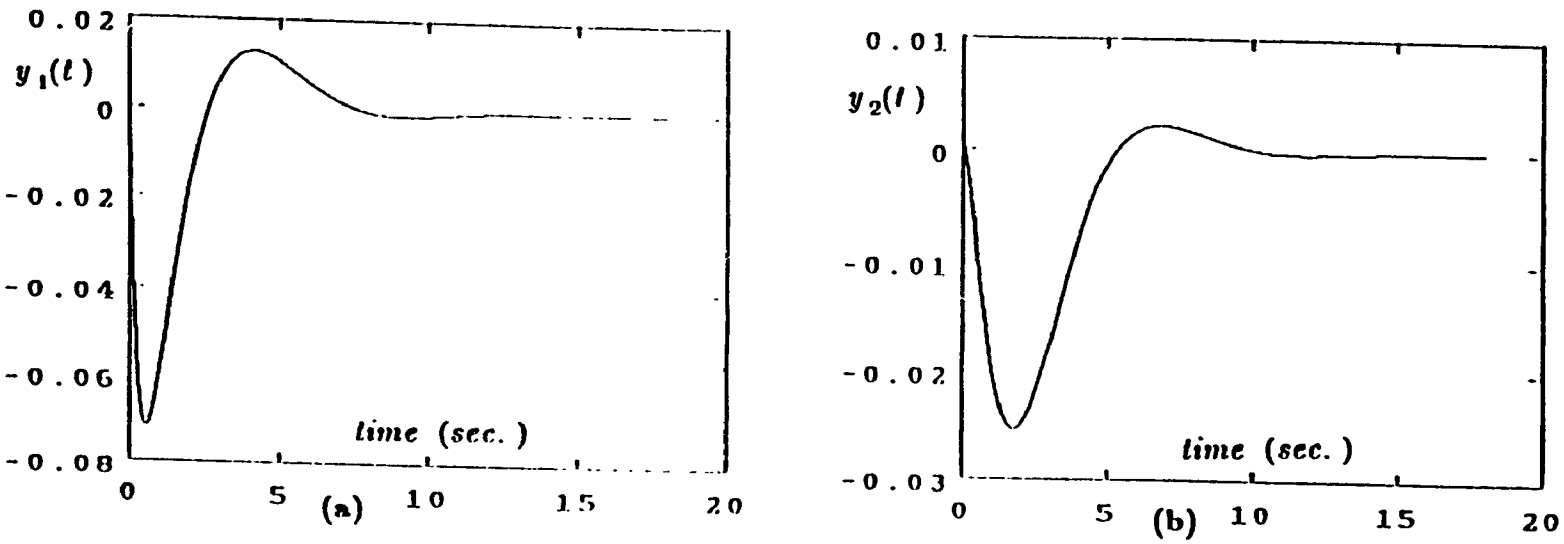

Fig.(5.6) Output responses of the closed-loop system in Example 5.1-ii to an exponential disturbance.

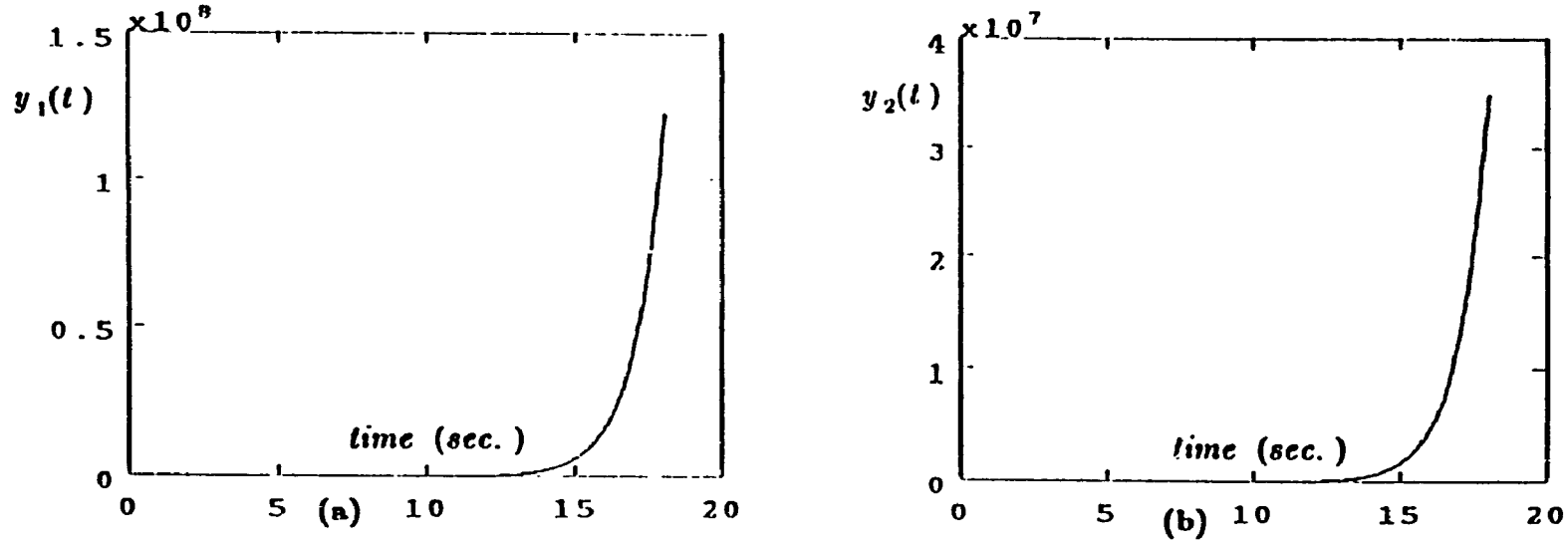

Fig.(5.7) Output responses of the open-loop system in Example 5.1-ii to combination of step and *exp* (*t*) disturbances.

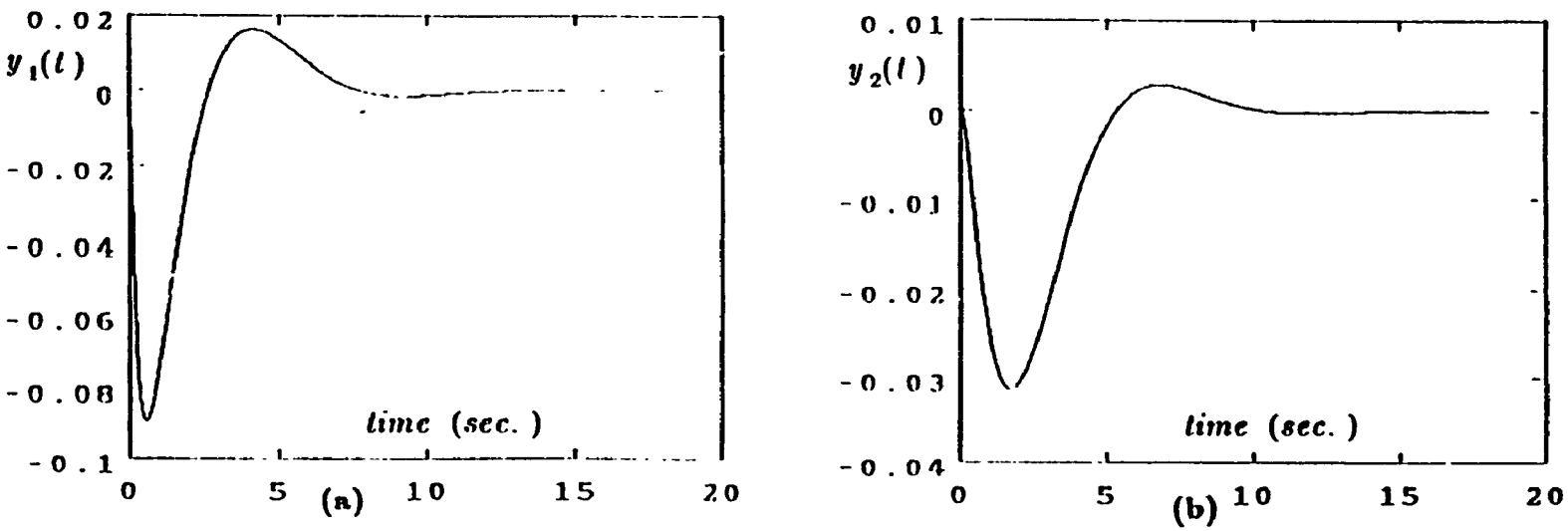

Fig.(5.8) Output responses of the closed-loop system in Example 5.1-ii to combination of step and *exp* (*t*) disturbances.

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CHAPTER VI

DYNAMIC OUTPUT FEEDBACK IN MULTIVARIABLE SYSTEMS

In this chapter, the effect of dynamic output feedback on linear time-invariant multivariable systems is studied and it is shown that a dynamic compensator introduces additional d.b.z.'s in the closed-loop which are located at the poles of the compensator. This important feature will be used to assign the poles of the dynamic compensator at prespecified locations in the complex plane, such that complete rejection of some disturbances is achieved in the steady state. The problem of assigning the poles of the augmented closed-loop system consisting of the plant and the compensator is also considered.

The layout of the chapter is as follows: Section 6.1 contains a brief review of some results presented in Chapter II for the design of dynamic output feedback to achieve pole assignment in multivariable systems. In Section 6.2, we show how a dynamic compensator introduces additional d.b.z.'s at the same locations as the poles of the compensator for systems described by $\sum_d \left[A, B, C, E\right]$. It is then shown that similar results can also be obtained for systems described by $\sum_d \left[A, B, C, D, E, F\right]$. Section 6.3, gives some general remarks concerning the problem of designing controllers consisting of state feedback and/or dynamic output feedback which achieve asymptotic regulation of all disturbances in the steady state and which stabilize the resulting closed-loop system by assigning all the closed-loop poles arbitrarily close to desired locations in the complex plane. It is also shown that we can design a dynamic output feedback compensator which is robust in the sense that it achieves the specifications of the servomechanism problem even in the presence of certain perturbations in the system parameters. In Section 6.4, a number of illustrative numerical examples are given to show the usefulness and efficiency of our approach in designing dynamic output feedback in multivariable systems, and finally a

brief discussion of the results is presented in Section 6.5.

6.1 POLE ASSIGNMENT USING DYNAMIC OUTPUT FEEDBACK

In this section, we will discuss the problem of designing dynamic output feedback compensators to achieve arbitrary pole assignment in multivariable systems with $n \geq m+l$, where n, m and l are the number of states, inputs and outputs respectively, i.e. the conditions for "almost" arbitrary assignment of all the poles by using constant output feedback is not satisfied. In Chapter II two approaches were discussed to handle this problem. Both methods design the dynamic output feedback such that the poles of the closed-loop system consisting of the compensator and the plant can be positioned arbitrarily close to the desired locations in the complex plane. These approaches compute the dynamic output feedback in two steps: The first step assigns a subset of the poles and the second assigns the remaining poles while preserving the previously assigned ones. The first approach uses Algorithm 2.3 which is based on a reformulation of the problem to one of eigenvalue assignment by constant gain output feedback. This algorithm is numerically reliable, but does not assign the poles of the compensator at prespecified locations in the complex plane. The second approach uses Algorithm 2.4 to compute a dynamic compensator with prespecified poles and achieves arbitrary pole assignment for the resulting closed-loop system. This advantage of Algorithm 2.4 will be used later to design a dynamic compensator with prespecified poles to achieves pole assignment as well as to introduce a model of the system's "environment", i.e. of the disturbances acting on the system. This aspect of the design is known as the "Internal Model Principle" [1], and is based on incorporating a model of the "outside world" in the controller structure. This is accomplished here by introducing additional d.b.z.'s (by means of dynamic output feedback) at the poles corresponding to the dynamics of the disturbances acting on the system. The mechanism by which the d.b.z.'s are introduced in the overall closed-loop system is illustrated in the next section.

6.2 DYNAMIC OUTPUT FEEDBACK AND DISTURBANCE BLOCKING ZEROS

In this section, we will show how dynamic output feedback used for pole assignment in multivariable systems described by $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$ introduces additional d.b.z.'s located at the poles of the compensator.

6.2.1 Dynamic Output Feedback in The System $\sum_d \left[A, B, C, E\right]$

Consider the nth-order, m-input, l-output, r-disturbance, linear, time-invariant multivariable system $\sum_d \left[A, B, C, E\right]$ described by its state-space equations

$$\dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) + \sum_{i=1}^{r} E_i\ d_i(t) \tag{6.2.1a}$$

$$\mathbf{y}(t) = C\ \mathbf{x}(t)$$

$$= \begin{bmatrix} C_1 \\ C_2 \\ \cdot \\ \cdot \\ \cdot \\ C_l \end{bmatrix} \mathbf{x}(t) \tag{6.2.1b}$$

where A, B, C and E_i are constant of appropriate dimensions. To show the effect of dynamic output feedback on the closed-loop d.b.z.'s between the outputs $\mathbf{y}(t)$ and the disturbances $d_i(t)$, we assume that the triples (A, E_i, C) are non-degenerate (i.e. the system (6.2.1a,b) has a finite

number of d.z.'s). Note that [2] for the case $l=r=1$, the maximum number of d.z.'s that system (6.2.1a,b) may possess is equal to $n-1$. We also assume that the pairs (A, B) and (A, E_i) are controllable and the pair (C_j, A) is observable. This assumption is made to simplify the presentation and is not necessary for achieving disturbance rejection. If the pairs (A, E_i) and/or (C_j, A) are uncontrollable and/or unobservable, then the number of d.z.'s between each output $y_j(t)$ for $j = 1, 2, \ldots, l$ and each disturbance $d_i(t)$, $i = 1, 2, \ldots, r$ of the system (6.2.1a,b) is less than $n-1$. This is due to cancellation of the poles and zeros which correspond to uncontrollable and/or unobservable modes of the system. Therefore, it is necessary for the pairs (A, E_i) and (C_j, A) to be completely controllable and observable for these to be $n-1$ d.z.'s between $y_j(t)$ and $d_i(t)$ [3].

Without loss of generality, we assume that the disturbance $d_i(t)$ for $i = 1, 2, \ldots, r$ may or may not be measurable but satisfies a differential equation of the form [4]

$$d_i^{(q_i)} + \alpha^i_{q_i} d_i^{(q_i-1)} + \ldots + \alpha^i_2 d_i + \alpha^i_1 d_i = 0 \quad , i = 1, 2, \ldots, r \tag{6.2.2}$$

Further, we assume that the characteristic roots of equation (6.2.2) are $\left\{\lambda^i_\kappa\right\}$, $\kappa = 1, 2, \ldots, q_i$, $i = 1, 2, \ldots, r$ lie in the right half of the complex plane, i.e. $\mathrm{Re}\left\{\lambda^i_\kappa\right\} \geq 0$. The initial conditions for (6.2.2) are assumed to be unknown.

Now, it is required to design a dynamic output feedback compensator, such that complete rejection of all disturbances satisfying (6.2.2) is achieved in the steady state and that the resulting closed-loop system is stable with all poles assigned at desired locations.

Let the control input $\mathbf{u}(t)$ to the system (6.2.1a,b) be given by the feedback control law

$$\mathbf{u}(t) = \mathbf{v}(t) - \hat{\mathbf{u}}_j(t) \tag{6.2.3}$$

where $\mathbf{v}(t) \in \mathbb{R}^m$ is the external input and $\hat{\mathbf{u}}_j(t) \in \mathbb{R}^m$ is the output of the dynamic compensator whose input is the jth output of the system (6.2.1a,b):

$$\dot{\mathbf{z}}_j(t) = F_j\, \mathbf{z}_j(t) + G_j\, y_j(t) \tag{6.2.4a}$$

$$\hat{\mathbf{u}}_j(t) = H_j\, \mathbf{z}_j(t) + J_j\, y_j(t) \tag{6.2.4b}$$

where $z_j(t) \in \mathbb{R}^{q_j}$ is the state vector of the compensator and q_j is the order of the dynamic compensator which introduces q_j d.b.z.'s between the output $y_j(t)$ and the disturbances $d_i(t)$ $,i = 1, 2, \ldots, r$, so that asymptotic regulation takes place. Note that [5] when the value of $q_j \geq (n-m)/m-1$, we can also use the dynamic compensator to "almost always" assign all the eigenvalues of the resulting closed-loop system arbitrarily close to $n+q_j$ desired locations in the complex plane.

By substituting for $y_j(t)$ from eqn.(6.2.1b) into eqns.(6.2.4a,b) we obtain

$$\dot{\mathbf{z}}_j(t) = F_j\, \mathbf{z}_j(t) + G_j\, C_j\, \mathbf{x}(t) \tag{6.2.5a}$$

$$\hat{\mathbf{u}}_j(t) = H_j\, \mathbf{z}_j(t) + J_j\, C_j\, \mathbf{x}(t) \tag{6.2.5b}$$

It should be noted that for multiple outputs $y_j(t)$, $j = 1, 2, \ldots, l$, the design procedure is sequential in nature in that we compute the required dynamic compensator for each output in turn so that asymptotic regulation takes place for all disturbances acting on the system.

On implementing the feedback law in eqn.(6.2.3), the resulting closed-loop system, consisting of the open-loop system (6.2.1a,b) and the dynamic compensator (6.2.5a,b), becomes

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}_j(t) \end{bmatrix} = \begin{bmatrix} \left[A - BJ_jC_j\right] & -BH_j \\ G_jC_j & F_j \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_j(t) \end{bmatrix}$$

$$+ \begin{bmatrix} B \\ 0 \end{bmatrix} \mathbf{v}(t) + \sum_{i=1}^{r} \begin{bmatrix} E_i \\ 0 \end{bmatrix} d_i(t) \tag{6.2.6a}$$

$$y_j(t) = \begin{bmatrix} C_j & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_j(t) \end{bmatrix} \qquad j = 1, 2, \ldots, l \tag{6.2.6b}$$

Let us now find the sets of d.z.'s between the output $y_j(t)$ and the disturbances $d_i(t)$,

$i = 1, 2, \ldots, r$. These are defined (see Chapter III) as those complex numbers λ for which the $(n+q_j+1)\times(n+q_j+1)$ matrix

$$\Gamma_i\left[\lambda\right] = \begin{bmatrix} \left[A-BJ_jC_j\right]-\lambda I_n & -BH_j & E_i \\ G_jC_j & F_j-\lambda I_{q_j} & 0 \\ C_j & 0 & 0 \end{bmatrix} \quad i = 1, 2, \ldots, r$$

has rank less than $\left[n+q_j+1\right]$. Next, we has a rank less than $\left[n+q_j+1\right]$. Next, we premultiply the last row of the above matrix by G_j and subtract the resulting q_j rows from the q_j rows immediately above the last row. Also, we premultiply the last row by BJ_j and add the resulting n rows to the first n rows. Note that these operations do not alter the rank of $\Gamma_i\left[\lambda\right]$. Therefore, we get

$$\Gamma_i\left[\lambda\right] = \begin{bmatrix} A-\lambda I_n & -BH_j & E_i \\ 0 & F_j-\lambda I_{q_j} & 0 \\ C_j & 0 & 0 \end{bmatrix} \quad i = 1, 2, \ldots, r$$

Rearranging the last q_j+1 columns and q_j+1 rows yields

$$\Gamma_i\left[\lambda\right] = \begin{bmatrix} A-\lambda I_n & E_i & -BH_j \\ C_j & 0 & 0 \\ 0 & 0 & F_j-\lambda I_{q_j} \end{bmatrix} \quad i = 1, 2, \ldots, r$$

which implies that

$$rank\ \Gamma_i\left[\lambda\right] = rank\ \begin{bmatrix} A-\lambda I_n & E_i \\ C_j & 0 \end{bmatrix} + rank\ \left[F_j-\lambda I_{q_j}\right] \quad i = 1, 2, \ldots, r$$

Therefore, all the values of λ for which rank $\Gamma_i\left[\lambda\right] < n+q_j+1$ are those complex numbers for which

(i)

$$rank\ \begin{bmatrix} A-\lambda I_n & E_i \\ C_j & 0 \end{bmatrix} < n+1$$

and/or

(ii)

$$rank\ \left[F_j-\lambda I_{q_j}\right] < q_j$$

Since (A, E_i) is controllable and (C_j, A) is observable, condition (i) is equivalent to λ being a d.z. of $\sum_d \left[A, B, C_j, E_i\right]$ which for a single-output, single-disturbance case is equivalent to λ being a d.b.z. of the open-loop system $\sum_d \left[A, B, C_j, E_i\right]$. Condition (ii) holds if and only if λ is an eigenvalues of F_j. Therefore, the closed-loop d.b.z.'s between $y_j(t)$ and $d_i(t)$ under the effect of the dynamic output feedback (6.2.4a,b) consist of the open-loop d.b.z.'s between the $y_j(t)$ and $d_i(t)$ together with the poles of the dynamic compensator (eigenvalues of F_j).

The above procedure allows us to introduce separately q_j additional d.b.z.'s between each output $y_j(t)$, $j = 1, 2, \ldots, l$ and the disturbances $d_i(t)$, $i = 1, 2, \ldots, r$ at the poles of the corresponding dynamic compensator. By continuing in the above manner for l outputs we get a dynamic compensator of order $q = \sum_{j=1}^{l} q_j$ defined by

$$\dot{\mathbf{z}}(t) = F\,\mathbf{z}(t) + G\,\mathbf{y}(t) \tag{6.2.7a}$$

$$\hat{\mathbf{u}}(t) = H\,\mathbf{z}(t) + J\,\mathbf{y}(t) \tag{6.2.7b}$$

where

$$F = diag\left\{F_1, F_2, \ldots, F_l\right\}, \quad G = diag\left\{G_1, G_2, \ldots, G_l\right\}$$

and

$$H = \left[H_1\ H_2\ \ldots H_l\right], \quad J = \left[J_1\ J_2\ \ldots J_l\right]$$

which introduces additional q_j, $j = 1, 2, \ldots, l$ d.b.z.'s between each output $y_j(t)$ and the disturbances.

6.2.2 Dynamic Output Feedback in The System $\sum_d \left[A, B, C, D, E, F\right]$

To consider the effect of dynamic output feedback on the d.b.z.'s of the system $\sum_d \left[A, B, C, D, E, F\right]$, consider an nth-order, m-input, l-output system with r disturbances described by

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) + \sum_{i=1}^{r} E_i d_i(t) \tag{6.2.8a}$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) + D\,\mathbf{u}(t) + \sum_{i=1}^{r} F_i d_i(t)$$

$$= \begin{bmatrix} C_1 \\ C_2 \\ . \\ . \\ C_l \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} D_1 \\ D_2 \\ . \\ . \\ D_l \end{bmatrix} \mathbf{u}(t) + \sum_{i=1}^{r} \begin{bmatrix} F_{1,i} \\ F_{2,i} \\ . \\ . \\ F_{l,i} \end{bmatrix} d_i(t) \tag{6.2.8b}$$

By using the feedback control law in (6.2.3) and the dynamic output feedback compensator defined in (6.2.4a,b), the following closed-loop system is obtained [6]:

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{z}}_j(t) \end{bmatrix} = \begin{bmatrix} \left[A - BM_jJ_jC_j\right] & -BM_jH_j \\ G_j\left[I - D_jM_jJ_j\right]C_j & \left[F_j - G_jD_jM_jH_j\right] \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_j(t) \end{bmatrix}$$

$$+ \begin{bmatrix} BM_j \\ G_jD_jM_j \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} \sum_{i=1}^{r} E_i - BM_jJ_j \sum_{i=1}^{r} F_{j,i} \\ G_j\left[I - D_jM_jJ_j\right] \sum_{i=1}^{r} F_{j,i} \end{bmatrix} d_i(t) \tag{6.2.9a}$$

$$y_j(t) = \left[\left[I - D_jM_jJ_j\right]C_j \quad -D_jM_jH_j\right] \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}_j(t) \end{bmatrix}$$

$$+ \left[D_jM_j\right] \mathbf{v}(t) + \left[\left[I - D_jM_jJ_j\right] \sum_{i=1}^{r} F_{j,i}\right] d_i(t) \tag{6.2.9b}$$

where $M_j = \left[I + J_jD_j\right]^{-1}$ and $\left[I + J_jD_j\right]$ is assumed to be nonsingular.

The d.z.'s of the augmented closed-loop system (6.2.9a,b) between the output $y_j(t)$ and each disturbance $d_i(t)$, $i = 1, 2, \ldots, r$, are defined as those complex numbers λ for which the $(n+q_j+1)\times(n+q_j+1)$ matrix

$$\Gamma_i\left[\lambda\right] = \begin{bmatrix} \left[A - BM_jJ_jC_j\right] - \lambda I_n & -BM_jH_j & \left[E_i - BM_jJ_jF_{j,i}\right] \\ G_j\left[I - D_jM_jJ_j\right]C_j & \left[F_j - G_jD_jM_jH_j\right] - \lambda I_{q_j} & G_j\left[I - D_jM_jJ_j\right]F_{j,i} \\ \left[I - D_jM_jJ_j\right]C_j & -D_jM_jH_j & \left[I - D_jM_jJ_j\right]F_{j,i} \end{bmatrix}$$

has a rank less than $\left[n+q_j+1\right]$.

We premultiply the last row of the above matrix by G_j and subtract the resulting q_j rows from the q_j rows immediately above the last row to get

$$\Gamma_i\left[\lambda\right]=\begin{bmatrix}\left[A-BM_jJ_jC_j\right]-\lambda I_n & -BM_jH_j & \left[E_i-BM_jJ_jF_{j,i}\right]\\ 0 & F_j-\lambda I_{q_j} & 0\\ \left[I-D_jM_jJ_j\right]F_{j,i} & -D_jM_jH_j & \left[I-D_jM_jJ_j\right]F_{j,i}\end{bmatrix}$$

Rearranging the last q_j+1 columns and q_j+1 rows yields

$$\Gamma_i\left[\lambda\right]=\begin{bmatrix}\left[A-BM_jJ_jC_j\right]-\lambda I_n & \left[E_i-BM_jJ_jF_{j,i}\right] & -BM_jH_j\\ \left[I-D_jM_jJ_j\right]C_j & \left[I-D_jM_jJ_j\right]F_{j,i} & -D_jM_jH_j\\ 0 & 0 & F_j-\lambda I_{q_j}\end{bmatrix}$$

from which it follows that the values of λ for which

$$rank\ \Gamma_i\left[\lambda\right]<n+q_j+1$$

are those values for which

(i)

$$rank\ \left[F_j-\lambda I_{q_j}\right]<q_j$$

and/or

(ii)

$$rank \begin{bmatrix} \left(A-BM_jJ_jC_j\right)-\lambda I_n & \left(E_i-BM_jJ_jF_{j,i}\right) \\ \left(I-D_jM_jJ_j\right)C_j & \left(I-D_jM_jJ_j\right)F_{j,i} \end{bmatrix} < n+1$$

It is easy to see that condition (i) holds for λ equal to the eigenvalues of F_j. Now in condition (ii), since $M_j = \left(I+J_jD_j\right)^{-1}$, it can be easily shown that

$$\left(I-D_jM_jJ_j\right)C_j = \left(I+D_jJ_j\right)^{-1}C_j$$

$$\left(I-D_jM_jJ_j\right)F_{j,i} = \left(I+D_jJ_j\right)^{-1}F_{j,i}$$

and

$$\left(A-BM_jJ_jC_j\right) = A-BJ_j\left(I+D_jJ_j\right)^{-1}C_j$$

$$\left(E_i-BM_jJ_jF_{j,i}\right) = E_i-BJ_j\left(I+D_jJ_j\right)^{-1}F_{j,i}$$

Thus, denoting the matrix in condition (ii) by $\Lambda_2\left(\lambda\right)$, we can write

$$\Lambda_2\left(\lambda\right) = \begin{bmatrix} \left(A-BJ_j\left(I+D_jJ_j\right)^{-1}C_j\right)-\lambda I_n & E_i-BJ_j\left(I+D_jJ_j\right)^{-1}F_{j,i} \\ \left(I+D_jJ_j\right)^{-1}C_j & \left(I+D_jJ_j\right)^{-1}F_{j,i} \end{bmatrix}$$

By premultiplying the last row of the above matrix by BJ_j and subtracting the resulting n rows from the n rows immediately above the last row, we get

$$rank\left(\Lambda_2\left(\lambda\right)\right) = rank \begin{bmatrix} A-\lambda I_n & E_i \\ \left(I+D_jJ_j\right)^{-1}C_j & \left(I+D_jJ_j\right)^{-1}F_{j,i} \end{bmatrix}$$

The matrix on the right-hand side can be written as

$$\begin{bmatrix} A-\lambda I_n & E_i \\ \left[I+D_jJ_j\right]^{-1}C_j & \left[I+D_jJ_j\right]^{-1}F_{j,i} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & \left[I+D_jJ_j\right]^{-1} \end{bmatrix} \begin{bmatrix} A-\lambda I_n & E_i \\ C_j & F_{j,i} \end{bmatrix}$$

implying that

$$rank\left[\Lambda_2\left[\lambda\right]\right] = rank \begin{bmatrix} A-\lambda I_n & E_i \\ C_j & F_{j,i} \end{bmatrix}$$

And, by assuming that the pair (A, E_i) is controllable and (C_j, A) is observable, it follows that the values of λ such that

$$rank \begin{bmatrix} A-\lambda I_n & E_i \\ C_j & F_{j,i} \end{bmatrix} < n+1$$

are the d.z.'s of the open-loop system (6.2.9a,b) between the output $y_j(t)$ and the disturbances $d_i(t)$. Therefore, the values of λ for which $rank\ \Gamma_i\left[\lambda\right]$ is less than $\left[n+q_j+1\right]$ correspond to the d.z.'s between the output $y_j(t)$ and disturbance $d_i(t)$ in the open-loop system together with the poles of the compensator (eigenvalues of F_j). Now, for a single-disturbance, single-output system $\sum\left[A, E_i, C_j, F_{j,i}\right]$ with (A, E_i) controllable and (C_j, A) observable all d.z.'s are also d.b.z.'s. Therefore, the closed-loop d.b.z.'s between $y_j(t)$ and $d_i(t)$ under the effect of the dynamic output (6.2.4a,b) are the open-loop d.b.z.'s between $y_j(t)$ and $d_i(t)$ together with the eigenvalues of $\left[F_j\right]$.

6.3 GENERAL REMARKS

In the rest of this section, for the sake of simplicity and to illustrate the main features of using dynamic output feedback in multivariable systems, we consider system $\sum_d\left[A, B, C, E\right]$

with two disturbances which may or may not be measurable, described by the following equations

$$\dot{\mathbf{x}}(t) = A\ \mathbf{x}(t) + B\ \mathbf{u}(t) + E_1\, d_1(t) + E_2\, d_2(t) \tag{6.3.1a}$$

$$\mathbf{y}(t) = C\ \mathbf{x}(t)$$

$$= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \mathbf{x}(t) \tag{6.3.1b}$$

We shall assume that the triples (A, B, C) and (A, E_i, C) are non-degenerate, i.e. system (6.3.1a,b) has a finite number of transmission and disturbance zeros. We also assume that the pairs (A, B), (A, E_i), are controllable and the pair (C_j, A) is observable.

Now to discuss the main results of this chapter, we consider the problem of finding a controller for the system (6.3.1a,b), such that the disturbances are rejected at the outputs in steady state and that the resulting closed-loop system is stable with all poles assigned at or arbitrarily close to desired locations in the complex plane.

The following situations can arise in solving this problem:

Remark 6.1: The first case occurs when the open-loop and closed-loop (with state feedback) transfer matrices relating the outputs to each disturbance ($d_1(t)$ and $d_2(t)$) have $n-l$ d.z.'s, i.e. conditions (5.2.2a,b) and (5.2.3a,b) are satisfied. This situation can be resolved in two stages: In the first stage, we reject the disturbances by suitably locating one or more closed-loop d.b.z.'s at desired locations by means of constant or dynamic state feedback. This controller can be constructed using either Algorithm 5.1. or 5.2. Then in the second stage, the resulting closed-loop system is stabilized and/or its transient performance is improved by means of dynamic output feedback by placing the closed-loop poles arbitrarily close to desired locations in the complex plane. This dynamic output feedback can be computed using either Algorithm 2.3 or 2.4.

Remark 6.2: In the second case, the open-loop transfer matrix between the outputs and one or

both of the disturbances has $n-l$ d.z.'s, i.e. condition (5.2.2a,b) is satisfied while condition (5.2.3a,b) is not. We solve this problem in two stages: First, we use a state feedback controller (constant or dynamic) to assign the required number of d.b.z.'s between the outputs and one of the disturbances (which satisfies the condition (5.2.2a)) at desired locations, to reject the disturbance in the steady state. This controller (constant or dynamic state feedback) can be calculated using either Algorithm 4.1 or 4.2. In the second stage, we can use the results obtained in Section 6.2 for designing dynamic output feedback with prespecified poles to introduce additional d.b.z.'s between each output and the disturbance that is not rejected in the first stage, such that asymptotic regulation takes place for this disturbance. In addition, the resulting closed-loop system is stabilized by assigning all the poles at or arbitrarily close to desired locations in the complex plane. Note that, in this case the minimum order of the dynamic compensator depends on the required number of additional d.b.z.'s that are introduced as well as the number of closed-loop poles that are to be assigned. This type of dynamic compensator can be computed by using Algorithm 2.4. It should be noted that in many problem, it may be required to position the d.b.z.'s in the right-half plane, in which case the resulting compensator would be unstable. However, the results of Section 6.2 may prove useful in rejecting step, ramp and sinusoidal disturbances in the steady state, for which the additional d.b.z.'s introduced are located at the origin or on the imaginary axis.

Remark 6.3: The third case includes the general problem for which the open-loop and closed-loop (with state feedback) transfer matrices relating the outputs to each disturbance do not have $n-l$ d.z.'s, i.e. conditions (5.2.2a,b) and (5.2.3a,b) do not hold. This problem was solved by Davison [7,8] , and is essentially the problem known in literature as "the servomechanism problem" [9,10](-it includes the problem of tracking some reference signal). It can be stated as follows: Find a linear time-invariant controller for the system (6.3.1a,b), such that the resulting controlled system is stable and the steady state error is zero (i.e. asymptotic rejection take place) for all disturbances acting on the system. In addition, the problem considered by Davison required that the controllers be "robust", i.e., asymptotic rejection of all disturbances should take place

even in the presence of certain perturbations in the parameters of the system i.e. matrices A, B, C, E. Such a controller consists of two parts: a "servocompensator" which is completely determined by the disturbances, and a "stabilizing compensator" which stabilizes the overall system. The robust controller obtained by Davison for solving the servomechanism problem is basically a dynamic compensator of order equal to the sum of the orders of the servocompensator and the stabilizing compensator. We can use the results of Section 6.2 to design a dynamic output feedback controller with prespecified poles which achieves the same requirements as those obtained by solving the servomechanism problem. Note that the order of the dynamic output feedback obtained by this approach will depend on the total number of d.b.z.'s required to be introduced to achieve complete rejection of all disturbances as well as the number of closed-loop poles that are to be assigned arbitrarily close to desired locations in the complex plane. This type of compensator with prespecified poles which has robust properties can be constructed by using Algorithm 2.4. Such a design will be robust in the same way as Davison's solution to the robust servomechanism problem. This is because in this scheme, the disturbance rejection property of the closed-loop system depends on the d.b.z.'s introduced by the dynamic compensator via its poles and not on the system parameters A, B, C . . etc.

It is worth pointing out that in some applications, the robust dynamic output feedback obtained by our approach may have lower order as compared to that obtained by solving the robust servomechanism problem [7-10]. This advantage appears in the case of constructing dynamic output feedback compensators to achieve asymptotic rejection of some specific disturbances from certain outputs e.g. step disturbance from the first output and a combination of step and ramp disturbances from the second output etc. This important feature of our approach will be illustrated by an example in the next section. It is also worth noting that in the servomechanism problem, the design of a robust dynamic output feedback compensator using our approach requires the outputs of the system to be measurable while the state may not necessarily be measurable.

Remark 6.4: The approach proposed in this thesis for disturbance rejection can be used to solve

the servomechanism problem i.e. the problem of asymptotically tracking some specified reference signals $\mathbf{y}_{ref}(t)$ and asymptotically rejecting a class of disturbances $\mathbf{d}(t)$. This can be done by using the error signal $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{y}_{ref}(t)$ in place of $\mathbf{y}(t)$ in the analysis. In other words, we assign blocking zeros between $\mathbf{e}(t)$ and $\mathbf{d}(t)$ and between $\mathbf{e}(t)$ and $\mathbf{y}_{ref}(\mathbf{t})$ at appropriate locations using feedback. Note that when $\mathbf{y}_{ref}(t) = 0$ for all t, this problem reduces to the asymptotic disturbance rejection problem.

6.4 NUMERICAL EXAMPLES

In this section, we illustrate the use of dynamic output feedback for disturbance rejection and pole assignment in multivariable systems by means of some numerical examples. The desired closed-loop poles have been selected for the purpose of illustration. The only constraints on the pole locations were stability and reasonable transient behaviour.

Example 6.1: In this example, we illustrate the performance of Algorithm 2.3 in designing dynamic output feedback to stabilize and assign all the resulting closed-loop poles at or arbitrarily close to some desired values. The system being considered is the 5th-order system [11] given in the previous chapter . The model that we considered is the one obtained after implementing a state feedback controller to reject the effect of disturbances in steady state. The parameters of this closed-loop system (with constant and dynamic state feedback) are given in the Chapter V (Eqns.(5.4.5a,b) and (5.4.8a,b), respectively).

(i) For the system given in eqns.(5.4.5a,b), we need a dynamic compensator of order one to stabilize the overall system and assign all the poles of the augmented closed-loop system at desired values, e.g. (-4, -5, -6, -7, -10 ± j10). The dynamic compensator that achieves our requirements was found by implementing Algorithm 2.3:

$$\dot{z}(t) = \begin{bmatrix} 0.5028921653E+04 \end{bmatrix} z(t)$$

$$+ \begin{bmatrix} -0.31049607826672E+08 & 0.1932730883E+04 \end{bmatrix} \mathbf{y}(t) \qquad (6.4.1a)$$

$$\hat{\mathbf{u}}(t) = \begin{bmatrix} -0.9161572475E+04 \\ 1.4313823045E+04 \\ 6.493273120E+04 \end{bmatrix} z(t)$$

$$+ \begin{bmatrix} 5.65776460263852E+08 & -3.5469817217E+04 \\ -0.88297788428924E+08 & 0.5501451858E+04 \\ -4.0033002815027E+08 & 2.4976818402E+04 \end{bmatrix} \mathbf{y}(t) \qquad (6.4.1b)$$

For the purpose of illustrating the characteristics of the resulting closed-loop system responses, we let $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1.0$. The responses at the outputs to each disturbance, i.e. exponential (exp(t) or exp(2t)), step, sinusoidal and combinations of these are shown in Figs.(6.1-6.5) for the open-loop system, and in Figs.(6.6-6.10) for the closed-loop system with controllers consisting of both state feedback and dynamic output feedback. It can be readily seen that in the closed-loop system all the specified disturbances are rejected completely in the steady state and the system has improved transient behaviour obtained by assigning all the closed-loop poles at the specified locations.

(ii) The unstable system given in eqns.(5.4.8a,b) needed a dynamic compensator of order two to stabilize and to assign all the closed-loop poles at the desired values i.e. -4, -5, -6, -7, -8, -9, -10.0 $\pm$ j10.0. The parameters of the dynamic compensator were found by applying Algorithm 2.3 and are shown in Table 6.1.

When $\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 1.0$, the responses at the outputs for each disturbance, i.e. exponential (exp(t) and exp(2t)), step, ramp, sinusoidal and combinations of these are shown in Figs.(6.11-6.16) for the open-loop system and Figs.(6.17-6.22) for the closed-loop system with controllers consisting of state and dynamic output feedback. From these responses, it can be seen

that asymptotic rejection takes place for all the specified disturbances acting on the system. In addition the resulting closed-loop has improved transient performance resulting from assigning all the closed-loop poles at the specified locations.

Example 6.2: For this example, the system being considered is the 3rd-order system given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \mathbf{d}(t) \tag{6.4.2a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \tag{6.4.2b}$$

For the purpose of illustration, suppose that the disturbances acting on the above system are unmeasurable and have the following form

$$\mathbf{d}(t) = \begin{bmatrix} \beta_1 \, exp(t) \\ \beta_2 \end{bmatrix}$$

where β_1 and β_2 are unknown values. Note that the given system has one d.b.z. at -2.0 between the outputs and the exponential disturbance. This implies that conditions (5.2.2a,b) are only satisfied for the open-loop system between the outputs and the first disturbance.

To demonstrate the steady state rejection of multiple disturbances, it is required to construct a stable controller having lower order which is not necessarily robust, such that the exponential disturbance is rejected in steady state at both outputs, while the step disturbance is rejected in steady state only at the first output. In addition, the resulting closed-loop system should be stable with all the closed-loop poles assigned at specified locations in the complex plane.

From Remark 6.2, we can solve this problem by constructing a controller consisting of a combination of state feedback and dynamic output feedback in two stages: In the first stage, we

use Algorithm 4.1 to find a constant state feedback matrix given by the following control law

$$\mathbf{u}(t) = \mathbf{u}_1(t) - \begin{bmatrix} 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -3.0 \end{bmatrix} \mathbf{x}(t) \tag{6.4.3}$$

such that the d.b.z between the outputs and the first disturbance is assigned at 1.0. This will reject the exponential disturbance at the system outputs in steady state. At the end of this stage the following closed-loop system is obtained

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{u}_1(t) + \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \mathbf{d}(t) \tag{6.4.4a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) \tag{6.4.4b}$$

which has one d.b.z. between the outputs and the first disturbance located at 1.0 .

In the second stage, we design dynamic output feedback of order one between output $y_1(t)$ and the control inputs $\mathbf{u}_1(t)$, to introduce an additional d.b.z located at 0.0 so that complete rejection of the step disturbance is achieved at $y_1(t)$ in steady state In addition, all the resulting closed-loop poles are assigned at desired locations e.g. at -2, -4, -1 $\pm$ j. The parameters of this compensator can be computed by applying Algorithm 2.4 (Step II):

$$\dot{z}(t) = \begin{bmatrix} 0.0 \end{bmatrix} z(t) + \begin{bmatrix} 1.0 & 0.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.5a}$$

$$\mathbf{u}_1(t) = \mathbf{v}(t) - \begin{bmatrix} 5.25 \\ 26.5 \end{bmatrix} z(t) - \begin{bmatrix} 5.5 & 0.0 \\ 33.5 & 0.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.5b}$$

By implementing the dynamic output feedback (6.4.5a,b) on the system in eqns.(6.4.4a,b), we get the overall closed-loop system

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} -5.5 & -4.5 & 0.0 & -5.25 \\ -5.5 & -5.5 & 1.0 & -5.25 \\ -39.0 & 39.0 & 3.0 & -31.75 \\ 1.0 & 1.0 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \\ 0 & 0 \end{bmatrix} \mathbf{d}(t) \tag{6.4.6a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z(t) \end{bmatrix} \tag{6.4.6b}$$

which has one d.b.z at 1.0 between the outputs and the first disturbance, and one d.b.z at 0.0 between output $y_1(t)$ and the second disturbance. In addition, the resulting closed-loop system is stable with all poles at the specified locations.

When $\beta_1 = \beta_2 = 1.0$, the responses at the outputs $y_1(t)$ and $y_2(t)$ to exponential disturbance are shown in Figs.(6.23a) and (6.23b) for the open-loop system and in Figs.(6.24a) and (6.24b) for the closed-loop system (with the controller consisting of both state and dynamic output feedback). It can be readily seen that in the closed-loop system the exponential disturbance is asymptotically rejected in the steady state at the system outputs and improved transient behaviour is obtained. For the step disturbance, the responses at the outputs $y_1(t)$ and $y_2(t)$ are shown in Figs.(6.25a) and (6.25b) for the open-loop system and in Figs.(6.26a) and (6.26b) for the closed-loop system (with the controller consisting of both state and dynamic output feedback). It is clearly seen that, the step disturbance is completely rejected only from output $y_1(t)$.

Example 6.3: We have selected this example to illustrate the special features of Algorithm 2.4 for designing a robust dynamic output feedback controller. The example chosen is a 3rd-order system described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{u}(t) + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \mathbf{d}(t) \tag{6.4.7a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}(t) \tag{6.4.7b}$$

with

$$\mathbf{d}(t) = \begin{bmatrix} \beta_1 \\ \beta_2 t \end{bmatrix}$$

where β_1 and β_2 are unknown constant values. Note that, the above system does not satisfy both conditions (5.2.2a,b) and (5.2.3a,b).

For the purpose of illustration, suppose that it is required to construct a robust controller so that (i) all step disturbances are completely rejected at the outputs in steady state and (ii) all ramp disturbances are rejected only from the second output $y_2(t)$ in steady state. In addition, we require that the resulting closed-loop system consisting of the open-loop system and the controller is stable with all closed-loop poles assigned at or arbitrarily close to specified desired values.

In order to solve the above problem using our approach (with dynamic output feedback), we first reject the step disturbance from the first output $y_1(t)$ by designing a dynamic output feedback controller of order one between the output $y_1(t)$ and the control input $\mathbf{u}(t)$. Note that, this compensator can be used to introduce an additional d.b.z at 0.0 between the output $y_1(t)$ and

the disturbances. This is to reject the step disturbance as well as to assign all the resulting closed-loop poles at the desired values, e.g. -2, -4, -1 ± j. The parameters of the dynamic compensator which achieves the above requirement can be computed by applying Algorithm 2.4. The compensator was found to be

$$\dot{z}_1(t) = \begin{bmatrix} 0.0 \end{bmatrix} z_1(t) + \begin{bmatrix} 1.0 & 0.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.8a}$$

$$\mathbf{u}(t) = \mathbf{u}_1(t) - \begin{bmatrix} 10.0 \\ -4.0 \end{bmatrix} z_1(t) - \begin{bmatrix} 8.0 & 0.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.8b}$$

On implementing the above compensator in the open-loop system in (6.4.7a,b), the following closed-loop system is obtained

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}_1(t) \end{bmatrix} = \begin{bmatrix} -8.0 & 0.0 & -7.0 & -10.0 \\ -16.0 & 0.0 & -16.0 & -16.0 \\ 0.0 & 1.0 & 0.0 & 4.0 \\ 1.0 & 0.0 & 1.0 & 0.0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_1(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{u}_1(t) + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 0 & 0 \end{bmatrix} \mathbf{d}(t) \tag{6.4.9a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_1(t) \end{bmatrix} \tag{6.4.9b}$$

which has one d.b.z at 0.0 between $y_1(t)$ and each disturbance ($d_1(t)$ and $d_2(t)$). In addition, the poles of the resulting closed-loop system are assigned at the specified desired values.

Now, to reject the combination of the step and ramp disturbances from the second output $y_2(t)$, we need to design a dynamic output feedback compensator of order two connected between the output $y_2(t)$ and the control input $\mathbf{u}_1(t)$ which will enable us to introduce two d.b.z.'s (0.0 and 0.0). Note that this compensator can also be used to preserve all the poles which were assigned before at -2, -4, -1 ± j as well as to assign the remaining closed-loop poles at, e.g. (-3, -5). The parameters of this compensator can be computed by applying Algorithm 2.4. The compensator is given by

$$\dot{\mathbf{z}}_2(t) = \begin{bmatrix} 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} \mathbf{z}_2(t) + \begin{bmatrix} 0.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.10a}$$

$$\mathbf{u}_1(t) = \mathbf{v}(t) - \begin{bmatrix} 20.0 & 5.0 \\ 16.0 & 16.0 \end{bmatrix} \mathbf{z}_2(t) - \begin{bmatrix} 0.0 & -4.5 \\ 0.0 & 8.0 \end{bmatrix} \mathbf{y}(t) \tag{6.4.10b}$$

By implementing the above compensator on the system in eqns.(6.4.9a,b), we get the overall closed-loop system

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{z}_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} = \begin{bmatrix} -8 & 0 & -2.5 & -10 & -20 & -5 \\ -16 & 0 & -15 & -16 & -56 & -26 \\ 0 & 1 & -8 & 4 & -16 & -16 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{v}(t) + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{d}(t) \tag{6.4.11a}$$

$$\mathbf{y}(t) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ z_1(t) \\ \mathbf{z}_2(t) \end{bmatrix} \tag{6.4.11b}$$

which has one d.b.z.'s at 0.0 between the output $y_1(t)$ and the disturbances and two d.b.z.'s at 0.0

and 0.0 between the output $y_2(t)$ and the disturbances. In addition, the poles of the resulting augmented closed-loop system are at the specified values i.e. -2, -3, -4, -5, $-1 \pm j$.

When $\beta_1 = \beta_2 = 1.0$ the responses at the outputs $y_1(t)$ and $y_2(t)$ to the step disturbances are shown in Figs.(6.27a) and (6.27b) for the open-loop system (6.4.7a,b), and in Figs.(6.28a) and (6.28b) for the closed-loop system (6.4.11a,b). It can be readily seen that the step disturbances are completely rejected from both outputs of the closed-loop system in steady state. For the ramp disturbances, the responses at the outputs $y_1(t)$ and $y_2(t)$ are shown in Figs.(6.29a) and (6.29b) for the open-loop system and in Figs.(6.30a) and (6.30b) for the closed-loop system. It can be seen that the ramp disturbance is completely rejected only from output $y_2(t)$.

6.5 CONCLUDING REMARKS

In this chapter, we have shown how dynamic output feedback can be used for pole assignment as well as for disturbance rejection in multivariable systems. In Section 6.1, it was shown that dynamic output feedback can be designed by using two alternative approaches: The first approach uses Algorithm 2.3 to compute an accurate dynamic compensator which achieves arbitrary pole assignment, while the second approach uses Algorithm 2.4 for arbitrary pole assignment as well as for assigning all the poles of the dynamic compensator at specified locations in the complex plane. Then in Section 6.2, for multivariable systems described by $\sum_d \left[A, B, C, E\right]$ and $\sum_d \left[A, B, C, D, E, F\right]$, we showed how dynamic output feedback introduces additional d.b.z.'s located at the poles of the compensator. The important feature of designing dynamic output feedback with prespecified poles was then used in Section 6.3 to solve the general problem of disturbance rejection. It should be pointed out that in some applications, the dynamic output feedback controller obtained using our approach has some advantages and flexibility over other approaches [7-10], e.g. in rejecting a specific class of disturbances from certain outputs. Finally in Section 6.4, we have given some examples to illustrate the main features of our approach which allow us to design dynamic output feedback to achieve steady-state

rejection of some disturbances acting on the system and to ensure closed-loop system stability by assigning all the system poles at desired locations in the complex plane.

$$F = \begin{bmatrix} -0.90568277763057E+06 & 5.55009929258726E+06 \\ -0.15424422439795E+06 & 0.94522185469363E+06 \end{bmatrix}$$

$$G = \begin{bmatrix} 3.8472739926937E+09 & -0.00008667432616E+09 \\ 0.65522041157581E+09 & -0.00001476045421E+09 \end{bmatrix}$$

$$J = \begin{bmatrix} 1.22863952356009E+10 & -0.00002769104562E+10 \\ 0.46398162185336E+10 & -0.00001045292926E+10 \\ 0.98274418282387E+10 & -0.00002213913413E+10 \end{bmatrix}$$

$$H = \begin{bmatrix} -0.28920688205495E+07 & 1.77231128561256E+07 \\ -0.10922514663015E+07 & 0.66934244095447E+07 \\ -0.23134468902850E+07 & 1.41771188918938E+07 \end{bmatrix}$$

Table 6.1 Parameters of the dynamic output feedback compensator for Example 6.1(ii).

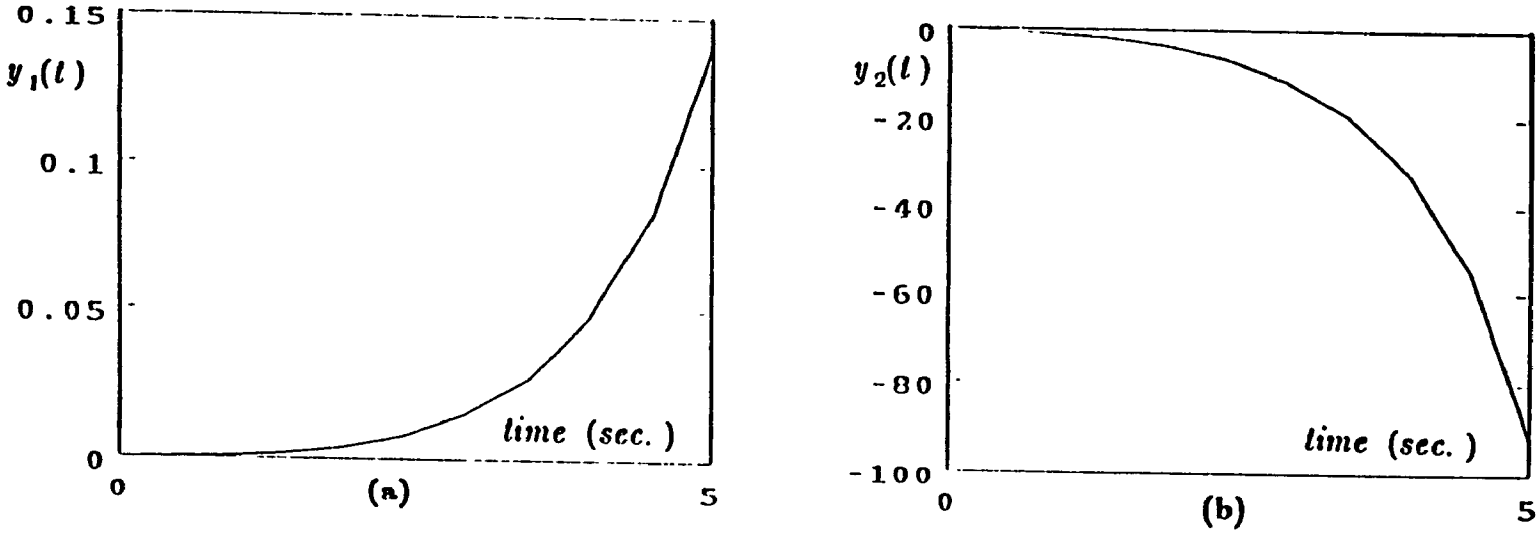

Fig.(6.1) Output responses of the open-loop system in Example 6.1-i to $exp(t)$ disturbance.

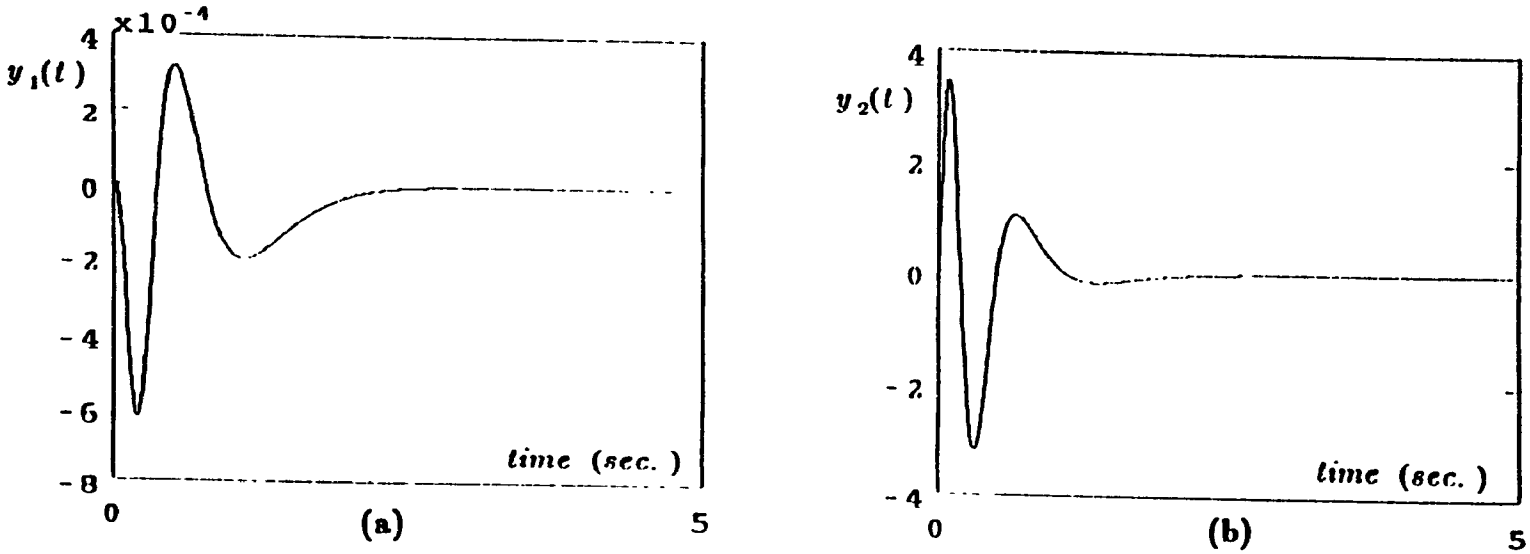

Fig.(6.6) Output responses of the closed-loop system in Example 6.1-i to $exp(t)$ disturbance.

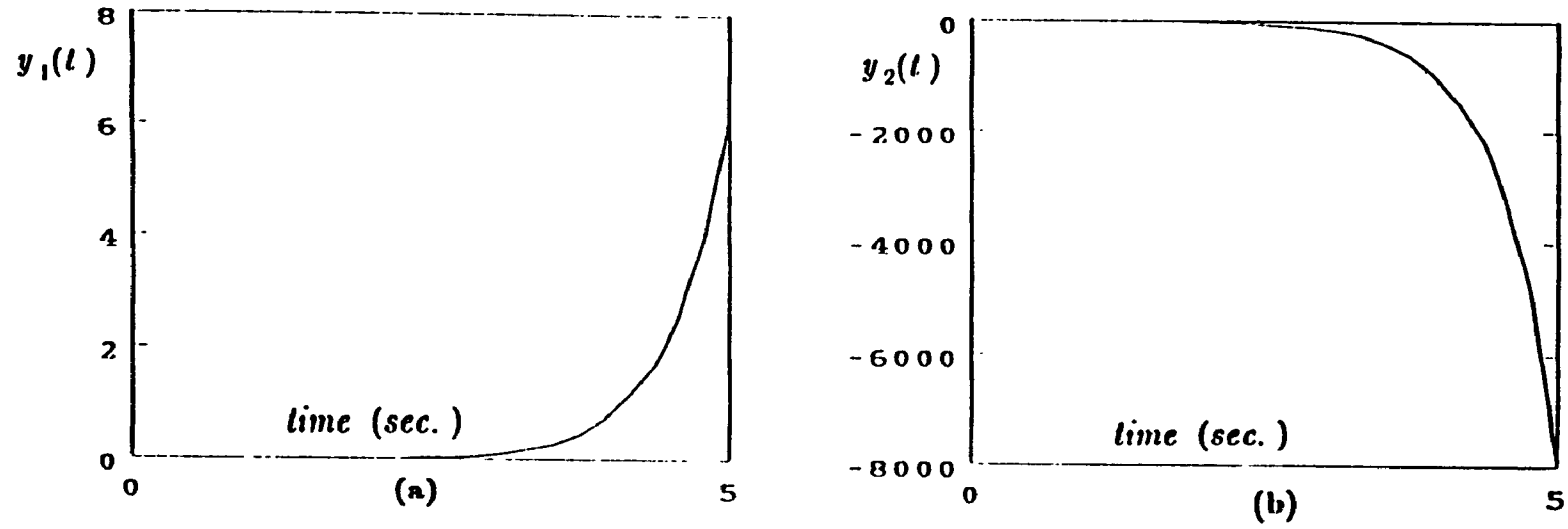

Fig.(6.2) Output responses of the open-loop system in Example 6.1-i to $exp(2t)$ disturbance.

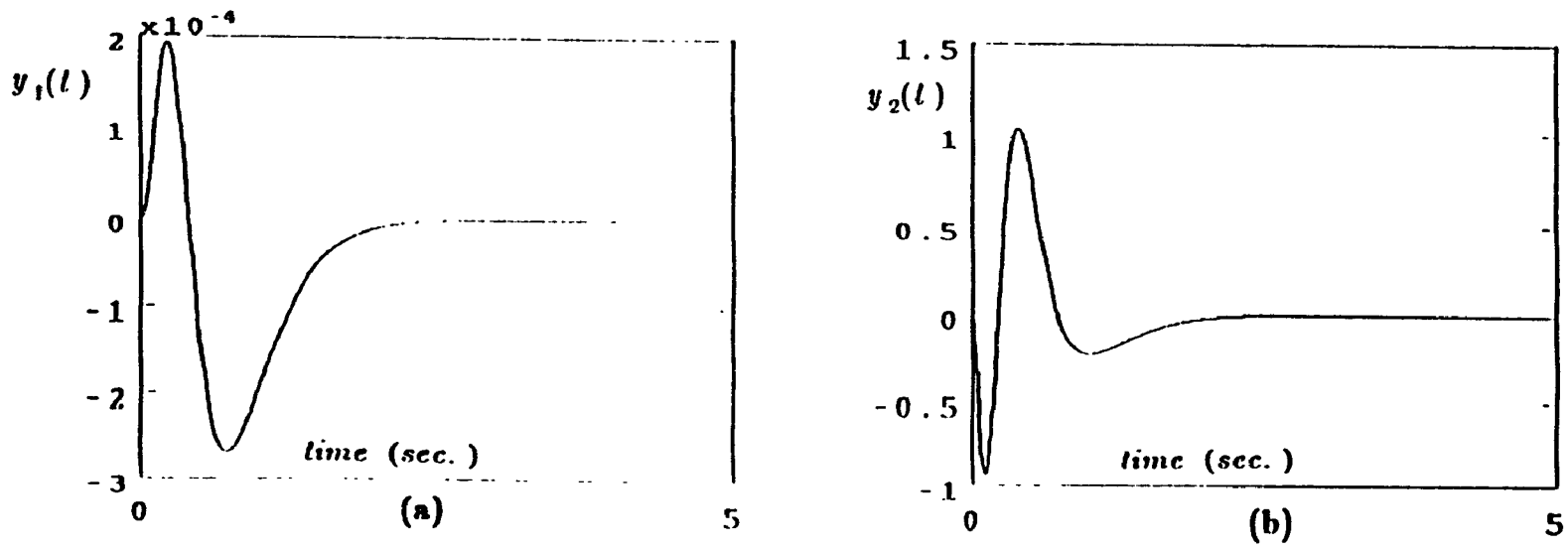

Fig.(6.7) Output responses of the closed-loop system in Example 6.1-i to $exp(2t)$ disturbance

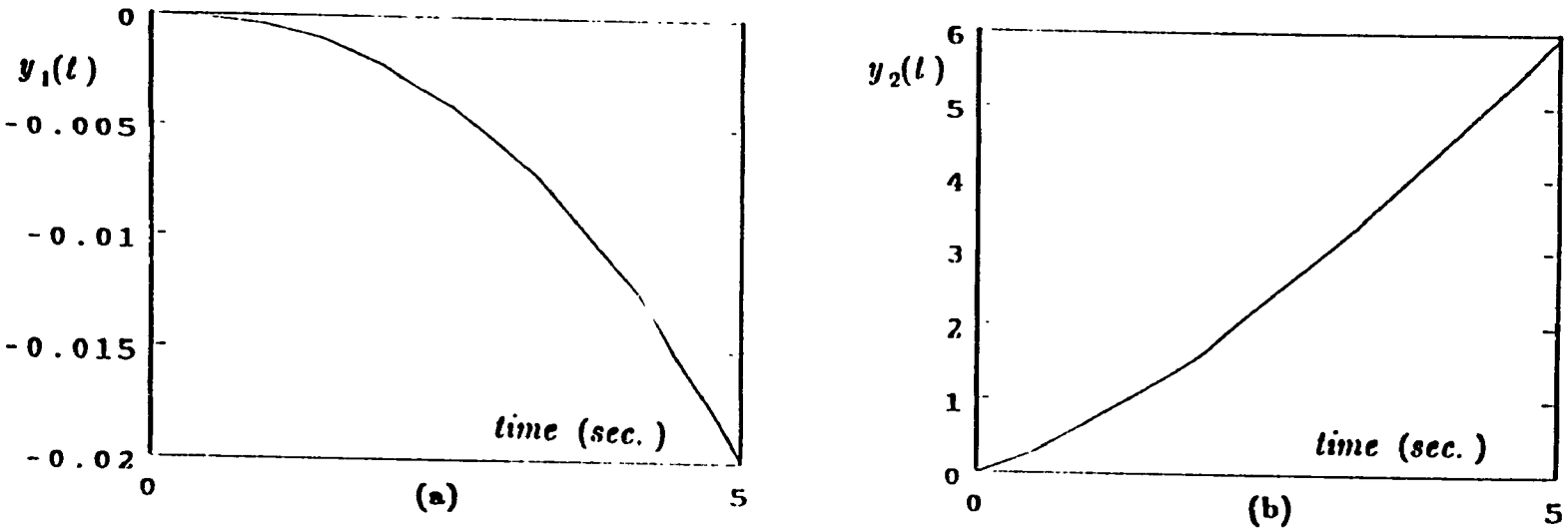

Fig.(6.3) Output responses of the open-loop system in Example 6.1-1 to step disturbance.

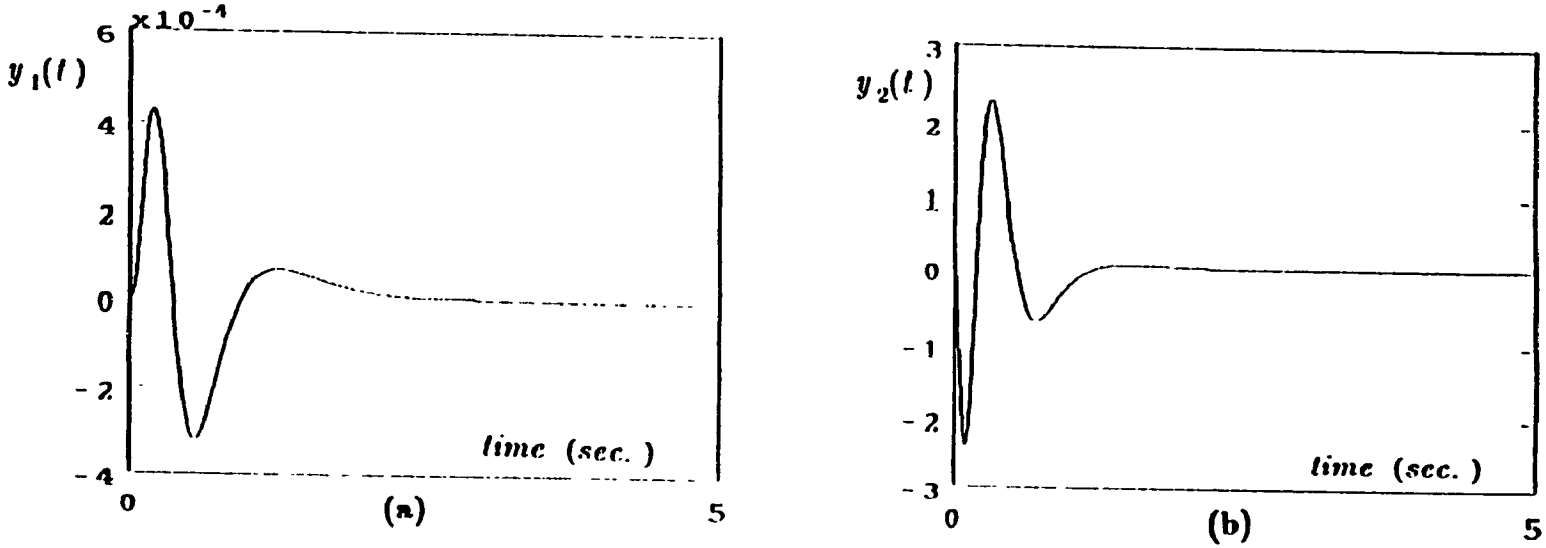

Fig.(6.8) Output responses of the closed-loop system in Example 6.1-1 to step disturbance.

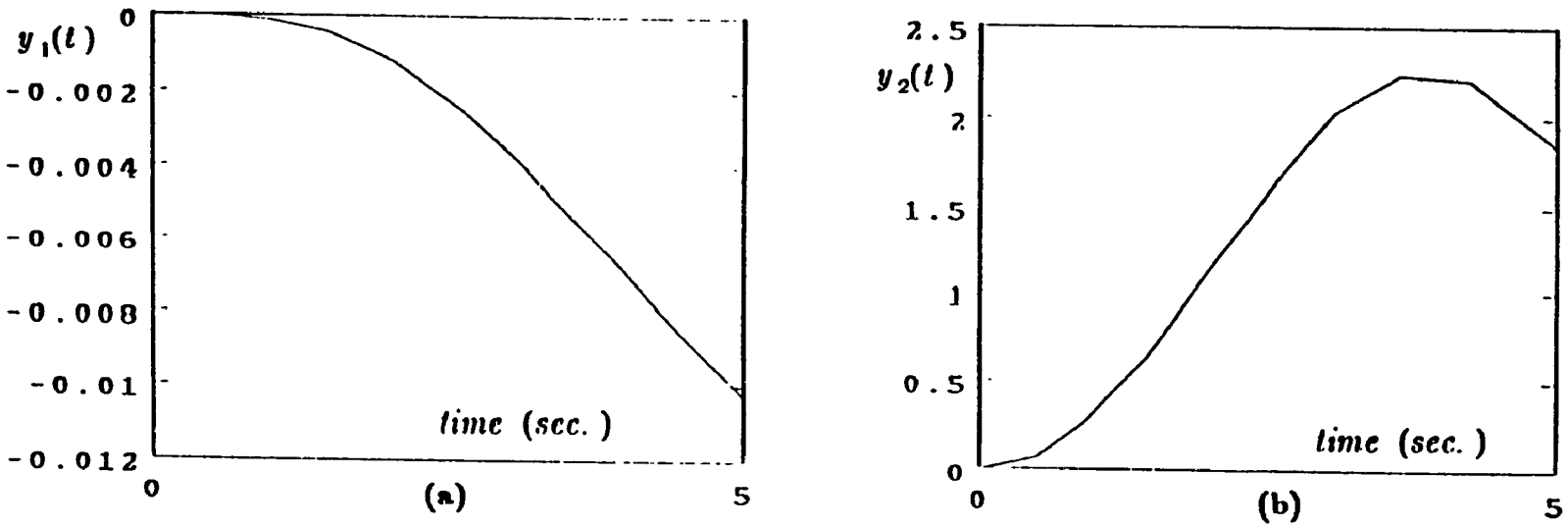

Fig.(6.4) Output responses of the open-loop system in Example 6.1-i to $sin\,(t)$ disturbance.

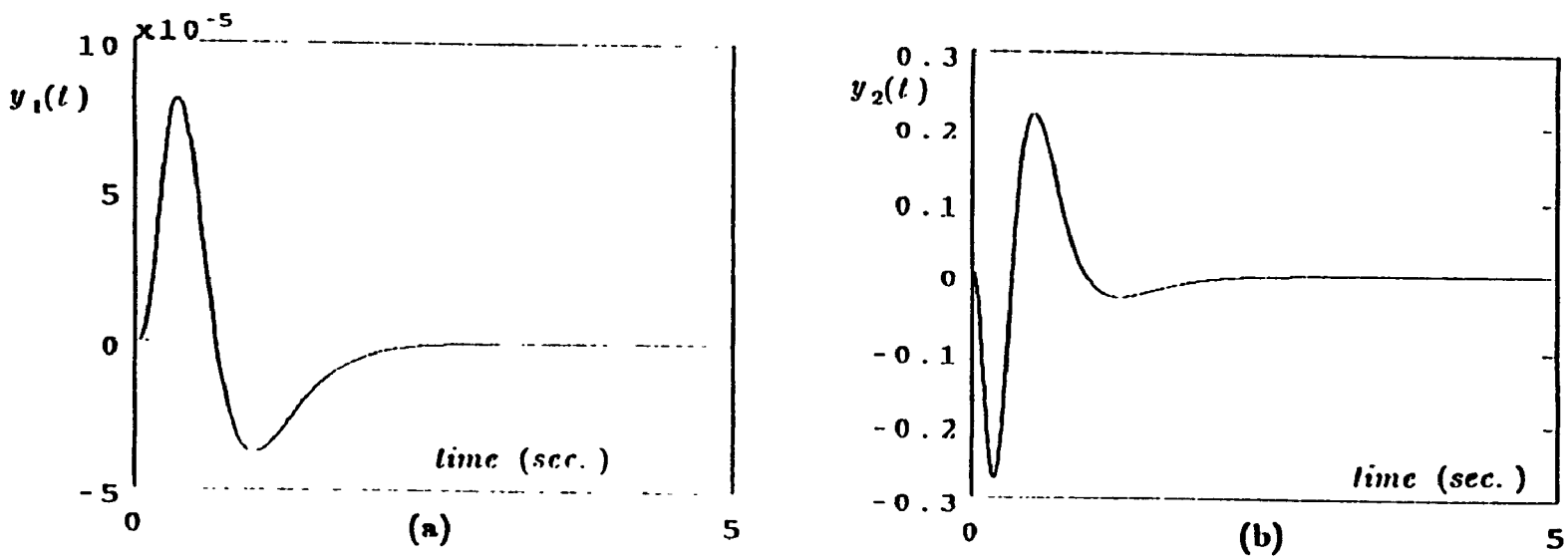

Fig.(6.9) Output responses of the closed-loop system in Example 6.1-i to $sin\,(t)$ disturbance.

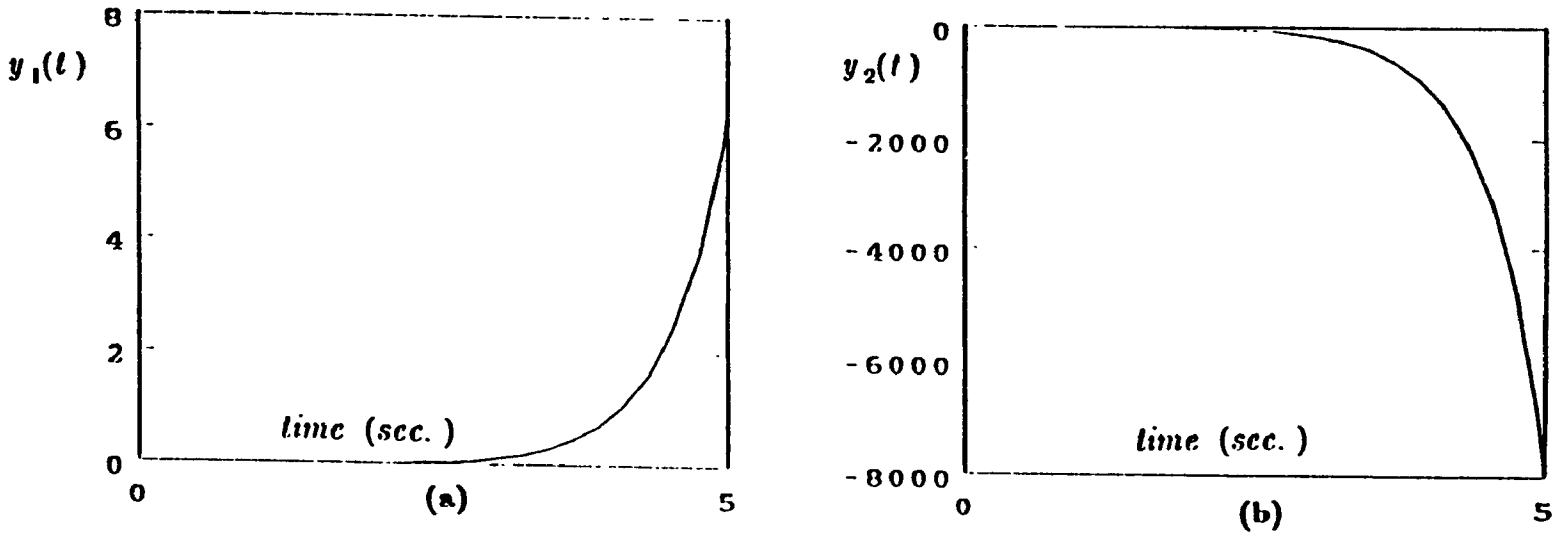

Fig.(6.5) Output responses of the open-loop system in Example 6.1-i to combination of $exp(t)$, $exp(2t)$, step and $sin(t)$ disturbances.

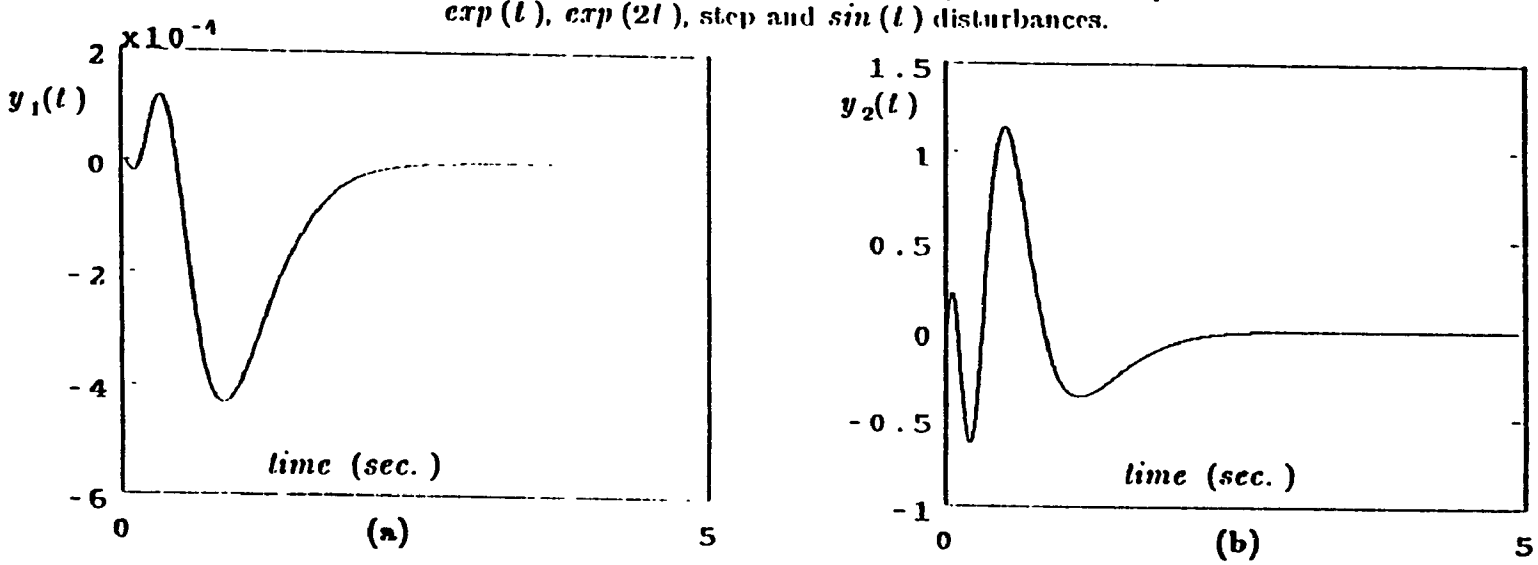

Fig.(6.10) Output responses of the closed-loop system in Example 6.1-i to combination of $exp(t)$, $exp(2t)$, step and $sin(t)$ disturbances.

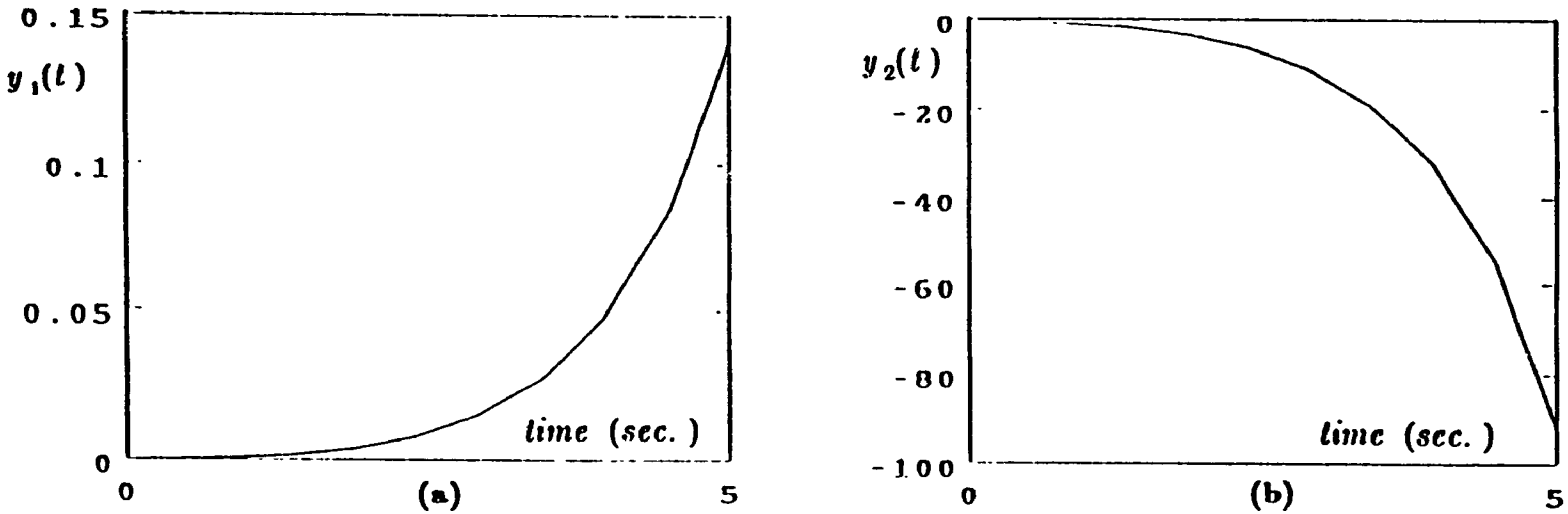

Fig.(6.11) Output responses of the open-loop system in Example 6.1-ii to $exp(t)$ disturbance.

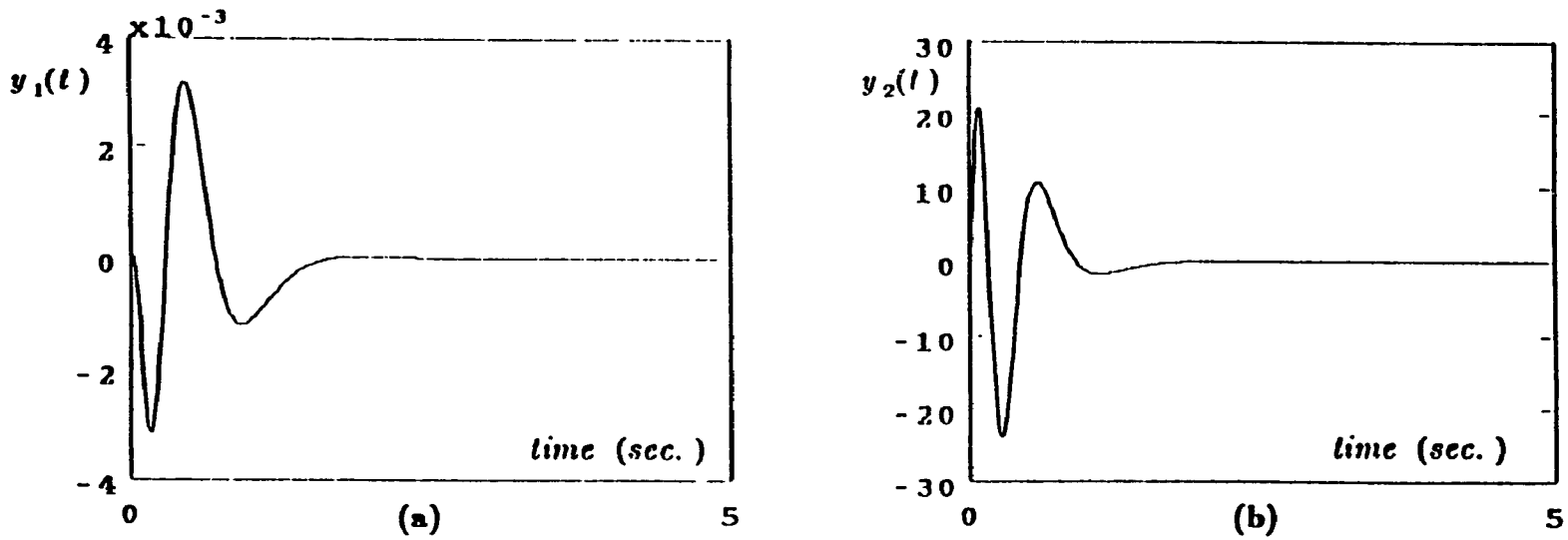

Fig.(6.17) Output responses of the closed-loop system in Example 6.1-ii to $exp(t)$ disturbance.

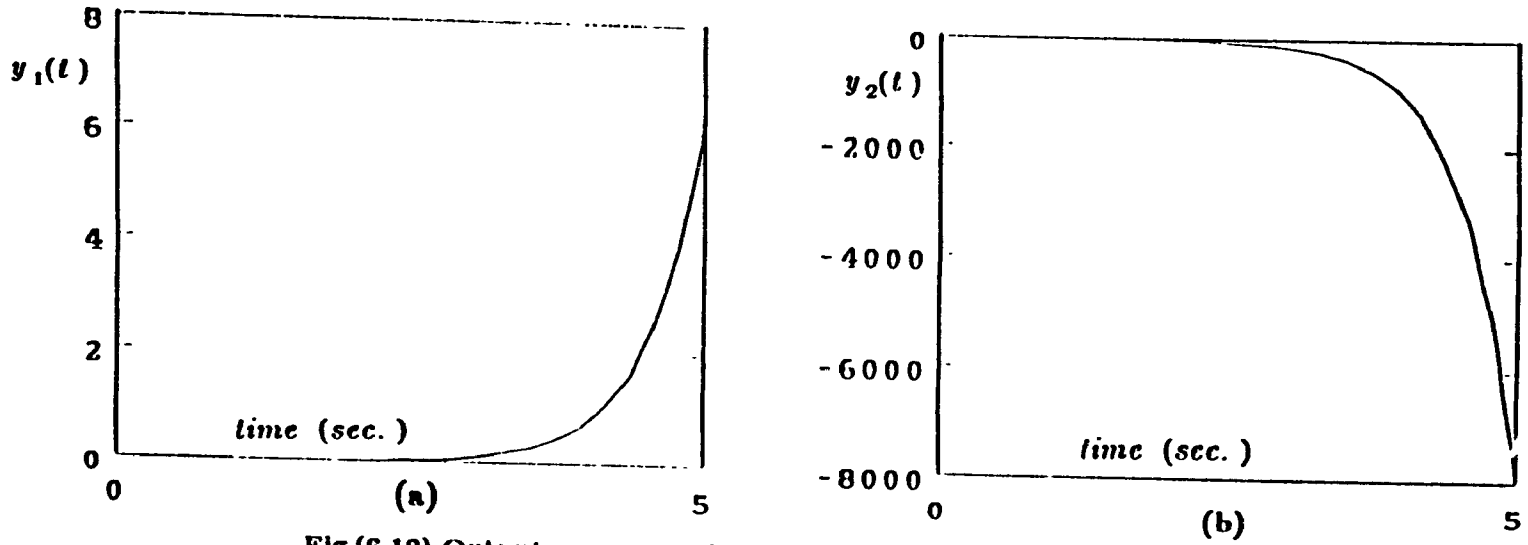

Fig.(6.12) Output responses of the open-loop system in Example 6.1-ii to $exp\,(2t)$ disturbance.

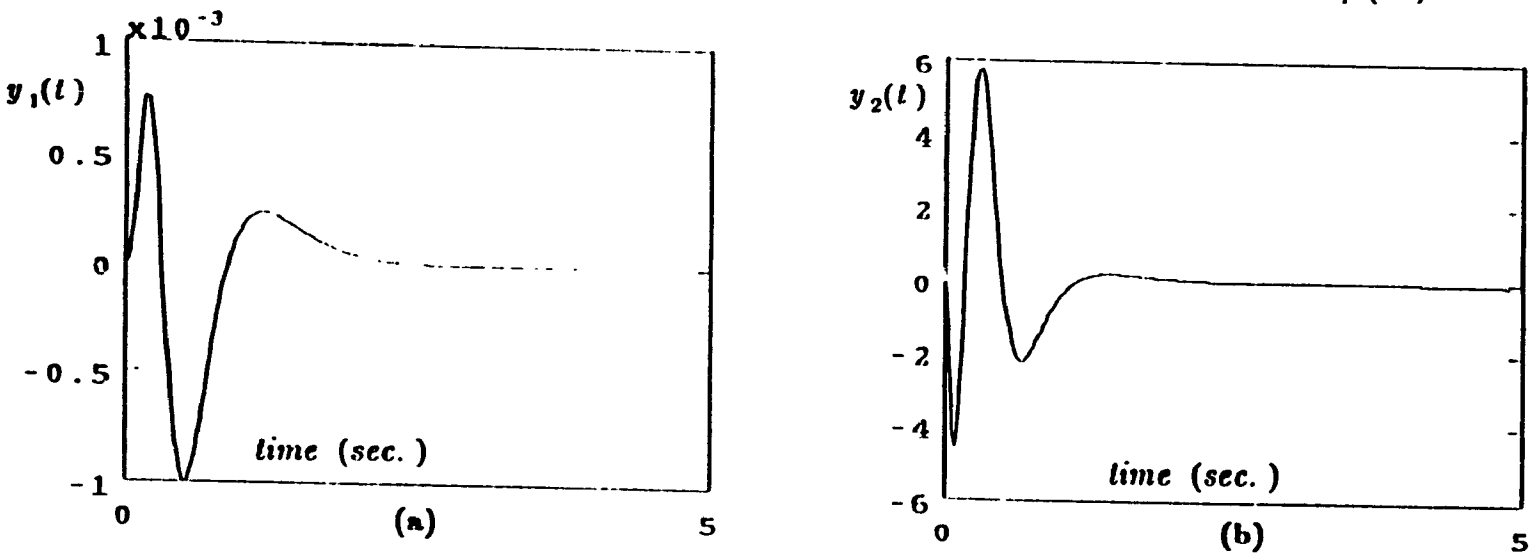

Fig.(6.18) Output responses of the closed-loop system in Example 6.1-ii to $exp\,(2t)$ disturbance.

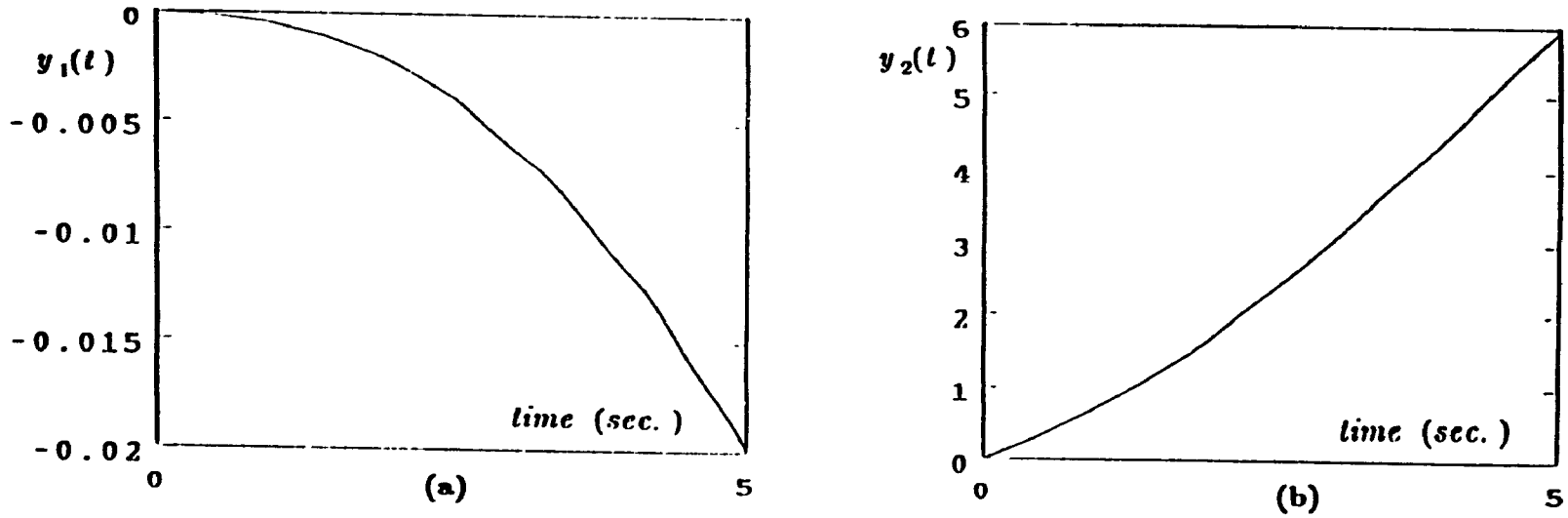

Fig.(6.13) Output responses of the open-loop system in Example 6.1-ii to step disturbance.

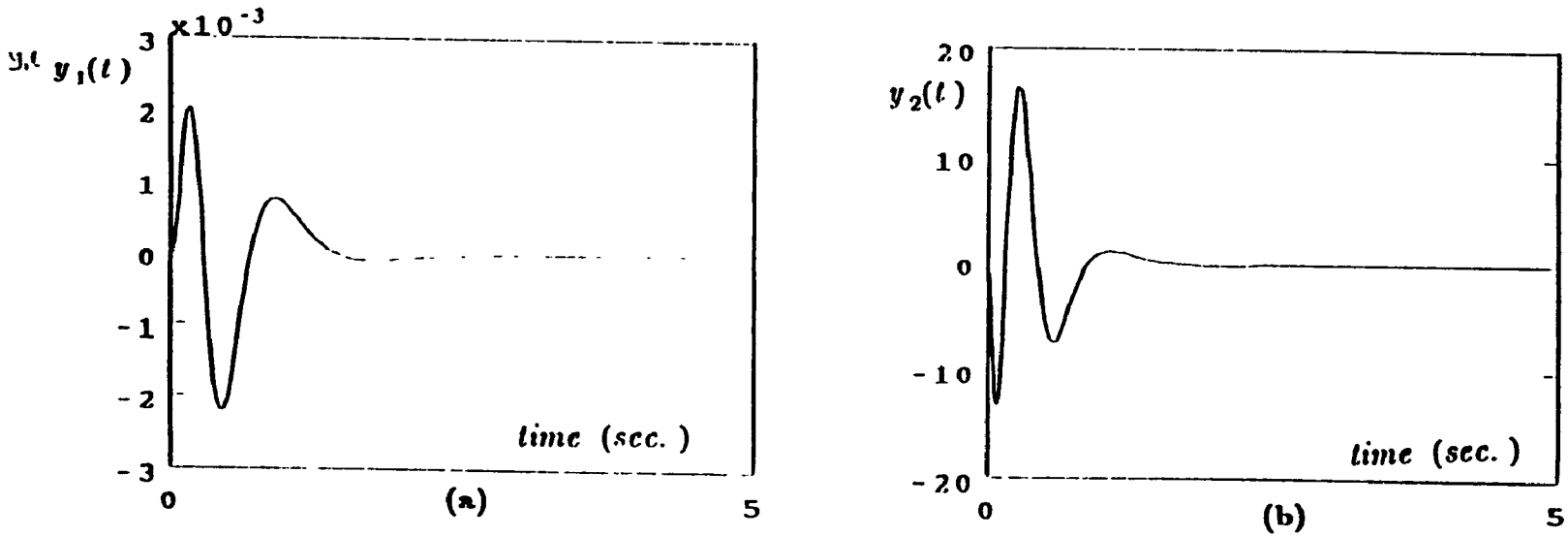

Fig.(6.19) Output responses of the closed-loop system in Example 6.1-ii to step disturbance.

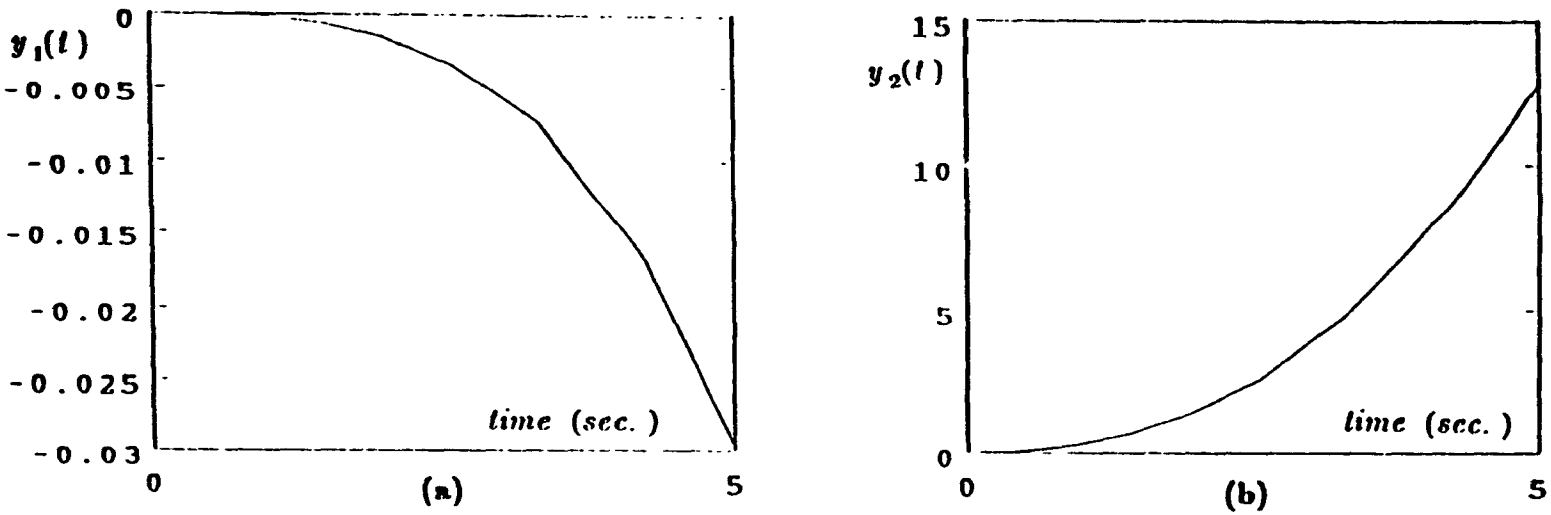

Fig.(6.14) Output responses of the open-loop system in Example 6.1-ii to ramp disturbance.

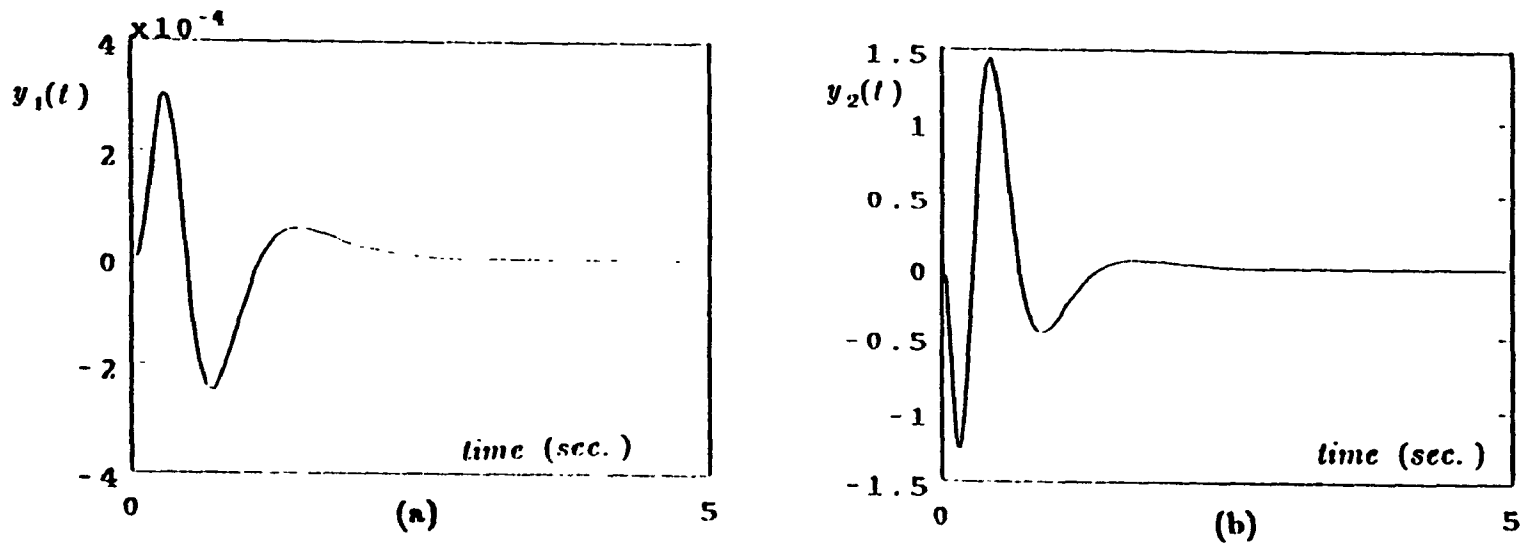

Fig.(6.20) Output responses of the closed-loop system in Example 6.1-ii to ramp disturbance.

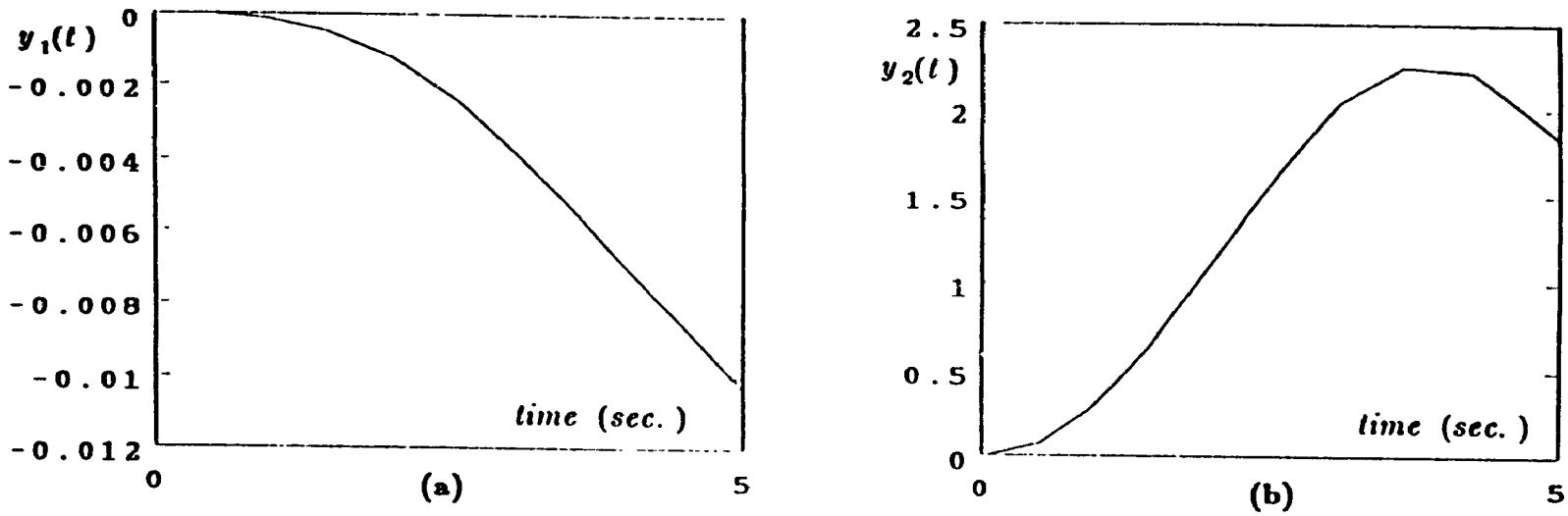

Fig.(6.15) Output responses of the open-loop system in Example 6.1-ii to $sin\,(t)$ disturbance.

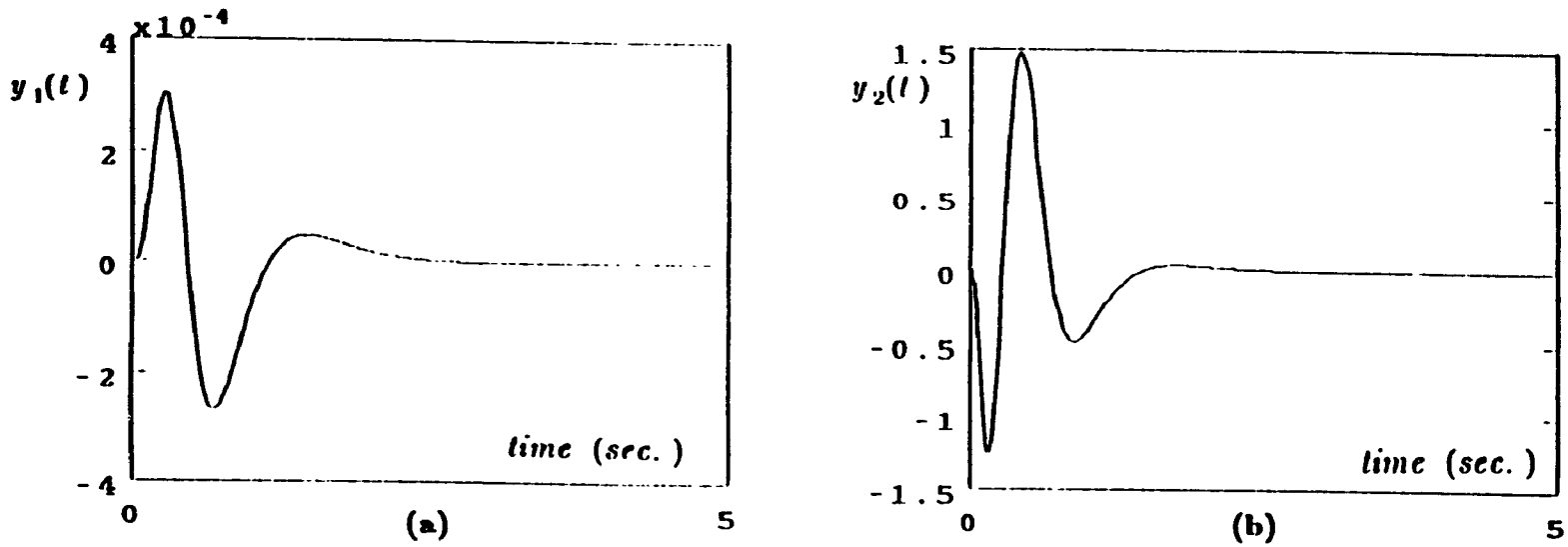

Fig.(6.21) Output responses of the closed-loop system in Example 6.1-ii to $sin\,(t)$ disturbance.

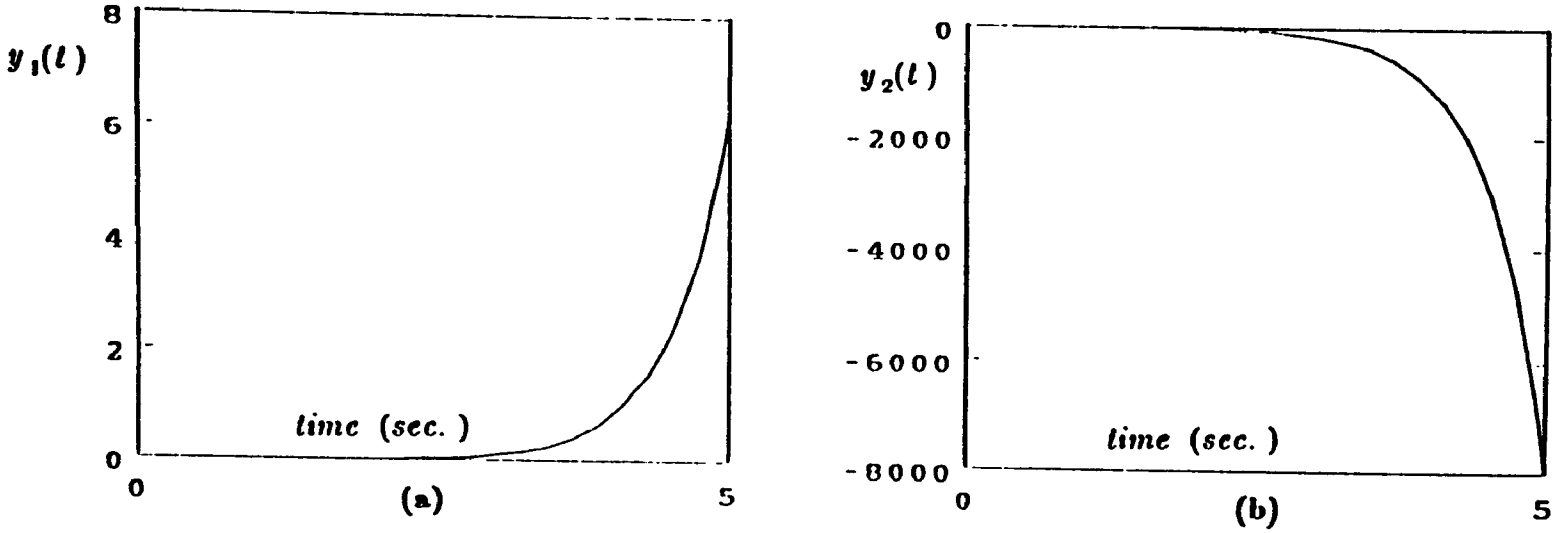

Fig.(6.16) Output responses of the open-loop system in Example 6.1-ii to combination of *exp* (t), *exp* ($2t$), step, ramp and *sin* (t) disturbances.

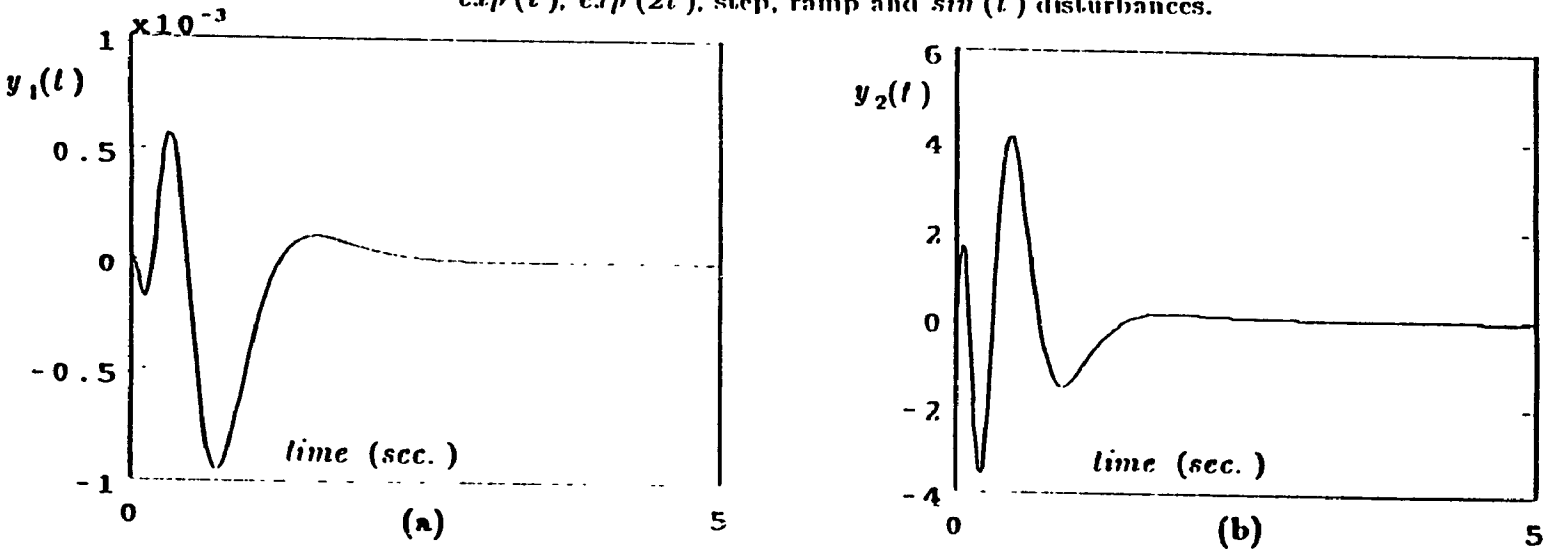

Fig.(6.22) Output responses of the closed-loop system in Example 6.1-ii to combination of *exp* (t), *exp* ($2t$), step, ramp and *sin* (t) disturbances.

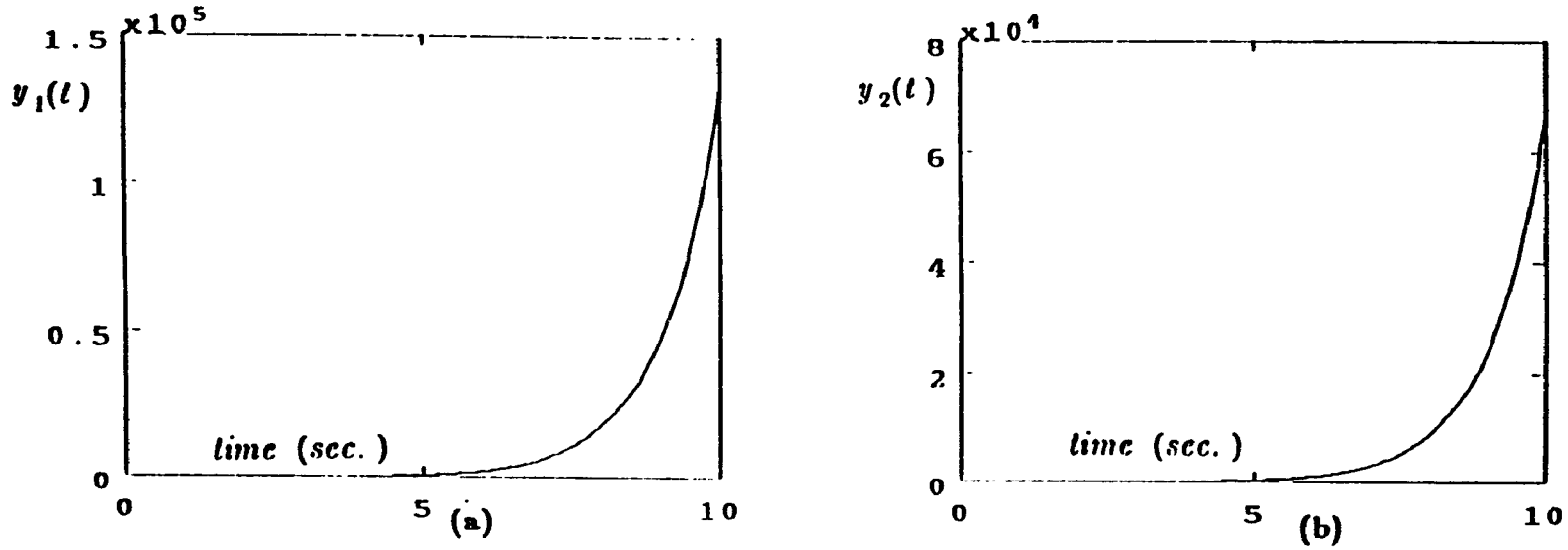

Fig.(6.23) Output responses of the open-loop system in Example 6.2 to $exp(t)$ disturbance.

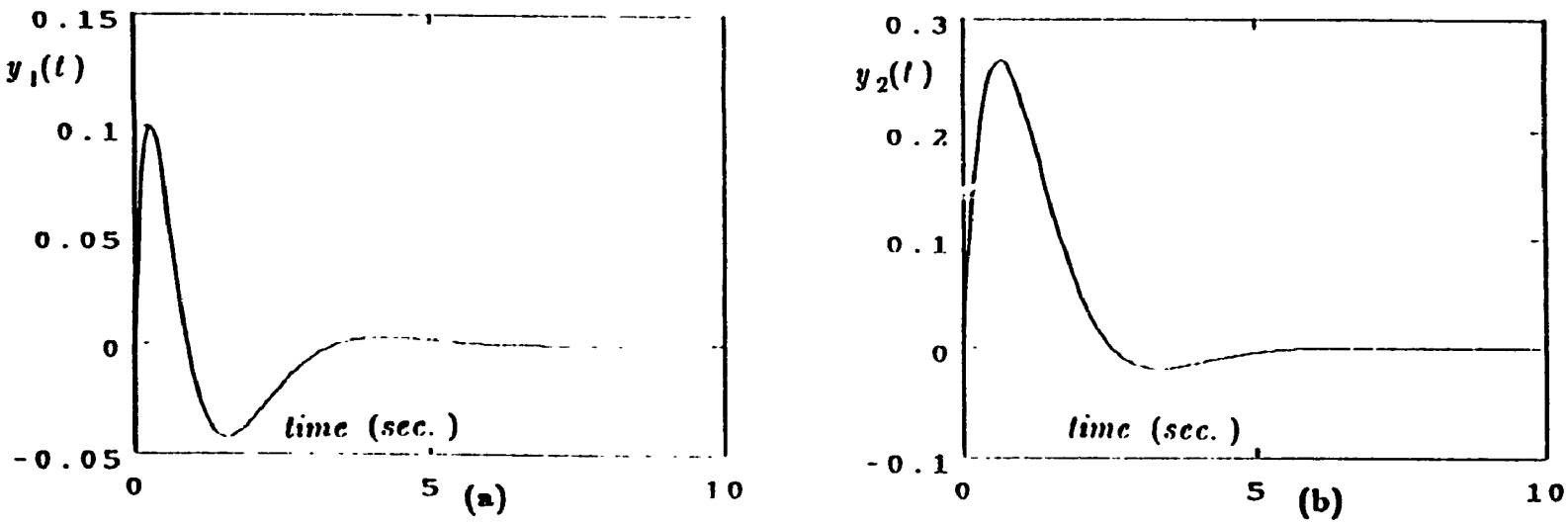

Fig.(6.24) Output responses of the closed-loop system in Example 6.2 to $exp(t)$ disturbance.

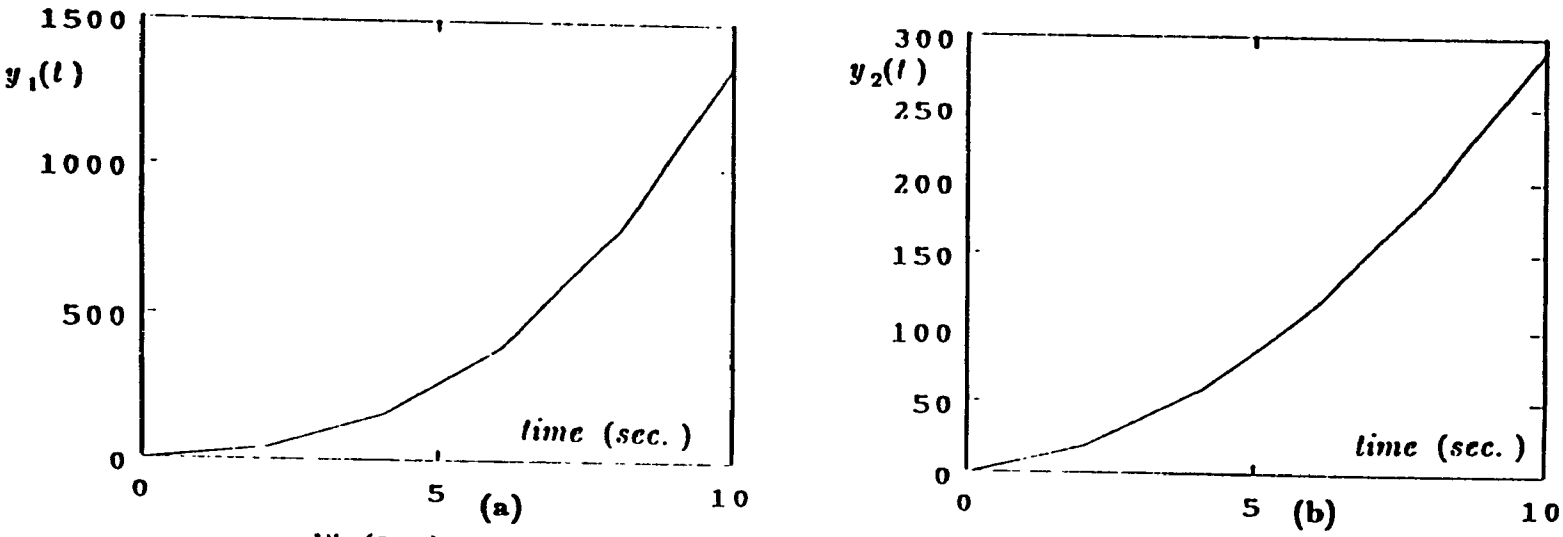

Fig.(6.25) Output responses of the open-loop system in Example 6.2 to step disturbance.

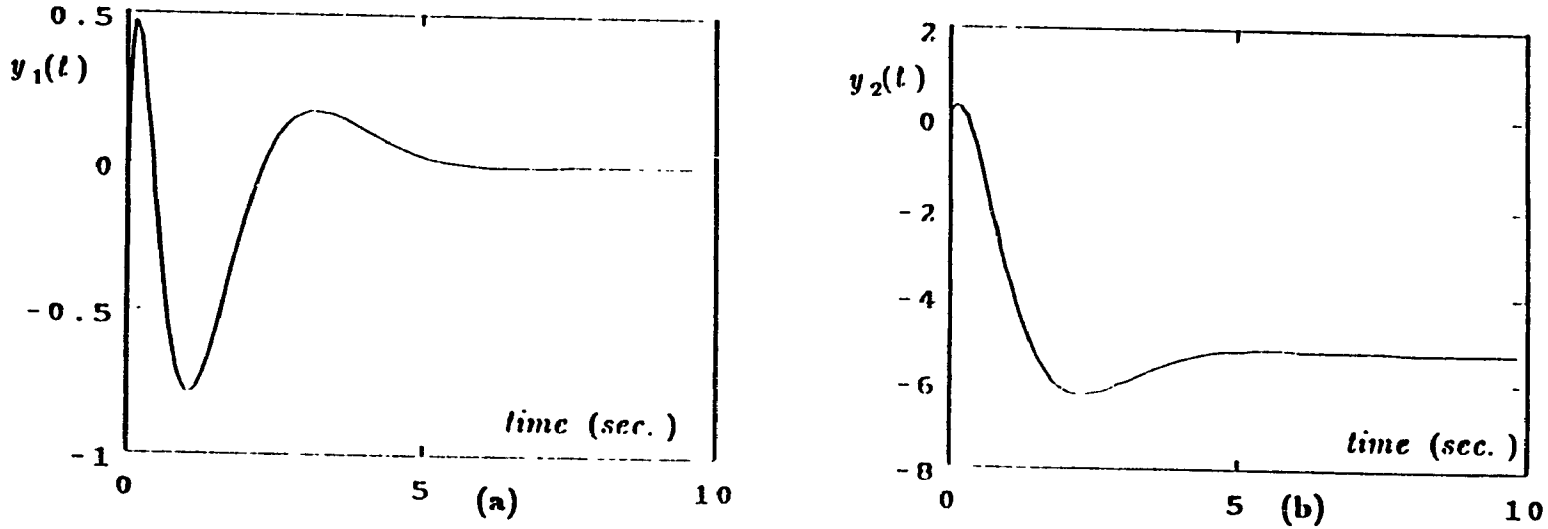

Fig.(6.26) Output responses of the closed-loop system in Example 6.2 to step disturbance.

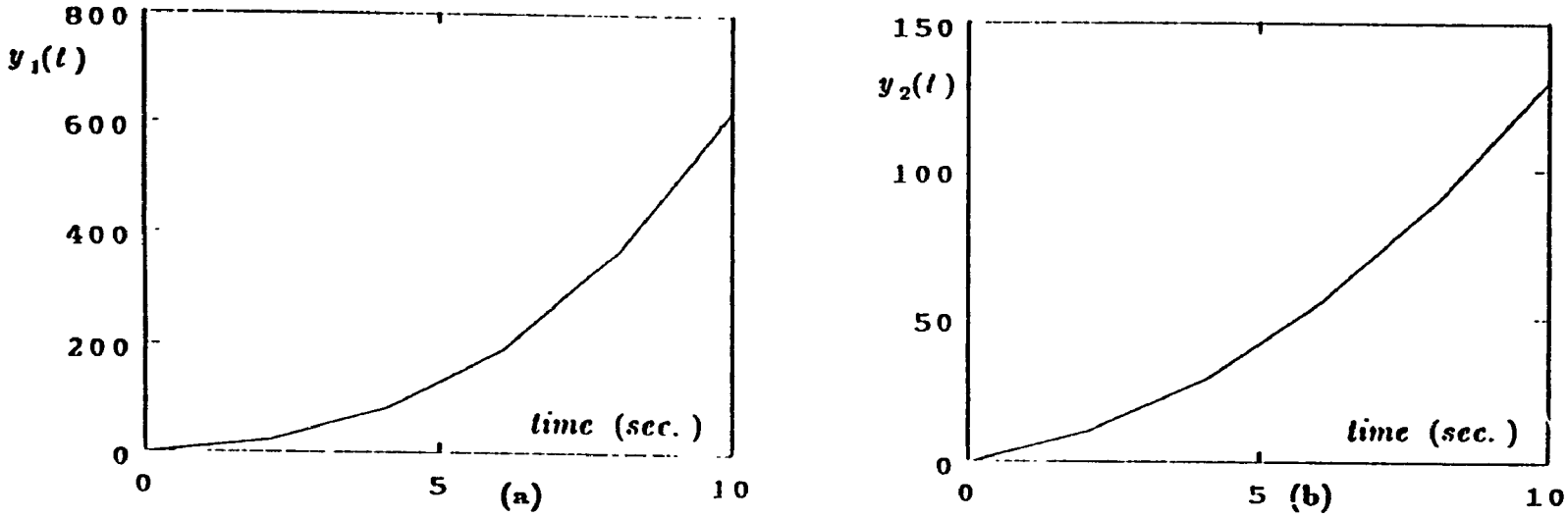

Fig.(6.27) Output responses of the open-loop system in Example 6.3 to step disturbance.

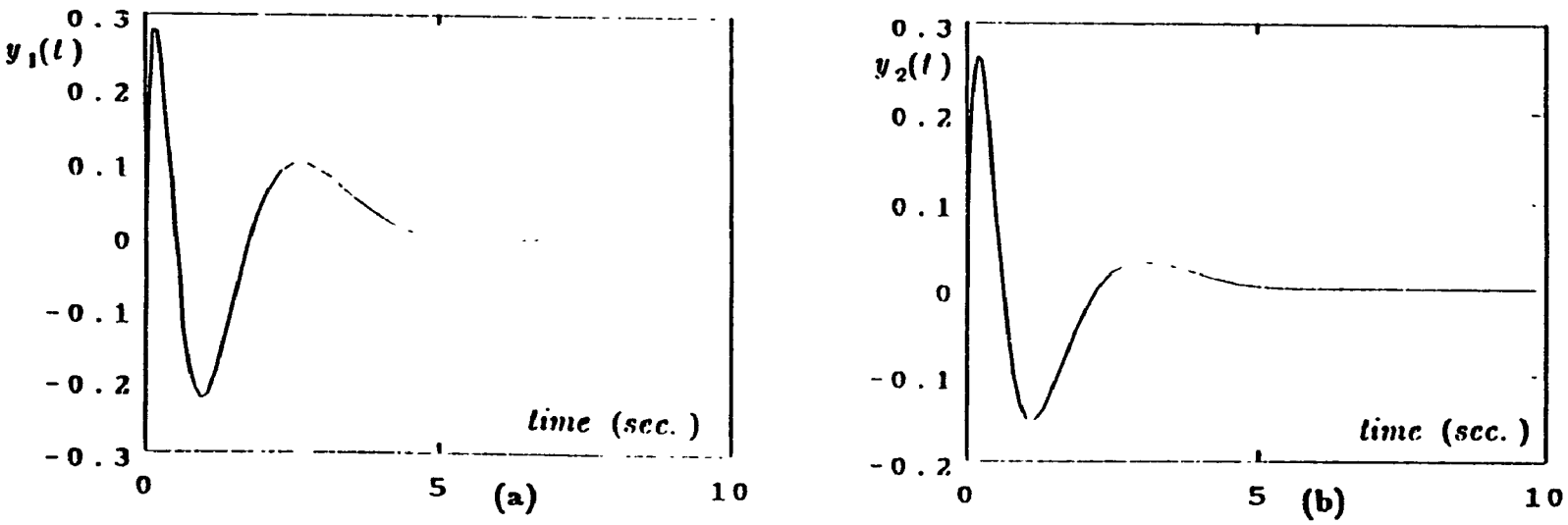

Fig.(6.28) Output responses of the closed-loop system in Example 6.3 to step disturbance.

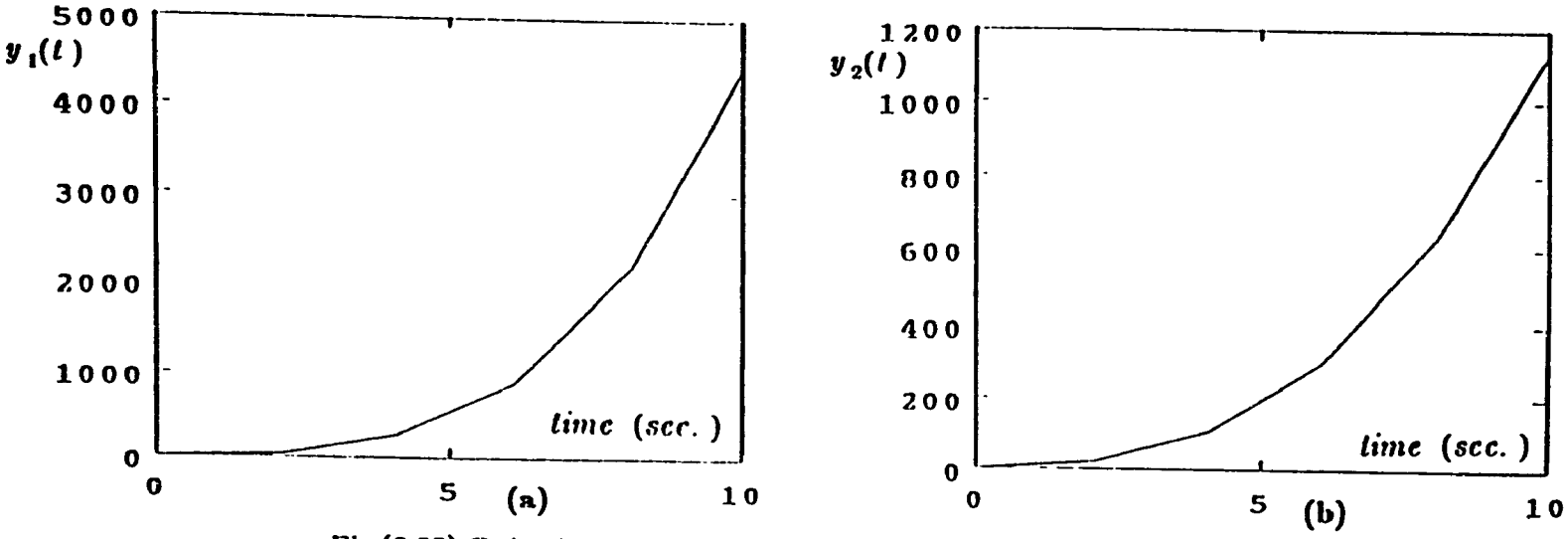

Fig.(6.29) Output responses of the open-loop system in Example 6.3 to ramp disturbance.

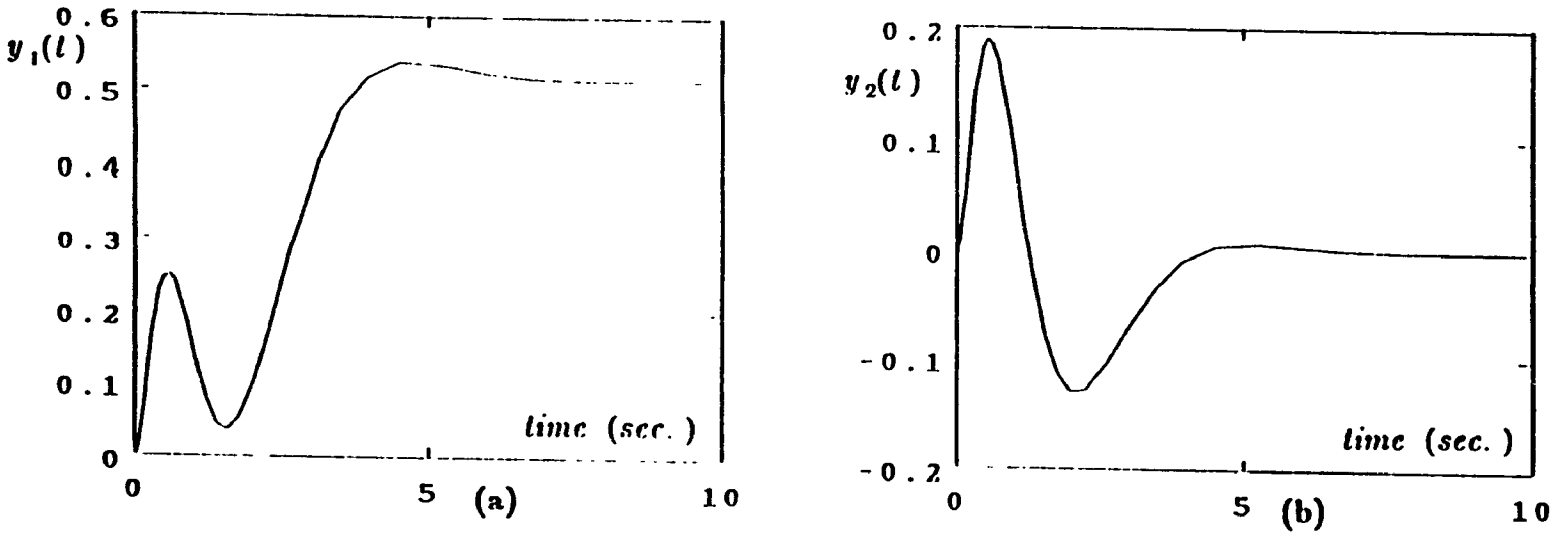

Fig.(6.30) Output responses of the closed-loop system in Example 6.3 to ramp disturbance.

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CHAPTER VII

CONCLUSIONS AND FUTURE WORK

An important problem in linear multivariable control is that of synthesizing controllers that make the outputs of a physical system respond in a desirable manner to reference inputs and disturbances. In this thesis, we have proposed one way of solving this problem. Our approach can be regarded as a logical extension of the pole assignment problem in that disturbance rejection is achieved by assignment of disturbance blocking zeros using the concepts of pole assignment.

The main contributions of the research described in this thesis have been to provide computational algorithms for designing feedback controllers that reject some or all disturbances acting on a linear time-invariant multivariable system. In addition, the resulting closed-loop system is stabilized and desired transient behaviour is obtained for the outputs by assigning the closed-loop poles at specified locations in the complex plane. We have shown that, the problem of disturbance rejection can be solved using the fact that by selecting suitable locations for the disturbance blocking zeros (d.b.z.'s) of the transfer function matrix between the outputs and the disturbances, some or all of the disturbances can be rejected at the outputs in steady state. These results have been used in Chapters IV and V to develop algorithms for designing constant or dynamic state feedback controllers to assign as many d.b.z.'s as required for disturbance rejection in a linear multivariable system described by $\sum_d \left[A, B, C, E\right]$ or $\sum_d \left[A, B, C, D, E, F\right]$. In developing the algorithms, extensive use was made of the factorization procedure for the transfer function matrix of a linear multivariable system, the concept of a minimal order inverse and the unity-rank dyadic design mechanism. A key property of disturbance zeros that was used in the algorithm is that there are zeros unaffected by output feedback and are only affected by that portion of the state feedback that is not contained in the outputs. The problem of stabilizing and/or improving the transient behaviour of the resulting closed-loop system obtained by imple-

menting state feedback was carried out by using an output feedback controller (constant or dynamic). Two alternative approaches were used to design the dynamic output feedback which assigns all closed-loop poles at or arbitrarily close to some desired values. The first approach is based on reformulation of the problem to that of the pole assignment by constant output feedback and then using the implicitly shift algorithm [1,2]. This approach is numerically reliable but does not allow for the assignment of the poles of the dynamic compensator at specified locations. To overcome this difficulty, a new approach was developed for pole assignment using dynamic output feedback compensator with prespecified poles. This approach was used later in Chapter VI to solve the problems of pole assignment as well as disturbance rejection in multivariable systems. It was shown that dynamic output feedback introduces additional d.b.z.'s located at the poles of the compensator. This important feature of designing dynamic output feedback with prespecified poles was used to solve the more general servomechanism problem [3,4].

The results presented in this thesis and related issues that may benefit from further investigation are summarized next. Algorithm 2.4 uses unity-rank dynamic compensator computed entirely in the frequency domain, by formation of a set of linear equations relating the parameters of the compensator to the desired closed-loop characteristic polynomial. The computation of the dynamic compensator via coefficients of the characteristic polynomials however may give rise to serious numerical difficulties thus resulting in unsatisfactory performance of the algorithm. An interesting problem that needs further investigation is the modification of this algorithm, such that it can be applied directly to a linear multivariable system described by its state-space equations. Such modification might result in some improvement in the numerical performance of the algorithm. Another interesting problem would be to determine how the numerical performance of the proposed algorithm for pole assignment varies with the use of different methods of computing transfer function matrices from state-space descriptions.

The assignment of d.b.z.'s for single-disturbance system by means of constant and dynamic state feedback was considered in Chapter IV and extended to system with multiple disturbances in Chapter V. In both the single and multiple disturbance cases, it was assumed that the given

open-loop has the maximum number of disturbance zeros, i.e. $n-l$ for $\sum_d \left[A, B, C, E\right]$ and n for $\sum_d \left[A, B, C, D, E, F\right]$. These assumptions were made to simplify the mathematical analysis and reduce the complexity of the proposed algorithms. We leave the problem of treating systems with fewer than the $n-l$ or n disturbance zeros as one to be investigated in the future. The proposed algorithms for assigning d.b.z.'s use unity-rank state feedback matrices. The use of such matrices may not be desirable from the numerical point of view especially for higher order systems. Better numerical performance may be achieved if the rank of the feedback matrices is not restricted to be equal to 1. In this thesis we have not concentrated specifically on developing numerically robust algorithms for d.b.z.'s assignment. Our main intention has been to show that disturbance rejection can be achieved by assigning d.b.z.'s and how this assignment problem can be solved using the techniques of pole (eigenvalue) assignment. The development of robust numerical algorithm is not a trail task and is proposed as an extensive project for future investigation.

It should be pointed out that the robustness of the assigned d.b.z.'s to uncertainties in the system model parameters has not been investigated in this thesis. We believe that it should be possible to obtain some measures of robustness of d.b.z.'s using concepts similar to those employed to study sensitivity of eigenvalues of a matrix. Such measures would allow us to incorporate such robustness features in the rejection of disturbances by assignment of d.b.z.'s using state feedback.

Finally, it is worth mentioning that the algorithms developed in this thesis are being applied to a practical problem of controlling the temperature and relative humidity for indoor environment in buildings. In a particular system of interest for example, the overall system consists of three subsystems: (i) Primary plants (boilers, chillers, heat pumps), (ii) Distribution systems (ductwork, fans), (iii) Environmental zones (interior spaces in buildings). The indoor environment control problem can be stated as the production of conditioned air in (i) distributed through

(ii) to the thermal comfort of the occupants of zone (iii). The overall system is acted upon by multiple disturbances. For example, the variations in temperature and mass flow rate of feed water to a boiler can be represented by step and exponential functions. The variations in outdoor ambient temperatures can be represented by sine/cosine functions. Also, the disturbances caused by changing occupancy patterns of spaces can be represented by step functions. Thus, by linearizing the indoor environment control problem about an operating point, state-space models with multiple disturbances can be developed as shown in [5]. To provide improved thermal comfort, it is required to design feedback controllers which will reject the multiple disturbances acting on the system. This can be easily achieved by applying the algorithms developed in this thesis.

7.1 REFERENCES

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