On existence and stability of absolutely continuous invariant measures in some chaotic dynamical systems

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ABSTRACT

On existence and stability of absolutely continuous invariant measures in some chaotic dynamical systems

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In this work, we study problems related to the existence and stability of absolutely continuous invariant measures (acim's) in chaotic dynamical systems. Since it is often necessary for a map to be eventually expanding in order to admit an acim, we start with the problem of showing eventual expansion for a family of piecewise linear maps of the unit interval. We conjecture that the piecewise linear map $f(x) = px$ for $x \in [0, 1/p)$ and $f(x) = sx - s/p$ for $x \in [1/p, 1]$, $p > 1$, $0 < s < 1$, which has an expanding, onto branch and a contracting branch, is eventually piecewise expanding. We prove this conjecture under additional assumptions on the slopes, in particular for values of p and s such that $\lceil -\frac{\ln(p(1-s)+s)}{\ln s} \rceil \neq \lceil -\frac{\ln p}{\ln s} \rceil$.

Next, we consider the problem of existence and stability of acim's for random maps with position dependent probabilities. We generalize some of the existing results in this direction by weakening the usual expansion criterion. Furthermore, we model the phenomenon of metastability by a position dependent random map

in one and higher dimensions and provide some examples.

Finally, we investigate the dependence on the parameters of acim's for a family of piecewise linear piecewise expanding maps (W-maps). We construct an example to show that the transitivity (lack of invariant intervals) of the maps does not imply the convergence of those measures to the absolutely continuous invariant measure for the limit map. We also explain how this family of maps exhibits metastable behaviour in a way that is similar to those in the existing literature.

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Introduction

In discrete dynamical systems, one is concerned with the asymptotic behaviour of trajectories under the iteration of a map. The existence of chaos (in particular, sensitivity to initial conditions) in deterministic dynamical systems makes it impossible to predict the long-term behaviour of these systems starting from a specific set of initial conditions. However it is possible to draw statistical conclusions about chaotic systems using ergodic theory. The main objects of study in this regard are invariant measures.

The following major result in ergodic theory, proved in 1931 by G. D. Birkhoff (see [9]), shows the importance of invariant measures in studying chaotic systems.

Theorem 0.1 (Birkhoff)**.** *Suppose* $\tau\colon (X,\mathcal{B},\mu) \to \tau(X,\mathcal{B},\mu)$ *is measure preserving, where* $\tau(X,\mathcal{B},\mu)$ *is* σ*-finite, and* $f \in L^1(\mu)$*. Then there exists a function* $f^* \in L^1$ *such that*

$$\frac{1}{n}\sum_{k=1}^{n-1} f(\tau^k(x)) \to f^*,\ \mu\text{-a.e.} \tag{1}$$

Furthermore, $f^* \circ \tau = f^*$ μ*-a.e. and if* $\mu(X) < \infty$*, then* $\int_X f^* d\mu = \int_X f d\mu$*.*

The importance of Birkhoff's Ergodic Theorem becomes more clear by the following corollary. It states that if μ is invariant under τ, then most initial conditions (according to μ) visit a given set E with asymptotic relative frequency equal to $\mu(E)$.

Corollary 0.1. *If τ is ergodic, then f^* is constant μ-a.e. and if $\mu(X) = 1$, then*

$$\frac{1}{n}\sum_{k=1}^{n-1} \chi_E(\tau^k(x)) \to \mu(E), \ \mu\text{-}a.e. \tag{2}$$

Invariant measures that are absolutely continuous with respect to the Lebesgue measure (referred to as acim's) often play a more important role in practice because they are consistent with our notion of length (area, volume in higher dimensions). Also, computer simulations of invariant measures only reveal the absolutely continuous invariant measures in general (for an explanation see [10], p.104).

Examples of acim's were known to Ulam and von Neumann (see [33]). Rényi was the first person to define a class of maps with an acim (see [31]) and his idea of using distortion estimates were used in more general works such as [1].

In 1973, Lasota and Yorke (see [25]) used bounded variation tools to prove the existence of acim's for piecewise monotonic and expanding maps of the interval. Since then, the study of acim's and their ergodic properties has been an active area of research. There have been numerous works on the existence of invariant measures for different classes of dynamical systems most of which use generalizations of the bounded variation technique of Lasota and Yorke. The existence of acim's and their properties is also well-known for Markov transformations (see e.g. [6]) or specific classes of maps such as the logistic family (see e.g. [27]). Except in rather special situations the techniques used in this field are refinements and generalizations of the techniques in the aforementioned works.

We remark that general results regarding the existence of acim's require the class of maps under consideration to satisfy a condition of expansion and a condition of regularity. In fact, to show the importance of the expansion condition, Lasota and Yorke (in [25]) constructed an example of a map on the interval that is expanding everywhere except at a single point but it does not admit a finite acim.

It often suffices to assume eventual expansion (rather than expansion of the first iterate) to show the existence of an acim. For piecewise linear and eventually expanding maps of the interval one can even express the invariant density of acim's in terms of an explicit formula (see [19]).

We show in Chapter 2[1] that even in the simple case where the map is piecewise linear and consists of an expanding branch and a contracting branch it is nontrivial to show eventual expansion. We show under additional assumptions that such maps are eventually expanding and hence admit an acim. This problem is of historical importance and has been studied in different forms, for example in [7] and [22]. It also appears in the context of ergodic theory of numbers (see for example [12]), and in the study of Lorenz-like maps [13].

We should mention that we recently learned about the main result of preprint [13] in which the authors prove the existence of an acim for a three-parameter family of piecewise linear Lorenz-like maps $f_{a,b,c}\colon [0,1] \to [0,1]$ defined as follows.

$$f_{a,b,c}(x) = \begin{cases} ax + 1 - ac, & 0 \leq x < c, \\ b(x - c), & c < x \leq 1, \end{cases} \tag{3}$$

where $ac + b(1-c) \geq 1$. They prove in particular that if $f_{a,b,c}(0) < f_{a,b,c}(1)$, then $f_{a,b,c}$ is eventually expanding. Since the family of maps that we consider

[1] This chapter, with minor modifications, has been published in the 2011 Aug/Sep issue of the American Mathematical Monthly (see [16]).

in Chapter 2 is contained in the $\{f_{a,b,c}\}$, it follows that our family is eventually expanding and Conjecture 2.1 is indeed true.

Besides the problem of existence of acims, a central problem in measure-theoretic dynamical systems is the stability of acim's with respect to a deterministic perturbation. In Chapter 3 and Chapter 4 we investigate the stability of acims for random maps with position dependent probabilities and for a class of continuous maps of the interval. In Chapter 3 we prove the stability of acim's of random maps in one and multi-dimensional settings under weak expansion and regularity conditions (conditions (A) and (B) or condition(C); see Chapter 3). Also, we use this result to model metastable behaviour with random maps. A metastable system is often created by a perturbation of a system with two or more invariant components. The perturbation is such that the trajectories can move from one component to the other through "holes".

Tokman et al. modeled metastability in [32] using deterministic maps. They considered a dynamical system consisting of two disjoint invariant sets which is perturbed so that trajectories can switch from one component to the other through holes. They showed, under some conditions (see [32]), that the acim's of the perturbded system converge to a convex combination of the acim's of the original map with weights proportional to the size of the holes.

In our random map model of metastability, we consider acim's of perturbations as the probabilities of escape through holes approach zero. In this way we do not need as many conditions on the system and our results hold in higher dimensional settings. In the setting of random maps, we show that the acim's of perturbations converge to a convex combination of the acim's of the original map with weights proportional to the probability of escape through the holes.

Metastability also appears in continuous dynamical systems such as the Lorenz-Model (see [34]).

In Chapter 4[2], we study the stability of acim's of a class of piecewise expanding piecewise linear transformations, called W-maps, under deterministic perturbations. One is often interested to know whether acim's of perturbations converge in the weak-* topology to the acim of the unperturbed transformation as the size of the perturbation approaches zero. If this property holds, the system is called acim-stable. One of the earliest and most general results on acim-stability appeared in the work of G. Keller (see [23]). He proved that piecewise monotonic and expanding maps satisfying a uniform Lasota-Yorke type inequality are acim-stable. Keller also provided an example of an acim-singular (not acim-stable) class of maps of the interval in which acim's of perturbations converge to a point measure. It was conjectured that the only way such an acim-singularity can occur for a continuous map of the interval with a fixed turning point is if small neighbourhoods of this turning point are invariant under the perturbations.

We provide a counter-example to this conjecture by constructing a three-parameter family of transitive W-maps whose acim's converge to a convex combination of the point measure at the fixed turning point and the acim of the unperturbed map.

Our family of W-maps also exhibits metastable behaviour in the following sense. As the original W-map is perturbed, the trajectory of almost every point spends a long time in a certain box (as defined in Chapter 4) around the turning point, switching between this box and its complement every now and then. In this way, our results regarding W-maps complements the metastability results of

[2]This chapter, with minor modifications, has been published in the Journal of Difference Equations and Applications (see [17])

Tokman et al., in the presence of a fixed turning point.

Chapter 1

Preliminaries

In this chapter we recall some standard definitions and results in ergodic and dynamical systems. Most of the material, including proofs, can be found in [9].

1.1 Review of measure theory

Let X be a set. In most cases we will assume that X is a compact metric space.

Definition 1.1. *A family $\mathcal{B}$ of subsets of X is called a* <u>*σ-algebra*</u> *if and only if:*

1. *$X \in \mathcal{B}$;*

2. *for any $B \in \mathcal{B}$, $X \setminus B \in \mathcal{B}$;*

3. *if $B_n \in \mathcal{B}$, for $n \in \mathbb{N}$, then $\cup_{n=1}^{\infty} B_n \in \mathcal{B}$.*

Elements of $\mathcal{B}$ are called *measurable sets*.

Definition 1.2. *A function $\mu\colon \mathcal{B} \to \mathbb{R}^+$ is called a* <u>*measure*</u> *on $\mathcal{B}$ if and only if:*

1. *$\mu(\emptyset) = 0$;*

2. *for any sequence* $\{B_n\}$ *of disjoint measurable sets,* $B_n \in \mathcal{B}$*,* $n \in \mathbb{N}$*,*

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n).$$

The triple $(X, \mathcal{B}, \mu)$ is called a *measure space.* If $\mu(X) = 1$, we say it is a *normalized measure space* or a *probability space.*

Definition 1.3. *A family* $\mathcal{A}$ *of subsets of* X *is called an* <u>*algebra*</u> *if:*

1. $X \in \mathcal{A}$*;*
2. *for any* $A \in \mathcal{A}$*,* $X \setminus A \in \mathcal{A}$*;*
3. *for any* A_1*,* $A_2 \in \mathcal{A}$*,* $A_1 \cup A_2 \in \mathcal{A}$*.*

For any family $\mathcal{J}$ of subsets of X there exists a smallest σ-algebra, $\mathcal{B}$, containing $\mathcal{J}$. We say that $\mathcal{J}$ generates $\mathcal{B}$ and write $\mathcal{B} = \sigma(\mathcal{J})$.

Theorem 1.1. *Given a set* X *and an algebra* $\mathcal{A}$ *of subsets of* X*, let* $\mu_1 \colon \mathcal{A} \to \mathbb{R}^+$ *be a function satisfying* $\mu_1(X)$ *be a function satisfying* $\mu_1(X) = 1$ *and*

$$\mu_1\left(\bigcup_n A_n\right) = \sum_n \mu_1(A_n)$$

whenever $A_n \in \mathcal{A}$*, for* $n = 1, 2, \ldots$*,* $\cup_{n=1}^{\infty} A_n \in \mathcal{A}$ *and* $\{A_n\}$ *disjoint. Then there exists a unique normalized measure* μ *on* $\mathcal{B} = \sigma(\mathcal{A})$ *such that* $\mu(A) = \mu_1(A)$ *whenever* $A \in \mathcal{A}$*.*

Definition 1.4. *Let* X *be a topological space. Then the smallest* σ*-algebra containing all open subsets of* X *is called the* <u>*Borel* σ*-algebra*</u> *of* X *and its elements,* <u>*Borel subsets*</u> *of* X*.*

Definition 1.5. *Let $(X, \mathcal{B}, \mu)$ be a measure space. The function $f\colon X \to \mathbb{R}$ is said to be* <u>*measurable*</u> *if for all $c \in \mathbb{R}$, $f^{-1}(c, \infty) \in \mathcal{B}$, or, equivalently, if $f^{-1}(A) \in \mathcal{B}$ for any Borel set $A \in \mathbb{R}$.*

If X is a topological space and $\mathcal{B}$ is the σ-algebra of Borel subsets of X, then each continuous function $f\colon X \to \mathbb{R}$ is measurable.

Definition 1.6. *Let $\mathcal{B}_n$ be a σ-algebra of Borel subsets of X, $n \in \mathbb{N}$. Let $n_1 < n_2 < \ldots n_r$ be integers and $A_{n_i} \in \mathcal{B}_{n_i}$, $i = 1, 2, \ldots, r$. We define a* <u>*cylinder*</u> *to be a set of the form*

$$C(A_{n_1}, \ldots, A_{n_r}) = \{(x_1, x_2, \ldots) \in X^{\mathbb{N}} : x_{n_i} \in A_{n_i}, 1 \leq i \leq r\}.$$

Definition 1.7. *Let $(X_i, \mathcal{B}_i, \mu_i)$, $i \in \mathbb{N}$ be normalized measure spaces. The* <u>*direct product*</u> *measure space $(X, \mathcal{B}, \mu) = \Pi_{i=1}^{\infty}(X_i, B_i, \mu_i)$ is defined by*

$$X = \Pi_{i=1}^{\infty} X_i \textit{ and } \mu(C(A_{n_1}, \ldots, A_{n_r})) = \Pi_{i=1}^{r} \mu_{n_i}(A_{n_i}).$$

It is easy to see that finite unions of cylinders form an algebra of subsets of X. By Theorem 1.1 it can be uniquely extended to a measure on $\mathcal{B}$, the smallest σ-algebra containing all cylinders.

1.2 Spaces of Functions and Measures

Let $\mathcal{F}$ be a linear space. A function $\|\cdot\|\colon \mathcal{F} \to \mathbb{R}^+$ is called a *norm* if it has the following properties:

1. $\|f\| = 0 \Longleftrightarrow f \equiv 0$

2. $\|\alpha f\| = |\alpha| \|f\|$

3. $\|f + g\| \leq \|f\| + \|g\|$,

for $f, g \in \mathcal{F}$ and $\alpha \in \mathbb{R}$. The space $\mathcal{F}$ endowed with a norm $\|\cdot\|$ is called a *normed linear space.*

Definition 1.8. *A sequence $\{f_n\}$ in a normed linear space is a* <u>*Cauchy sequence*</u> *if for any $\epsilon > 0$, there exists an $N \geq 1$ such that for any $n, m \geq N$,*

$$\|f_n - f_m\| < \epsilon.$$

Every convergent sequence is a Cauchy sequence.

Definition 1.9. *A normed linear space $\mathcal{F}$ is complete if every Cauchy sequence converges, i.e., if for each Cauchy sequence $\{f_n\}$ there exists $f \in \mathcal{F}$ such that $f_n \to f$. A complete normed space is called a* <u>*Banach space*</u>*.*

Let $(X, \mathcal{B}, \mu)$ be a normalized measure space.

Definition 1.10. *Let $1 \leq p < \infty$. The family of real-valued measurabale functions (or rather a.e.-equivalence classes of them) $f \colon X \to \mathbb{R}$ satisfying*

$$\int_X |f(x)|^p d\mu < \infty \tag{1.1}$$

is called the $L^p(X, \mathcal{B}, \mu)$ space and is denoted by $L^p(\mu)$ when the underlying space is clearly known, and by L^p where both the space and the measure are known.

The integral in (1.1) is assigned a special notation

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu \right)^{\frac{1}{p}},$$

and is called the L^p *norm* of f. L^p with the norm $\|\cdot\|_p$ is a complete normed space, i.e., a Banach space.

The space of almost everywhere bounded measurable functions on $(X, \mathcal{B}, \mu)$ is denoted by L^∞. Functions that differ only on a set of μ-measure 0 are considered to represent the same element of L^∞. The L^∞ norm is given by

$$\|f\|_\infty = \operatorname{ess\,sup} |f(x)| = \inf\{M : \mu\{x : f(x) > M\} = 0\}.$$

The space L^∞ with the norm $\|\cdot\|_\infty$ is a Banach space.

Definition 1.11. *The space of bounded linear functionals on a normed space $\mathcal{F}$ is called the adjoint space of $\mathcal{F}$ and is denoted by $\mathcal{F}^*$. The weak convergence in $\mathcal{F}$ is defined as follows: a sequence $\{f_n\}_1^\infty \subset \mathcal{F}$ converges weakly to an $f \in \mathcal{F}$ if and only if for any $F \in \mathcal{F}^*$, $F(f_n) \to F(f)$ as $n \to \infty$. Similarly, a sequence of functionals $\{F_n\}_1^\infty \subset \mathcal{F}^*$ converges in the weak-* topology to a functional $F \in \mathcal{F}^*$ if and only if for any $f \in \mathcal{F}$, $F_n(f) \to F(f)$ as $n \to \infty$.*

Theorem 1.2. *Let $1 \le p < \infty$ and let q satisfy*

$$\frac{1}{p} + \frac{1}{q} = 1, \ (\frac{1}{\infty} = 0).$$

Then L^q is the adjoint space of L^p.

If $f \in L^p$, $g \in L^q$, then fg is integrable and the Hölder inequality holds:

$$\int_X |fg| d\mu \le \|f\|_p \|g\|_q.$$

Let $g \in L^q$. We define a functional F_g on L^p by setting

$$F_g(f) = \int_X fgd\mu$$

$$\|F_g\| = \sup_{f \neq 0} \left\{ \frac{|F_g(f)|}{\|f\|} \right\}.$$

Clearly, F_g is linear.

Proposition 1.1. *Each function $g \in L^q$ defines a bounded linear functional F_g on L^p with $F_g(f) = \int_X fgd\mu$ and $\|F_g\| = \|g\|_q$.*

Theorem 1.3 (Riesz Representation Theorem [14])**.** *Let F be a bounded linear functional on L^p, $1 \le p < \infty$. Then there exists a function g in L^q such that*

$$F(f) = \int_X fgd\mu.$$

Furthermore, $\|F\| = \|g\|_q$.

We will use the following types of convergence in L^p spaces.

1. Norm (or strong) convergence:

$$f_n \to f \text{ in } L^p\text{-norm} \iff \|f_n - f\|_p \to 0, n \to \infty.$$

2. Weak convergence: $f_n \to f$ weakly in L^p, $1 \le p < \infty \iff$

$$\forall g \in L^q,\ \int f_n g d\mu \to \int fgd\mu, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

3. Pointwise convergence:

$$f_n \to f \text{ almost everywhere (a.e.) } \iff f_n(x) \to f(x)$$

for almost every $x \in X$.

The following results give several characterizations of these types of convergence and connections between them:

Theorem 1.4. *Let a sequence $\{f_n\}_{n=1}^{\infty}$, $f_n \in L^1$, $n = 1, 2, \ldots$ satisfy*

1. $\|f_n\|_1 \leq M$ *for some* M*;*

2. $\forall \epsilon > 0\ \exists \delta > 0$ *such that for any* $A \in \mathcal{B}$*, if* $\mu(A) < \delta$ *then for all* n*,*

$$\left| \int_A f_n d\mu \right| < \epsilon.$$

Then, $\{f_n\}$ contains a weakly convergent subsequence, i.e., $\{f_n\}$ is weakly compact.

Corollary 1.1. *If there exists $g \in L^1$ such that $f_n \leq g$ for $n = 1, 2, \ldots$, then $\{f_n\}$ is weakly compact.*

Theorem 1.5 (Scheffé's Theorem [6])**.** *If $f_n \geq 0$, $\int f_n d\mu = 1$, $n = 1, 2, \ldots$ and $f_n \to f$ a.e. with $\int f d\mu = 1$, then $f_n \to f$ in L^1-norm.*

Theorem 1.6. *If $f_n \to f$ weakly in L^1 and almost everywhere, then $f_n \to f$ in L^1-norm.*

We now consider spaces of continuous and differentiable functions. Let X be a compact metric space.

Definition 1.12. $C^0(X) = C(X)$ *is the space of all continuous real functions* $f\colon X \to \mathbb{R}$, *with the norm*

$$\|f\|_{C^0} = \sup_{x \in X} |f(x)|.$$

Definition 1.13. $\mathcal{M}(X)$ *denotes the space of all measures* μ *on* $\mathcal{B}(X)$. *The norm, called the* *<u>total variation norm</u>* *on* $\mathcal{M}(X)$, *is defined by*

$$\|\mu\| = \sup_{A_1 \cup \cdots \cup A_N = X} \{|\mu(A_1)| + \cdots + |\mu(A_N)|\},$$

where the supremum is taken over all finite partitions of X.

A more frequently used topology on $\mathcal{M}(X)$ is the *weak topology of measures*, which we can define with the aid of the following result [14]:

Theorem 1.7. *Let* X *be a compact metric space. Then the adjoint space of* $C(X)$, $C^*(X)$, *is* $\mathcal{M}(X)$.

Definition 1.14. *The* *<u>weak topology of measures</u>* *is a topology of weak convergence on* $\mathcal{M}(X)$, *i.e.,*

$$\mu_n \to \mu \Longleftrightarrow \int_x g d\mu_n \to \int_x g d\mu, \text{ for any } g \in C(X).$$

In view of Theorem 1.7 this is sometimes referred to as the topology of weak-* convergence.

Theorem 1.8. *The weak topology of measures is metrizable and any bounded (in norm) subset of* $\mathcal{M}(X)$ *is compact in the weak topology of measures.*

We now present two important corollaries of Theorem 1.7.

Corollary 1.2. *Two measures μ_1 and μ_2 are identical if and only if*

$$\int_X g d\mu_1 = \int_X g d\mu_2$$

for all $g \in C(X)$.

Corollary 1.3. *The set of probability measures is compact in the weak topology of measures.*

For excellent accounts on the weak topology of measures, the reader is referred to [5] and [29].

Definition 1.15. *Let ν and μ be two measures on the same measurable space $(X, \mathcal{B})$. We say that ν is* <u>*absolutely continuous*</u> *with respect to μ if for any $A \in \mathcal{B}$, such that $\mu(A) = 0$, it follows that $\nu(A) = 0$. We write $\nu << \mu$.*

A useful condition for testing absolute continuity is given by

Theorem 1.9. *$\nu << \mu$ if and only if given $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\nu(A) < \epsilon$.*

The proof of this theorem can be found in [14].

If $\nu << \mu$, then it is possible to represent ν in terms of μ. This is the essence of the Radon-Nikodym Theorem (Theorem 1.10).

Theorem 1.10 (Radon-Nikodym)**.** *Let $(X, \mathcal{B})$ be a space and let ν and μ be two normalized measures on $(X, \mathcal{B})$. If $\nu << \mu$, then there exists a unique $f \in L^1(X, \mathcal{B}, \mu)$ such that for every $A \in \mathcal{B}$,*

$$\nu(A) = \int_A f d\mu.$$

f is called the Radon-Nikodym derivative and is denoted by $\frac{d\nu}{d\mu}$.

Definition 1.16. *Let X be a compact metric space and let μ be a measure on $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of subsets of X. We define the support of μ as the smallest closed set of full μ measure, i.e.,*

$$supp(\mu) = X \setminus \bigcup_{\substack{O\text{-}open \\ \mu(O)=0}} O.$$

It is worth noting that two mutually singular measures may have the same support.

Let $\mathcal{M}(X)$ denote the space of measures on $(X, \mathcal{B})$. Let $\tau \colon X \to X$ be a measurable transformation (i.e., $\tau^{-1}(A) \in \mathcal{B}$ for $A \in \mathcal{B}$). τ induces a transformation τ_* on $\mathcal{M}(X)$ by means of the definition: $(\tau_*\mu)(A) = \mu(\tau^{-1}A)$. Since τ is measurable, it is easy to see that $\tau_*\mu \in \mathcal{M}(X)$. Hence, τ_* is well-defined.

Definition 1.17. *Let $(X, \mathcal{B}, \mu)$ be a normalized measure space. Then $\tau \colon X \to X$ is said to be nonsingular if and only if $\tau_*\mu << \mu$, i.e., if for any $A \in \mathcal{B}$ such that $\mu(A) = 0$, we have $\tau_*\mu(A) = \mu(\tau^{-1}A) = 0$.*

Proposition 1.2. *Let $(X, \mathcal{B}, \mu)$ be a normalized measure space, and let $\tau \colon X \to X$ be nonsingular. Then, if $\nu << \mu$, $\tau_*\nu << \tau_*\mu << \mu$.*

Proof. Since $\nu << \mu$, $\mu(A) = 0 \Rightarrow \nu(A) = 0$. Since τ is nonsingular, $\mu(A) = 0 \Rightarrow \mu(\tau^{-1}A) = 0 \Rightarrow \nu(\tau^{-1}A) = 0$. Thus, $\tau_*\nu << \tau_*\mu$. Since τ is nonsingular, we obtain $\tau_*\mu << \mu$. $\square$

Definition 1.18. *Let $(X, \mathcal{B}, \mu)$ be a normalized measure space. Let*

$$\mathcal{D} = \mathcal{D}(X, \mathcal{B}, \mu) = \{f \in L^1(X, \mathcal{B}, \mu) : f \geq 0 \text{ and } \|f\|_1 = 1\}$$

denote the space of <u>*probability density*</u> *functions. A function* $f \in \mathcal{D}$ *is called a density function or simply a density.*

If $f \in \mathcal{D}$, then

$$\mu_f(A) = \int_A f d\mu << \mu$$

is a measure and f is called the *density of* μ_f and is written as $d\mu_f/d\mu$.

1.3 Functions of bounded variation

Let $[a, b] \subset \mathbb{R}$ be a bounded interval and let λ denote Lebesgue measure on $[a, b]$. For any sequence of points $a = x_0 < x_1 < \cdots < x_n = b$, $n \geq 1$, we define a partition $\mathcal{P} = \{I_i = [x_{i-1}, x_i) : i = 1, \ldots, n\}$ of $[a, b]$. The points $\{x_0, x_1, \ldots, x_n\}$ are called *end-points of the partition* $\mathcal{P}$. Sometimes we will write $\mathcal{P} = \mathcal{P}\{x_0, x_1, \ldots, x_n\}$.

Definition 1.19. *Let* $f \colon [a, b] \to \mathbb{R}$ *and let* $\mathcal{P} = \mathcal{P}\{x_0, x_1, \ldots, x_n\}$ *be a parition of* $[a, b]$*. If there exists a positive number* M *such that*

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M$$

for all partitions $\mathcal{P}$*, then* f *is said to be of* <u>*bounded variation*</u> *on* $[a, b]$*.*

Definition 1.20. *Let* $f \colon [a, b] \to \mathbb{R}$ *be a function of bounded variation. the number*

$$V_{[a,b]} f = \sup_{\mathcal{P}} \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}$$

is called the total variation or, simply, the <u>*variation*</u> *of* f *on* $[a, b]$*.*

Theorem 1.11. *Let $f\colon [a,b] \to \mathbb{R}$ have a continuous derivative f' on $[a,b]$. Then*

$$V_{[a,b]} f = \int_a^b |f'(x)| d\lambda(x).$$

Let us define the indefinite integral $\int(\Phi)$ of a function $\Phi \in L^1$ by

$$\int(\Phi)(y) = \int_{x \leq y} \Phi(x) d\lambda(x).$$

Theorem 1.12. *For $f \in L^1$,*

$$V(f) = \sup_{\Phi} |\int f \Phi d\lambda|,$$

where the supremum extends over all $\Phi \in L^1$ with $\| \int(\Phi) \|_\infty \leq 1$ and $\int \Phi d\lambda = 0$.

1.4 Perron-Frobenius Operator

Definition 1.21. *Let $I = [a,b]$, $\mathcal{B}$ be the Borel σ-algebra of subsets of I and let λ denote the normalized Lebesgue measure on I. Let $\tau\colon I \to I$ be a nonsingular transformation. We define the Frobenius-Perron operator $P_\tau\colon L^1 \to L^1$ as follows: For any $f \in L^1$, $P_\tau f$ is the unique (up to a.e. equivalence) function in L^1 such that*

$$\int_A P_\tau f = \int_{\tau^{-1}A} f d\lambda$$

for any $A \in \mathcal{B}$.

The validity of this definition, i.e., the existence and the uniqueness of $P_\tau f$, follows by the Radon-Nikodym Theorem.

Proposition 1.3. *$P_\tau \colon L^1 \to L^1$ enjoys the following properties:*

1. *(Linearity) $\forall f, g \in L^1$ and $\alpha, \beta \in \mathbb{R}$, $P_\tau(\alpha f + \beta g) = \alpha P_\tau f + \beta P_\tau g$ a.e.*

2. *(Positivity) If $f \in L^1$ and $f \geq 0$, then $P_\tau f \geq 0$.*

3. *(Preservation of Integrals) $\int_I P_\tau f d\lambda = \int_I f d\lambda$.*

4. *(Contraction property) $\forall f \in L^1$, $\|P_\tau f\| \leq \|f\|$.*

5. *(Composition property) If $\tau, \sigma \colon I \to I$ are nonsingular, then $P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f$. In particular, $P_{\tau^n} f = P_\tau^n f$.*

6. *(Adjoint property) If $f \in L^1$ and $g \in L^\infty$, then*

$$\int_I (P_\tau f) \cdot g d\lambda = \int_I f \cdot U_\tau g d\lambda,$$

where $U_\tau \colon L^\infty \to L^\infty$ is called the Koopman operator and is defined by $U_\tau g = g \circ \tau$.

The following proposition shows the connection between fixed points of P_τ and τ-invariant measures.

Proposition 1.4. *Let $\tau \colon I \to I$ be nonsingular. Then $P_\tau f^* = f^*$ a.e., if and only if the measure $\mu = f^* \cdot \lambda$, defined by $\mu(A) = \int_A f^* d\lambda$, is τ-invariant, i.e., if and only if $\mu(\tau^{-1}A) = \mu(A)$ for all measurable sets A, where $f^* \geq 0$, $f^* \in L^1$ and $\|f^*\|_1 = 1$.*

There is an extremely useful representation for the Frobenius-Perron operator for a large class of one-dimensional transformations .

Definition 1.22. *Let $I = [a, b]$. The transformation $\tau\colon I \to I$ is called* <u>*piecewise monotonic*</u> *if there exists a partition of I, $a = a_0 < a_1 < \cdots < a_q = b$, and a number $r \geq 1$ such that*

1. *$\tau|_{(a_{i-1}, a_i)}$ is a C^r function, $i = 1, \ldots, q$ which can be extended to a C^r function on $[a_{i-1}, a_i]$, $i = 1, \ldots, q$, and*

2. *$|\tau'(x)| > 0$ on (a_{i-1}, a_i), $i = 1, \ldots, q$.*

If in addition, $|\tau'(x)| \geq \alpha > 1$ whenever the derivative exists, then τ is called piecewise monotonic and expanding.

Proposition 1.5. *If $\tau\colon I \to I$ is piecewise monotonic, then*

$$P_\tau f(x) = \sum_{y \in \{\tau^{-1}(x)\}} \frac{f(y)}{|\tau'(y)|} = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{\tau'(\tau_i^{-1}(x))} \chi_{\tau(a_{i-1}, a_i)}(x).$$

1.5 Some theorems on the existence of acim's

We consider the interval $I = [a, b]$ with normalized Lebesgue measure λ on I. Let $\mathcal{T}(I)$ denote the class of transformations $\tau\colon I \to I$ that satisfy the following conditions

1. τ is piecewise monotonic and expanding, i.e., there exists a partition $\mathcal{P} = \{I_i = [a_{i-1}, a_i], i = 1, \ldots, q\}$ of I such that $\tau|_{I_i}$ is C^1 and $|\tau_i'(x)| \geq \alpha > 1$ for any i and for all $x \in (a_{i-1}, a_i)$;

2. $g(x) \equiv \frac{1}{\tau'(x)}$ is a function of bounded variation, where $\tau'(x)$ is the appropriate one-sided derivative at the endpoints of $\mathcal{P}$.

Theorem 1.13 (Lasota-Yorke). *Let $\tau \in \mathcal{T}(I)$. Then it admits an absolutely continuous invariant measure whose density is of bounded variation.*

Theorem 1.14 (Folklore Theorem). *Suppose τ is piecewise monotonic and expanding and satisfies the following conditions.*

1. *(smoothness) For each $i = 1, 2, \ldots, N$, $\tau|_i$ has a C^2-extension to the closure of I_i, $\bar{I}_i$.*

2. *(local invertibility) For each $i = 1, 2, \ldots, N$, τ is strictly monotone on $\bar{I}_i$ and therefore determines a 1-to-1 mapping of $\bar{I}_i$ onto some closed subinterval $\tau(\bar{I}_i)$ of I.*

3. *(Markov property) For each $J \in \mathcal{P}$, there is a subset $\mathcal{P}(J)$ of $\mathcal{P}$ such that $\tau(J) = \bigcup\{\bar{K} : K \in \mathcal{P}(J)\}$.*

4. *(aperiodicity) for each $J \in \mathcal{P}$, there exists a positive integer q such that $\tau^q(\bar{J}) = \bar{I}$.*

Then it has an ergodic and hence unique (actually exact) invariant probability measure μ equivalent to λ with density function $d\mu/d\lambda$ which can be chosen as a piecewise continuous function with the discontinuities only at endpoints of intervals in $\mathcal{P}$, and satisfying

$$1/M \leq \frac{d\mu}{d\lambda} \leq M$$

for some $M > 0$.

1.6 Functions of bounded variation and existence results in $\mathbb{R}^N$

The main tool in proving the existence of acim's in $\mathbb{R}^N$ is the multidimensional notion of variation defined using derivatives in the distributional sense (see [14]):

Definition 1.23. *Let $f \in L_1(\mathbb{R}^N)$ with bounded support. The* <u>*total variation*</u> *of f is defined by*

$$V(f) = \int_{\mathbb{R}^N} \|Df\| d\lambda_N = \sup \left\{ \int_{\mathbb{R}^N} f\, div(g) d\lambda_N : g = (g_1, ..., g_N) \in C_0^1(\mathbb{R}^N, \mathbb{R}^N) \right\},$$

where Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^N, \mathbb{R}^N)$ is the space of continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}^N$ having compact support.

We will use the following property of variation which is derived from [14], Remark 2.14:

Proposition 1.6. *If $f = 0$ outside a closed domain A whose boundary, ∂A, is Lipschitz continuous, $f|_A$ is continuous, $f|_{int(A)}$ is C^1, then*

$$V(f) = \int_{int(A)} \|Df\| d\lambda_N + \int_{\partial A} |f| d\lambda_{N-1},$$

where λ_{N-1} is the $(N-1)$-dimensional measure on the boundary of A.

In the multidimensional setting we shall always consider the Banach space (see [14], Remark 1.12),

$$BV(S) = \{f \in L_1(S) : V(f) < +\infty\},$$

with the norm $\|f\|_{BV} = V(f) + \|f\|_1$.

Now we describe the setting for Theorem 1.15 showing the existence of acim's in the multidimensional case [21].

Let S be a bounded region in $\mathbb{R}^N$ and let λ_N be the Lebesgue measure on S. Let $\tau\colon S \to S$ be a piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of S : $\mathcal{P} = \{S_1, ..., S_q\}$ and $\tau_i = \tau|_{S_i}$, $i = 1, ..., q$. Let each S_i be a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure. We assume that the faces of ∂S_i meet at angles bounded uniformly away from 0. Let $D\tau_i^{-1}(x)$ be the derivative matrix of τ_i^{-1} at x.

Fix $1 \le i \le q$. Let Z denote the set of singular points of ∂S_i. Let us construct for an $x \in Z$ the largest cone with vertex at x and which lies completely in S_i. Let $\theta(x)$ denote the vertex angle of this cone. Then define

$$\beta(S_i) = \min_{x \in Z} \theta(x).$$

Since the faces of ∂S_i meet at angles bounded away from 0, we have $\beta(S_i) > 0$. Let $\alpha(S_i) = \pi/2 + \beta(S_i)$ and

$$a(S_i) = |\cos(\alpha(S_i))|.$$

Now we will construct a C^1 field of segments L_y, $y \in \partial S_i$, every L_y being a central ray of a regular cone contained in S_i, with vertex angle at y greater than or equal to $\beta(S_i)$.

We start at points $y \in Z$ where the minimal angle $\beta(S_i)$ is attained, defining L_y to be central rays of the largest regular cones contained in S_i. Then we extend this field of segments to the C^1 field we want, making L_y short enough to avoid overlapping. Let $\delta(y)$ be the length of L_y, $y \in \partial S_i$. By compactness of ∂S_i we have

$$\delta(S_i) = \inf_{y \in \partial S_i} \delta(y) > 0.$$

Now, we shorten the L_y of our field, making them all of length $\delta(S_i)$.

Suppose there exists $0 < \sigma < 1$ such that for all $i = 1, \dots, q$,

$$\|D\tau^{-1}\| < \sigma.$$

We have the following theorem from [21]:

Theorem 1.15. *Let $\tau \colon S \to S$, $S \subset \mathbb{R}^N$ be a piecewise C^2 expanding transformation. If $\sigma(1 + 1/a) < 1$, then τ admits an absolutely continuous invariant measure.*

Chapter 2

On piecewise expanding maps of the interval

A piecewise differentiable function f is expanding if $|f'(x)| > 1$ for all x at which the derivative exists. f is said to be eventually expanding if there exists $N \in \mathbb{N}$ such that f^N (the N-fold composition of f with itself) is expanding.

Eventually expanding maps play an important role in dynamical systems theory. For example, most theorems on existence of absolutely continuous invariant measures require the map to be expanding or eventually expanding. Very often proofs for general maps are reduced to the eventually expanding situation. However, showing that a map is eventually expanding is far from trivial. As a simple example, let f be a piecewise linear function on the unit interval $[0, 1]$ with two increasing branches, one of which has slope greater than one and the other less than one. This is one of the simplest maps one can define that is not expanding, but it seems to be rather difficult to show that it is eventually expanding. In this paper we conjecture that f is eventually expanding if the first branch is onto, and the second branch is touching the x-axis. We provide a partial proof of the conjecture. This family of maps was investigated in [12] by different methods. For $p \leq 2$ (p being the slope of the first branch) its natural extension was constructed and proved to be Bernoulli. Similar, but different types of maps were analyzed in [7] and [22] and shown to admit absolutely continuous invariant measures.

As we mentioned in the Introduction, we recently learned about the main result of preprint [13] in which Conjecture 2.1 is proven to be true.

2.1 A region where f is eventually expanding

Let $f\colon [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} px, & 0 \leq x < \frac{1}{p}, \\ s(x - \frac{1}{p}), & \frac{1}{p} \leq x \leq 1. \end{cases} \tag{2.1}$$

Figure 2.1 shows the graph of f for $p = 7/2$ and $s = 1/2$.

Conjecture 2.1. *For all $(s,p) \in (0,1) \times (1,\infty)$, f is eventually expanding.*

For a real number x, let $\lceil x \rceil = \min\{n \in \mathbb{Z} |\ n \geq x\}$ and $\lfloor x \rfloor = \max\{n \in \mathbb{Z} |\ n \leq x\}$.

Theorem 2.1. *For all $(s,p) \in (0,1) \times (1,\infty)$ such that*

$$\left\lceil -\frac{\ln(p(1-s)+s)}{\ln s} \right\rceil \neq \left\lceil -\frac{\ln p}{\ln s} \right\rceil,$$

f is eventually expanding.

Proof. Consider a positive integer N and for any $x \in [0,1]$, consider the sequence $x, f(x), f^2(x), \ldots, f^{N-1}(x)$. There are only finitely many values of x such that

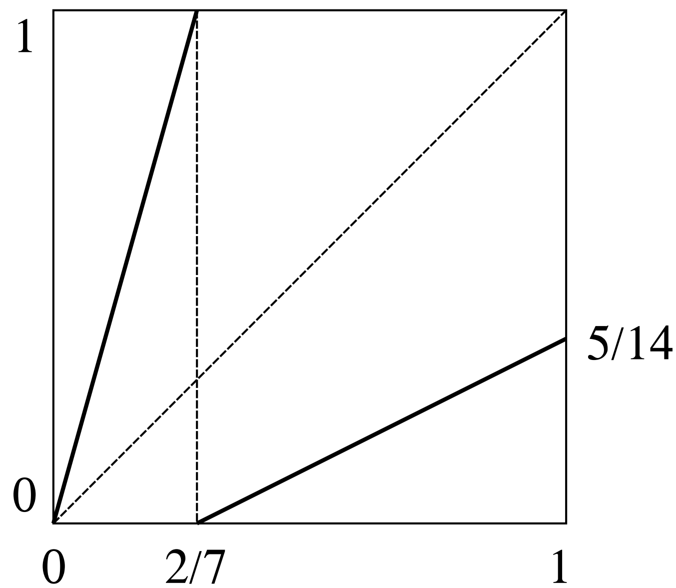

Figure 2.1: Graph of $y = f(x)$ for $p = 7/2$ and $s = 1/2$.

for some i with $0 \leq i < N$, $f^i(x) = 1/p$. These finitely many values of x divide the interval $[0,1]$ into finitely many intervals. If J is one of these intervals, then one can verify by induction that for all $i < N$, $f^i(J)$ is an interval that does not contain $1/p$, so it is contained in either $[0, 1/p)$ or $(1/p, 1]$. If $f^i(J) \subset [0, 1/p)$, then for all $x \in J$, $f^{i+1}(x)$ is obtained from $f^i(x)$ by applying the first branch of f, and if $f^i(J) \subset (1/p, 1]$ then $f^{i+1}(x)$ is obtained from $f^i(x)$ by applying the second branch of f. In the first case we say that f is expanding on J at step $i+1$ and in the second case we say that it is contracting. It is easy to see that f^N is linear on J, with slope $p^m s^n$, where m and n are the numbers of steps at which f is expanding and contracting on J. Note that $0 \leq n, m \leq N$ and $m + n = N$.

Let $A = \{(s,p) : \lceil -\ln p / \ln s \rceil \neq \lceil -\ln(p(1-s)+s)/\ln s \rceil\}$. Suppose there exists $(s,p) \in A$ such that for every $N \in \mathbb{N}$, f^N is not expanding. Then for every N there exists an interval $J \subset I$, as described above, on which f is linear with slope

$p^m s^n \le 1$. This implies

$$m \le cN, \text{ where } c = \frac{-\ln s}{\ln p - \ln s}.$$

J expands m times and contracts n times during N iterations; hence J must contract consecutively $\lceil n/(m+1)\rceil$ times during N iterations of f. That is, there exists $i \in \mathbb{N}$ such that $f^{i+k}(J) \subset (1/p, 1]$, for $k = 0, 1, \ldots, \lceil n/(m+1)\rceil - 1$.

Since $m \le cN$ and m is an integer, $m \le \lfloor cN \rfloor$, and therefore

$$\left\lceil \frac{n}{m+1} \right\rceil \ge \left\lceil \frac{N - \lfloor cN \rfloor}{\lfloor cN \rfloor + 1} \right\rceil = \left\lceil \frac{\frac{1}{c} - \frac{\lfloor cN \rfloor}{cN}}{1 + \frac{1-(cN)}{cN}} \right\rceil \ge \left\lceil \frac{\frac{1}{c} - 1}{1 + \epsilon_N} \right\rceil,$$

where (cN) denotes the fractional part of cN and $\epsilon_N = (1 - (cN))/cN \ge 0$. Note that $\epsilon_N \to 0$ as $N \to \infty$. Therefore, there exists $N_0 \in \mathbb{N}$ such that for every $N > N_0$, $\lceil (1/c - 1)/(1 + \epsilon_N) \rceil = \lceil 1/c - 1 \rceil = \lceil -\ln p / \ln s \rceil$. Taking $N = N_0 + 1$, we conclude that there is an interval J that, in the first N iterations of f, has $j = \lceil -\ln p / \ln s \rceil$ consecutive contractions. This means that there is some i such that for all $x \in f^i(J)$, $f^k(x) > 1/p$ for $k = 0, 1, \ldots, j-1$. Letting x be any point in the interval $f^i(J)$, we find that

$$\frac{1}{p} < f^{j-1}(x) = s^{j-1}x - \frac{s}{p}\left(\frac{1 - s^{j-1}}{1-s}\right) \le s^{j-1} - \frac{s}{p}\left(\frac{1 - s^{j-1}}{1-s}\right),$$

which means that

$$j - 1 < -\frac{\ln(p(1-s)+s)}{\ln s}.$$

Since j is an integer, this implies that

$$\left\lceil -\frac{\ln(p(1-s)+s)}{\ln s} \right\rceil \geq j = \left\lceil -\frac{\ln p}{\ln s} \right\rceil .$$

But since $0 < s < 1$ and $p > 1$, $p(1-s)+s < p$, so

$$\left\lceil -\frac{\ln(p(1-s)+s)}{\ln s} \right\rceil \leq \left\lceil -\frac{\ln p}{\ln s} \right\rceil$$

and therefore

$$\left\lceil -\frac{\ln(p(1-s)+s)}{\ln s} \right\rceil = \left\lceil -\frac{\ln p}{\ln s} \right\rceil .$$

Therefore, $(s,p) \notin A$, a contradiction. $\square$

The complement of the set A in the proof of Theorem 2.1 is given by $\lceil -\ln p / \ln s \rceil = \lceil -\ln(p(1-s)+s)/\ln s \rceil$. If we solve this equation for p, we get the regions

$$1 + \frac{1}{s} + \cdots + \frac{1}{s^k} < p \leq \frac{1}{s^{k+1}}, \text{ for } k \in \mathbb{N} \cup \{0\} .$$

Hence, the curves on the boundary of the region A are of the form $p = 1 + \frac{1}{s} + \cdots + \frac{1}{s^k}$ and $p = \frac{1}{s^{k+1}}$.

2.2 Other regions where f is eventually expanding

We will refer to $\{(s,p) \in (0,1) \times (1,\infty) |\ f \text{ is eventually expanding}\}$ as the "good" region. We show that the good region contains all points with small enough p:

Proposition 2.1. *If $1 < p \leq 2$ and $0 < s < 1$, then f is eventually expanding.*

Proof. Let N be the least positive integer such that $p^{N-1}s > 1$. Note that $N \geq 2$ and $p^{N-2}s \leq 1$. Consider the first N iterations of f. As in the proof of Theorem 2.1, f^N is piecewise linear on $[0, 1]$, and if J is one of the intervals on which f is linear, then the slope of f on J is $p^m s^n$, where m and n are the numbers of expansions and contractions of J under f. We claim now that we always have $n \leq 1$, so the slope is at least $p^{N-1}s > 1$ and therefore f is eventually expanding. To prove this claim, suppose that $n \geq 2$. Then there must be some i and j such that $0 \leq i < j \leq N-1$, f is contracting on J at steps $i+1$ and $j+1$, and f is expanding on J at step $k+1$ whenever $i < k < j$. In other words, $f^i(J) \subset (1/p, 1]$, $f^j(J) \subset (1/p, 1]$, and if $i < k < j$ then $f^k(J) \subset [0, 1/p)$. This means that if $x \in f^i(J)$ then $x > 1/p$, $f^k(x) < 1/p$ for $k = 1, \ldots, j-i-1$, and $f^{j-i} > 1/p$. But then

$$f^{j-i}(x) = p^{j-i-1}s\left(x - \frac{1}{p}\right) \leq p^{N-2}s\left(x - \frac{1}{p}\right) \leq x - \frac{1}{p} \leq 1 - \frac{1}{p} = \frac{p-1}{p} \leq \frac{1}{p},$$

which is a contradiction. $\square$

Denote the boundary curves of the region A by:

$$\begin{aligned}
\gamma_k^L(s) &= \frac{1}{s^k}, \\
\gamma_k^U(s) &= 1 + \frac{1}{s} + \frac{1}{s^2} + \cdots + \frac{1}{s^k} = \frac{1 - s^{k+1}}{s^k(1-s)},
\end{aligned}$$

where $k \in \mathbb{N} \cup \{0\}$. The following lemma shows that for $p \in [\gamma_{n-1}^U(s), \gamma_n^U(s))$, the nth image of 1 is the first image of 1 to fall into $[0, 1/p)$.

Lemma 2.1. *For $n \geq 0$, $f^j(1) \geq 1/p$ for $j = 0, 1, \ldots, n$ if and only if $p \geq \gamma_n^U(s)$. If $p = \gamma_n^U(s)$, then $f^n(1) = 1/p$.*

Proof. We prove it by induction. If $n = 0$, then the equivalence to be proven says that $1 \geq 1/p$ if and only if $p \geq 1$ and that is clearly true. For the induction step, assume that the statement is true for $n = k - 1$. Then to prove the statement for $n = k$ it suffices to show that for $p \geq \gamma_{k-1}^U(s)$, $f^k(1) \geq 1/p$ if and only if $p \geq \gamma_k^U(s)$. So suppose that $p \geq \gamma_{k-1}^U(s)$. Then by inductive hypothesis, $f^i(1) \geq 1/p$ for $i = 0, \ldots, k-1$. Therefore

$$f^k(1) = s^k - \frac{s}{p}\left(\frac{1-s^k}{1-s}\right),$$

and we have

$$f^k(1) \geq \frac{1}{p} \Longleftrightarrow s^k - \frac{s}{p}\left(\frac{1-s^k}{1-s}\right) \geq \frac{1}{p} \Longleftrightarrow p \geq \frac{1}{s^k}\left(\frac{1-s^{k+1}}{1-s}\right) = \gamma_k^U(s).$$

A similar argument shows that $f^n(1) = 1/p$ if $p = \gamma_n^U(s)$. $\square$

Consider any $s \in (0,1)$ and any $k \geq 2$. For $p \geq \gamma_{k-1}^U(s)$ we have

$$f^k(1) = s^k - \frac{s}{p}\left(\frac{1-s^k}{1-s}\right),$$

which clearly increases as p increases. Also, if $p = \gamma_{k-1}^U(s)$ then $f^{k-1}(1) = 1/p$ and therefore $f^k(1) = 0$, and if $p = \gamma_k^U(s)$ then $f^k(1) = 1/p$. It follows that there is a unique $p \in (\gamma_{k-1}^U(s), \gamma_k^U(s))$ such that $f^k(1) = 1/p^2$. We denote this unique value of p by $\gamma_k^M(s)$. Clearly if $\gamma_{k-1}^U(s) \leq p < \gamma_k^M(s)$ then $f^k(1) < 1/p^2$, and if $p > \gamma_k^M(s)$ then $f^k(1) > 1/p^2$. We can find a formula for $\gamma_k^M(s)$ by setting the formula for $f^k(1)$ above equal to $1/p^2$. Solving the resulting quadratic equation we get

$$\gamma_k^M(s) = \frac{1 - s^k + \sqrt{(1-s^k)^2 + 4(1-s)^2 s^{k-2}}}{2(1-s)s^{k-1}}.$$

Proposition 2.2. *Suppose $k \geq 2$. If $1/2 < s < 1$ and*

$$\gamma_{k-1}^U(s) \leq p \leq \gamma_k^M(s) = \frac{1 - s^k + \sqrt{(1-s^k)^2 + 4(1-s)^2 s^{k-2}}}{2(1-s)s^{k-1}},$$

then f is eventually expanding.

Proof. Let N be a positive integer, and consider N iterations of f. As usual, let J be an interval on which f^N is linear, with slope $p^m s^n$, where m and n are the numbers of expansions and contractions of J in the N iterations of f. Since $\gamma_{k-1}^U(s) \leq p \leq \gamma_k^M(s)$, we have $f^k(1) \leq 1/p^2$. It follows that J can never have more than k consecutive contractions, and if it has k consecutive contractions and those contractions are followed by at least two more steps, then both of those steps must be expansions.

The sequence of contractions and expansions of J can be described by a string of c's and e's, where the ith letter is a c if J contracts at step i and an e if it expands. This string can be broken up into blocks of the form $c^i e$, where $0 \leq i \leq k-1$, or $c^k ee$, except possibly for a final block consisting of up to k c's, perhaps followed by an e. If we associate with each block of the form $c^i e^j$ the factor $s^i p^j$, then the product of all of these factors is $p^m s^n$, the slope of f^N on J.

For a block of the form $c^i e$ with $0 \leq i \leq k-1$, the corresponding factor is $s^i p \geq s^{k-1} p$, and for a block of the form $c^k ee$ the factor is $s^k p^2$. Since $p \geq \gamma_{k-1}^U(s) = 1 + \cdots + 1/s^{k-1} > 1/s^{k-1}$, we have $s^{k-1}p > 1$. And $sp \geq s^{k-1}p > 1$, so $s^k p^2 = (sp)(s^{k-1}p) > s^{k-1}p$. Thus, for all blocks except the last, the factor is at least $s^{k-1}p$, which is greater than 1. The factor for the last block is at least s^k.

The length of the last block is at most $k+1$, and the length of every other block is at most $k+2$, so the number of blocks is at least $N/(k+2)$. Therefore the slope of f^N on J is at least

$$(s^{k-1}p)^{N/(k+2)-1}s^k.$$

Since $s^{k-1}p > 1$, this will be larger than 1 for sufficiently large N, so f is eventually expanding. $\square$

Corollary 2.1. *If $1 < p$ and $\frac{1}{2} \leq s < 1$, then f is eventually expanding.*

Proof. Suppose f is not eventually expanding. Then by Theorem 2.1, there is some $k \geq 1$ such that $\gamma_{k-1}^U(s) < p \leq \gamma_k^L(s)$. If $k = 1$ then this means $1 < p \leq 1/s \leq 2$, contradicting Proposition 2.1. Now suppose $k \geq 2$. Since $p \leq \gamma_k^L(s) = 1/s^k$, we have $s^k \leq 1/p$, and therefore

$$f^k(1) = s^k - \frac{s}{p}\left(\frac{1-s^k}{1-s}\right) \leq \frac{1}{p} - \frac{s}{p}\left(\frac{1-1/p}{1-s}\right) = \frac{1}{p^2} - \left(\frac{2s-1}{1-s}\right)\left(\frac{p-1}{p^2}\right) \leq \frac{1}{p^2}.$$

Therefore $p \leq \gamma_k^M(s)$, so we have a contradiction with Proposition 2.2. $\square$

The following proposition shows other parts of the good region.

Proposition 2.3. *If $\frac{1}{p^k} < s \leq \frac{1}{p^{k-1}(p-1)}$, $k \geq 2$, then f is eventually expanding.*

Proof. If $s \leq \frac{1}{p^{k-1}(p-1)}$, then $f(1) = s(1-1/p) \leq 1/p^k$. It follows that in $k+1$ iterations, any interval can contract at most once. So on any of the intervals on which f^{k+1} is linear, the slope is at least $sp^k > 1$. $\square$

Therefore, if we let $\eta_k^U(p) = 1/(p^{k-1}(p-1))$ and $\eta_k^L(p) = 1/p^k$ where $k \geq 2$, then for (s,p) satisfying $\eta_k^L(p) < s \leq \eta_k^U(p)$, f is eventually expanding.

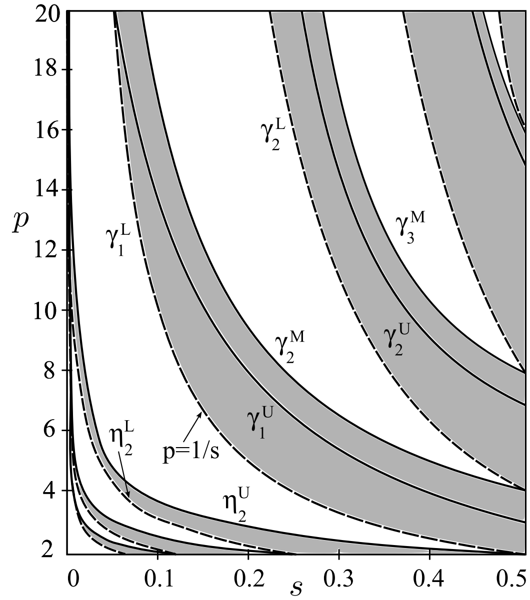

Figure 2.2: The shaded regions are the regions where f has been proven to be eventually expanding for $p > 2$ and $s < 1/2$. If $1 < p \leq 2$ or $s \geq 1/2$, f is eventually expanding by Proposition 2.1 and Corollary 2.1. It is conjectured that f is eventually expanding whenever $1 < p$ and $0 < s < 1$.

Figure 2.2 shows the regions where we have proven f to be eventually expanding (darker regions) for $p > 2$ and $s < 1/2$. Note that f is also eventually expanding if $1 < p \leq 2$ or $s \geq 1/2$ as shown by Proposition 2.1 and Corollary 2.1.

2.3 Exactness and other properties.

A function $f\colon I \to I$ is said to be exact or locally eventually onto if for every open interval $J \in I$ there exists N such that $f^N(J) = I$.

Proposition 2.4. *The map f defined by (2.1) is exact (or locally eventually onto) if it is eventually expanding.*

Proof. By assumption, there exists N such that f^N is piecewise expanding. Since both branches of f touch the x axis, all branches of f^j touch the x axis, for any $j \in \mathbb{N}$. Since f^N is piecewise expanding, any given interval $J \in [0, 1]$ grows under action of f^N until its image covers a discontinuity point. Thus, there exists an integer k such that $f^{kN+1}(J)$ contains the fixed point 0. Since the branch that contains the fixed point is onto and expanding, some iterate of $f^{kN+1}(J)$ under f eventually covers all of $[0, 1]$. $\square$

If the map f is eventually expanding, the whole rich theory of such maps applies to it. In particular f admits an absolutely continuous invariant measure μ [9, 25]. Similarly as in Proposition 2.4 it can be proven that μ is unique and the system $\{f, \mu\}$ is exact in the measure-theoretical sense. An explicit formula for the density of μ can be obtain using methods of [19].

A point x is called periodic under f if there exists $N \in \mathbb{N}$ such that $f^N(x) = x$. In this case, x is said to be repelling if $|(f^N)'(x)| > 1$ and attracting if $|(f^N)'(x)| < 1$. The following property of f has been noticed by M. Misiurewicz.

Proposition 2.5. (Misiurewicz) *All periodic points of f are repelling.*

Proof. Let us fix an $N \geq 1$. All branches of f^N are increasing and touch the x axis. The slope of f^N at 0 is $p^N > 1$. Thus, no branch with a slope smaller than or equal to 1 can intersect the diagonal. Thus, any fixed point of f^N is repelling. $\square$

Chapter 3

Metastable Systems as Random Maps

3.1 Introduction

One-dimensional metastable systems were recently studied [32] in the framework of piecewise expanding maps on two disjoint ergodic sets. Under small deterministic perturbations, the asymptotic dynamics of the merged metastable system is captured by the absolutely continuous invariant measure (acim) on the combined ergodic sets. The main result of [32] shows that this combined acim can be approximated by a convex combination of the two disjoint acim's with weights depending on the respective measures of the holes. The method of [32] invokes the usual bounded variation technique that applies naturally in a setting where the slopes of the original map are > 2. For maps with slopes only > 1 in magnitude, the BV technique encounters difficulties as the partitions needed for the approximating family of maps have elements that go to zero in measure and hence render the standard BV inequalities ineffective in establishing precompactness of the family of probability density functions associated with the family of approximating maps. To handle this problem, the authors of [32] introduce some additional conditions on the maps they consider.

In this chapter we take a different approach to modeling metastable behaviour. We consider two piecewise expanding maps: one is the original map, τ_1, defined on two disjoint invariant sets of $\mathbb{R}^N$ and the other, τ_2, is a deterministically perturbed version of τ_1, which allows passage between the two disjoint invariant sets of τ_1 via holes. We model such a system by means of a random map based on τ_1 and τ_2, to which we associate position dependent probabilities that reflect the switching between the maps. A typical orbit spends a random amount of time governed by the dynamics of either τ_1 or τ_2, then switches to the other map. Suppose p_1, the probability of using map τ_1, is close to 1, then with very high probability the orbit spends a lot of time under the influence of τ_1, that is, it stays in either one or the other of the two disjoint sets invariant under τ_1. Since $p_1 < 1$, there is a small but positive probability of switching from τ_1 to τ_2. When this happens, the dynamics comes under the control of τ_2, which allows movement between the disjoint invariant sets. Unlike the model in [32] where the hole sizes shrink to 0, the hole sizes in our random map model stay fixed. (Their measures in a skew product interpretation of random maps converge to 0, see [2], so one could argue that both models are in a way similar.) What changes are the probabilities of switching from one map to the other. As p_1 approaches 1, the orbits are almost completely defined by τ_1 and therefore remain in one or the other of the two disjoint invariant sets for a very long time. This behaviour is the hallmark of metastable dynamics. Our main result establishes a result similar in spirit to that in [32]: we prove that, as the probability of using τ_1 converges to 1, the dynamics is captured by an acim that is a convex combination of the acim's on the two disjoint invariant sets. Furthermore, we calculate the weights of the respective acim's from a formula analogous to the one derived in [32].

In the billiards problem metastable behaviour is attributed to small physical holes in the boundary between the tables. From the perspective of random maps, the holes can be large with the probabilities of switching controlling the metastable behaviour. This allows for the consideration of situations where there are no actual physical holes, but where balls can "leap" from one table to the other.

In Sections 3.2 and 3.3 we recall the definition of a position dependent random map and collect some existence and continuity results in 1 and N dimensions. In Section 3.4 we present a random map model for a metastable system with two ergodic components. We show that there exists a unique acim which is a convex combination of the acim's on the two ergodic sets where the weights in the combination are calculated from a formula similar to the one in [32]. In Section 3.5 we present the generalization of this result for a metastable system with more than two ergodic components. A deterministic model of such situation is discussed in [15]. Section 3.6 contains examples.

3.2 Position Dependent Random Maps and Their Properties

Let $(I, \mathfrak{B}, \lambda)$ be a measure space, where λ is an underlying measure. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one, differentiable, non-singular transformations on a common partition $\mathcal{P}$ of I : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k|_{I_i}$, $i = 1, ..., q$,

$k = 1, ..., K$ ($\mathcal{P}$ can be found by considering finer partitions). We define the transition function for the random map

$$T = \{\tau_1, ...\tau_K; p_1(x), ...p_K(x)\}$$

as follows:

$$\mathbb{P}(x, A) = \sum_{k=1}^{K} p_k(x)\chi_A(\tau_k(x)), \tag{3.1}$$

where A is any measurable set and $\{p_k(x)\}_{k=1}^K$ is a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^K p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in I$ and χ_A denotes the characteristic function of the set A. We define $T(x) = \tau_k(x)$ with probability $p_k(x)$ and $T^N(x) = \tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)$ with probability $p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)$. The transition function $\mathbb{P}$ induces an operator $\mathbb{P}_*$ on measures on $(I, \mathfrak{B})$ defined by

$$\begin{aligned}\mathbb{P}_*\mu(A) &= \int_I \mathbb{P}(x, A)d\mu(x) = \sum_{k=1}^{K} \int_I p_k(x)\chi_A(\tau_k(x))d\mu(x) \\ &= \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x)d\mu(x) = \sum_{k=1}^{K}\sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x)d\mu(x)\end{aligned} \tag{3.2}$$

We say that the measure μ is T-invariant iff $\mathbb{P}_*\mu = \mu$, i.e.,

$$\mu(A) = \sum_{k=1}^{K} \int_{\tau_k^{-1}(A)} p_k(x)d\mu(x), \qquad A \in \mathfrak{B}. \tag{3.3}$$

If μ has density f with respect to λ, then $\mathbb{P}_*\mu$ has also a density which we denote by $P_T f$. By change of variables, we obtain

$$\begin{aligned} \int_A P_T f(x) d\lambda(x) &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_{\tau_{k,i}^{-1}(A)} p_k(x) f(x) d\lambda(x) \\ &= \sum_{k=1}^{K} \sum_{i=1}^{q} \int_A p_k(\tau_{k,i}^{-1} x) f(\tau_{k,i}^{-1} x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} d\lambda(x) \ , \end{aligned} \tag{3.4}$$

where $J_{k,i}$ is the Jacobian of $\tau_{k,i}$ with respect to λ, $J(\tau) = \frac{d\tau_*(\lambda)}{d\lambda}$. Since this holds for any measurable set A we obtain an a.e. equality:

$$(P_T f)(x) = \sum_{k=1}^{K} \sum_{i=1}^{q} p_k(\tau_{k,i}^{-1} x) f(\tau_{k,i}^{-1} x) \frac{1}{J_{k,i}(\tau_{k,i}^{-1})} \chi_{\tau_k(I_i)}(x) \tag{3.5}$$

or

$$(P_T f)(x) = \sum_{k=1}^{K} P_{\tau_k} (p_k f)(x) \tag{3.6}$$

where P_{τ_k} is the Perron-Frobenius operator corresponding to the transformation τ_k (see [9] for details). We call P_T the Perron-Frobenius operator of the random map T.

3.3 Continuity theorems

3.3.1 Existence and Continuity theorems in one dimension

Let $(I, \mathfrak{B}, \lambda)$ be a measure space, where λ is normalized Lebesgue measure on $I = [a, b]$. Let $\tau_k : I \to I$, $k = 1, ..., K$ be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of I : $\mathcal{P} = \{I_1, ..., I_q\}$ and $\tau_{k,i} = \tau_k|_{I_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Let $\{p_k(x)\}_{k=1}^{K}$ be a set of position dependent measurable probabilities, i.e., $\sum_{k=1}^{K} p_k(x) = 1$, $p_k(x) \geq 0$, for any $x \in I$. Assume in addition that p_k is piecewise differentiable on $\mathcal{P}$.

Denote by $V(\cdot)$ the standard one-dimensional variation of a function, and by $BV(I)$ the space of functions of bounded variations on I equipped with the norm $\|\cdot\|_{BV} = V(\cdot) + \|\cdot\|_1$.

Let $g_k(x) = \frac{p_k(x)}{|\tau_k'(x)|}$, $k = 1, \ldots, K$. We assume the following conditions:

Condition (A): $\sum_{k=1}^{K} g_k(x) < \alpha < 1$, $x \in I$, and

Condition (B): $g_k \in BV(I)$, $k = 1, \ldots, K$.

Let $T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$ be a random map with position dependent probabilities satisfying conditions (A) and (B). We define $\mathcal{P}^N$ as a maximal common monotonicity partition for all maps defining T^N. For $w = (k_1, \ldots, k_{N-1}, k_N) \in \{1, ..., K\}^N$ we define

$$g_w = \frac{p_{k_N}(\tau_{k_{N-1}} \circ ... \circ \tau_{k_1}(x)) \cdot p_{k_{N-1}}(\tau_{k_{N-2}} \circ ... \circ \tau_{k_1}(x)) \cdots p_{k_1}(x)}{|(\tau_{k_N} \circ \tau_{k_{N-1}} \circ ... \circ \tau_{k_1})'(x)|}.$$

The following results are proved in [3]:

Lemma 3.1. *Let T satisfy conditions (A) and (B). Then for any $f \in BV(I)$ and $M \in \mathbb{N}$,*

$$\|P_T^M f\|_{BV} \le A_M \|f\|_{BV} + B_M \|f\|_1, \tag{3.7}$$

where $A_M = 3\alpha^M + W_M$, $B_M = \beta_M(2\alpha^M + W_M)$, $\beta_M = \max_{J \in \mathcal{P}^M} (\lambda(J))^{-1}$, $W_M \equiv \max_{J \in \mathcal{P}^M} \sum_{w \in \{1,\ldots,K\}^M} V_J g_w$.

Theorem 3.1. *Let T be a random map which satisfies conditions (A) and (B). Then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(I)$, see [9].*

We now present the Continuity Theorem in one dimension. A similar theorem was proved in proposition 2 of [20] under stronger conditions. Our aim is to show that it holds under the weaker conditions (A) and (B).

Theorem 3.2 (Continuity Theorem 1-dim)**.** *Let* $T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$ *be a random map with position dependent probabilities satisfying conditions* (A) *and* (B)*. Let* $\{p_1^{(n)}, \ldots, p_K^{(n)}\}_{n=1}^{\infty}$ *be a sequence of sets of probabilities such that* $p_k^{(n)} \to p_k$ *as* $n \to +\infty$*,* $k = 1, \ldots, K$*, in the piecewise* C^1 *topology on the fixed partition* $\mathcal{P}$*. Let* $T^{(n)} = \{\tau_1, \ldots, \tau_K; p_1^{(n)}, \ldots, p_K^{(n)}\}$*,* $n = 1, 2, \ldots$ *be a sequence of random maps. For* n *large,* $T^{(n)}$ *has an invariant density* $f^{(n)}$ *and the sequence* $\{f^{(n)}\}_{n=1}^{\infty}$ *is precompact in* L^1*. Moreover, all limit points of* f^* *of this sequence are fixed points of* P_T*.*

Proof. We will prove the theorem in three steps. In the first step we show that an inequality similar to inequality (3.7) of Lemma 3.1 holds uniformly for all $T^{(n)}$ with n large enough. In order to achieve this, we need to show that for large enough n conditions (A) and (B) are satisfied uniformly.

Suppose $\alpha < \gamma < 1$, where $\sum_{k=1}^{K} g_k(x) < \alpha < 1$. First, choose ϵ such that $\sum_{k=1}^{K} \frac{\epsilon}{|\tau_k'(x)|} < \gamma - \alpha$. Then choose N_1 such that for $n > N_1$ and $1 \leq k \leq K$, $p_k - \epsilon \leq p_k^{(n)} \leq p_k + \epsilon$. Then

$$\sum_{k=1}^{K} \frac{p_k^{(n)}(x)}{|\tau_k'(x)|} \leq \sum_{k=1}^{K} \frac{p_k + \epsilon}{|\tau_k'(x)|} = \sum_{k=1}^{K} \frac{p_k(x)}{|\tau_k'(x)|} + \sum_{k=1}^{K} \frac{\epsilon}{|\tau_k'(x)|} \leq \alpha + (\gamma - \alpha) = \gamma < 1.$$

Therefore, condition (A) holds uniformly for all $n > N_1$, with α replaced by γ. Regarding condition (B), note that

$$|V_J g^{(n)} - V_J g| \leq \int_J |(g^{(n)})' - g'| d\lambda \to 0 \text{ as } n \to \infty.$$

It follows that there exists a constant C_1 and an integer N_2 such that for all $n > N_2$, $V_J g^{(n)} < C_1$ for any interval $J \subset I$.

Now consider $W_1^{(n)} = \max_{J\in\mathcal{P}} \sum_{k=1}^{K} V_J g_k^{(n)}$. From the above statement it follows that $W_1^{(n)}$ is also uniformly bounded for n sufficiently large. That is, there exists C_2 and integer N_3 such that for all $n > N_3$, $W_1^{(n)} < C_2$. Let $N_4 = \max\{N_1, N_2, N_3\}$ and $C = \max\{C_1, C_2\}$. It is shown in [3] that $W_M^{(n)} \leq M\alpha^{M-1} W_1^{(n)}$, hence for $n > N_4$, $W_M^{(n)} < M\gamma^{M-1}C$. Therefore, for $n > N_4$, inequality (3.7) holds uniformly with α replaced by γ.

In the next step we show that the sequence of invariant densities $\left\{f^{(n)}\right\}$ is uniformly bounded in $BV(I)$. Without loss of generality consider $\left\{f^{(n)}\right\}_{n=N_4+1}^{\infty}$ instead of $\left\{f^{(n)}\right\}$. Moreover since inequality (3.7) is now satisfied uniformly for all n, we drop the superscript of (n) and write A_M, B_M for $A_M^{(n)}$, $B_M^{(n)}$ respectively. Also assume M is large enough so that $A_M = 3\gamma^M + W_M < 1$.

To summarize, we have shown that there exists M such that for any $f \in BV(I)$ and $n \in \mathbb{N}$:

$$\|P_{T^{(n)}}^{M} f\|_{BV} \leq A_M \|f\|_{BV} + B_M \|f\|_1, \tag{3.8}$$

where $A_M = 3\gamma^M + W_M < 1$, $B_M = \beta_M(2\gamma^M + W_M)$, $\beta_M = \max_{J\in\mathcal{P}^M}(\lambda(J))^{-1}$, $W_M \equiv \max_{J\in\mathcal{P}^M} \sum_{w\in\{1,\ldots,K\}^M} V_J g_w$.

Using inequality (3.8) repeatedly, one can show that each $f^{(n)}$ is a limit point of the sequence of averages $\left\{\frac{1}{m}\sum_{j=0}^{m-1} P_{T^{(n)}}^{Mj} 1\right\}$ and

$$\|f^{(n)}\|_{BV} \leq 1 + \frac{B_M}{1 - A_M}.$$

Therefore $\left\{f^{(n)}\right\}$ is a bounded set in $BV(I)$ and hence it has a limit point f^* in L^1.

In the final step show that f^* is invariant under P_T:

$$\begin{aligned}
\|P_T f^* - f^*\|_1 &\leq \|P_T f^* - P_{T^{(n)}} f^*\|_1 + \|P_{T^{(n)}} f^* - P_{T^{(n)}} f^{(n)}\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1 \\
&= \|\sum_{k=1}^{K} P_{\tau_k}(p_k f^*) - \sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^*)\|_1 \\
&+ \|\sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^*) - \sum_{k=1}^{K} P_{\tau_k}(p_k^{(n)} f^{(n)})\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1 \\
&\leq \sum_{k=1}^{K} \|f^*(p_k - p_k^{(n)})\|_1 + \sum_{k=1}^{K} \|(f^* - f^{(n)}) p_k^{(n)}\|_1 \\
&+ \|P_{T^{(n)}} f^{(n)} - f^{(n)}\|_1 + \|f^{(n)} - f^*\|_1
\end{aligned}$$

The third summand is 0 by definition of $f^{(n)}$. The other three converge to 0 since $f^{(n)} \to f^*$ and $p_k^{(n)} \to p_k$ as $n \to \infty$ in L^1 and L^∞, respectively. $\square$

3.3.2 Existence and Continuity theorems in higher dimensions

We now prove the Continuity Theorem in $\mathbb{R}^N$. Let S be a bounded region in $\mathbb{R}^N$ and λ_N be Lebesgue measure on S. Let $\tau_k : S \to S$, $k = 1, ..., K$ be piecewise one-to-one and C^2, non-singular transformations on a partition $\mathcal{P}$ of S : $\mathcal{P} = \{S_1, ..., S_q\}$ and $\tau_{k,i} = \tau_k|_{S_i}$, $i = 1, ..., q$, $k = 1, ..., K$. Let each S_i be a bounded closed domain having a piecewise C^2 boundary of finite $(N-1)$-dimensional measure. We assume that the faces of ∂S_i meet at angles bounded uniformly away from 0. We will also assume that the probabilities $p_k(x)$ are piecewise C^1 functions on the partition $\mathcal{P}$. Let $D\tau_{k,i}^{-1}(x)$ be the derivative matrix of $\tau_{k,i}^{-1}$ at x. We assume:

Condition (C):

$$\max_{1\leq i\leq q}\sum_{k=1}^{K} p_k(x)\|D\tau_{k,i}^{-1}(\tau_{k,i}(x))\| < \sigma < 1.$$

The main tool of this section is the multidimensional notion of variation defined using derivatives in the distributional sense (see [14]):

$$V(f) = \int_{\mathbb{R}^N} \|Df\| = \sup\{\int_{\mathbb{R}^N} f\mathrm{div}(g)d\lambda_N : g = (g_1, ..., g_N) \in C_0^1(\mathbb{R}^N, \mathbb{R}^N)\},$$

where $f \in L_1(\mathbb{R}^N)$ has bounded support, Df denotes the gradient of f in the distributional sense, and $C_0^1(\mathbb{R}^N, \mathbb{R}^N)$ is the space of continuously differentiable functions from $\mathbb{R}^N$ into $\mathbb{R}^N$ having a compact support. We will use the following property of variation which is derived from [14], Remark 2.14: If $f = 0$ outside a closed domain A whose boundary is Lipschitz continuous, $f_{|A}$ is continuous, $f_{|\mathrm{int}(A)}$ is C^1, then

$$V(f) = \int_{\mathrm{int}(A)} \|Df\|d\lambda_N + \int_{\partial A} |f|d\lambda_{N-1},$$

where λ_{N-1} is the $(N-1)$-dimensional measure on the boundary of A. In this section we shall consider the Banach space (see [14], Remark 1.12),

$$BV(S) = \{f \in L_1(S) : V(f) < +\infty\},$$

with the norm $\|f\|_{BV} = V(f) + \|f\|_1$.

Theorems 3.3 and 3.4 were established in [3]. We refer the reader to [3] for proofs of these theorems. The functions $a(\cdot)$ and $\delta(\cdot)$ which appear in these theorems are defined as in Section 1.6. We remark here that for a random map

$T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$, the functions a and δ are independent of the probabilities $\{p_1, \ldots, p_K\}$.

Theorem 3.3. *If T is a random map which satisfies condition (C), then*

$$V(P_T f) \leq \sigma(1 + 1/a)V(f) + (M + \frac{\sigma}{a\delta})\|f\|_1, \tag{3.9}$$

where $a = \min\{a(S_i) : i = 1, \ldots, q\} > 0$, $\delta = \min\{\delta(S_i), : i = 1, \ldots, q\} > 0$, $M_{k,i} = \sup_{x \in S_i}(Dp_k(x) - \frac{DJ_{k,i}}{J_{k,i}} p_k(x))$ *and* $M = \sum_{k=1}^{K} \max_{1 \leq i \leq q} M_{k,i}$.

Theorem 3.4. *Let T be a random map which satisfies condition (C). If $\sigma(1 + 1/a) < 1$, then T preserves a measure which is absolutely continuous with respect to Lebesgue measure. The operator P_T is quasi-compact on $BV(S)$, see [9].*

Now we present the multi-dimensional version of theorem 3.2. The proof of this theorem is similar to the proof of the one-dimensional Continuity Theorem hence we will only sketch the proof here.

Theorem 3.5 (Continuity Theorem in $\mathbb{R}^N$)**.** *Let $T = \{\tau_1, \ldots, \tau_K; p_1, \ldots, p_K\}$ be a random map with position dependent probabilities, satisfying condition (C). Also assume that $\sigma(1 + 1/a) < 1$. Let $\{p_1^{(n)}, \ldots, p_K^{(n)}\}_{n=1}^{\infty}$ be a sequence of sets of probabilities such that $p_k^{(n)} \to p_k$ as $n \to +\infty$, $k = 1, \ldots, K$, in the piecewise C^1 topology on the fixed partition $\mathcal{P}$. Let $T^{(n)} = \{\tau_1, \ldots, \tau_K; p_1^{(n)}, \ldots, p_K^{(n)}\}$, $n = 1, 2, \ldots$ be a sequence of random maps. For m large, $T^{(n)}$ has an invariant density $f^{(n)}$ and the sequence $\{f^{(n)}\}_{n=1}^{\infty}$ is precompact in L^1. Moreover, all limit points f^* of this sequence are fixed points of P_T.*

Proof. The main part of the proof is to establish an inequality similar to the inequality (3.9) uniformly for all n larger than some integer N_1. As a result of applying Theorem 3.3 to $T^{(n)}$ we obtain:

$$V(P_{T^{(n)}}f) \leq \underbrace{\sigma^{(n)}(1+1/a^{(n)})}_{A^{(n)}} V(f) + \underbrace{(M^{(n)} + \frac{\sigma^{(n)}}{a^{(n)}\delta})}_{B^{(n)}} \|f\|_1, \tag{3.10}$$

where

$$a^{(n)} = \min\{a^{(n)}(S_i) : i = 1, \dots, q\} > 0,$$

$$\delta^{(n)} = \min\{\delta(S_i), : i = 1, \dots, q\} > 0,$$

$$M_{k,i}^{(n)} = \sup_{x \in S_i}(Dp_k^{(n)}(x) - \frac{DJ_{k,i}}{J_{k,i}} p_k^{(n)}(x)),$$

and

$$M^{(n)} = \sum_{k=1}^{K} \max_{1 \leq i \leq q} M_{k,i}^{(n)}.$$

Note that $a^{(n)}$ and $\delta^{(n)}$ do not depend on probabilities, so the superscript (n) can be dropped. In order to show that inequality (3.10) holds uniformly it suffices to choose N_1 large enough that $\sigma^{(n)}(1+1/a) < 1$ for all $n > N_1$. This is easily achievable since $p_k^{(n)} \to p_k$ for all $k = 1, \dots, K$. The uniform boundedness of $\{f^{(n)}\}$ in BV and the invariance of its limit points under P_T follow in a similar way to the one-dimensional case. Note that in this case it is not necessary to consider a higher power of the map $T^{(n)}$ as opposed to the one-dimensional case. $\square$

3.4 A model of metastability for a system with two ergodic components

Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be an N-dimensional random map with position dependent probabilities $p_1(x) = 1$ and $p_2(x) = 0$ satisfying the conditions of the previous section. Note that since $p_2(x) = 0$ for all x, T is essentially the same map as τ_1. Let us suppose that the domain of T is $I = I_1 \cup I_2$, where I_1 and I_2 are invariant under τ_1. Also, suppose τ_1 has exactly two ergodic measures μ_1, and μ_2 with densities f_1 and f_2 on I_1 and I_2, respectively. The map τ_2 differs from τ_1 on the sets $H_{1,2} \subset I_1$ and $H_{2,1} \subset I_2$, where $H_{1,2} = I_1 \cap \tau_2^{-1}(I_2)$ and $H_{2,1} = I_2 \cap \tau_2^{-1}(I_1)$. We assume that

$$\mu_1(H_{1,2}) > 0 \quad \text{and} \quad \mu_2(H_{2,1}) > 0\ . \tag{3.11}$$

Now consider a sequence of random maps $T^{(n)} = \left\{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\right\}$, perturbations of T, where only the probabilities are changed. Let

$$p_1^{(n)} = (1 - p_{2,1}^{(n)})\chi_{H_{2,1}} + (1 - p_{1,2}^{(n)})\chi_{H_{1,2}} + \chi_{I \setminus H_{2,1} \cup H_{1,2}} \tag{3.12}$$

$$p_2^{(n)} = 1 - p_1^{(n)}, \tag{3.13}$$

with $p_{1,2}^{(n)}$, $p_{2,1}^{(n)} > 0$, independent of x. Our main result is the following theorem.

Theorem 3.6. *If $p_{1,2}^{(n)}, p_{2,1}^{(n)} \to 0$ and $\lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}$ exists, then the acim's of the $n-$dimensional random maps $T^{(n)}$ converge to the measure $\mu = \alpha_1\mu_1 + \alpha_2\mu_2$, where*

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}\ .$$

Proof. Let $\mu_{T^{(n)}}$ be an acim of $T^{(n)}$ (we do not assume it to be unique). Let $f^{(n)}$ be the invariant density of $\mu_{T^{(n)}}$. By (3.11) we have $\mu_{T^{(n)}}(H_{1,2}) > 0$ and $\mu_{T^{(n)}}(H_{2,1}) > 0$. Then,

$$\begin{aligned}\mu_{T^{(n)}}(I_1) = \int_I \mathbb{P}(x, I_1) d\mu_{T^{(n)}} &= 1 \cdot \mu_{T^{(n)}}(I_1 \setminus H_{1,2}) + (1 - p_{1,2}^{(n)}) \cdot \mu_{T^{(n)}}(H_{1,2}) \\ &+ 0 \cdot \mu_{T^{(n)}}(I_2 \setminus H_{2,1}) + p_{1,2}^{(n)} \mu_{T^{(n)}}(H_{2,1}).\end{aligned}$$

Hence,

$$\frac{\mu_{T^{(n)}}(H_{1,2})}{\mu_{T^{(n)}}(H_{2,1})} = \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}. \tag{3.14}$$

By Theorem 3.5, $\left\{f^{(n)}\right\}_{n\geq 1}$ is precompact in L^1 and if f^* is a limit point, then f^* is of the form $\alpha_1 f_1 + \alpha_2 f_2$ for some $0 \leq \alpha_1, \alpha_2 \leq 1$, $\alpha_1 + \alpha_2 = 1$. In terms of the corresponding measures, there exists a subsequence n_k such that:

$$\begin{aligned}
\mu_{T^{(n_k)}}(H_{1,2}) &\rightarrow \alpha_1\mu_1(H_{1,2}) + \alpha_2\mu_1(H_{2,1}) = \alpha_1\mu_1(H_{1,2}) \qquad (3.15)\\
\mu_{T^{(n_k)}}(H_{2,1}) &\rightarrow \alpha_1\mu_2(H_{1,2}) + \alpha_2\mu_2(H_{2,1}) = \alpha_2\mu_2(H_{2,1}) \qquad (3.16)
\end{aligned}$$

Applying (3.14), (3.15) and (3.16), we get

$$\frac{\alpha_1}{\alpha_2} = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{k\to\infty} \frac{p_{2,1}^{(n_k)}}{p_{1,2}^{(n_k)}}.$$

□

Additional information about the spectrum of operators $P_{T^{(n)}}$ is provided in the following theorem based on results of [24].

Theorem 3.7. *Let us assume that 1 is an eigenvalue of P_T of multiplicity 2. For arbitrarily small $\delta > 0$, there exists an n_δ such that for $n \geq n_\delta$ the spectrum of $P_{T^{(n)}}$ intersected with $\{z : |z-1| < \delta\}$ consists of two eigenvalues of multiplicity 1: 1 and r_n, $|r_n| \leq 1$, $r_n \neq 1$ and $r_n \to 1$, as $n \to \infty$.*

Proof. The family $P_{T^{(n)}}$, $n \geq 1$, satisfies the assumptions of Corollary 1 of [24] which implies the above statement. □

3.5 A generalization for a system with L ergodic components

Let $T = \{\tau_1, \tau_2; p_1, p_2\}$ be an N-dimensional random map with position dependent probabilities $p_1(x) = 1$ and $p_2(x) = 0$. So T is essentially the same map as τ_1. Suppose τ_1 has L ergodic components $I_1, \ldots, I_L$, $\cup_{i=1}^{L} I_i = I$. Suppose there are $L-1$ "holes" $\{H_{i,j}\}_{1\le j\le L,\ j\ne i}$ in each component I_i. Map τ_2 is defined as a piecewise expanding map which has the following properties

$$\tau_2(H_{i,j}) \subset I_j \ , \ \text{for} \ \ i,j \in \{1,\ldots,L\}$$

and $\tau_2 = \tau_1$ outside the holes.

Let $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$ be a sequence of random maps such that

$$1 - p_1^{(n)} = p_2^{(n)} = \sum_{i=1}^{L} \sum_{j\ne i} p_{i,j}^{(n)} \chi_{H_{i,j}}, \tag{3.17}$$

$0 < p_{i,j}^{(n)} < 1$ and

$$p_{i,j}^{(n)} = h(n) a_{i,j} + o(h(n)) \ , \tag{3.18}$$

for some function h such that $\lim_{n\to\infty} h(n) = 0$. Let $\mu_T^{(n)}$ denote the invariant measure of $T^{(n)}$. Then for every $1 \le k \le L$,

$$\mu_{T^{(n)}}(I_k) = \int \mathbb{P}(x, I_k) d\mu_{T^{(n)}}(x) = \int_{\tau_1^{-1}(I_k)} p_1^{(n)}(x) d\mu_{T^{(n)}} + \int_{\tau_2^{-1}(I_k)} p_2^{(n)}(x) d\mu_{T^{(n)}} \ .$$

It follows that for every $1 \le k \le L$,

$$\sum_{j \neq k} p_{k,j}^{(n)} \mu_{T^{(n)}}(H_{k,j}) = \sum_{i \neq k} p_{i,k}^{(n)} \mu_{T^{(n)}}(H_{i,k}). \tag{3.19}$$

The left hand side of the equation (3.19) can be interpreted as the amount of $\mu_{T^{(n)}}$-measure that leaves the component I_k and the right hand side as the amount of $\mu_{T^{(n)}}$-measure that enters the component I_k. Intuitively, these two quantities are equal because $\mu_{T^{(n)}}$ is preserved under $T^{(n)}$.

Let us define $q_{i,j} = a_{i,j}\mu_i(H(i,j))$ for $j \neq i$, $q_{i,i} = 1 - \sum_{j \neq i} q_{i,j}$ for $1 \le i \le L$, and

$$Q = [q_{i,j}]_{1 \le i,j \le L} \ . \tag{3.20}$$

By the Continuity Theorem for random maps, there exists a subsequence n_k such that $\mu_{T^{(n_k)}} \to \mu_T = \sum_{i=1}^{L} \alpha_i \mu_i$. Therefore, $\mu_{T^{(n_k)}}(H_{i,j}) \to \alpha_i \mu_i(H_{i,j})$. Hence, the equations (3.19), for $n = n_k$, can be written as

$$\sum_{j \neq k} a_{k,j} \alpha_k \mu_k(H_{k,j}) = \sum_{i \neq k} a_{i,k} \alpha_i \mu_i(H_{i,k}) + o(1) \ ,$$

or

$$(1 - q_{k,j})\alpha_k = \sum_{i \neq k} q_{i,k} \alpha_i + o(1) \ ,$$

which in matrix form is

$$\alpha Q = \alpha + o(1) \ ,$$

where $\alpha = (\alpha_1, \ldots, \alpha_L)$. If, for $w = (w_1, \ldots, w_L)$ the solution of the equation

$w = wQ$ is stable under small perturbations, then, $\alpha = (\alpha_1, \ldots, \alpha_L)$ satisfies

$$\alpha \cdot Q = \alpha.$$

The conditions for stability of the eigenvectors for probability matrices are well known, see for example [11].

We have proved the following theorem

Theorem 3.8. *Let $T^{(n)}$ be a sequence of random maps satisfying assumptions of Section 3.4 but such that map τ_1 has $L \geq 2$ ergodic components. Let probabilities $p_{i,j}^{(n)}$, $1 \leq i, j \leq L$ satisfy assumptions (3.18). If the matrix Q defined in (3.20) has stable left 1-eigenvector, then the invariant measures $\mu_{T^{(n)}}$ converge as $n \to \infty$ to the measure $\sum_{i=1}^{L} \alpha_i \mu_i$, where $\alpha Q = \alpha$, and μ_i is the τ_1-invariant measure on the i-th ergodic component.*

3.6 Examples

Example 3.1.

We now present a simple Markov map example on the interval [0,1]. Consider the maps τ_1 and τ_2 as shown in figure 3.1.

Both maps are Markov on the partition $\mathcal{P} = \{J_1 = [0, 0.1], J_2 = [0.1, 0.5], J_3 = [0.5, 0.95], J_4 = [0.95, 1]\}$. Let $|J|$ denote the Lebesgue measure of the set J. Then $|J_1| = 0.1$, $|J_2| = 0.4$, $|J_3| = 0.45$, $|J_4| = 0.05$. τ_1 and τ_2 have slopes of the same magnitude on $J_1, \ldots, J_4$. They are $s_1 = 5$, $s_2 = 5/4$, $s_3 = 10/9$, $s_4 = 10$, respectively. The ergodic components of τ_1 are $I_1 = J_1 \cup J_2$ and $I_2 = J_3 \cup J_4$. The holes are $H_{1,2} = J_1$ and $H_{2,1} = J_4$.

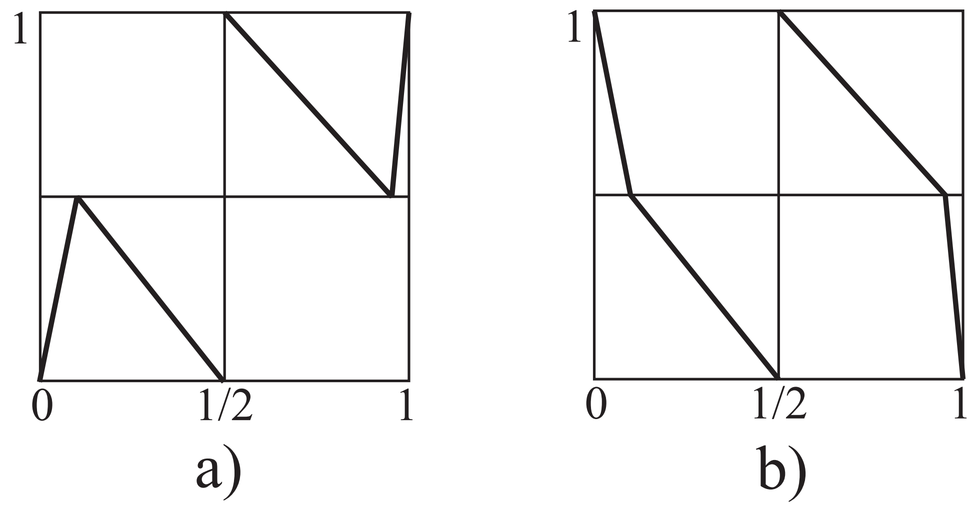

Figure 3.1: Maps τ_1 and τ_2.

Our aim is to compute the acim's of the random maps $T = \{\tau_1, \tau_2, 1, 0\}$ and $T^{(n)} = \{\tau_1, \tau_2, p_1^{(n)}, p_2^{(n)}\}$, where $p_1^{(n)}$ and $p_2^{(n)}$ are defined as in equation (3.12) and (3.13). To this end, we will first compute the invariant densities of τ_1 and τ_2.

The matrices corresponding to Perron-Frobenius operators for τ_1 and τ_2 are

$$M_{\tau_1} = \begin{bmatrix} 1/5 & 1/5 & 0 & 0 \\ 4/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 9/10 \\ 0 & 0 & 1/10 & 1/10 \end{bmatrix} \quad , \quad M_{\tau_2} = \begin{bmatrix} 0 & 0 & 1/5 & 1/5 \\ 4/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 9/10 \\ 1/10 & 1/10 & 0 & 0 \end{bmatrix} .$$

Any invariant density of τ_1 or τ_2 is piecewise constant on the partition $\mathcal{P}$. Moreover, if we denote the value of the invariant density on J_i by f_i, $1 \leq i \leq 4$, then (f_1, f_2, f_3, f_4) is the left eigenvector of the Perron-Frobenius matrix corresponding to eigenvalue 1. For τ_2, one easily checks that $(2/3, 2/3, 4/3, 4/3)$ is the unique normalized invariant density. On the other hand, τ_1 has two ergodic components with acim's μ_1 and μ_2 which are simply the normalized Lebesgue measure on I_1

and I_2, respectively. Any acim of τ_1 is of the form $t\mu_1 + (1-t)\mu_2$, $0 \le t \le 1$.

It follows from equation (3.6) that the invariant density of T is the same as the invariant density of τ_1.

For the random map $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$, equation (3.6) implies that the invariant density $(f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, f_4^{(n)})$ satisfies

$$\begin{aligned} \left(f_1^{(n)}, f_2^{(n)}, f_3^{(n)}, f_4^{(n)}\right) &= \left((1-p_{1,2}^{(n)})f_1, f_2^{(n)}, f_3^{(n)}, (1-p_{2,1}^{(n)})f_4^{(n)}\right) M_{\tau_1} \\ &\quad + \left(p_{1,2}^{(n)} f_1^{(n)}, 0, 0, p_{2,1}^{(n)} f_4^{(n)}\right) M_{\tau_2}, \end{aligned} \tag{3.21}$$

which yields $f_1^{(n)} = f_2^{(n)}$, $f_3^{(n)} = f_4^{(n)}$ and $p_{2,1}^{(n)} f_4^{(n)} = 2p_{1,2}^{(n)} f_1^{(n)}$. So the unique normalized invariant density for $T^{(n)}$ is

$$f^{(n)} = \frac{2}{p_{2,1}^{(n)} + 2p_{1,2}^{(n)}} \left(p_{2,1}^{(n)}, p_{2,1}^{(n)}, 2p_{1,2}^{(n)}, 2p_{1,2}^{(n)}\right).$$

Suppose $\lim_{n\to\infty} p_{2,1}^{(n)}/p_{1,2}^{(n)} = l$. Then $f^{(n)} \to (2/(2+l))(l, l, 2, 2)$. It follows that the invariant measure $\mu_{T^{(n)}} \to \alpha_1\mu_1 + \alpha_2\mu_2$, where $\alpha_1 = (2l)/(l+2)$ and $\alpha_2 = 4/(l+2)$. Moreover,

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{2}l = \frac{0.05}{0.1}l = \frac{\mu_2(H_{2,1})}{\mu_1(H_{1,2})} \lim_{n\to\infty} \frac{p_{2,1}^{(n)}}{p_{1,2}^{(n)}}.$$

The Perron-Frobenius operator for the random map $T^{(n)}$ corresponds to the matrix (already shown in (3.21))

$$M_{T^{(n)}} = \begin{bmatrix} 1/5-(1/5)p_{1,2}^{(n)} & 4/5 & 0 & (1/10)p_{1,2}^{(n)} \\ 1/5 & 4/5 & 0 & 0 \\ 0 & 0 & 9/10 & 1/10 \\ (1/5)p_{2,1}^{(n)} & 0 & 9/10 & 1/10-(1/10)p_{2,1}^{(n)} \end{bmatrix} ,$$

with eigenvalues: 1, $r_1^{(n)} = 1/2-(1/20)p_{2,1}^{(n)}-(1/10)p_{1,2}^{(n)}+a$, $r_2^{(n)} = 1/2-(1/20)p_{2,1}^{(n)}-(1/10)p_{1,2}^{(n)}-a^{(n)}$ and 0, where

$$a^{(n)} = (1/20)\sqrt{100+16p_{2,1}^{(n)}+24p_{1,2}^{(n)}+(p_{2,1}^{(n)})^2+4p_{2,1}^{(n)}p_{1,2}^{(n)}+4(p_{1,2}^{(n)})^2} \ .$$

For $p_{1,2}^{(n)}$ and $p_{2,1}^{(n)}$ close to 0, we have $r_1^{(n)}$ close to 1 and $r_2^{(n)}$ close to 0. For example, if $p_{1,2}^{(n)} = p_{2,1}^{(n)} = 0.01$, then $r_1^{(n)} \sim 0.9995$ and $r_2^{(n)} \sim -0.0025$. The eigenvector corresponding to $r_1^{(n)}$ is $v \sim [-0.749265, -0.751139, 0.375571, 0.373698]$.

Example 3.2.

We present a random map with 3 ergodic components of the original map τ_1, see figure 3.2. Consider maps τ_1 and τ_2 on a set $I=[0,1]$: τ_1 has three ergodic components $I_1=[0,1/3]$, $I_2=[1/3,2/3]$ and $I_3=[2/3,1]$, $\cup_{i=1,2,3}I_i = I$. On each components normalized Lebesgue measure μ_i, $i=1,2,3$, is τ_1-invariant. There are 2 holes in each component. They are

$$H_{1,2}=[1/9,2/9], H_{1,3}=[2/9,1/3] \subset I_1 \ ;$$

$$H_{2,1}=[1/3,4/9], H_{2,3}=[5/9,2/3] \subset I_2 \ ;$$

$$H_{3,1}=[2/3,7/9], H_{3,2}=[7/9,8/9] \subset I_3 \ .$$

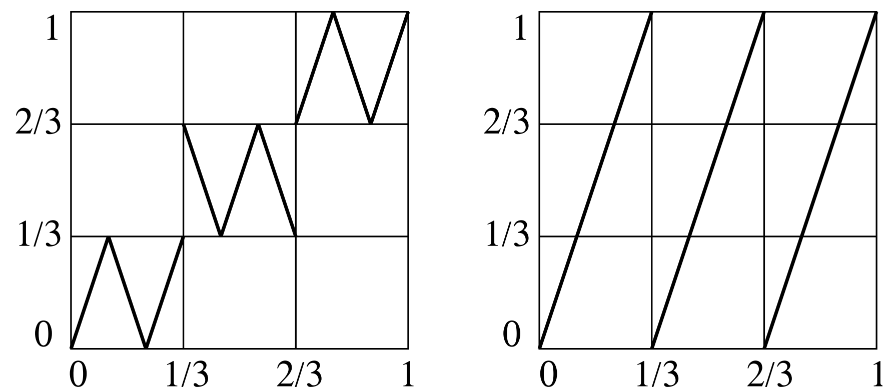

Figure 3.2: Maps τ_1 and τ_2 for Example 3.2 with 3 ergodic components.

Map τ_2 is defined as a piecewise expanding map (shown in Fig. 3.2). It has the following properties

$$\tau_2(H_{i,j}) \subset I_j \ , \ \text{for} \ \ i,j \in \{1,2,3\}$$

and $\tau_2 = \tau_1$ outside the holes.

We define the probabilities that each of the holes will be used by

$$p_{i,j}^{(n)} = h(n)a_{i,j} + o(h(n)), \quad 1 \le i,j \le 3 \ ,$$

where h is such that $\lim_{n\to\infty} h(n) = 0$ and the matrix $A = [a_{i,j}]_{1\le i,j\le 3}$ is given by

$$A = \begin{bmatrix} 0 & 0.3 & 0.5 \\ 0.7 & 0 & 0.2 \\ 0.1 & 0.1 & 0 \end{bmatrix} .$$

The position dependent probability of applying the map τ_2 is defined by

$$p_2^{(n)}(x) = \sum_{i=1,2,3} \sum_{j\neq i} p_{i,j}^{(n)} \chi_{H_{i,j}}(x) \ , \quad x \in I. \tag{3.22}$$

The probability of applying map τ_1 is defined by $p_1^{(n)}(x) = 1 - p_2^{(n)}(x)$, $x \in I$.

Consider the random map $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$. Let $\mu_{T^{(n)}}$ be its invariant measure. By the Continuity Theorem, $\mu_{T^{(n)}} \to \alpha_1\mu_1 + \alpha_2\mu_2 + \alpha_3\mu_3$ as $p_{i,j} \to 0$, $i \neq j$. Since $\mu_i(H_{i,j}) = 1/3$ for $i \neq j$, $1 \leq i, j \leq 3$, by (3.20) we have

$$Q = \frac{1}{30}\begin{bmatrix} 22 & 3 & 5 \\ 7 & 21 & 2 \\ 1 & 1 & 28 \end{bmatrix}.$$

Therefore, the normalized vector $\alpha = (\alpha_1, \alpha_2, \alpha_3) = \frac{1}{78}(16, 11, 51)$.

Example 3.3.

We consider a two dimensional Markov map example with τ_1 having 4 ergodic components. We will use the notation of Example 3.2. The space I is a unit square of the plane $\mathbb{R}^2$. It is divided into 4 identical subsquares I_1, I_2, I_3, I_4 and each of them is further divided into 9 identical smaller subsquares: $I_1 = \cup_{i=1}^{9} S_i$, $I_2 = \cup_{i=10}^{18} S_i$, $I_3 = \cup_{i=19}^{27} S_i$, $I_4 = \cup_{i=28}^{36} S_i$, as in figure 3.6.

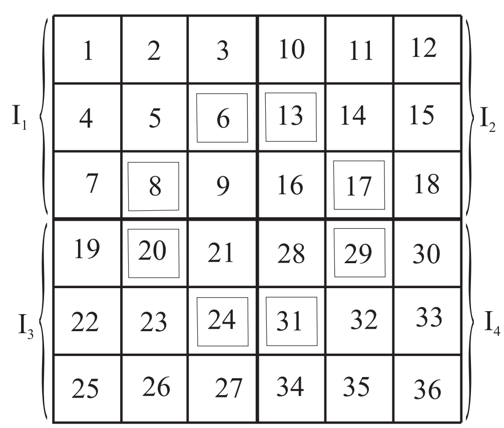

Figure 3.3: The Markov partition for map τ_1 of Example 3.3.

We define τ_1 restricted to each of I_i, $i = 1, 2, 3, 4$, as the same Markov map transforming each square S_j onto four squares S_k in such a way that the corresponding adjacency matrix of the map τ_1 restricted to I_i is

$$M = \frac{1}{4}\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix} .$$

The matrix M_{τ_1} corresponding to τ_1 is the block matrix with 4 matrices M along the diagonal. The map τ_1 has 4 ergodic components. For each component the normalized acim μ_i , $i = 1, 2, 3, 4$, invariant for τ_1 restricted to I_i, can be represented by the vector

$$[\mu_i(1), \mu_i(2), \mu_i(3), \mu_i(4), \mu_i(5), \mu_i(6), \mu_i(7), \mu_i(8), \mu_i(9)]$$
$$= [0.05357, 0.16071, 0.10714, 0.08036, 0.25, 0.16964, 0.02679, 0.08929, 0.0625] .$$

The squares $S_6, S_8, S_{13}, S_{17}, S_{20}, S_{24}, S_{29}, S_{31}$, are designated as holes. We have $S_6 = H_{1,2}$, $S_8 = H_{1,3}$, $S_{13} = H_{2,1}$, $S_{17} = H_{2,4}$, $S_{20} = H_{3,1}$, $S_{24} = H_{3,4}$, $S_{29} = H_{4,2}$, $S_{31} = H_{4,3}$. We have $\mu_1(S_6) = \mu_3(S_{24}) = 0.16964$, $\mu_1(S_8) = \mu_2(S_{17}) = 0.08929$,

$\mu_2(S_{13}) = \mu_4(S_{31}) = 0.08036$, $\mu_3(S_{20}) = \mu_4(S_{29}) = 0.16071$.

We define τ_2 to be the Markov map on I which realizes the transfers. On squares which are not holes it is equal to τ_1. On each of the squares which is a hole τ_2 is a linear map transferring this square onto four squares in appropriate component I_j. The matrix M_{τ_1} has most of its rows the same as the matrix M_{τ_1}, except for rows $6, 8, 13, 17, 20, 24, 29, 31$ which have elements $(6, 10)$, $(6, 11)$, $(6, 13)$, $(6, 14)$, $(8, 19), (8, 20), (8, 22), (8, 23)$, $(13, 5), (13, 6)$, $(13, 8), (13, 9)$, $(17, 29), (17, 30), (17, 32), (17, 33)$, $(20, 4)$, $(20, 5)$, $(20, 7)$, $(20, 8)$, $(24, 31)$, $(24, 32)$, $(24, 34)$, $(24, 35)$, $(29, 14)$, $(29, 15)$, $(29, 17)$, $(29, 18)$, $(31, 20)$, $(31, 21)$, $(31, 23)$, $(31, 24)$, equal to $1/4$ and all other elements 0.

Let h be such that $\lim_{n\to\infty} h(n) = 0$. We define the matrix of transfer probabilities between I_i and I_j as

$$P^{(n)} = \left[p_{i,j}^{(n)}\right]_{1\le i,j\le 4} = h(n)\cdot A \quad , \quad \text{where} \quad A = \begin{bmatrix} 0 & 0.4 & 0.5 & 0 \\ 0.3 & 0 & 0 & 0.8 \\ 0.7 & 0 & 0 & 0.5 \\ 0 & 0.6 & 0.6 & 0 \end{bmatrix} .$$

We define position dependent probabilities $p_1^{(n)}, p_2^{(n)}$ as in (3.17). The random map $T^{(n)} = \{\tau_1, \tau_2; p_1^{(n)}, p_2^{(n)}\}$ has matrix $M_{T^{(n)}}$ with rows the same as the rows of M_{τ_1}

except for rows $6, 8, 13, 17, 20, 24, 29, 31$ defined by

$$\mathrm{row}(6, M_{T^{(n)}}) = (1 - p_{1,2}^{(n)})\mathrm{row}(6, M_{\tau_1}) + p_{1,2}^{(n)} \cdot \mathrm{row}(6, M_{\tau_2}) \ ,$$
$$\mathrm{row}(8, M_{T^{(n)}}) = (1 - p_{1,3}^{(n)})\mathrm{row}(8, M_{\tau_1}) + p_{1,3}^{(n)} \cdot \mathrm{row}(8, M_{\tau_2}) \ ,$$
$$\mathrm{row}(13, M_{T^{(n)}}) = (1 - p_{2,1}^{(n)})\mathrm{row}(13, M_{\tau_1}) + p_{2,1}^{(n)} \cdot \mathrm{row}(13, M_{\tau_2}) \ ,$$
$$\mathrm{row}(17, M_{T^{(n)}}) = (1 - p_{2,4}^{(n)})\mathrm{row}(17, M_{\tau_1}) + p_{2,4}^{(n)} \cdot \mathrm{row}(17, M_{\tau_2}) \ ,$$
$$\mathrm{row}(20, M_{T^{(n)}}) = (1 - p_{3,1}^{(n)})\mathrm{row}(20, M_{\tau_1}) + p_{3,1}^{(n)} \cdot \mathrm{row}(20, M_{\tau_2}) \ ,$$
$$\mathrm{row}(24, M_{T^{(n)}}) = (1 - p_{3,4}^{(n)})\mathrm{row}(24, M_{\tau_1}) + p_{3,4}^{(n)} \cdot \mathrm{row}(24, M_{\tau_2}) \ ,$$
$$\mathrm{row}(29, M_{T^{(n)}}) = (1 - p_{4,2}^{(n)})\mathrm{row}(29, M_{\tau_1}) + p_{4,2}^{(n)} \cdot \mathrm{row}(29, M_{\tau_2}) \ ,$$
$$\mathrm{row}(31, M_{T^{(n)}}) = (1 - p_{4,3}^{(n)})\mathrm{row}(31, M_{\tau_1}) + p_{4,3}^{(n)} \cdot \mathrm{row}(31, M_{\tau_2}) \ .$$

The $T^{(n)}$-invariant measure $\mu_{T^{(n)}}$ has been obtained using Maple. We define the vector $\alpha^{(n)} = [\mu_{T^{(n)}}(I_1), \mu_{T^{(n)}}(I_2), \mu_{T^{(n)}}(I_3), \mu_{T^{(n)}}(I_3)]$. Then,

$$\alpha^{(n)} = \frac{1}{126509}\,[25416, 52668, 14130, 34295] + O(h(n)) \ .$$

The matrix Q is defined as in (3.20). The left 1-eigenvector of Q is equal to $\lim_{n\to\infty} \alpha^{(n)}$.

For $\epsilon := h(n)$ close to 0 the matrix corresponding to Frobenius–Perron operator of $T^{(n)}$ has, except 1, three other eigenvalues close to 1 but different from 1. For $\epsilon = 10^{-3}$ they are 0.9997176900 , 0.9998399077 and 0.9998924535. For $\epsilon = 10^{-4}$ we obtained 0.9999717673, 0.9999839914, 0.9999892419.

Chapter 4

Singular limits of absolutely continuous invariant measures for families of transitive maps

4.1 Introduction

The existence of chaos in deterministic systems has been known for a long time. In such systems it is impossible to make accurate predictions of the long-term behaviour of trajectories. However, it may be possible to make statistical predictions with the use of invariant measures. Of such measures, the ones that are absolutely continuous with respect to the Lebesgue measure play the most important role. In particular, these measures are physically meaningful (in the sense that we explained in the Introduction). Consider a system with a unique absolutely continuous invariant measure . In practice, due to measurement errors, one is really dealing with a perturbation of the system. It is natural to ask whether the acim of the perturbed system is in some sense close to the acim of the unperturbed system. We will consider here one-dimensional dynamical systems and show that even in very simple systems the question of this type of stability is difficult.

When we deal with piecewise expanding maps, we know that for each of them an acim exists, as was proved by Lasota and Yorke [25]. Moreover, if the map is transitive, then this measure is unique (it follows immediately from the results of [26]). Consider the case when there is an invariant interval such that the trajectory of almost every point falls into this interval, and the map restricted to this interval is transitive. Then there is also a unique acim, and it is supported by this invariant interval. Keller in [23] used this property to construct an example in which such an interval exists for some interval of parameters, and as the parameter converges to a limit value, those intervals become shorter and shorter. Then the weak-$*$ limit of acim's is a measure concentrated at one point, while the limit map is transitive and has an acim with the support equal to the whole phase space. He conjectured that this is the only mechanism in which the continuity of the acim's can be violated. We are showing here that other mechanisms can exist.

The chatper is organized as follows. In Section 4.2 we briefly describe Keller's example. In Section 4.3 we construct our own example. Then we study it in Section 4.4, where we compute the invariant density, and in Section 4.5, where we compute limit measures. In Section 4.6 we look at what happens if the slopes on laps (intervals of monotonicity) are constant, similarly as in the Keller's example. Finally, in Section 4.7 we review what we did, and pose some additional questions.

4.2 Keller's example

Keller [23] showed that a large class of piecewise expanding maps, namely those that admit uniform Lasota-Yorke bounds, are acim-stable (in the sense of weak-$*$ convergence of acim's). However, many simple dynamical systems exist that do not fall into this category. Keller's example mentioned in the preceding section looks as follows. Consider a 3-parameter family $\{W_{a,b,r} : 1/2 \le a, b \le 1;\ 0 < r < 1/2\}$ of maps of the interval $[0,1]$ into itself, defined on $[0,1/2]$ by

$$W_{a,b,r}(x) = \begin{cases} a\left(1 - \frac{x}{r}\right) & \text{if } 0 \le x \le r, \\ \frac{2b}{1-2r}(x-r) & \text{if } r \le x \le 1/2, \end{cases}$$

and on $(1/2, 1]$ by $W_{a,b,r}(x) = W_{a,b,r}(1-x)$ (see Figure 4.1).

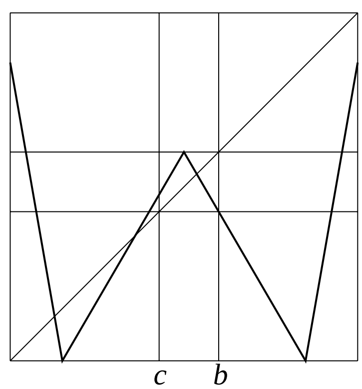

Figure 4.1: Map $W_{6/7,3/5,3/20}$.

Those maps are piecewise expanding and, if $1/2 < b \le 1 - 2r$, then the trajectory of almost every point falls into the invariant interval $[c, b]$, where $c = 2b(1-b-r)/(1-2r)$, on which the map is transitive. Thus, for any sequence (a_n, b_n, r_n) converging to $(a, 1/2, 1/4)$, if $1/2 < b_n \le 1 - 2r_n$ for all n, the acim's of W_{a_n,b_n,r_n} converge to the measure concentrated at 1/2. On the other hand, $W_{a,1/2,1/4}([0,1]) = [0,a]$ and on $[0,a]$ this map is transitive. Therefore it has an acim with the support $[0,a]$. Keller conjectured that for continuous maps of the interval, the only way such acim-instability can occur is if small neighborhoods of the orbit of a periodic turning point of the unperturbed map are invariant under the perturbed maps.

The acim-instability of a dynamical system is closely related to sensitive dependence on parameters defined by M. Misiurewicz in [27]. It is shown in [27] that the popular class of logistic maps has sensitive dependence on parameters which implies they are not acim-stable. However, there the acim-instability is based on the fact that for most of the maps there is no acim, and instead we consider Sinai-Ruelle-Bowen (or *physical*) measures, that are often concentrated on attracting periodic orbits.

Let us also mention that by the result of Raith [30], if the family of maps consists of unimodal maps with constant slope, then we have acim-stability. When we say "slope," we mean the absolute value of the slope.

4.3 Construction of transitive W-maps

Let $I = [0,1]$ and let $T : I \to I$ be a continuous map. Let $\mathcal{P}$ be a partition of I given by the points $0 = a_0 < a_1 < \cdots < a_n = 1$. For $i = 1, \ldots, n$ let $I_i = [a_{i-1}, a_i]$ and denote the restriction of T to I_i by T_i. If T_i is a homeomorphism from I_i onto some connected union of intervals of $\mathcal{P}$, i.e., some interval $[a_{j(i)}, a_{k(i)}]$, then T is said to be Markov. The partition $\mathcal{P} = \{I_i\}_{i=1}^n$ is referred to as a Markov partition with respect to T. If each T_i is also linear on I_i, we say T is a piecewise linear

Markov transformation. For a piecewise linear Markov transformation we define the incidence matrix $A_T = (a_{ij})_{i,j=1}^n$ induced by T and $\mathcal{P}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } I_j \subset T(I_i), \\ 0 & \text{otherwise.} \end{cases}$$

In Keller's example perturbations of the map left a small neighborhood of 1/2 (the turning fixed point of the unperturbed map) invariant. Therefore the measure piled up at 1/2 as the size of the perturbation decreased. In our example we allow for the leak of the measure from small neighborhoods of 1/2. We have some nearly invariant interval surrounding 1/2; we will call it the *box*, because we are thinking about the graph of the map. We define perturbations of the map such that the measure can escape through some small interval centered at 1/2, contained in the box. If Keller's conjecture were true, we would see a convergence of acim's of perturbed maps to the acim of the limiting map. However, by controlling how fast the measure escapes out of the box and how fast it comes back into it, we prove that the measure can still pile up at 1/2. We define perturbations based on three parameters a, b and c, as shown in Figure 4.2. Parameter a represents the size of the box. Parameter b is the size of the opening through which measure escapes. In this way we control how much of the measure escapes out of the box. Parameter c is the height of the peak that sticks out of the box; it controls how long the measure stays out of the box.

More precisely, we define a 3-parameter family $W(a, b, c)$ of piecewise linear maps of the unit interval as follows. If $0 \leq x \leq 1/2$ then

$$W(a,b,c)(x) = \begin{cases} (1-4x) & \text{if } 0 \leq x \leq \frac{1}{4}, \\ \frac{2-2a}{1-2a}\left(x - \frac{1}{4}\right) & \text{if } \frac{1}{4} \leq x \leq \frac{1-a}{2}, \\ \frac{2a}{a-b}\,x - \frac{(1-a)(a+b)}{2(a-b)} & \text{if } \frac{1-a}{2} \leq x \leq \frac{1-b}{2}, \\ \frac{2c}{b}\,x + \frac{1+a}{2} + c - \frac{c}{b} & \text{if } \frac{1-b}{2} \leq x \leq \frac{1}{2}; \end{cases}$$

and if $1/2 < x \leq 1$ then $W(a,b,c)(x) = W(a,b,c)(1-x)$ (see Figure 4.2). In particular, we have $W(a,b,c)(0) = 1$, $W(a,b,c)(1/4) = 0$, $W(a,b,c)((1-a)/2) = (1-a)/2$, $W(a,b,c)((1-b)/2) = (1+a)/2$, $W(a,b,c)(1/2) = (1+a)/2 + c$.

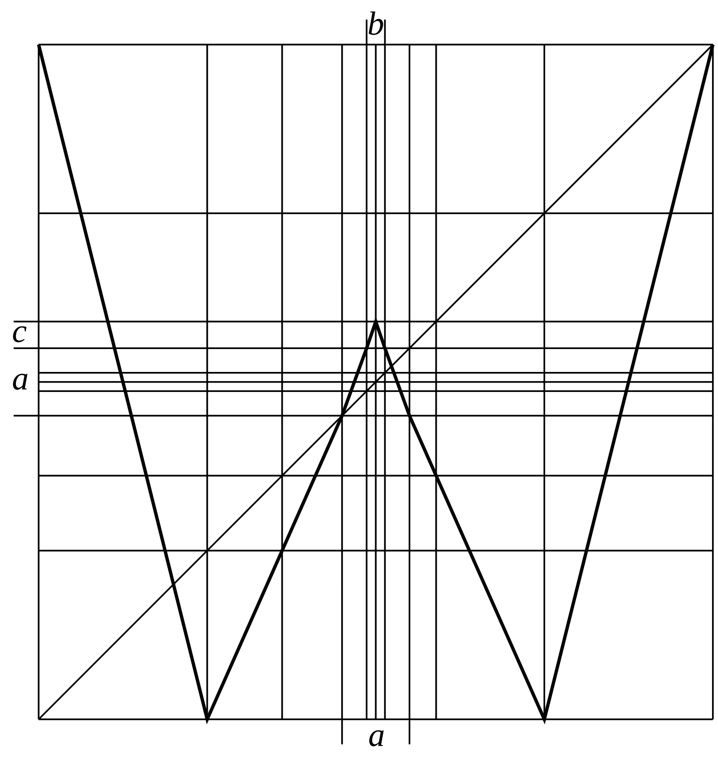

Figure 4.2: Graph of $W(a,b,c) = W_2(a,b)$ for $a = 1/10$, $b = 11/405$, $c = 16/405$, and its Markov partition.

Let

$$s_1 = 4, \quad s_2 = \frac{2-2a}{1-2a}, \quad s_3 = \frac{2a}{a-b}, \quad s_4 = \frac{2c}{b} \tag{4.1}$$

denote the slopes of $W(a, b, c)$ on the consecutive pieces of $[0, 1/2]$ on which the slope is constant.

Lemma 4.1. *If $0 < b < a < 1/2$ and $b < c \le (1-a)/2$, then the map $W(a, b, c)$ is transitive. Likewise, the map $W(0, 0, 0)$ is transitive.*

Proof. Assume that $0 < b < a < 1/2$ and $b < c \le (1-a)/2$. Then all the slopes are larger than some constant $\alpha > 2$. Suppose an interval J of length $|J|$ is contained in a lap of $W(a, b, c)$. Then $W(a, b, c)(J)$ either contains a lap, or contains an interval K contained in a lap, with $|K| > (\alpha/2)|J|$. Since $\alpha/2 > 1$, this proves that for some n the interval $W^n(a, b, c)(J)$ contains a lap. Then $W^{n+1}(a, b, c)(J) \supset [0, 1/2]$, and $W^{n+2}(a, b, c)(J) = [0, 1]$. This proves transitivity of $W(a, b, c)$.

For $W(0, 0, 0)$ the situation is a little more complicated because the slopes of the second and third laps are equal to 2. However, if $K = W(0, 0, 0)(J)$ is contained in the union of the first and second laps or in the union of the third and fourth laps, then (because the slope of the first and fourth laps is 4) the length of $W^2(0, 0, 0)(J)$ is equal to $\max(4p, 2q)$ for some non-negative p, q with $p+q = |K|$. The function $p \mapsto 4p$ is increasing, while the function $p \mapsto 2(|K|-p)$ is decreasing. Therefore the minimum of $\max(4p, 2q)$ occurs at the point where $4p = 2q$, that is, $p = (1/3)|K|$. This proves that $\max(4p, 2q) \ge (4/3)|K|$. Thus, the only reason why the proof from the preceding paragraph may not work for $W(0, 0, 0)$ is that $W^k(0, 0, 0)(J)$ contains $1/2$ in its interior for some k. However, $1/2$ is a fixed point, and its left-hand-sided neighborhood grows under the action of $W(0, 0, 0)$ until some image contains the second lap. Then the next image contains the interval $[0, 1/2]$, and again we get transitivity. $\square$

We will show that there exists a sequence (a_n, b_n, c_n) converging to $(0, 0, 0)$ such that the unique acim's of $W(a_n, b_n, c_n)$ converge to the measure concentrated at $1/2$ rather than the acim of $W(0, 0, 0)$. With other choices of (a_n, b_n, c_n), other behaviours are possible, as described in Theorem 4.1. We will choose the sequence (a_n, b_n, c_n) so that the maps $W(a_n, b_n, c_n)$ are Markov. More precisely, we require that $1/2$ is mapped to a point on the third lap, then for some time the trajectory stays on the second lap, being repelled from the fixed point $(1-a)/2$, until it gets to $1/4$. The number n is such that $W^{n+1}(a_n, b_n, c_n)(1/2) = 1/4$. The point symmetric to $W(a, b, c)(1/2)$ with respect to $1/2$ is $(1-a)/2 - c$ and the slope on the interval $[1/4, (1-a)/2]$ is s_2. Thus we get the equation

$$c \cdot \left(\frac{2-2a}{1-2a}\right)^n = \frac{1-a}{2} - \frac{1}{4}.$$

The solution to this equation is

$$c = c_n(a) = \frac{1-2a}{4} \left(\frac{1-2a}{2-2a}\right)^n. \tag{4.2}$$

When we specify a_n and b_n, then we will take $c_n = c_n(a_n)$.

Let us denote $W_n(a, b) = W(a, b, c_n(a))$. This map is a Markov map on $n + 8$ subintervals $\{I_i\}_{i=1}^{n+8}$. The first subinterval is $[0, 1/4]$. then there come n subintervals of $[1/4, (1-a)/2]$ determined by the images of $1/2$, then 4 subintervals of the box, 2 subintervals of $[(1+a)/2, 3/4]$, and finally $[3/4, 1]$ (see Figure 4.2).

4.4 Invariant density

Generally, the density of an acim for a map of the interval cannot be written in a closed form. However, for a piecewise linear Markov map this density can be calculated. Let T be a piecewise linear Markov map with incidence matrix $A_T = (a_{ij})_{i,j=1}^k$. Define $M_T = (m_{ij})_{i,j=1}^k$ by $m_{ij} = a_{ij}/|T_i'|$. If T admits a unique invariant density, then the invariant density is piecewise constant on the intervals of the Markov partition and is given by the left eigenvector of the matrix M_T corresponding to eigenvalue 1 (for a reference see [9]). This vector is normalized so that the total measure is 1.

Let A_n be the incidence matrix for $W_n(a,b)$. Then the entry a_{ij} of A_n is equal to 1 in the following cases:

- $1 \le j \le n$ and $i \in \{1, j+1, n+7, n+8\}$,
- $j = n+1$ and $i \in \{1, n+1, n+6, n+8\}$,
- $n+2 \le j \le n+5$ and $i \in \{1, n+2, n+5, n+8\}$,
- $j = n+6$ and $i \in \{1, n+3, n+4, n+8\}$,
- $n+7 \le j \le n+8$ and $i \in \{1, n+8\}$.

The slopes $|T_i'|$, according to (4.1), are

- 4 if $i \in \{1, n+8\}$,
- $(2-2a)/(1-2a)$ if $2 \le i \le n+1$ or $i \in \{n+6, n+7\}$,
- $2a/(a-b)$ if $i \in \{n+2, n+5\}$,
- $2c_n(a)/b$ if $i \in \{n+3, n+4\}$.

This gives us the following equations for our eigenvector $(x_1, x_2, \ldots, x_{n+8})$. If $1 \le j \le n$ then

$$x_j = \frac{x_1}{4} + \frac{1-2a}{2-2a} x_{j+1} + \frac{1-2a}{2-2a} x_{n+7} + \frac{x_{n+8}}{4}, \tag{4.3}$$

$$x_{n+1} = \frac{x_1}{4} + \frac{1-2a}{2-2a} x_{n+1} + \frac{1-2a}{2-2a} x_{n+6} + \frac{x_{n+8}}{4}, \tag{4.4}$$

if $n+2 \le j \le n+5$ then

$$x_j = \frac{x_1}{4} + \frac{a-b}{2a} x_{n+2} + \frac{a-b}{2a} x_{n+5} + \frac{x_{n+8}}{4}, \tag{4.5}$$

$$x_{n+6} = \frac{x_1}{4} + \frac{b}{2c_n(a)} x_{n+3} + \frac{b}{2c_n(a)} x_{n+4} + \frac{x_{n+8}}{4}, \tag{4.6}$$

if $n+7 \le j \le n+8$ then

$$x_j = \frac{x_1}{4} + \frac{x_{n+8}}{4}. \tag{4.7}$$

Set

$$x_1 = 1. \tag{4.8}$$

Then from (4.7) we get

$$x_{n+7} = x_{n+8} = \frac{1}{3}. \tag{4.9}$$

Next, from (4.5) we get

$$x_{n+2} = x_{n+3} = x_{n+4} = x_{n+5} = \frac{a}{3b}. \tag{4.10}$$

Further, from (4.6) we get

$$x_{n+6} = \frac{1}{3} + \frac{a}{3c_n(a)}. \tag{4.11}$$

Finally, from (4.4) we get

$$x_{n+1} = 1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_n(a)}. \tag{4.12}$$

Now, in order to compute $x_2, x_3, \ldots, x_n$, we rewrite (4.3) as

$$x_{j+1} = \frac{2-2a}{1-2a} x_j - \frac{3-4a}{3-6a}.$$

From this, we get by induction

$$x_j = 1 - \frac{4a}{3} + \frac{4a}{3}\left(\frac{2-2a}{1-2a}\right)^{j-1}.$$

Taking into account (4.2), we get for $1 \le j \le n+1$

$$x_j = 1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_{j-1}(a)}. \tag{4.13}$$

Note that for $j = n+1$ this agrees with (4.12).

Now we have to find the normalizing factor

$$C = \sum_{j=1}^{n+8} |I_j| x_j. \tag{4.14}$$

The lengths of intervals I_j of our Markov partition are:

- $1/4$ if $j \in \{1, n+8\}$,
- $c_{j-2} - c_{j-1}$ if $2 \le j \le n$,
- c_{n-1} if $j = n+1$,

- $(a-b)/2$ if $j \in \{n+2, n+5\}$,
- $b/2$ if $j \in \{n+3, n+4\}$,
- $c_n(a)$ if $j = n+6$,
- $(1-2a)/4 - c_n(a)$ if $j = n+7$.

Let us look at various parts of the sum (4.14) and their limits as a, b go to 0 and n goes to infinity (so $c_n(a) \to 0$). We have

$$|I_1|x_1 + \sum_{j=n+6}^{n+8} |I_j|x_j = \frac{1}{4}\cdot 1 + c_n(a)\cdot\left(\frac{1}{3} + \frac{a}{3c_n(a)}\right) + \left(\frac{1-2a}{4} - c_n(a)\right)\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{3} \to \frac{5}{12}, \tag{4.15}$$

$$\sum_{j=n+2}^{n+5} |I_j|x_j = \left(2\cdot\frac{a-b}{2} + 2\cdot\frac{b}{2}\right)\cdot\frac{a}{3b} = \frac{a^2}{3b}, \tag{4.16}$$

and

$$|I_{n+1}|x_{n+1} = c_{n-1}(a)\cdot\left(1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_n(a)}\right) \to 0. \tag{4.17}$$

Finally,

$$\sum_{j=2}^{n} |I_j|x_j = \sum_{j=2}^{n} (c_{j-2}(a) - c_{j-1}(a))\cdot\left(1 - \frac{4a}{3} + \frac{a(1-2a)}{3c_{j-1}(a)}\right).$$

We will compute this sum in two steps. First,

$$\sum_{j=2}^{n} (c_{j-2}(a) - c_{j-1}(a))\cdot\left(1 - \frac{4a}{3}\right) = (c_0(a) - c_{n-1}(a))\frac{3-4a}{3} \to \frac{1}{4}. \tag{4.18}$$

Next, since $c_{j-2}(a) - c_{j-1}(a) = c_{j-1}(a)/(1-2a)$, we have

$$\sum_{j=2}^{n} (c_{j-2}(a) - c_{j-1}(a))\cdot\frac{a(1-2a)}{3c_{j-1}(a)} = (n-1)\frac{a}{3}. \tag{4.19}$$

Thus, we get

$$\liminf \sum_{j=2}^{n} |I_j| x_j = \frac{1}{4} + \liminf \frac{na}{3}, \quad \limsup \sum_{j=2}^{n} |I_j| x_j = \frac{1}{4} + \limsup \frac{na}{3}. \quad (4.20)$$

Now we see that the behaviour of the invariant density as a, b go to 0 and n goes to infinity depends on the behaviour of the quantities a^2/b and na. However, it turns out that only a^2/b matters.

Lemma 4.2. *Let $0 < b_n < a_n < 1/2$ and $b_n < c_n \leq (1-a_n)/2$ with $a_n, b_n, c_n \to 0$ as $n \to \infty$, and $c_n = c_n(a)$. If $na_n \to \alpha$ on a subsequence, with $\alpha \in (0, \infty]$, then $(a_n^2/b_n)/(na_n) \to \infty$ on the same subsequence.*

Proof. By (4.2), we have $c_n < 2^{-n-2}$, so $1/b_n > 2^{n+2}$. Therefore

$$\frac{a_n^2/b_n}{(na_n)^2} > \frac{2^{n+2}}{n^2} \to \infty$$

as $n \to \infty$. Thus, if $na_n \to \alpha > 0$ on a subsequence, then $(a_n^2/b_n)/(na_n) \to \alpha \cdot \infty = \infty$ on the same subsequence. □

Using the same methods, it is very easy to find the density of the acim for $W(0,0,0)$. We get a Markov partition into 4 intervals: $[0, 1/4]$, $[1/4, 1/2]$, $[1/2, 3/4]$ and $[3/4, 1]$. The density on the first two intervals is $3/2$, and on the last two $1/2$.

4.5 Limit measures

Now we investigate what happens with the acim's μ_n for $W_n(a_n, b_n)$ as n goes to infinity and a_n, b_n go to 0. We denote by μ the acim for $W(0,0,0)$ and by $\delta_{1/2}$ the Dirac delta measure at $1/2$.

Theorem 4.1. *Let $0 < b_n < a_n < 1/2$ and $b_n < c_n(a_n) \leq (1-a_n)/2$ with $a_n, b_n \to 0$ and $a_n^2/b_n \to \beta \in [0,\infty]$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \mu_n = \frac{2}{2+\beta}\mu + \frac{\beta}{2+\beta}\delta_{1/2} \tag{4.21}$$

in the weak-$*$ topology.

Proof. We can write μ_n as the sum of three measures: $\nu_n + \sigma_n + \tau_n$, defined as follows. They are all absolutely continuous with respect to the Lebesgue measure, and their densities are:

- for ν_n:

  x_j/C on I_j for $j = 1, n+1, n+6, n+7, n+8$,

  $(3-4a)/(3C)$ on I_j for $j = 2, \dots, n$,

  0 on I_j for $j = n+2, \dots, n+5$,

- for σ_n:

  x_j/C on I_j for $j = n+2, \dots, n+5$,

  0 on all other I_j,

- for τ_n:

  $a(1-2a)/(3Cc_{j-1}(a))$ on I_j for $j = 2, \dots, n$,

  0 on all other I_j,

where $a = a_n$, and I_j, x_j and C depend on n.

Consider now three cases, depending on the value of β.

Case I: $\beta = 0$. Then by (4.16), $\sigma_n \to 0$. Moreover, by Lemma 4.2, $na_n \to 0$, so by (4.19) $\tau_n \to 0$. Therefore the limit of the measures μ_n is the same as the limit of measures ν_n. By (4.15), (4.16), (4.17) and (4.20), the limit of C as $n \to \infty$ is $5/12 + 1/4 = 2/3$, and thus the density of ν_n is $3/2$ on $[0, 1/4]$, $1/2$ on $I_{n+7} \cup I_{n+8}$ (and this interval converges to $[1/2, 1]$), and $(3-4a)/2$ on $\bigcup_{j=2}^{n} I_j$ (and this interval converges to $[1/4, 1/2]$. The total measure on remaining intervals converges to 0, and thus $\nu_n \to \mu$. This proves (4.21) in this case.

Case II: $\beta \in (0, \infty)$. The only difference between this case and the preceding one is that this time σ_n converges to a positive constant times $\delta_{1/2}$. This changes the constant by which we divide x_j's to get the density of ν_n. By (4.16) and the computations from Case I we know that

$$\lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \nu_n([0,1]) = \frac{\beta/3}{2/3} = \frac{\beta}{2}.$$

Thus the constant mentioned above is $\beta/(2+\beta)$, and the limit of the measures ν_n is $2/(2+\beta)$ times μ instead of just μ. This proves (4.21) in this case.

Case III: $\beta = \infty$. Then

$$\lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \nu_n([0,1]) = \lim_{n\to\infty} \sigma_n([0,1]) / \lim_{n\to\infty} \tau_n([0,1]) = \infty,$$

so

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \sigma_n = \delta_{1/2}.$$

This proves (4.21) in this case. □

The above theorem does not yet prove that the example we claimed we built really exists. Namely, we have to show that the sequences (a_n) and (b_n) satisfying its conditions exist. We can also settle the question whether in such examples we can have slopes bounded independently of n.

Theorem 4.2. *For every $\beta \in [0, \infty]$ there exist sequences (a_n) and (b_n) satisfying the assumptions of Theorem 4.1 and such that for sufficiently large n all slopes of the maps $W_n(a_n, b_n)$ are in $(2, 4]$.*

Proof. The slopes of $W_n(a_n, b_n)$ are 4, $(2 - 2a_n)/(1 - 2a_n)$, $2a_n/(a_n - b_n)$ and $2c_n(a_n)/b_n$. Under the assumptions of Theorem 4.1, they are all larger than 2. Additional conditions guaranteeing that they are not larger than 4 are

$$a_n \leq \frac{1}{3}, \quad b_n \leq \frac{a_n}{2}, \quad c_n(a_n) \leq 2b_n.$$

Thus, we need to show that we can find sequences (a_n) and (b_n) of positive numbers convergent to 0, with $a_n^2/b_n \to \beta$ and

$$2b_n \leq a_n \leq \frac{1}{3}, \quad \frac{c_n(a_n)}{2} \leq b_n < c_n(a_n) < \frac{1 - a_n}{2}$$

when n is sufficiently large.

We define numbers β_n as follows. If $\beta = 0$ then $\beta_n = 1/n$. If $\beta \in (0, \infty)$ then $\beta_n = \beta$ for all n. If $\beta = \infty$, then $\beta_n = n$. Then we define continuous functions $f_n : [0, 1/2) \to \mathbb{R}$ by

$$f_n(a) = \frac{5a^2}{4c_n(a)}.$$

Note that $f_n(0) = 0$ and if $a > 0$ then $f_n(a) > 5a^2 2^n$. For all values of β we have $\sqrt{\beta_n/5} \cdot 2^{-n/2} \to 0$, so for sufficiently large n there exists $a_n \in (0, \sqrt{\beta_n/5} \cdot 2^{-n/2})$ such that $f_n(a_n) = \beta_n$, and we have $a_n \to 0$. Therefore $c_n(a_n) < (1 - a_n)/2$ for sufficiently large n.

Set $b_n = (4/5)c_n(a_n)$. Then $c_n(a_n)/2 < b_n < c_n(a_n)$. Moreover, $a_n^2/b_n = f_n(a_n) = \beta_n$, so $a_n^2/b_n \to \beta$. We have

$$\frac{a_n}{b_n} = \frac{\beta_n}{a_n} > \frac{\beta_n}{\sqrt{\beta_n/5} \cdot 2^{-n/2}} = \sqrt{5\beta_n} \cdot 2^{n/2} \to \infty.$$

Therefore $2b_n \leq a_n$ for sufficiently large n. Thus, the sequences (a_n) and (b_n) satisfy all properties they were supposed to satisfy. □

4.6 Maps with constant slopes on laps

In this section we study the special case when the slope of $W(a, b, c)$ is constant on each lap of the map (see Figure 4.3). This means that

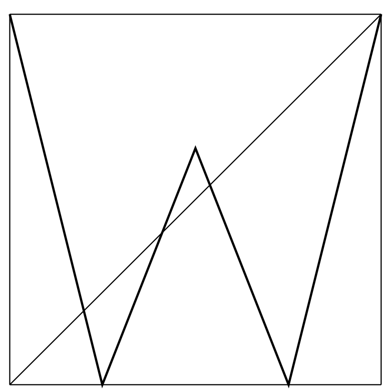

Figure 4.3: Map $W(a, b, c)$ with constant slopes on laps.

$$\frac{2 - 2a}{1 - 2a} = \frac{2a}{a - b} = \frac{2c_n(a)}{b},$$

that is,

$$b = \frac{a^2}{1 - a}, \quad c_n(a) = \frac{a^2}{1 - 2a}. \tag{4.22}$$

In view of (4.2), we get an equation

$$\left(\frac{2a}{1-2a}\right)^2 = \left(\frac{1-2a}{2-2a}\right)^n.$$

For $a = 0$, the left hand side of this equation is 0, while the right hand side is positive. For $a = 2^{-(n+2)/2}$ the left hand side is larger than 2^{-n}, while the right hand side is smaller than 2^{-n}. Therefore it has a solution $a_n \in (0, 2^{-(n+2)/2})$. Then we set $b_n = a_n^2/(1-a_n)$.

Let us check whether the assumptions of Theorem 4.1 are satisfied. Since $a_n \in (0, 2^{-(n+2)/2})$, we get $a_n < 1/2$ and $a_n \to 0$. Then $0 < b_n < a_n$ and $b_n < c_n(a_n)$ follow immediately from (4.22). The inequality $c_n(a_n) \le (1-a_n)/2$ is equivalent to $a_n \le 1/3$, so it is satisfied for all $n \ge 2$. We have $a_n^2/b_n = 1-a_n \to 1$. Therefore, by Theorem 4.1 we get

$$\mu_n \to \frac{2}{3}\mu + \frac{1}{3}\delta_{1/2}.$$

Thus, even in this simple case the limit of the acim's of the maps $W_n(a_n, b_n)$ is not the acim for $W(0,0,0)$.

4.7 Discussion and questions

Let us review our example. As we mentioned in Section 4.3, parameters a, b and c play different roles. The size b of the hole in the box, compared to the size a of the box, determine how fast the measure leaks from the box. The parameter c controls how long the part of the measure that left the box stays outside. However, according to Theorem 4.1, only the ratio a^2/b plays any role in determining the limit behaviour. This is due to the additional assumption that $b < c$. It is a technical assumption, used in Lemma 4.1 to make the slopes larger than 2. In fact, that lemma is probably also true without this assumption; while some slopes may be even less than 1, for an appropriate iterate of the map they should become larger than 2. Thus, we are left with the question: why does it seem that the size of c is irrelevant in the limit behaviour of acim's? The answer is in Lemma 4.2. For this lemma to hold, we need $n^2 b$ to converge to 0, and if c is too small then n is too large. Thus, the heuristic arguments are correct.

Let us now pose a couple of questions. The first one is whether it is important in our example that the maps are Markov (or even Markov with this specific Markov partition). While the "common sense" suggests that everything should be similar in the non-Markov case, estimates of the density of the acim do not seem to be simple.

The second question is about unimodal maps. As we mentioned in Section 4.2, if the family of the maps consists of unimodal maps with the constant slope, in this family we have acim-stability. However, there is an interesting family of unimodal maps, for which the acim-stability is unknown. It is defined as follows (see Figure 4.4).

$$A(a,b)(x) = \begin{cases} \frac{1-a}{b}\, x + a & \text{if } 0 \le x \le b, \\ \frac{1}{1-b}(1-x) & \text{if } b \le x \le 1. \end{cases}$$

Consider the map $A(1/2, 1/2)$. It seems that this map is acim-stable in this family. This example is the simplest example one can make whose acim-stability seem not to follow from any of the existing techniques. We remark that this map

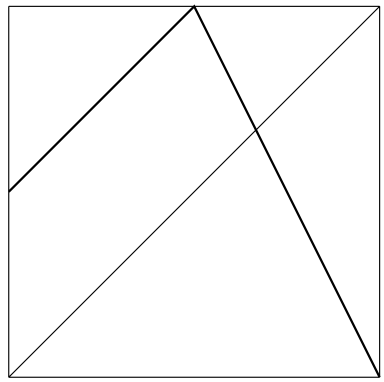

Figure 4.4: Map $A(1/2, 1/2)$.

is not a "good" map as defined in [4]. A unimodal map is good in this sense if its critical point is not periodic or it is periodic of period n and $\inf |(f^n)'| > 2$.

Chapter 5

Conclusions

In this work we tackled the problem of existence and stability of acim's in some chaotic dynamical systems in one and higher dimensions.

First, we considered the problem of eventual expansion of maps of the unit interval. Since this property is a common assumption in most theorems on the existence of acim's, it is important that one be able to verify whether a map is eventually expanding or not. We constructed a family of piecewise linear maps defined on two laps, one expanding and one contracting, and we showed that under additional assumptions these maps are eventually expanding. We conjectured that such maps are eventually expanding in general (without additional assumptions on the slopes). As described in the Introduction, the validity of this conjecture follows from a recent preprint (see [13]); however, the methods of proof are different. We used elementary mathematics to prove the eventual expansion of our family of maps. It is conceivable that this problem and its proof could be generalized to other piecewise linear maps of the interval or even slightly nonlinear ones by approximation. These results could also be investigated in a higher dimensional setting. In general, the problem of eventual expansion has rarely been studied and there is room for much improvement.

Next we turned to the problem of stability. In Chapter 3, we showed that random maps with position dependent probabilities in a multidimensional setting are acim-stable under weak expansion conditions (in the sense that maps with low probabilities could be non-expanding). We used this result to model metastable systems which often appear as a result of perturbation of a a dynamical system with several invariant components. It was shown that the acim's of perturbations converge to a convex combination of ergodic acim's of the unperturbed map with weights proportional to probabilities of escape through the holes.

We also studied the stability of a class of continuous, piecewise linear maps of the interval called W-maps. The main characteristic of W-maps is that they contain a periodic turning point. It has been well-known that perturbations near such points create difficulties in obtaining stability results; however, to our knowledge there has not been any stability results for this type of systems up to now. In Chapter 4, a three-parameter family of W-maps where constructed to show that transitivity of continuous maps of the interval containing a fixed turning point does not imply their acim-stability. There are still many open questions left unanswered regarding the stability of maps with periodic turning points. Section 4.7 discusses a few such questions.

W-maps, as described in Chapter 4, also exhibit a metastable behaviour. It would be interesting to explore this property in higher dimensional setting and for other maps of the interval with periodic turning points.

In conclusion, this work has attempted to shed light on some of the darker corners of the area of dynamical systems as far as existence and stability of acim's are concerned.

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