Comparison theorems for the principal eigenvalue of the Laplacian

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Science (Mathematics) at

Concordia University

Montreal, Quebec, Canada

June 2011

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Yasmine Raad

Entitled: Comparison theorems for the principal eigenvalue of the Laplacian

and submitted in partial fulfillment of the requirements for the degree of

Master of Science (Mathematics)

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ABSTRACT

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We study the Faber - Krahn inequality for the Dirichlet eigenvalue problem of the Laplacian, first in $\mathbb{R}^N$, then on a compact smooth Riemannian manifold M. For the latter, we consider two cases. In the first case, the compact manifold has a lower bound on the Ricci curvature, in the second, the integral of the reciprocal of an isoperimetric estimator function of the Riemannian manifold is convergent. In all cases, we show that the first eigenvalue of a domain in $\mathbb{R}^N$, respectively M, is minimal for the ball of the same volume, respectively, for a geodesic ball of the same relative volume in an appropriate manifold M^*. While working with the isoperimetric estimator, the manifold M^* need not have constant sectional curvature. In $\mathbb{R}^N$, we also consider the Neumann eigenvalue problem and present the Szegö - Weinberger inequality. In this case, the principal eigenvalue of the ball is maximal among all principal eigenvalues of domains with same volume.

Acknowledgments

I would like to thank very much my supervisor Dr. Alina Stancu for all the guidance and help that she has provided. I am very lucky that I had the chance to work with her in such a relaxing and friendly environment. Our meetings on a regular basis, her patience and knowledge helped very much in completing this thesis.

Contents

Notations

- $\mathbb{R}^+$ denotes all positive real numbers.
- $\mathbb{R}^N$ denotes the N-dimensional Euclidean space.
- For any measurable subset E of $\mathbb{R}^N$, $|E|$ denotes the N-dimensional Lebesgue measure.
- For any $x \in \mathbb{R}^N$, $\|x\|$ denotes the standard, Euclidean norm of x in $\mathbb{R}^N$.

Let Ω be a bounded domain (open, connected) in $\mathbb{R}^N$, and let $u : \Omega \to \mathbb{R}$ be an arbitrary smooth function on Ω. Then

- $\partial\Omega = \bar{\Omega} \setminus \Omega$ denotes the boundary of Ω.
- ∇u denotes the gradient of u in Ω.
- $\Delta u = \operatorname{div}(\nabla u)$ is the Laplacian of u in Ω.
- Let $A = \{a \in \mathbb{R}, |\{u > a\}| = 0\}$ be the set of essential upper bounds and $B = \{b \in \mathbb{R}, |\{u < b\}| = 0\}$ be, respectively, the set of essential lower bounds. Then

$$\operatorname{ess\,sup}(u) = \begin{cases} \inf A, & \text{if } \ A \neq \emptyset, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1}$$

and

$$\operatorname{ess\,inf}(u) = \begin{cases} \sup B, & \text{if } \ B \neq \emptyset, \\ -\infty, & \text{otherwise.} \end{cases} \tag{2}$$

- $C(\Omega)$ denotes the space of continuous functions on Ω.
- $C(\overline{\Omega})$ denotes the space of continuous functions on $\bar{\Omega} = \Omega \cup \partial\Omega$.
- $C^k(\Omega)$ denotes the space of k-times continuously differentiable functions on Ω, for $1 \leq k < \infty$.

- $C^k(\overline{\Omega})$ denotes the space of k-times continuously differentiable functions on $\overline{\Omega}$ whose all derivatives up to the order k have continuous extensions to $\overline{\Omega}$.

- C^∞ denotes the space of infinitely differentiable functions, or also called smooth functions.

- $\mathcal{D}(\Omega)$ denotes the space of $C^\infty(\Omega)$ functions with compact support in Ω.

- $L^p(\Omega)$ denotes the space of measurable functions that are p-integrable; its norm is denoted by $\| \, . \, \|_{p,\Omega}$.

- $W^{k,p}(\Omega)$ denotes the Sobolev space of order k of functions in $L^p(\Omega)$ whose all (distribution) derivatives up to order k are in $L^p(\Omega)$.

- $W_0^{k,p}(\Omega)$ denotes the closure of $D(\Omega)$ in $W^{k,p}(\Omega)$.

- $H^m(\Omega) = W^{m,2}(\Omega)$.

- $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

Let (M, g) be a smooth Riemmanian manifold with Riemannian metric g. Let Ω be a bounded domain in M and let u be a smooth real function on Ω. Then

- $V(\Omega)$ denotes the n-dimensional volume with respect to g, also referred to as the volume, of a domain $\Omega \subseteq M$.

- $A(\Omega)$ denotes the $(n-1)$-dimensional volume of $\partial\Omega$ induced by g, also called the surface area of $\Omega \subseteq M$.

- $|\nabla u|$ denotes the norm of the gradient of u whether it is in $\mathbb{R}^N$ or with the metric on the Riemannian manifold (M, g).

Chapter 1

Introduction and Prerequisites

1.1 Introduction

Our aim was to understand the use of symmetrization in proving isoperimetric inequalities. Studying the Schwarz symmetrization led naturally to the Faber - Krahn and Szegö - Weinberger inequalities for a class of elliptic operators in $\mathbb{R}^N$ which included the Laplacian. These inequalities compare the principal eigenvalue of the operator on a bounded domain $\Omega \subset \mathbb{R}^N$ to the corresponding eigenvalue of the operator on a ball of $\mathbb{R}^N$ with the same volume as Ω. At the core of the proofs lie the Schwarz symmetrization of the eigenfunction on Ω. Along the way, we consider a couple of variations of the eigenvalue problem of the Laplace operator on bounded Euclidean domains from $-\Delta u = \lambda u$ to $-\Delta u = \lambda P u$, with P any positive continuous function, with both Dirichlet and Neumann boundary conditions. Following the work of Kesavan, we give our own proofs of two inequalities related to this new problem, and generalizing Faber - Krahn, respectively Szego - Weinberger for the Laplace operator in $\mathbb{R}^N$, see Proposition 2.3.1, and Proposition 2.4.1.

Next, we consider the Laplace - Beltrami operator on a smooth N-dimensional Riemannian manifold (M, g) without boundary. There exists a classical result of

Chavel [3], going back to the 20's, which states roughly the following. If for any $\Omega \subseteq M$ open domain whose volume equal to the volume of a geodesic ball $B_k(r)$ in an N-dimensional manifold M^* with constant sectional curvature k, we have that $Area(\partial\Omega) \geq Area(\partial B_k(r))$, then the first Dirichlet eigenvalue of the Laplacian on Ω is greater than the first Dirichlet eigenvalue of the Laplacian on $B_k(r)$. Given an arbitrary Riemannian manifold, this hypothesis is hard to check. It is, however, implied by a certain Ricci curvature bound and we choose to present here the proof of the Faber - Krahn inequality on (M, g) under this Ricci curvature bound, which also implies the compactness of M. Once again one symmetrizes the eigenfunction corresponding to $\lambda(\Omega)$ which, even if on (M, g) versus $(M^* = S^N, g^*)$, resembles closely the Schwarz symmetrization in $\mathbb{R}^N$. Here S^N is the unit sphere in $\mathbb{R}^{N+1}$ with the induced metric. Finally, we asked ourselves whether M^* really needs to be a space of constant curvature or is it only a matter of convenience implied by the above symmetrization. There are very few results in this direction. We encountered a comment made by Bérard in [1], page 96, about the validity of the Faber - Krahn inequality on a compact Riemannian manifold M where the principal eigenvalue of the Laplacian on a domain $\Omega \subset M$ is compared with the corresponding principal eigenvalue of a domain in a manifold M^* which has revolution symmetry, but, in general, it is not S^N. The hypothesis needed here is based on the isoperimetric profile of the manifold M and, unfortunately, in general, it is not easy to check either. The proof of Theorem 3.4.1 is the result of our own work to conclude Bérard's assertion.

The paper is structured as follows. We continue this chapter with the basics of the Schwarz symmetrization. In Chapter 2, we concentrate on eigenvalue problems for domains in $\mathbb{R}^N$, while Chapter 3 deals entirely with the Dirichlet eigenvalue problem on Riemannian manifolds.

1.2 Schwarz symmetrization

For any measurable subset E of $\mathbb{R}^N$, denote by $|E|$ its N-dimensional Lebesgue measure, and by E^* the ball centered at the origin and having the same Lebesgue measure as E. We will denote by ω_N, the volume of the unit ball in $\mathbb{R}^N$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded set and let $u : \Omega \to \mathbb{R}$ be a measurable function. For $t \in \mathbb{R}$, given the following level sets

$$\{u > t\} = \{x \in \mathbb{R}^N; u(x) > t\}, \tag{1.1}$$

the distribution function of u is defined by

$$\mu_u(t) = |\{u > t\}|. \tag{1.2}$$

This is a decreasing function that takes the value 0 for $t \geq \operatorname{ess\,sup}(u)$, and the value $|\Omega|$ for $t \leq \operatorname{ess\,inf}(u)$. Therefore the range of this function is $[0, |\Omega|]$.

Furthermore, $u^{\#}$, the decreasing rearrangement of u, is given as

$$\begin{aligned} u^{\#} : [0, |\Omega|] &\to \mathbb{R} \\ s &\to u^{\#}(s) = \begin{cases} \operatorname{ess\,sup}(u), & \text{if} \quad s = 0, \\ \inf\{t\,;\, \mu_u(t) < s\}, & \text{if} \quad s > 0. \end{cases} \end{aligned} \tag{1.3}$$

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded set. Let $u : \Omega \to \mathbb{R}$ be a measurable function. Its Schwarz symmetrization u^*, also known as the spherically symmetric decreasing rearrangement, is the function*

$$\begin{aligned} u^* : \Omega^* &\to \mathbb{R} \\ x &\mapsto u^*(x) = u^{\#}(\omega_N \|x\|^N). \end{aligned} \tag{1.4}$$

Properties of the Schwarz symmetrization [10]

a. The function u^* is radially symmetric and decreasing.

b. The functions u, $u^{\#}$, u^* are all equimeasurable, i.e. they all have the same distribution function. More precisely,

$$|\{u > t\}| = |\{u^{\#} > t\}| = |\{u^* > t\}|, \quad \forall t \in \mathbb{R}. \tag{1.5}$$

c. If $F : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $F(u) \in L^1(\Omega)$, or $F(u) \geq 0$, then

$$\int_\Omega F(u(x))\,dx = \int_{\Omega^*} F(u^*(x))\,dx. \tag{1.6}$$

In particular,

$$\int_\Omega u(x)\,dx = \int_{\Omega^*} u^*(x)\,dx, \tag{1.7}$$

when u is integrable over Ω. Moreover, u and u^* have the same L^p norms, i.e. $\|u\|_{p,\Omega} = \|u^*\|_{p,\Omega^*}$.

d. If $\psi : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then

$$(\psi(u))^* = \psi(u^*). \tag{1.8}$$

e. For any measurable set $E \subset \Omega$, we have

$$\int_E u(x)\,dx \leq \int_0^{|E|} u^{\#}(s)\,ds = \int_{E^*} u^*(x)\,dx. \tag{1.9}$$

The equality holds above if and only if $(u|_E)^* = u^*|_{E^*}$.

f. The Hardy - Littlewood inequality: For $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, where p and q are such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then

$$\int_\Omega f(x)g(x)\,dx \leq \int_0^{|\Omega|} f^\#(s)g^\#(s)\,ds = \int_{\Omega^*} f^*(x)g^*(x)\,dx. \tag{1.10}$$

g. The Pólya - Szegö inequality: For $1 \leq p < \infty$, and $u \in W_0^{1,p}(\Omega)$, with $u \geq 0$, then

$$\int_{\Omega^*} |\nabla u^*|^p\,dx \leq \int_\Omega |\nabla u|^p\,dx. \tag{1.11}$$

Moreover, $u^* \in W_0^{1,p}(\Omega^*)$.

Chapter 2

The Eigenvalue Problem in $\mathbb{R}^N$

2.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For $0 < \alpha < \beta$, define $\mathcal{M}(\alpha, \beta, \Omega)$ to be the set of all $N \times N$ matrices $A = (a_{ij}(x))$, whose coefficients are functions on Ω and they satisfy the ellipticity condition

$$\alpha \|\xi\|^2 \leq A\xi \cdot \xi \leq \beta \|\xi\|^2, \quad \forall \xi \in \mathbb{R}^N, \tag{2.1}$$

for almost every $x \in \Omega$.

Define the second order elliptic differential operator $\mathfrak{L}$ corresponding to such a matrix by

$$\mathfrak{L}(u) = -\mathrm{div}\,(A\nabla u), \tag{2.2}$$

and consider, for a given $f \in L^2(\Omega)$, the elliptic boundary value problem:

$$\begin{cases} \mathfrak{L}(u) = f & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

A function $u \in C^2(\overline{\Omega})$ that satisfies (2.3) is called a classical solution of the problem. Assume that u is a classical solution of (2.3) and take a test function $\varphi \in \mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ is the set of smooth functions on Ω with compact support. Multiplying both sides of the equation in (2.3) by φ, and integrating over Ω, we get:

$$\begin{aligned} \int_\Omega \mathfrak{L}(u)\,\varphi &= \int_\Omega f\varphi, \\ \int_\Omega -\mathrm{div}\,(A\nabla u)\,\varphi &= \int_\Omega f\varphi. \end{aligned} \tag{2.4}$$

By applying Green's theorem A1.2, as $\varphi \in \mathcal{D}(\Omega)$ and thus $\varphi = 0$ on $\partial\Omega$, we have

$$\begin{aligned} -\int_{\partial\Omega} A\nabla u\,\varphi + \int_\Omega A\nabla u \cdot \nabla\varphi &= \int_\Omega f\varphi \\ \int_\Omega A\nabla u \cdot \nabla\varphi &= \int_\Omega f\varphi. \end{aligned} \tag{2.5}$$

As $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, equality (2.5) holds for any $v \in H_0^1(\Omega)$, and therefore

$$\int_\Omega A\nabla u \cdot \nabla v = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega). \tag{2.6}$$

Note that, if $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$, then $u \in H_0^1(\Omega)$.

Defining the bilinear form

$$a(u,v) = \int_\Omega A\nabla u \cdot \nabla v, \quad \forall u, v \in H_0^1(\Omega), \tag{2.7}$$

a solution $u \in H_0^1(\Omega)$ of (2.3) is also a solution of

$$a(u,v) = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega). \tag{2.8}$$

We call $u \in H_0^1(\Omega)$ as above a *weak solution*. The existence and uniqueness of a weak solution is long known, see for example [9].

2.2 The Rayleigh Quotient

Theorem 2.2.1 ([9]). *Consider the eigenvalue problem:*

$$\begin{cases} \mathfrak{L}(u) = \lambda u & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{2.9}$$

There exist an orthonormal basis $\{\varphi_n^A\}_n$ *of* $L^2(\Omega)$ *and a sequence of positive real numbers* $\{\lambda_n^A\}_n$ *with*

$$0 < \lambda_1^A < \lambda_2^A \leq \lambda_3^A \leq ... \leq \lambda_n^A ... \to \infty, \tag{2.10}$$

$$\begin{cases} \mathfrak{L}(\varphi_n^A) = \lambda_n^A \varphi_n^A & in \quad \Omega \\ \varphi_n^A \in H_0^1(\Omega), \end{cases} \tag{2.11}$$

and

$$a(\varphi_n^A, v) = \lambda_n^A (\varphi_n^A, v), \quad \forall v \in H_0^1(\Omega), \tag{2.12}$$

where $(.\,,.)$ *is the standard inner product in* $L^2(\Omega)$.

Proof. Let $f \in L^2(\Omega)$, and let $u = \mathcal{G}f \in H_0^1(\Omega)$ be the weak solution of (2.3) where

$$\begin{aligned} \mathcal{G} : L^2(\Omega) &\longrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega) \\ f &\longrightarrow \mathcal{G}f. \end{aligned} \tag{2.13}$$

Thus

$$\int_\Omega A\nabla(\mathcal{G}f) \cdot \nabla v = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega). \tag{2.14}$$

As Ω is bounded, by Rellich-Kondrasov Theorem (A2.1), the inclusion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, [9]. Furthermore, $\mathcal{G}$ is self-adjoint, as, for $g \in L^2(\Omega)$ we get:

$$\int_\Omega \mathcal{G}f \cdot g = \int_\Omega ug = \int_\Omega gu = a(w, u) = a(u, w) = \int_\Omega fw = \int_\Omega f \cdot \mathcal{G}g, \tag{2.15}$$

since $w = \mathcal{G}g$ for some $g \in L^2(\Omega)$. Therefore, since $\mathcal{G}$ is a compact self-adjoint linear operator on $L^2(\Omega)$, by the Spectral Theorem (A3.1), there exist at most countably many non-zero eigenvalues $\{\mu_n^A\}_{n=1}^{\infty}$ decreasing to zero, and the spectrum of $\mathcal{G}$ is $\sigma(\mathcal{G}) = \{0\} \cup \bigcup_{n=1}^{\infty}\{\mu_n^A\}$.

Consequently, there exists an orthonormal basis $\{\varphi_n^A\}_n$ of $L^2(\Omega)$ such that

$$\mathcal{G}\varphi_n^A = \mu_n^A \varphi_n^A. \tag{2.16}$$

By setting $\lambda_n^A = (\mu_n^A)^{-1}$, we get the sequence of positive real numbers $\{\lambda_n^A\}_n \nearrow \infty$, such that

$$\begin{aligned} \varphi_n^A &= \lambda_n^A \mathcal{G}\varphi_n^A \\ \text{or} \quad \varphi_n^A &= \mathcal{G}(\lambda_n^A \varphi_n^A). \end{aligned} \tag{2.17}$$

Furthermore, plugging the latter into the equation (2.14), we have:

$$\int_\Omega A\nabla\varphi_n^A \cdot \nabla v = \int_\Omega \lambda_n^A \varphi_n^A v, \quad \forall v \in H_0^1(\Omega), \tag{2.18}$$

which means that $\mathfrak{L}(\varphi_n^A) = \lambda_n^A \varphi_n^A$ in Ω, in the sense of distributions. Since the range of $\mathcal{G}$ is in $H_0^1(\Omega)$, then $\varphi_n^A \in H_0^1(\Omega)$.

For the proof of (2.12), as $f = \lambda u$, we have that:

$$a(u, v) = \int_\Omega fv = \int_\Omega \lambda uv = \lambda \int_\Omega uv = \lambda(u, v), \tag{2.19}$$

where $(.,.)$ is the standard inner product in $L^2(\Omega)$. In particular, for $u = \varphi_n^A$, we obtain that

$$a(\varphi_n^A, v) = \lambda_n^A(\varphi_n^A, v), \quad \forall v \in H_0^1(\Omega), \tag{2.20}$$

concluding the proof of the theorem. □

We will now introduce the Rayleigh quotient which provides a variational characterization of the eigenvalues of the operator $\mathfrak{L}$. We will see that λ_1^A, called the principal eigenvalue, is simple - which explains the strict inequality between λ_1^A and λ_2^A in (2.10), and that its corresponding eigenfunction has constant sign.

Definition 2.2.1. *For any $v \neq 0$, $v \in H_0^1(\Omega)$, we define the Rayleigh quotient of v to be:*

$$R_A(v) = \frac{a(v,v)}{\|v\|_{2,\Omega}^2}. \tag{2.21}$$

Theorem 2.2.2 ([9], [10])**.** *For any integer $n \geq 1$, let V_n be the subspace of $H_0^1(\Omega)$ spanned by $\{\varphi_1^A, \varphi_2^A, ..., \varphi_n^A\}$, the corresponding eigenfunctions of the operator $\mathfrak{L}$. Consider $V_0 = \emptyset$ and denote by W an arbitrary n-dimensional subspace of $H_0^1(\Omega)$. Then*

$$\forall n \geq 1: \quad \lambda_n^A = R_A(\varphi_n^A) = \min_{0 \neq v \perp V_{n-1}} R_A(v) \tag{2.22}$$

$$= \max_{0 \neq v \in V_n} R_A(v) \tag{2.23}$$

$$= \min_{\substack{W \subset H_0^1(\Omega) \\ dim\, W = n}} \max_{v \in W} R_A(v). \tag{2.24}$$

In particular,

$$\lambda_1^A = \min_{0 \neq v \in H_0^1(\Omega)} R_A(v) = R_A(\varphi_1^A). \tag{2.25}$$

Proof. Note that, by the definition of the Rayleigh quotient, if φ_n^A is the n-th eigenfunction of $\mathfrak{L}$, we have

$$R_A(\varphi_n^A) = \frac{a(\varphi_n^A, \varphi_n^A)}{\|\varphi_n^A\|_{2,\Omega}^2} = \frac{\lambda_n^A(\varphi_n^A, \varphi_n^A)}{\|\varphi_n^A\|_{2,\Omega}^2} = \lambda_n^A, \quad \forall n \geq 1. \tag{2.26}$$

We will start by proving (2.22) (and implicitly(2.25)). Let $v \in H_0^1(\Omega)$ such that

$v \perp \{\varphi_1^A, \varphi_2^A, ..., \varphi_{n-1}^A\}$. By the Fourier expansion, v can be written in $L^2(\Omega)$ as

$$v = \sum_{k=n}^{\infty} \alpha_k \varphi_k^A, \tag{2.27}$$

where $\alpha_k = \int_\Omega v\varphi_k^A \, dx$. On the other hand, consider the set $\{(\lambda_k^A)^{-1/2}\varphi_k^A\}_{k\geq n}$. These latter functions form an orthonormal basis for $H_0^1(\Omega)$ endowed with the inner-product $(u, v)_H = \int_\Omega A\nabla u \cdot \nabla v \, dx$. This is easily seen from

$$\begin{aligned}
((\lambda_k^A)^{-1/2}\varphi_k^A, (\lambda_h^A)^{-1/2}\varphi_h^A)_H &= \int_\Omega A(\lambda_k^A)^{-1/2}\nabla\varphi_k^A \cdot (\lambda_h^A)^{-1/2}\nabla\varphi_h^A \\
&= (\lambda_k^A)^{-1/2}(\lambda_h^A)^{-1/2} \int_\Omega A\nabla\varphi_k^A \cdot \nabla\varphi_h^A \\
&= (\lambda_k^A)^{-1/2}(\lambda_h^A)^{-1/2}(\lambda_k^A) \int_\Omega \varphi_k^A\varphi_h^A \\
&= (\lambda_k^A)^{1/2}(\lambda_h^A)^{-1/2} \int_\Omega \varphi_k^A\varphi_h^A \\
&= \delta_{kh}.
\end{aligned} \tag{2.28}$$

If $u \in H_0^1(\Omega)$ such that $(u, \varphi_k^A)_H = 0$, for all $k \geq n$, then

$$0 = \int_\Omega A\nabla u \cdot \nabla\varphi_k^A = \lambda_k^A \int_\Omega u\varphi_k^A, \quad \forall k. \tag{2.29}$$

Therefore

$$\int_\Omega u\varphi_k^A = 0, \quad \forall k, \tag{2.30}$$

which gives that $u = 0$, as $u \in L^2(\Omega)$ and $\{\varphi_k^A\}_k$ is an orthonormal basis of $L^2(\Omega)$, and thus complete in $L^2(\Omega)$.

Set $v_l = \sum_{k=n}^{l} \alpha_k\varphi_k^A$. Then $v_l \to v$ in $L^2(\Omega)$, and in $H_0^1(\Omega)$, and the Fourier expansion of v in $H_0^1(\Omega)$ is

$$v = \sum_{k=n}^{\infty} (\lambda_k^A)^{-1/2}\beta_k\varphi_k^A, \tag{2.31}$$

where

$$\begin{aligned} \beta_k &= \int_\Omega A\nabla v \cdot \nabla\big((\lambda_k^A)^{-1/2}\varphi_k^A\big)\,dx = (\lambda_k^A)^{-1/2}\int_\Omega A\nabla v\cdot\nabla\varphi_k^A\,dx \\ &= (\lambda_k^A)^{1/2}\int_\Omega v\varphi_k^A\,dx = (\lambda_k^A)^{1/2}\alpha_k. \end{aligned} \tag{2.32}$$

Thus, v can be written in $H_0^1(\Omega)$ as

$$v = \sum_{k=n}^{\infty}\alpha_k\varphi_k^A. \tag{2.33}$$

and therefore

$$\lim_{v_l\to v} R_A(v_l) = R_A(v). \tag{2.34}$$

Using now the properties of the orthonormal basis i.e. $(\varphi_k^A, \varphi_n^A) = \int_\Omega \varphi_k^A\varphi_n^A = \delta_{kn}$, and the fact that $\{\lambda_i\}_i$ is an increasing sequence, we get

$$\begin{aligned} R_A(v_l) &= \frac{\displaystyle\int_\Omega A\nabla v_l\cdot\nabla v_l}{\displaystyle\int_\Omega v_l^2} \\ &= \frac{\displaystyle\sum_{k=n}^{l}\alpha_k^2\int_\Omega A\nabla\varphi_k^A\cdot\nabla\varphi_k^A}{\displaystyle\sum_{k=n}^{l}\alpha_k^2} \\ &= \frac{\displaystyle\sum_{k=n}^{l}\alpha_k^2\lambda_k^A\int_\Omega\varphi_k^A\varphi_k^A}{\displaystyle\sum_{k=n}^{l}\alpha_k^2} \\ &= \frac{\displaystyle\sum_{k=n}^{l}\alpha_k^2\lambda_k^A}{\displaystyle\sum_{k=n}^{l}\alpha_k^2} \ge \lambda_n^A\,\frac{\displaystyle\sum_{k=n}^{l}\alpha_k^2}{\displaystyle\sum_{k=n}^{l}\alpha_k^2} = \lambda_n^A. \end{aligned} \tag{2.35}$$

Thus, for any $v \neq 0, v \in H_0^1(\Omega)$, we have that $R_A(v) \geq \lambda_n^A$. Since the minimum is attained for φ_n, then (2.22) and (2.25) are proved.

Let now $v \in V_n$. Then $v = \sum_{k=1}^{n} \alpha_k \varphi_n^A$, thus

$$R_A(v) = \frac{\sum_{k=1}^{n} \alpha_k^2 \lambda_k^A}{\sum_{k=1}^{n} \alpha_k^2} \leq \lambda_n^A. \tag{2.36}$$

Therefore $\max_{0 \neq v \in V_n} R_A(v) \leq \lambda_n^A$, but as this maximum is attained for φ_n^A, then (2.23) is proved.

Finally, consider W, any n-dimensional subspace of $H_0^1(\Omega)$. There exists $\varphi_0^A \in W$ such that $\varphi_0^A \perp \varphi_i^A$, $\forall i = 1, \ldots, n-1$. Hence, by proceeding as in the first part of the proof, we get $R(\varphi_0^A) \geq \lambda_n^A$, thus $\max_{v \in W} R_A(v) \geq \lambda_n^A$. As this holds for any n-dimensional subspace, we infer that $\min_{dim\, W=n} \max_{v \in W} R_A(v) \geq \lambda_n^A$. On the other hand, we have seen that for the n-dimensional subspace V_n, the minimum is attained. Therefore equality (2.24) holds and the proof of the theorem is now complete.

□

Lemma 2.2.1 ([9])**.** *If $0 \neq u \in H_0^1(\Omega)$ satisfies $R_A(u) = \lambda_1^A$, then u is an eigenfunction corresponding to λ_1^A.*

Proof. Let $0 \neq u \in H_0^1(\Omega)$ such that $R(u) = \lambda_1^A$, and let $v \in H_0^1(\Omega)$ an arbitrarily selected element. By taking $t \in \mathbb{R}_+^*$, we have that $u + tv \in H_0^1(\Omega)$ and

$$R(u + tv) \geq \min_{0 \neq w \in H_0^1(\Omega)} R(w) = \lambda_1^A = R(u), \tag{2.37}$$

i.e.

$$\frac{\int_\Omega A\nabla(u + tv) \cdot \nabla(u + tv)}{\int_\Omega (u + tv)^2} \geq \frac{\int_\Omega A\nabla u \cdot \nabla u}{\int_\Omega u^2} = \lambda_1^A. \tag{2.38}$$

By normalizing u, we may further assume, without any loss of generality, that $\int_\Omega u^2 = 1$, and thus $\int_\Omega A\nabla u \cdot \nabla u = \lambda_1^A$. Multiplying both sides of the above inequality by the denominator of the left hand side, we obtain

$$\int_\Omega A\nabla(u+tv)\cdot\nabla(u+tv) \geq \lambda_1^A \int_\Omega (u+tv)^2. \tag{2.39}$$

The latter is equivalent to

$$\int_\Omega A\nabla u\cdot\nabla u + 2t\int_\Omega A\nabla u\cdot\nabla v + t^2\int_\Omega A\nabla v\cdot\nabla v \geq \lambda_1^A\left[\int_\Omega u^2 + 2t\int_\Omega uv + t^2\int_\Omega v^2\right], \tag{2.40}$$

and, furthermore, as $\displaystyle\int_\Omega A\nabla u\cdot\nabla u = \lambda_1^A\int_\Omega u^2$, to

$$2t\int_\Omega A\nabla u\cdot\nabla v + t^2\int_\Omega A\nabla v\cdot\nabla v \geq \lambda_1^A\left[2t\int_\Omega uv + t^2\int_\Omega v^2\right]. \tag{2.41}$$

Dividing by $2t$, then letting $t \to 0$, we obtain

$$\int_\Omega A\nabla u\cdot\nabla v = \lambda_1^A\int_\Omega uv. \tag{2.42}$$

As equation (2.42) holds for any function $v \in H_0^1(\Omega)$, we infer that u satisfies the equality $a(u,v) = \displaystyle\int_\Omega fv = \int_\Omega \lambda uv$. Hence u satisfies the problem (2.9) and, thus, it is an eigenfunction of λ_1^A.

□

Theorem 2.2.3 ([1])**.** *The eigenfunction φ_1^A associated with the eigenvalue λ_1^A, also called the principal eigenfunction, is of constant sign in Ω. We may choose φ_1^A to be positive in Ω.*

Proof. If $\varphi_1^A \in H_0^1(\Omega)$, then $|\varphi_1^A| \in H_0^1(\Omega)$. Note now that $R(\varphi_1^A) = R(|\varphi_1^A|) = \lambda_1^A$. Thus, by the previous lemma, $|\varphi_1^A|$ is an eigenfunction associated with λ_1^A and, by the elliptic regularity theory [6], $|\varphi_1^A| \in C^2(\Omega)\cap C^0(\overline{\Omega})$. As $\mathcal{L}(|\varphi_1^A|) = \lambda_1^A|\varphi_1^A| \geq 0$, then, by the Weak Maximum Principle (A4.1), we have $\min_{\overline{\Omega}}(|\varphi_1^A|) = \min_{\partial\Omega}(|\varphi_1^A|)$.

Thus, as the eigenfunction cannot be constant, the minimum of $|\varphi_1^A|$ in $\overline{\Omega}$ is achieved only on $\partial\Omega$. Due to the Dirichlet condition $|\varphi_1^A| = 0$ on $\partial\Omega$, or $\min_{\Omega} |\varphi_1^A| = \min_{\partial\Omega} |\varphi_1^A| = 0$, and we infer that $|\varphi_1^A|$ is strictly positive in Ω. Therefore $\varphi_1^A > 0$ everywhere on Ω, or $\varphi_1^A < 0$ everywhere on Ω. Without any loss of generality, we may consider $\varphi_1^A > 0$ on Ω.

□

Theorem 2.2.4 ([1])**.** *The principal eigenvalue λ_1^A is simple.*

Proof. Let us assume that λ_1^A is not simple. Then consider two orthogonal eigenfunctions φ_0^A and φ_1^A corresponding to the principal eigenvalue. Their existence is guaranteed by Theorem 2.2.1. Moreover, by the previous theorem, we may consider both of the eigenfunctions to be positive in Ω. Therefore $\int_\Omega \varphi_0^A \varphi_1^A > 0$, contradicting the fact that the two eigenfunctions are orthogonal.

□

Remark 2.2.1. *Note that, in our work, Ω is assumed to be connected. Otherwise the value of each eigenvalue must be taken as the minimum among all corresponding eigenvalues of each connected component, [8].*

2.3 The Faber - Krahn Inequality

In the special case where $A = Id$, we have $\mathcal{L} = -\Delta$, the Laplace operator. Lord Rayleigh conjectured in [11] that, in this case, the principal eigenvalue of the disc is minimal among all plane domains of equal area, that is

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega), \tag{2.43}$$

where λ_1 denotes the eigenvalue corresponding to the eigenvalue problem of the Laplace operator, whereas before we used λ_1^A to denote the principal eigenvalue of the general operator $\mathfrak{L}$ defined in (2.2).

The inequality (2.43) was proved independently by Faber and Krahn in the 20's and it is known as the Faber - Krahn inequality, see [10]. Using Pólya - Szegö inequality, Kesavan proved the following, more general result, which holds for the entire class of elliptic operators defined earlier.

Theorem 2.3.1 ([10]). *With the previous notations, we have the following inequalities:*

$$\alpha\lambda_1(\Omega^*) \leq \alpha\lambda_1(\Omega) \leq \lambda_1^A(\Omega), \tag{2.44}$$

where $A \in \mathcal{M}(\alpha, \beta, \Omega)$, $0 < \alpha < \beta$, *is defined as in Section 2.1.*

Proof. Let φ_1^A be the eigenfunction corresponding to the eigenvalue λ_1^A. Then we have:

$$\begin{align}
\lambda_1^A(\Omega) = R_A(\varphi_1^A) &= \frac{a(\varphi_1^A, \varphi_1^A)}{\|\varphi_1^A\|_{2,\Omega}^2} \tag{2.45}\\
&= \frac{\int_\Omega A\nabla\varphi_1^A \cdot \nabla\varphi_1^A}{\|\varphi_1^A\|_{2,\Omega}^2} \tag{2.46}\\
&\geq \frac{\int_\Omega \alpha|\nabla\varphi_1^A|^2}{\|\varphi_1^A\|_{2,\Omega}^2} \tag{2.47}\\
&= \frac{\alpha\int_\Omega |\nabla\varphi_1^A|^2}{\|\varphi_1^A\|_{2,\Omega}^2}, \tag{2.48}
\end{align}$$

where we have (2.45) by the definition of the Rayleigh quotient, (2.46) by the definition of the bilinear form $a(u, v)$, and (2.47) by the ellipticity condition (2.1), which reduces here to

$$A\nabla\varphi_1^A \cdot \nabla\varphi_1^A \geq \alpha|\nabla\varphi_1^A|^2. \tag{2.49}$$

On the other hand,

$$\begin{aligned}\frac{\int_\Omega |\nabla\varphi_1^A|^2}{\|\varphi_1^A\|^2_{2,\Omega}} \geq \min_{0\neq v\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2}{\|v\|^2_{2,\Omega}} &= \min_{0\neq v\in H_0^1(\Omega)} \frac{\int_\Omega \nabla v\cdot\nabla v}{\|v\|^2_{2,\Omega}} \\ &= \min_{0\neq v\in H_0^1(\Omega)} R(v) \\ &= \lambda_1(\Omega), \end{aligned} \tag{2.50}$$

where $R(v)$ is the Rayleigh quotient corresponding to $A = Id$, so $a(u,v) = \int_\Omega \nabla u\cdot\nabla v$. Thus, we proved the second inequality of (2.44).

To prove the first inequality of (2.44), consider φ_1 to be the eigenfunction corresponding to the eigenvalue λ_1. By Theorem 2.2.3, we have $\varphi_1 > 0$. Thus, we can apply Pólya - Szegö inequality and obtain

$$\int_{\Omega^*} |\nabla\varphi_1^*|^2 \leq \int_\Omega |\nabla\varphi_1|^2. \tag{2.51}$$

Recall that the Schwarz symmetrization preserves the L^2 - norm, thus

$$\|\varphi_1\|^2_{2,\Omega} = \|\varphi_1^*\|^2_{2,\Omega^*}. \tag{2.52}$$

Therefore we get:

$$\frac{\int_{\Omega^*} |\nabla\varphi_1^*|^2}{\|\varphi_1^*\|^2_{2,\Omega^*}} \leq \frac{\int_\Omega |\nabla\varphi_1|^2}{\|\varphi_1\|^2_{2,\Omega}}. \tag{2.53}$$

Since $\varphi_1^* \in H_0^1(\Omega^*)$, then

$$\frac{\int_{\Omega^*} |\nabla\varphi_1^*|^2}{\|\varphi_1^*\|^2_{2,\Omega^*}} = R(\varphi_1^*). \tag{2.54}$$

Thus

$$\lambda_1(\Omega^*) = \min_{0\neq v\in H_0^1(\Omega^*)} R(v) \leq R(\varphi_1^*) = \frac{\int_{\Omega^*} |\nabla\varphi_1^*|^2}{\|\varphi_1^*\|^2_{2,\Omega^*}} \leq \frac{\int_\Omega |\nabla\varphi_1|^2}{\|\varphi_1\|^2_{2,\Omega}} = R(\varphi_1) = \lambda_1(\Omega),$$

which completes the proof. $\square$

We will now give our own proof to a problem found in Kesavan's book, [10].

Proposition 2.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $P : \Omega \to \mathbb{R}$ be a strictly positive continuous function. The eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda P u & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.55}$$

admits an increasing sequence $\{\lambda_{n,P}(\Omega)\}_n$ of positive eigenvalues which tends to infinity and the first eigenvalue $\lambda_{1,P}(\Omega)$ admits an eigenfunction of constant sign. Furthermore,

$$\lambda_{1,P}(\Omega) = \min_{0 \neq v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega P v^2 dx}. \tag{2.56}$$

Finally, if P^* denotes the Schwarz symmetrization of P, we have

$$\lambda_{1,P}(\Omega) \geq \lambda_{1,P^*}(\Omega^*). \tag{2.57}$$

Proof. Define in $L^2(\Omega)$ the inner product

$$(u, v)_P = \int_\Omega P u v \, dx, \tag{2.58}$$

where P is the function given above.

For the elliptic boundary value problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.59}$$

follow the same argument as in the introduction of this chapter, and use a test function v, to get

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v dx. \tag{2.60}$$

Therefore, if $f = \lambda P u$, we have

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v dx = \lambda \int_\Omega P u v dx = \lambda (u,v)_P, \quad \forall v \in H_0^1(\Omega). \tag{2.61}$$

Using the same operator $\mathcal{G}$ as in the proof of Theorem 2.2.1, the similar argument with $(u,v)_P$ instead implies that $\mathcal{G}$ is a compact self-adjoint linear operator. Relying again on the Spectral Theorem, the problem (2.55) admits an increasing sequence $\{\lambda_{n,P}(\Omega)\}_n$ of positive eigenvalues that tends to infinity.

Define the Rayleigh quotient

$$R_P(u) := \frac{a(u,u)}{\|u\|_{2,P}^2} = \frac{\int_\Omega -\Delta u \cdot u}{\int_\Omega P u^2} = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega P u^2}. \tag{2.62}$$

As the eigenfunctions form an orthonormal set of $L^2(\Omega)$: $(\varphi_h, \varphi_k)_P = \displaystyle\int_\Omega P \varphi_h \varphi_k = \delta_{hk}$, the set $\{\varphi_k\}_k$ forms an orthonormal basis for $H_0^1(\Omega)$ endowed with the inner-product $(u,v)_H = \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx$, as

$$(\varphi_h, \varphi_k)_H = \int_\Omega \nabla \varphi_h \cdot \nabla \varphi_k = \int_\Omega \lambda P \varphi_h \varphi_k = \lambda \delta_{hk}, \tag{2.63}$$

where $\lambda = \lambda_h = \lambda_k$, for $h = k$.

Following further the same reasoning as in the proof of Theorem 2.2.2, we get

$$\lambda_{1,P}(\Omega) = \min_{0 \neq v \in H_0^1(\Omega)} R_P(v) = \min_{0 \neq v \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega P v^2}. \tag{2.64}$$

We will now prove the last inequality of the theorem. Let ϕ denote the eigenfunction of $\lambda_{1,P}$. By slightly adjusting the proof of Theorem 2.2.3 to the present eigenvalue problem, it can be shown that this first eigenfunction has constant sign on

Ω. Using Pólya - Szegö inequality, we have that:

$$\int_{\Omega} |\nabla \varphi_1|^2 \geq \int_{\Omega^*} |\nabla \varphi_1^*|^2. \tag{2.65}$$

By Hardy - Littlewood inequality, we also have:

$$\int_{\Omega} P\varphi_1^2 \leq \int_{\Omega^*} P^* \varphi_1^{*2}, \tag{2.66}$$

thus

$$\frac{1}{\int_{\Omega} P\varphi_1^2} \geq \frac{1}{\int_{\Omega^*} P^* \varphi_1^{*2}}, \tag{2.67}$$

and, therefore,

$$\frac{\int_{\Omega} |\nabla \varphi_1|^2}{\int_{\Omega} P\varphi_1^2} \geq \frac{\int_{\Omega^*} |\nabla \varphi_1^*|^2}{\int_{\Omega^*} P^* \varphi_1^{*2}}. \tag{2.68}$$

On the other hand,

$$\frac{\int_{\Omega^*} |\nabla \varphi_1^*|^2}{\int_{\Omega^*} P^* \varphi_1^{*2}} \geq \min_{0 \neq v^* \in H_0^1(\Omega^*)} \frac{\int_{\Omega^*} |\nabla v^*|^2}{\int_{\Omega^*} P^* v^{*2}} = \lambda_{1,P^*}(\Omega^*), \tag{2.69}$$

and

$$\lambda_{1,P}(\Omega) = R_P(\varphi_1) = \frac{\int_{\Omega} |\nabla \varphi_1|^2}{\int_{\Omega} P\varphi_1^2}, \tag{2.70}$$

thus

$$\lambda_{1,P}(\Omega) \geq \lambda_{1,P^*}(\Omega^*). \tag{2.71}$$

□

2.4 The Szegö - Weinberger Inequality

While, in the previous section, we discussed the eigenvalue problem associated with the Laplace operator with homogeneous Dirichlet boundary conditions, here we will focus on the eigenvalue problem of the Laplacian with homogeneous Neumann boundary condition.

Thus, for $\Omega \in \mathbb{R}^N$ a smooth bounded domain, consider the problem:

$$\begin{cases} -\Delta u = \mu u & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.72}$$

where ν is the outer unit normal of $\partial\Omega$.

A similar argument as in the Dirichlet boundary value problem implies that there exists an increasing sequence of eigenvalues

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n \to \infty. \tag{2.73}$$

Since, in this case, the first eigenvalue is zero, we consider μ_1 the lowest non-zero eigenvalue to be the principal eigenvalue. Let ψ_0 be a constant function denoting the eigenfunction corresponding to μ_0. Note that here the eigenfunctions are elements of $H^1(\Omega)$ with mean value zero.

By using again the Rayleigh quotient, we have the variational description of μ_1 as

$$\mu_1 = \min_{\substack{0 \neq v \in H^1(\Omega) \\ \int_\Omega v = 0}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2}. \tag{2.74}$$

This follows in the same manner as its corresponding counterpart from the Dirichlet problem described in detail earlier.

Our next objective is the isoperimetric inequality

$$\mu_1(\Omega) \leq \mu_1(\Omega^*), \tag{2.75}$$

which was proved in the 50's by Szegö, for simply connected domains, and, in general, by Weinberger. In preparation for the proof, we need the following three lemmas where we followed the outline of [10].

Lemma 2.4.1. *Let g be a continuous, nonnegative real-valued function on $\mathbb{R}^+$. For each i, $1 \leq i \leq N$, let $P_i : \mathbb{R}^N \longrightarrow \mathbb{R}$ be defined by*

$$P_i(x) = g(\|x\|)\frac{x_i}{\|x\|}. \tag{2.76}$$

Then, we can choose the origin in $\mathbb{R}^N$ such that

$$\int_\Omega P_i(x)\,dx = 0, \quad \forall i : \quad 1 \leq i \leq N. \tag{2.77}$$

Proof. Let $B(0,\rho)$ be a ball in $\mathbb{R}^N$, centered at the origin, containing Ω, and define on it the map $F : B(0,\rho) \longrightarrow \mathbb{R}^N$ by

$$y \longrightarrow F(y) = \int_\Omega g(\|x-y\|)\,\frac{x-y}{\|x-y\|}\,dx. \tag{2.78}$$

Taking the inner product of $F(y)$ with y, we obtain:

$$F(y)\cdot y = \int_\Omega g(\|x-y\|)\,\frac{(x\cdot y - \|y\|^2)}{\|x-y\|}\,dx. \tag{2.79}$$

By considering $y \in \partial B(0,\rho)$, we have $\|y\| = \rho$, and

$$x\cdot y - \|y\|^2 \leq \|x\|\cdot\|y\| - \|y\|^2 \leq 0, \, \forall x \in \Omega. \tag{2.80}$$

Thus $F(y)\cdot y \leq 0$, $y \in \partial B(0,\rho)$ which we will use to show that there exists a $y_0 \in B(0,\rho)$ such that $F(y_0) = 0$.

To do so, we will use Brouwer's fixed point theorem which states that every continuous function f from a convex compact subset K of a Euclidean space to K itself has a fixed point, i.e. $\exists\, x_0 \in K : f(x_0) = x_0$. Indeed, assume that $F(y) \neq 0$, $\forall y \in B(0,\rho)$. Define the function G such that $G : B(0,\rho) \to B(0,\rho)$ with $G(y) :=$

$\rho \dfrac{F(y)}{\|F(y)\|}$, $\forall y \in B(0,\rho)$. As G is a continuous function from $B(0,\rho)$ to itself, it has a fixed point by the Brouwer's fixed point theorem. Consequently, in our case, $\exists\, y_0 \in B(0,\rho) : G(y_0) = \rho \dfrac{F(y_0)}{\|F(y_0)\|} = y_0$. Hence $\|y_0\| = \rho$ and therefore $\rho \dfrac{F(y_0)\cdot y_0}{\|F(y_0)\|} = \|y_0\|^2 > 0$, contradicting $F(y)\cdot y \le 0$, $\forall y \in \partial B(0,\rho)$. Hence the existence of an element y_0 such that $F(y_0) = 0$ is established.

Now by taking the origin to be y_0, we get

$$0 = F(y_0) = \int_\Omega g(\|x-y_0\|)\frac{x-y_0}{\|x-y_0\|}\,dx = \int_\Omega P(x-y_0)\,dx, \tag{2.81}$$

where $P(x-y_0) = (P_1(x-y_0),\ldots,P_i(x-y_0),\ldots,P_N(x-y_0))$. Which gives that $\int_\Omega P_i(x-y_0)\,dx = 0$, $\forall i = 1,...,N$, and this completes the proof of the first lemma.

□

Lemma 2.4.2. *Let $g:\mathbb{R}\to\mathbb{R}$ be a non-increasing function, and let $Q:\Omega\to\mathbb{R}$ be a strictly positive continuous function. The following statements hold:*

a. $\displaystyle\int_\Omega g(\|x\|)\,dx \le \int_{\Omega^*} g(\|x\|)\,dx.$

b. $\displaystyle\int_\Omega Qg(\|x\|) \le \int_{\Omega^*} Q^* g(\|x\|).$

If g is a non-decreasing function, the above inequalities are reversed.

Proof. a. Set $\tilde{g}(x) = g(\|x\|)$, respectively $h = \tilde{g}|_\Omega$. Then, we have

$$\int_\Omega g(\|x\|) = \int_\Omega \tilde{g}(x) = \int_\Omega h(x) = \int_{\Omega^*} h^*(x), \tag{2.82}$$

where the last equality follows from property e. of the Schwarz symmetrization listed in Chapter 1.

To complete the proof, it remains to show that $h^*(x) \le g(\|x\|)$, $\forall x \in \Omega^*$. As

$h^*(x)$ and $g(\|x\|)$ are radially non-increasing functions, it suffices to show that

$$|\{x \in \Omega^*; h^*(x) > t\}| \leq |\{x \in \Omega^*; g(\|x\|) > t\}|. \tag{2.83}$$

Obviously, $\{h^* > t\}$ is a ball centered at the origin included in Ω^*. Now let $t \in \mathbb{R}$ and note that

$$\begin{aligned} |\{h^* > t\}| = |\{x \in \Omega^*; h^*(x) > t\}| &= |\{x \in \Omega; h(x) > t\}| \\ &= |\{x \in \Omega; \tilde{g}(x) > t\}| \\ &= |\{x \in \Omega; g(\|x\|) > t\}| \\ &\leq |\{x \in \mathbb{R}^N; g(\|x\|) > t\}|. \end{aligned} \tag{2.84}$$

As g is a non-increasing function, then $\{x \in \mathbb{R}^N; g(\|x\|) > t\}$ is also a ball centered at the origin, and therefore we conclude that

$$|\{h^* > t\}| \leq |\Omega^* \cap \{x \in \mathbb{R}^N; g(\|x\|) > t\}| = |\{x \in \Omega^*; g(\|x\|) > t\}|. \tag{2.85}$$

b. To prove this part, use the same notations as in part a. and note that we have

$$\int_\Omega Qg(\|x\|) = \int_\Omega Qh \leq \int_{\Omega^*} Q^* h^* \leq \int_{\Omega^*} Q^* g(\|x\|). \tag{2.86}$$

The first inequality is due to the Hardy-Littlewood theorem (1.10), while the second inequality is due to part a. as $h^* \leq g(\|x\|)$.

A similar argument applies to the case of g non-decreasing.

□

Lemma 2.4.3. *If $\Omega = B_R$ is a ball of radius R in $\mathbb{R}^N$ centered at the origin, then the principal eigenvalue of the Neumann problem (2.72) is*

$$\mu_1(B_R) = \frac{\displaystyle\int_{B_R} \left(w'(\|x\|)^2 + \frac{N-1}{\|x\|^2} w(\|x\|)^2\right) dx}{\displaystyle\int_{B_R} w(\|x\|)^2 \, dx}, \tag{2.87}$$

for a real function w which can be found explicitly.

Proof. Using spherical coordinates, and the method of separation of variables, we write $u(x) = w(\|x\|)v(\varpi)$, where $\varpi \in S^{N-1}$. Then w satisfies

$$w''(r) + \frac{N-1}{r} w'(r) - \frac{N-1}{r^2} w(r) + \mu_1(B_R) w(r) = 0, \quad \text{for } 0 < r < R, \tag{2.88}$$

and $w(0) = w'(R) = 0$. The solution of this equation is given by

$$w(r) = c r^{\frac{N}{2}-1} J_{\frac{N}{2}}(\sqrt{\mu_1(B_R)} r), \tag{2.89}$$

where c is a constant, J_n is the Bessel function of the first kind, and of order n. The equation satisfied by w can be written as

$$\mu_1(B_R) w(r) = -\frac{1}{r^{N-1}} \frac{d}{dr}\left(r^{N-1} w'(r)\right) + \frac{N-1}{r^2} w(r). \tag{2.90}$$

Multiplying all terms by w, and integrating over B_R, using the notation $dA(r)$ for the surface area element of the sphere of radius r induced by the Euclidean metric of $\mathbb{R}^N$, we get

$$\begin{aligned}
\mu_1(B_R)\int_{B_R} w^2(\|x\|)dx &= \int_0^R \int_{\partial B_r} -\frac{w(r)}{r^{N-1}}\frac{d}{dr}\big(r^{N-1}w'(r)\big)\,dA(r)\,dr \\
&+ \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx \\
&= \int_0^R -\frac{w(r)}{r^{N-1}}\frac{d}{dr}\big(r^{N-1}w'(r)\big)\int_{\partial B_r} dA(r)\,dr \\
&+ \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx \\
&= \int_0^R -\frac{w(r)}{r^{N-1}}\frac{d}{dr}\big(r^{N-1}w'(r)\big)|\,\partial B_r|\,dr \\
&+ \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx \\
&= N\omega_N \int_0^R -\frac{d}{dr}(r^{N-1}w'(r))w(r)dr \\
&+ \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx, \\
&= N\omega_N \int_0^R r^{N-1}w'(r)^2 dr + \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx \\
&= \int_0^R \int_{\partial B_r} w'(r)^2 dA(r)\,dr + \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx \\
&= \int_{B_R} w'(\|x\|)^2 dx + \int_{B_R} \frac{N-1}{\|x\|^2} w(\|x\|)^2 dx. \qquad (2.91)
\end{aligned}$$

Above, we used integration by parts, the fact that $|\,\partial B_r| = N\omega_N r^{N-1}$, $\omega_N := |B_1|$, and the conditions $w(0) = w'(R) = 0$. Hence the third lemma is also proved.

Note that here we may choose w such that $w'(r) > 0$ in $(0, R)$, just like we chose the principal eigenfunction to be positive in the Dirichlet problem. Therefore w is a non-negative increasing function. $\square$

Theorem 2.4.1. *With the previous notations of this section, for any bounded domain $\Omega \subset \mathbb{R}^N$, we have*

$$\mu_1(\Omega) \leq \mu_1(\Omega^*). \tag{2.92}$$

Proof. We will show that

$$\mu_1(\Omega) \leq \frac{\int_\Omega G(\|x\|)\,dx}{\int_\Omega g(\|x\|)^2\,dx} \leq \frac{\int_{\Omega^*} G(\|x\|)\,dx}{\int_{\Omega^*} g(\|x\|)^2\,dx} = \mu_1(\Omega^*), \tag{2.93}$$

where $g(r)$ and $G(r)$ are functions whose precise definition will appear shortly in a natural way.

Define the function

$$g(x) = \begin{cases} w(r) & \text{for} \quad 0 < r < R \\ w(R) & \text{for} \quad r \geq R, \end{cases} \tag{2.94}$$

where R is the radius of Ω^*, and w is the function defined by (2.89). Thus g is a non-decreasing, non-negative function. Set

$$P_i(x) = g(\|x\|)\frac{x_i}{\|x\|}, \quad \forall i: \ 1 \leq i \leq N. \tag{2.95}$$

By Lemma 2.4.1, we can choose the origin such that $\int_\Omega P_i(x) = 0$, $\forall i: \ 1 \leq i \leq N$. As $\mu_1(\Omega)$ is the minimum over all non-zero functions $v \in H_0^1(\Omega)$ such that $\int_\Omega v = 0$, and as all P_i's satisfy these conditions, then

$$\mu_1(\Omega) \leq \frac{\int_\Omega |\nabla P_i|^2 dx}{\int_\Omega P_i^2 dx}, \quad \forall i: \ 1 \leq i \leq N, \tag{2.96}$$

or

$$\mu_1(\Omega) \int_\Omega P_i^2 dx \leq \int_\Omega |\nabla P_i|^2 dx, \quad \forall i: \ 1 \leq i \leq N. \tag{2.97}$$

Taking the summation over all i's, we get

$$\sum_{i=1}^{N}\Big(\mu_1(\Omega)\int_{\Omega}P_i^2\Big) \leq \sum_{i=1}^{N}\int_{\Omega}|\nabla P_i|^2, \tag{2.98}$$

then

$$\mu_1(\Omega) \leq \frac{\int_{\Omega}\sum_{i=1}^{N}|\nabla P_i|^2}{\int_{\Omega}\sum_{i=1}^{N}P_i^2}. \tag{2.99}$$

On the other hand,

$$\sum_{i=1}^{N}P_i^2 = \sum_{i=1}^{N}\left(\frac{g(\|x\|)^2\cdot x_i^2}{\|x\|^2}\right) = \frac{g(r)^2}{r^2}\sum_{i=1}^{N}x_i^2 = g(r)^2, \tag{2.100}$$

with $r = \|x\| = \sqrt{x_1^2 + ... + x_N^2}$.

Furthermore,

$$\left(\sum_{i=1}^{N}|\nabla P_i|^2\right)(x) = g'(r)^2 + \frac{N-1}{r^2}g(r)^2 \overset{\text{def}}{:=} G(r), \tag{2.101}$$

where note here the definition of the function G of (2.93).

To show the first equality of (2.101), we evaluate

$$\begin{aligned}
\frac{\partial P_i}{\partial x_i}(x) &= \frac{\partial g(r)}{\partial x_j}\cdot\frac{x_i}{r} + \frac{g(r)}{r}\cdot\delta_{ij} + g(r)\cdot x_i\cdot\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) \\
&= \frac{\partial g(r)}{\partial r}\cdot\frac{\partial r}{\partial x_j}\cdot\frac{x_i}{r} + \frac{g(r)}{r}\cdot\delta_{ij} - g(r)\cdot x_i\cdot\frac{1}{r^2}\frac{\partial r}{\partial x_j} \\
&= g'(r)\cdot\frac{x_i}{r}\frac{\partial r}{\partial x_j} + \frac{g(r)}{r}\cdot\delta_{ij} - \frac{g(r)\cdot x_i}{r^2}\frac{\partial r}{\partial x_j} \\
&= \left(g'(r) - \frac{g(r)}{r}\right)\frac{x_i}{r}\cdot\frac{\partial r}{\partial x_j} + \frac{g(r)}{r}\cdot\delta_{ij}.
\end{aligned} \tag{2.102}$$

Moreover,

$$\begin{aligned}
\nabla P_i(x) &= \left(\left(g'(r) - \frac{g(r)}{r}\right)\frac{x_i}{r}\right)\Big(\frac{\partial r}{\partial x_1}, ..., \frac{\partial r}{\partial x_j}, ..., \frac{\partial r}{\partial x_N}\Big) + \frac{g(r)}{r}(0, ..., 1, ..., 0) \\
&= \frac{x_i}{r}\left(g'(r) - \frac{g(r)}{r}\right)\cdot\nabla r + \frac{g(r)}{r}\cdot e_i. \qquad (2.103)
\end{aligned}$$

Note that

$$\begin{aligned}
\|e_i\|^2 = 1, \quad \frac{\partial r}{\partial x_j} = \frac{x_j}{r},\ \nabla r = (\frac{\partial r}{\partial x_1}, ..., \frac{\partial r}{\partial x_N}) = (\frac{x_1}{r}, ..., \frac{x_N}{r}) = \frac{x}{r}, \\
|\nabla r|^2 = \frac{\|x\|^2}{r^2} = 1, \quad \text{and} \quad \nabla r \cdot e_i = \frac{x_i}{r}. \qquad (2.104)
\end{aligned}$$

Thus, by substituting the latter identities into $|\nabla P_i(x)|^2$, we obtain

$$\begin{aligned}
|\nabla P_i(x)|^2 &= \frac{x_i^2}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)^2 |\nabla r|^2 \qquad (2.105) \\
&+ \left(\frac{g(r)}{r}\right)^2 \|e_i\|^2 + 2\frac{x_i}{r}\left(g'(r) - \frac{g(r)}{r}\right)\frac{g(r)}{r}\cdot(\nabla r\cdot e_i) \\
&= \frac{x_i^2}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)^2 + \left(\frac{g(r)}{r}\right)^2 + 2\frac{x_i^2}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)\frac{g(r)}{r}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{i=1}^N |\nabla P_i(x)|^2 &= \frac{\sum_{i=1}^N x_i^2}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)^2 + \left(\frac{g(r)}{r}\right)^2 \sum_{i=1}^N 1 \\
&+ 2\frac{\sum_{i=1}^N x_i^2}{r^2}\left(g'(r) - \frac{g(r)}{r}\right)\frac{g(r)}{r} \\
&= \left(g'(r) - \frac{g(r)}{r}\right)^2 + N\left(\frac{g(r)}{r}\right)^2 + 2\left(g'(r) - \frac{g(r)}{r}\right)\frac{g(r)}{r} \\
&= g'(r)^2 + (N-1)\left(\frac{g(r)}{r}\right)^2, \qquad (2.106)
\end{aligned}$$

concluding the proof of the first inequality of (2.93),

$$\mu_1(\Omega) \le \frac{\int_\Omega G(\|x\|)dx}{\int_\Omega g(\|x\|)^2 dx}.$$

To prove the second inequality of (2.93), compute $G'(r)$. For $0 < r < R$,

$$G'(r) = -2\Big(\mu_1(B_R)g(r)g'(r) + \frac{N-1}{r^3}\Big(rg'(r) - g(r)\Big)^2\Big), \tag{2.107}$$

as for

$$\begin{aligned} G'(r) &= \Big(g'(r)^2 + (N-1)\Big(\frac{g(r)}{r}\Big)^2\Big) \\ &= 2g'(r)g''(r) + 2\frac{N-1}{r^2}g(r)g'(r) - 2\frac{N-1}{r^3}g(r)^2. \end{aligned} \tag{2.108}$$

Now by substituting $g''(r)$ from the equation (2.88), we get that

$$\begin{aligned} G'(r) &= -2g'(r)\Big(\frac{N-1}{r}g'(r) - \frac{N-1}{r^2}g(r) + \mu_1(B_R)g(r)\Big) + 2\frac{N-1}{r^2}g(r)g'(r) \\ &- 2\frac{N-1}{r^3}g(r)^2 \\ &= -2\Big(\frac{N-1}{r}g'(r)^2 - 2\frac{N-1}{r^2}g(r)g'(r) + \mu_1(B_R)g(r)g'(r) + \frac{N-1}{r^3}g(r)^2\Big) \\ &= -2\Big(\mu_1(B_R)g(r)g'(r) + \frac{N-1}{r^3}\big(r^2g'(r)^2 - 2rg(r)g'(r) + g(r)^2\big)\Big) \\ &= -2\Big(\mu_1(B_R)g(r)g'(r) + \frac{N-1}{r^3}\Big(rg'(r) - g(r)\Big)^2\Big). \end{aligned} \tag{2.109}$$

As g is non-decreasing, by Lemma 2.4.2, we get that

$$\int_\Omega g(\|x\|)^2 \ge \int_{\Omega^*} g(\|x\|)^2. \tag{2.110}$$

Note that $G'(r) \le 0$, for $0 < r < R$, as all the terms inside the brackets in (2.109) are non-negative. In particular, we know that g is non-negative, and that μ_1 is positive;

$g'(r)$ is also non-negative, as g is non-decreasing. Therefore G is non-increasing, which again by Lemma 2.4.2, gives

$$\int_{\Omega} G(\|x\|)\, dx \leq \int_{\Omega^*} G(\|x\|)\, dx. \tag{2.111}$$

So, we can conclude the second inequality of (2.93). This also completes the proof of the theorem, as the last equality follows from Lemma 2.4.3 and the fact that Ω^* is the ball B_R.

□

Inspired by Kesavan's problem (Proposition 2.3.1), we conjectured and proved the following proposition. We believe that the result is known to the specialists in the field, however we did not find any reference to it in the literature.

Proposition 2.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $Q : \Omega \to \mathbb{R}$ be a strictly positive continuous function. The eigenvalue problem*

$$\begin{cases} -\Delta u = \mu Q u & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{2.112}$$

admits an increasing sequence $\{\mu_{n,Q}(\Omega)\}_n$ of positive eigenvalues which tends to infinity. Furthermore,

$$\mu_{1,Q}(\Omega) = \min_{\substack{0 \neq v \in H^1(\Omega) \\ \int_\Omega Qv dx = 0}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega Q v^2 dx}, \tag{2.113}$$

and

$$\mu_{1,Q}(\Omega^*) = \frac{\int_{\Omega^*} \left(w'(\|x\|)^2 + \frac{N-1}{\|x\|^2} w(\|x\|)^2 \right) dx}{\int_{\Omega^*} Q^*(x) w(\|x\|)^2 dx}, \tag{2.114}$$

where w is determined from the separation of variables $u(x) = w(\|x\|) v(\varpi)$ and Q^* denotes the Schwarz symmetrization of Q.

Finally, we have

$$\mu_{1,Q}(\Omega) \le \mu_{1,Q^*}(\Omega^*). \tag{2.115}$$

Proof. The proof involves many steps; many of them were seen earlier in similar situations.

The existence of the increasing sequence of positive eigenvalues is given by the same argument as in the proof Theorem 2.2.1. To evaluate $\mu_{1,Q}(\Omega)$ as the minimum of the Rayleigh quotient, proceed like in the proof of Proposition 2.3.1.

We will now derive equation (2.114). By a change of variables $u(x) = w(\|x\|)v(\varpi)$, we get that w satisfies the differential equation

$$w''(r) + \frac{N-1}{r}w'(r) - \frac{N-1}{r^2}w(r) + \mu_{1,Q}(\Omega^*)Q^*(x)w(r) = 0,\ \ 0 < \forall r < R, \tag{2.116}$$

and $w(0) = w'(R) = 0$. And thus, as in the proof of Lemma 2.4.3, we will have

$$\mu_1(\Omega^*)\int_{\Omega^*} Q^*(x)w^2(\|x\|)dx = \int_{\Omega^*} w'(\|x\|)^2dx + \int_{\Omega^*} \frac{N-1}{\|x\|^2}w(\|x\|)^2dx, \tag{2.117}$$

which implies (2.114).

To finish the proof of the proposition, we will show that

$$\mu_{1,Q}(\Omega) \le \mu_{1,Q^*}(\Omega^*). \tag{2.118}$$

By denoting

$$w'(\|x\|)^2dx + \frac{N-1}{\|x\|^2}w(\|x\|)^2dx := G(\|x\|), \tag{2.119}$$

it will be enough to prove

$$\mu_{1,Q}(\Omega) \le \frac{\int_\Omega G(\|x\|)dx}{\int_\Omega Q(x)g(\|x\|)^2dx} \le \frac{\int_{\Omega^*} G(\|x\|)dx}{\int_{\Omega^*} Q^*(x)g(\|x\|)^2dx} = \mu_{1,Q}(\Omega^*). \tag{2.120}$$

The last equality is equation (2.114). The first inequality is proved in the same way as in the proof of Theorem 2.4.1, here using the function P of Lemma 2.4.1 to be such that

$$P(x-y_0)=(QP_1(x-y_0),...,QP_i(x-y_0),...,QP_N(x-y_0)). \tag{2.121}$$

Then using Rayleigh's eigenvalue characterization,

$$\mu_1(\Omega)\leq\frac{\int_\Omega\sum_{i=1}^N|\nabla P_i|^2}{\int_\Omega\sum_{i=1}^N QP_i^2}, \tag{2.122}$$

where

$$\int_\Omega\sum_{i=1}^N QP_i^2=\int_\Omega Q\sum_{i=1}^N P_i^2=\int_\Omega Qg(\|x\|)^2, \tag{2.123}$$

and

$$\int_\Omega\sum_{i=1}^N|\nabla P_i|^2=\int_\Omega G(\|x\|). \tag{2.124}$$

The second inequality is obtained by using Lemma 2.4.2, which implies

$$\int_\Omega Qg(\|x\|)\geq\int_{\Omega^*}Q^*g(\|x\|), \tag{2.125}$$

as g is non-decreasing. Therefore the proof is concluded.

□

Summing up all previous work, we can conclude the following corollary which compares $\mu_1(\Omega)$ with $\lambda_1(\Omega)$.

Corollary 2.4.1 ([10]). *Let $\lambda_1(\Omega)$ be the principal Dirichlet eigenvalue of the Laplacian as in*

$$\begin{cases}-\Delta u=\lambda u & in \quad \Omega\\ u=0 & on \quad \partial\Omega,\end{cases} \tag{2.126}$$

and let $\mu_1(\Omega)$ be the principal Neumann eigenvalue of the Laplacian as in (2.72), then

$$\mu_1(\Omega) < \lambda_1(\Omega). \tag{2.127}$$

Proof. It is enough to prove that

$$\mu_1(\Omega^*) < \lambda_1(\Omega^*), \tag{2.128}$$

as we already know by the Szegö - Weinberger inequality that

$$\mu_1(\Omega) \leq \mu_1(\Omega^*), \tag{2.129}$$

and, by the Faber - Krahn inequality, that

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega). \tag{2.130}$$

Let R be the radius of Ω^*. The first Dirichlet eigenfunction φ_1 associated with $\lambda_1(\Omega^*)$ satisfies

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in} \quad \Omega^* \\ \quad \varphi_1 = 0 & \text{on} \quad \partial\Omega^*. \end{cases} \tag{2.131}$$

Note that, in this case, $\varphi_1 = \varphi_1^*$, as we are working in Ω^*, therefore φ_1 is radially decreasing. The fact that φ_1 is radial gives, by using spherical coordinates, the following differential equation

$$\varphi_1''(r) + \frac{N-1}{r}\,\varphi_1'(r) + \lambda_1(\Omega^*)\varphi_1(r) = 0, \tag{2.132}$$

with $r = \|x\|$. Differentiate this equation with respect to r, then set $v = \varphi_1'$ to get

$$v''(r) + \frac{N-1}{r}\,v'(r) - \frac{N-1}{r^2}\,v(r) + \lambda_1(\Omega^*)v(r) = 0. \tag{2.133}$$

Now, on one hand, φ_1 is decreasing, thus $v(R) < 0$. On the other hand, by using the differential equation (2.132) for φ_1, we get that $v'(R) = \varphi_1''(R) > 0$. Therefore $v(R)v'(R) < 0$. We also have $v(0) = \varphi_1'(0) = 0$. Proceeding one more time as in the proof of Lemma 2.4.3, we obtain

$$\lambda_1(\Omega^*) \int_{B_R} v^2(\|x\|)\, dx = N\omega_N \int_0^R -\frac{d}{dr}(r^{N-1}v'(r))v(r)dr + \int_{B_R} \frac{N-1}{\|x\|^2} v(\|x\|)^2 dx.$$

Evaluating by parts the first integral on the right hand side, with the conditions just obtained above $v(0) = 0$ and $v(R)v'(R) < 0$, we can conclude the following inequality

$$\lambda_1(\Omega^*) > \frac{\int_{B_R} \left(v'(\|x\|)^2 + \frac{N-1}{\|x\|^2} v(\|x\|)^2\right) dx}{\int_{B_R} v(\|x\|)^2 dx}. \tag{2.134}$$

Moreover, as $v = \varphi_1' < 0$, by applying Lemma 2.4.1 on $P_i(x) = v(\|x\|)\dfrac{x_i}{\|x\|}$, and following the same argument as in the first part of the proof of Theorem 2.4.1, we obtain that

$$\frac{\int_{B_R} \left(v'(\|x\|)^2 + \frac{N-1}{\|x\|^2} v(\|x\|)^2\right) dx}{\int_{B_R} v(\|x\|)^2 dx} \geq \mu_1(\Omega^*), \tag{2.135}$$

so the proof is complete.

$\square$

Chapter 3

The Dirichlet Eigenvalue Problem on Compact Riemannian Manifolds

3.1 Introduction

Let (M, g) be an N-dimensional complete, compact, connected Riemannian manifold with no boundary, endowed with the Riemannian metric g. Let dV be the volume element of M.

We denote by $L^2(M)$ the space of real, measurable functions f on M such that

$$\int_M f^2 dV < +\infty. \tag{3.1}$$

For any $f, h \in L^2(M)$, we denote by

$$(f, h) = \int_M fh\, dV, \quad \text{and} \quad \|f\|^2 = (f, f), \tag{3.2}$$

the usual inner product and the induced norm, respectively, on this space.

3.2 The Dirichlet Eigenvalue Problem for the Laplacian

Let $\Omega \subset M$ be a domain on M and consider the following eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in} \quad \Omega \\ \quad u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.3}$$

We start with some properties of the real numbers λ for which there exists a nontrivial solution of the problem (3.3). In this respect, we have the following theorem:

Theorem 3.2.1 ([1], [4]). *Let (M, g) be a compact Riemannian manifold, and let Ω be a domain of M. The eigenvalues of the problem (3.3) form an infinite increasing sequence*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \to \infty, \tag{3.4}$$

where each eigenvalue is repeated as much as its multiplicity. Associated with this set of eigenvalues, there exists a set of eigenfunctions, $\{\varphi_n\}_{n=1}^{\infty}$, which is an orthonormal basis of $L^2(\Omega)$; moreover $\varphi_n \in C^\infty(\Omega)$ for each n.

For each eigenvalue, the eigenspace is finite dimensional. In addition, the eigenspaces associated with distinct eigenvalues are orthogonal in $L^2(\Omega)$.

Proof. The proof of this theorem involves many classical theorems of spectral theory. To start, consider $D := \{f \in C^\infty(\Omega) \cup C^0(\bar{\Omega}) : f = 0 \text{ on } \partial\Omega\}$, subspace of $C^\infty(\Omega)$, which is dense in $L^2(\Omega)$. Regard Δ as an unbounded operator in $L^2(\Omega)$ with domain D. We have, by Green's theorem A1.2, that the Laplace operator Δ is

Symmetric : that is for any $\varphi, \psi \in D$, we have

$$(\Delta\varphi, \psi) = (\varphi, \Delta\psi); \tag{3.5}$$

Positive : that is for any $\varphi \in D$, we have

$$(\Delta\varphi, \varphi) \geq 0. \tag{3.6}$$

Then, by Friedrichs' Theorem A3.2, this positive symmetric operator admits a positive self-adjoint extension (D_d, Δ_d), called the Friedrichs extension. The positivity of Δ_d is implied by the positivity of Δ, and the former gives that the spectrum of (D_d, Δ_d) is contained in $\mathbb{R}^+$. The compactness of $\bar{\Omega}$ implies that the inclusion $D_d \hookrightarrow L^2(\Omega)$ is compact, and this means that for $\lambda \notin \mathbb{R}^+$, the resolvent $(\lambda - \Delta_d)^{-1}$ is a compact operator in $L^2(\Omega)$. Now that we constructed a compact self-adjoint operator, we can apply the spectral theorem A3.1 on the Hilbert space $L^2(\Omega)$.

As for the orthogonality, let φ_1 and φ_2 be two distinct eigenfunctions associated with two different eigenvalues λ_1 and λ_2. Then, by Green's formula (A.4), we have:

$$0 = \int_\Omega (\varphi_1 \Delta\varphi_2 - \varphi_2 \Delta\varphi_1)\, dx = \int_\Omega (\varphi_1 \lambda_2 \varphi_2 - \varphi_2 \lambda_1 \varphi_1)\, dx = (\lambda_2 - \lambda_1) \int_\Omega \varphi_1 \varphi_2 \, dx,$$

which, as $\lambda_1 \neq \lambda_2$, gives that $\int_\Omega \varphi_1\varphi_2 = 0$, thus φ_1 and φ_2 are orthogonal in $L^2(\Omega)$, and the proof is complete.

$\square$

Remark 3.2.1. *Note that $\varphi_i \in C^\infty(\Omega)$ eigenfunction implies easily that the corresponding eigenvalue must be strictly positive. Indeed, by taking $h = f = \varphi$ in the Green's formula (A.1), we get*

$$\int_\Omega \varphi \Delta \varphi \, dV + \int_\Omega |\nabla\varphi|^2 dV = 0 \tag{3.7}$$

$$\int_\Omega \varphi(-\lambda\varphi) dV + \int_\Omega |\nabla\varphi|^2 dV = 0 \tag{3.8}$$

$$-\lambda \|\varphi\|^2 + \int_\Omega |\nabla\varphi|^2 dV = 0 \tag{3.9}$$

and thus

$$\lambda = \|\varphi\|^{-2} \int_\Omega |\nabla\varphi|^2 dV \geq 0. \tag{3.10}$$

If $\lambda = 0$, then $\nabla\phi = 0$ a.e. but, as φ is a C^∞ function, we have that $\varphi = constant = 0$ which cannot be an element of a basis of $L^2(\Omega)$. Hence $\lambda > 0$.

Denote now by $W^{1,2}(\Omega)$ the space of all $L^2(\Omega)$ functions with $|\nabla f| \in L^2(\Omega)$, and by $H_0^1(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

Define the following bilinear form by

$$\mathfrak{D}[f,h] = \int_\Omega (\nabla f, \nabla h)\, dV, \quad \forall f,h \in H_0^1(\Omega). \tag{3.11}$$

Taking $\{\varphi_i\}_{i=1}^\infty$, the orthonormal basis of $L^2(\Omega)$, we have that, for each $f \in L^2(\Omega)$,

$$f = \sum_{i=1}^\infty (f,\varphi_i)\varphi_i, \tag{3.12}$$

and

$$\|f\|^2 = \sum_{i=1}^\infty (f,\varphi_i)^2. \tag{3.13}$$

Theorem 3.2.2 (Lord Raleigh, [4])**.** *Let $f \in L^2(\Omega)$, $f \neq 0$. Then*

$$\lambda_1 \leq \frac{\mathfrak{D}[f,f]}{\|f\|^2} \tag{3.14}$$

with equality if and only if f is an eigenfunction of λ_1.

If $\{\varphi_i\}_{i=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$ formed by the eigenfunctions of λ_i, and if f satisfies

$$(f,\varphi_1) = ... = (f,\varphi_{k-1}) = 0, \;\; for\;\; k \geq 2, \tag{3.15}$$

then we have

$$\lambda_k \leq \frac{\mathfrak{D}[f,f]}{\|f\|^2} \tag{3.16}$$

with equality if and only if f is an eigenfunction of λ_k.

Proof. Let $f \in L^2(\Omega)$, $f \neq 0$. Set

$$\alpha_i = (f, \varphi_i) \tag{3.17}$$

and note that the hypothesis condition (3.15) becomes

$$\alpha_1 = ... = \alpha_{k-1} = 0, \text{ for } k \geq 2. \tag{3.18}$$

Now, for a fixed $k \geq 1$, and for $r \geq k$, we have

$$\begin{aligned}
0 \quad &\leq \quad \mathfrak{D}\Big[f - \sum_{i=1}^{r} \alpha_i \varphi_i, f - \sum_{i=1}^{r} \alpha_i \varphi_i\Big] && (3.19)\\
&= \quad \mathfrak{D}\Big[f - \sum_{i=k}^{r} \alpha_i \varphi_i, f - \sum_{i=k}^{r} \alpha_i \varphi_i\Big] && (3.20)\\
&= \quad \mathfrak{D}[f,f] - 2\sum_{i=k}^{r} \alpha_i \mathfrak{D}[f, \varphi_i] + \sum_{i,j=k}^{r} \alpha_i \alpha_j \mathfrak{D}[\varphi_i, \varphi_j] && (3.21)\\
&= \quad \mathfrak{D}[f,f] + 2\sum_{i=k}^{r} \alpha_i (f, \Delta\varphi_i) - \sum_{i,j=k}^{r} \alpha_i \alpha_j (\varphi_i, \Delta\varphi_j), && (3.22)
\end{aligned}$$

where in the last step we used that, for any $f, h \in L^2(\Omega)$,

$$\mathfrak{D}[f,h] = -(f, \Delta h), \tag{3.23}$$

Furthermore, as φ_i is an eigenfunction of λ_i, we have

$$(f, \Delta\varphi_i) = \int_\Omega f\Delta\varphi_i = -\int_\Omega f\lambda_i\varphi_i = -\lambda_i \int_\Omega f\varphi_i = -\lambda_i(f, \varphi_i) = -\lambda_i\alpha_i. \tag{3.24}$$

Moreover, as the set of eigenfunctions is an orthonormal basis of $L^2(\Omega)$, then we also have

$$(\varphi_i, \Delta\varphi_j) = \int_\Omega \varphi_i \Delta\varphi_j = -\lambda_j \int_\Omega \varphi_i\varphi_j = -\lambda_j(\varphi_i, \varphi_j) = -\lambda_i\delta_{ij}. \tag{3.25}$$

Thus, equation (3.22) becomes

$$\mathfrak{D}[f,f] - 2\sum_{i=k}^{r} \alpha_i^2\lambda_i + \sum_{i,j=k}^{r} \alpha_i\alpha_j\lambda_j\delta_{ij} = \mathfrak{D}[f,f] - \sum_{i=k}^{r} \alpha_i^2\lambda_i. \tag{3.26}$$

Therefore

$$\mathfrak{D}[f,f] - \sum_{i=k}^{r} \alpha_i^2\lambda_i \geq 0, \tag{3.27}$$

or, equivalently,

$$\sum_{i=k}^{r} \alpha_i^2\lambda_i \leq \mathfrak{D}[f,f], \tag{3.28}$$

from which we can conclude that

$$\sum_{i=k}^{\infty} \alpha_i^2\lambda_i < +\infty, \tag{3.29}$$

and

$$\mathfrak{D}[f,f] \geq \sum_{i=k}^{\infty} \alpha_i^2\lambda_i \geq \lambda_k \sum_{i=k}^{\infty} \alpha_i^2 = \lambda_k \sum_{i=k}^{\infty} (f, \varphi_i)^2 = \lambda_k\|f\|^2. \tag{3.30}$$

As the choice of $k \geq 1$ was arbitrary, the first inequality of the theorem is proved.

As for the case of the equality, if f is an eigenfunction for some λ, then $\Delta f = -\lambda f$, and

$$\mathfrak{D}[f,f] = -(f, \Delta f) = -(f, -\lambda f) = \lambda(f,f) = \lambda\|f\|^2. \tag{3.31}$$

For the second direction, suppose that we have equality for some $f \neq 0$ as in the hypothesis (3.15), that is

$$\lambda_k = \frac{\mathfrak{D}[f,f]}{\|f\|^2}. \tag{3.32}$$

It means that all inequalities in (3.30) become equalities. Thus

$$\alpha_j = 0,\ \forall j : \lambda_j > \lambda_{k+l}, \tag{3.33}$$

where l is the multiplicity of λ_k, with $l < +\infty$ by Theorem 3.2.1. As

$$\alpha_j = 0,\ \forall j < k, \tag{3.34}$$

as well, we have

$$\alpha_j = 0,\ \forall j \neq k, ..., k+l. \tag{3.35}$$

and f will be such that

$$f = \sum_{i=k}^{k+l} \alpha_i \varphi_i, \tag{3.36}$$

with $\alpha_i = (f, \varphi_i),\ \forall i = k, ..., k+l$, and φ_i the linearly independent eigenfunctions associated with the eigenvalue λ_k with its multiplicity l. Therefore, as f is a linear combination of the eigenfunctions of λ_k, then f itself is an eigenfunction of λ_k.

$\square$

3.3 The Faber - Krahn Inequality on Compact Manifolds with Pinched Ricci Curvature

We will first introduce some additional notations. Let

$$R(M, g) = \inf\{Ric(u, u) : u \in TM\}, \tag{3.37}$$

where Ric is the Ricci curvature of (M, g), and TM is the tangent bundle of M. Therefore for (S^N, g^*), the N-dimensional sphere of radius 1 in $\mathbb{R}^{N+1}$ with the induced metric, we have $R(S^N, g^*) = N - 1$.

Theorem 3.3.1 ([2]). *Let (M, g) be a complete, N-dimensional, smooth Riemannian manifold with no boundary. Suppose that the smallest eigenvalue of the Ricci curvature of (M, g) is greater than or equal to $(N-1)$, i.e. $R(M, g) \geq (N-1)$.*

Let Ω be an open in M, such that $\partial\Omega$ is a smooth submanifold of M, and let Ω^* be a geodesic ball of the canonical sphere S^N, such that the relative volume of Ω^* in S^N is equal to the relative volume of Ω in M, i.e. $V(\Omega)/V(M) = V(\Omega^*)/V(S^N)$.

Then, the first eigenvalue of the Dirichlet problem in Ω is greater than or equal to the first eigenvalue problem in Ω^*, i.e. $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.

Equality holds if and only if (M, g, Ω) is isometric to (S^N, g^*, Ω^*).

Proof. Let β be such that

$$\beta = V(M)/V(S^N), \tag{3.38}$$

thus the hypothesis becomes $V(\Omega) = \beta V(\Omega^*)$. Note that the condition $R(M, g) \geq (N-1)$ on the complete manifold M implies, by Bonnet - Myers Theorem A5.1, that M is compact and that $V(M)$ is finite. Let u be an eigenfunction associated with $\lambda_1(\Omega)$. The idea of the proof is to construct a function u^* on Ω^* such that

$$\frac{\displaystyle\int_\Omega |\nabla u|^2\, dV}{\displaystyle\int_\Omega u^2\, dV} \geq \frac{\displaystyle\int_{\Omega^*} |\nabla u^*|^2\, dV}{\displaystyle\int_{\Omega^*} u^{*2}\, dV}. \tag{3.39}$$

As Theorem 3.2.2 gives us that

$$\lambda_1(\Omega^*) = \inf_{\substack{0 \neq f \in L^2(\Omega^*) \\ f|_{\partial\Omega^*}=0}} \frac{\displaystyle\int_{\Omega^*} |\nabla f|^2 dV}{\displaystyle\int_{\Omega^*} f^2 dV}, \tag{3.40}$$

where the infimum is attained when f is an eigenfunction of $\lambda_1(\Omega^*)$, the inequality (3.39) will imply $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.

Thus, for $0 < r \leq m = \sup\{u(x) : x \in \Omega\}$, define the level sets

$$B(r) = \{u > r\} = \{x \in \Omega : u(x) > r\}, \tag{3.41}$$

with $B(0) = \Omega$. We will denote the measure, or volume of $B(r)$, by $V(r)$, i.e.

$$V(r) = V(B(r)) = V(\{u > r\}) = \int_{\{u>r\}} dV, \tag{3.42}$$

and the volume of $\partial B(r)$ by $A(r)$, i.e.

$$A(r) = A(\partial B(r)) = A(\partial\{u > r\}) = \int_{\partial\{u>r\}} dA. \tag{3.43}$$

For convenience, denote $G(r) = \int_{B(r)} |\nabla u|^2 \, dV$ and, respectively, $H(r) = \int_{B(r)} u^2 \, dV$. Define the geodesic concentric balls $B^*(r)$ of (S^N, g^*) such that

$$V(B(r)) = \beta V(B^*(r)). \tag{3.44}$$

Define, like before, the volume of $B^*(r)$ to be $V^*(r)$, i.e.

$$V^*(r) = V(B^*(r)) = \int_{B^*(r)} dV, \tag{3.45}$$

so that the condition on the geodesic balls becomes

$$V(r) = \beta V^*(r). \tag{3.46}$$

Similarly, define the volume of $\partial B^*(r)$ by $A^*(r)$, i.e.

$$A^*(r) = A(\partial B^*(r)) = \int_{\partial B^*} dA. \tag{3.47}$$

For simplicity, we have used the same notations dV and dA on S^N even if these measures are now induced by the metric g^*.

Define the function $u^* : B^*(0) = \Omega^* \to \mathbb{R}$ such that

$$u^*|_{\partial B^*(r)} = r = u|_{\partial B(r)}, \quad \forall r: \;\; 0 \le r \le m. \tag{3.48}$$

It is worth noting that this symmetrization procedure generalizes naturally the symmetrization on $\mathbb{R}^N$. Note that $u^*|_{\partial\Omega^*} = u^*|_{\partial B^*(0)} = 0$, and that $u^* \in H_0^1(\Omega^*)$. The last claim is immediate for the regular points of u, where u^* is C^∞, and it suffices to check that u^* is continuous as it passes through the values of r corresponding to the critical points of u. Indeed, at a critical point $p \in \Omega$, one can use local coordinates on M, to conclude $\frac{\partial u}{\partial x_i}(p) = 0$ and, due the fact that u is an eigenfunction of the Dirichlet problem of the Laplacian, $\sum_{i,j=1}^N \frac{\partial^2 u}{\partial x_i \partial x_j} < 0$ on a neighborhood of p. Hence, by the implicit function theorem, the set of critical values p of u, Γ, is at most an $(N-1)$-submanifold, therefore $V(\Gamma) = 0$ and u^* has no jump discontinuities. Finally, by Sard theorem, the set of critical points of u forms a set of measure zero on M.

Define, by analogy,

$$G^*(r) = \int_{B^*(r)} |\nabla u^*|^2 \, dV, \tag{3.49}$$

and

$$H^*(r) = \int_{B^*(r)} u^{*2} \, dV. \tag{3.50}$$

Note that we have

$$V'(r) = -\int_{\partial B(r)} |\nabla u|^{-1} dA. \tag{3.51}$$

Indeed, as $dV = |\nabla u|^{-1} dA\, dt$, then

$$\begin{aligned} V'(r) &= \frac{d}{dr}\Big(\int_{B(r)} dV\Big) \\ &= \frac{d}{dr}\Big(\int_r^\infty \int_{\partial B(t)} |\nabla u|^{-1} dA\, dt\Big) \\ &= -\int_{\partial B(r)} |\nabla u|^{-1} dA. \end{aligned} \tag{3.52}$$

Following the same argument, we get

$$G'(r) = \Big(\int_{B(r)} |\nabla u|^2 dV\Big)' = -\int_{\partial B(r)} |\nabla u|\, dA. \tag{3.53}$$

Similarly, as u^* is additionally radial by construction, we also have

$$V^{*\prime}(r) = -|\nabla u^*|^{-1} \int_{\partial B^*(r)} dA = -|\nabla u^*|^{-1} A^*(r), \tag{3.54}$$

and

$$G^{*\prime}(r) = -|\nabla u^*| \int_{\partial B^*(r)} dA = -|\nabla u^*| A^*(r). \tag{3.55}$$

The Cauchy-Schwartz inequality applied to the functions $|\nabla u|^{-1/2}$ and $|\nabla u|^{1/2}$ on the level sets of u implies

$$\begin{aligned} A^2(r) &= \Big(\int_{\partial B(r)} dA\Big)^2 \\ &\leq \int_{\partial B(r)} |\nabla u|^{-1} dA \cdot \int_{\partial B(r)} |\nabla u| dA \\ &= V'(r) \cdot G'(r), \end{aligned} \tag{3.56}$$

while, directly,

$$V^{*\prime}(r) \cdot G^{*\prime}(r) = [-|\nabla u^*|^{-1} A^*(r)] \cdot [-|\nabla u^*| A^*(r)] = A^{*2}(r). \tag{3.57}$$

We will now apply Gromov's isoperimetric inequality (A.11),

$$A(\partial\Omega) \geq \beta A(\partial\Omega^*), \tag{3.58}$$

thus

$$A(r) = A(\partial B(r)) \geq \beta A(\partial B^*(r)) = \beta A^*(r). \tag{3.59}$$

On the other hand, by construction, we have

$$V(r) = \beta V^*(r), \tag{3.60}$$

therefore, from (3.56) and (3.57), we obtain

$$\begin{aligned}
\beta^2 V^{*\prime}(r) \cdot G^{*\prime}(r) = \beta^2 A^{*2}(r) \leq A^2(r) &\leq V'(r) \cdot G'(r) \\
\beta V'(r) \cdot G^{*\prime}(r) &\leq V'(r) \cdot G'(r) \\
\beta \left(-\int_{\partial B(r)} |\nabla u|^{-1} dA \right) \cdot G^{*\prime}(r) &\leq -\left(\int_{\partial B(r)} |\nabla u|^{-1} dA \right) \cdot G'(r) \\
-\beta G^{*\prime}(r) &\leq -G'(r).
\end{aligned} \tag{3.61}$$

Integrating from r to m, where recall that $m = \sup\{u(x) : x \in \Omega\}$, we obtain that

$$\beta G^*(r) \leq G(r). \tag{3.62}$$

On the other hand, we will now show that

$$H(r) = \beta H^*(r). \tag{3.63}$$

Indeed

$$
\begin{aligned}
H^*(r) &= \int_{B^*(r)} u^{*2} dV, \quad \text{and use the co-area formula (A7.1)}, \\
&= \int_0^r \int_{\partial B^*(\rho)} \rho^2 \frac{1}{|\nabla u^*|} dA\, d\rho, \quad \text{as } u^* \text{ is radial and } u^*|_{\partial B^*(r)} = r, \\
&= \int_0^r \rho^2 \frac{1}{|\nabla u^*|} \int_{\partial B^*(\rho)} dA\, d\rho \\
&= \int_0^r \rho^2 \frac{1}{|\nabla u^*|} A^*(\rho)\, d\rho \\
&= -\int_0^r \rho^2 V^{*\prime}(\rho)\, d\rho, \quad \text{as } \; V^{*\prime}(\rho) = -\frac{1}{|\nabla u^*|} A^*(\rho), \\
&= -\int_r^m t^2 \frac{1}{\beta} V'(t) dt, \quad \text{by a change of variable as } \; r \le u \le m, \\
&= -\frac{1}{\beta} \int_r^m t^2 \int_{\partial B(t)} -\frac{1}{|\nabla u|}\, dA\, dt \\
&= \frac{1}{\beta} \int_r^m \int_{\partial B(t)} \frac{t^2}{|\nabla u|}\, dA\, dt \\
&= \frac{1}{\beta} \int_{B(r)} u^2 dV, \quad \text{again by the co-area formula}, \\
&= \frac{1}{\beta} H(r). \qquad (3.64)
\end{aligned}
$$

Thus

$$
\frac{G(r)}{H(r)} \ge \frac{G^*(r)}{H^*(r)}, \quad \forall r, \qquad (3.65)
$$

which, for $r = 0$, gives

$$\frac{G(0)}{H(0)} \geq \frac{G^*(0)}{H^*(0)}. \tag{3.66}$$

Thus, as $B(0) = \Omega$ and $B^*(0) = \Omega^*$,

$$\frac{\displaystyle\int_\Omega |\nabla u|^2 dV}{\displaystyle\int_\Omega u^2 dV} \geq \frac{\displaystyle\int_{\Omega^*} |\nabla u^*|^2 dV}{\displaystyle\int_{\Omega^*} u^{*2} dV}. \tag{3.67}$$

Note that equality in (3.67) implies equality in Gromov's Levy inequality. The latter was shown to occur if and only if (M, g, Ω) is isometric to (S^N, g^*, Ω^*), concluding the proof of the theorem.

□

3.4 The Faber - Krahn Inequality on Compact Manifolds without Curvature Bound

In this section we will follow mostly [1]. We continue to denote by (M, g) a compact, connected, smooth, N-dimensional Riemannian manifold with no boundary, and by Ω a smooth domain in M. We seek to replace the hypothesis $R(M, g) \geq N - 1$ by a global condition on M. This will bring in a comparison of $\lambda_1(\Omega)$ with the corresponding eigenvalue of a domain Ω^* in a symmetrized manifold M^* which, in general, will not have the full symmetry group of S^N.

Definition 3.4.1. *We call the isoperimetric function of a Riemannian manifold* (M, g) *the real function on* $[0, 1]$ *defined by*

$$h(\beta) = h(M, g; \beta) = \inf_\Omega \left\{ \frac{A(\partial\Omega)}{V(M)} : \Omega \subseteq M, V(\Omega) = \beta V(M) \right\}. \tag{3.68}$$

Note from the definition that the isoperimetric function is non-negative and that

$h(\beta) = h(1-\beta)$ for all $\beta \in [0,1]$.

Definition 3.4.2. *We call an isoperimetric estimator of* (M, g) *any function* $H : [0,1] \to \mathbb{R}^+$ *such that* $h(\beta) \geq H(\beta),\ \forall \beta \in [0,1]$.

Consider a Riemannian manifold (M, g) equipped with an isoperimetric estimator $H(\beta)$. We will construct a symmetric Riemannian manifold M^*, of the same dimension as M, having H as its isoperimetric function.

Let $M^* := S^{N-1} \times (0, L)$ endowed with $g^* = a^2(s)\, d\theta^2 + ds^2$, where $\theta \in S^{N-1}$, $s \in (0, L)$, $d\theta^2$ is the canonical Riemannian metric on S^{N-1}, ds^2 is the arclength element on $(0, L)$, and $a^2(s)$ stands for a strictly positive, smooth function on $(0, L)$, continuous on $[0, L]$, with $a(0) = a(L) = 0$. Note that (M^*, g^*) is a Riemannian manifold with revolution symmetry, but not necessarily complete.

Let $B(N, s) \subseteq M^* \cup \{N\}$ be the ball of radius s and center N, the North pole, that is the point corresponding to $S^{N-1} \times \{0\}$. Denote by $Rvol(s)$ the relative volume of the ball $B(N, s) = \{N\} \cup \left(S^{N-1} \times (0, s)\right)$ with respect to the volume of the entire manifold M^*. Thus, we have

$$Rvol(s) = \frac{V(B(N,s))}{V(M^*)} = \frac{V(S^{N-1})}{V(M^*)} \int_0^s a^{N-1}(t)\, dt. \tag{3.69}$$

As $Rvol(s) \in [0,1]$, we may evaluate any estimator of M^* at $Rvol(s)$. Due to the structure of M^*, note that the isoperimetric function on the balls $B(N, s)$ of $M^* \cup \{N\}$, i.e.

$$h^*(Rvol(s)) := \frac{A(\partial B(N,s))}{V(M^*)}, \tag{3.70}$$

is actually the isoperimetric function of M^* itself.

In fact, we want to implement the main feature of the construction by choosing the estimator H of (M, g) to be the isoperimetric function of the balls $B(N, s)$, in other words

$$H(Rvol(s)) := \frac{A(\partial B(N,s))}{V(M^*)}. \tag{3.71}$$

As $A(\partial B(N,s)) = V(S^{N-1})\, a^{N-1}(s)$, the previous equality can be written as

$$H(Rvol(s)) = \frac{A(\partial B(N,s))}{V(M^*)} = \frac{V(S^{N-1})a^{N-1}(s)}{V(M^*)} = \frac{dRvol(s)}{ds}, \quad s \in (0,L). \tag{3.72}$$

Therefore, given a Riemannian manifold (M,g) equipped with an isoperimetric estimator H, we can summarize the construction of (M^*, g^*) as follows. Determine the function $Rvol(s)$ from the differential equation (3.72) with the initial condition $Rvol(0) = 0$. Equivalently, this gives $Rvol(s)$ implicitly as

$$s = \int_0^{Rvol(s)} \frac{dv}{H(v)}. \tag{3.73}$$

From the latter, since s is the arclength parameter, we can find the value of L by

$$L = \int_0^1 \frac{dv}{H(v)}. \tag{3.74}$$

We can see that $h(0) = h(1) = 0$ (using the property $h(\beta) = h(1-\beta)$), thus $H(0) = H(1) = 0$. Therefore the two previous equations make sense only when the integrals converge. The convergence of the integral (3.74) implies the compactness of M^*, by extending the notation M^* to $\left(S^{N-1} \times (0,L)\right) \cup \{N, S\}$, where N and S correspond to the points $S^{N-1} \times \{0\}$ and $S^{N-1} \times \{L\}$. Note, however, that for M^* to be smooth, we need $a'(0) = 1$ and $a'(L) = -1$. Otherwise, M^* will be a manifold with conical singularities at the North and South poles, respectively.

Before proving the main result, we will present an example of M^*.

Example 3.4.1 ([1])**.** *Let (M,g) be a two-dimensional Riemannian manifold equipped with the isoperimetric estimator $H(\beta) = \sqrt{\beta(1-\beta)}$. By equation (3.73), and by a* *change of variable $u = \sqrt{\beta}$, we get*

$$s = \int_0^{Rvol(s)} \frac{d\beta}{\sqrt{\beta(1-\beta)}} = \int_0^{\sqrt{Rvol(s)}} \frac{2\,du}{\sqrt{1-u^2}} = 2\arcsin\sqrt{Rvol(s)}. \tag{3.75}$$

Thus, we have

$$Rvol(s) = \sin^2\frac{s}{2}. \tag{3.76}$$

Similarly by equation (3.74), we obtain $L = \pi$. To compute $a(s)$, apply equation (3.69), and the fact that here $N = 2$, to get

$$Rvol(s) = \frac{V(S^1)}{V(M^*)}\int_0^s a(t)dt, \tag{3.77}$$

then

$$\frac{d}{ds}\sin^2\frac{s}{2} = \frac{d}{ds}\Big(\frac{V(S^1)}{V(M^*)}\int_0^s a(t)dt\Big), \tag{3.78}$$

or

$$a(s) = \frac{V(M^*)}{V(S^1)}\frac{d}{ds}\Big(\sin^2\frac{s}{2}\Big) = \frac{V(M^*)}{V(S^1)}\left(\sin\frac{s}{2}\,\cos\frac{s}{2}\right). \tag{3.79}$$

Thus

$$a(s) = \frac{V(M^*)}{V(S^1)}\left(\frac{1}{2}\sin s\right) = \frac{V(M^*)}{4\pi}\sin s. \tag{3.80}$$

We thus have $M^* = S^1 \times (0,\pi)$, and $g^* = a^2(s)\,d\theta^2 + ds^2$ with $a(s)$ defined as above. It was proved in [1] that the choice of $V(M^*)$ is arbitrary, as the Raleigh quotient does not depend on $V(M^*)$. Here, a good choice for $V(M^*)$ is $V(M^*) = 4\pi$, as, in this case, the couple (M^*, g^*) will be exactly (S^2, can) whose isoperimetric estimator is $h(S^2, can; \beta) = \sqrt{\beta(1-\beta)}$.

Influenced by the work of Bérard, and using his notations and definitions in [1], we proved the following generalized theorem of the Faber-Krahn inequality.

Theorem 3.4.1. *Let (M, g) be a compact Riemannian manifold with no boundary,* *equipped with an isoperimetric estimator H such that the integrals (3.73) and (3.74) converge. Let Ω be a domain in M, and let Ω^* be a ball in M^* centered at N such that $V(\Omega)/V(M) = V(\Omega^*)/V(M^*)$, where $M^* = \big(S^{N-1} \times (0, L)\big) \cup \{N, S\}$. Then $\lambda_1(\Omega) \geq \lambda_1(\Omega^*)$.*

Proof. Let $\gamma = \dfrac{V(M)}{V(M^*)}$. By the hypothesis on the convergence of $\dfrac{1}{H}$, we have that M^* is compact, and thus $V(M^*)$ is finite. So is $V(M)$, as M is compact.

Let u be the eigenfunction associated with the eigenvalue $\lambda_1(\Omega)$. Consider the level sets $\Omega_t = \{x \in \Omega : u(x) \geq t\}$, with $0 \leq t \leq m = \sup u$, and $\Omega_0 = \Omega$. Denote by $G(t)$ and $F(t)$, respectively, the following

$$G(t) = \int_{\Omega_t} |\nabla u|^2 dV, \tag{3.81}$$

and

$$F(t) = \int_{\Omega_t} u^2 dV. \tag{3.82}$$

Define Ω_t^* the balls on M^* centered at N such that $V(\Omega_t)/V(M) = V(\Omega_t^*)/V(M^*)$, i.e. $V(\Omega_t) = \gamma V(\Omega_t^*)$. Furthermore, define the function $u^* : \Omega_0^* = \Omega^* \to \mathbb{R}$ radially such that

$$u^*|_{\partial \Omega_t^*} = t = u|_{\partial \Omega_t}, \ \forall t: \ 0 \leq t \leq m. \tag{3.83}$$

Note that $u^*|_{\partial \Omega_0^*} := u^*|_{\partial \Omega^*} = 0$, and that $u^* \in H_0^1(\Omega^*)$.

Analogously to (3.81) and (3.82), define the functions $G^*(t)$ and $F^*(t)$ by

$$G^*(t) = \int_{\Omega_t^*} |\nabla u^*|^2 dV, \tag{3.84}$$

and

$$F^*(t) = \int_{\Omega_t^*} u^{*2} dV. \tag{3.85}$$

We have $V'(\Omega_t) = -\int_{\partial\Omega_t} |\nabla u|^{-1} dA$, as

$$\begin{aligned}
V'(\Omega_t) &= \frac{d}{dt}\big(V(\Omega_t)\big) = \frac{d}{dt}\Big(\int_{\Omega_t} dV\Big) \\
&= \frac{d}{dt}\Big(\int_t^\infty \int_{\partial\Omega_\tau} |\nabla u|^{-1} dA d\tau\Big) \\
&= -\int_{\partial\Omega_t} |\nabla u|^{-1} dA. \qquad (3.86)
\end{aligned}$$

Similarly, we get

$$G'(t) = -\int_{\partial\Omega_t} |\nabla u| dA. \qquad (3.87)$$

Now, as u^* is radial, by the same method, we get

$$V'(\Omega_t^*) = -|\nabla u^*|^{-1} A(\partial\Omega_t^*), \qquad (3.88)$$

and

$$G^{*\prime}(t) = -|\nabla u^*| A(\partial\Omega_t^*). \qquad (3.89)$$

Therefore, by applying the Cauchy-Schwartz inequality to the functions $|\nabla u|^{1/2}$ and $|\nabla u|^{-1/2}$ on $\partial\Omega_t$, we get

$$A^2(\partial\Omega_t) = \Big(\int_{\partial\Omega_t} dA\Big)^2 \leq V'(\Omega_t) \cdot G'(t), \qquad (3.90)$$

while

$$A^2(\partial\Omega_t^*) = V'(\Omega_t^*) \cdot G^{*\prime}(t), \qquad (3.91)$$

as u^* is radial.

While on the compact manifolds with pinched Ricci curvature, we had Gromov's isoperimetric inequality implying

$$A(\partial\Omega) \geq \gamma A(\partial\Omega^*), \qquad (3.92)$$

here we will derive it using the isoperimetric estimator. If Ω is a domain in M with $V(\Omega) = \beta V(M)$, then

$$\begin{aligned}
A(\partial\Omega) &\geq h(\beta)V(M), \text{ by the definition of the function } h,\\
&\geq H(\beta)V(M), \text{ by the definition of the function } H,\\
&= \frac{A(\partial\Omega^*)}{V(M^*)}\cdot V(M), \text{ by (3.71)},\\
&= \gamma A(\partial\Omega^*).
\end{aligned} \tag{3.93}$$

Thus, we have $A(\partial\Omega_t) \geq \gamma A(\partial\Omega_t^*)$ and, by (3.90), (3.91), and the co-area formula, we obtain that

$$\gamma G^*(t) \leq G(t), \tag{3.94}$$

and

$$F(t) = \gamma F^*(t). \tag{3.95}$$

We omitted the details which can be found in the proof of Theorem 3.3.1. In particular, for the derivation of the last two equalities, we refer to the equations (3.61), (3.62) and (3.64).

Accordingly,

$$\frac{G(t)}{F(t)} \geq \frac{\gamma G^*(t)}{\gamma F^*(t)} = \frac{G^*(t)}{F^*(t)},\ \forall\, t \in [0, m], \tag{3.96}$$

in particular for $t = 0$,

$$\frac{G(0)}{F(0)} \geq \frac{G^*(0)}{F^*(0)}, \tag{3.97}$$

from which we conclude that

$$\frac{\displaystyle\int_\Omega |\nabla u|^2 dV}{\displaystyle\int_\Omega u^2 dV} \geq \frac{\displaystyle\int_{\Omega^*} |\nabla u^*|^2 dV}{\displaystyle\int_{\Omega^*} u^{*2} dV}. \tag{3.98}$$

Finally, u being an eigenfunction of $\lambda_1(\Omega)$, by the variational characterization of eigenvalues (Theorem 3.2.2), we have

$$\lambda_1(\Omega) = \frac{\displaystyle\int_\Omega |\nabla u|^2 dV}{\displaystyle\int_\Omega u^2 dV}, \tag{3.99}$$

and

$$\frac{\displaystyle\int_{\Omega^*} |\nabla u^*|^2 dV}{\displaystyle\int_\Omega u^{*2} dV} \geq \lambda_1(\Omega^*). \tag{3.100}$$

Therefore

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*), \tag{3.101}$$

and the Faber - Krahn inequality is then proved. $\square$

Appendix A

Referenced Results

A1 Green's Formulas

Theorem A1.1 (Green's formula I, [4]). *Let (M, g) be an N-dimensional Riemannian manifold without boundary. Consider two functions f and h in $C^2(M)$ and $C^1(M)$, respectively. Then*

$$\int_M \{h\Delta f + (\nabla h, \nabla f)\} dV = 0. \tag{A.1}$$

If both functions are of class C^2 over M, then

$$\int_M \{h\Delta f - f\Delta h\} dV = 0. \tag{A.2}$$

Theorem A1.2 (Green's formula II, [4]). *Let M be as above and, additionally, oriented. Let Ω be a domain in M with boundary of class C^1 and let ν be the outward normal unit vector field along $\partial\Omega$. Consider two functions f and h in $C^2(M)$ and $C^1(M)$, respectively. Then*

$$\iint_\Omega \{h\Delta f + (\nabla f, \nabla h)\} dV = \int_{\partial\Omega} h(\nu, \nabla f) dA. \tag{A.3}$$

If both f and h are C^2 on M, then

$$\iint_{\Omega} \{h\Delta f - f\Delta h\} dV = \int_{\partial\Omega} \{h(\nu, \nabla f) - f(\nu, \nabla h)\} dA. \tag{A.4}$$

A2 Rellich - Kondrasov Theorem

Theorem A2.1 ([9])**.** *Let Ω be an open bounded subset of $\mathbb{R}^N$ with boundary of class C^1. For $1 \le p < N$, define p^* as $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{N}$. Then the following inclusions are compact:*

a. If $p < N$, $W^{1,p}(\Omega) \to L^q(\Omega)$, $1 \le q \le p^*$,

b. If $p = N$, $W^{1,p}(\Omega) \to L^q(\Omega)$, $1 \le q \le \infty$,

c. If $p > N$, $W^{1,p}(\Omega) \to C(\overline{\Omega})$.

If Ω is any bounded domain, then the inclusions are valid for $W_0^{1,p}(\Omega)$.

A3 Spectral Results

Theorem A3.1 (Spectral Theorem [5])**.** *Suppose that $T : H \to H$ is a non-zero self-adjoint compact operator from a Hilbert space H to itself. Then*

1 - There exists at least one eigenvalue $\lambda \in \{\pm\|T\|\}$.

2 - There are at most countably many non-zero eigenvalues, $\{\lambda_n\}_{n=1}^N$, where $N = \infty$ is allowed. Unless T is finite rank, N will be infinite.

3 - The eigenvalues λ_n's may be arranged so that $|\lambda_n| \ge |\lambda_{n+1}|$, for all n. If $N = \infty$, then $\lim_{n\to\infty} |\lambda_n| = 0$. In particular, any eigenspace of T corresponding to a non-zero eigenvalue is finite dimensional.

4 - The eigenvectors $\{\varphi_n\}_{n=1}^{\infty}$ can be chosen to form an orthonormal basis such that $H = \overline{span\{\varphi_n\}} \oplus Nul(T)$.

5 - Using $\{\varphi_n\}_{n=1}^{N}$, we have

$$T\psi = \sum_{n=1}^{N} \lambda_n(\psi, \varphi_n)\varphi_n, \ \forall \psi \in H. \tag{A.5}$$

6 - The spectrum of T is $\sigma(T) = \{0\} \cup \cup_{n=1}^{\infty}\{\lambda_n\}$.

Theorem A3.2 (Friedrichs extension [12]). *Let A be a positive symmetric operator, and let $q(\varphi, \psi) = (\varphi, A\psi)$, for $\varphi, \psi \in D(A)$, where $D(A)$ is a domain of the operator A. Then q is a closable quadratic form and its closure $\hat{q}$ is the quadratic form of a unique self-adjoint operator $\hat{A}$, $\hat{A}$ is a positive extension of A, and the lower bound of its spectrum is the lower bound of q. Furthermore, $\hat{A}$ is the only self-adjoint extension of A whose domain is contained in the form domain of $\hat{q}$.*

A4 Weak Maximum Principle

Theorem A4.1. *[6] Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, and let L be a linear elliptic second order differential operator on Ω of the form*

$$Lu = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{k=1}^{N} b_k(x)\frac{\partial u}{\partial x^k} + c(x)u. \tag{A.6}$$

with continuous coefficients. It is assumed that the coefficients a_{ij} satisfy the uniform ellipticity condition and that $a_{ij} = a_{ji}$. Suppose that

$$c = 0 \ \ \textit{and} \ \ Lu \geq 0 \ \ (\textit{resp.} \leq 0) \ \textit{in} \ \Omega \tag{A.7}$$

for some $u \in C^2(\Omega) \cup C^0(\overline{\Omega})$. Then, unless u is constant in Ω, the maximum (resp.

minimum) of u in $\overline{\Omega}$ is not attained on Ω, in other words

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u \quad (\textit{respectively} \;\; \min_{\bar{\Omega}} u = \min_{\partial\Omega} u). \tag{A.8}$$

A5 Bonnet - Myers Theorem

Theorem A5.1. *[4] Let (M, g) be a complete Riemannian N-dimensional manifold, $N \geq 2$, such that there exists a constant $k > 0$ for which*

$$Ric\,(u, u) \geq k(N-1)\, g(u, u), \tag{A.9}$$

for all $u \in TM$. Then M is compact and the manifold's diameter, $d(M)$, satisfies

$$d(M) \leq \pi/\sqrt{k}. \tag{A.10}$$

A6 Gromov's Isoperimetric Inequality

Theorem A6.1. *[7] Let M be a compact N-dimensional Riemannian manifold, $N \geq 2$, equipped with a Riemannian metric g. Let Ω be a domain in M with smooth boundary. Let Ω^* be a geodesic ball of the canonical sphere S^N, such that the relative volume of Ω^* in S^N is equal to the relative volume of Ω in M, and let $R(M, g) = \inf\{Ric\,(u, u) : u \in TM\}$. If $R(M, g) \geq N - 1 = R(S^N, g^*)$, then we have*

$$\frac{A(\partial\Omega)}{V(M)} \geq \frac{A(\partial\Omega^*)}{V(S^N)}, \tag{A.11}$$

with equality if and only if the triplet (M, g, Ω) *is isometric to* (S^N, can, Ω^*).

A7 The Co-Area Formula

Theorem A7.1. *Let* Ω *be an open domain on a manifold* M *and let* u *be a real valued Lipschitz function on* Ω. *Then, for any real valued integrable function* f *on* Ω, *we have*

$$\int_{\Omega} f dV = \int_{\inf f}^{\sup f} \int_{\{u=t\}} f |\nabla u|^{-1} dA \, dt. \tag{A.12}$$

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