APOS Theory as a Framework to Study the Conceptual Stages of Related Rates Problems.

Mathew Tziritas

A Thesis in the Department of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements

for the Degree of Masters in the Teaching of Mathematics at

Concordia University

Montreal, Quebec, Canada

September 2011

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis prepared

By: Mathew Tziritas

Entitled: APOS theory as a framework to study the conceptual stages of related rates problems

And submitted in partial fulfillment of the requirements for the degree of

Master in the Teaching of Mathematics

Complies with the regulations of the University and meets the accepted standards with respect to originality and quality.

Signed by the final examining committee:

H. Proppe	____	Chair
N. Hardy	____	Examiner
H. Proppe	____	Examiner
A. Sierpinska & A. Reyes	____	Supervisors

Approved by ____

Chair of Department or Graduate Program Director

____

Dean of Faculty

Date ____

Abstract

APOS Theory as a Framework to Study the Conceptual Stages of Related Rates Problems.

Mathew Tziritas, M.A.
Concordia University, 2011

A study was done in an attempt to use the APOS theory of learning and teaching mathematics to develop and test a teaching cycle for the improvement of students' conceptual understanding of related rates problems. "APOS" is an acronym that stands for Action, Process, Object, and Schema, and refers to both a theory of teaching and learning and a research methodology in mathematics education. "Related rates problems" refers to problems in Calculus that require finding the rate of change of one value, given the rate of change of a related value.

Part of APOS research methodology is a "genetic decomposition" of the concepts to be learned by the students in terms of the mental constructions that such learning requires. In the present study, the genetic decomposition focused on the mental constructions required for student success during the initial conceptual stages of related rates problems learning. The decomposition was constructed using the author's knowledge of the subject. The genetic decomposition was used to construct an Action – Discussion – Exercise (ACE) teaching cycle which was then tested on two groups of students. Finally, students were asked to solve related rates problems during an individual interview with the author. Data from students' involvement in the ACE cycle as well as their work during the interview process were then used to suggest changes to the genetic decomposition and the ACE cycle. These suggestions constitute the results of the study. Their purpose is to improve the starting point for further iterations of experimentation of teaching related rates problems.

Acknowledgments

I would like to thank all those who supported me and to my advisors, Dr. Reyes and Dr. Sierpinska for their patience and wisdom.

Table of Contents

1 Introduction

Related rates is a topic taught in most Differential Calculus courses in North America. Historically, the related rates problems were the original cornerstone of differential calculus (Boyer, 1949). Most modern texts, especially those used at the college introductory level, however, only introduce related rates problems near the end of the chapter on differentiation [(Thomas, Weir, & Hass, 2010); (Stewart, 2008)]. The purpose of these problems is to showcase the nature and ability of differentiation while being geared towards using them as an application of the derivative and not as an epistemological tool.

The chapters of the texts that are concerned with related rates problems give a solution procedure that is generally broken down into six steps;

1. Drawing a diagram
2. Labeling the diagram and summarizing the problem mathematically
3. Identifying relevant geometric relationships
4. Differentiating the geometric relationship
5. Substituting given information into the differential to solve
6. Interpreting the results.

There hasn’t been much research into the teaching and learning of geometric related rates problems. What is known well, however, is that students have difficulty solving these problems. There are several studies that demonstrate the particular difficulties that students have when attempting to solve related rates problems. They point to difficulties with the conceptual stages, especially the first two stages, where students have to sketch and summarize the problem mathematically, but also with some aspects of the third stage where they must identify geometric relationships, and with the last step where they are required to interpret their results in terms of the initial geometric/physical situation (Martin, 2000; White and

Mitchelmore, 1996). In particular, students have trouble engaging in transformational reasoning (Engelke, 2004). This hinders students' ability to coordinate change in one variable with change in another. Similarly, it can be said that student lack an "abstract-general" concept of variable (White & Mitchelmore, 1996).

Students' difficulties with fundamental concepts of calculus such as the derivative, limits, variables, and functions also hinder their ability to successfully model and solve related rates problems. For the purposes of this thesis, however, we have designed a teaching experiment that was intended to deal with conceptual difficulties involved in steps 1, 2 and 3 only. Therefore, concepts of limits and derivatives will not intervene in our sequence of teaching activities for students. Derivatives will intervene only in the final interview, where we will test whether our teaching activities have helped students in solving related rates problems about which they learned in their regular Calculus course in the time between our teaching activities and the final interview session.

1.1 Background

1.1.1 MATH 203 at Concordia University

The research was done at Concordia University in Montreal, Quebec, Canada. This university has a differential calculus course with course code MATH 203. The textbook used by the students is Thomas Calculus (Thomas, Weir, & Hass, 2010), with related rates problems taught from the text during the second half of the semester. MATH 203 is a first course in differential calculus offered at Concordia University. Differential and integral calculus are normally offered at the pre-University level in the Quebec college system. This course is not part of a bachelor's level study; it is a pre-university level course offered to those candidates who need it as a prerequisite for an academic program of their choice. Many of the students are mature students, seeking to go into fields such as the sciences and engineering but lacking the prerequisite mathematics courses for admission into these programs. The course outline, provided in Appendix 5, covers in more detail the specifics of the course.

1.1.2.2 Difficulties Specific to the Concordia University elementary Calculus course

The final examination for MATH 203 usually contains one geometric related rates problem. Students have particular difficulty with this part of the examination, so much so that in the semester preceding this study, the related rates problem on the final examination had to be removed from the grading scheme because students' performance on the question was very low. The question was similar to those in past finals, and those presented in the text. It is clear that at a local level, geometric related rates problems are cause for much frustration in students.

1.2 Research Question

The general purpose of this study is to explore the possibility of designing a sequence of teaching-learning activities intended to help students during the initial conceptual stages of related rates problems. The teaching experiment was based on a "genetic decomposition" of those stages. Genetic decomposition is an epistemological instrument developed in the frame of APOS theory. "APOS" stands for Action, Process, Object, and Schema, and refers to both a theory of teaching and learning and a research methodology in mathematics education. APOS theory originated in the research of an American mathematician and mathematics educator, Ed Dubinsky, on undergraduate students' learning of mathematics (Calculus, Linear Algebra, Abstract Algebra); see, e.g. (Dubinsky E. , 1991); (Dubinsky, Breidenbach, Hawks, & Nichols, 1992). The psychological assumptions of the APOS theory are grounded in Piaget's constructivism (Piaget, 1972).

Part of APOS research methodology is a “genetic decomposition” of the concepts to be learned by students in terms of the mental constructions that such learning requires. In the present study, the genetic decomposition focused on the mental constructions required for student success during the initial conceptual stages of related rates problems learning. The genetic decomposition was then used to produce a cycle of teaching instruments, the so-called ACE cycle: Activity-Class Discussion-Exercise.

The ACE teaching cycle is meant to promote reflective abstraction (Piaget, 1972) in order to build relevant mental constructions. APOS methodology consists in an iterative process beginning with the design and implementation of instruction in the form of a genetic decomposition and an ACE teaching cycle, the analysis of data, and a theoretical analysis leading back to the first step. This study aims at doing one full iteration of this process. The general purpose of this study is to:

- Construct a genetic decomposition of the mental construction necessary for the student to succeed at the initial conceptual stages of related rates problems solution process (not including the stage of interpretation of the result in terms of the initial physical/geometric situation).
- Use the genetic decomposition to construct and implement an ACE cycle (focused on those initial conceptual stages) based on the genetic decomposition.
- Assess students’ understanding of related rates problems after they have completed the ACE cycle.
- Use the collected data to revise the genetic decomposition and the ACE cycle.

This research is exploratory in nature, because we are not aware of the existence of previous APOS studies of related rates problems; therefore it appears to be a first iteration of such study.

This study is also, in the broadest sense, a use-inspired basic research (Schoenfeld, 2007). The research searches for a fundamental understanding while simultaneously considering usefulness and application. The study could also be classified under the category of a "phase one design experiment" (Schoenfeld, 2007) .

1.3 Structure of the Thesis

The thesis is structured as follows. Chapter 2, which follows the present Introduction, will describe the theoretical framework used in the research. Here the APOS framework will be described in general, followed by how it has been modified and refined for this particular study. Also in this chapter, the study and its methodology will be situated in the context of research in mathematics educational research in general.

Chapter 3 will provide an epistemological and historical background for the genetic decomposition of related rates problems which will be presented in Chapter 4.   Chapter 3 will also review relevant research on related rates problems in mathematics education. Chapter 4, as mentioned, will present the genetic decomposition of the initial conceptual stages of solving related rates problems and, based on this decomposition, the methodology of the research will be discussed. The genetic decomposition is used to produce the research instrument, the ACE cycle. The procedure used to test the ACE cycle on students is also defined.

Chapter 5 will give a description of the results of testing the ACE cycle on the students. The result of the students' responses to the ACE cycle and an interpretation of these responses will be given.

In Chapter 6, the results will be used to discuss changes to the genetic decomposition. This chapter ends with the changes to the ACE cycle for use in further iterations of the study, as inspired by the genetic decomposition.

The thesis contains moreover a list of references, and several appendices with all the transcripts and scans of students' work.

2. Theoretical Framework and Methodology of Research

2.1. APOS theory

APOS theory is a framework for instructional research in mathematics education. It proposes to conduct research in three steps:

- Theoretical analysis of the content to be taught and learned;
- Design and implementation of instruction; and
- Collection and Analysis of Data.

These steps are expected to be repeated until satisfactory results regarding students' learning and understanding of the content are obtained. We say that the research process is cyclical: analysis of the data leads to the collection of more data or to a redesign of the instructional methods.

2.1.1 Theoretical analysis

The primary goal of the theoretical analysis is to attempt to construct a description of the specific mental constructions that a learner might make when developing his understanding of the concept to be taught. This type of analysis is called the genetic decomposition of the content. The researcher's own understanding of the topic greatly influences the genetic decomposition and therefore many iterations of the framework are required to eventually incorporate the actual learners' understandings and background.

Underlying the theoretical analysis step of the research are psychological assumptions based primarily on the work of Piaget. In particular, Piaget's concept of reflective abstraction is usually invoked by Dubinsky (1991) as the main inspiration for the theory. Piaget's ideas are reformulated to fit the context of college level Mathematics.

The following statement describes what it means to understand something in mathematics:

> *An individual's mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (Asiala, Brown, DeVries, Dubinsky, Matthews, & Thomas, 1996, p. 7)*

Therefore, just knowing the definition of a mathematical concept is not enough. The learner must be able to call upon the concept in suitable situations when solving problems. This perspective becomes very specific to mathematics in the next section.

Four levels of understanding are assumed, called action, process, object, and schema. The process of understanding of a concept begins with actions that the learner is familiar with from previous studies. At this level, the learner must be told which actions to use and in what sequence. At the process level, the learner is already able to choose the actions to perform and is able to decide in which order to perform them. We say that the learner has "interiorized the actions to form a process. When the learner has identified the different concrete instances of the process as a new object (has "encapsulated" the processes into a mathematical object), we say that his or her understanding attained the object level. Below, we explain the mental processes involved in each level, based on (Dubinsky & Mcdonald, 2002)

Action – An action is characterized specifically by the individual having an external perception of the mathematical concept. That is to say, the individual can only carry out transformations via specific external cues and detailed step by step procedure.

Process – The process of interiorization occurs when the individual begins to reflect upon the action which he or she is performing. An individual who is at the process level of understanding can "reflect on, describe, or even reverse the steps" of a transformation on previously learned objects without actually performing those steps. Here the student may view the process as an input-output process.

Object – The process of encapsulation involves reflection on a particular set of processes until the individual can construct transformations on the mathematical concept. Once this is achieved the concept is said to be at the object level.

Schema – Objects and processes are interconnected in the individual's minds to construct schemas. Schemas are less understood as they vary from person to person since the various connections can differ. Schemas can be thematized into objects so that they can be used to build new mathematical objects and can sometimes be called higher schemas.

Development of a schema occurs using a process called reflective abstraction. This process utilizes two mechanisms: projection unto a higher level of abstraction and reflection aimed at reconstruction and reorganization into larger systems. The process of reflective abstraction is the means by which concepts can evolve from actions to processes to objects and finally into schemes. These processes are termed interiorization, encapsulation and thematization, respectively.

Interiorization - Transformation of an action is the process by which a physical series of actions can be performed in the mind without the need to be prompted or having to perform every individual step. Once achieved, it can be said that a given action has been interiorized into a process.

Encapsulation – A process is encapsulated when the given mathematical concept exists without the need to perform any specific actions or steps. At this stage the concept gains invariant properties. Once this is achieved, the concept can be transformed and new actions can be learned using the encapsulated mathematics process, now said to be at the object level.

Thematization – Thematization is the process by which multiple objects, processes, and actions, form a coherent body, called a scheme, where concepts can be manipulated and related to one another.

2.1.2 Design and Implementation of Instruction

This is the second part of the APOS based research cycle, where the theoretical analysis influences the creation of specific instructional treatments. Underlying the design of instruction is the assumption that learning is a non-linear method process. The student first develops partial understanding, repeatedly returns to the same idea, and periodically summarizes and tries to pull his or her ideas together. APOS theory assumes moreover that learning is fundamentally dependent on cognitive conflicts whose overcoming requires a "re-equilibration" of previously developed mental constructions (Piaget, 1985). Cognitive conflicts may arise when the learner's ideas are verbalized and confronted with contrasting ideas of others. Therefore, in research based on APOS theory, students are usually organized into groups where they can work cooperatively and are encouraged to reflect on procedures that they perform. This is indented to thrust the students into an environment where their mental constructions can disequilibrate, or start to contradict each other in the learner's mind. The effort to overcome those contradictions may lead to the formation of new mental constructions.

An instructor-guided classroom discussion, which follows the collaborative group work, is meant to clarify and consolidate these new constructions in the learners' minds. This is further enforced by take-home exercises.

We have thus described the "ACE" teaching cycle: from activities to classroom discussions to exercises:

Activities – Activities are generally done in a computer lab where students work in teams. The students perform activities which are designed to foster specific mental constructions which are suggested by the genetic decomposition.

Class Discussion – Here students continue to work in groups and are given an opportunity to reflect on the work they did in the lab and do further problems. The instructor attempts to stimulate the students to have inter-group discussion, and will occasionally provide definitions, explanations, and overviews to tie together what the students have been thinking about.

Exercises – Standard exercises are given to the students to perform, usually in groups, outside of class. These exercises are designed to reinforce the concepts learned in class and to use the mathematics learned in class.

2.1.3 Data

Data must contain information about the students and the course(s) taken. Sometimes the study is on students who have previously learned the mathematics or students who have studied an ACE teaching cycle on the same topic. The most important thing is that data be collected on the students' relationship with the particular material.

The data is, first of all, the source of precise objective information about the students' behaviour when faced with the material. Based on this data, the researchers build plausible hypotheses about the mental constructions the students might be making and how much of what mathematics are they learning and using. The purpose is to confront the actual students' behaviour and possible thinking with the conjectures about mental constructions involved in learning the material obtained through the theoretical analysis.

The data is collected in the form of written answers to specially designed question sets, along with in-depth interviews of students about their answers to these questions. The interview process is often the most difficult. The questions and general interview process must be designed with the purpose of testing the hypotheses set forth by the current version of the genetic decomposition: do these hypotheses support or lead to a revision of the theoretical perspective.

It is not expected that the data will give clear yes or no answers to the theoretical conjectures about student learning. Rather, a spectrum of mental constructions, closer or farther from those predicted by the theoretical analysis, is more likely to emerge from the work with students.

2.2 Adaptation of APOS Theory

2.2.1 Adaptation of Genetic Decomposition

This being the first iteration of the APOS framework, with no previous genetic decompositions for the conceptual stages of related rates problems at our disposal, a far more generalized and simplified adaptation of mental construction framework is used. It is similar to the genetic decomposition of the chain rule obtained in (Clark, et al., 1997). Since these authors had no access to any previous genetic decomposition for the chain rule, they used a more vague description of certain mental constructions. For example, instead of choosing action, process, or object, the authors described the students' function concept as process or object. The development of the schema through the processes of interiorization, encapsulation, and thematization is generally overlooked in the generalized and simplified models.

The genetic decomposition used here will not decisively choose directly from action, process or object but will use only two categories: "action or process", and "process or object". This allows this preliminary genetic decomposition to be more flexible. Since this is the first iteration of the APOS framework, this binary decomposition will be open to more refinement upon iteration.

An action or process mental construction is where the student is incapable of performing transformations on processes encapsulated into objects or to reflect abstractly on a mathematical concept. The student is incapable of reversing the process, or transforming the concept for use in more complex mathematical situations. The student is therefore restricted to viewing the concept as an input-output process and can only use the concept in response to particular cues. For example, when students are learning the concept of function, at one stage they conceive of them as sequences of things to do: compute values, draw a graph, decide using some tests where the function is increasing, etc. But they have trouble of conceiving functions as objects that can be added, subtracted, multiplied, inversed, etc. Another example could be students' use of the Pythagorean Theorem. At the pre-object level, students can use the Pythagorean equation $a^2 + b^2 = c^2$ to calculate a side of a right-angled triangle; they treat the equation as an input-output process. But they do not conceive of the equation as a necessary condition of the assumption that the triangle is right-angled, and it would not occur to them to ask if the converse of the theorem is also true. They cannot manipulate the concept abstractly in the aim of understanding perpendicular movement of objects or to determine the length of a line in the Cartesian plane unless explicitly prompted to do so.

The process or object level mental construction, on the other hand, is one in which the student is capable or beginning to become capable of performing transformations on the encapsulated processes. Here, even if the student is able to only view the concept as input-output process, he or she does not need to be prompted and is beginning to form the encapsulation process. The student is able to transform the concept for use in different mathematical situations. For example, the student would be able to use the Pythagorean Theorem to conceptualize and construct the unit circle, without explicit instruction or hint to use this theorem.

2.2.2 Adaptation of the ACE Cycle

In this study, several modifications of the ACE cycle have been introduced, because of the size and small number of participants in the study. The activities are not to be done in a lab, on the computer. Instead, they will be performed on paper. Also, only one group will be tested at a time, so there will be no inter-group discussion. Students in MATH 203 often have jobs and may live far from the campus, therefore the take-home assignments will be done individually.

2.3 Methodology

Alan H. Schoenfeld, (Schoenfeld, 2007) discusses research methods in mathematics education as a whole. The paper contains three parts. The third part of the paper describes how mathematics education research can lead to developing meaningful curriculum adjustments and educational interventions. It describes a sequence of research activities that leads to eventual educational intervention. The sequence has three phases: preliminary studies and design experiments, contextual studies, and large-scale validation studies.

Design experiments are low-risk experiments aimed at exploring possible contributions to learning theory as well as the applicability of certain theories. These exploratory design experiments are to have pragmatic roots and are intended to test theory through interactive study. The present study is this type of phase one exploratory design research.

3. Epistemological Analysis of Related Rates Problems

3.1 History of Related Rates Problems

Related rates problems used to be an integral part of differential calculus courses (see, e.g., (Ritchie, 1837) ; (Connell, 1845), as quoted in a brief historical account of the related rates problems by Austin, Barry & Berman (2000)). The related rates problems showcased the nature of the derivative and highlighted the key concepts underlying the notion of derivative. In more recent textbooks, however, their role has been shifted from epistemologically essential to merely didactic. They appear in “end-of-the-chapter” remarks or exercises, to show applications of the notions of function and rate of change and of the chain rule. For example, in (Kleine, 1977) , they appear in the chapter devoted to the chain rule, and serve to show the applications of this rule in solving physical problems (e.g. the problem of comparing the rates at which the height of a liquid is increasing when poured into containers of different shapes (Kleine, 1977, p. 181)).

One reason behind this marginalisation of related rates problems could be the historical shift of the fundamental concept of Calculus from being the derivative (which was natural from the perspective of Newton’s interest in physical problems of motion) to this concept being the limit. Founding Calculus on the notion of limit appeared more natural to the nineteenth century mathematicians, in particular A. Cauchy (Cauchy, 1821/1968), interested in the arithmetization and formalization of Calculus and not in solving problems of motion in physics (Boyer, 1949).

In the above mentioned early 19th century textbook, intended for pre-university students, William Ritchie (1832-1837), professor of Natural Philosophy at London University, planned to use a “discovery method” to teach Calculus:

The plan adopted in the present work is founded on the same process of thought by which we arrive at actual discovery, namely, by proceeding step by step from the simplest particular examples till the principle unfolds itself in all its generality. (Ritchie, 1837, p. vii)

For Ritchie, "the object of the differential calculus is to determine the ratio between the rate of variation of the independent variable and that of the function into which it enters" (Ritchie, 1837, p. 11) . This reads like a reference to the inverse of what we now call "the differential quotient", whose limit we name the derivative of the function, which supports the idea advanced above that related rates problems hold a central epistemological position when *derivative* is taken as the fundamental concept of Calculus. Ritchie's approach is more subtle, however. He explains the idea in the languages of both "fathers" of Calculus: the language of infinitesimal increments of Newton (p. 12), and the language of differentials of Leibniz (p. 13). In the language of Newton, the central problem of differential calculus indeed appears to be finding a derivative of a function. In the language of Leibniz, however, the problem, as stated by Ritchie, translates into a related rates problem.

In the language of Leibniz, if x denotes the independent variable, and u the "function into which it enters", then $dx\ \ (du)$, called "the differential", is a symbol for "the rate at which x, (u) varies uniformly". Ritchie's "object of Calculus" therefore, in this language, is to find the ratio of two rates of change: dx: du. The example that Ritchie uses to illustrate the idea is:

If the side of a square increase uniformly, at what rate does the area increase when the side becomes x? (Ritchie, 1837, p. 11)

In the language of Leibniz, Ritchie (p. 13) leads the reader to the statement,

$$dx : du \ :: 1 : 2x$$

whence he derives $du = 2x\, dx$.

With Ritchie's references to Newton and Leibniz, his discovery approach in teaching qualifies also as "genetic", in the sense that the path to discovery prepared for the students is a simplified version of the historical invention of the concepts by mathematicians.

Problems in his text are some of the earliest forms of related rates problems capturing this way of looking at Calculus. Although he uses abstract type problems to begin with, later his text contains problems concerned with the rate of change of areas and volumes. For example:

> *If the sides of an equilateral triangle increase uniformly at the rate of 3ft per second, at that rate is the area increasing when the first side becomes 10 feet? (Ritchie, 1837, p. 47)*

These problems showcase the earliest types of geometric related rates problems. It is clear from the placement of related rates problems at the beginning of Richie's text that he intended related rates problems to be good explanatory exercises that helped the student to understand the nature of Calculus.

Another notable text quoted by Austin, Barry and Berman (2000) was James Connell's 1845 "The elements of the differential and integral calculus" (Connell, 1845). The textbook also takes the genetic approach:

> *Like Ritchie, Connell complained that the differential calculus was enveloped in needless mystery for all but a select few; he, too, proposed to reform the teaching of calculus by returning to its Newtonian roots. (Austin, Barry, & Berman, 2000, p. 4)*

In Connell's text again the notion of rate becomes the lowest common denominator for Calculus. He develops the notion of differentiation from the notion of rates in the context of problems of motion. Here is an example of such problem:

> *A stone dropped into still water produces a series of continually enlarging concentric circles; it is required to find the rate per second at which the area of*

one of them is enlarging, when its diameter is 12 inches, supposing the wave to be then receding from the centre at the rate of 3 inches per second. (Connell, 1845, p. 14), cited by Austin, Barry & Berman (2000)

These types of problems are almost identical to ones found in modern textbooks. They play a different role in them, however, as mentioned above.

Modern Calculus textbooks introduce the concept of limit as the central concept of differentiation and the definition of the derivative is given as a limit of the difference quotient in its most formal analytical sense, thus relegating the notion of derivative to a secondary role in Calculus. For example, the “Thomas’ Calculus” (Thomas, Weir, & Hass, 2010), or Stewart’s “Calculus” (Stewart, 2008) textbooks contain similar geometric related rates problems to those found in the texts by Richie or Connell, yet have them as an illustration of applications of the derivative. This can be seen in the phrasing these texts use when describing related rates problems. In the introductory paragraph to Stewart’s related rates section he writes;

The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time. (Stewart, 2008, p. 182)

Here it can be seen that these related rates problems are meant to make use of differentiation. Although they are intended to show some of the properties of differentiation as they are in the *Differentiation Rules* chapter of the text, rates are not the primary method for conveying the concept of differentiation itself. This is further reinforced by the fact that related rates problems are first seen only at the end of the chapter on differentiation as a separate sub-topic.

3.2 Modern Geometric Related Rates Problems

Modern related rates problems are almost exclusively geometric in nature. Geometric related rates problems are defined as "problems for which the relationship between the variables, upon which the rate is based, is described by a geometric equation." (Martin, 2000) These relationships are usually the volumes or areas of geometric figures related to one of the dimensions or a relationship between lengths and velocities of objects in a given geometric pattern. The geometric shape and equations used are usually confined to Pythagorean and trigonometric relations as well as volumes and areas of simple shapes such as circles, squares, and cones.

In order to analyze the status of geometric related rates problems in modern texts, we will analyze the two textbooks already mentioned above, the "Thomas' Calculus" (Thomas, Weir, & Hass, 2010) and "Calculus" by Stewart (Stewart, 2008). These books were chosen as they are the most widely used books at the college and university levels geographically near to Concordia University. There are two main aspects of the books that must be looked at to understand the modern status of related rated problems. First, the solution methods as proposed by the authors are analyzed to understand how the texts instruct the student to solve these types of problems.

Both texts have their related rates sections near the end of the third chapter, the chapter on differentiation. Stewart's text has the related rates section in the ninth section of the third chapter, while Thomas has it in the seventh section of the third chapter. Both texts refer to related rates as a type of problem in their introductory paragraphs and both begin with examples of geometric related rates problems. We will now look at each book separately, focusing on the first aspect of the analysis, namely solution methods.

3.2.1 Thomas' Calculus Solution Method for Related Rates Problems

Thomas' Calculus breaks the solution process into six steps. There are several worked out problems given in the chapter to illustrate these steps. A copy of the related rates section of this text can be found in Appendix 4.2. We state the steps and briefly comment on each of them:

The first step is:

1. Draw a picture and name the variables and the constants. Use t for time. Assume that all variables are differentiable functions of time.

There are several things that are important to note about the first step. First of all, Thomas distinguishes between variable and constant during the first stage of the solution process. He also assumes that the variables are functions of time. He does not suggest any specific functional notation for distinguishing a variable from a constant but he uses functional notation in several diagrams in the chapter. Unfortunately there is little to no consistency in his notations as some example problems have variables labeled with a single letter symbol, for example "y", while others are explicitly labeled as functions of time, for example "y(t)". He also assumes that variables are differentiable over time, suggesting that learners do not worry about the differentiability of functions involved in this type of problems.

Thomas' first step also asks for a "picture", which might mislead some students to draw some realistic representation of the real life situation described in the problem (e.g. a sketch of ripples on a lake as a stone is being thrown into it). What the author expects, however, is a diagram, or a geometric figure (Figure 1).

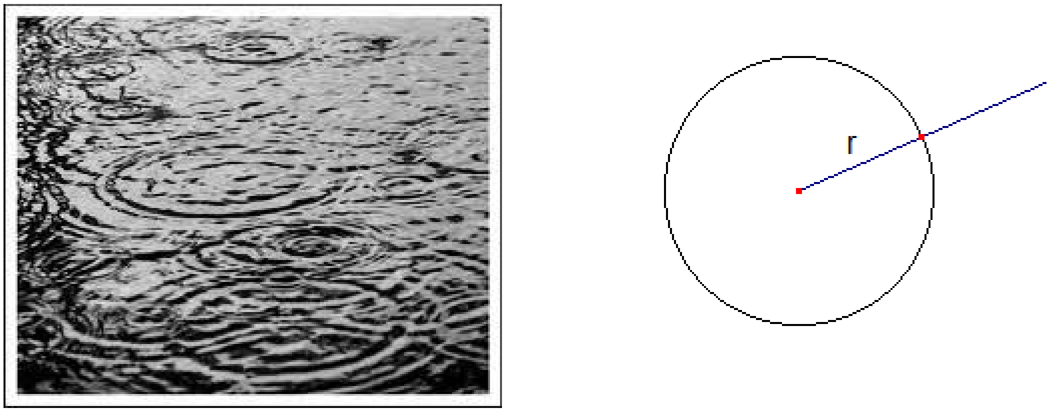


Figure 1 - The difference between a picture and a diagram.

The second step is:

2. Write down the numerical information (usually a rate, expressed as a derivative)

Here the author asks the solver to express the written rate information in differential notation. In all the subsequent examples he does not simply express the information in an algebraic form but places it on the diagram next to the variable for which the derivative represents the rate.

The third step is:

3. Write down what you are asked to find.

The author asks for the rate in question to be simply written down, yet in examples the rate is always placed next to the variable for which the rate relates.

These first three steps could therefore almost be combined into one:

1-3. Draw a diagram, labeling all variables, constants, given rates, and unknown rates.

One of the difficulties with these first three steps is that they do not indicate how to handle the fact that these problems ask for rates at a specific instant. In all the diagrams given in examples the specific time is written near the diagram in some fashion. Placing this near the diagram is not mentioned in the steps.

The fourth step is:

4. Write an equation that relates the variables. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to calculate to the variables whose rates you know.

The first thing to notice is that all mention of constants has been dropped. Yet many problems require the relationship between variables and constants. The first example given is one of a cylindrical tank where the radius is a constant which must be used in the relationship between volume and the height of water in the tank. Step 4 then mentions that more than two equations may be needed in order to find a single equation relating the variables whose rates you want to variables whose rate you know. Unfortunately this is not the reason why multiple equations are needed. Often, multiple equations are needed to reduce a function of multiple variables to a function of one variable so that the student's limited differentiation techniques can be used. These are problems where two separate variables are multiplied, for example a cone problem where the formulas for the area or volume are functions of both the height and the radius.

The fifth step is:

5. Differentiate with respect to t. Then express the rate you want in terms of the rate and variables whose values you know.

Here it should be noted that Thomas does not specify the type of differentiation, chain rule or implicit differentiation. In his examples he often uses the definition of the chain rule to differentiate.

Step six is:

6. Evaluate. Use known values to find the unknown rate.

Here Thomas uses the word “value” which he used in step 5 to describe the given rates and variables. He makes no mention of how to treat constants or the use of them. He also interchanges between “the rate you want” and “unknown rate”. He is therefore not making a clear distinction between constants and variables and is calling all the quantities in the problem “values”.

3.2.2 Stewart Calculus Solution Method for Related Rates Problems

Stewart’s textbook breaks down the solution process for related rates into seven steps. A copy of the related rates section if this text can be found in Appendix 4.1. We state the steps and briefly comment on each of them:

1. Read the problem carefully.

   This step is more a didactic advice than part of a method of solving the specific type of problems.

2. Draw a diagram if possible.

   Here Stewart asks only for a diagram of the situation. Using symbols to label elements of the diagram is identified as a step in its own right; it is the next step. In the example problems, Stewart uses both more or less realistic pictures, and diagrams, sometimes mixing the two types of representation in one figure. For example, in some figures, the geometric figure may have points drawn as cars or as a person running. In the first example he gives he chooses not to draw a diagram.

   What is not mentioned in this step is the use of arrows to show the direction of movement in problems. In half of the given examples, Stewart uses arrows to show the movement of points. He only does this when he chooses to give diagrams without pictures resembling real world objects.

3. Introduce Notation. Assign symbols to all quantities that are functions of time.

In his wording of the step, Stewart avoids the use of the words "variable" and "constant" and asks the student to label with symbol only those quantities that vary with time. Constants are placed on the diagram using their numeric value, not a letter symbol.

4. Give the given information and the required rate in terms of derivatives.

This step not only asks the student to identify the rates but to use a specific notation to describe the given and then unknown rates. He does not place this information anywhere near the diagram. Instead he places it in sentence form as the first sentence of the solution. He also mentions in his examples the moment at which the rate is being asked.

5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution.

Once again Stewart stays away from identifying variables and constants and simply calls everything a quantity. He does not specify what "if necessary" entails. The idea is to avoid using the product or quotient rule, taking multiple derivatives and performing substitutions after differentiation. In every worked out problem, he chooses the equation based on the rates written down in step 4.

6. Use the chain rule to differentiate both sides with respect to time.

Here he does not remind the student that all symbols, by step 2, are functions of time. He does not use the definition of the chain rule; he simply differentiated implicitly in his problems.

7. Substitute the given information into the resulting equation and solve for the unknown rate.

In several of his problems all information required but not given in the problem must be found using the original geometric relationships. He also does not ask the student to algebraically isolate the unknown rate prior to substituting. This once again avoids problems where the unknown rate may be a function of a variable which is not a function of time.

3.3 Current Research on Related Rates Problems

Students' difficulties with the related rates problems (and other problems in mathematics more generally) are commonly attributed to their tendency to apply procedures without understanding the reasons behind them. Researchers have therefore been trying to see if teaching focused on a conceptual understanding of the problems would help. For example Paul White and Michael Mitchelmore (1996) designed a 24-hour concept-based "supplemental instruction" sessions for first year Calculus students: students were given 4 hours of additional instruction for a six week period. During these supplemental sessions, students investigated rates of change using graphs of physical situations and examining secants and tangent lines. This was an attempt to improve students' ability to deal with the abstract concept of rate of change. Students were then tested, using applied word problems that involved rate of change. In the literature review, the study cites several papers demonstrating that students' difficulty with rates of change in word problems is well documented.

The study found a marked improvement in the student's ability to symbolize rates of change in non-complex situations. Unfortunately, the instruction did not improve students' ability to symbolize rates of change in complex situations. This inability to deal with rates of change was attributed to students' inability to treat variables as quantities to be related, but rather as symbols to be manipulated, or to a "manipulation focus" in the authors' terms. The authors attributed it to the students' "abstract-apart" concept of variable, rather than an "abstract-general" concept of variable. If a student has an abstract-apart concept of variable, he/she can only perform procedural tasks involving variables. In order to link a variable to conceptual knowledge about a complex system, a student must have an abstract-general concept of variable:

> *"Students showing the manipulation focus have a concept of variable that is limited to algebraic symbols; they have learned to operate with symbols without any regard to their possible contextual meaning. We call such concepts 'abstract-apart'…. [A]bstract-apart ideas are formed without any true abstraction (and, for that matter, without any generalization).*
>
> *We call concepts that are formed by a generalizing --> synthesizing --> abstracting sequence … 'abstract-general'. Only abstract-general ideas can be linked in conceptual knowledge….*
>
> *When symbols represent abstract-apart concepts, as in the manipulation focus, they are not related to any mathematical objects at a lower level of abstraction that could give them meaning. Relationships between the symbols can therefore only be superficial, that is, they can only be based on what the symbols look like; and learned rules can only be applied on the basis of the visible symbols available. An abstract-apart concept might be adequate to deal with routine symbolic procedures, but the limitations of such purely procedural knowledge … become apparent when the symbols have a specific contextual meaning. In particular, symbolizing a rate of change in complex problem situations requires the existence of abstract-general concepts because the symbols used to represent general variables and derivatives have to be related to the specific variables and rate of change that occur in that situation (necessarily at a lower level of abstraction). Modeling a given situation using algebra may represent an even higher level of abstraction than the several abstract-general concepts that may be invoked. (White & Mitchelmore, 1996)*

Tami Martin's 2000 study (2000) was another attempt at understanding students' difficulties in solving geometric related rates problems. The author of the study divided the solution process for related rates problems into seven steps and classified each step as either procedural or conceptual:

1. Sketch the situation and label the sketch with variables or constants. *Conceptual*
2. Summarize the problem statement by defining the variables and rates involved in the problem (words to symbols translation) and identifying the given and requested information. *Conceptual*
3. Identify the relevant geometric equations. *Procedural*/*Conceptual*
4. Implicitly differentiate the geometric equation to transform a statement relating measurements to a statement relating rates. *Procedural*
5. Substitute specific values of the variables into the related-rates equation and solve for the desired rate. *Procedural*
6. Interpret and report results. *Conceptual*

These steps are similar to those found in both Thomas' Calculus and Stewart's Calculus. Martin defines conceptual and procedural knowledge as follows:

- Procedural Knowledge: "The ability to note, select, and apply the appropriate concrete, numerical, or symbolic procedures required to solve a problem and verify and justify the correctness of these procedures".
- Conceptual Knowledge: "The ability to identify examples and non-examples of a concept; to use, connect, and interpret various conceptual representations; to know, apply, distinguish, and integrate facts, definitions and principles; and to interpret assumptions and relationships in a mathematical setting."

With particular reference to a student's concept of variable, steps that involve conceptual knowledge cannot be accomplished without an abstract-general concept of variable (White & Mitchelmore, 1996). Martin refers to this distinction between an abstract-apart and an abstract-general concept of variable, as highlighting a notable difference between conceptual and procedural knowledge.

The specific aims of Martin's study were to characterize student's ability to solve geometric related rates problems and to identify which type of understanding leads to students' success and/or failure. The research questions were, in particular:

1. How do university students perform on related rates problems?
2. How do students perform on the conceptual and the procedural steps of the solution method?
3. How is the performance on each step related to their performance on others and to general success in related rates problems?

The study used two written instruments to collect data:

1. Three open-ended problems to judge overall performance on related rates problems
2. Problems targeting one specific step of the overall solution process.

Students performed the worst on questions relating to steps one and two of the solution process. This represents students' difficulty translating from a verbal to a symbolic representation. For Martin, this was an indication that students have an abstract-apart notion of variable. Students had particular difficulty representing the rate in the notation of derivatives. The use of the second instrument also demonstrated that students have difficulty distinguishing between quantities that are variable and those that are constant in time. In particular, they had trouble identifying the rate as being constant.

Question relating to step three (writing an equation representing geometric relations) had the highest success rate. One reason for this could be that it is easy for students to memorize and recall geometric formulae. Question relating to steps four and five had just over 50% success rates. This shows that even though the students have had a week of practice with implicit differentiation, their abilities were still relatively weak.

Students performed quite poorly on related rates problems as a whole. There was a greater correlation between procedural knowledge and success, rather than conceptual knowledge and success. This study also showed that students have trouble applying previously learnt geometric and algebraic knowledge to Calculus problems. Student also had much trouble translating prose to symbol and understanding symbolic representations. The study ends by concluding that there is evidence to support the fact that being able to summarize the problem statement increases student's overall performance.

As already mentioned above in the case of White & Mitchelmore's study (1996), researchers have experimented with teaching innovations focused on improving students' use of conceptual knowledge when solving related rate problems. Another example of such attempt is Engelke's research (2004). Engelke used a computer program to help students succeed in solving related rates problems. The computer based activities did improve students' notion of rate and specifically their ability to related variables in geometric related rate type situations. Again, however, her research led mainly to confirming or refining the diagnosis of sources of students' difficulties. For her, the primary culprit or "conceptual barrier" is covariational reasoning. Her research demonstrated that students' inability to relate variables that are functions of time, in a dynamical way, prevents them from succeeding during the initial phases of related rates problems. Covariational reasoning is defined as the ability to coordinate related changes between objects and variables inside a related rates problem (Engelke & CadwalladerOlsker, 2011); (Engelke, 2007).

3.4 Summary of Epistemological Analysis

Geometric related rates problems are a standard topic taught in a first course in Calculus. These problems involve geometric situations where the solver must find some unknown rate using geometric relationships and differentiation. The solution process can be broken down into 6 steps, with the first three focussing on a conceptual understanding of the problem. This conceptual portion of the problem poses much difficulty to students. This difficulty arises primarily from the inability to relate variables and constants abstractly with the help perhaps of a diagram of the situation. Textbook authors and researchers propose various hypotheses about the sources of the difficulty and different ways of helping the students to identify and relate the variables, constants, and rates, alongside the diagram of the situation during the conceptual steps.

4. Methodology

4.1 Genetic Decomposition of the Conceptual Steps in Solving Relates Rates Problems

The following genetic decomposition was constructed by solving a wide range of related rates problems and reflecting on the mental processes at play during the initial, conceptual stages of the solution process. This genetic decomposition outlines the conceptual scheme required to complete the conceptual stages of a given geometric related rates problem, prior to the step involving differentiation.

The mental constructions are listed below, followed by an explanation of each specific part of the scheme. In this preliminary genetic decomposition, the mental constructions have been generalized to either action or process, or process or object. For further reference in the thesis, the mental constructions identified in the genetic decomposition are labeled as GD1, GD2, GD3, and GD4.

- GD1. Process or Object level concept of the use of the distinction between variables and constants.
- GD2. Process or Object level concept of recognizing variables as functions of time.
- GD3. Action or Process concept of geometric relationships.
- GD4. Action or Process concept of geometric representations of related rates systems.

In many related rates problems there are letter representations of quantities that are constant. It is important that the students recognize that these variables are constant and that they are not functions of time. It has been shown that students have difficulty distinguishing between variables and constants (Martin, 2000). If they do not make this distinction they will not be able to properly differentiate any geometric equations constructed using these constants. This concept, GD1, must be at the process or object level since the student is not prompted in any way by the question to make this distinction. A student who is at an action level of the distinction between variables and constants would not be able to distinguish between variables and constants unless prompted to do so. Failure to make such a distinction would be fatal during the conceptual part of the solution process. Also, a student with a pre-object level concept of this distinction will not be able to appropriately model the situation and manipulate the labelled quantities correctly.

Mental construction GD2 of the genetic decomposition is justified by the need for students to recognize quantities that are functions of time. Once again the students are not prompted in any way to identify certain variables as functions of time. Knowing which variables are functions of time and which are not is necessary during the conceptual steps of the problem, as setting up relevant equations and relating variables to their respective time derivatives is crucial to success. Since this concept must be abstract, and students are not directly prompted in any way, this mental construction must be at the process or object level. A student at the action level must be directly prompted to identify variables that are functions of time. Students must also perform transformations on the variables during later stages of the solution process. It is therefore crucial that the concept of certain labeled quantities being functions of time be at least at the process level.

In the related rates problems used in this research a diagram is always provided. The student is therefore prompted by the diagram as to which geometric relationships to use. The student may also require knowledge of geometric relationship as an input-output process. This concept must therefore exist at an action or process level. Mental construction GD3 is this action or process level concept of recognizing and applying the geometric relationships. It is assumed that the student has knowledge of the basic trigonometry and has access to any volume of area formula that is needed.

In order to solve related rates problems students must recognize the geometric nature of the problem. When asked to find a certain rate, students must appropriately label the given diagram and understand that geometric relationships between these variables exist. The student must use this information to construct a geometric equation representing the relationships. Since the students should be familiar with basic area, volume and Pythagorean formulae, and is prompted by a diagram, this concept is at an action or process level. The actual geometric representation does not need to be abstractly manipulated in order for the student to correctly relate the variables and constants.

4.2 The ACE Cycle

The framework of the ACE teaching cycle has been used in designing activities for the early stages of learning geometric related rates by students. The sequence of activities has been divided into three parts. . First, the students were asked to perform three guided activities. Next, these activities were discussed in a classroom setting where two classroom exercises were done with the instructor's aid. Finally, students were given a problem to solve at home.

The activities were sequenced by order of difficulty. The difficulty of the problems was assessed based on the number of variables that are functions of time, the presence of constants, the number of variables and constants that are to be identified and labeled on the diagrams, and the number of geometric relationships. All activities designed to create a learning situation where students' existing mental constructions would disequilibrate and therefore students would experience the need to re-forge them into constructions hopefully closer to those outlined in the genetic decomposition. The activities were to be done in small groups with little intervention by the instructor.

Since the ACE cycle research focuses on the initial conceptual steps of solving related rates problems, prior to the differentiation step, the designed activities do not engage students in solving complete related rates problems. No notation or concepts that use the notion of rates or derivative will be used. The activities focus on the distinction between variable and constants, the notion of variables being functions of time and the problems geometric representation. This is intended to help students to appropriately set up geometric relationships using function notation, a diagram, and geometric relations, when, later, they are faced with related rates problem. The students were asked to solve full geometric related rates problems during the interview process.

The two classroom exercises are meant to be done with more inter-group conversation with instructor intervention and feedback. The exercises are similar to the activities but are more complex and are designed to reinforce the same mental constructions. These exercises are of the highest level of difficulty with multiple geometric relationships, variables, and constants.

The take home assignment is designed to be a simple synopsis of the salient points presented in the classroom work. This problem is less exploratory than the activities, and less complex than the classroom discussion problems. It is designed as a way for students to put the ideas together and to verbally summarize their understanding of these types of problems.

The ACE activities focused on the use of functional notation as well as a dynamical representation of geometric relationships, as well as the composition of functions in an attempt to help students build the mental constructions outlined in the genetic decomposition. This dynamical method is a simple frame by frame view using tables and in some cases the addition of multiple diagrams to demonstrate the system at subsequent times. The activities also asked students to describe different aspects of the given situation verbally, in an attempt to reveal their understanding of the situation. The activities will be described in detail in sections 4.2.1.1-5. They are also provided in the form in which they were presented to students in Appendix 1.A.

In order to focus on the difference between variables and constants, as well as discern which variables represented functions of time, students were asked to label diagrams using functional notation. Variables that represented constants were labelled using only letters, for example "r" for the constant radius of a cylinder (activity 3 &4), while variables that represented functions used functional notation, for example "h(t)" for the height of water in a cylinder (Activities 3 & 4). The use of these different notations is intended to disequilibrate the student's use of single letters indiscriminately as both constants and functions. In addition, students were asked to complete tables with time as an independent variable and variables and geometric equations as the dependent columns. For example, given a specific time in seconds they must calculate the area of an expanding circle (Activity 1). This mental construction follows GD1 & GD2 of the genetic decomposition.

The activities also ask students to construct geometric relations of the situation. This includes labeling a diagram of the situation and completing a geometric equation. The diagrams of the situation are given and the equations are explicitly asked of the student. This prompting of the student is consistent with mental constructions GD3 and GD4. Each activity includes a diagram of the situation and asks the student to represent the geometric equation algebraically.

The activities are very similar and progress in order of difficulty. The difficulty is gauged by the number of variables, constants and the number of geometric equations required describing the situation. Activity 5 differs slightly in that it is more concise and prompts the student much less. This is meant to promote discussion between the instructor and the students. The following table outlines the number of variables that are function of time, constants, and geometric relationships present in each activity.

Table 1 - Relative Difficulty of ACE Tasks Based on the # of Variable, Constants, and Equations

Activity	Variables	Constants	Geometric Equations
1	2	0	1
2	3	0	1
3	4	1	2
4	5	1	3
5	2	1	1

We now proceed to describing each activity in detail.

4.2.1. Activity 1

The first activity is an expanding circle problem. Here the radius of a circle is a function of time and is increasing at a constant rate. This activity represents the simplest form of geometric related rate problem with one equation and no constants. Here two functions of time, area and radius, are related using one geometric equation. The activity focuses on almost all the constructions proposed by the genetic decomposition. The problem focuses on the concept of variable as a function of time by allowing the students to construct a table relating the radius, the area and time. The activity also asks students to represent the area as a function of time using composition of functions. This allows students to develop the notion of representing related rates using a geometric equation. This activity, as presented to the students, can be found in Appendix 1.A.1.

4.2.2 Activity 2

The second activity is an expanding right triangle. Here the students must manage three variables, denoted $x(t)$, $y(t)$ and $z(t)$, the base, the height and the hypotenuse of the triangle, respectively. Here the students are given the rules of the functions $x(t)$ and $y(t)$. They must use these functions to fill in a table, focusing not only on the values of x and y over time, but also using the Pythagoras theorem to calculate the hypotenuse. The students are also asked to redraw the triangle at different times and are asked to describe the relationship between the resulting triangles. This is intended to foster the concept that the three sides are functions of time, and are related by the Pythagorean equation.

4.2.3 Activity 3

The third activity is a volume transfer problem. Here liquid is pouring from one tank to another. The radii of the tanks are constant and are represented by the letter r. The other three variables, the heights of the water in the tanks and the volume of the water in the tanks, are represented using functional notation. This is meant to introduce the concept that certain quantities do not vary with time. Students are asked once again to fill in a table with rows representing different moments in time. They are also asked to label the diagram and complete geometric relationships algebraically.

4.2.4 Activity 4 (Class Discussion 1)

This problem is similar to Activity 3 except that the bottom tank is replaced with a conical shape. This adds an additional geometric equation to the problem. Here the students must understand that the height and radius of the water in the bottom tank are related via similar triangles. This problem also includes several additional descriptive questions where students are asked to describe relationships between variables. This is intended to promote discussion with the instructor.

4.2.5 Activity 5 (Class Discussion 2)

The final activity involves a car passing in front of a camera. Here the geometric relationship is a cosine relation between the angle the car makes with the camera and a perpendicular line to the cars path and the position of the lens. This activity is shorter than the others in terms of the number of specific questions, and is intended to provide a closing discussion to the ACE cycle, time permitting. The questions are also more general and open ended in order to promote discussion. In this problem there is no time relation shown via a table or the use of multiple diagrams. Instead, the problem simply asks for the diagram to be labelled using functions of time for variables and simple letters for constants. The student is also asked to provide the geometric relationship. The final part of the question asks the student to describe a particular aspect of the situation, which is to note that the angle goes to zero when the car is directly in front of the lens. This is a situational way of having the student explain what is happening in the problem. This exercise can be found in Appendix 1.A.6.

4.3 Take Home Exercise

The take home assignment is meant to reinforce the mental constructions fostered during the ACE cycle. The activities targeted these mental constructions through the use of functional notation for variables that were functions of time, and the use of tables and multiple diagrams to demonstrate geometric relationships in a dynamical method. The take home assignment reinforces these ideas by asking students to expand a diagram of a ball falling in front of a lamppost by adding future positions of the ball and its shadow. As in the ACE activities, students are asked to label the diagram with functional notation. They are not given the specific label but asked to pick their own letters. They are asked to use this functional notation to describe the situation geometrically. Finally they are asked to describe the situation in their own words. This allows them to not only reflect on the situation, but to convey their conceptual understanding of it. The full problem, as given, can be found in Appendix 1.A.6.

4.4 The Interview

For the purpose of the interview, two standard related rates problems were used. They were slightly altered problems from Thomas' Calculus (Thomas, Weir, & Hass, 2010)**.** The problems were altered to match the formatting of the ACE cycle questions. The problems were also given an additional section asking the students to describe the situation in their own words. This was added in order to explore the student's conceptual understanding of the problem.

The first problem was a problem involving a ladder sliding down a wall. Here the length of the ladder is kept constant as the ladder slides down a wall. The distance between the wall and the base of the ladder is constantly increasing while the distance between the top of the ladder and ground is decreasing. The student is given the rate by which the ladder is sliding away from the wall and is asked to use that information to find the rate at which the ladder is sliding down the wall, the rate at which the area produced by the ladder, the wall and the ground is changing, and the rate at which the angle produced by the ladder and the ground is changing. The student must use geometric relationships between the length of the ladder and the distances between the ladder and the ground, and the ladder and the wall, in order to differentiate and solve for the required rates. An exact copy of this problem can be found in the Appendix 1.B.1.

The second problem was one similar to activities 3 and 4 of the ACE cycle. Here a coffee percolator, modeled by a cone, pours coffee at a constant rate into a coffee pot, modelled as a cylinder. The student is first asked to describe in his own words what is happening in the problem. The student must then determine how fast the height of the coffee is rising in the coffee pot, given the rate at which the volume is increasing. This question can be solved without Calculus since the height is increasing at a constant rate, and therefore the student can simply find the average rate of increase. The last part of the second problem asks the student to find the rate at which the height of the coffee is lowering inside the conical filter at a particular point. This question is very difficult as the radius of the filter is not given and therefore the solution is a function of the radius. This question is intended to be of a high level of difficulty being the final question of the interview. Its similarity with ACE activities 3 and 4 should allow the student to at least explore different solution possibilities. For an exact copy of question 2 see Appendix 1.B.2.

4.5 Research Procedures

Students were recruited from the MATH 203 Calculus course during the first few weeks of the semester. Students were told that they would be participating in a study that focussed on related rates, a topic they would learn during week 8 of their 13-week class schedule. Students were offered a free tutoring session as well as a small increase in their grade as compensation. Four students volunteered for the study.

The students were divided into two groups of two students. Each group participated in the ACE cycle during a one-hour session. This session took place before the students were presented with the related rates topic during their regular class hours. This took place between week 6 and 7 of their regular class schedule. The students solved the problems in groups of two, each group working individually. The problems were printed on large paper (0.61m x 1.22m), and were to be worked on by the students on a round table over a one hour session. The students had access to writing tools, calculators, rulers, and wrote their answers directly on the large sheets. An instructor was present and discussion between the instructor and the students took place during the class discussion portion of the ACE cycle. The instructor also clarified any issues that the students had with the activities. The session was audio recorded and transcribed, and their written work on the large paper was scanned. The students were each given the take home exercise at the end of the one-hour session. The students then returned to their regular class schedule to learn the related rates portion of their course during week 8 of their Calculus course.

They were told to complete the take-home assignments individually over the following weeks. They were to submit their take home exercises when they returned for an interview during weeks 9 or 10 of their regular class schedule. The interviews were audio recorded and transcribed, and the students' work scanned. No follow up was done after the interview process.

5. Results

5.1 Subjects' Response to the Research

Four students volunteered for the study. For purposes of anonymity they will be called students S1, S2, S3 and S4. They formed groups of two and performed the ACE cycle at different times during weeks 5 and 6 of the semester. The two groups will be called group G1 and group G2, with G1 containing students S1 and S2, and G2 containing students S3 and S4. Each student then returned during week 9 of the semester (3 weeks after completing the ACE cycle) to perform the interview questions. The full transcripts of the interviews and the ACE cycles, as well as the written work from both the ACE cycle and the interviews can be found in Appendix 2 and Appendix 3 respectively.

Relevant excerpts from the study are quoted and discussed in the proceeding sections. These excerpts are chosen based on their ability to demonstrate the need for changes in both the genetic decomposition and in the ACE cycle. Interpretation in terms of the framework is discussed later in this chapter, while discussion on changes to the decomposition and the ACE cycle will be discussed in Chapter 6.

Activity 5 was only completed by one group and was extremely rushed by the other. It was therefore omitted from the analysis.

5.1.1 Student's Response to ACE Cycle.

5.1.1.2 Activity 1

Both groups completed the first activity in around 5 minutes. During the activity it was evident that, for each group of two students, one of the students' conceptions of function was not as strong as the other. During Question 1.5 (Find the function of the area in terms of the radius $r(t)$: $A\big(r(t)\big) = \cdots$) and Question 1.6 (Given the function given above, $r(t) = 2 + t$, find a function for the area in terms of t: $A(t) = \cdots$), students S3 and S2 had trouble performing the simple composition of functions. Student S4 performs the composition but refered to it as an action that he learned last semester.

> *S4: We did this last semester, functions. Are you supposed to plug that …and then…*

Student S2 had trouble even with verbalizing that the radius is a function of time even though this information is explicitly given in the question.

> *I: No. Here radius is a function of what?*
>
> *S2: Radius times pi.*

Students S1 and S2 had a good understanding of the fact that labeling the radius as $r(t)$ signifies that the radius is a function of time.

> *S1: By the function of the radius which is affected by time. So we are at the left an area as a function of time, and the formula for our area is the same but now our radius is a function of time. We substitute our r with r(t). Does that work?*
>
> *I: You both agree with that?*
>
> *S2: Yeah*
>
> *S1: More or less*

On the other hand students S3 and S4 show no evidence that they see the functional notation as anything more that another way of describing the radius.

5.1.1.2 Activity 2

Both groups completed the first six parts of Activity 2 (Right-angled triangle with sides *x*, *y* and *z*) quite easily. Both groups seems to be comfortable to the given functions of time for *x* and *y*. The group of S1 and S2 actually created the function before arriving at Question 2.6 (Verify that the functions $x(t) = 11 - t$ and $y(t) = t$ output the length values of x and y for a specific triangle t) where they were given the functions and asked to verify them.

> *S1: So x(t) is eleven minus t, and y(t) equals t, so basically y is t. (During Question 2.6)*
>
> *S1: OK, that's like exactly what we just did. Construct a function, 11 minus t squared, plus t squared. (During Question 2.7)*

Student S2 again demonstrates strength in completing the composition of functions.

> *S2: It would just be 11, right, using the Pythagorean formula, 11 minus t squared, plus t squared.*
>
> *S2: Square root that.*
>
> *S1: Yeah, we don't need to simplify that, it would just remain the same.*
>
> *I: Yeah, you don't need to simplify*
>
> *S1: And it's basically, it works for this specific triangle. Or rather for these.*

Student S1 agrees with the composition and understands that the new function can be used for any of the triangles.

In Question 2.8 (Draw a triangle labeling the sides $x(t)$, y(t), and $z(t)$, with t now being time starting at $t = 1\ second$), the students encounter their first misconception. Both groups decide that they must calculate the value of the hypotenuse when $t = 1$. This suggests that students understand $\{1\}$ as the entire domain and not just the lower bound. Here, even after being prompted by the instructor, S1 still insists on solving for *t* equals 1.

> *I: It's the same x and y.*
>
> *S1: So if we are going to substitute right away, z would equal... square root of 11 minus 1 squared plus 1 squared.*
>
> *S1: That would simply equal one.*
>
> *I: Well, I asked you to label them x of t, y of t and z of t.*

S1: x of t is eleven minus one, y of t equals one. Z of t is root of 11 minus 1 squared minus 1 squared which brings us back to z of t equals root of 101.

Also, in the other group, student S3 refers to time as being exactly one and student S4 agrees to plug 1 for *t* into the function.

S3: These are some nice triangles [draws triangle]. So we label it using the functions? So we've established our z(t) function. But t is 1 right, so it's the square root of 101.

S4: So we just put in the numbers. So it's the same as that [Question 2.3]

S3: But I guess we have to figure it out.

S4: Yeah, use the functions, so y(1).

In the last question, Question 2.9 (What can you say about this triangle [obtained in Question 2.8] in relation to triangles #1, #2, and #3 [values of the sides for *t* equal 1, 2, 3 and 4] given in the table [in Question 2.4]), both groups have trouble describing what is happening. They understand that the lengths of the sides are changing with time but they cannot properly describe the situation. They describe each of the sides of the triangle as either decreasing or increasing, instead of relating the sides using the Pythagoras equation.

S3: As t increases, y increases, x decreases, and z decreases.

S4: As time increases.

S1: I'd say that triangles 1,2,3,4 are the same triangle growing through time at a rate of one unit per second. Or, uh, shrinking. Do you agree?

S2: Yeah, looks like it.

Only student S2 questions, after some time, the fact that the hypotenuse does not simply decrease as time increases.

S2: Actually, I don't know if they are shrinking or not? Like if the hypotenuse is shrinking... I don't know if the area is actually shrinking.

Only then does he describe each side has having a different rate of change. He still does not mention that the lengths are related through the Pythagoras relation.

5.1.1.3 Activity 3

In Activity 3 (about a system of two cylindrical connected tanks), as in the previous activities, the students have no trouble labeling the diagram using function notation. Unlike in Activity 2, they do grasp the dynamical nature of the problem. Both groups recognize and use the fact that the total volume in both tanks is always equal to 9.

> *S2: Just the first term is the same. We don't know the radius, so we can't actually calculate the actual volume. But they always have to be equal to 9, because there is no water escaping.*
>
> *S1: so we should have a function here instead of a number right.*
>
> *S1: This is exactly what I was thinking.*

Unfortunately the students began to show confusion when presented with the several composition of functions questions. The discussion on the notation and substitution took over 10 minutes with both groups and neither group seemed to understand that variables were function for time. They saw the exercise as an algebra activity.

> *I: So what's the difference in the questions? What's the difference between this notation and that notation? There's no difference?*
>
> *S1: Well I mean we combine them together.*
>
> *…*
>
> *S1: Um, perfectly honest, it's your notation here of h(t) and H(t), told me right away that it's a combination of two functions, and I have these two functions right here, as h(t) being 9 minus t and H(t) being t.*
>
> *I: So this notation and that notation is the same thing, this one just indicates some sort of composition, and this one does not.*

S1: Pretty much

…

I: So to write it using function notation we write it as h(H(t))

S3: Why? How does that make sense?

I: Well because this is just saying that there is a relationship between the two of them.

S4: You could plug in a number for the h.

I: This is saying that the big H is specifically a function of the little h. So when H is three, we have this. It's as if we had f(x). I see it is difficult for you to assimilate this as a whole.

S4: I always had a problem with that, seeing too many alphabets.

S3: I enjoy notation… I thought there is a lot of interesting notation

The students were mostly uncomfortable with the function notation and had trouble with the algebra and the purpose of the questions.

5.1.1.4 Activity 4

In Activity 4 (A system of two tanks one pouring water into the other, with the top one being cylindrical and the bottom one – conical) the students were first asked to describe the geometry of the cone using the assumption that the radius is twice the height. Both groups spent a lot of time discussing this relation with the instructor. The discussion focused mostly on what it means to "describe the relationship geometrically". The students were split on their understanding of how the overall relation between the height and the radius holds true for any water level. S1, for example, could not understand the relationship, and, even after much discussion, saw it simply as a substitution of the constants for the total cone, $2r = h$ with the function notation used for the water, $2r(t) = h(t)$.

S1: Yeah, but here it's the radius of the water, which is at different levels at each point, so like that relationship doesn't necessarily help us out. If we knew the height of the water it would.

I: Here we have the height and the radius and the height is twice the radius. How do we write that? Then the water that's at some point here, the radius is r(t) and the height is h(t). Makes a triangle. Is this angle the same?

S2: The proportions stay the same.

S1: So the relationship is pretty much what you just wrote.

I: If your two triangles have the same angles then all the triangles are in the same proportion.

S1: So it's just 2r(t)=h(t)

Student S3 responds almost identically to S1. Eventually

S3: So if we have our h there and our r there we can just substitute h(t) for h and r(t) for r (in the volume equation).

…

S3: This doesn't explain this relationship geometrically. This is our function.

Students S4 and S2, on the other hand, demonstrate the relationship using a diagram and attempt to explain it to their partners but are unsuccessful.

As in Activity 3, a long discussion ensues about the composition of function. The students have to be guided through the algebra and are therefore not focusing on the variables being functions of time. At the end of the discussion, S1 even admits that he was unfamiliar with the way in which functions were being used in this activity.

S1: I don't think that this concept of combining functions was ever presented to us like that.

In the final part of the activity, S1 attempts to describe the volume of the cylindrical tank in terms of $r(t)$, even though the radius is constant.

I: Without including volume. What's the radius of this tank, of the water in the tank?

S1: r(t)

I: Does it ever change?

S2: No the height will change but the radius will never change.

I: So is it possible to relate them?

S2: No

I: Why could you relate them here then? Here you have the height at different levels…. Why not?

S1: You would need like a third dimension.

He later admits that he has trouble identifying which quantities are variable and which are constant.

5.1.1.5 Take Home Assignment

The *Take Home assignment* was about a ball being dropped from a certain height above the ground and a certain distance from a lamp post. All students could appropriately draw the ball at two later times. Students also properly traced the shadow as getting closer to the lamppost. The students' responses began to diverge when asked to label the diagram using function notation where applicable. S1 labeled the distance to the shadow as a function of time yet the height of the ball as simply "H". S3 did the opposite, labeling the height of the ball as a function of time, but not the distance of the shadow to the lamp post. The other two students, S2 and S4 did not use function notation appropriately at all. Student S2 went as far as label the distance between the ball and the lamp post, which is constant, as $f(a)$, and the initial distance between the shadow and the lamp post, also constant, as $f(d)$. Only students S1 and S3 used the Pythagoras equation to relate the variables geometrically. Both only found one relationship, with S3 solving for the height of the ball and S1 for the distance between the ball and the shadow. None of the student described the relationship as being related through Pythagoras. They describe the ball as "moving towards the lamp post" or just state that they are "related".

5.1.2 Results from the Interview.

The students were given two related rates problems to solve. The problems were modifications of the typical end-of-the-chapter problems from Thomas (Thomas, Weir, & Hass, 2010). In describing the results from the interview, we will focus on the early conceptual stages of the students' solution processes. With the students we went through the entire solution process.

5.1.2.1 Interview Question 1

Question 1 was about a long ladder sliding down a wall. All students described the situation without the use of the Pythagorean Theorem. They described the ladder as sliding down and away from the wall in terms of being "related" or "proportional". Students S1, S2 and S4 then all answered the second question by immediately stating that the ladder is sliding away from the wall at the same speed as it is falling down the wall.

> *S3: So this is eight metres. [labels distance from wall to ladder] And it's moving at 3m/s away, and so down as well. So explain what is happening in your own words… As the ladder slides away from the wall, it also decreases proportionally or the height of the ladder decreases proportionally to the length form the wall. As the ladder drops the base extends further away from the wall.*

In answering Question 1b (How fast is the top of the ladder moving down the wall at this point?), S3 said:

> *S3: How fast is the ladder moving down the wall…I would say 3m/s.*
>
> *I: Why would you say that?*
>
> *S3: Because the bottom is going to be moving at the same speed as the top.*
>
> *I: Why?*

S3: Well short of the physics involved with gravity and such… because it's all one unit, so I would assume that it would be falling and moving at the same pace, at 3m/s…. Um, yeah…. Is that incorrect?

S4 does not set up a geometric relationship at all. Instead he simply solves for the height of the top of the ladder (labeled *y*). He has to be prompted heavily to set up the geometric equation.

I: So there is some sort of relationship.

S4: Right.

I: What is it? How did you find y? This (x) was eight. How did you get it?

S4: I used the Pythagorean Theorem.

I: Ok, so. Can you relate x and y using that? Just if it wasn't eight. Just some relationship between x and y.

…

I: Rewrite it for any y. This is true always

S4: Right, this is true always

As for labeling, none of the student labeled the diagram until they were prompted by the instructor to do so or until they reached the third part of the question (At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment). Only S4 used function notation, and only to describe the vertical and horizontal distances. None of the students spontaneously set up geometric relationships amongst the variables. Students also did not attempt to draw the diagram at various future positions.

5.1.2.1 Interview Question 2

The interview Question 2 was about a coffee percolator. None of the students were able to attempt solving the problem, because of lack of time. Students could only say that the level in the coffee pot was rising but could not describe the change in terms of geometric relationships, with the exception of S3 who described the two heights as "proportionate". The students did not label the diagram in any appropriate manner, with the exception of S1 labeling the radius of the water in the filter as "r" and S3 labeling the total height of the filter with an "h". None of the students used any function notation or attempted to draw the situation at subsequent times.

5.2 Interpretation of the Results in Terms of the APOS Framework

5.2.1 The use of Function Notation in the ACE Cycle, Mental Constructions GD1 and GD2.

Originally, the use of function notation was intended to target mental constructions GD1 and GD2. This did not appear to be the case as the students felt uncomfortable with the notation and had trouble using it, especially in the context of composition of functions. Students spent most of the time on Activities 3 and 4 discussing the algebra and notation involved with the composite functions. Students' poor algebra skills and unfamiliarity with the notation did not allow them to use it fluently.

In Activity 4, S2 tried to describe the radius of the cylindrical tank as $r(t)$, later admitting to having trouble identifying which quantities were variables. This indicates that for him, the use of function notation in the previous activities did not promote much reflective abstraction on mental construction GD1. Rather, it would seem that he was merely using function notation because he was prompted to do so throughout the activities.

With the exception of S1 during Activity 4, students demonstrated that they understood which values were constant and which were variable when prompted during the activities. They also had little difficulty filling out the tables and describing the motion involved in the problems, which reinforces that once prompted, they could distinguish between the two. This would indicate that their ability to distinguish between variable and constant was only at the action or pre-process level.

The ACE cycle was unsuccessful in disequilibrating the students' ability to distinguish between variables and constants, or to recognize variables as functions of time, or, even if students experienced some mental conflicts, these did not provoke them to revise their conceptions and construct new ones. This was apparent by their inability to demonstrate these concepts without being prompted during the Take Home exercise and the Interview. Having mental constructions GD1 and GD2 at the action level made appropriate labeling of the diagrams impossible unless prompted by the instructor.

5.2.2 Dynamical Representations, Mental Constructions GD3 and GD4.

The use of tables and diagrams in the ACE cycle were intended to target mental constructions GD3 and GD4. This dynamical method did not disequilibrate the students' rather qualitative conceptions of relationships among variables and they did not feel the necessity of more precise geometric models. Although students were able to complete the tables and draw the diagrams, they were incapable of describing the situations using, for example, the Pythagoras relation. The students could only describe the change over time loosely as increasing, decreasing, or "proportionate". Also, the students failed to use any similar techniques during the interview process.

6. Discussion, Conclusion, and Recommendations

6.1 Genetic Decomposition and ACE Cycle

The results from the ACE cycle and the interview process are used to propose changes to the genetic decomposition and the ACE cycle. These changes can then be used to construct a second iteration of this study.

6.1.1 Changes to the Genetic Decomposition

The results suggest three changes to the genetic decomposition. The first two are refinements to mental constructions GD1 and GD2. The third is the fusion of GD3 and GD4 to form a new mental construction focusing on the geometry of related rates problems. The changes are described and justified bellow.

6.1.1.1 Mental Constructions GD1 and GD2

It was clear from the results that the students had action level concepts of GD1 and GD2. The ACE cycle failed to disequilibrate these concepts and as a result students could not complete the labeling step of the solution process during the interview. Changes to the ACE cycle will be suggested in the following section in order to try and better construct GD1 and GD2.

It was also apparent from the results that the use of function notation was not helpful for the students. The concept of variable as function of time will therefore be absorbed by GD3 and GD4 and will be part of the understanding of geometric representations of related rates problems. This change will be explained in the following section. This change will help to make changes to the ACE cycle that will focus more on the geometry of the problem.

6.1.1.3 Mental Constructions GD3 and GD4 Focus on Object Level Concept of Specific Geometric Relationships

Students' unfamiliarity and difficulty in using the functional notation made it difficult for the ACE cycle to focus on their ability to relate the given variables geometrically. Activities 3 and 4, originally intended to focus a great deal on the cultivation of GD4, turned into long explanations and discussions about the composition of functions.

On the other hand, looking at the results from the first Interview question, it can be seen that students' misconceptions about certain geometric situations can greatly hinder their ability to understand the problem. All four students believed that the ladder would slide down the wall at the same speed as it was sliding along the ground. This would suggest that an action or process concept of geometric representations is insufficient. Students must have an object level conception of specific geometric relationships, as they must use construct transformations of these relationships. An inability to do so may be the main cause of the conceptual barrier they all encountered during the first ladder problem.

Students also had difficulty during the last question of Activity 2. When asked to relate the sides of a triangle, which were functions of time related via the Pythagorean Theorem, students described the situation as purely linear. This could be explained by the common and well-known "illusion of linearity" obstacle (de Bock, van Dooren, Janssens, & Verschaffel, 2007). The fact that this obstacle was revealed in the students is further evidence that students' knowledge of specific geometric relationships can contain misconceptions, or simply a lack of knowledge, that can be detrimental to their conceptual understanding of geometric related rates problems.

- GD2, GD3 and GD4 are combined to become: GD2* Object level concept of specific geometric relationships.

This new mental construction focuses on specific geometric situations. This allows students to preform necessary transformations on the geometric relationships, allowing them to construct a more accurate representation of specific dynamic relationships. For example, students would develop a well-defined object level understanding of the Pythagorean Theorem. This object level mental construction includes the use of appropriate variables on diagrams and the use of these variables to construct a geometric relationship.

This would also include appropriate labeling of diagrams, again specific to certain geometric situations. Students would know to use variable labels in a particular manner for a particular geometric situation. Students did not have much difficulty distinguishing between variable and constants. Instead they were having difficulty applying variables to diagrams.

This object level mental construction will also allow the students to use covariational reasoning, allowing abstract thought about the relationships between the variables and constants as they vary in time. This allows for the illumination of mental construction GD2.

6.1.1.4 Summary of Changes to the Preliminary Genetic Decomposition

Table 2 summarizes the recommended changes to be used in a second iteration of the APOS framework.

Table 2. Summary of Changes to the Genetic Decomposition

Preliminary Genetic Decomposition	**Genetic Decomposition after 1st Iteration**
GD1- Process or Object level concept of the distinction between variables and constants. GD2- Process or Object concept of variables as functions of time. GD3- Action or Process level concept of geometric relationships. - GD4- Action or Process concept of geometric representations of related rates systems.	GD1* - Process or Object level concept of the distinction between variable and constant. GD2* – Object level concept of specific geometric representations of related rates systems, which includes: -Appropriate variable labels -Appropriate geometric relationships using those variables. -Covariational reasoning amongst variables

6.1.2 Changes to the ACE Cycle

The results have shown that students are uncomfortable with function notation, the ACE cycle was too long, and students have misconceptions and lack of covariational reasoning for particular geometric systems. These results suggested changes to the genetic decomposition.

6.1.2.1 Length of ACE cycle

Both groups took much longer than expected to complete the ACE cycle. The ACE cycle was designed to take approximately one and a half hours. Instead, both groups went significantly over time and had to either rush or omit Activity 5. This was most likely due to the long conversations that took place during the discussions about function notation and the composition of function. Perhaps the removal of the function notation could adequately shorten the ACE cycle. That being said, future iterations of the ACE cycle should aim to shorten the number of questions given during the ACE cycle to better match the prescribed time.

6.1.2.2 Removal of Function Notation

It was originally thought that students would have difficulty distinguishing between variables that are functions of time and those that are constant. This led to the use of functional notation, such as $x(t)$, during the ACE cycle in order to try and help student recognize and relate variables that are functions of time. This was intended to also help foster their geometric understanding of related rates problems (GD4). This did not happen, as the students found the functional notation difficult to work with. Not only did the functional notation not help students discern between variables that are functions of time and those that are constant but it seemed that students were, for the most part, capable of distinguishing between the two when asked. The exception to this was student S1, who had trouble in Activity 4. The function notation did not seem to help him as he was using the function notation for the constant radius of the cylindrical tank. The function notation therefore only served as a distraction that led to a drawn out discussion during Activities 3 and 4 on the composition of functions and helping the students with the necessary algebra.

The use of function notation did not seem to disequilibrate students' concept of variable as being functions of time. With the exception of S4's work on Interview question 1, the interview diagrams were labeled and questions attempted without the use of function notation. The use of this notation therefore does not help the students to discern between variables and constants as much as it distracts them in their work during the ACE cycle. The use of functional notation should therefore be replaced with activities that ask students to discern between variables and constants in a different manner. Most likely a method with less complex notation should be used. For example, students could simply be asked to list variables used in the diagrams as either being constant or functions of time.

6.1.2.3 Focus on Particular Geometric Relationships.

The original design assumed that students were "familiar" with geometric relationships such as volume, area and the Pythagoras formulas. This assumption is not entirely correct as can be seen with the students' response to the study and the subsequent changes to the genetic decomposition. The ACE cycle should follow with changes to the types of geometric relationships studied as well as the types of descriptive questions being asked.

It can be seen from the interview, and from Activity 2, that the students were incapable of describing the geometric relationships between the sides of a right-angled triangle. This led to a change in GD3 and GD4 of the genetic decomposition. The new mental construction GD2* requires the students to build an object level mental construction of particular geometric relationships. This would require that the ACE cycle contain more questions targeted at promoting covariational reasoning and the elimination of particular geometric related rates problems. This can be done by focusing on particular geometric relationships throughout the ACE cycle. For example, in the original ACE cycle, students are only asked to describe a Pythagorean relationship during the second activity and not during the class discussion. This does not allow for discussion to take place about the misconceptions they have about the relationship between the variables. The entire ACE cycle should therefore focus on particular geometric relationships, and those relationships should be used in all three components of the ACE cycle.

Since the ACE cycle will focus on particular geometric relationships, it may not be necessary to remove the dynamical representations used in during the ACE cycle. Although the students did not use these methods during the interview, the purpose was to disequilibrate their specific constructions and not to necessarily teach them a technique. These methods of using multiple diagrams and tables may prove useful once there is a clear focus on specific geometric relationships during the entire ACE cycle.

6.1.2.4 Less Labeling Prompting during Class Discussion.

This change follows suit with the changes in the genetic decomposition. Originally, the use of function notation was used with the intention of disequilibrating students' concept of variables and constants and function of time. The students' response to the ACE cycle also showed that students had little trouble labeling the diagram and understanding which variable is a function of time and which is a constant. Therefore with the elimination of the function notation, the students do not need to be prompted to label specific quantities with function notation and constants with single letters. Instead, they can be asked to label the diagram how they chose, and during the class discussion, these choices can be discussed. This will also allow for the ACE cycle to show instances where students use inappropriate labeling techniques. This was not possible with the heavy prompting in the current ACE activities.

6.2 Related Rates Problems in General

In this study the APOS framework was used to construct a genetic decomposition of the schemas needed to understand the early conceptual staged of related rates problems. Although students responded well to the ACE cycle in terms of completing the problems, they were totally unable to solve the related rates problems. This could be explained by the fact that the genetic decomposition did not focus enough on the basic geometric nature of the problems. The simple question of “why does this situation require Calculus?” is overlooked as it is assumed the students understand that these problems cannot be solved, for the most part, without Calculus. This became clear during the interview process when the students were asked to find the speed at which the ladder slides down the wall. Their first intuition was to try and solve the problem empirically, by using “common sense”. This inability to understand that the variables are related geometrically most likely comes from a lack of covariational reasoning skills (Engelke, 2007). Students’ difficulties in modeling the situation in the problem are not rooted in the distinction between variables and constants or the recognition of a variable as a function of time, but in the relation of these with the given geometric situation.

The use of APOS theory to study the conceptual steps of related rates problems encounters several difficulties. The most prominent is that related rates problems require the application of mathematical concepts to real world situations. This application requires the student to model real world situations using geometric representations, variables, and the tools of Calculus. APOS theory has been used to construct schema for students’ concept of specific abstract mathematical concepts such as function (Dubinsky E. , 1991) and the chain rule (Clark, et al., 1997). APOS theory has proved to be useful in these cases as a web of concepts can be constructed, looking at student’s concept of function, variable, derivative, etc. Related rates problems, however, involve modeling and problem solving abilities. The action, process, object, mental constructions do not seem to be an appropriate framework for studying and fostering the development of those abilities.

A combination of frameworks may be helpful in understanding students' difficulties with related rates problems. A simultaneous look at three different aspects of this type of problems could be useful for future work. First, the role of related rates problems in the teaching of Calculus should be analysed. History has pushed this topic to the back of the differentiation section of Calculus textbooks. Their role has been reduced to an example of the application of differentiation. Using related rate problems in such a manner may be in itself a root of the difficulty that students' have in solving the problems. The sheer removal of problems that were at the origin of the invention of Calculus from their place as a tool for students' understanding the purpose and concepts behind basic differentiation to the rear end of the course may be inherent to students' difficulties. A second aspect is related with students' difficulty with the conceptual stages of related rates problem. This would focus on students' inability to understand the general nature of the problem. These conceptual stages rely on student use of covariational reasoning as well as an understanding that the problems are indeed Calculus problems and that the rates must be related accordingly. The third aspect is students' overall difficulties with related rates problems. This would include a look at obstacles students encounter during the entire solution method. These obstacles may include, but are not limited to, difficulties with algebra, functions, composition of functions, differentiation, the chain rule, modeling, geometry and general issues with problem solving.

References

Asiala, M., Brown, A., DeVries, D. J., Dubinsky, E., Matthews, D., & Thomas, K. (1996). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education. In J. Kaput, *Research in Collegiate Mathematics Education 2, CBMS Isssues in Mathematics Education 6* (pp. 1-32). Rhode Island: The American Mathematics Society.

Austin, B., Barry, D., & Berman, D. (2000). The lengthening shadow: the story of related rates. *The Mathematics Magazine, 73*(1), 3-12.

Boyer, C. B. (1949). *The history of the calculus and its conceptual development.* New York: Hafner.

Cauchy, A.-L. (1821/1968). *Cours d'Analyse de l'Ecole Royale Polytechnique.* Darmstadt: Wissenschaftliche Buchgesellschaft.

Clark, J., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John, D., et al. (1997). Construting a schema: The case of the chain rule. *The Journal of Mathematical Behavior, 16*(4), 345-364.

Connell, J. (1845). *The elements of the differential and integral calculus: with numerous examples and familiar illustrations, designed for the use of schools and private students.* London: Longman, Brown, Green and Longman.

de Bock, D., van Dooren, W., Janssens, D., & Verschaffel, L. (2007). *The illusion of linearity. From analysis to improvement.* New York: Springer.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall, *Advanced mathematical thinking* (pp. 95-126). Dortrecht: Kluwer.

Dubinsky, E., & Mcdonald, M. A. (2002). APOS: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research. In *The Teaching and Learning of Mathematics at University Level* (pp. 275-282).

Dubinsky, E., Breidenbach, D., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, 247-285.

Engelke, N. (2004). Related rates problems: Identifying conceptual barriers. *Proceedinsg fo teh 26th annual conference of the North-American chapter of the International Group for the Psychology of Mathematics Education* (pp. 455-462). Toronto, ON: OISE.

Engelke, N. (2007). A framework to describe the solution process for related rates problems in calculus. *Proceedinsg of the 29th annual conference of the North-American chapter of the International Group for the Psychology of Mathematics Education* (pp. 441-448). Stateline (Lake Tahoe), NV: University of Nevada, Reno.

Engelke, N., & CadwalladerOlsker, T. (2011). *Supplemental instruction and related rates problems.* Retrieved July 31, 2011, from SIGMAA/RUME (Special Interest Group of the Mathematical Assocaition of America - Research in Undergraduate Mathematics Education): http://www.http://sigmaa.maa.org/rume/crume2011/RUME2011_FinalSchedule_files/PreliminaryReportsShortPapers/Engelke_Proceedings.pdf

Kleine, M. (1977). *Calculus. An intuitive and physical approach. Second edition.* New York: John Wiley and Sons.

Martin, T. (2000). Calculus students' abilities to solve geometric related rates problems. *Mathematics Education Research Journal*, 74-91.

Piaget, J. (1972). *The principles of genetic epistemology.* London: Routledge & Kegan Paul.

Piaget, J. (1985). *The equilibration of cognitive structures: The central problem of intellectual development.* Chicago: University of Chicago Press.

Ritchie, W. (1837). *Principles of the differential and integral calculus.* London: John Taylor.

Schoenfeld, A. (2007). Method. In F. K. Lester Jr., *Second Handbook of Research on Mathematics Teaching and Learning: a Project of the National Council of Teachers of Mathematics, Volume 2* (pp. 69-108). Information Age Publishing.

Stewart, J. (2008). *Calculus: Early Transcendentals 6th Edition.* Toronto: Brooke / Cole.

Thomas, G. J., Weir, M. D., & Hass, J. (2010). *Thomas' Calculus, 12th Edition.* Don Mills, ONT: Pearson.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27*(1), 79-95.

Appendices

Appendix 1.A – ACE Cycle Questions

1.A.1 Activity 1 (Group work 1)

Question 1.1

Sketch a circle and label its radius 4m. Find the area of the circle.

Question 1.2

Sketch another circle and label the radius r(t).

Question 1. 3

Suppose the radius of the circle is expanding by the function r(t)=2+t in meters. Find the radius of the circle at t=1second. Find the area of the circle at this time.

Question 1.4

Fill the following table using the above information:

Time seconds (t)	Radius r(t) = 2+t	Area
1		
2		
3		
	5	
		81π

Question 1.5

Find a function of the area in terms of the radius r(t)

A(r(t))=

Question 1.6

Using the function given above r(t)=2+t, find a function for the area in terms of t

A(t)=

1.A.2 Activity 2 (Group Work 2)

Question 2.1

Draw a right angle triangle and label the sides x, y and z, with z being the hypotenuse.

Question 2.2

Suppose the value of x was 10 and the value of y was 1, find the length of the value z. Place this value in table labeling this triangle as triangle #1.

Question 2.3

Now draw another right angle triangle and again label the sides x, y, and z. Now suppose that this triangle, label it triangle #2, has x=9 and y=2. Find z for this triangle.

Question 2.4

Place the information you have found into the following table, then complete the table.

	X	Y	Z
#1	10	1	
#2	9	2	
#3	8	3	
#4			

Question 2.5

Can you tell me the value of z for the 8th triangle keeping with this pattern?

Question 2.6

Verify that the functions x(t)=11-t and y(t)=t output the length values of x and y for a specific triangle t.

Question 2.7

Using the functions x(t)=11-t and y(t)=t construct a function z(t).

<u>Question 2.8</u>

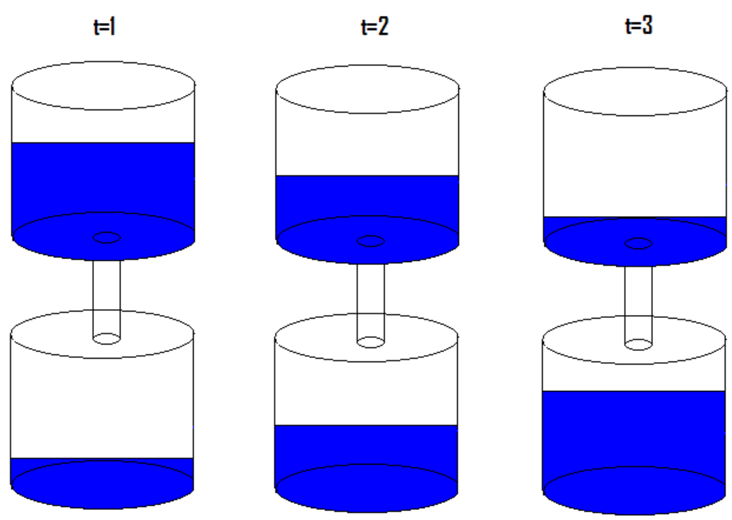
Draw a triangle labeling the sides x(t), y(t), and z(t), with t now being time starting at t=1 second.

Question 2.9

What can you say about this triangle in relation to triangles #1,#2,#3,and #4, from the table on the left?

1.A.3 Activity 3 (Group Work 3)

Note: Question 3.3 does not exist.



Question 3.1

Consider the diagram above:

A) Label the radius of the tanks r.

B) Label the height of the water in the top tanks h(t), and the height of the water in the bottom tanks H(t).

Question 3.2

Using the table below, filling the missing values with V_1 being the volume of water in the top tank, and V_2 being the water in the bottom tank.

Time	Volume in top Cylinder V_1	Height of water in top tank h(t)	Volume in bottom Cylinder V_2	Height of water In bottom tank H(t)
1 second	$8\pi r^2$		$1\pi r^2$	
2 second	$7\pi r^2$		$2\pi r^2$	
3 second	$6\pi r^2$			

Question 3.4

Create a function of time for the height of the water in the top tank and the bottom tank.

h(t)=

H(t)=

Question3.5

Create a function that relates the volume of the second tank to that of the first tank.

V_2=

Question 3.6

Create a function relating the height of the water in the first tank to it's volume.

$V_1(h(t))$=

Question 3.7

Do the same for the second tank

$V_2(H(t))$=

Question 3.8

Find a relationship between the height of the first tank and that of the second.

h((H(t))=

Question 3.9

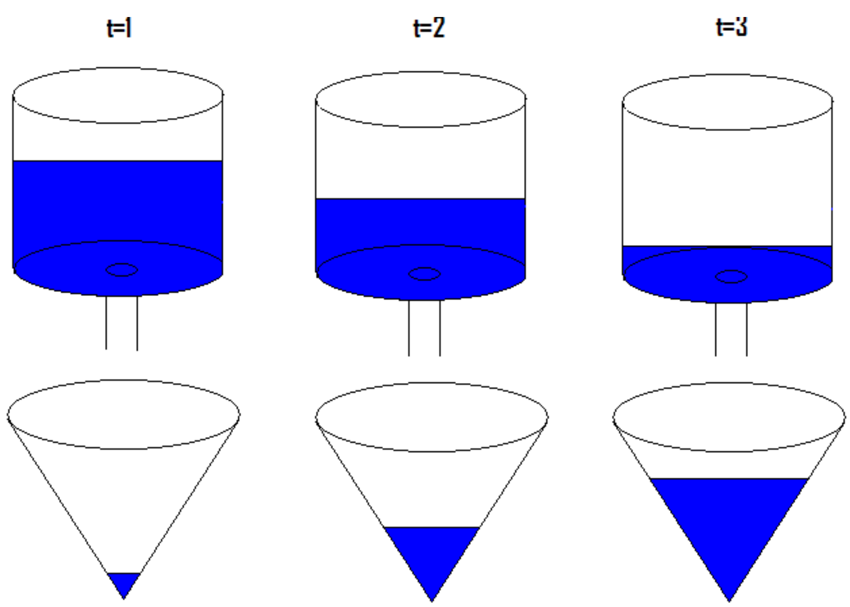
Using the Functions h(t) and H(t), find a function of time for V_1 and V_2

V_1(t)=

V_2(t)=

1.A.4 Activity 4 (Class Discussion 1)

Consider the following setup, a tank with a circular radius empties into a cone with a circular base with the same radius. The height of the cone is twice the radius of the cone.



Question 4.1

Label the height of the water in the cylindrical tanks, h(t)

Label the height of the water in the water in the cones H(t)

Label the radius of the water in the cone R(t)

Question 4.2

Give a relationship between the radius, R(t), and the height of the water, H(t) in the cone.

Question 4.3

Explain this relationship geometrically.

Question 4.4

Find a function Volume of the water in the cone as a function of H(t) and R(t), only R(t), and only H(t)

V(H(t),R(t))=

V(H(t))=

V(R(t))=

Question 4.5

Is there a relationship between the radius of the cylindrical tank, r, and the height of the water possible? Why or Why not?

Question 4.6

Suppose the radius of the tank is 1. Fill in the following table:

Time	Volume in Cylinder	h(t)	Volume in Cone	H(t)	R(t)
1	8π		1 π		
2	7π				
3	6 π				

Question 4.7

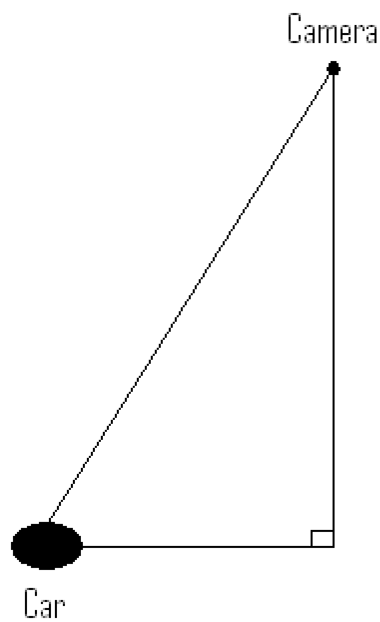
Using the data from the table, verify that 2R(t)=H(t).

Question 4.8

Can you create a function to relate the h(t) with H(t)? What about h(t) with R(t)?

1.A.5 Activity 5 (Class Discussion 2)

Consider the following: You are videotaping a race car from a stand 100m from the track, following a car that is moving at some speed. (see diagram below)



Question 5.1

Label the distance between you and the car d, the distance between the car and the point on the track that is directly in front of you x, and the angle between the car and the vertical θ.

Question 5.2

What variables are functions of time? Which are constant? Label those that are function of time using functional notation. (ex: f(t))

Question 5.3

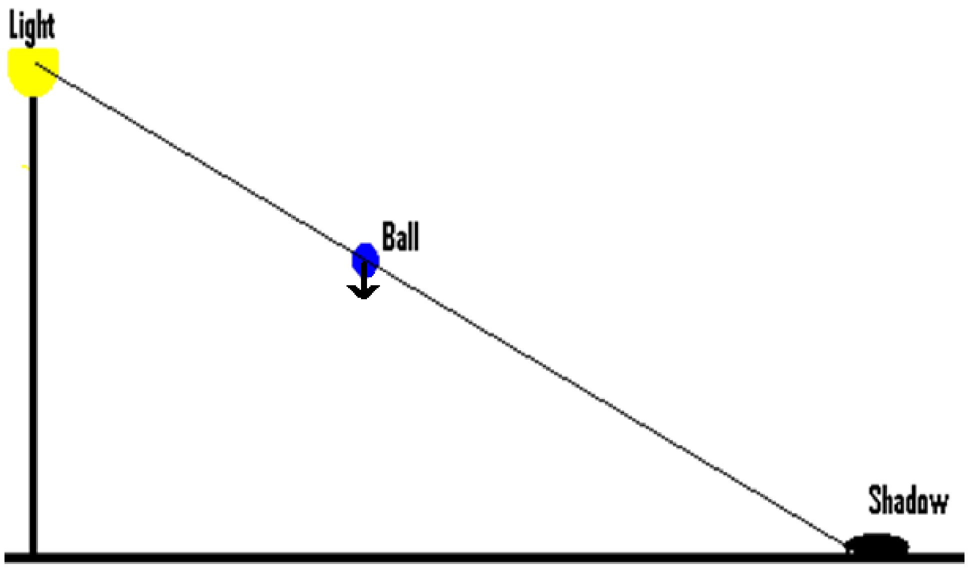
Find geometric relationships between these variables.

Question 5.4

What is the value of θ(t) when the car is directly in front of you? What is the value of x(t)?

1.A.6 Activity 6 (Take-Home Assignment)

A ball is dropped from a height h, a distance d away from a lamp post.



Question 6.1

Draw the ball at two later times and label then time 1 and 2.

Question 6.2

Trace a line and find the shadow of the ball at times 1 and 2.

Question 6.3

Label the distance of the shadow from the lamppost, the distance of the ball from the lamppost, the height of the ball, and the distance from the ball to its shadow, using functional notation where necessary. (ex: f(t))

Question 6.4

Using the above labeling, find a geometric relationship between the distance of the ball and the shadow, the shadow and lamppost and the height of the ball.

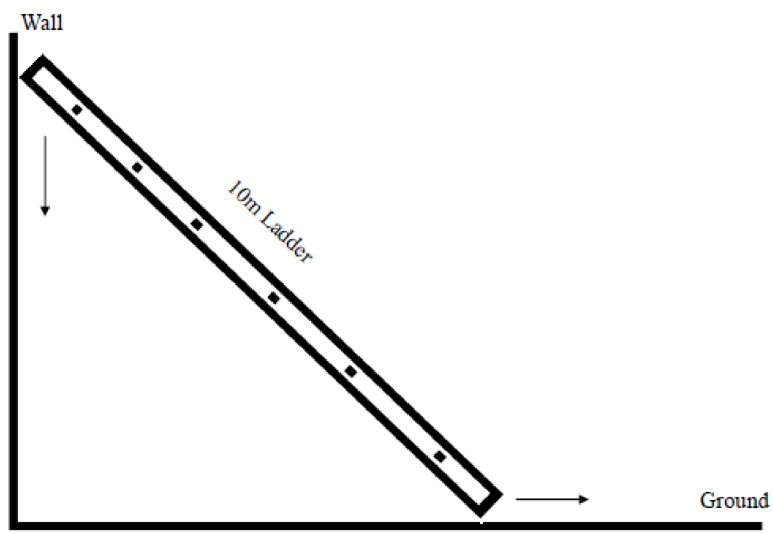
Question 6.5

Summarize, using as few sentences and in your own words, what is happening in this problem.

Appendix 1.B – Interview Problems

1.B.1 Problem 1

A 10m long ladder is sliding down a wall as seen in the following diagram. At the instant shown the bottom of the ladder is 8m from the vertical wall and is moving away from that wall at 3 m/s.



Question 1a

Explain in your own words what is happening in this problem.

Question 1b

How fast is the top of the ladder moving down the wall at this point.

Question 1c

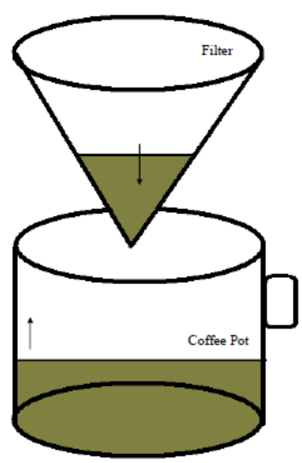
At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment.

Question 1d

At what rate is the angle between the wall and the ground changing.

1.B.2 Problem 2

A coffee percolator pours coffee into a cylindrical coffee pot from a conical filter. The coffee machine makes coffee at a rate of 5cm3/min. The filter and the pot both have a radius of 3cm. The filter's height is twice its radius.



Question 2a

Describe in your own words what is happening in this problem.

Question 2b

How fast is the height in the pot rising.

Question 2c

How fast is the height in the filter decreasing when it is 3cm high?

Appendix 2A – Group Work Transcripts

Students 1, 2, 3 and 4 are labeled S1, S2, S3, and S4 respectively. The interviewer is labeled as I. Comments that are unspoken are given in italics. Portions that are quoted in the analysis are highlighted for easy reference.

2.A.1 Students S1 and S2 Transcript

S1 & S2 Activity 1

Question 1.1 and Question 1. 2

S1: well for the first part I sketched circle and labeled the radius 4m and the area is πr^2, I think.

S2: I need you to explain this for me

I: Well you can read the question (2)

Question 1.3 and Question1. 4

S2: So this is 2 plus 1 right?

[…]

S1: R(1)=2+1 so R(1) is 3.

S2: ok

S1: so at time 1 r is three, r(3) will be five… area at time one is going to be 9pi, area at time 2 will be 16pi and the area at time 3 will be 25pi…. And from question one we know area at time 1 is…. Metres squared.

S2: the first one is 28.27

S1: metres squared

S2: the last one is 78.54. This would also be three seconds right.

S1: and for this… the radius would have to be nine, which would give a time of eight.

S2: ok

Question1. 5

S1: So A of r of t equals, so a of r is pi r squared, r of t is 2 +t, therefore, A of r or t is going to be pi times r t squared. Does that seem right.

S2: Yeah I think so.

S1: Combining the two, functions.

Question 1.6

S1: So A of t… Do you get this?

S2: not yet.

S1: r of t equals 2 plus t and… Area t equals pi r t squared?

I: Perhaps you should look at the table.

[…]

I: you want the area as a function of time.

S1: Other than going A(1)… is still pi r squared, and r is 2+t. So it would be π 2+t squared. Does that make sense?

I: Does that work?

S1: Yeah it would cuz you are simply replacing r by the

S2: Radius

S1: By the function of the radius, which is affected by time. So we are at the left an area as a function of time, and the formula for our area is the same but now our radius is a function of time. We substitute our r with r(t). Does that work?

I: You both agree with that?

S2: yeah

S1: More or less

S1 & S2 Activity 2

Question 2.1

Draw triangle and put in hypotenuse. Don't comment on anything.

Question 2.2

S1: This one is going to be root 85?

S2: Yeah

S1: So the hypotenuse of triangle 2 is root 85.

Question 2.3

Start by filling in table

S2: So we are just supposed to keep on descending?

I: Yeah

S1: So 5 would be 6, 6 would be 7, 8 would be 3. So find the value of z..

S1: Would be 65?

S2: So it would be 3, this would be 8, so z of the either triangle would be 3 squared + eight squared, equals root 73, just like triangle # 3.

Question 2. 4

S1: function x(t)…

Lots of writing

S2: So you can just compare it to the chart

S1: I guess

S2: Minus one would be ten.

S1: So x(t) is eleven minus t, and y(t) equals t, so basically y is t.

S2: It would just be 11, right, using the Pythagorean formula, 11 minus t squared plus t squared.

[…]

S2: Square root that.

S1: Yeah, we don’t need to simplify that, it would just remain the same.

I: Yeah, you don’t need to simplify

S1: And it’s basically, it works for this specific triangle. Or rather for these.

Question 2.6

S1: OK, that’s like exactly what we just did. Construct a 78unction, 11 minus t squared, plus t squared.

Question2.7 and 2.8

I: It’s the same x and y.

S1: So if we are going to substitute right away, z would equal... square root of 11 minus 1 squared plus 1 squared.

S1: that would simply equal one.

I: well, i asked you to label them x of t, y of t and z of t.

S1: x of t is eleven minus one, y of t equals one. Z of t is root of 11 minus 1 squared minus 1 squared which brings us back to z of t equals root of 101.

Question2.9

S1: I’d say that triangles 1, 2, 3, 4 are the same triangle growing through time at a rate of one unit per second. Or, uh, shrinking. Do you agree?

S2: Yeah, looks like it.

Time

S2: Actually, i don't know if they are shrinking or not? Like if the hypotenuse is shrinking... I don't know if the area is actually shrinking.

S1: let's make our own column on the chart. In this case the area is growing, you are right. [...] ok, well i didn't even mention area here.

S2: The x and the y sides are switching kinda. Moving towards each other, one unit at a time.

S1: This one is growing.

S2: They are proceeding in opposite directions.

S1: Basically, at this point here, y can be replaced with time. So it's not that the triangle is shrinking, it's just the base that is shrinking.

S2: Yeah.

S1: So it's the same triangle but the base is shrinking.

I: I think we can stop here for this one.

S1 & S2 Activity 3

Question 3.1

Label diagrams, no discussion

Question 3.2

Read question

I: Volume of a cylinder is one the board.

S1: So the volume of the cylinder is 8 pi r squared.

[...]

S1: Now volume in top cylinder, this is the volume of the liquid?

I: Yeah, the volume of the liquid, and the diagram gives three separate times. This would be 1,2, and 3 seconds

S1: So if the volume at time 1 is8 pi r squared, then it's going to equal pi r squared h. [...]

S2: So you just have to solve for h, to find height.

S1: Wouldn't that just simply be this divided by h? Does it work that way? Height of water in the top tank is pi r squared 8 divided by h.

S2: I'm not sure yet.

I: The volume can also be written as h pi r squared.

S2: Ok. so I guess that would just be the height. These values are just the height if you switch the formula around, these values are just the height. So it would just be 8,7,6.

S1: Yup.

S2: So this would probably be 3 pi r squared. So this had got to be 9, which has to be the total height. Cuz the height of the first base is just one. you know, they are just switching, like the triangular problem.

I: the water if pouring from the top tank to the bottom tank.

S1: k, but the volume in the top cylinder should be...oh, it bottom cylinder. As this is shrinking by one, this is growing by one. So at time 3, pi r squared....and we are back to...

S1: The drawings are not to scale are they?

I: No, they are not to scale.

S2: So i guess it's just three.

I: So the total volume of the liquid never changes.

S2: Yup

S1: and then here we are back to substitution.

S2: It would be the same as this, just 1, 2, 3. It could be the same as the first column.

S1: It's always going to be, that they are directly related like that. I'm comfortable with that, I mean , I see the pattern. What bothers me is that height and volume are always going to be the same value.

S2: Just the first term is the same. We don't know the radius, so we can't actually calculate the actual volume. But they always have to be equal to 9, cuz there is no water escaping.

S1: so we should have a function here instead of a number right.

Question 3.4

S1: This is exactly what I was thinking.

S2: Every second it goes down by one, for the other, each second it increases by one.

S1: Yup. For the height of water in the top tank so... I think we are back to what i was doing here. If this is your volume... We need a function for the height.

S2: I think for the first one it will just be 9 minus the time, and for the second one it would be 9 plus the time.

S1: We need to find the function of height so that we can look at it as a fucntion of time, and the volume would be the height devided by pi r squared? Is that how we are seeing this. I mean, if we are going old school here, if v= h tims pi times r then h would equal v over pi r squared.

I: That algebra is correct, yes.

S1: My algebra is my weakest point.

I: I think Dylan is on the right track

S2: Yeah that’s how we got the heights; it's like the last question. So here h of t would just be t, since it's just the height, just like the one before, and for the first one it would just be 9 minus t to get the height valua as time gois on.

S1: But is one the only...one is the only...and one isn't a function

S2: It's just equal to t. Just seems like thats the pattern. the height is the same as the time.

S1: I see that correlation, it just seems wrong.

I: why?

S2: Seems too simple?

S1: It seems like, yeah, unless this particular problem was designed to have that common factor that this will always equal time. It just seems that for the bottom tank there should be a function letting us know what’s going on there. Unless the function is just going to simplify down to one.

S2: It's just like the triangle problem from before (Activity 2), one side was just increasing, and y just equalled t. It's just the same thing.

S1: so you are going to have specifically designed cylinders to drinp at exactly one unit per second.

I: Looks like it. Well how do you check if a function works?

S1: Process it out. We still don't have our r value.[...] totally, that the function that proves it. If h is V over pi r squared, in this case, if we look at this one, h is three, and the volume is 3 pi r squared, if you solve, h is just three. So h of t simply equals t, which is a really roundabout way of exactly what you said.

I: So I guess someone is faster at finding patters.

S2: This is the same as the previous problem.

I: The problems are related

S1: I can pick out the problem, I just don’t trust it. Cuz like trusting without testing is how I failed the midterm last semester.

I: What about the next one?

S2: It would just be 9 minus t.

S1: Yup.

I: Try to convince him Dylan.

S2: I’m still trying to picture the formula in my head… convince me, how did you arrive at that?

S2: It’s just the pattern that…it’s difficult to explain.

S1: We know the total volume is 9, and in this particular time the height is nine.

S2: When you subtract from 9 you come up with the height of the water in the top tank. That's just the way that you come up with those numbers

S1: So the only think that is not laid out with us plainly is the total being 9. It's obvious when you look at it. So the height of the water in the top tank is 9 minus t.

Question 3.5

S1: So the volume in tank 2 is always t times pi r squared. This is not combining the function, like we did before?

S2: I don't think so.

S1: Would it simply be 9 minus t times pi r squared?

S2: That doesn't have the volume of the first tank in it.

I: You are kind of solving another problem. The question just wants volume 1 and volume 2.

S2: It would just be total volume minus the volume of the first tank.

S1: So what I wrote there is the total volume of the second tank…

S2: Total volume of both tanks. If we say that the total volume is pi r 2, of all the liquid.

S1: But we are looking at the volume of the second cylinder, which at time one is always this and at time two it's nanana, etc. It's always time times pi r squared.

Poor understanding of function

S1: Oh, what you are saying is the total volume minus the volume of this one is the other one.

S2: And the total volume is 9 pi r squared, since we know the total height.

S1: So 9 pi r squared…

S2: And not just minus V1. Just leave it like that.

S1: But yeah, ok. In that case V1 is 9 minus t pi r squared.

I: But you don't need any time

S1: But I want to know what V1 is.

I: Well you will have your chance.

S1: But what I am saying is correct, no?

Questions 3.6, 3.7, 3.8, and 3.9

S1: We know that volume 1 is this. This is out h(t), this is V_1. Is it the same answer and I am just over complicating this? I'm sticking with this.

S2: Seems good. That would just be t.

S1: Yup. And is this what I did here.

S1: So here we are simply combining the functions. Are we just back to 9 minus t. Cuz is we are putting this into that, we are substituting t for t. Does that make sense.

S2: yeah

S1: It's just 9 minus time. Yup, I'm sticking with that.

I: So what's the difference in the questions? What's the difference between this notation and that notation? There's no difference?

S1: Well I mean we combine them together and…. The part that confuses me is that our time unit is progressing at the exact same rate as our level units, of height unit.

I: Which?

S1: Of the top tank. Well the top tank is shrinking sat the same rate as tima and the bottom tank is growing at the same rate as time.

I: What does this mean here?

S1: That the height of the bottom tank, is it always a direct function of time. Because they are interchangeable. Whatever the time is, that will be the height of the bottom tank.

I: So they are interchangeable, you both agree with that.

S1: Yeah.

I: So what's the difference in notation? What is this asking for? They are both little h, but what the difference in the notation. Here you are asked for little h as a function of what?

S1: Time.

I: And here?

S1: Of the height of the bottom one.

I: What's the relationship between the height of the bottom on and time?

S1: they are identical.

I: So what do you think this is asking for?

S1: The rate by which the top one shrinks.

I: And what the bottom one.

S1: t

I: and what's t?

S1: In this case, the height of the bottom one. In this particular wording of the question. But it could just be time.

I: Why would you not write 9 minus H(t). Why would you write 9 minus t, if they are equivalent? What makes you chose t?

S1: Um, perfectly honest, it's your notation here of little h of H of t, told me right away that it's a combination of two functions, and I have these two functions right here, as h(t) being 9 minus t and H(t) being t.

I: So this notation and that notation is the same thing, this one just indicates some sort of composition, and this one does not.

S1: Pretty much

I so how would I ask you, If I wanted you to write little h is equal to 9 minus big H? Let's say I wanted that relation in functional notation. If I wanted to ask you.

S1: If you wanted to write h=9-H, how would we write that? In this form?

I: Using functional notation, yeah.

S1: I mean the only two things that are related, that is, being the same unit, is the height and the time of the bottom one. So it could be 9 minus h or it could be 9 minus t.

S2: would it be h(h(t))? Would that be the same thing?

I: So it's a function of itself?

S2: Oh sorry, this is 9 minus t and this is 9 minus h.

I: Let's move onto the last question

S2: Yeah we already got this. 9 minus t and t times pi r squared.

I: We should move on to the next activity.

S1 & S2 Activity 4

Questions 4.1, 4.2, and 4.3

I: So now we are going to relate radius with time?

[…]

I: Is it asking for us to label the water's radius in the cone?

S2: It's going to be the same.

I: Well for the overall cone the height is twice the radius.

S1: Yeah, but here it's the radius of the water, which is at different levels at each point, so like that relationship doesn't necessarily help us out. If we knew the height of the water it would.

I: Here we have the height and the radius and the height is twice the radius. How do we write that? Then the water that's at some point here, the radius is r(t) and the height is h(t). Makes a triangle. Is this angle the same?

S2: the proportions stay the same.

S1: So the relationship is pretty much what you just wrote.

I: If you two triangles have the same angles then all the triangles are the same proportion.

S1: So it's just 2r(t)=h(t)

I: Yeah. But convince yourself.

S1: How much time do you have?

I: It's just similar triangles. You can think of in many different ways. Ok.

S2: So we don't have to explain the relationship geometrically any more.

S1: if 2r=h then r=h/2. And it will maintain that relationship, no matter what value these take.

S2: Cuz the angles are staying the same.

Question 4. 4

I: so here is where you might get confused with the notation. So I mean, what you were saying before was wrong. When I say H(t), it means that I want to represent H as a function of time, which meant that the domain of the function is time. So if I have a value of time I can find a value for H. If I have H(h), this means that it is a composition of functions but you don't have to do the full decomposition, it's just asking, if I know what h(t) is then I can find H(t), regardless of the time. So this would just be asking for H=9-h. I understand that you know that h(t) is equal to time, but that is what H(t) is asking for. You want it as a function of what's in here. So what this question is asking is, I want volume, such that if I know the radius which happens to be a function of time, and height. If I know the height and I know the radius, what's the volume?

S1: But does that mean it with time.

I: there doesn't have to be any time's in it. Other than in the brackets of h. So if I give you a particular ha and an r, can you tell me what V is? With some function?

S2: It's just the same formula as the volume.

I: It's the same except that it these specific r and h.

S1: So you can just replace r with r(t).

I: this is a generic formula for any cone, now you are saying that it is this particular r and h which are varying with time. But you don't need to know how they are varying with time. You are just representing the fact that some cone exists where the radius are some functions of time. So before, you were trying to solve for too much stuff. It's just a notation thing. It's the same formula, but if I give you a value for r or h you can find the volume. Try and do the others. You were always trying to work all the way back to time. But you do not need to do that.

I: So the whole way, I jumped to this conclusion, which is the correct answer since the time is the same unit as the height. So if the cylinders where a different size and the height was not changing at one per second, then this would not have worked out as nicely.

I: no. If this was some other function of time then you would have to substitute differently. It's just about what the function is asking for. I want to know what this value is if you give me this. It's domain range. You were always working back to time. You don't need to know what this function is.

S1: Going to have to bust out the triangle on this one.

S2: so the radius is equal to one half of the height, right? Form the instructions of the question. So then you can just replace the r(t) with the 1/2h(t).

I: You want value just as a function of just height.

S1: so I mean, unless I am just totally missing the point here. Is it simply.. but we are just talking about notation.

S2: But you have to express it as h(t).

S1: Express r and h(t)

I: this one says you need r to get volume. I want you to show that you only need h to get volume.

S2: And r is equal to ½ h. Right. So just put that in.

S1: Or we replace r completely and it’s pi times 1/2 of h square. So the only thing we are missing to get the formula is h, that’s what you are talking about.

I: The fact that you are writing h(t) instead of just h is to let yourself know that it is in itself a function of time. There lies your composition of function. If you had height as a function of time then you could find volume as a function of time, but you don’t have to.

S1: this remains the same. And now 2 h(t), no 2r(t).

I: you just multiplied by 2?

S1: Taking the whole thing?

I: Well if you didn't multiply by 2 it would be the same thing. The volume is this but it's also this, a multiple of 2. Two cones, one with the height twice as high with the same radius...

S2: You mean the height of the water in the cylindrical tank, or the cone?

Question 4.5

I: This part is all with the cylindrical tank. Here you related the height of the water in the cone. Is it possible to relate the hight and the radius of the cylinder?

S1: Yeah, cuz where h maxes out..

I: It's asking you if you could get something relating little h, from the top one, so that it is some function of r. Is that possible? Before for the bottom one you wrote H(t)=2R(t). So it is asking if this is possible for the top cylinder.

[...]

I: what's the basis of this relationship here, for the cone? why could you relate when the water was here.

S1: because the volume calculation is constant. I see this as being a limit of its size. This is definitely a relationship between the height of the water and the radius, because the radius is going to determine...

I: Without including volume. What's the radius of this tank, of the water in the tank?

S1: r(t)

I: does it ever change.

S2: No the height will change but the radius will never change.

I: So is it possible to relate them.

S2: No

I: why could you relate them here then? Here you have the height at different levels......No, is correct, but why no?

S1: you would need like a third dimension.

Time

Question 4.6

I: So we know that the radius is constant and that it is always equal to 1

S2: It's just 8

I: What about you.

S2: Cuz the pi would cancel out

S1: the height equals volume over pi r squared, and it's given that the volume is 8 times pi, and it's given that the volume is 1. So we have 8 times pi divided by 8 times one, which gives a height of 8.

I: You could plug 1 in here and you get volume is 8 pi. Getting the same answer. In this case the height is 8.

S2: Then you have 7 and 6.

[..]

I: Once again, the tan empties into the cone.

S2: I'm trying to figure out the heights of the water in the cone.

I: you've already convinced yourself that the volume in the cone is that.

S2: Sort of.

I: Is it not a total volume thing again? Total volume is 9 pi. The water can't go anywhere.

[...]

I: How are you trying to solve for the height

S2: Solving for h using the volume formula.

I: Using this one.

S2: I just set the volume to pi and then solved for h.

I: like this?

S2: It would be 1/3, the 3 would be on the bottom.

I: You want it like this.

S2: Yeah, never mind, I made a mistake, it would be 3.

I: Why would you go back to this column equation? What are you looking for there? Height?

S2: Yeah.

I: And what do you have, Volume right?

S2: Yeah.

I: Do we have an equation with just height and volume in it?

S1: We have the height of the top cylinder.

I: You have volume and we want the height.

S1: So we can replace the volume...For the case of time is 1, h is 3 pi over pi r squared. Does that make sense? We don't know r.

I: Can you not replace r with h? This 2r=h thing, is that always true.

S1: Yeah, for the cone.

I: So you guys are stuck with the same r.

S1: So 3 pi over h over 2 squared.

I: So you are replacing r with h over 2

S1: Yeah

I: Can you solve this.(3V/pi(h/2)^2=h)

S1: No. I mean it's not going to help us narrow down what h is.

I: Really... (Use some algebra to solve). Why do you think that you cannot solve for h here? Do you think it is algebraically impossible?

S1: No, algebraically, i mean you just showed that we can isolate h. But when I see this form here, I'm like no, obviously, assign one value on one side, the other side is gonna have to match it.

I: So what if we have x=4/x. Can you solve for x here?

S1: Now that I've seen your method. But my algebra is my weakest part of my calculus. If I see that I would keep putting values into h until I arrive at this equals this. and that's not what gets you points.

I: What about you. You are looking over here.

S2: Yeah.

I: I mean you guys have already solved for V(h(t)), over there. So...

S1: 1/3 pi time h/2 squared time h(t)

I: Yeah here we have volume just has a function of height. ad of course, you get the same thing. That's the purpose of writing it like that, since you can now get volume if you have only h, and you can get h if you have only volume. If you have the original function here.

S1: I don't think that this concept of combining functions was ever presented to us like that.

I: How would you write this. How a bot if I just write V(h). All the t is representing is that height is a function of time. That's all it is.

S1: In this case, does that work out because height is moving at the same rate as time.

I: It doesn't matter.

Time

I: Here you are just using functional notation to show that h is a function of time while r is not

S1: So h is you dependant and r is you independent variable?

I: Well, no.

S1: We because r is it's own thing, it doesn't care what else is going on. Whereas the value of h is going to depend...

I: The value of r isn't going to do anything. It's just that you don't know what it is, right. But it's not something that is variant. For example; here it is 1, so it's 1 forever.

S1: I think my hangup on this one is identifying which part is moving and which part isn't.

I: Also, I think your hang-up is the fact that when you see this notation, you automatically have to represent volume as a function of time, and you don't need to.

Time

Question 4.7

I: So how would you find the radius as a function of time? So we found the first height is cube root of 12. Now that you know the height what are your options for finding the radius.

S1: It's going to be working it back into.....

D: You just divide it by 2.

I: That's always true.

S1: Yeah, I'm totally down with that.

Question 4.8

S1: I probably could but we would be here another 30 min

I: It is just asking if you can, not to actually do it. Before we said we could not relate h and r of the cylinder.

S1: would it not be volume...

I: h(t) is over to volume 1 times pi r squared. and H(t) is..

S1: Cube root 12

I: I mean just in general. Volume over pi r squared over 3. Can you relate h and H. This radius is always one. So this is just Volume 1 over pi. This one here r is always

S1: h/2

I: so you can just write that....It's exactly as before. How could you relate h and H. You have this a s a function of volume 1 and here H as a function of V2. So you would have to relate the volumes. Can you relate volume one to volume 2?

S1: Nope. they are different shapes.

I: remember before, we related the volume of the cylinders, can we not do that here?

S1: Nope. I'm sticking with no

I: Well total volume equals volume 1 plus volume 2. Do you know the total volume here.

S1: 9

I: just 9

S1: 9 cm cubed

I: Take a look at the table, is there a total amount of volume

S1: 9pi

I: that is always true.

S2: 9 pi minus volume 1.

I: So you could do it.

2.A.2 Students S3 and S4 Transcript

S3 & S4 Activity 1

Questions 1.1 and 1.2

S3: Alright, the first one; sketch a circle with radius 4m. Find the area of the circle. Here's a circle, radius is half.

S4: Yup, that would be 4.

S3: The area… radius squared times pi, is that it? Which is 4 squared time pi. [starts to use calculator]

I: You can keep it in terms of pi.

S4: Ok, 16 pi.

Sa: Draw a circle and lable the radius r(t). Ok, so we are doing the same thing, but we are lable this [radius] r(t).

S3: That was an easy one.

S4: I know there is something to this.

Question 1.3

S4: So this now becomes t plus 2. Find it at 1 second

S3: So, r(1) is 2 plus 1

S4: So the area is 9 pi.

Question 1.4

S3: At one, radius is 3 and the area is 9 pi. At 2, this would equal 4, and the area is 16. At 3…

S3: 25 π *recognizes pattern quickly*

S3: They give you the 5 again.

S4: So the time is 3.

S3: If the area is 81pi, then the radius is 9, which means the time would be nine minus 1 which is 8.

Question 1.5

S1: We know that the area equals the radius squared time pi…

S2: So you put, r(t) as 1…no

S1: Is this using the same function as in part 3?

I: No. Here radius is a function of what?

S2: Radius times pi.

I: Well it's just a function of time. It's just a notational thing.

St: yeah, makes sense.

Question 1.6

S1: would it not be t=-2?

S2: We did this last semester, functions. Are you supposed to plug that …and then…

S1: Ok, r(t) equals 2+t, we can use this equation here and sub it in for r(t)

S3 & S4 Activity 2

S4: I like triangles.

Question 2.1

They draw a triangle and lable sides x,y,z no discussion

Question 2.2

S3: So we sub them in, this is one..

S4: The other one is ten.

S3: So Pythagorean theorem.

S4: 10 squared plus 1 squared would be square root of 11.

S3: No that doesn't make any sense. It's 100 plus 1. Is that right?

S4: It's 101

S3: 100 plus 1 equals c squared. So it's 101 square root.

Question 2.3

Draws triangle

S3: So a squared plus b squared is c squares. 81 + 4, is square root of 85. Triangle 2.

Question 2.4

S3: So if we have 64 +9, we have 73, so this would be root 73.

S4: So I see the pattern, this is 7 and 4, 49 and 16.

St: Which is root 65?

Table completed quickly

Question 2.5

S3: So it would be like 6,7…5,6,7,8. So it would be 3,8..

S4: 3 , 8, 9. It would be 4.

S3: did we not just have that. No. ...64. Can I find the value? Yes I can find it, root 73.

Question 2.6

About a minute passes as they think about the question

S3: what does this mean? Oh, and a triangle t. So x(t) is 11-t. So here, let's say we take triangle 2, we assume the t value is 2, so x(2) is 9.

S4: And y is equal to t.

S3: So they are the same value. So the functions are correct.

S4: I got that that makes sense.

Question 2.7

S3: Well what it works out to is the Pythagorean theorem. So if we do x(t) squared, plus y(t) squared equals.. and then if we sub out points in…But it's going to have to equal z(t) square..

S4: So we can put this under the square root?

S3: No I think we have to sub our points in. Well wouldn't the t's cancel and it would just be 11?

S4: Square root 11.

S3: Well I don't know. This could become 11-t squared, plus t squared, ok that makes more sense. it's the square root of that, equals z(t). Can we then take that to 11-t, no cuz it's all one.

Question 2.8

S3: Do we use the same function.

I: Yeah, it's all the same.

S3: These are some nice triangles *[draws triangle].* So we lable it using the functions? So we've establishes our z(t) function. But t is 1 right, so it's the square root of 101.

S4: So we just put in the numbers. So it's the same as that [question 3]

S3: But I guess we have to figure it out though out function.

S4: Yeah use the functions, so y(1).

Question 2.9

S3: As t increases, y increases, x decreases, and z decreases.

S4: As time increases.

S3: This triangle, in relation to these guys… well it's the exact same proportions… because it's the exact same triangle.

S4: Well these two are the same but this one is different…

S3: But it's the same function, same proportions.

S4: So they are all triangles of the same function

S3: Guess we can say that. So, as t increase, z(t) decreases, x(t) decreases, and y(t)

S3 & S4 Activity 3

Question 3.1

S3: So we will assume this is here… easy enough.

Label h(t), r, and H(t).

S3: I feel like this sheet is gonna be a problem.

Question 3.2

S4: So with time at one, the volume is 8 pi r squared.

S3: Right. The height of the water, that's h(t). I don't know the volume of a cylinder.

I: Its pi r squared h. IT's the area of the base times the height.

S3: That question was silly.

S3: So we can work this here…It's eight

S4: Eight, yeah.

S3: V equals pi r squared, then the height is going to be eight.

S4: If you got this, and you got h.

S3: Then h has to be eight.

Time

S3: Volume of top of cylinder. So in V2..the patter is 8-1, 7-2, and 6-3. That wasn't so hard.

Question 3.4

S3: So the height of water h(t) equals 8 pi r squared minus t.

S4: Divided by t

S3: I think it's minus t, because as time progresses…. No it would be the volume…

S4: Function of time…

S3: Our volume is this, what do we know about the height.

S4: Height is volume divided by pi radius square. H(t) is volume over pi r square.

S3: But that's not in reference to t. We have to figure out…as time increases we have the height decreasing. I think what it will be is h(t) equals 9 minus t times pi r squared. That will give us the correct values.

S4: yup

I: It’s for h, not for volume.

S3: oh. Ok, so the height is this value, the eight.

S4: Yeah.

S3: I don’t know. Eight minus…

I: I think you should focus not on the volume, but just on the height and the time.

S3: So….

I: So you want a function that you could plug in the time and find the height.

> S3: So it’s just 9 minus t. Wow, that’s so much easier than we made it. So for the bottom tank, it’s H(t)=t. all right.

Question 3.5

S4: There’s seven.

S3: The volume of the first tank specifically.

S4: 7-6-5

S3: It equals the absence of it to equal 9. So for V2, it would be 9 minus…

S4: 9 minus the first one, this, of the first tank.

S3: 9 minus V_1?

I: Well it isn’t just 9, it’s the volume.

S3: Create a function… so.

S4: You have to put time in that function?

I: No. Just V_2 and V_1. Just relate them.

S3: Equals 9…

I: It has to do with the fact that there is only so much water.

S3: Right. So 9 pi r squared minus V_2…

I: Perhaps if you think of it this way; $V_1 + V_2$ always equal what?

S3: Nine.

I: Just nine?

S3: 9 pi r squared.

Write $V1+V2=9\pi r2$

S4: Ok then move that… there

Put $V2=9\pi r2\text{-}V1$

Question 3.6

S3: So it equals 8 times pi r squared.

S4: So this is V1, is height t, which is 8…

S3: would it not be nine minus t times pi r squared

S4: It would be times eight

S3: Yeah, but it's not always 8.

S4: It would be the number.. pi r squared times this number will always be the number…Whatever the number.

S3: there has to be a t in it.

I: there has to be an h(t) in it

S4: This times this gives you the total.

I: But what is this?

S3: H(t)

S4: So h(t) times pi r squared. I like that, big yes.

S3: Silly.

Question 3.7

S3: So wouldn't it be the exact same thing?

S4; Yup. Should be.

S3: Maybe not…Change in case (capital H). That's it.

Question 3.8

S3: This is h(H(t)). We know these things, so we can just say t at 9 minus t.. I think that's correct.

S4: No wait

S3: It would be 9 minus t.

I: You want h as a function of H, so try using the same logic as here.

Time

S3: So h, which is 9 minus t.

S4: but 9-1 is 8

S3: that's not right. It's like…

I: Think of it like this; h(t) + H(t) has to equal what?

S3: 9

S4: 9

S3: So 9 equals.

S4: H2 plus H1..small h plus big H equals…

S3: So H(t) plus h(t) equals 9

I: You manipulate this to isolate V1, not you have to isolate h(t). Since its h(t) as a function of H(t). Why is this here a function of V2?

S3: Did you get that?

S4: I did. This is the function we are looking for. So for every h…

I: right now you have h(t) + H(t) =9

S3: right

I: We want h(H(t)) so that if we have a value for H(t) we can find h(t).

S3: So it would be 9 minus H(t)

Interviewer writes equation on blackboard: $h(t)=9-H(t)$

I: So to write it using function notation we write it as h(H(t))

S3: Why? How does that make sense?

I: well because this is just saying that there is a relationship between the two of them.

S4: You could plug in a number for the h.

I: this is saying that the big H is specifically a function of the little h. So when H is three, we have this. It's as if we have f(x). I see it is difficult for you to assimilate this as a whole.

S4: I always has a problem with that, seeing too many alphabets.

S3: I enjoy notation… I do thought, there is a lot of interesting notation.

I: what are you are saying is that they are not just functions of t, they have their own relationship as well.

Question 3.9

S3: V1, which is this guy here… and we have h, which is this guy… so V(t)=(9-t)(pir^2). This would be right because for any value of t, you would end up with the right value.

S4: we have to use out functions here [h(t), H(t)]

S3: so this is out function here, at h(t). We are trying to find it at t. Are we not on the right track here?

I: Yeah. Why is that correct? Sandro is trying to use value from the table while Steve is using substitution. Both you are both getting the right answer. How do they relate?

S3: I don't know.

I: Can you see the raltaionshsip here.

Interviewer writes out functions and performs substitution on the black board

S3: No. Well I can see the relationship. But the reasoning for the relationship, I don't know.

Substitution is re-explained by interviewer.

S3: Yeah ok, I can see that.

S4: Um, yeah, you can get one function and substitute it into the other.

An example of composition of two basic function of time, x(t), y(t). is given on the board

I: If x(t) and y(t), then there has to be a relationship between the two of them. Here we have volume and height as functions of time, and we know volume is a function of height. Therefore volume is a function of time.

S3: Alright, so for V2(t), this would just be tπr2.

I: Right

S4: Ok. Fresh course on functions. Why didn't we do this in our class?

S3 & S4 Activity 4

I: We are going to do a similar situation but with a cone on the bottom. The volume of a cone is... (Writes equation on board). IT's similar to the volume of a cylinder but it's divided by three.

Question 4.1

Label diagram

S3: this is going to be h(t)

S4: This here is 2...

I: What did you write here?

S4: No that’s not right, it should be h, the height h, is two times the radius.

S3: So the radius of the water which is here is R(t)

Questions 4.1 and 4.2

S3: The height of the cone is twice the radius

S4: We do the radius r(t), the height of the water...

S3: the height of the cone is twice the radius of the cone, give a relationship between radius R(t) and the height of the water h(T) in the cone.... explain this relationship geometrically...oh man

I: you got it there.

S3: What does that have to do with the water necessarily? We can find the relatioship between the height of the cone and the radius, but how do we know how much water is in it?

I: Sorry?

S3: All we know so far is that...

I: what is this representation here? h=2r. Why did you write that down?

S4: Cuz that's my height.

S3: but this is the height of the cone.

I: Where is that there? So you have the cone..

S3: But is that the height of the water or the height of the cone?

I: The general height of the cone, we'll call it whatever, H, and this r. what did you say before...

S3: 2r equals h.

I: So 2r equals h. So now you have, at any given point, theres water, and the height of that water is H(t), and the radius is r(t).

S3: So that same relationship would exist because it's all proportial for a cone

S4: right

I: Well, you could look at it as these are the heights and these are the radii, so what is it that you always have?

S3: The same relationship between..

S4: The same right triangle with the same

ST: So if we have our h there and our r there we can just substitute h(t) for h and r(t) for r (in the valume equation).

I: This is true all the time, no mattwer what level of water exists. You agree that when it id full you have this relatonships. Evben if you are at any goven point, the height will always be tqice the radius, and this angle will always be the same and these will always be similar triangles.

S3: This doesn't explain this relationship geometrically. This is our function.

I: what's the function?

S3: didn't we just determine that? Well, yeah, ok, so we can go like 2r(t)=h(t), therefore... oh that's pretty straight forward there.

I: The thing with similar triangles is that, let's say this is any given r(t) right, and this is any h(t), at some point 1, and some other point, second. So if you remember from similar triangles you could always right that r(1)/ r(2) is always equal to h(1)/h(2). this relationship always has to hold then, that 2r(t)=2h(t)

S4: right, right.

I: I mean you don't have to provide a strong proof, you just have to understand that it's similar triangles.

S3: right, so as it increases or decreases the proportions stay the same.

I: yeah the same rations. So for any... r or h, the ration is always the same. So r over h is always equal to one half. All the triangles no matter where you make them is always equal to...

S3: where does that one come from?

I: which one?

S3: well if you have 2r=h...

I: I just divided here

S3: Oh right.

I: we are over explaining it here. Just saying that the triangles are proportinal is fine. I think you got it from the beginning.

Sa: Yeah, but I had to see the picture to be sure

Question 4.4

S3: Ok, so the volume of water in the cone. So, what is h(t).

I: so volume, just to remind you is... (writes volume equations on board)

S4: thank you.

S3: ok, so find a function...

S4: function that gives volume....

S3: This is a relationship between volume, H(t) and R(t).

Time

I: So I mean, what did we do in the previous problems. So the volume qas $V=\pi r^2 h$ and we didn't touch the pi or the r squared.

S4: we replace the h with the function of h.

I: h(t)

S3: so if we have v here, what is it, V=1/3 pi r2 h, what it is, we have H(t) is... there is a comma, I don't really know how to deal with that...

I: The difference now is like...there is a big difference. Here, in the first proble, the r was always the same, it would never change. But now here, r and h are changing. Like you seein these pictures that the r and the h are increasing.

S3: So could we substitute so that it would be like 1/3 pi times whatever r with this relationship...

I: But we don't need to look at that relationship yet, because we can straight up just use R(t) and H(t). So what is the radius here.

S3: Radius is R(T).

I: And what i the height?

S3: H(t). so we can just drop it in.

I: yup

S3: Seems more complicated than that.

I: Well then you are over thinking it.

S3: well we are in a little room with people watching us.

Time

S3: so if that is that...

I: Here comes the more difficult part. So like you said it's hard to deal with this. Here you have a function with two variables. The most important part is that you cannot do any calculus with a function of two variables. Well that you have learned. Have you learnt any multivariable calculus?

S4: No

I: so how can we make volume just a function of one variable?

S3: Well, we know how these two relate.

I: Right, so we have...

S3: well can't you do this, just, we know H(t)=2R(t), so can’t we just substitute in 4R(t), or like whatever for H(t). We can just drop those bad boys in. We've determined here that are relationship is that H(t) = 2 R(t), so for this we are trying to find for H(t), and we don't know R(t).

I: just look at this one here. And what you wrote before, H=2R. So can you eliminate all the R's here?

S4: I'm thinking...

I: So H/2=R. So can we eliminate the R's by replacing them with H's?

S4: Sure, you can replace the R with H/2 square, right, and you got rid of all your R's.

I: that’s exactly what we are doing, but instead now we have functions.

S4: We are eliminating one variable.

S3: Pretty much, by making it another one.

S4: ok

S3: So it would be this

S4: And here we do the opposite

S3: Right. So H(t) is 2R(t). So we are replacing H(t), with R(t)m so we'll end up having 2 R(t)'s in this equation.. So we have 1/3 R(t) ... times R(t)... there's gotta be a square here.

I: Well what are you replacing here?

S3: We are replacing R(t).

I: Yeah

S3: so R(t) equals H(t)/2

I: So this is...

S3: Wrong

I: So we have one third pi, and what are we replacing...

S3: It's going to be H(t) square over a third times pi times R(T).

I: Why is this R(t)?

S3: This it is R(t)

I: Well didn't we just replace that with H(t)=R(t)/2? The whole point is replacing R(t) with H(t)/2, and that's it.

S4: Yeah.

I: It's exactly the same.

S3: So this would be H(t). Is that what you are saying?

I: Yeah. It's exactly the same as with the variables.

S4: So we are keeping the pi r squared, we are replacing the H with 2R.

I: The whole point is to eliminate to get one variable. Here we have 2 variables, can't do any calculus, so we've eliminated one variable. So we are trying to do the same thing here.

S3: So we are keeping this R(t) and we are replacing H(t) and we've determined that H(t)=2R(t). Can we then bring that 2 out?

I: well we can simplify.

S3: So you would have R(t) tripled?

I: Yeah. R(t) cubed.

S3: what happens to the 2?

I: yeah so...which 2

S3: we have a 2 here

I: that just comes out.

S3: K, perfect.

I: It's just algebraic simplification at this point.

Question 4.5

I: So here you have related the height of the water with the radius of the water, right. What about here.

S3: Can we determine that?... I'm gonna say ,no.

I: What the radius of the tank. Here we are just looking at the cylinder. Let's say the radius is 5, what will it be here?

S4: 5

I: And here.

S4: 5

I: And the heights are always the same as well?

S4: No.

I: So can a function be related to a constant?

S3: No

I: There's no way to write h(t)=.

St: No

S4: No, no you can't.

I: Why not? Because radius is a constant, right?

S4: Right. The radius is the same, while the height is changing.

I: So there is a difference between things that are constant and things that are variables.

S3: The radius is constant while the height changes, therefore no relationship.

S4: There's a relationship...

S3: The is one in the cone because we've determined that at any point the height is twice the radius

I: So as before, the whole reason for using the notation is that you can show which variables are changing with time and which are constant. So before when we had the tanks by themselves he had them as h(t), and we could say the volume is a function of h(t), knowing that this quantity changes as time goes forward. This one does not do that. If I don't use functional notation and I just write $V=\pi r^2 h$, you don’t have any information as to which variables are function of time.

S3: All right.

I: But with the cone, as the water rises, both the height and the radius changes. But here, both things are constant. This is important in calculus, because when you take the derivative of things that are constant, they are just zero.

S4: Right

I: If you take the derivative of a function...

S3: In that case wouldn't it be 2?

I: No, the derivative of, let's say you have...

S3: So R is a variable and not a constant?

I: Exactly. Only in this case.

S3: We know pi, is a constant.

I: The important thing is that even though we use letters as labels, some things are variables and some of them are functions of time.

S3: Is that what related rates are? Is this like and intro to that.

I: This is designed to be an intro to related rates.

Question 4.6

I: This is similar to the previous activity. Before what was the radius of the tank always? It was 8 pi r squared or something like 7 pi r squared. If r is one, that's just not there anymore.

S3: Suppose the radius is one, fill in the following table. When time is one, volume of the cylinder in... h(t) equals. Can you repeat what you said just a second ago?

I: Yeah, so i mean the 4 in the regular tanks was pi r squared, but r is one here so i mean, it's just not written.

S4: So it's 8.

S3: H of t equals 8. Our h(t) is just our h value

I: It's the same as before; it's just now that r is one. It's still a constant it's just that now it has been given an actual quantity. If r was three then there would be a three there. The top tank is the same as before, with the radius being one, so the height is just 8,7,6.OK, now what about the volume of the cone?

S3: K, the volume of the cone.

I: what principle did we use here, the fact that the total volumes are always equal. Here the total volume is how much.

S4: 9 pi.

I: So how much is the total volume here have to be.

S3: So h(t)...

S4: Now we can find a function for the height of the water in the cone.

I: Well you want to find the height.

S3:The volume of the cone is this guy.

I: So here what do you have.

S3: these guys. pi over 12 equals h(t) cubed.

I: What about here. So if you have the volume and you want to fund h you can just work backwards.

S3: We have that the volume is pi.

I: so the volume is $(1/12)\pi h^3$.

S3: Correct.

I: So if the volume is 1 pi, and it has to equal this...

S3: Right so you just solve for h(t).

I: Not so bad. h(t)=... Eliminate the π's

S4: And cube it.

I: How would you get the r(t).

S3: Same thing but with this guy.

At this point they both clearly understand the difference between the functions and which function to use where to solve for a particular variable. They are answering quickly and finishing my sentences.

I: You can get r and h. So since r and h are related, you only need one of them to find volume and if you have volume then you can find either one of them.

S4: Right

S3: Yup

I: So what's r. Volume equals... So what's the volume

S3 : Cube root of three halves.

I: Ok

Question 4.7

I: Here you can check the relationship 2r=h

S4: So if you use this this will be two time this.

I: So then is true.

Question 4.8

I: So what are we relating here?

S3: So related H(t) to h(t) and relate little h(t) and the radius

I: You have a relation between the two volumes right?

S3: Do we?

I: You got it here.

S3: Right

S4: Total volume

I: So volume one plue volume 2 is equal to what?

S3: 9pi.

I: So instead of writing the volume alone you want a relationship between h(t) and H(t). So volume one is what...pi time 1 squared...

S3: π times r square h(t).

I: Instead of r squared i can write 1.

S4: Yup, squared

I: Times h(t)

S4: h(t) right?

I: And volume two, is the volume of the cone.

St: 1/3 π r is going to be one

I: No.

S4: Cuz the radius is changing now. So it's the function r.

I: But you have it here as a function of r and here as a function of…

S4: h.

I: So we have π/12..

Sa: As a function of h(t).

I: And that's it?

S3: Cubed.

S4: We just have to replace is with those.

I: And here you can isolate h(t).

Time

I: Yup, that's the relation between h(t) and H(t)

S: Yeah so it would be like...9 pi...$H(t)^3$ .....

I: It's equal to π times h(t).

S3: And here we can just divide by π.

I: There is a π missing. Never mind.

S3: So that works. You got it

S4: Yup, good job.

I: You have h(t) and H(t) related

S3: That makes sense.

I: Can you now relate H(t) and r(t).

S4: Sure we can, just use the function of r(t) one instead.

I: Yup, instead of using this use this one and you can once again isolate for h(t) or r(t).

S4: Yup

I: Do it

Time

S3: Just isolate for h(t). Yup.

I: There you are.

S3: All right.

They both seem very pleased with this activity.

S3 & S4 Activity 5

I: This is a camera and there is a car coming and you just have to answer the question.

Question 5.1

S3: This is 100 m.

S4: Ah, makes sense.

S3: this is d, this x.

I: Label the distance between you and the track, the car...yeah ok

S3: and this is the date here.

S4: Distance between the […] and the vertical

Question 5.2

S3: The only thing that is a function of time is this, is the car, x, is a function of time.

I: And this, and the angle.

S3: Right, but the only thing that is constant here is the distance from the track.

S3: So we have x(t), theta(t), and

I: I think we should label this, i think you and the car should be d

S3: Yeah eh, either way this is 100 m, then this is d(t). The constants are the distance to the track

S4: I don't get it, the difference is constant?

S3: the only thing that is moving is the car, so when the car moves the hypotenuse is changing and the angle is changing, but the distance to the track is never changing.

S4: Oh, ok, i get it.

Question 5.3

S3: So it's the Pythagorean theorem. So do we write these as constants or as functions

I: As functions.

S3: So x(t) squared plus 100 squared equals d(t) squared

I: Not theta and the others

S3: Right. Theta...uh...what relationship is involved with this angle

S4: x...

S3: What is determining the tangle?

S4: The tangent?

I: Which is what?

S4: Opposite over adjacent. So it's x/100

S3: Ok, alright.

I: It's the tangent though. The tangent of theta is x over 100.

St: Oh right, tangent of theta is ...

Question 5.4

S3: Both zero

S4: Explain

S3: Let's say the car is right in from of you, what happens to the angle

S4: The angle goes to zero.

S3: Exactly, and what about x

S4: Zero

I: OK, that’s it

S3: We should have started with this one.

S4: The third page, or I mean this one (#4) killed me. I feel like I didn’t learn anything in algebra

I: That's why they are so difficult.

Appendix 1.B – Interview Transcripts

1.B.1 S3 Interview

Student is given volume and Pythagorean formulas on the black board.

Problem 1

Question 1a

S3: So this is eight metres. [labels distance from wall to ladder] And it’s moving at 3m/s away, and so down as well. So explain what is happening in your own words… As the ladder slides away from the wall, it also decreases proportionally or the height of the ladder decreases proportionally to the length form the wall. As the ladder drops the base extends further away from the wall.

Question 1b

S3: How fast is ladder moving down the wall…. I would say 3m/s.

I: Why would you say that?

S3: Because the bottom is going to be moving at the same speed as the top.

I: Why?

S3: Well short of the physics involved with gravity and such… cuz it's all one unit, so I would assume that it would be falling and moving at the same pace, at 3m/s…. Um, yeah… Is that incorrect?

I: Yeah

S3: Um…

Re-reads question

S3: So if this is eight, we could use Pythagoras to solve for the distance here to here. I don't see what that's going to do for me in this particular problem.

Time

I: What happens if we just label it x.

S3: Oh, like this, this guy… ok then let's lable this guy y and so the position of the… the height would be y(t), no, it would be a function of the x. Well, if the bottom is moving at 3m/s, the top would be moving at 3m/s, but it doesn't seem that that's the case.

I: What relationships can you construct between x and y?

S3: Um.

I: You told me that when x was 8 y was 6, what were you using there?

S3: Pythagoras. Well like, what… I don't know how that's going to figure out how fast it's falling down the wall.

Time

I: What is x and y?

S3: x is distance, and y is also distance.

I: And the question is asking for what?

S3: A speed

I: Is there a relationship between distance and speed?

S3: Yeah It's distance over time equals the speed. But we don't have any time.

I: Can you relate x and y. Using Pythagoras.

Time

S3: x Squared plus y squared is t squared or whatever, I guess 10 squared. So y squared equals 6.

I: What about in general.

S3: Then it would be like, in this situation here, x is 8, in general it's x minus 2 equals y.

I: What if instead of eight you use just x.

S3: So it would be like y equals the square root of h, ill it h, the ladder guy.

I: Well the ladder is always 10.

S3: So y equals the square root of x squared minus 10 squared.

I: So this is always true.

S3: Yeah. But how does that help me with how fast the ladder is moving down the wall? So that is my distance. So speed equals distance over time.

I: Just dividing

S3: its change, I got the delta going on here

I: What is delta?

S3: It would be x2-x1 over t2-t1. But my distance is this guy right.

Time

I: But this 3m/s is instantaneous change…

S3: Derivatives

I: Right

S3: Derivative with respect to x, yeah. So how fast the ladder is moving would just be my derivative of y.

I: What would you do here?

S3: I would say like 3 equals... or were not supposed to use this, let's forget this.

Time

I: Well, I wouldn't forget it.

Time

I: This three, what is it. It's a speed so it's delta v over delta t, it's not equal to x or y. Here you have y equals root 10 squared minus x squared. We can just plug in eight because we get 6m a distance at a particular point. What are we looking for here?

S3: As this changes, how fast it that changes, the instantaneous change

I: Which is what?

S3: Delta y by delta t.

I: That's what we are looking for.

S3: We should write it as dy over dt.

I: What's the difference?

S3: dx over dt is instantaneous. Ok, but I have an x value in here as opposed to a t.

I: When you are differentiating and you want to get a dy by dt it's a different kind of differentiating.

S3: Yeah it's implicit. So is this just an implicit differentiation? Like that's pretty tough man.

I: Well nobody said it was easy.

S3: I know this is my equation for how this moves. Can I say f(x) equals root 10 squares minus x squared? Then If I were to implicitly differentiate, I'd… I don't know.

I: What's the implicit derivative of…

S3: I could find the derivative of this, but it's not respect to t. So I would have to find it like… d.. um I don't know..

I: Well what’s the derivative of y with respect to t.

S3: But I got an x in here.

I: But that wouldn’t be implicit

S3: Yeah, that would just be regular.

I: What would you get by regular differentiation? Dy/dx…

Does the chain rule with no difficulty

I: What’s that equal to?

S3: dy/dx

I: but you want…

S3: dy/dt

I: Can we move this over here? (dx)

S3: dy equals …..Times dx

I: Can we divide both sides by dt?

Time

I: So know you have an equation…

S3: Change in y in relation to change in time.

I: What does that represent?

S3: The speed in which the ladder falls down the walls.

I: dx/dt?

S3: Difference in the distance on the ground over time. Distance from the wall. This here would be like dx and this here dy. (Labels velocities)

I: x?

S3: x is the distance between the wall and the end of the ladder on the ground.

I: Now in this particular case that we have here, where.. Well reread the question.

Rereads question

I: What is it asking for?

S3: How does this dy take place.

I: dy/dt?

S3: Right. Can I just take that three and put it in?

I: Put it in where?

S3: I would put it in as my… I don't know…As my dx/dt…no

I: What's dy/dt?

S3: My speed.

I: Speed of what?

S3: The speed of it falling

I: And dx/dt?

S3: Moving out here.

I: And the 3?

S3: This thing. (points to speed of ladder)

I: Are those the same?

S3: Yes

Time

S3: This is going be like this time… this one over dt is it not?

I: Haven't you already divided by dt?

S3: Right I already did that here. Right then I can sub in three as dx/dt.

Time

I: What do you also need?

S3: x

I: What is x?

S3: At this current point it's 8.

Time

I: So it's not 3.

S3: Where does that relation even come into effect? That doesn't even make any sense to me. That like if something is moving away from the wall at one speed it's dropping from the wall at a different speed?

I: Yeah

S3: why? How do account for that, isn't it the same force acting on everything?

I: Well, think about an extreme case of it up against the wall. The top of the ladder would be moving much slower.

S3: Right. Yeah.

I: The other extreme case. If the ladder is against the ground the bottom of the ladder is sliding very little for how far…

S3: Right

The solution to the problem is given to show how that comes out in the calculus. Basically re-writing what he did onto the blackboard but with functional notation y(t), ect.

I: why is the problem hard for you?

S3: Making the step from the algebraic understanding into finding the derivative and the differentiation of it. Understanding how…

I: How to differentiate?

S3: Not so much how to differentiate, but in this case, what steps you use where to differentiate. I can understand up the point where we have y(t) equals whatever. But I would never be able to get this on my own.

Problem 2

Question 2a

S3: OK, filter height is twice it's radius, ok. This part is twice of it, and then...how does this relate. What does that mean, does that mean it's dropping from this point to here?

I: at this point...

S3: So this is where your coffee is being made?

I: Yeah.

S3: Ok. I don't need to worry about that yet. (talking about radius height relation) As it makes coffee the amount of coffee empties proportionally to the amount that it fills in the cup. And how that changes... I gotta figure that out.

Question 2b

S3: How fast is the pot rising? So this could be my y here, y(t) would then be how fast this rises.

I: y(t) here

S3: Or h(t), whatever.

I: I'm just asking.

S3: The pot is rising… let's call this y, the amount of coffee in the pot, therefore the instantaneous rate of change of y with respect to t is how fast the pot is growing therefore what does that equal. Well that’s going to equal...I don't know. It's going to be the instantaneous change in the radius which will be half of the height...

I: Perhaps you should write down some of these relationships

S3: Yeah, maybe. [writes down volume equations] This is same up here as it is down here. [He is talking about the radius.] Ok...

I: Well what are they asking for?

S3: They are asking for the rate for which the coffee pot fills. And that is rising, um. Ok, what I know is that my coffee maker machine...This is gonna be my change. I'm trying to think about the first problem. Then my speed gets plugged in for my x value in the algebraic relationship. Now for the coffee pot. So the coffee pot is increasing...we have pi r square h...

I: what is this value here? $5cm^3/s$?

S3: It's the amount that's dropping in.

I: What amount? It's a notion of what?

S3: Volume per minute. So is that it? Can i just say that.

I: No.

S3: That's the amount that's dropping in

I: Which is volume. The question is asking...

S3: How fast. The time?

I: The level means the actual height.

S3: Right. The change in y.

I: So it's a notion of.

S3: t. time.

I: Let's change this word from level to height.

S3: That's not it, i don't understand the entire concept. Ah, just like how the height changes with respect to the volume per minute.

I: Right.

Time

I: So what is 5? In the previous example 3 was dx/dt.

S3: Is that it? Ok so the time is equal to the change in this, the change in height of the cylinder over time.

I: The height?

S3: No, change in volume over time.

I: How would you write that?

S3: I don't know. Dv/dt? or would it be dv/dx?

I: What is x.

S3: I don't know what x is.

I: Well it's in minute

S3: Well the v is d of d of t of pi r squared h. (using differential operator) Do I even use that now. You got so many variables here.

I: Well, what's r?

S3: r is 3. 3 cm. Can i just incorporate pi now

I: you can just pull it out.

S3: I don't what I am doing. Where are we going with this. I don't get it. I don't get how to relate these things. dv/dt over pi? Times 1 over pi times h equals 9?

I: What about the d over dt.

S3: I don't even know if that's where I need to be headed.

I: Why did you put h into denominator?

S3: because over here we got pi times 8, so we can divide this whole side by pi times 8. equals dt/dt of 9, which is just zero. this guy i zero over here. So, but...I know really

I: What did have over here (dv/dt=d/dt(pi*3^2*h)

S3: We'll there's a bunch of ones.

I: But this one here. r and pi are just constants. So

S3: So, I'm looking for the derivative of height in relation to the derivative time. Change in height....

I: What if we write volume is equal to 9 pi h. This value here is dv/dt. (5cm^3/min). So what are you looking for here?

S3: how fast the height.

I: What that?

S3: dh/dt

I: so you have dv/dt, but you don't have...

S3: So can I go um dv/dt equals 9 pi dh/dt, and then substitute 5 for dv/dt and solve. (solves for dh/dt) So 5/9pi equals dh/dt.

I: Does that make sense to you?

S3: Generally. Whatever that equals. So that is the change in height of the coffee pot, ok. Can I apply that same theorem to my c here? (Part c)

S3: So my cone size is v= 1/3 pi r^2 h. So If I'm looking at a similar situation...But in this case my rate of coffee is decreasing, so this (5) is going to be a negative value. (starts doing calculations)

I: Is r still 3 here?

S3: Is it not? Both have a radius of 3.

I: but what radius is this. (volume formula)

S3: The pot and filter have the same radius..

I: What volume are you looking for? Volume of the cone?

S3: Yeah of the cone.

I: Of the whole cone?

S3: of right eh. The filter height is twice it's radius. How do I work that is there. I thought I was doing it right.

I: Well it's not so bad. I think you're just missing...

S3: Yeah, but how do i put that in? Do i replace the height with 2h?

I: The height is twice the radius.

S3: right..

I: What does that mean algebraically?

S3: Um, that r will equal h/2, always, for the entire cone, as opposed to, like the whole thing, oh wait, cuz that will continue, this relationship, that this will always be half, like that (writes r=h/2 and 2r=h)

I: See you're still doing ok.

S3: Where does that fir into this. I'm trying to find the change of dv/dt...I'm trying to find the instantaneous change of the...as this decreases.

I: What are you trying to find?

S3: I'm trying to find dv/dt for this area here. I know my speed it's leaving.

I: Isn't 5 dv/dt?

S3: Yeah I know. Let me finish, I mean that I want to find my dh/dt but i know my dv/dt because..

I: What about here where you wrote the relation between h and r.

S3: Yeah, i know i just don't see where...

I: What happens if you write h and h/2 (replacing r). Is that the same thing?

S3: Yeah, it's the exact same thing.

I: Ok, so now you have it like this.

S3: So then could I just go like, 1/3 pi 2r square oh o wait 2h squared times h.

I: r is 2h.

S3: Oh wait, 1/3 pi h/2 squared times h. That will be equal to v. If I go dv over dt equals 1/3 pi dh over dt times...wait wait...dh/dt of 1/2 squared time dh/dt. Does that make sense?

I: What is this?

S3: This is the 2 cuz i have to find the differential of h.

I: What is h here? I think the square is in your way here. Perhaps if you look at it like this.

[Interviewer points at volume of a cone equation]

S3: Oh, we just bring that in. That makes a lot more sense. Where you get 4

I: The 2.

S3: Oh yeah the 2 squared. I'm on the right track here.

Time

I: So what do you understand about this now? What is this?

S3: The instantaneous change of the volume of the cone over time, the volume of the coffee. the volume of coffee changing over time equals 1/3 pi h cubed over 4. Oh, but I haven't differentiate this side yet. Um... ok well yeah.

I: So what's the derivative ok h cubed?

S3: 3 h squared. and then, ok. dv/dt equals derivative dh over dt 1/3 pi h cubed over 4. This all becomes zero, well not all of it...

I: If you have the derivative of 3x^2 the three becomes zero?

S3: No (laughs) It becomes three times 1/3 which is just pi time h squared and then the 4 is the same as timesing by 1/4, so it would be 1/4 pi h squared, well c'mon it's gotta be that.

I: Well you took a derivative here. And v became dv/dt, this has to have some dh/dt.

S3: But this is my derivative of that is it not? This is my dh/dt.

I: What do you understand from this dh/dt. (operator)

S3: That is change in height over the change in time.

I: Is it acting on this or what is it doing? (talking about finals lines where dh/dt disappear)

S3: It's instantaneous change.

I: What did you do after this part here? You basically said this (dh/dt) time this (1/3 pi h^3 /4) becomes.

S3: This. That is my change in height.

I: So this says take the derivative of what's after it.

S3: Yeah. Find the derivative of this, boom. Ok, cuz what I've been doing here is v, which becomes dv/dt. So I'm taking the derivative of this side and this side. The derivative of this side is 1/2 πh squared. So I found the relationship that exists here and then... Now, I know my change in volume, and I'm asked how fast the height is decreasing then I just go like. [replaces and solves algebra] It's gonna be negative in this place is it not? This doesn't really make sense...Won't it end up being seven or six or something... (adds a negative in front of h)

I: I think we can stop here.

1.B.2 S4 Interview

Problem 1

Question 1a

Labels the floor as the x-axis and puts in a vertical measurement of 8m.

I: You can explain oral what is happening here. There is not a lot of space to write there.

S4: Well I know

I: You can write on the scrap paper

Continues to write

S4: So as the ladder falls, there's a velocity that goes this way.

Points to the right.

S4: This is like physics am I right? With gravity it is coming down and we then we have velocity along the x axis.

I: OK

Question 1b

S4: How far is it moving down the wall, hum.

I: How fast?

S4: Yeah, how fast. Is the top of the ladder even on the wall? Ok. Well I got 10, 8…

Uses Pythagoras

Time

I: Is there any relationship between the x axis and the y axis?

S4: Yeah right, um, you would need to, so I'm thinking you would need the speed that this is going down. Right, so you need to know.. You could find your y-component here right? Like if I put y there, I'm thinking, ok, this is it what I'm thinking, r squared is equal to y square plus ten squared. 10 squared equals y square plus 8 squared. Solve for y, do you have a calculator

I: It's six

S4: Six,

I: y is six

S4: Oh you're doing for me, you're my calculator. We know this is the velocity this way. This is three per second.

Time

M: What happens if this is not 8? If it just any value.

S: If this is any value?

I: Yeah. What would you label it?

S4: x

I: Is there a relationship between this x here and this y?

S4: A relationship between this x and this y?

I: Yeah

S4: It's tangent, the x is tangent to the y.

I: What else? Is there another relationship?

Time

I: You said this was eight right? And you could find y right?

S4: Right

I: So there is some sort of relationship.

S4: Right

I: What is it? How did you find y? This (x) was eight. How did you get it?

S4: I used the Pythagorean Theorem.

I: Ok, so. Can you relate x and y using that? Just if it wasn't eight. Just some relationship between x and y.

Time

I: What happened here if it's not 8?

S4: And this wouldn't have a value of six.

I: No, just y.

S4: then you would have two variables.

I: ok. You'd be able to relate them tough, if I give you any x, you could give me y?

S4: Without the y you are saying?

I: If I give you any x, you can find the y?

S4: yeah

I: If I give you any y you can find the x?

S4: Right. With one variable you could find the other.

I: Exactly.

S4: That's a relationship. Ok, right, I see what you are saying.

I: So what is that relationship?

S4: Like a function?

I: Yeah sure. You can call it a function. Let's say I gave you a specific y, how would you find x?

S4: You give me a specific y.. I'd isolate the x.

I: Isolate x where?

S4: Isolate x here. And I have a y, so I have this, and then this.

I: So this is y for any y.

S4: Yeah, right

I: Rewrite it for any y. this is true always

S4: Right, this is true always

I: What is x and y

S4: x and y are length.

I: and what is this, 3 m/s

S4: Velocity.

I: Is there some way to relate velocity and length?

S4: Time. Velocity is change in x over time.

I: Which velocity?

S4: Velocity of the top of the ladder.

I: That's the change in x over time?

S4: You don't have time, time is not given (looks at given values)

I: Time is not given.

S4: You have to do the derivative, like that

I: What is that?

S4: Find the rate of change over time of x

I: And that's the velocity of x.

S4: What is the velocity? We don't know. As this comes down, the height over the time will determine the velocity. This divided by time.. We have the velocity here.. Right

I: So this dx/dt here is equal to… unknown?

S4: That's what I don't know

I: It's the velocity?

S4: It's the y-component.

I: So the velocity of your y-component is dx/dt?

S4: It's dx/dt, 6 over dt

I: 6 is dx?

S4: 6 is x. My mind right now, to be honest, I'm thinking physics. I'm thinking gravity. I'm thinking time. Gravity.. I don't know, I'm thinking some sort of physics equation.

I: You don't need to use gravity

S4: No, no gravity, nothing like that. Static friction? In my mind I'm mixing everything up.

I: ok

S4: Is it coming 3m/s here, it is the same. If this is 3m/s then it is not the same coming down right

I: Why not?

S4: Why can't it be 3m/s. If it's 3m's going down here. It's 3m/s. Isn't it the reciprocal?... If it's 3m/s going this way, why wouldn't it be 3m/s going this way. What does the length have to do with it?

I: Well let's this of an extreme case. Let's say the ladder was almost near the ground. Would it not be moving faster along than down? Or if the ladder was right up against the wall?

S4: and started sliding?

I: Would it not being going faster here.

S4: Hum, ok. I don’t see that one

Question 1c

S4: Area is ½ um, height times width. This is what we are doing, this is a function, ok. I think I would do a lot better if I could use my textbook. Can I use my textbook?

I: No

S4: Then I’m no good.

I: Well what is the height?

S4: The height is six.

I: Just in general.

S4: The width is this. The height is metres, the width in metres.

I: what variable did you label here?

S4: This is f(h) for height.

I: What did you draw over here?

S4: Y over here. This is the y, this is the x. I was thinking function of height, or function of width.

I: But you also label this x and y. Does that also make sense to you?

S4: Yeah sure.

I: Can you use these?

S4: Yeah sure, comes up as same thing. Now you need to find the rate of change. You need to find the derivative of this. OK, respect to him.

I: What derivative?

S4: Well you are looking for a rate of change. In order to find a rate of change.. first step you change this into y prime. Dy... How is it again, y over…. So you’d take the… this is u, so it’s x is one, dy over dx equals 1 over dx over dx…

I: Why are they over dx?

S4: Because we are looking for rate of change, no? I don't know. I am trying to apply what I learnt to this, I don't know if it makes sense or not? I don't know. Um…

Time

I: what are you looking at now?

S4: I'm just looking at the thing.

I: So it's not one.

S4: No. Yeah, it is one… OK

I: What is this equal to here?

S4: One half dy/dx, is one half…is one half. So this would be something like the function of the rate of change.

I: This here is y'

S4: Yeah

I: What happened to this A here.

S4: What happened to the A. We were taking the derivative of A. What happened to the A?

Time

I: You don't know? It disappeared?

Question 1d

S4: So you're looking at this, the rate of change of the angle. OK, let's start with the rate of change is the x.. this is y…

Time

S4: It's the change of y over change in x

I: Which angle?

S4: As this changes, this decreases over this as this increase.

I: What if you just wanted to know what the value of the angle is

S4: This angle, it's your y/x. It could be this over this; it could be this over that. It could be that over this.

I: Just y/x

S4:Is it a trig function. Tan.

I: What's a trigonometric function?

S4: It's a relationship between the three sides and how you divide it.

I: You said tangent before

S4: Isn't tangent opposite over adjacent?

I: ok

S4: I know that much. So let's say 6 over 8. I want it to be cosine, its 8 over…

I: What does that solve?

S4: Tan of your angle. You were asking me how I would find the angle right now.

I: Right

S4: So I would probably use one of these.

I: OK

S4: But you are looking at the rate of change of the angle.

I: Fair enough.

S4: Which I cannot give you. I am trying to give you. But I don't know.

Problem 2

Question 2a

S4: The coffee is pouring through the filter, the volume is increasing. Right?

Time

S4: So you have to take into account the fact that….um, ok, how fast is the height in the pot rising? Well.

I: Do you know the equation for the volume of a cylinder?

S4: It's one third…

I: You know them.

S4: No

I: A cone is 1/3 pi r squared h. Volume of a cylinder is pi r squared h.

S4: pi r squared?

I: yeah.

Question 2b

Time

I: So what is the question asking for?

S4: How fast is the height of the pot rising?

Time

S4: This is in minutes, I could put it in..naw it doesn't matter.

I: What are you looking for here? So before when it said at what rate is the angle changing you wrote dθ by dt right.

S4: Yeah.

I: Why?

S4: That's the rate of change of the angle

I: what do we need here…? If I reword the question to say "at what rate is the height changing". Does that make a difference to you?

S4: No, I know what it is asking. It's asking me to apply a concept which I am not good at. I'm really struggle.

S4: The general concept I understand. I know you have to find a rate, but they are asking for the rate of the height, or how fast. So the coffee machine makes coffee at a rate or, well this, and we need to plug in…

I: That's the rate of what?

S4: The machine. The machine makes coffee at 5 cm per minute.

I: 5 what? It is a length?

S4: It's a volume. The volume is that. This is being used here. Ok

I: The whole thing is the coffee machine.

S4: So this arrow is going here, ok.

I: So you label this h. Why?

S4: Height.

I: Height of what.

S4: Height of the coffee in the pot. Is two times this radius, no. The radius is here is twice the filters height, is twice its radius.

I: What does that mean?

S4: The filter's height is twice the radius. So the height here is 2 times the radius. OK

I: For the bottom. Here you have a notion of volume, and here you have something that related volume to height.

S4: That's volume as well.

I: What's this r?

S4: Radius

I: Radius of what. What height are we looking for?

S4: This one. This, no?

I: What's r?

S4: Radius

I: Of what?

S4: The coffee pot

I: Is it changing, is it the same all the time?

S4: It's constant

I: At what?

S4: Like what….

I: Where is the radius on this diagram?

S4: 3cm

I: Always

S4: Yeah it doesn't change?

I: What are we looking for

S4: Height. How fast.. the rate of change. The volume per minute

I: Well this is the volume per minute. We are looking for.

S4: Rate of the height. How fast is the height of the pot rising…

I: What were you doing over here? In 1c.

S4: Looking for the derivative.

I: Why.

S4: I don't know. No, it's the rate of change…

I: how we right this. Is it v equals?

S4: No it's dv/dt.

I: Does that make sense to you.

S4: Yeah, it really does.

I: You look eye opened.

S4: I am. This is the volume, this is the rate of change.

I: How can we go from one to the other?

S4: Tell me.

I: You tell me.

S4: I need to somehow find the derivative of that.

Time

I: What are the variables here? Is pi a variable.

S4: No it's constant. Height is a variable. And radius, which is constant.

I: What about volume?

S4: No the volume is changing.

I: Can you take the derivative of volume?

S4: Perhaps, I think if you do this. Um, r squared.

I: What's the derivative of r.

S4: r^2 would be 2r.

I: Why would it be 2r. What's the derivative of pi squared?

S4: I don't know.

I: Well why are you changing it from r to three or from three to 3?

S4: Why? Memory lapse. It's because I have to leave it as a variable to restart the problem. You know what I'm saying.

I: Does that not mean that you can take the derivative of a constant?

S4: You don't take the derivative of something that is constant, right?

I: What the rate of change of something that is constant?

S4: Zero. So the only derivative we are taking is the h. Is it that? Ok.

I: What's this?

S4: The rate of change of volume over time.

I: Do you know that?

S4: Um, it's the 5.

S4: This is what we are looking for. Right. So... there. *Smiles, and laughs.*

Question 2c

S4: This here is 3cm high. So we are looking at this. This formula, this, because we are only dealing with the filter. The height is decreasing when it's 3cm high. I'm thrown off by the extra value though.

I: What do you mean?

S4: When it's 3cm high. How fast is the height decreasing….ok This is constant

I: r

S4: No, r is not constant now. Again we are goanna have the rate f change, and again we have 2 pi r. the radius is changing, but we are looking at height. They are both not constant. I don't know what to do from there on. So dr/dt, and then this is dh/dt, but, where do I plug in the three?

I: What is this here?

S4: Is it the chain rule, not the product rule

I: When do you use the product rule?

S4: When you got two rate of changes, let's say. When you've got two variables.

I: So this is what here

S4: I got to do substitution again. So this is u, u is r squared. We'll go 1 plus , what is it, 2r, dh/dt.

I: So here you too the derivative with respect to what here?

S4: hum.

I: This is what. Uv', this is what?

S4: h’

I: h’, and this is one.

S4: And it dh/dt? No, it’s one it’s one.

I: And this is 2r as well.

S: 2r is the derivative of u.

I: And this is?

S4: 1

M: 1 again? What this in you formula (product rule), dh/dt

S4: It’s just h, like that.

I: ok

S4: And so, we just plug in everything.

I: What are we looking for here.

S4: The height.

I: The height.

S4: Well yeah, how fast is the height increasing?

Pause

S4: The rate of the height

I: so why did you.. here the derivative of v by itself became dv/dt, but here the derivative of h is just 1. Why the difference?

S4: it shouldn’t eh. The derivative will be dh/dt?

I: why are you taking derivative by time?

Erases dh/dt

S4: I’m going to start again. Um r squared… is 2r…

Pause

I: So what's this 2r here.

S4: I mean I'm applying the product rule.

I: what do you know?

S4: we know the volume, it's given. The rate of change of volume. R, r is, radius is changing. What do I get? And the height is changing. You got 2 unknown. One variable had to be figured out.

I: At this particular instant can you find the radius? What the height is 3.

S4: well the height is two times the radius. Filter's height is twice its radius.

I: is that for the whole filter?

Pause

S4: it would apply to the whole thing. Let's say whether or not the height is here…that's the relationship; it's always going to be twice its radius. So whether the radius is here or here, the height is always going to be twice.

I: what the radius there?

S4: We don't know, we just know its 2r. We know the height is 2r.

I: What's the height?

S4: 3

I: So what's the radius?

S4: 3 over 2. So you could plug it in there. And we can isolate for that.

1.B.3 S1 Individual Work

Problem 1

Question 1a

S1: Well it's showing that the rate that the top of the ladder is sliding down the wall is going to be related to the base of the ladder sliding on the ground.

Time

S1: Also the distance it drops will probably proportional to the distance it slides.

Time

Question 1b

S1: How fast is the top of ladder moving down at this point? 3m/s.

Question 1c

S1: How fast is the area changing at this moment? Would it be 3m/s or would it be 3 metres squared? Well go with 3 metres squared.

Question 1d

S1: At what rate is the angle between the wall and the ground… Well the angle isn't changing

I: It's the angle and the ladder, it's a typo.

S1: Ok, that makes more sense. At what rate is angle between the ladder and the ground changing?

Time

S1: So ladder and the ground, would be hypotenuse and adjacent. That's cosine right?

I: yeah, cosine is adjacent over hypotenuse.

S1: I don't know if that's going to help me. The length of h is going to stay the same, the length of a is going to change.

Time

S: Cosine theta, a(t)…The rate is going to be a degree measurement, like so many degrees per second…

Time

Return to Question 1b and Question 1c

S1: 8x8 is 64. Wait I have to figure out…Well it's not going to be 3m/s divided by the height. Or we could actually find this angle and then take a second snapshot and compare them. Would that be the long way of finding it?

I: It would an empirical way of finding it. There's a tool called calculus though.

S1: This is my problem

I: Where would you like to take the next snapshot?

S1: Further down.

I: How much further down?

S1: It could be anywhere right.

I: But where would be the best?

S1: At the following second?

I: OK

S1: We know it's going 3m/s. If I assume… ok, so if this is the angle of t zero I would have to find the angle at t one. If t one is a second later, this length is going to be 11m.

Time

I: you can use the inverse cosine relationship.

S1: ok, it's going to be like…

I: The inverse cosine function is on the calculator.

S1: So the angle at t zero is …this will be 10, this will be 11, and this will be 3, so… math error..

I: Why is it a math error?

Time

S1: Is this triangle possible?

S1: No

Time

S1: So this means that the rate it is falling is not equal to the base that it is sliding away?

Time

I: This is still not a possible triangle. So what does that mean?

S1: That it hit the ground before the following second. If this was 11m away, then this wouldn't be touching the wall anymore.

I: Where would the ladder be here?

S1: It would no longer be a triangle.

I: Does your answer to question b make sense?

S1: Well I mean… I still stand by this, I just don't' think it will complete a full second before it…so then like…

I: How high is this at the beginning? You said it was 6m. There is a contradiction, it can't do both. Do you recognize that? IT can go 3m/s here and 6 plus m in one second here. So here you say it's proportional.

S1: The rate that the ladder is falling.

I: You wrote that it's equivalent.

S1: Well I'm definitely standing by six here. That is correct. So the relationship is going to be…

I: How is this speed related to this distance a that you've drawn.

S1: Well a is growing by 3m/s.

I: "a" is a measure of what?

S1: Distance

I: 3 is a measure of what?

S1: Velocity.

I: Are distance and velocity related in any way?

S1: Yup

I: How?

Time

I: Is velocity the rate at which distance changes.

S1: Definitely.

I: Does rate and derivative have any meaning to you?

S1: I'm sure it does. What it is…

I: So here you are taking these snapshots right. Do you have to take a full second later?

S1: No, and a full second later it's no longer a triangle.

I: so less than one second?

S1: You definitely could. You'd only be able to take….well for it to remain a triangle; you could use 1/3 of a second.

I: What if we use this snapshot idea for question b. Could we do that at 1/3 of a second later.

S1: Make a new triangle at t_1. This would be 9, this would be 10 and this would be three.

I: why?

S1: Because 100 minus 81 is 9.

I: 100 minus 81 is 9?

S1: No 19, so root 19.

I: so that's a third of a second. So how fast is it moving? Goes from this to that in a third of a second. So how fast is it going?

S1: 4.95m/s

I: Does that make sense to you?

S1: Yeah, I'd say I trust that.

I: What about c? Could you do the same thing?

Time

S1: So let's call it delta area…

I: Is that per second.

S1: It's per third of a second. So then, it's this times three. If that's how much the area changed over one third of a second… is it still going to be the same for the second third of a second and the third of a second? Will it keep the same rate.

I: Well can we say that it's the same during the entire first third of a second?

S1: …Which is exactly why we need calculus… Instead of doing little calculations of every infinitely small time moment.

I: that's the idea. You want instantaneous, you want the very next snapshot. That the limit as time goes to zero. What you're doing here is the average slope, where area is like your y function and time is your x. You're saying you want the very next snapshot so you want the limit as the time goes to zero. That's the idea of calculus; this is limited accuracy, where calculus would give you an exact value.

S1: So you would have to find the actual the formula of area as a function of time and find the limit…

I: I can show you after. You have a good sense of the physical relationship.

S1: I know what's going on, I just can't…. formulate… I can see what's going on and what we are trying to get from it, it's just when it comes to formulating it.

Problem 2

Given volume of cone and cylinder

Question 2a

S1: Right off the bat we know that the radius of both of these is 3cm. The cone has a height double of that. The coffee machine makes coffee at $5cm^3/s$. We're talking about this here?

I: This is the machine.

S1: Yeah. But is that how fast this is filling or this

I: It's how fast this *(the cone)* is producing coffee.

S1: OK. So it's draining. The level is shrinking.

S1: Describe in your own words… I believe the height of the coffee pot is going to grow at a rate proportionate to the shrinking volume of the filter.

Question 2b

S1: I'd like to say it's just 5cm cubed but…. That would be the case if it was cylinder to cylinder. So if this volume is one third of this volume.. I mean you can't say this is growing three times as fats. I mean it's not getting bigger.

Time

S1: So we are going to have to set up the height of the pot at time zero and height of the pot at time one.

Time

S1: We are going to assume the height of the filter is 3cm.

I: Why are you assuming that?

S1: Because visually it looks like that…

I: The diagram is not to scale. Does it really matter where the coffee is being poured in by. Let's say it was a hose at the same rate. Would it make a difference how fast the level is raising?

S1: So it's more like we are looking for the height of the pot in reference to the volume changing over time.

Time

S1: The rate of this… Ah so this rate of 5cm^3/min has nothing to do with the shape. It could be any random shape. We should call it time one at a minute… or…maybe I'm going off the wrong path here. We can call this time one and call this time two. Volume at time one and time two…

Time

I: What did you put in for r?

S1: Three cm… and then…

Time

I: So what's the problem?

S1: Calculate the height at time one, which I just decided the volume to be 5 cm cubed. Then to get one second later, it will be 10cm cubed. But I don't think it's going to give me the right info. It will just give me the change between second one and two.

I: Change of what?

S1: In height, it gives me the change in height. This leads me to believe that my calculations went wrong somewhere.

I: why?

S1: Well then does that work and all that it means is that the rate of change is accelerating as the height decreases.

I: Which height are you looking at?

S1: For the height of the coffee pot to rise at a steady incremental rate like that, although the cone is getting smaller near the bottom, so this would have to speed up.

I: What would have to speed up?

S1: The height of the cone.

I: I don't understand.

S1: the height would change….would it be faster or slower?....(does some calculations)

I: What's that?

S1: This would be if we took a time three.

I: What did you calculate?

S1: The pot. Was trying to prove that it's steady increments. And if this is at fixed increments then this one can't be fixed increments.

I: Is it fixed increments?

S1: Yup.

I: What's the question asking?

S1: How fast is the height in the pot rising? … By 0.17 cm/min. doesn't seem right.

I: why not?

S1: this value just seems too small.

I: In comparison to what?

S1: To the 5 cm^3.

Time

I: Is there any way that you could check that this is correct?

S1: Well you'd have to set up a formula of height of the coffee pot vs. height of the cone.

I: Why?

Time

I: What if it was just a hose pouring at 5cm^3/min?

S1: I mean, then we're back to filling at fixed increments, the height is progressing at fixed increments.

I: So why is this different?

S1: this would be the same. Where this would differ from the hose is that the rate that this drops is not going to be steady.

I: Well that's question c.

S1: I can't think about this without thinking about that. My brain wants to figure everything out simultaneously.

Question 2c

S1: For this, height of and volume of cone…We should first calculate the value of the volume, or we don't actually need it, we can just go with V_1 and V_2. So hi equals V1 and h_2 equals V_2… and we want to find the limit as h approaches 3, 3 plus because we are coming in as it's falling. Limit as h approached 3+ of ….

I: There's no h in there.

S1: Oh, h is got to be in here.

I: Just saying you have no h at all, but your limit is as h goes to 3.

S: What do you mean?

I: you are taking the limit of a function as a variable approaches something but that function doesn't even have that variable in it. Isn't that a problem?

S1: Is the whole limit as h goes to 3 … does that even belong here?

I: how did you previously calculate rate of change.

S1: Taking time snapshots.

I: So what limit do you want? How close is time zero and time one.

S1: As close as we possibly could.

I: So the limit as what?

Time

S1: Totally lost. If we are doing the whole snapshot thing, we are choosing a snapshot at 3 cm, are we going to do one at 2.9 cm? Just arbitrarily, to give it a second point? Or is this simply going to be like, 3cm minus time 2?

1.B.4 S2 Individual Work

Problem 1

I: Here is the first problem, read it, and try and do as many parts as you can.

S2: OK

S2: You can use the scrap paper if you like since it may take extra space to answer these

I: Ok

S2: you can draw on this though

Question 1a

I: so you need to just write it down, to explain verbally?

S2: you can explain verbally

D: ok

S2: so the ladder is falling down the wall, the distance where the ladder is touching the ground is constantly increasing and the distance on the wall is constantly shrinking at the same speed. So then the ladder would be moving down the wall at three metres per second.

Question 1c

I: You trying to find the area now

S2: Yeah. So it would be varying by whatever the initial height is, minus three metres per second then whatever the base is, plus three metres per second. Just divided it by two.

I: How would you do that?

S2: how would you write it out? How would you write a formula for that?

I: I want to know how fast… so what would be the units of this area that's decreasing?

S2: it's going to be metres squared, right

I: Just metres squared is the rate?

S2: Metres squared per second.

Long silence (~2 min)

I: So you found this (pointing to area formula)

S2: yeah

S2: so it would be the eight plus three metres per second

I: what's the units of eight? *(Notices as he writes 8m+3m/s)*

S2: eight metres

I: So you can add things with different units?

S2: Um, nope.

I: So what is the rate that you are looking for?

S2: the change of the area

I: Over what?

S2: Over time

I: Can that be expresses, here you wrote A equals this *(A=bh/2)* and here A also equals that *(A=(8+3t)(6-3t)/2).* So A=bh/2 is just the area right

I: What's the units of that?

S2: metres squared

I: and what you are looking for here that (A=(8+3t)(6-3t)/2) is in

S2: metres squared per second

I: so are they both just A. If I want to know the rate of something

S2: So I can write *A (A=(8+3t)(6-3t)/2)* as a function of time. *Writes (A(t)=(8+3t)(6-3t)/2)*

I: ok

S2:ok

I: And this is A just as base times height.

S2: yup

S2: So do you have to change the way these terms are expressed?

I: um which terms? The eight and the three?

D: the metres just so that they have.

He seems quite lost at this point.

I: The thing is that this is a rate. You are looking for a rate, so you need to do calculus; you need to take a derivative. That's the only way. There no other way of getting it to work. I think you should try working back at this one again. *(first problem)*

S2: starting with this again. *(first problem)*

We stay with the area concept

I: Yeah, that's just the area right? No you want the area as a rate. What is, I mean, you never sort of understood derivative as being a rate. For example change in area.

S2: yeah

I: Or change in time as a DA by DT, or A prime. So you would need area as a function of something.

S2: *(quickly)* Area as a function of time

I: Essentially yeah. And then you would need to take the derivative.

I: So I mean what's this quantity here. You wrote b and h there, what's this quantity?

S2: Base

I: Just… label it, where is it?

S2: on this *(points at diagram)*

I: where is it? Show me where it is.

S2: It's this area over here.

I: is it always eight? Is it moving?

S2: it's increasing. So eight plus three metres per second

I: Ok. So what happens if you…if I want to know how far it was one second later?

S2: uh

I: could you tell me that?

S2: just divide the

I: like 1 second later with it moving 3 metres per second.

S2: yeah it would be three metres further

I: and one second more

S2: It would be 3 metres closer to the bottom.

I: could you find it for any time?

S2: um

I: if I say two seconds after, two seconds before

S2: yup

I: how are you doing that how would you… if I ask you two seconds later, could you tell me how far it is?

S2: it would be two time three.

I: What's the two?

S2: Time

I: So could you write base as a function of time?

Pause

I: could you find the value of the base for any time I would give you?

S2: yes

I: So the base is some function of time.

S2: yup

I: What about the height, is the height also a function of time.

S2: yup

I: Just some function of time.

S2: ok, so the base is gonna be eight plus three times t.

I: ok, and so the height is also a function of time.

S2: yup

I: just label it as a function of time. We know that we could figure it out if we wanted to.

S2: so just label this as the height.

I: yeah, height is a function of time.

I: you that you could find it right

S2: yeah

I: what related the base and the height? How did you find this initial root thirty-six?

S2: subtracting this.

I: you are pretty quick with Pythagoras

chuckles

S2: subtracting this from that*. (hypotenuse squared and base squared)*

I: so could you always relate height and base? Like if I give you any height you could find the base?

S2: yeah

I: how? How does that relationship work? Just with h and t and b. could you just in general relate them?

S2: they always have to add up to the hypotenuse.

I: what is that relationship?

S2: it would be h(t) squared plus b(t) squared, the square root of that.

I: equals what?

S2: equals the…10

I: ok

I: feel free to write it.

S2: in the page section

I: anywhere you want. It might be useful.

Writes 10=root((8+3t)^2+(root(36)-3t)^2)

S2: ok

I: ok. You can stay away from square root, using just 100

S2: oh yeah

I: So now what do you think?

S2: Ummmmm. Still on number C

I: yeah, I don't think you've solve it yet. So you've written down a formula right, area equals base times height divided by two.

S2: yep.

I: and we know both base times height are function of time right? So that mean area has to also…..

S2: be a function of time as well. Yup

I: so I mean, could you write out what the area is

S2: you could put these into the area formulae. *(8+3t and root(36)-3t)*Just substitute the base and the height, for the values in the formula.

I: and that would give you what?

S2: It would give you the area at any time.

I: Try and work from here. See what you can come up with.

Pause 1min

S2: Not sure if I'm going down the right path.

I: what are you doing?

S2: I'm multiplying it out.

I: ok

Multiplies out factored trinomial

D: Is there any further to go than that?

I: what's the question asking for?

S2: the rate.

I: and what does that give you

S2: gives you the area

I: is that equivalent to rate? What the difference between…what does rate mean?

S2: it means that…the change

I: change of what. You want to find the change of

S2: the change of the area

I: And this is

S2: This is just the area

I: So how do you go from one to the other?

Time

I: If I take the derivative of something what does that give me.

S2: Yeah, that gives you the rate.

I: Rate of what. Let's say I have y equals three x, and I find the derivative, what does that mean? What does it mean y prime equals three?

S2: umm, is the rate of change of what.

I: Of the y… with respect to…what

S2: For this it would be time. *Referring to area equation.*

I: Does that make sense to you?

S2: It makes sense now yeah.

I: Now? By what um…what was the problem would you say?

S2: Ummmm, forgetting that the derivative is the rate.

I: so when you saw rate you didn't automatically think derivative

S2: no

I: Ok, fair enough

S2: Should I use the quotient rule for um taking the derivative of this?

I: Well it's just divided by two. That's like multiplying by half in front…That make sense to you?

S2: I don't intuitively understand it.

I: ok

S2: But it makes sense according to what I've learned.

I: So what does this say to you? So if I want the rate at this point, what point would that be?

S2: What point is it at?

I: It says, "what rate is the area triangle changing at this moment." What moment? What does that mean "at this moment"?

S2: It doesn't specify what moment, or where it's moving at.

I: It's still part of the same uh

S2: At the instant where the distance is eight. The base.

I: If this is eight, what does that mean about time?

S2: If this is at eight metres what does that mean about the time?

I: Yeah, the time we are at according to this time we set up here. If the base is eight then the time is what?

Pause

I: You told me before that when the base is eleven the time is at one second, and at minus one second it’s at five, so what is the time.

S2: zero?

I: So that’s the instant right?

S2: Yeah

I: So what the rate of the area changing at that time?

S2: At minus 3 metres squared per second.

I: That makes sense to you?

S2: yeah.

I: OK.

Question 1d

I: What about D).

S2: This is the problem I was having with the take home question as well. I couldn’t figure out the relationship of the angle to the time.

I: Well what is it asking for? Which angle?

S2: The one on the ground. *(The right angle)*

I: It’s between the ladder and the ground. That’s a typo.

S2: OK

I: I think we want the ladder and the ground. I’m pretty sure the rate this angle *(right angle)* is changing is zero.

S2: yeah.

I: Unless the wall is breaking. So which angle are you looking for?

S2: This one right here. *(Draws angle)*

I: So what's the problem? You don't understand the trig relationships?

S2: um, not that well. Like I know that both these angles have to add up to the same amount.

I: If you have this angle here, call this *(triangle on blackboard)* ABC then you have sine of this angle is equal to C/A

S2: wouldn't it be A/C

I: Um, yeah you're right. We also have cos. I mean those are the only relationships, I mean we also have tangent. I mean you know that these angles add up to one eighty.

S2: yeah

I: Maybe these will help you here.

Pause

I: So what do you think?

S2: I haven't figured out a way to determine the rate yet.

I: How did you find the rate in the one before?

S2: Based on the Area?

I: We found the rate how?

S2: We used the Pythagoras equation.

Pause

S2: So the rate that this angle is changing would be this diminishing by three metres per second, divided by ten.

I: The only way you can relate those sides to that angle… What you're talking about id as if the ladder was doing some sort of circular…and you knew that distance.

Pause.

I: What do you think you would have to do to solve this problem?

S2: um

I: perhaps you don't know how to do it

S2: Find some kind of relationship between this angle and that angle?

I: OK, and then?

S2: uh.

I: why between those two angles?

S2: because they always have to equal a certain amount.

I: ok. That's good. Let's move on.

S2: what's the answer?

I: I'll tell you after. I'll try and show you how it all works out.

Problem 2

I: Here is a similar problem to the one we did. The thing is that we didn't talk about rates at all; we just talked about the physicality of the problem. So this is a similar problem, cone into cylinder. I can remind you of the formulas. Volume of a cone is one third pi r squared h and the volume of a cylinder is pi r squared h. A cone fits into a cylinder three times.

S2: you mean if they are the same radius.

I: Yeah, it's a cool proof. If I have time I'll show you.

S2: What was it though, what you said?

I: Oh, it's because you see the volume of a cone is one third…

S2: oh, ok.

Time

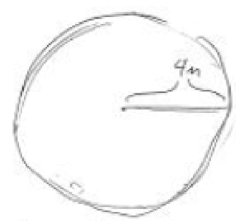
S2: So what is happening is the coffee is draining out.

Time

Appendix 3.A – Student's Group Work during Class Activities

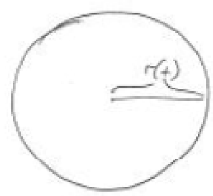
3.A.1 S1 and S2 Group Work.

1.Sketch a circle and label its radius 4m. Find the area of the circle.



$A = r^2\pi$
$= 4^2(\pi)$
$= 16\pi$

2. Sketch another circle and label the radius r(t).



3.Suppose the radius of the circle is expanding by the function r(t)=2+t in meters. Find the radius of the circle at t=1second. Find the area of the circle at this time.

$r(t) = 2+t$ $\quad A = r^2\pi$
$r(1) = 2+1$ $\quad = 3^2\pi$
$= 3$ $\quad = 9\pi$

4.Fill the following table using the above information:

Time	Radius r(t)	Area
1	3	9π
2	4	16π
3	5	25π
3	5	25π
8	9	81π

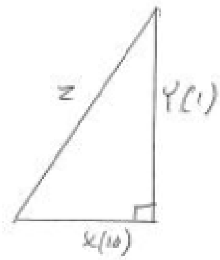
5.Find a function of the area in terms of the radius r(t)

A(r(t))= $(r(t))^2\pi$

6.Using the function given above r(t)=2+t, find a function for the area in terms of t

A(t)= $(2+t)^2\pi$

1.Draw a right angle triangle and label the sides x, y and z, with z being the hypotenuse.

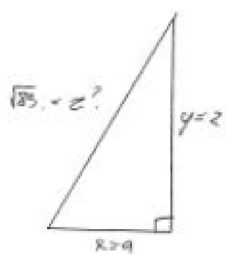


2.Suppose the value of x was 10 and the value of y was 1, find the length of the value z. Place this value in table labeling this triangle as triangle #1.

$10^2 + 1^2 = \sqrt{\ }$ $\qquad 100 + 1 = c^2$

$\sqrt{101} = c$

3.Now Draw another right angle triangle and again label the sides x, y, and z. Now suppose that this triangle, label it triangle #2, had x=9 and y=2. Find z for this triangle.



$a^2 + b^2 = c^2$

$9^2 + 2^2 = c^2$

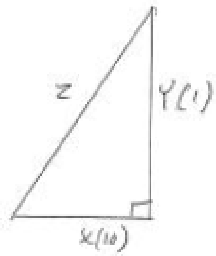
$81 + 4 = 35^2$

$\sqrt{85} = z$

4.Place the information you have found into the following table, then complete the table.

triangle (t)	X	Y	Z
#1	10	1	$\sqrt{101}$
#2	9	2	$\sqrt{35}$.
#3	8	3	$\sqrt{73}$.
#4	7	4	$\sqrt{65}$.

1.Draw a right angle triangle and label the sides x, y and z, with z being the hypotenuse.

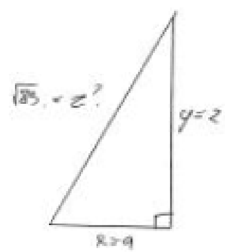


2.Suppose the value of x was 10 and the value of y was 1, find the length of the value z. Place this value in table labeling this triangle as triangle #1.

$10^2+1^2=\sqrt{81}$ $\quad 100+1=c^2$

$\cancel{\quad}=\sqrt{100}$ $\quad \sqrt{101}=c$

3.Now Draw another right angle triangle and again label the sides x, y, and z. Now suppose that this triangle, label it triangle #2, had x=9 and y=2. Find z for this triangle.



$a^2+b^2=c^2$

$9^2+2^2=c^2$

$81+4=85^c$

$\sqrt{85}=z$

4.Place the information you have found into the following table, then complete the table.

triangle (t)	X	Y	Z
#1	10	1	$\sqrt{101}$
#2	9	2	$\sqrt{85}$.
#3	8	3	$\sqrt{73}$.
#4	7	4	$\sqrt{65}$.

5.Can you find the value of z for the 8th triangle keeping with this pattern?

Yes. $\sqrt{73}$

6. Verify that the functions x(t)=11-t and y(t)=t output the length values of x and y for a specific triangle t.

$x(t) = 11 - t$ $\quad$ $y(t) = t$

$x(2) = 11 - 2$ $\quad$ $y(2) = 2$.

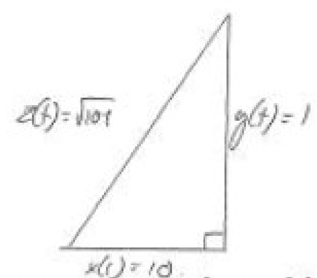
$= 9$.

∴ The functions are correct.

7.Use the functions x(t)=11-t and y(t)=t to construct a function z(t).

z(t)= $(x(t))^2 + (y(t))^2$ $\cancel{=}$ $\sqrt{(11-t)^2 + (t)^2} = z(t)$.

8.Draw a triangle labeling the sides x(t), y(t), and z(t), with t now being time starting at t=1 second.



$z(t) = \sqrt{(11-t)^2 + (t)^2}$

$z(1) = \sqrt{(11-1)^2 + (1)^2}$

$= \sqrt{100 + 1}$

$= \sqrt{101}$

$y(t) = t^2$

$y(1) = 1$

$x(t) = 11 - t$

$x(1) = 10$

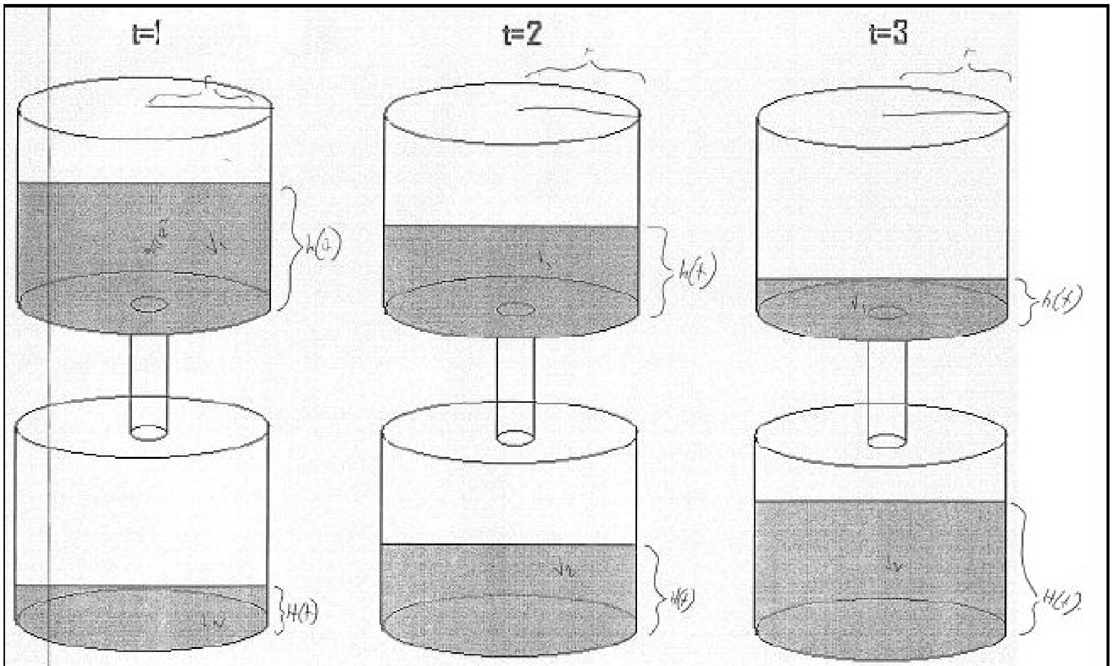
9.What can you say about this triangle in relation to triangles #1, #2, #3, and #4 from the table on the left?

∴ The triangles are product of the same function.

. As (t) increases, z(t) decreases

x(t) decreases

y(t) increases.



1.Consider the diagram above:
A) Label the radius of the tanks r.
B) Label the height of the water in the top tanks h(t), and the height of the water in the bottom tanks H(t)

2. Using the table below, fill in the missing values with V_1 being the volume of water in the top tank, and V_2 being the volume of water in the bottom tank.

Time	Volume in top cylinder V_1	Height of water in top tank h(t)	Volume in ~~top~~ bottom cylinder V_2	Height of water in bottom tank H(t)
1	$8\pi r^2$	8.	$1\pi r^2$	1
2	$7\pi r^2$	7	$2\pi r^2$	2
3	$6\pi r^2$	6.	$3\pi r^2$.	3

$(9-t)\pi r^2$

$h(t)+H(t)=9$

4. Create a function of time for the height of the water in the top tank and the bottom tank

h(t)= $(9-t)$

H(t)= t

5. Create a function that relates the volume of the second tank to that of the first tank

V_2= $9\pi r^2 - V_1$

6. Create a function relating the height of the water in the first tank to it's ~~height~~ Volume

V_1(h(t))= $V_1(8) = \pi r^2 \cdot 8$ $\quad h(t)\cdot\pi r^2 = V_1(h(t))$

7. Do the same for the second tank

V_2(H(t))= $H(t)\cdot\pi r^2$

8. Find a relationship between the height of the first tank and that of the second.

$h(t)$ $\quad h_1(H(t))$

h(H(t))= $9-t+9-1=8$

$h(t)+H(t)=9$

$h(t)=9-H(t)$

$h(H(t))=9-H(t)$

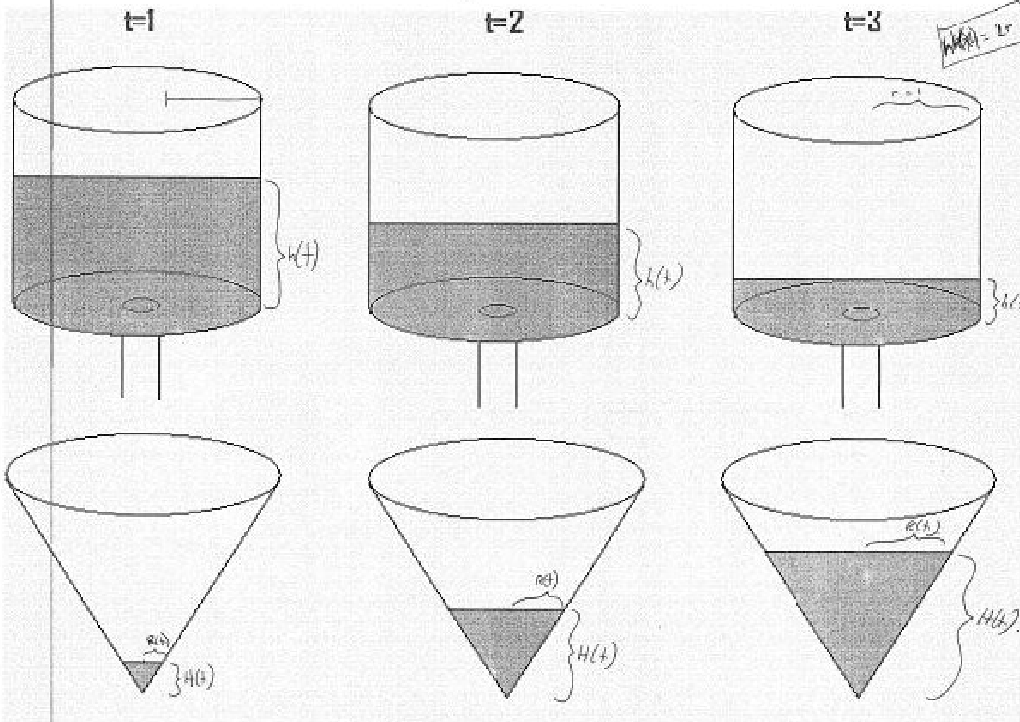
9. Using the Functions h(t) and H(t), find a function of time for V_1 and V_2

V_1(t)= $(9-t)\pi r^2$

V_2(t)= $(t)\pi r^2$

$H(t)+h(t)=$

Consider the following setup, a tank with a circular radius empties into a cone with a circular base with the same radius. The height of the cone is twice the radius of the cone.



$V = \frac{1}{3}\pi r^2 h$

-Label the height of the water in the cylindrical tanks, h(t)

-Label the height of the water in the water in the cones H(t)

-Label the radius of the water in the cone R(t)

-Give a relationship between the radius, R(t), and the height of the water, H(t) in the cone.

$2(R(t)) = H(t)$ $\frac{H(t)}{2} = R(t)$. $8\pi r^2$

Explain this relationship geometrically.

The triangle is proportional.

Find a function for the volume of the water in the cone as a function of H(t) and R(t), only R(t), and only H(t)

V(H(t),R(t))= $\frac{1}{3}\pi(R(t))^2(H(t))$

V(H(t))= $\cancel{\frac{1}{3}\pi(\ldots)(\ldots)}$ $\cancel{\frac{1}{3}\pi(2R(t))(R(t))}$ $\frac{1}{3}\pi\left(\frac{H(t)}{2}\right)^2(H(t)) \rightarrow \frac{\pi}{12}(H(t))^3$

V(R(t))= $\frac{1}{3}\pi(R(t))^2(2R(t)) \rightarrow \frac{2}{3}\pi(R(t))^3$

-Is there a relationship between the radius of the cylindrical tank, r, and the height of the water possible? Why or Why not?

No, the radius is constant while ~~and~~ height changes.
∴ No relationship.

-Suppose the radius of the tank is 1. Fill in the following table:

Time	Volume in Cylinder	h(t) πr^2	Volume in Cone	H(t)	R(t)
1	8π	8	1π	$\sqrt[3]{12}$	$\sqrt[3]{\frac{3}{2}}$
2	7π	7	2π		
3	6π	6	3π		

Using the data from the table, verify that 2R(t)=H(t).

-Can you create a function to relate the h(t) with H(t)? What about h(t) with R(t)?

$(\pi(1)^2 h(t)) + \left(\frac{1}{12}\pi H(t)^3\right) = 9\pi$

$= 9\pi - \frac{1}{12}\pi H(t)^3$

$h(t) = 9 - \frac{1}{12}H(t)^3$

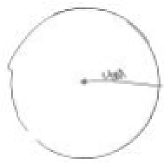
$\pi h(t) + \frac{2}{3}\pi R(t)^3 = 9\pi$

$\pi h(t) = \frac{9\pi - \frac{2}{3}\pi R(t)^3}{\pi}$

$h(t) = 9 - \frac{2}{3}R(t)^3$

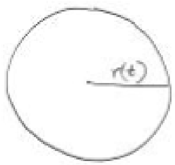
3.A.2 S3 and S4 Group Work

1.Sketch a circle and label its radius 4m. Find the area of the circle.



$A=\pi R^2$
$A=\pi(4)^2$
$A=50.26$

2. Sketch another circle and label the radius r(t).



3.Suppose the radius of the circle is expanding by the function r(t)=2+t in meters. Find the radius of the circle at t=1second. Find the area of the circle at this time.

$r(t)=2+t$
$=2+1$
$r(1)=3$

4.Fill the following table using the above information:

Time	Radius r(t)	Area
1	3	$A=\pi(9)$ 28.27 m^2
2	4	$A=\pi(16)$ 50.26 m^2
3	5	$A=\pi(25)$ 78.54 m^2
3	5	$A=\pi(25)=78.54$ m^2
7	9	81π

5.Find a function of the area as a function of the radius r(t).

$A(R)=\pi R^2$ $R(t)=2+t$ $\therefore A(r(t))$ $\pi(R(t))^2$

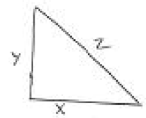
A(r(t))= $\pi(R(t))^2$

6.Using the function given above, r(t)=2+t, find a function for the area as a function of t.

$r(t)=2+t$ $\pi R(t)^2$
$A(R)=\pi R^2$

A(t)=
$A(t)=\pi R^2$
$R=2+t$
$A(t)=\pi(2+t)^2$
$\boxed{A(t)=\pi(2+t)^2}$

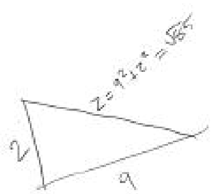
1.Draw a right angle triangle and label the sides x, y and z, with z being the hypotenuse.



2.Suppose the value of x was 10 and the value of y was 1, find the length of the value z. Place this value in table labeling this triangle as triangle #1.

$z=\sqrt{101}$

3.Now Draw another right angle triangle and again label the sides x, y, and z. Now suppose that this triangle, label it triangle #2, had x=9 and y=2. Find z for this triangle.



4.Place the information you have found into the following table, then complete the table.

triangle (t)	X	Y	Z	A
#1	10	1	$\sqrt{101}$	5
#2	9	2	$\sqrt{85}$	9
#3	8	3	$z=8^2+3^2=\sqrt{73}$	12
#4	7	4	$\sqrt{65}$	

5.Can you find the value of z for the 8th triangle keeping with this pattern?

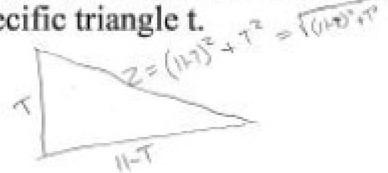
5-6
6-5
7-4
8-3

x=3
y=8

$3^2+8^2=\sqrt{73}$

6. Verify that the functions x(t)=11-t and y(t)=t output the length values of x and y for a specific triangle t.

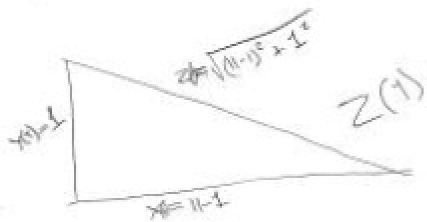
X(T)=11-T
Y(T)=T



7.Use the functions x(t)=11-t and y(t)=t to construct a function z(t).

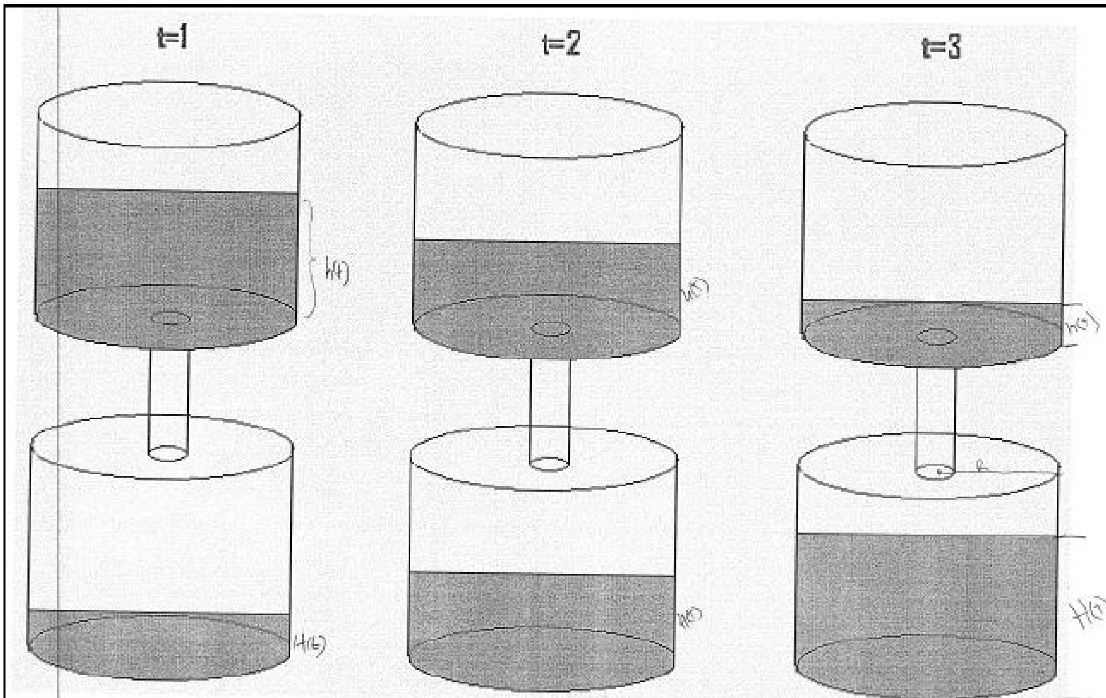
z(t)=$\sqrt{(11-T)^2+T^2}$

8.Draw a triangle labeling the sides x(t), y(t), and z(t), with t now being time starting at t=1 second.



9.What can you say about this triangle in relation to triangles #1, #2, #3, and #4 from the table on the left?

the triangles #1,2,34
Are the Same triangle
with the base Shrinking by the rate
of 1 unit/sec



1. Consider the diagram above:

A) Label the radius of the tanks r.

B) Label the height of the water in the top tanks h(t), and the height of the water in the bottom tanks H(t)

2. Using the table below, fill in the missing values with V_1 being the volume of water in the top tank, and V_2 being the volume of water in the bottom tank.

Time	Volume in top cylinder V_1	Height of water in top tank h(t)	Volume in ~~top~~ Bottom cylinder V_2	Height of water in bottom tank H(t)
1	$8\pi r^2$	8	$1\pi r^2$	1
2	$7\pi r^2$	7	$2\pi r^2$	2
3	$6\pi r^2$	6	$3\pi r^2$	3

4. Create a function of time for the height of the water in the top tank and the bottom tank

h(t)= (9 − T)

H(t)= $\boxed{T}$

5. Create a function that relates the volume of the second tank to that of the first tank

V_2= 9πR² − (9−TπR²) T

6. Create a function relating the height of the water in the first tank to it's volume.

V_1(h(t))= ((9−T)πR²)

7. Do the same for the second tank

V_2(H(t))= (TπR²)

8. Find a relationship between the height of the first tank and that of the second.

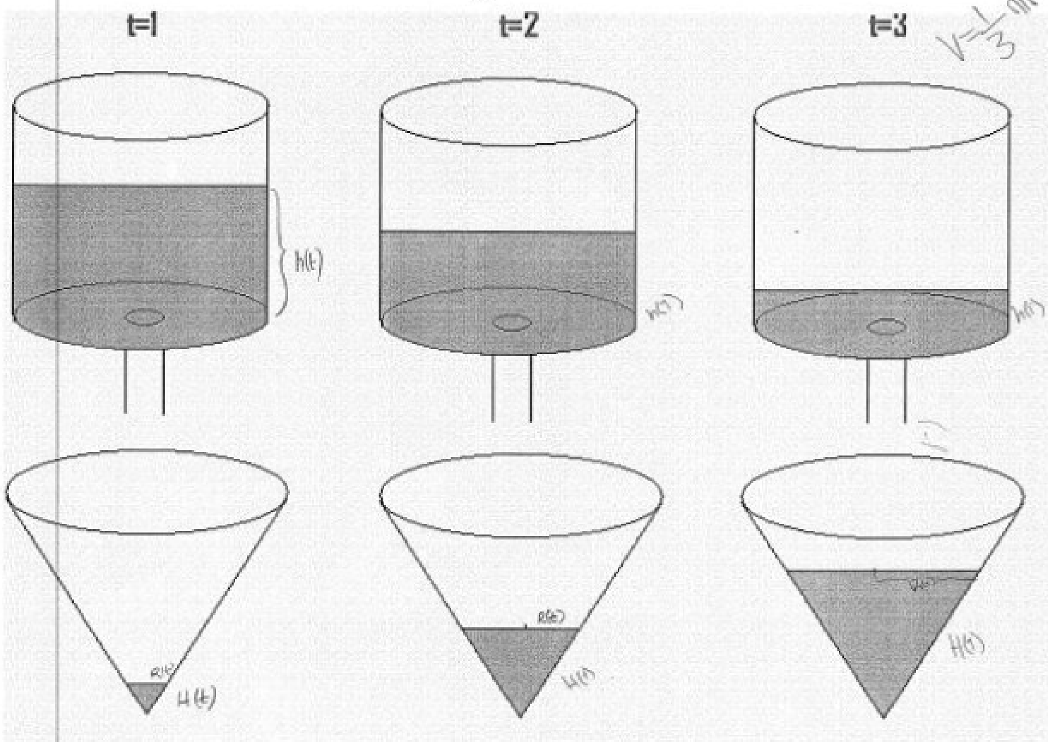
h(H(t))= 9−T   h(h(t))   h=9−H

9. Using the Functions h(t) and H(t), find a function of time for V_1 and V_2.

V_1(t)=

V_2(t)=

Consider the following setup, a tank with a circular radius empties into a cone with a circular base with the same radius. The height of the cone is twice the radius of the cone.



-Label the height of the water in the cylindrical tanks, h(t)

-Label the height of the water in the water in the cones H(t)

-Label the radius of the water in the cone R(t)

$V = \frac{1}{3}\pi r^2 h$

-Give a relationship between the radius, R(t), and the height of the water, H(t) in the cone.

h

$8\pi = h\pi r^2$

$2R(t) = H$

.Explain this relationship geometrically.

$2\pi =$

$= \frac{1}{3}\pi h$

Find a function for the volume of the water in the cone as a function of H(t) and R(t), only R(t), and only H(t)

V(H(t),R(t))= $V=\frac{1}{3}\pi(R(t))^2(H(t))$

V(H(t))= $\frac{1}{3}\pi(\frac{1}{2}H_1)^2(H_1)$

V(R(t))= $\frac{1}{3}\pi(R_1)^2(2R_1)$

-Is there a relationship between the radius of the cylindrical tank, r, and the height of the water possible? Why or Why not?

-Suppose the radius of the tank is 1. Fill in the following table:

Time	Volume in Cylinder	h(t)	Volume in Cone	H(t)	R(t)
1	8π	$h(t)=\frac{V}{\pi r^2}$ $\frac{8\pi}{\pi 1}$ 8	1π	$H=\sqrt[3]{12}$	$R(t)=\frac{H(t)}{2}$
2	7π	7	2π		
3	6π	6	3π		

Using the data from the table, verify that 2R(t)=H(t).

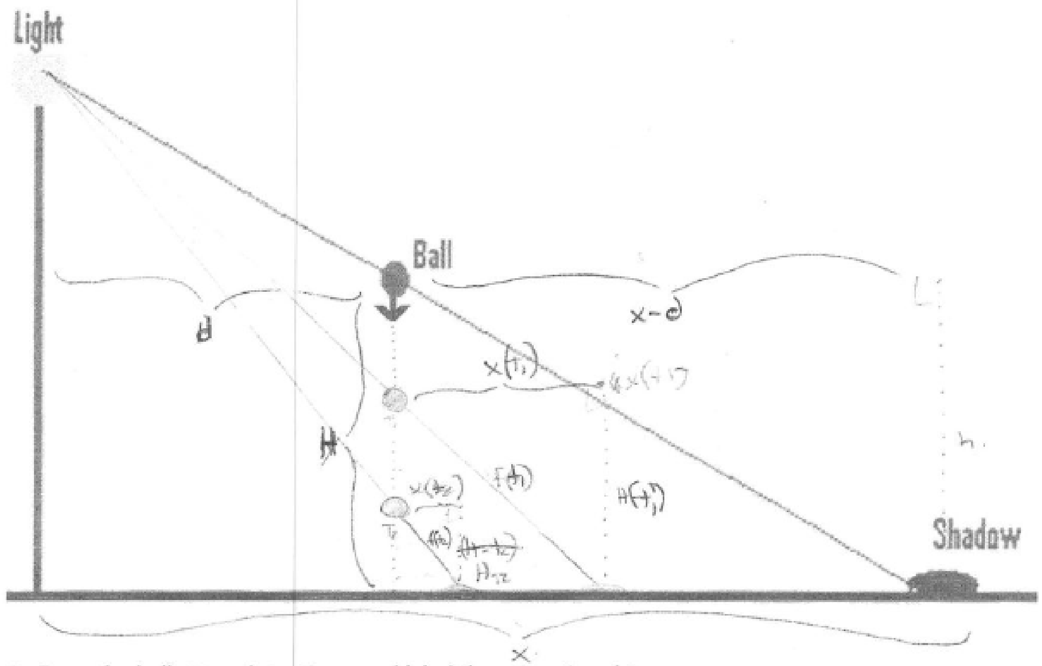
$H=\frac{3\pi}{\pi(\frac{H}{2})^2}$ $V=\frac{1}{3}\pi r^2 h$ $h=\frac{3V}{\pi r^2}$ $H=\frac{3\pi}{\pi R^2}$

-Can you create a function to relate the h(t) with H(t)? What about h(t) with R(t)?

Appendix 3.B – Student's Individual Work during Interview and Take Home Activity

3.B.1 S1 Work During Interview and Take Home assignment

A ball is dropped from a height h, a distance d away from a lamp post.



1 - Draw the ball at two later times and label then time 1 and 2.

2 - Trace a line and find the shadow of the ball at times 1 and 2.

3 - Label the distance of the shadow from the lamppost, the distance of the ball from the lamppost, the height of the ball, and the distance from the ball to its shadow, using functional notation where necessary. (ex: f(t))

4 –Using the above labeling, find a geometric relationship between the distance of the ball and the shadow, the shadow and lamppost and the height of the ball.

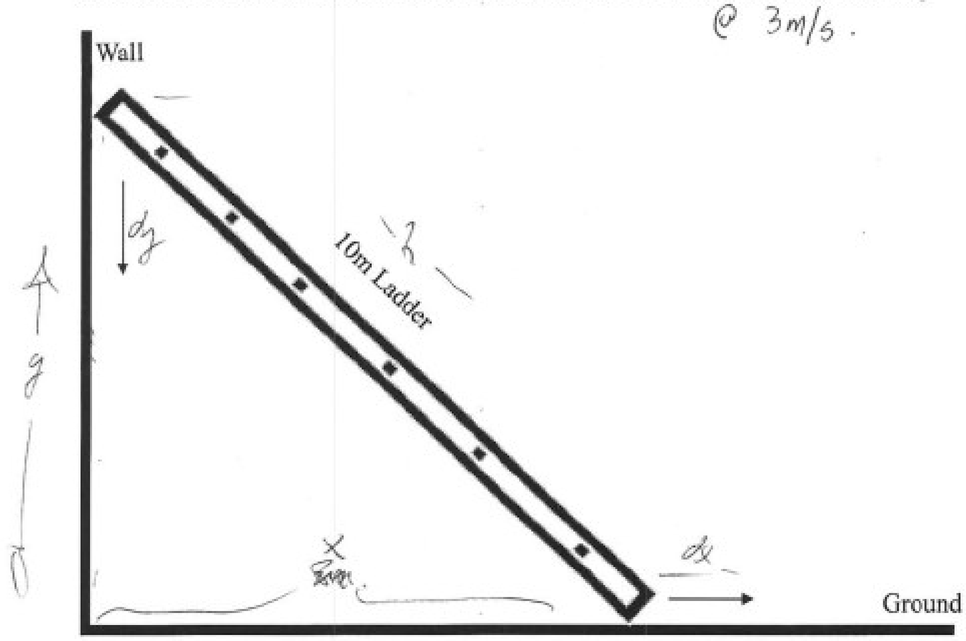
a) $x(t)^2 + h(t)^2 = f(t)^2 \rightarrow \sqrt{x(t)^2 + h(t)^2} = f(t)$

b)

5 - Summarize, using as few sentences and in your own words, what is happening in this problem.

As the ball drops, the distance between the shadow and the lamppost decreases proportionally to the height of the ball.

A 10m long ladder is sliding down a wall as seen in the following diagram. At the instant shown the bottom of the ladder is 8m from the vertical wall and is moving @ 3m/s.



A) Explain in your own words what is happening in this problem.

B) How fast is the top of the ladder moving down the wall at this point.

C) At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment.

D) At what rate is the angle between the wall and the ground changing.

A) As the ladder

B) $x^2+y^2=c^2$

$8^2+y^2=10^2$

$x-2=y$.

$\cancel{S=\frac{\Delta y}{\Delta t}}$

$y^2=10^2-8^2$

$=100-64$

$y=6=36$

$y=\sqrt{10^2+x^2}$

$3=\sqrt{10^2+x^2}$

$d=(10^2+x^2)^{1/2}$

$=\frac{1}{2}(10^2+x^2)^{-1/2}\cdot(2x)$ $\quad 3=\frac{\Delta x}{\Delta t}$

$=\frac{\Delta y}{\Delta t}$

$=\frac{dy}{dt}$

$f(x)=\sqrt{10^2+x^2}$ =

$\frac{dy}{dx}=$

d

$\frac{1}{2}(100+x^2)^{-1/2}\cdot 2x$

$\frac{dy}{dx}=\frac{x}{\sqrt{100+x^2}}$

$dy=\left(\frac{x}{\sqrt{100+x^2}}\right)(dx)$

$\frac{dy}{dt}=\left(\frac{x}{\sqrt{100+x^2}}\right)\left(\frac{dx}{dt}\right)$

$=\left(\frac{x}{\sqrt{100+x^2}}\right)(3\ m/s)$

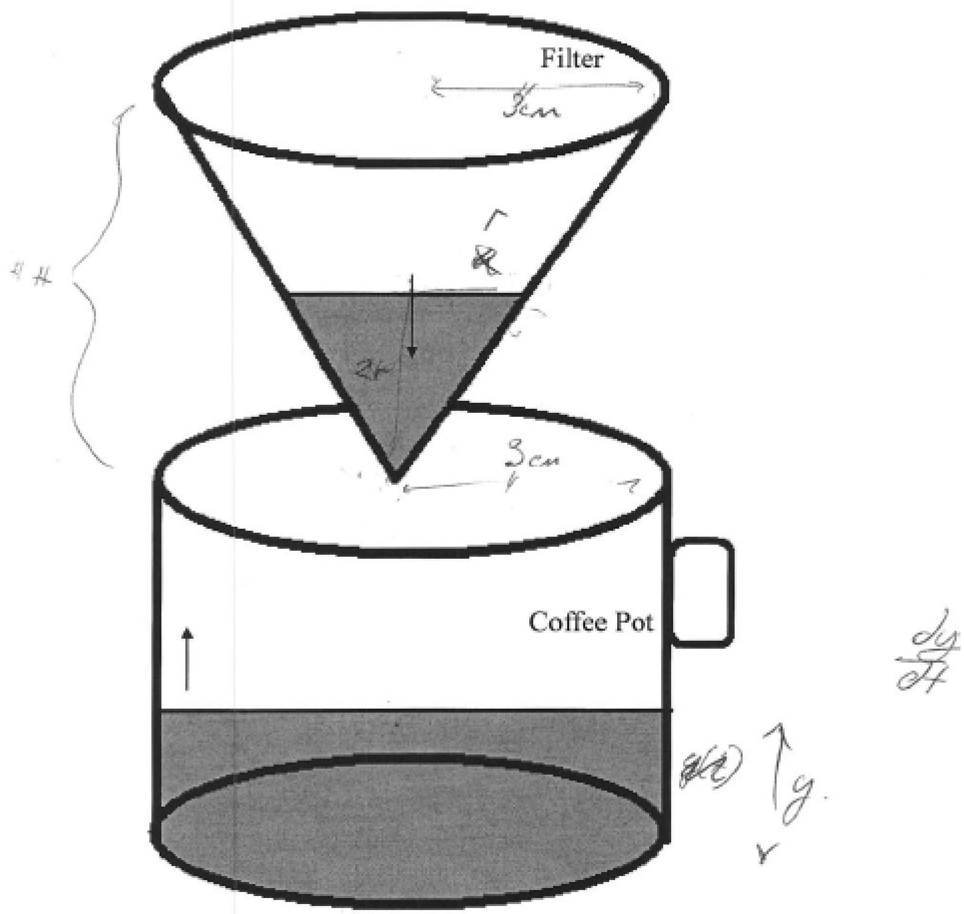
$=\left(\frac{8}{\sqrt{100+8^2}}\right)(3\ m/s)$

$=\frac{24}{\sqrt{164}}$

$\frac{dy}{dt}=\frac{24}{}$

$3x^2$

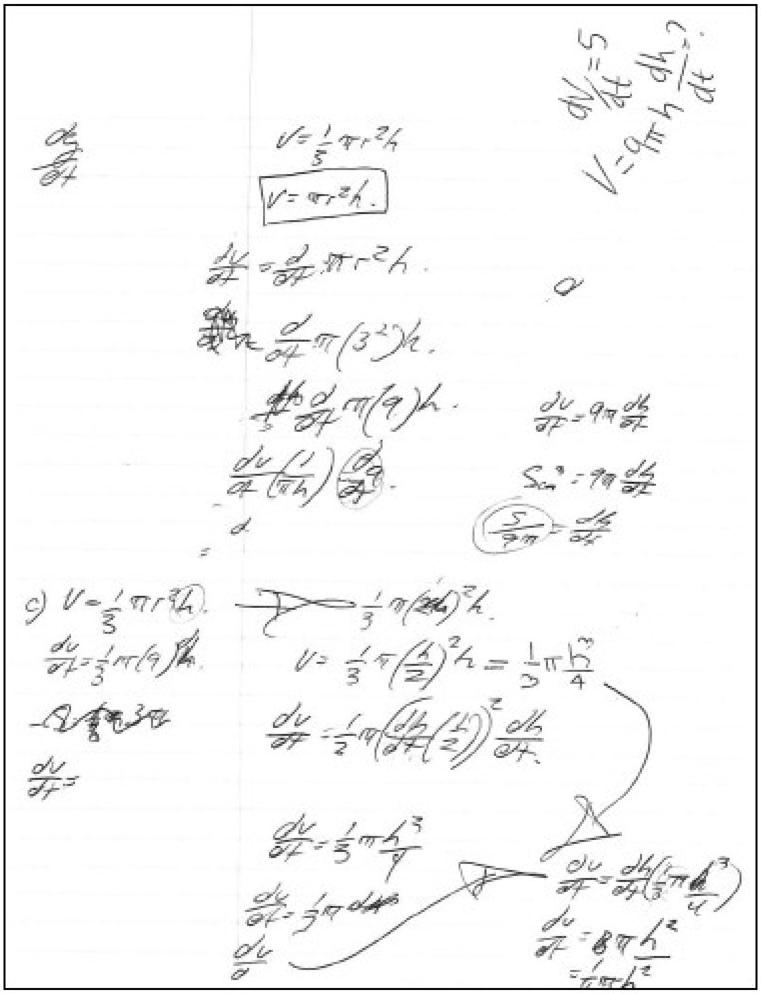
A coffee percolator pours coffee into a cylindrical coffee pot from a conical filter. The coffee machine makes coffee at a rate of $5cm^3/min$. The filter and the pot both have a radius of 3cm. The filter's height is twice it's radius.



A) Describe in your own words what is happening in this problem.

B) How fast is the ~~level~~ height in the pot rising.

C) How fast is the ~~level~~ height in the filter decreasing?



$\frac{dV}{dt}=5$

$V=9\pi h \frac{dh}{dt}=?$

$V=\frac{1}{3}\pi r^2 h$

$\boxed{V=\pi r^2 h.}$

$\frac{dv}{dt}=\frac{d}{dt}\pi r^2 h.$

$=\frac{d}{dt}\pi(3^2)h.$

$=\frac{d}{dt}\pi(9)h.$

$\frac{dv}{dt}\left(\frac{1}{\pi h}\right)\frac{d}{dt}$

$\frac{dv}{dt}=9\pi\frac{dh}{dt}$

$5\,cm^3=9\pi\frac{dh}{dt}$

$\frac{5}{9\pi}=\frac{dh}{dt}$

c) $V=\frac{1}{3}\pi r^2 h.$ $\frac{1}{3}\pi(\frac{h}{2})^2 h.$

$\frac{dv}{dt}=\frac{1}{3}\pi(9)h.$

$V=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h=\frac{1}{3}\pi\frac{h^3}{4}$

$\frac{dv}{dt}=\frac{1}{2}\pi\left(\frac{dh}{dt}\left(\frac{h}{2}\right)\right)^2\frac{dh}{dt}.$

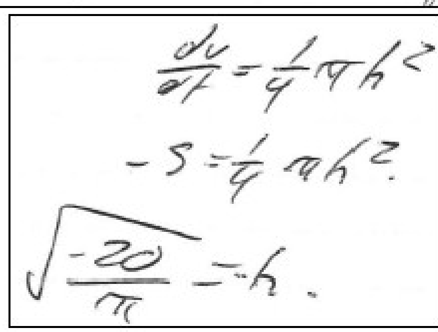
$\frac{dv}{dt}=$

$\frac{dv}{dt}=\frac{1}{3}\pi\frac{h^3}{4}$

$\frac{dv}{dt}=\frac{dh}{dt}\left(\frac{1}{3}\pi\frac{h^3}{4}\right)$

$\frac{dv}{dt}=8\pi h^2$

$=\frac{1}{4}\pi h^2$



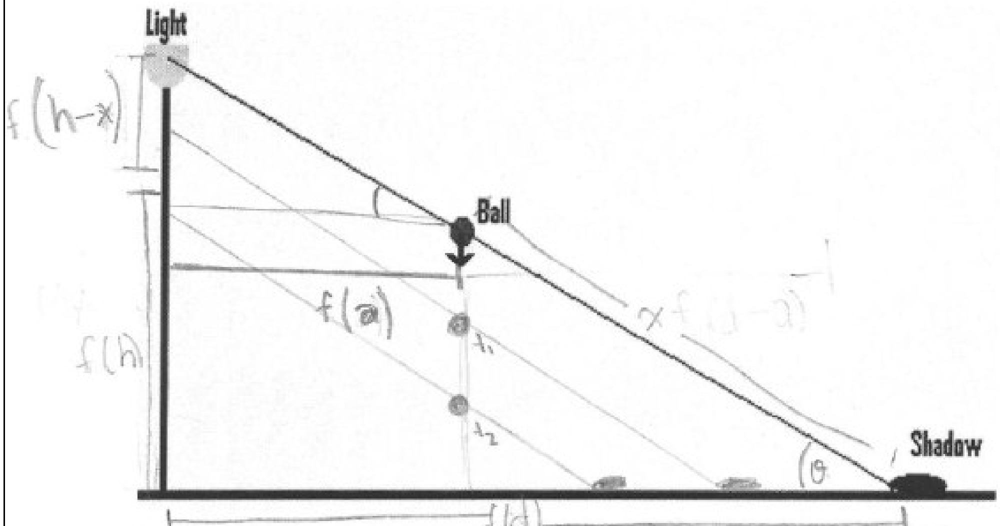
$$\frac{dv}{dt}=\frac{1}{4}\pi h^2$$

$$-5=\frac{1}{4}\pi h^2.$$

$$\sqrt{\frac{-20}{\pi}}=h.$$

3.B.2 S2 Work During Interview and Take Home assignment

A ball is dropped from a height h, a distance d away from a lamp post.



1 - Draw the ball at two later times and label then time 1 and 2. f(d-a)

2 - Trace a line and find the shadow of the ball at times 1 and 2.

3 - Label the distance of the shadow from the lamppost, the distance of the ball from the lamppost, the height of the ball, and the distance from the ball to its shadow, using functional notation where necessary. (ex: f(t))

4 –Using the above labeling, find a geometric relationship between ① the distance of the ball and the shadow, ② the shadow and lamppost and the height of the ball.

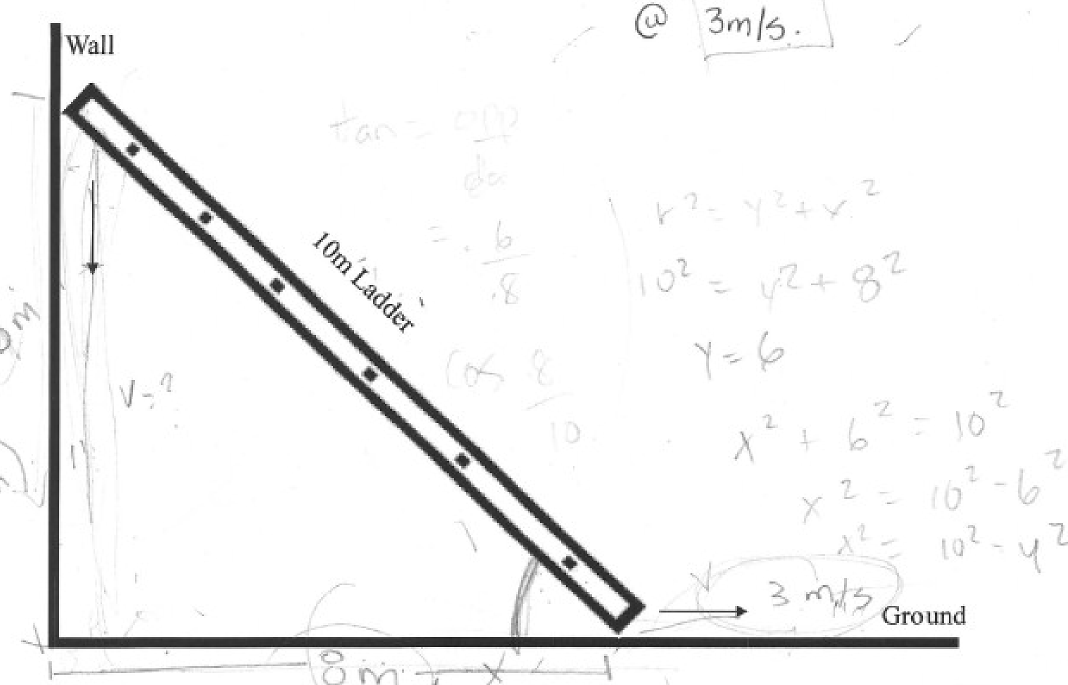
② $f(x) = f\left(\frac{h-x}{f(d)}\right) - f(h)$; ① $f(x) = \frac{f(h)}{f(d-a)}$

5 - Summarize, using as few sentences and in your own words, what is happening in this problem.

With time as a rate of change, the shadow of the ball approaches the lamppost.

A 10m long ladder is sliding down a wall as seen in the following diagram. At the instant shown the bottom of the ladder is 8m from the vertical wall and is moving

@ 3m/s.



A) Explain in your own words what is happening in this problem.

As the top ladder is sliding downWARD, there is a velocity along the x axis.

B) How fast is the top of the ladder moving down the wall at this point.

$V = \frac{\Delta x}{t}$ $\quad \frac{dx}{dt}$ $\quad \frac{dx}{dt}\ 6m =$

C) At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment.

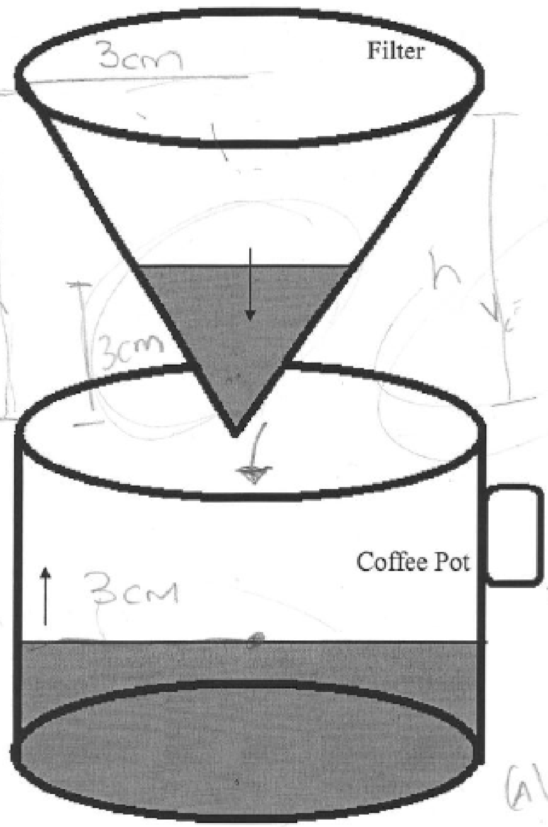
$A = \frac{1}{2}hw$ $\quad \frac{1}{2} x y$ $\quad \frac{1}{2}\ 1\left(\frac{dy}{dx}\right) + 1\frac{dx}{dx} = \frac{1}{2}\frac{dy}{dx} + \frac{1}{2}$

D) At what rate is the angle between the wall and the ~~ground~~ ladder changing.

$y' = \frac{1}{2}\frac{dy}{dx} + \frac{1}{2}$

$\frac{d\theta}{dt} = \Delta y$

A coffee percolator pours coffee into a cylindrical coffee pot from a conical filter. The coffee machine makes coffee at a rate of $5cm^3/min$. The filter and the pot both have a radius of 3cm. The filter's height is twice it's radius.



$V = 5cm^3/min$

$h = 2r$

$V_c = \frac{1}{3}\pi r^2 h$

$V_{cyl} = \pi r^2 h$

(A) coffee pot volume is increasing.

A) Describe in your own words what is happening in this problem.

B) How fast is the height in the pot rising.

C) How fast is the height in the filter decreasing when it 3cm high?

(B)

$V = \frac{1}{3}\pi r^2 h$

$\frac{dv}{dt} = 5cm^3/min$

$V_{cyl} = \pi r^2 h$

$V = \pi\, 3^2 h$

$\frac{dv}{dt} = \pi\, 3^2 \frac{dh}{dt}$

$5cm^3/min = \pi \cdot 3^2 \cdot \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{5cm^3/min}{\pi \cdot 3^2}$

u v

(C) $V = \frac{1}{3}\pi r^2 h$

$uv' + u'v$

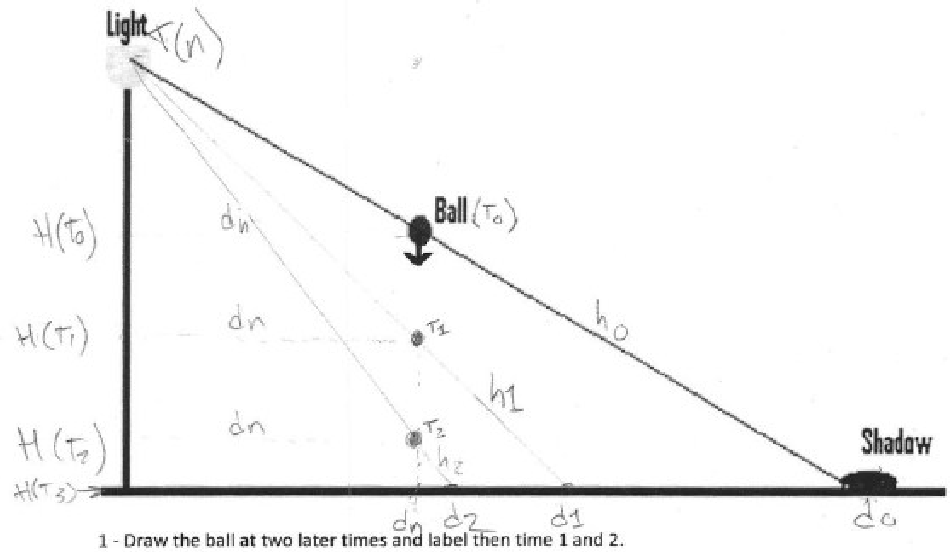
$\frac{dv}{dt} = \frac{1}{3}\pi \cdot r^2 \cdot 1 + 2r \cdot \frac{dh}{dt}$

r

$= \frac{1}{3}$

3.B.3 S3 Work During Interview and Take Home assignment

A ball is dropped from a height h, a distance d away from a lamp post.



1 - Draw the ball at two later times and label then time 1 and 2.

2 - Trace a line and find the shadow of the ball at times 1 and 2.

3 - Label the distance of the shadow from the lamppost (T_0), the distance of the ball from (T_n) the lamppost, the height of the ball, and the distance from the ball to its shadow, using functional notation where necessary. (ex: f(t))

4 –Using the above labeling, find a geometric relationship between the distance of the ball and the shadow, the shadow and lamppost and the height of the ball.

$h_1 = \sqrt{H(T_1)^2 + (d_1 - d_n)^2}$

$h_2 = \sqrt{(H(T_2))^2 + (d_2 - d_n)^2}$

$\boxed{\lim H = 0 \quad d_0 \to d_n}$

$\lim_{H \to 0} \Delta d = 0$

$\lim_{H \to 0} d_0 = d_n$

5 - Summarize, using as few sentences and in your own words, what is happening in this problem.

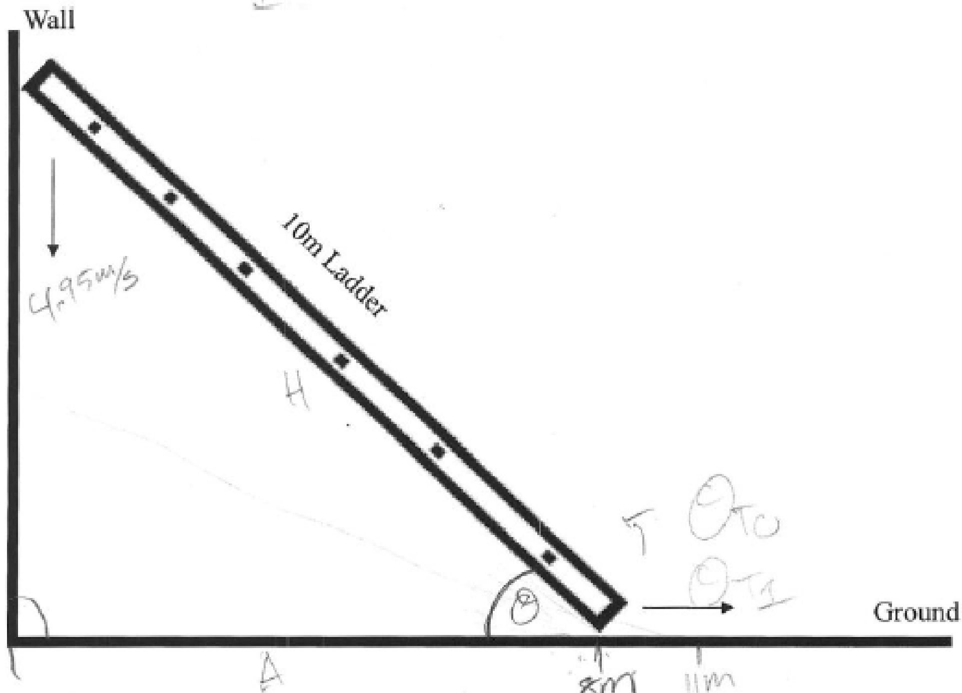
as the ball approaches $H=0$, the shadow approaches d_n

the length of h will always be proportionate to the Height, and distance to the shadow from d_n.

the shrinking rates are related.

the accute angles change at a rate which is related to the change in side lengths.

A 10m long ladder is sliding down a wall as seen in the following diagram. At the instant shown the bottom of the ladder is 8m from the vertical wall and is moving away from that wall at 3 m/s.



A) Explain in your own words what is happening in this problem.

the top of the Ladder is falling at a rate connected to the rate @ which the base is sliding away, the height it drops will be related to the dist it slides.

B) How fast is the top of the ladder moving down the wall at this point.

3 m/s

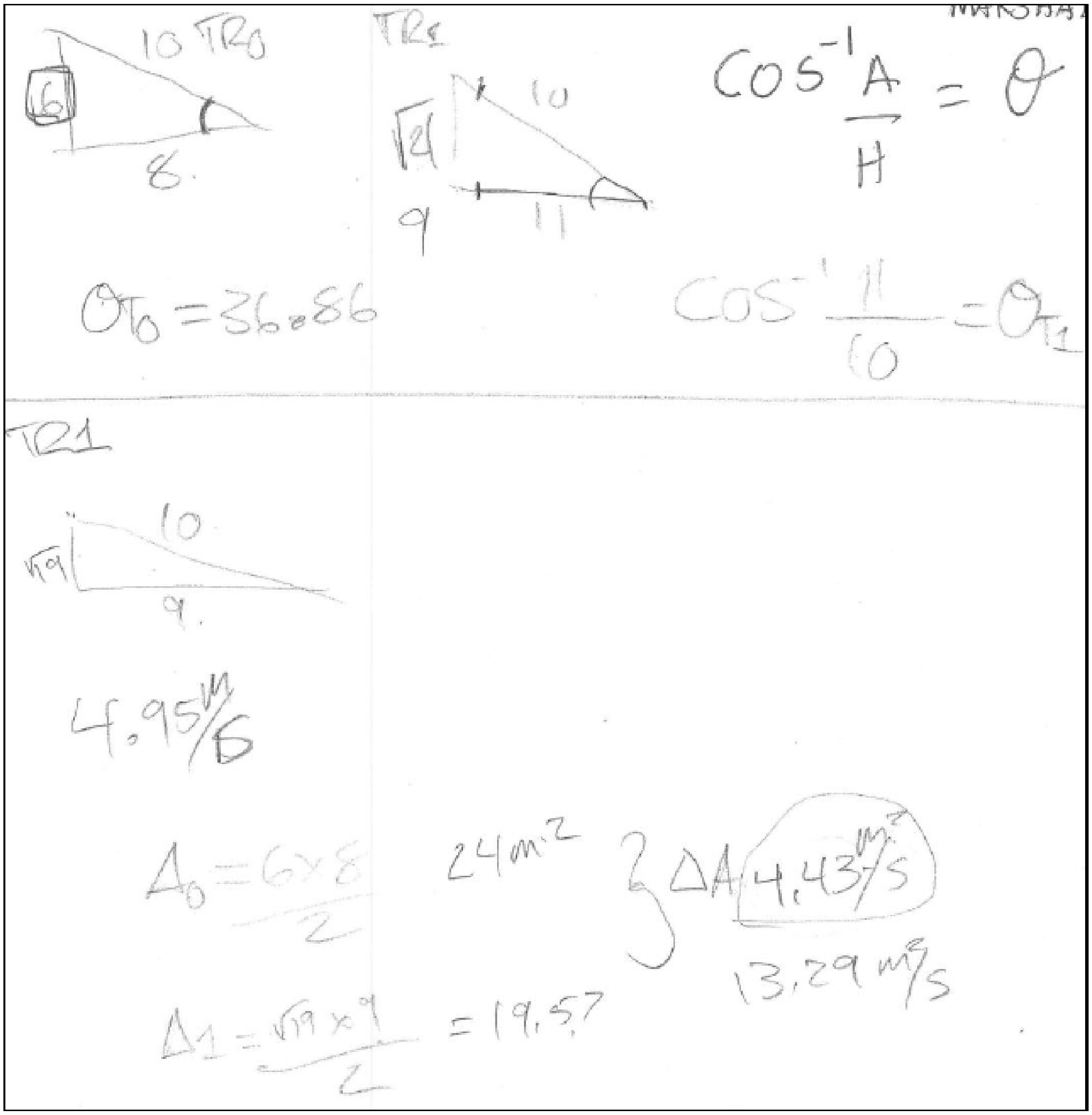
C) At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment.

3 m/s $\frac{3m^2}{2}/s$

D) At what rate is the angle between the ~~wall~~ ladder and the ~~ground~~ ground changing.

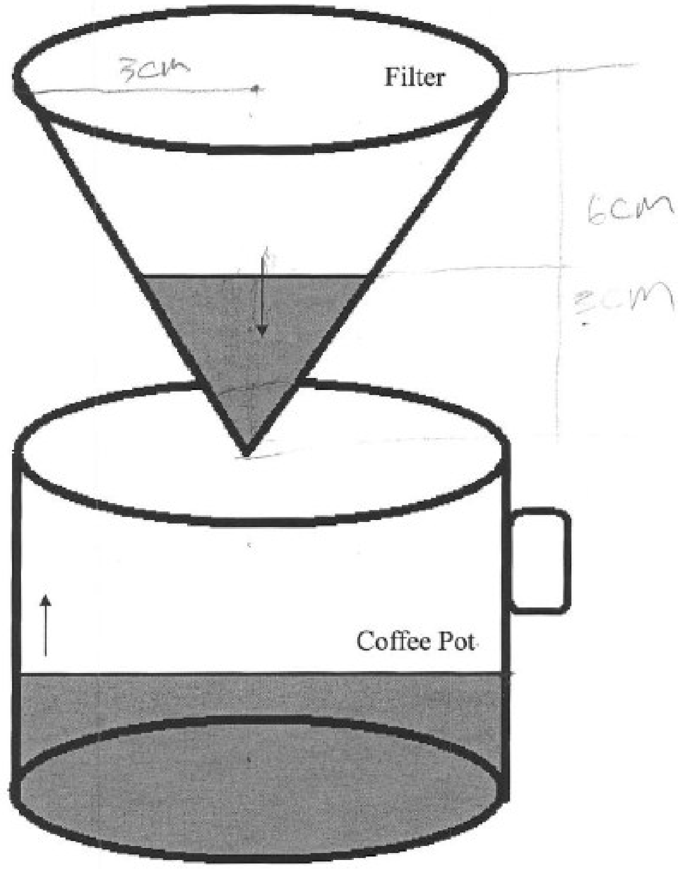
$\cos\theta = \frac{A}{H}$

$\cos\theta = \frac{A(t)}{H}$ $\cos\theta_0 = \frac{A_{T0}}{H}$ $\cos\theta_{T1} = \frac{A_{T1}}{H}$



10 TR0
6
8.
TR1
10
√19
11
9
COS⁻¹ A/H = θ
θT0 = 36.86
COS⁻¹ 11/10 = θT1
TR1
10
√19
9.
4.95 m/s
A0 = 6×8/2
24 m²
ΔA 4.43 m²/s
13.29 m²/s
Δ1 = √19 × 9/2 = 19.57

A coffee percolator pours coffee into a cylindrical coffee pot from a conical filter. The coffee machine makes coffee at a rate of $5cm^3/min$. The filter and the pot both have a radius of 3cm. The filter's height is twice it's radius.



A) Describe in your own words what is happening in this problem.

B) How fast is the height in the pot rising. 0.1768 cm/min (?)

C) How fast is the height in the filter decreasing when it is 3cm high?

T

$3cm - T_2$

$h_{cone} = \frac{V_1}{\frac{1}{3}\pi r^2}$ $h_2 = \frac{V_2}{\frac{1}{3}\pi r^2}$ $\lim_{h \to 3^+}$ $V \lim_{h \to 3^+} \frac{5cm^3}{\frac{1}{3}\pi r^2 h}$

h

A) Height in C.P. will grow at a rate proportionate to the shrink Vol of the machine.

$H(T_0) =$ $V_{cyl} = \pi r^2 \boxed{h}$

$H(T_1)$ $h = \frac{V}{\pi r^2}$

$h_1 = \frac{5cm^3 \; V(T_1)}{\pi r^2}$

$h_2 = \frac{T_2 \; V(T_2)}{3 \; \pi r^2}$

$h = \frac{5cm^3}{\pi r^2}$ $\frac{10cm^3}{\pi r^2}$ $\frac{15cm^3}{\pi r^2}$

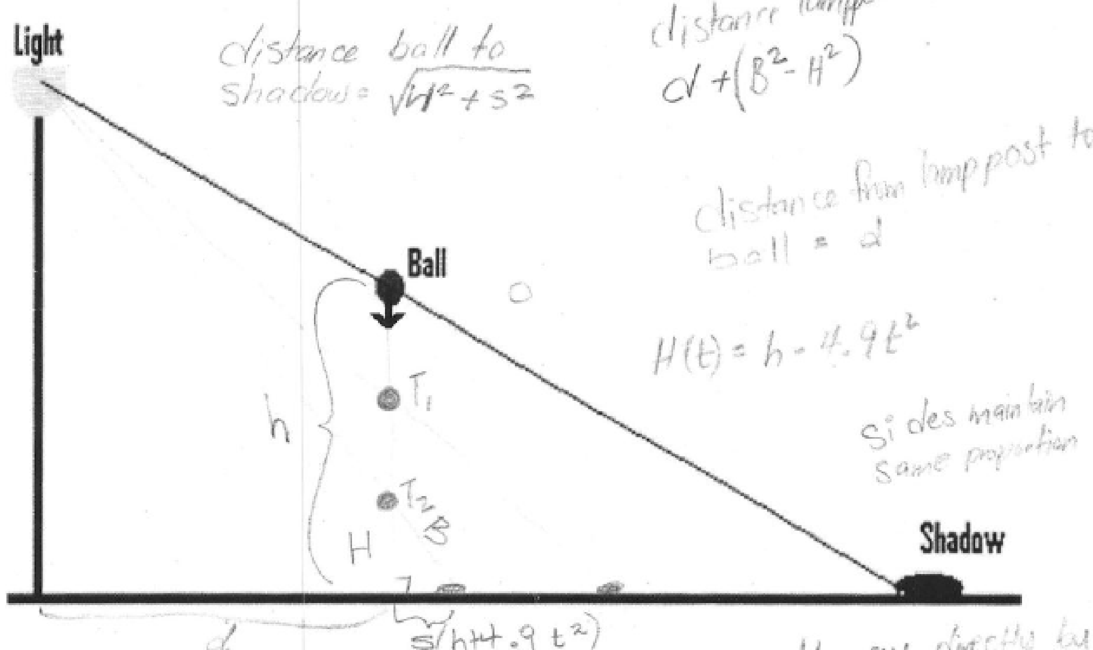
$\boxed{0.1768} \rightarrow \boxed{0.3536} \longrightarrow \boxed{0.53}$

cm/min →

3.B.3 S4 Work During Interview and Take Home assignment

A ball is dropped from a height h, a distance d away from a lamp post.

distance lamppost to shadow =



1 - Draw the ball at two later times and label then time 1 and 2.

both vary directly by $h - 4.9^2$?

2 - Trace a line and find the shadow of the ball at times 1 and 2.

3 - Label the distance of the shadow from the lamppost, the distance of the ball from the lamppost, the height of the ball, and the distance from the ball to its shadow using functional notation where necessary. (ex: f(t))

4 –Using the above labeling, find a geometric relationship between the distance of the ball and the shadow, the shadow and lamppost and the height of the ball. triangles similar

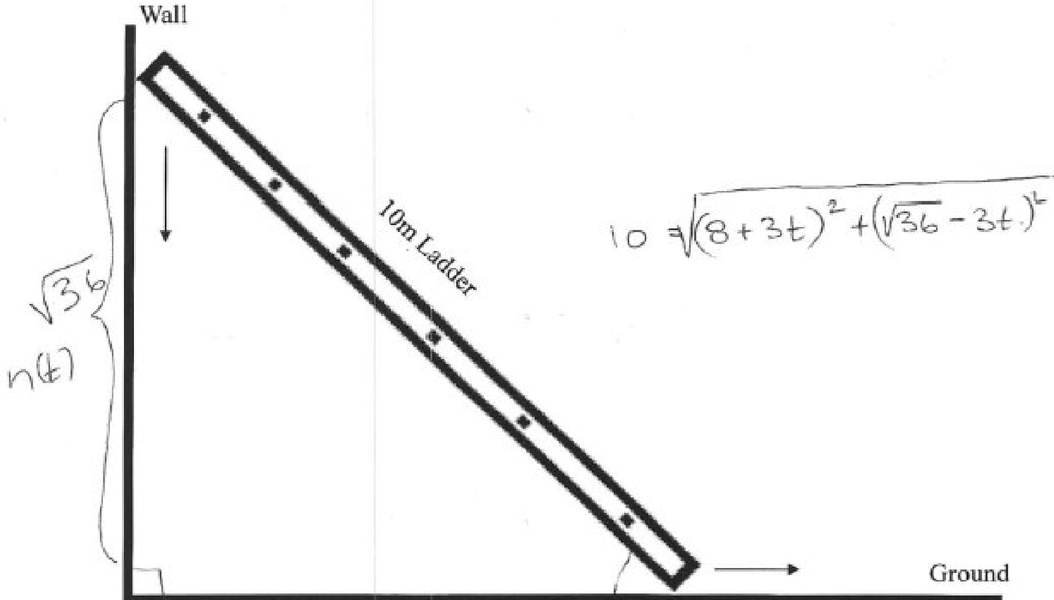
d ball&shadow $= (h - 4.9t^2)^2 +$

d shadow&lampposts

5 - Summarize, using as few sentences and in your own words, what is happening in this problem.

As ball proceeds to the ground it is moving further from

A 10m long ladder is sliding down a wall as seen in the following diagram. At the instant shown the bottom of the ladder is 8m from the vertical wall and is moving away from that wall at 3 m/s.



base $8\,m + 3\,m/s$ $b(t) = 8 + 3(t)$

A) Explain in your own words what is happening in this problem.

B) How fast is the top of the ladder moving down the wall at this point.

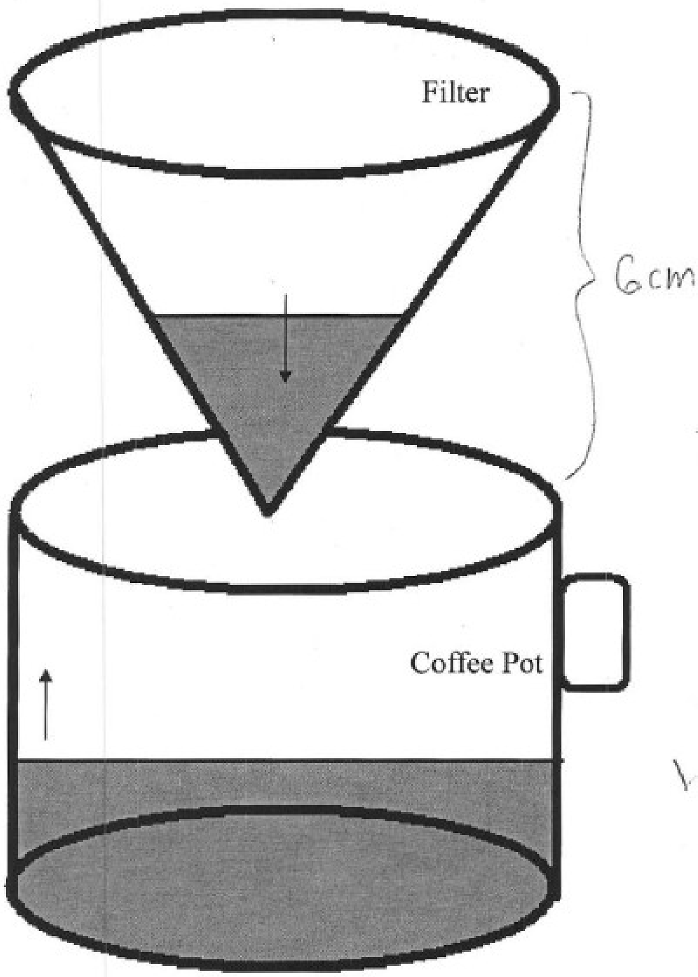
C) At what rate is the area of the triangle formed by the ladder and the two walls changing at this moment.

$A = \frac{(8+3t)(6-3t)}{2}$ $A = \frac{3h}{2}$ $\cancel{(8 - 3m/s)}$

$= \frac{48 - 24t + 18t - 9t^2}{2}$ $A(t) = \frac{(8m + 3m/s)(\sqrt{36} - 3m/s}{2}$

D) At what rate is the angle between the ~~wall~~ ladder and the ground changing.

A coffee percolator pours coffee into a cylindrical coffee pot from a conical filter. The coffee machine makes coffee at a rate of $5\text{cm}^3/\text{min}$. The filter and the pot both have a radius of 3cm. The filter's height is twice it's radius.



$V = \frac{1}{3}\pi r^2 h$

$V = \pi r^2 h$

$h = \frac{V}{\pi r^2}$

A) Describe in your own words what is happening in this problem.

B) How fast is the height in the pot rising.

C) How fast is the height in the filter decreasing when it is 3cm high?

Appendix 4. Textbooks

Appendix 4.1 Stewart's Calculus Related Rates Section

(Stewart, 2008, pp. 182-186)

(b) Find $C'(200)$ and explain its meaning. What does it predict?
(c) Compare $C'(200)$ with the cost of manufacturing the 201st yard of fabric.

28. The cost function for production of a commodity is

$$C(x) = 339 + 25x - 0.09x^2 + 0.0004x^3$$

(a) Find and interpret $C'(100)$.
(b) Compare $C'(100)$ with the cost of producing the 101st item.

29. If $p(x)$ is the total value of the production when there are x workers in a plant, then the *average productivity* of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

(a) Find $A'(x)$. Why does the company want to hire more workers if $A'(x) > 0$?
(b) Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity.

30. If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.
(a) Find the sensitivity.
(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

31. The gas law for an ideal gas at absolute temperature T (in kelvins), pressure P (in atmospheres), and volume V (in liters) is $PV = nRT$, where n is the number of moles of the gas and $R = 0.0821$ is the gas constant. Suppose that, at a certain instant, $P = 8.0$ atm and is increasing at a rate of 0.10 atm/min and $V = 10$ L and is decreasing at a rate of 0.15 L/min. Find the rate of change of T with respect to time at that instant if $n = 10$ mol.

32. In a fish farm, a population of fish is introduced into a pond and harvested regularly. A model for the rate of change of the fish population is given by the equation

$$\frac{dP}{dt} = r_0\left(1 - \frac{P(t)}{P_c}\right)P(t) - \beta P(t)$$

where r_0 is the birth rate of the fish, P_c is the maximum population that the pond can sustain (called the *carrying capacity*), and β is the percentage of the population that is harvested.
(a) What value of dP/dt corresponds to a stable population?
(b) If the pond can sustain 10,000 fish, the birth rate is 5%, and the harvesting rate is 4%, find the stable population level.
(c) What happens if β is raised to 5%?

33. In the study of ecosystems, *predator-prey models* are often used to study the interaction between species. Consider populations of tundra wolves, given by $W(t)$, and caribou, given by $C(t)$, in northern Canada. The interaction has been modeled by the equations

$$\frac{dC}{dt} = aC - bCW \qquad \frac{dW}{dt} = -cW + dCW$$

(a) What values of dC/dt and dW/dt correspond to stable populations?
(b) How would the statement "The caribou go extinct" be represented mathematically?
(c) Suppose that $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$. Find all population pairs (C, W) that lead to stable populations. According to this model, is it possible for the two species to live in balance or will one or both species become extinct?

3.8 RELATED RATES

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate of increase of the radius.

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

EXAMPLE 1 Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

■ According to the Principles of Problem Solving discussed on page 54, the first step is to understand the problem. This includes reading the problem carefully, identifying the given and the unknown, and introducing suitable notation.

SOLUTION We start by identifying two things:

the *given information:*

the rate of increase of the volume of air is 100 cm^3/s

and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathematically, we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative dV/dt, and the rate of increase of the radius is dr/dt. We can therefore restate the given and the unknown as follows:

$$\textit{Given:} \qquad \frac{dV}{dt} = 100 \text{ cm}^3/\text{s}$$

$$\textit{Unknown:} \qquad \frac{dr}{dt} \quad \text{when } r = 25 \text{ cm}$$

■ The second stage of problem solving is to think of a plan for connecting the given and the unknown.

In order to connect dV/dt and dr/dt, we first relate V and r by the formula for the volume of a sphere:

$$V = \tfrac{4}{3}\pi r^3$$

In order to use the given information, we differentiate each side of this equation with respect to t. To differentiate the right side, we need to use the Chain Rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now we solve for the unknown quantity:

■ Notice that, although dV/dt is constant, dr/dt is *not* constant.

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $dV/dt = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $1/(25\pi) \approx 0.0127$ cm/s. □

EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

FIGURE 1

SOLUTION We first draw a diagram and label it as in Figure 1. Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground. Note that x and y are both functions of t (time, measured in seconds).

We are given that $dx/dt = 1$ ft/s and we are asked to find dy/dt when $x = 6$ ft (see Figure 2). In this problem, the relationship between x and y is given by the Pythagorean Theorem:

$$x^2 + y^2 = 100$$

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

and solving this equation for the desired rate, we obtain

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$

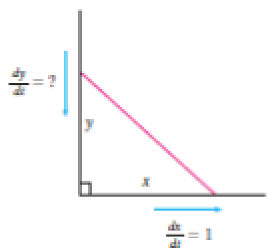


FIGURE 2

When $x = 6$, the Pythagorean Theorem gives $y = 8$ and so, substituting these values and $dx/dt = 1$, we have

$$\frac{dy}{dt} = -\frac{6}{8}(1) = -\frac{3}{4}\text{ ft/s}$$

The fact that dy/dt is negative means that the distance from the top of the ladder to the ground is *decreasing* at a rate of $\frac{3}{4}$ ft/s. In other words, the top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ ft/s. □

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

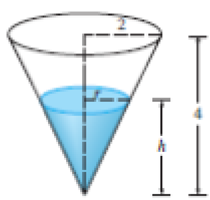


FIGURE 3

SOLUTION We first sketch the cone and label it as in Figure 3. Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $dV/dt = 2$ m^3/min and we are asked to find dh/dt when h is 3 m. The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but it is very useful to express V as a function of h alone. In order to eliminate r, we use the similar triangles in Figure 3 to write

$$\frac{r}{h} = \frac{2}{4} \qquad r = \frac{h}{2}$$

and the expression for V becomes

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

so

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3$ m and $dV/dt = 2\ \text{m}^3/\text{min}$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}\cdot 2 = \frac{8}{9\pi}$$

The water level is rising at a rate of $8/(9\pi) \approx 0.28$ m/min. □

■ Look back: What have we learned from Examples 1–3 that will help us solve future problems?

STRATEGY It is useful to recall some of the problem-solving principles from page 54 and adapt them to related rates in light of our experience in Examples 1–3:

1. Read the problem carefully.
2. Draw a diagram if possible.
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 3).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

WARNING A common error is to substitute the given numerical information (for quantities that vary with time) too early. This should be done only *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 3 we dealt with general values of h until we finally substituted $h = 3$ at the last stage. (If we had put $h = 3$ earlier, we would have gotten $dV/dt = 0$, which is clearly wrong.)

The following examples are further illustrations of the strategy.

EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

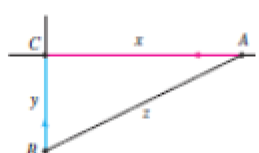


FIGURE 4

SOLUTION We draw Figure 4, where C is the intersection of the roads. At a given time t, let x be the distance from car A to C, let y be the distance from car B to C, and let z be the distance between the cars, where x, y, and z are measured in miles.

We are given that $dx/dt = -50$ mi/h and $dy/dt = -60$ mi/h. (The derivatives are negative because x and y are decreasing.) We are asked to find dz/dt. The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2$$

Differentiating each side with respect to t, we have

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

When $x = 0.3$ mi and $y = 0.4$ mi, the Pythagorean Theorem gives $z = 0.5$ mi, so

$$\frac{dz}{dt} = \frac{1}{0.5}[0.3(-50) + 0.4(-60)]$$

$$= -78 \text{ mi/h}$$

The cars are approaching each other at a rate of 78 mi/h. □

EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

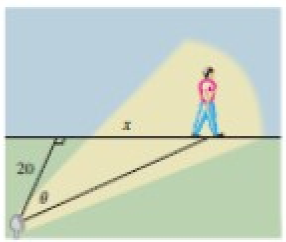


FIGURE 5

SOLUTION We draw Figure 5 and let x be the distance from the man to the point on the path closest to the searchlight. We let θ be the angle between the beam of the searchlight and the perpendicular to the path.

We are given that $dx/dt = 4$ ft/s and are asked to find $d\theta/dt$ when $x = 15$. The equation that relates x and θ can be written from Figure 5:

$$\frac{x}{20} = \tan\theta \qquad x = 20\tan\theta$$

Differentiating each side with respect to t, we get

$$\frac{dx}{dt} = 20\sec^2\theta\,\frac{d\theta}{dt}$$

so

$$\frac{d\theta}{dt} = \frac{1}{20}\cos^2\theta\,\frac{dx}{dt} = \frac{1}{20}\cos^2\theta\,(4) = \frac{1}{5}\cos^2\theta$$

When $x = 15$, the length of the beam is 25, so $\cos\theta = \frac{4}{5}$ and

$$\frac{d\theta}{dt} = \frac{1}{5}\left(\frac{4}{5}\right)^2 = \frac{16}{125} = 0.128$$

The searchlight is rotating at a rate of 0.128 rad/s. □

3.8 EXERCISES

1. If V is the volume of a cube with edge length x and the cube expands as time passes, find dV/dt in terms of dx/dt.

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.
   (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

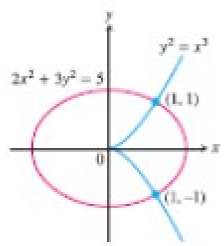
3. Each side of a square is increasing at a rate of 6 cm/s. At what rate is the area of the square increasing when the area of the square is 16 cm^2?

4. The length of a rectangle is increasing at a rate of 8 cm/s and its width is increasing at a rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

Appendix 4.2 Thomas's Calculus Related Rates Section.

(Thomas, Weir, & Hass, 2010, pp. 155-160)

48. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.

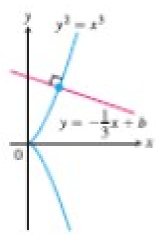


49. Verify that the following pairs of curves meet orthogonally.

a. $x^2 + y^2 = 4, \quad x^2 = 3y^2$

b. $x = 1 - y^2, \quad x = \frac{1}{3}y^2$

50. The graph of $y^2 = x^3$ is called a **semicubical parabola** and is shown in the accompanying figure. Determine the constant b so that the line $y = -\frac{1}{3}x + b$ meets this graph orthogonally.



T In Exercises 51 and 52, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related? Explain the relationship geometrically in terms of the graphs.

51. $xy^3 + x^2y = 6$ **52.** $x^3 + y^2 = \sin^2 y$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps in Exercises 53–60.

a. Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.

b. Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P.

c. Use the slope found in part (b) to find an equation for the tangent line to the curve at P. Then plot the implicit curve and tangent line together on a single graph.

53. $x^3 - xy + y^3 = 7, \quad P(2, 1)$

54. $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

55. $y^2 + y = \frac{2 + x}{1 - x}, \quad P(0, 1)$

56. $y^3 + \cos xy = x^2, \quad P(1, 0)$

57. $x + \tan\left(\frac{y}{x}\right) = 2, \quad P\left(1, \frac{\pi}{4}\right)$

58. $xy^3 + \tan(x + y) = 1, \quad P\left(\frac{\pi}{4}, 0\right)$

59. $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

60. $x\sqrt{1 + 2y} + y = x^2, \quad P(1, 0)$

3.8 Related Rates

In this section we look at problems that ask for the rate at which some variable changes when it is known how the rate of some other related variable (or perhaps several variables) changes. The problem of finding a rate of change from other known rates of change is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r,

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to directly measure the rate of increase of the volume (the rate at which air is being pumped into the balloon) than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

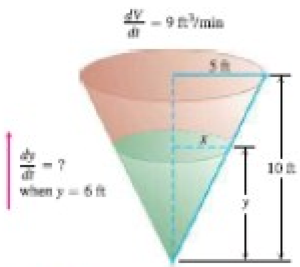


FIGURE 3.32 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 1).

EXAMPLE 1 Water runs into a conical tank at the rate of $9\ \text{ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

Solution Figure 3.32 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t.

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6\ \text{ft} \qquad \text{and} \qquad \frac{dV}{dt} = 9\ \text{ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 3.32 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \qquad \text{or} \qquad x = \frac{y}{2}.$$

Therefore, find

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants.* Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables.* You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t.* Then express the rate you want in terms of the rates and variables whose values you know.
6. *Evaluate.* Use known values to find the unknown rate.

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

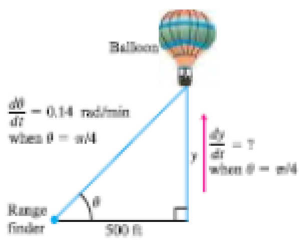


FIGURE 3.33 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.33). The variables in the picture are

   $\theta =$ the angle in radians the range finder makes with the ground.

   $y =$ the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find.* We want dy/dt when $\theta = \pi/4$.
4. *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan\theta \qquad \text{or} \qquad y = 500\tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500(\sec^2\theta)\frac{d\theta}{dt}$$

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

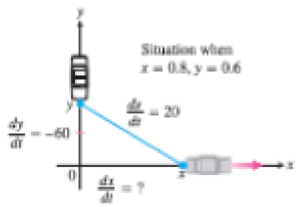


FIGURE 3.34 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

Solution We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3.34). We let t represent time and set

$$x = \text{position of car at time } t$$
$$y = \text{position of cruiser at time } t$$
$$s = \text{distance between car and cruiser at time } t.$$

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \quad y = 0.6 \text{ mi}, \quad \frac{dy}{dt} = -60 \text{ mph}, \quad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, we use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

EXAMPLE 4 A particle P moves clockwise at a constant rate along a circle of radius 10 ft centered at the origin. The particle's initial position is (0, 10) on the y-axis and its final destination is the point (10, 0) on the x-axis. Once the particle is in motion, the tangent line at P intersects the x-axis at a point Q (which moves over time). If it takes the particle 30 sec to travel from start to finish, how fast is the point Q moving along the x-axis when it is 20 ft from the center of the circle?

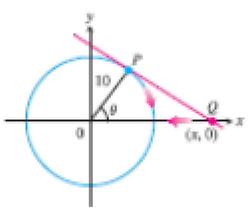


FIGURE 3.35 The particle P travels clockwise along the circle (Example 4).

Solution We picture the situation in the coordinate plane with the circle centered at the origin (see Figure 3.35). We let t represent time and let θ denote the angle from the x-axis to the radial line joining the origin to P. Since the particle travels from start to finish in 30 sec, it is traveling along the circle at a constant rate of $\pi/2$ radians in 1/2 min, or π rad/min. In other words, $d\theta/dt = -\pi$, with t being measured in minutes. The negative sign appears because θ is decreasing over time.

Setting $x(t)$ to be the distance at time t from the point Q to the origin, we want to find dx/dt when

$$x = 20 \text{ ft} \quad \text{and} \quad \frac{d\theta}{dt} = -\pi \text{ rad/min}.$$

To relate the variables x and θ, we see from Figure 3.35 that $x \cos\theta = 10$, or $x = 10 \sec\theta$. Differentiation of this last equation gives

$$\frac{dx}{dt} = 10 \sec\theta \tan\theta \frac{d\theta}{dt} = -10\pi \sec\theta \tan\theta.$$

Note that dx/dt is negative because x is decreasing (Q is moving towards the origin).

When $x = 20$, $\cos\theta = 1/2$ and $\sec\theta = 2$. Also, $\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{3}$. It follows that

$$\frac{dx}{dt} = (-10\pi)(2)\left(\sqrt{3}\right) = -20\sqrt{3}\pi.$$

At the moment in question, the point Q is moving towards the origin at the speed of $20\sqrt{3}\pi \approx 108.8$ ft/min. ■

EXAMPLE 5 A jet airliner is flying at a constant altitude of 12,000 ft above sea level as it approaches a Pacific island. The aircraft comes within the direct line of sight of a radar station located on the island, and the radar indicates the initial angle between sea level and its line of sight to the aircraft is 30°. How fast (in miles per hour) is the aircraft approaching the island when first detected by the radar instrument if it is turning upward (counterclockwise) at the rate of 2/3 deg/sec in order to keep the aircraft within its direct line of sight?

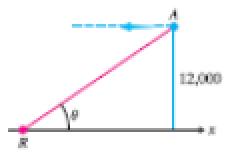


FIGURE 3.36 Jet airliner A traveling at constant altitude toward radar station R (Example 5).

Solution The aircraft A and radar station R are pictured in the coordinate plane, using the positive x-axis as the horizontal distance at sea level from R to A, and the positive y-axis as the vertical altitude above sea level. We let t represent time and observe that $y = 12{,}000$ is a constant. The general situation and line-of-sight angle θ are depicted in Figure 3.36. We want to find dx/dt when $\theta = \pi/6$ rad and $d\theta/dt = 2/3$ deg/sec.

From Figure 3.36, we see that

$$\frac{12{,}000}{x} = \tan\theta \quad \text{or} \quad x = 12{,}000 \cot\theta.$$

Using miles instead of feet for our distance units, the last equation translates to

$$x = \frac{12{,}000}{5280} \cot\theta.$$

Differentiation with respect to t gives

$$\frac{dx}{dt} = -\frac{1200}{528} \csc^2\theta \frac{d\theta}{dt}.$$

When $\theta = \pi/6$, $\sin^2\theta = 1/4$, so $\csc^2\theta = 4$. Converting $d\theta/dt = 2/3$ deg/sec to radians per hour, we find

$$\frac{d\theta}{dt} = \frac{2}{3}\left(\frac{\pi}{180}\right)(3600) \text{ rad/hr.} \qquad \text{1 hr = 3600 sec, 1 deg = } \pi/180 \text{ rad}$$

Substitution into the equation for dx/dt then gives

$$\frac{dx}{dt} = \left(-\frac{1200}{528}\right)(4)\left(\frac{2}{3}\right)\left(\frac{\pi}{180}\right)(3600) \approx -380.$$

The negative sign appears because the distance x is decreasing, so the aircraft is approaching the island at a speed of approximately 380 mi/hr when first detected by the radar. ■

EXAMPLE 6 Figure 3.37(a) shows a rope running through a pulley at P and bearing a weight W at one end. The other end is held 5 ft above the ground in the hand M of a worker. Suppose the pulley is 25 ft above ground, the rope is 45 ft long, and the worker is walking

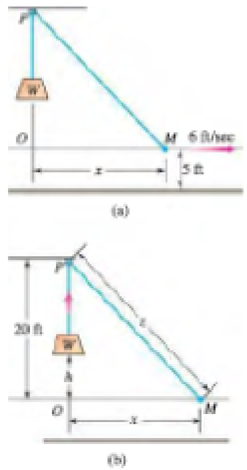


FIGURE 3.37 A worker at M walks to the right pulling the weight W upwards as the rope moves through the pulley P (Example 6).

rapidly away from the vertical line PW at the rate of 6 ft/sec. How fast is the weight being raised when the worker's hand is 21 ft away from PW?

Solution We let OM be the horizontal line of length x ft from a point O directly below the pulley to the worker's hand M at any instant of time (Figure 3.37). Let h be the height of the weight W above O, and let z denote the length of rope from the pulley P to the worker's hand. We want to know dh/dt when $x = 21$ given that $dx/dt = 6$. Note that the height of P above O is 20 ft because O is 5 ft above the ground. We assume the angle at O is a right angle.

At any instant of time t we have the following relationships (see Figure 3.37b):

$$20 - h + z = 45 \qquad \text{Total length of rope is 45 ft.}$$
$$20^2 + x^2 = z^2 \qquad \text{Angle at } O \text{ is a right angle.}$$

If we solve for $z = 25 + h$ in the first equation, and substitute into the second equation, we have

$$20^2 + x^2 = (25 + h)^2. \tag{1}$$

Differentiating both sides with respect to t gives

$$2x\frac{dx}{dt} = 2(25 + h)\frac{dh}{dt},$$

and solving this last equation for dh/dt we find

$$\frac{dh}{dt} = \frac{x}{25 + h}\frac{dx}{dt}. \tag{2}$$

Since we know dx/dt, it remains only to find $25 + h$ at the instant when $x = 21$. From Equation (1),

$$20^2 + 21^2 = (25 + h)^2$$

so that

$$(25 + h)^2 = 841, \quad \text{or} \quad 25 + h = 29.$$

Equation (2) now gives

$$\frac{dh}{dt} = \frac{21}{29} \cdot 6 = \frac{126}{29} \approx 4.3 \text{ ft/sec}$$

as the rate at which the weight is being raised when $x = 21$ ft. ■

Exercises 3.8

1. **Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.
2. **Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.
3. Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt.
4. Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt.
5. If $y = x^2$ and $dx/dt = 3$, then what is dy/dt when $x = -1$?
6. If $x = y^3 - y$ and $dy/dt = 5$, then what is dx/dt when $y = 2$?
7. If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?
8. If $x^2y^3 = 4/27$ and $dy/dt = 1/2$, then what is dx/dt when $x = 2$?
9. If $L = \sqrt{x^2 + y^2}$, $dx/dt = -1$, and $dy/dt = 3$, find dL/dt when $x = 5$ and $y = 12$.
10. If $r + s^2 + v^3 = 12$, $dr/dt = 4$, and $ds/dt = -3$, find dv/dt when $r = 3$ and $s = 1$.
11. If the original 24 m edge length x of a cube decreases at the rate of 5 m/min, when $x = 3$ m at what rate does the cube's
    a. surface area change?
    b. volume change?
12. A cube's surface area increases at the rate of 72 in^2/sec. At what rate is the cube's volume changing when the edge length is $x = 3$ in?

Appendix 5 – MATH 203, Concordia University – Course Outline

Lecture	Topics		Supplementary Problems	
1	1.1	Representations of functions	p.20:	1, 9, 25, 29, 43, 45
	1.2	A catalogue of functions	p.34	1, 9, 13, 15
(Review of	1.3	New functions from old	p.43:	11, 17, 33, 35, 43
functions)	1.5	Exponential functions	p.58:	3, 7, 15, 19
2	1.6	Inverse and logarithmic functions	p.70:	9, 11, 21, 37 39, 53
(Review	App.D	Trigonometric functions (review)	p.A32	3, 9, 13, 49, 65 69
Continued)				
3	2.1	The tangent and velocity problems	p.87:	5, 7
	2.2	Limit of a function	p.96:	7, 19, 29
	2.3	Calculating limits	p.106:	11, 17, 27, 43, 61
		Omit 2.4		
	2.6	Limits at infinity, horizontal asymptotes	p.140:	17, 19, 27, 29, 43
4	2.5	Continuity	p.128:	3, 21, 19, 37
	2.7	Derivatives and rates of change	p.150:	5, 22, 29, 37
	2.8	Derivative as a function; higher derivatives	p.162:	21, 25, 35, 51
5	3.1	Derivatives of polynomials and exp.	p.180:	9, 17, 29, 49
	3.2	Product and quotient rules	p.187:	3, 13, 19, 27
	3.3	Derivatives of trigonometric functions	p.195:	3, 5, 11, 23
6	3.4	Chain Rule	p.203:	9, 11, 13, 29, 35, 53
	3.5	Implicit differentiation	p.213:	9, 17, 25, 35
7		**Class Test**		
	3.6	Derivatives of logarithmic functions	p.220:	5, 15, 23, 41, 45
8	3.7	Rates of change in the sciences	p.230:	7, 15, 29
	3.8	Exponential growth/decay	p.239	3, 9, 11, 19
	3.9	Related rates	p.245:	3, 7, 11, 15
	3.10	Linear approximations, differentials	p.252:	1, 11, 13, 15, 17, 33, 37
9	4.1	Maximum/minimum values	p.277:	22, 31, 37, 43, 51, 61
	4.2	Mean Value Theorem	p.285:	3, 11, 13
10	4.3	Shape of graphs	p.295:	9, 11, 15, 19, 21
	4.4	Indeterminate forms; L'Hôpital's Rule	p.304:	9, 11, 15, 17, 49
11	4.5	Summary of curve sketching	p.314:	5, 11, 19, 43, 49
		Omit 4.6		
12	4.7	Optimization problems	p.328:	13, 17, 23, 27, 35
13		**Review - tutorials**		