# Numerical Bifurcation Analysis of Delay Differential Equations

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ABSTRACT

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The intended purpose of this research was to study the bifurcation analysis of delay differential equations and to develop a computer program to compute the bifurcation diagram of delay differential equations. In this thesis, we discuss: computation of stable and unstable, stationary and periodic solutions of delay differential equations; detection of stationary bifurcation points, Hopf bifurcation points, secondary periodic bifurcation points including period doubling bifurcation points; and techniques of switching automatically onto branches of periodic solutions and tracing out such branches. Numerical results of a model from physiology and other examples will be shown.

## Acknowledgement

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CONTENTS

# CHAPTER ONE

## Introduction

Differential Equations serve as mathematical descriptions for many physical problems and phenomena. Differential equations are equations involving the derivative(s) of some unknown function(s). Typical elementary examples have the form

(1.1.1) $u'(t) = f(t, u(t))$

where $u'$ is the derivative of the unknown function $u$, and $f$ is a given continuous function. The goal is to find the unknown function. Differential equations have contributed immeasurably to the advancement of physics and engineering. They also have a significant role in mathematical biology, in mathematical ecology, in chemical reactions and nuclear reactions, in physiology and in economics. A differential equation of the form (1.1.1) is known as <u>Ordinary Differential Equation</u>, in which the rate of change of the unknown function $u$ at time $t$ depends on the function value $u$ at time $t$. Many physical problems cannot be modelled by ordinary differential equations because the past history of the system is essential to the present rate of change of system. Differential equation modelling this kind of physical problem are known as <u>Retarded Functional Differential Equations</u>. The equation

(1.1.2) $u'(t) = f(t, u(t), u(t-\tau))$,

where $\tau$ is a positive constant, is a special kind of retarded functional differential equation, known as a Delay Differential Equation.

Usually, parameters are present in models of physical systems. As a parameter varies, a family of solutions arises and branching of solutions is possible. In 1981, a computer program for the automatic bifurcation analysis of autonomous system of ordinary differential equations without delays was developed by Doedel[7]. This program, named AUTO, can trace out branches of stable and unstable, stationary and periodic solutions; can detect stationary bifurcation points, Hopf bifurcation points, secondary periodic bifurcation points including period doubling bifurcation points; it can also switch automatically onto branches of periodic solutions and trace out such branches. (See definitions on page 18 and Figure 2.7).

In this thesis the development of a computer program, DLAY, for the automatic bifurcation analysis of delay differential equations is reported. The program DLAY can compute stable and unstable, stationary and periodic solution of differential equations of the form

$$u'(t) = f(u(t),u(t-\tau),\lambda) \tag{1.1.3}$$

where $\tau$ is a constant delay and $\lambda$ is a parameter.
As the parameter $\lambda$ varies, the program DLAY can detect stationary bifurcation points and Hopf bifurcation points while it is computing a branch of stationary solutions.

Taking a Hopf bifurcation point as a starting point of periodic solution branch, DLAY can trace out such a branch and in the meantime it can detect secondary periodic bifurcation points including period doubling bifurcations. With an appropriate technique to switch branches at secondary periodic bifurcation points, DLAY can compute cascading bifurcations of periodic branches.

In Chapter 2 of this thesis, Keller's general pseudo arclength continuation technique for solution branches is discussed. General results from bifurcation theory including the Crandall and Rabinowitz Bifurcation Theorem and the Hopf Bifurcation Theorem are also discussed.

In Chapter 3, definitions, examples and basic theory of retarded functional differential equations are given. In the last section of Chapter 3, Hadeler's technique for computing bifurcation diagram for delay differential equations is discussed.

Chapter 4 and 5 are devoted to our method of computing the bifurcation diagram for delay differential equations. In Chapter 4, computation of steady state solution branches is discussed. The detection of stationary bifurcation point and a branch switching technique for stationary solutions are presented. In the last section of Chapter 4, we discuss systems of delay differential equations.

In Chapter 5, the detection of Hopf bifurcation points is discussed. Further we consider the detection of secondary periodic bifurcation points including period doubling bifurcations. A method for switching onto periodic solution branches and tracing out such branches is presented also.

Numerical results analysed by the program DLAY are presented in Chapter 6.

CHAPTER TWO

# Continuation of Solutions and Bifurcation Theory

## 2.1 Continuation of solutions

Consider the operator equation

(2.1.1) $G(u, \lambda) = 0,$

where $\lambda$ is a parameter and G is a nonlinear mapping from one Hilbert space into another. As the parameter, $\lambda$, ~~varies,~~ one expects branches of solutions. Usually these branches cannot be continued at turning points if $\lambda$ is treated as the continuation parameter. (see Figure 2.1).

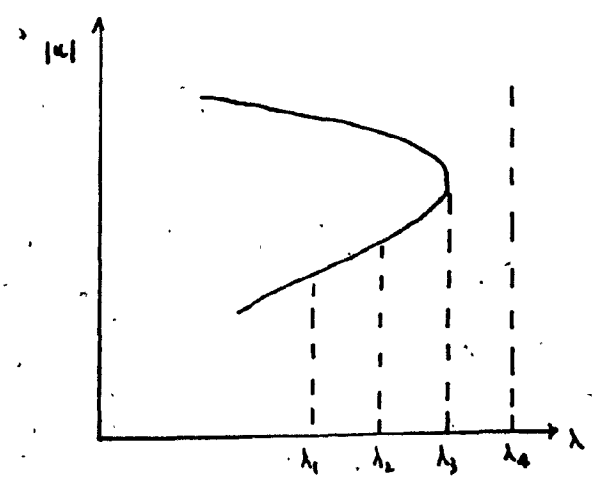



Fig. 2.1 Continuation parameter $\lambda$ does not work at turning points

Instead of using $\lambda$, one can use the arclength of the solution branch as the continuation parameter. In other words, instead of walking along the $\lambda$-axis, we walk along

the solution branch. To compute the actual arclength of the solution branch would be tedious. A pseudo arclength continuation technique, which works also at turning points, was suggested by [20] and has been using by various authors [2,5,7,8,14,21,36]. By pseudo arclength between two consecutive solutions on the branch, we mean the projection of the difference of the two solution vectors onto the tangential vector at the prior solution vector. (see Figure 2.2).

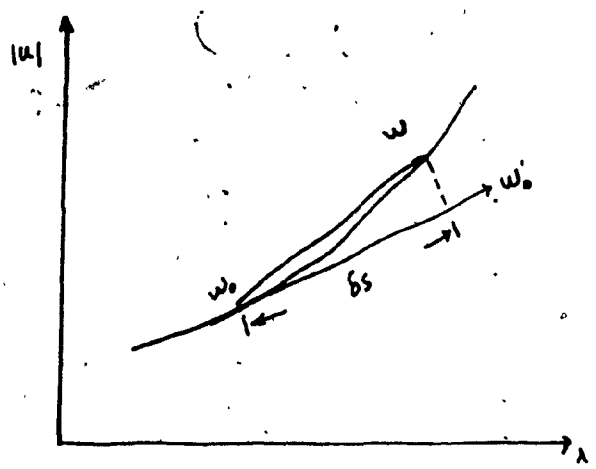



Fig. 2.2 Pseudo arclength $\delta s$

For the continuation of a solution branch, we require the pseudo arclength between the current solution on the branch and the next one to equal a prespecified value, say, $\delta s$. As can be seen in Figure 2.3, this technique works equally well at turning points.

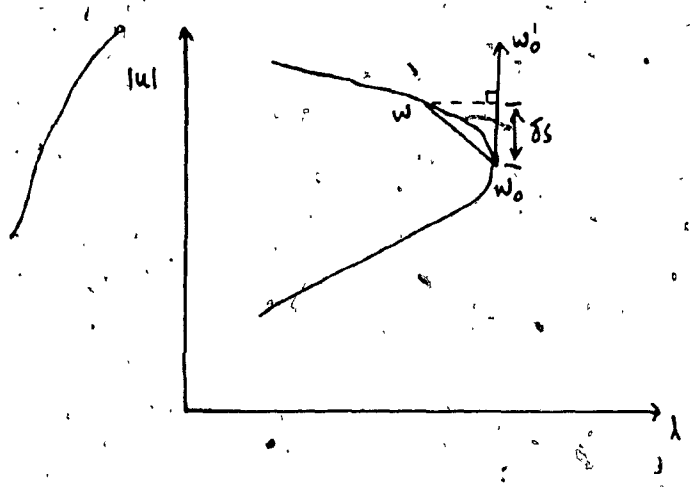



Fig. 2.3 Pseudo arclength continuation works at turning points

To show the existence of a branch of solutions, we first recall the basic features of the general procedure in [20]. Let $w=(u,\lambda)$. If there exists some parametrized branch $w(s)$ of solutions to (2.1.1) then under appropriate smoothness assumptions we have

$$G'(w(s))\ w'(s) = 0.$$

Thus the derivative $G'$ always has a null space along the branch. Assume now that we have a solution $w_0$ of (2.1.1), i.e. $G(w_0)=0$, and that the nullspace of $G'(w_0)$ is one dimensional and spanned by a vector $w_0'$. Let $w_0'^*$ be the adjoint element such that $w_0'^* w_0'=1$. Then the inflated problem

(2.1.2a) $\quad G(w) = 0$

(2.1.2b) $\quad w_0'^* (w-w_0) - s = 0$

which we write more compactly as

$$H(w,s) = 0,$$

has the solution $w=w_0$ when $s=0$. Further the derivative

$$H_w(w_0,0) = \begin{bmatrix} G'(w_0) \\ w_0'^* \end{bmatrix}$$

is nonsingular since $H_w(w_0,0)\phi = 0$ iff $G'(w_0)\phi = 0$ and $w_0'^*\phi = 0$; but $w_0'^* \neq 0$, hence $\phi = 0$.

Now the implicit mapping theorem as stated below guarantees the existence of a branch of solutions $w(s)$ for small $s$.

<u>Theorem</u>: (2.1.1) ( <u>Implicit Mapping Theorem</u> )

Let X and Y be Banach spaces and let H be a continuously differentiable transformation from an open set D in X x Y with values in X. Let $(w_0,s_0)$ be a point in D for which $H(w_0,s_0)=0$ and for which $H_w(w_0,s_0)$ is non-singular. Then there is a neighbourhood N of $s_0$ and a continuous function w mapping N into X such that $w(s_0)=w_0$ and $H(w(s),s)=0$ for all s in N.

The pseudo arclength continuation technique described above applies in a very general setting. In fact, the continuation of both stationary and periodic solutions to

systems of differential equations can be treated in the same framework. An application of this technique for systems of ordinary differential equations without delays can be found in [7]. In the later chapters of this thesis, we implement this technique for delay differential equations.

## 2.2 The Crandall & Rabinowitz Bifurcation Theorem

Bifurcation theory involves the study of equations whose solutions branch, or bifurcate, as a parameter in these equations varies. This parameter is usually called the bifurcation parameter. Such equations often occur in models used in mechanics, fluid dynamics, elasticity, population dynamics, physiology and in many other areas. The bifurcation of the solutions to these equations means, physically, that the system is in a situation of change of state. For example, in the nonlinear equations modeling the buckling of a rod that is subjected to increasing pressure, the applied pressure is the bifurcation parameter. When the pressure reaches certain values, the solutions to these model equations bifurcate. The bifurcation points correspond to critical pressures that can cause the rod to buckle in one way or the other, depending on which solution branch is followed.

Since many models of physical events consist of equations whose solutions bifurcate, bifurcation theory is of considerable practical importance. Complicated behaviour

can occur when branches of solutions split to form secondary bifurcations. The branches originating from secondary bifurcation points may split to form tertiary bifurcations, and this process can sometimes continue indefinitely to form a cascading bifurcation of solutions with infinitely many branches.

For many problems of the form $F(u;\lambda) = 0$, there is a distinguished state, $u_0$, which is a solution for all values of $\lambda$ and corresponds to a special configuration of the physical system. It is called the basic solution. The choice of the basic solution is usually obvious from physical considerations. Frequently, it can be obtained explicitly. For instance, consider the model equations for the buckling of a uniform, thin, inextensible rod subject to axial thrust:

$$u'' + \lambda \sin(u(x)) = 0, \quad x \in [0,1],$$
$$u'(x) = 0 \text{ at } x = 0 \text{ and } x = 1,$$

where $u(x)$ is the angle made between the axis of the rod and the tangential direction at displacement $x$ from an end point of the rod, and $\lambda$ is the bifurcation parameter proportional to the applied thrust on the rod. A basic solution $u(x) \equiv 0$ for all $\lambda$ is obtained easily from the model. Physically, it means that the state of no buckling is always a solution to the model for any applied thrust. Obviously, beyond certain value of the applied thrust, no buckling is no longer a

physical or stable solution to the model and at that particular value of applied thrust, bifurcation appears.

Bifurcation points on the basic solution are referred to as primary bifurcation points, and the branches of solutions that bifurcate from these points, other than the basic solution are called primary states. Any solutions other than the basic solution which bifurcate from a primary state are called secondary states, and the corresponding bifurcation points are called secondary bifurcation points. In Figure (2.4), a bifurcation diagram of solutions to a hypothetical nonlinear equation, $F(u,\lambda) = 0$, is shown.

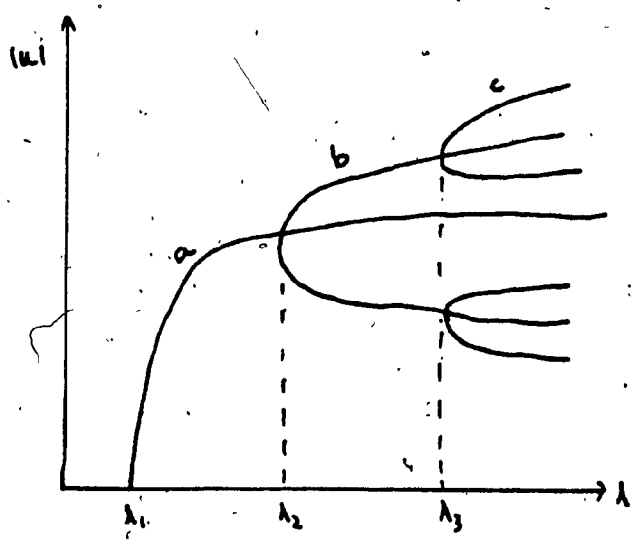



Fig. 2.4 Bifurcation diagram for a hypothetical equation $F(u,\lambda) = 0$.

As a bifurcation parameter is varied, a solution can follow a path in the cascading bifurcation or can jump from branch to branch. How a solution jumps basically depends on the change of stability of the solutions on the branches and it is why bifurcation is closely related to the loss of

stability of the solution of the system.

To be more precise, we recall the basic bifurcation theorem as given in [3]. Let W and Y be real Banach spaces, $\Omega$ an open subset of W and $G : \Omega \to Y$ be a continuous map. Suppose there is a simple arc C in $\Omega$ given by $C = \{w(t) : t \in I\}$, where I is an interval, such that $G(w) = \emptyset$ for $w \in C$. If there is a number $p \in I$ such that every neighborhood of w(p) contain zeros of G not lying on C, then w(p) is called a <u>bifurcation point</u> for the equation $G(w) = \emptyset$ with respect to the curve C. (see Figure 2.5).

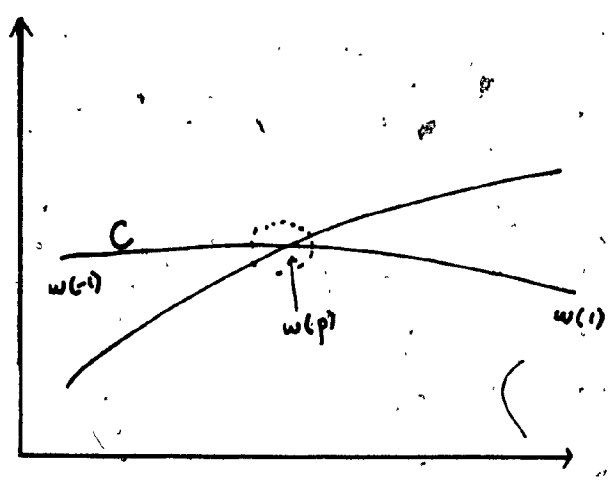



Fig. 2.5 w(p) is a bifurcation point with respect to the curve C.

Roughly speaking, a bifurcation point is the "intersection" point of two solution branches. The <u>multiplicity</u> of a bifurcation point is m if there are m+1 simple arcs intersecting at the bifurcation point. A bifurcation point of multiplicity one is called a simple bifurcation. A

general theorem for the existence of a simple bifurcation has been given by Crandall & Rabinowitz[3]. This theorem is as follows

Theorem(2.2.1)

( The Crandall and Rabinowitz Bifurcation Theorem )

Let W,Y be Banach spaces, $\Omega$ an open subset of W and $G:\Omega\to Y$ be twice continuously differentiable. Let $w:[-1,1]\to\Omega$ be a simple continuously differentiable arc in $\Omega$ such that $G(w(t)) \equiv 0$ for $|t|\leq 1$. Suppose

(a) $w'(0) \neq 0$

(b) $\dim N(G'(w(0))) = 2$, $\operatorname{codim} R(G'(w(0))) = 1$

(c) $N(G'(w(0)))$ is spanned by $w'(0)$ and $v$, and

(d) $G''(w(0))(w'(0),v) \notin R(G'(w(0)))$.

Then $w(0)$ is a bifurcation point of $G(w) = 0$ with respect to $C = \{w(t):t\in[-1,1]\}$ and in some neighbourhood of $w(0)$ the totality of solutions of $G(w) = 0$ form two continuous curves intersecting only at $w(0)$.

Here dim and codim are abbreviations for dimension and codimension respectively( the codimension of a subspace Z of Y is the dimension of Y/Z), and N(T), R(T) denote the null space and the range of a linear operator T. G' and G" are the first and second Fréchet derivatives of G. G'(.) is a linear operator from W into Y and G"(.) is a bilinear operator from W x W into Y. For example G"(w(0))(w(0),v) is the value of the bilinear operator G"(w(0)) at

$(w'(0),v) \in W \times W$.

An example is given below to show how this theorem can be applied to find a simple bifurcation point.

<u>Example</u>: Let $W = R^2 x R$, $Y = R^2$, $G:W \to Y$ such that

$$G(x) = \begin{bmatrix} u(1-u)-uv-\lambda(1-e^{-4u}) \\ \\ -v+2uv \end{bmatrix}$$

where $x = (u,v;\lambda)^T \in W$.

Let $w:[-1,1] \to W$ be such that $w(s) = (0,0;\lambda(s))^T$, $\lambda(s)$ not constant. Then $G(w(s)) = 0$ for $|s| \leq 1$, and

$$G'(x) = \begin{bmatrix} 1-2u-v-4\lambda e^{-4u} & -u & e^{-4u}-1 \\ \\ 2v & 2u-1 & 0 \end{bmatrix}$$

The second derivative of G, $G''(x)$ is

$$\begin{bmatrix} (-2+16\lambda e^{-4u}, -1, -4e^{-4u}) & (-1,0,0) & (-4e^{-4u},0,0) \\ \\ (0, 2, 0) & (2,0,0) & (0,0,0) \end{bmatrix}$$

$$G'(w(s)) = \begin{bmatrix} 1-4\lambda & 0 & 0 \\ \\ 0 & -1 & 0 \end{bmatrix}$$

$$G''(w(s)) = \begin{bmatrix} (2, -1, -4) & (-1, 0, 0) & (-4, 0, 0) \\ (0, 2, 0) & (2, 0, 0) & (0, 0, 0) \end{bmatrix}$$

Let $w(0) = (0, 0; 0.25)^T$ and $w'(0) = (0,0;1)^T \neq 0$.

$$G'(w(0)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

has a two dimensional null space which is spanned by $w'(0) = (0,0,1)^T$ and $v = (1,0,0)^T$, and

$$G'(w(0))^T = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

has a one dimensional null space.

Finally $G''(w(0))(w'(0),v)$

$$= \begin{bmatrix} (2,-1,-4) & (-1, 0, 0) & (-4, 0, 0) \\ (0, 2, 0) & (2, 0, 0) & (0, 0, 0) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= (-4 \quad 0)^T$$

$$\notin R(G'(w(0))) = \{(0, y)^T : y \in R\}$$

Hence by the Crandall and Rabinowitz Bifurcation Theorem, $w(0) = (0, 0; 0.25)^T$ is a simple bifurcation of $G(x) = 0$ with respect to the simple arc $w(s) = (0, 0; \lambda(s))^T$.

## 2.3 The Hopf Bifurcation Theorem

To detect Hopf bifurcation points, the Crandall and Rabinowitz Bifurcation theorem does not apply directly. In this section, the Hopf Bifurcation Theorem for ordinary differential equations without delays is discussed. The Hopf Bifurcation Theorem for retarded functional differential equations will be discussed in Section 3.2.

Consider an autonomous system of differential equations of the form

$$(2.3.1) \qquad u'(t) = f(u(t), \lambda), \quad t \geq 0, \quad u, f \in R^n.$$

Here $\lambda$ is the bifurcation parameter. Such systems arise in many areas, especially in the study of chemical reactions, in population dynamics and in mathematical biology.

A function $u(t) \in R^n$, $t \geq 0$, is said to be a <u>solution</u> to

(2.3.1) if it satisfies (2.3.1) for some $\lambda$.

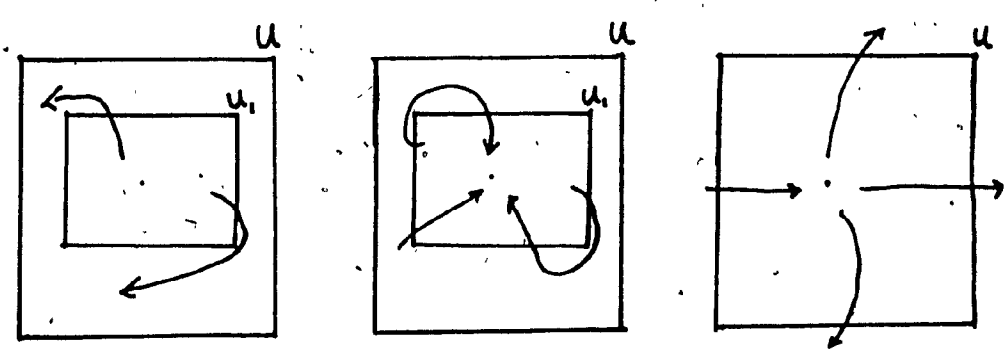
A solution, $u(t) \equiv u$, to (2.3.1) is said to be a <u>steady state</u> or <u>stationary</u> solution to (2.3.1) if $f(u,\lambda) = 0$.

A stationary solution u in $R^n$ is said to be <u>stable</u> if for every neighbourhood U of u in $R^n$, there is a neighbourhood $U_1$ of u in $R^n$ such that every solution u(t) with u(0) in $U_1$ is defined and in U for all $t>0$.

In addition to the above properties, if $u(t) \to u$ as $t \to \infty$ then u is said to be <u>asymptotically stable</u>.

A stationary solution that is not stable is called <u>unstable</u>.

Figure 2.6 illustrates the stability of a stationary solution.




(a) stability (b) Asymptotic stability (c) Instability

Fig. 2.6 Stability of a stationary solution.

A solution, u(t), to (2.3.1) is said to be a <u>p-periodic</u>

solution to (2.3.1) if $u(t)$ is a solution to (2.3.1) and $u(t) = u(t+\rho)$ for all $t$.

Generally one is interested in both steady state and periodic solutions to (2.3.1). As the bifurcation parameter, $\lambda$, varies, one expects branches of both types of solutions.

A <u>steady state bifurcation point</u> is the bifurcation point of two branches of steady state solutions.

The bifurcation point of two branches of periodic solutions is called a <u>secondary periodic bifurcation point</u>.

These two types of bifurcation points can be detected by applying the Crandall and Rabinowitz Bifurcation Theorem directly.

A <u>Hopf bifurcation point</u> is the point where a branch of steady states and a branch of periodic solutions intersect. (see Figure 2.7)

The Hopf bifurcation refers to the development of periodic orbits("self-oscillations") from a equilibrium point, as the bifurcation parameter crosses a critical value. (see Figure 2.8). To detect a Hopf bifurcation along a branch of stationary solutions of a one-parameter family of Ordinary Differential Equations

$$(2.3.2) \qquad u'(t) = f(u(t), \lambda) \qquad u, f \in R^n,$$

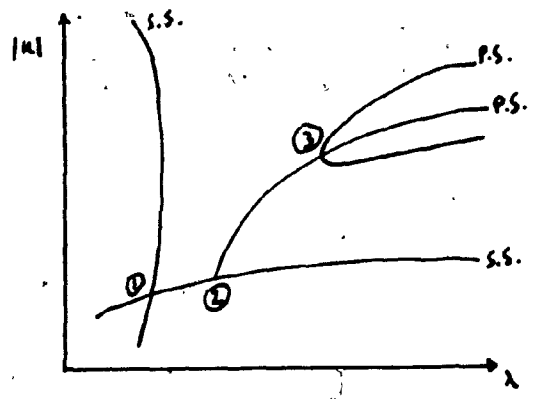



Fig. 2.7

s.s. means stationary solutions
p.s. means periodic solutions
① is a steady state bifurcation point
② is a Hopf bifurcation point
③ is a secondary periodic bifurcation point

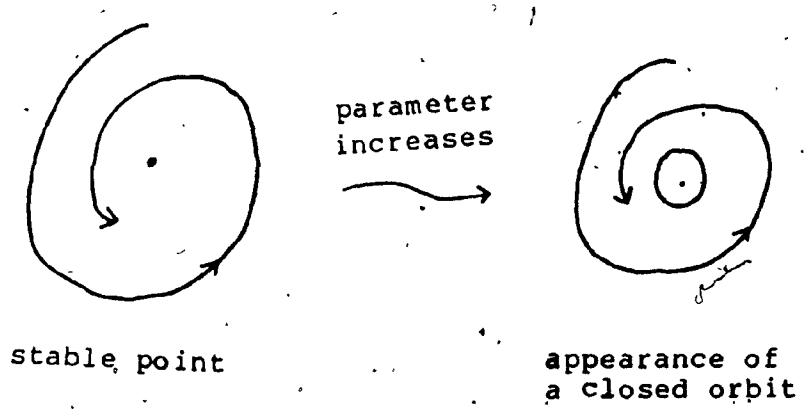



Fig. 2.8 Hopf Bifurcation

let $(u_*, \lambda_*)$ be a point on a stationary solution branch. Look for small amplitude periodic solutions

$$(2.3.3) \qquad u(t) = u_* + \varepsilon\phi(t)$$

with $\phi(t) = \phi(t+\rho)$ for all t, where $\rho$ is the period of the

solution.

Substitute (2.3.3) into (2.3.2) to obtain

$$(2.3.4) \qquad \varepsilon\phi'(t) = f(u_* + \varepsilon\phi(t), \lambda_*)$$

Taylor expanding about $u_*$, we have

$$(2.3.5) \qquad \varepsilon\phi'(t) = f(u_*, \lambda_*) + \varepsilon f_u^* \phi(t) + O(\varepsilon^2)$$

where $f_u^* \equiv f_u(u_*, \lambda_*)$.

For $\varepsilon$ small, $\phi(t)$ approximately satisfies

$$(2.3.6) \qquad \phi'(t) = f_u^* \phi(t).$$

Note that $f_u^*$ is an $n$ by $n$ matrix of constants. Thus a necessary condition for (2.3.2) to have small amplitude periodic solution is that the linear constant coefficient differential equation (2.3.6) have a periodic solution. In 1942, Hopf's pioneering paper[15] appeared, giving the basic results on time periodic bifurcation, that is, existence and uniqueness, symmetry properties, and stability of the solutions. The Hopf Bifurcation Theorem is the simplest result which guarantees the bifurcation of a family of time periodic solution of a system of differential equations, from a family of equilibrium solutions.

<u>Theorem</u> (2.3.1): ( <u>The Hopf Bifurcation Theorem</u> )

Consider a one-parameter family of Ordinary Differential Equations

$$u'(t) = f(u(t), \lambda) \qquad \lambda \in R,\ u \in R^n.$$

Suppose that $f(u_0, \lambda_0) \equiv 0$ and $f$ admits the linearization

$$y'(t) = A(\lambda)y(t)$$

where $A(\lambda) = \frac{\partial f}{\partial u}$.

Assume that $A(\lambda_0)$ has a pair of complex conjugate eigenvalues $z(\lambda)$ and $\bar{z}(\lambda)$ such that

$$(2.3.7) \quad \frac{d}{d\lambda}(\mathrm{Re}\ z(\lambda_0)) \neq 0,\ \mathrm{Re}\ z(\lambda_0) = 0 \text{ and } \mathrm{Im}\ z(\lambda_0) \neq 0.$$

Then there are periodic orbits bifurcating from the stationary solution $(u_0, \lambda_0)$ which has period close to $\frac{2\pi}{\mathrm{Im}\ z(\lambda_0)}$.

The hypothesis for the above theorem, simply means that a pair of complex conjugate eiganvalues $z(\lambda)$, $\bar{z}(\lambda)$ is crossing the imaginary axis at an acute angle. (see Figure 2.9).

The Hopf Bifurcation Theorem in infinite dimensional spaces was proved by Crandall and Rabinowitz [4] in 1977. Another version of the Hopf Bifurcation Theorem which applies to Delay Differential Equations was proved by Hale[11,12] and will be discussed in Chapter 3.

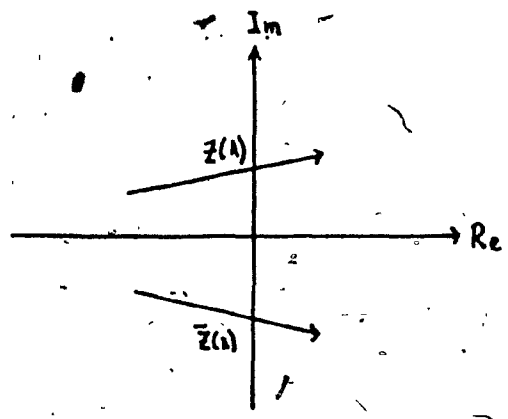



Fig. 2.9 Existence of Hopf Bifurcation

# CHAPTER THREE

## Delay Differential Equations

### 3.1 Retarded Functional Differential Equations

Suppose $r \geq 0$ is a given real number, $R=(-\infty,\infty)$, $R^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $C([a,b], R^n)$ is the Banach space of continuous functions mapping the interval $[a,b]$ into $R^n$. If $[a,b] = [-r,0]$, we let $C = C([-r,0], R^n)$, and designate the norm of an element $\phi$ in $C$ by $|\phi| = \sup_{-r\leq\theta\leq 0}|\phi(\theta)|$. If $\sigma\in R$, $A\geq 0$ and $x\in C([\sigma-r,\sigma+A],R^n)$, then for any $t\in[\sigma,\sigma+A]$, we let $x_t\in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-r\leq \theta \leq 0$. If $D$ is a subset of $R \times C$, $f : D \to R^n$ is a given function and "'" represents the right-hand derivative, we say that the relation

$$x'(t) = f(t,x_t) \tag{3.1.1}$$

is a Retarded Functional Differential Equation on $D$.

Equation (3.1.1) is a very general type of equation and includes

ordinary differential equations $(r=0)$

$$x'(t) = f(t,x(t)),$$

differential difference equations

$$x'(t) = f(t,x(t),x(t-\tau_1(t)),\ldots,x(t-\tau_p(t)))$$

with $0 \leq \tau_j(t) \leq r$, $j=1,2,\ldots,p$,

delay differential equations

$$x'(t) = f(t,x(t),x(t-\tau))$$

with $\tau$ constant in R, as well as

integro-differential equations

$$x'(t) = \int_{-r}^{0} g(t,\theta,x(t+\theta))\, d\theta.$$

We say equation (3.1.1) is <u>linear</u> if $f(t,\phi) = L(t,\phi)+h(t)$, where $L(t,\phi)$ is linear in $\phi$; <u>linear homogeneous</u> if $h\equiv 0$ and <u>linear nonhomogeneous</u> if $h \not\equiv 0$. We say equation (3.1.1) is <u>autonomous</u> if $f(t,\phi)=g(\phi)$ where g does not depend on t.

## 3.2 Basic Theory

For the existence and uniqueness of solutions with initial value of equation (3.1.1), we have the Existence and Uniqueness Theorems which can be found in [11]. Here we state these two theorems without proof.

<u>Theorem</u> (3.2.1) (Existence)

Suppose $\Omega$ is an open subset in R x C and $f\in C(\Omega,R^n)$. If $(\sigma,\phi)\in\Omega$, then there is a solution of equation (3.1.1)

passing through $(\sigma,\phi)$.

<u>Definition</u>: Let $\Omega$ is an open subset in $R \times C$ and $f: C \to R^n$. If for some $k \geq 0$

$$|f(t,\phi)-f(t,\bar{\phi})| \leq k|\phi - \bar{\phi}|$$

whenever $(t,\phi)$ and $(t,\bar{\phi})$ in $\Omega$, we say $f$ is <u>Lipschitz continuous</u> on $\Omega$ with Lipschitz constant $k$.

<u>Theorem</u> (3.2.2) (Uniqueness)

Suppose $\Omega$ is an open subset in $R \times C$, $f \in C(\Omega, R^n)$ and $f(t,\phi)$ is Lipschitz continuous in $\phi$ in each compact set in $\Omega$. If $(\sigma,\phi) \in \Omega$, then there is a unique solution of equation (3.1.1) through $(\sigma,\phi)$.

Apart from the Existence and Uniqueness Theorems in Hale's book, the Hopf Bifurcation Theorem for retarded functional differential equations is also presented. This theorem gives the conditions for which nonconstant periodic solutions of autonomous equations can arise. In the following the Hopf Bifurcation Theorem is stated without proof. The detection of Hopf Bifurcations along a stationary solution branch of delay differential equations will be discussed in Section 5.1 in this thesis.

Consider a one parameter family of retarded functional differential equations of the form

(3.2.1) $x'(t) = F(x_t, \lambda)$

where $F(\phi,\lambda)$ has continuous first and second derivatives in $\phi$, $\lambda$ for $\phi \in C$ and $\lambda \in R$, and $F(\emptyset,\lambda) = \emptyset$ for all $\lambda$. Define $L : R \times C \to R^n$ by

(3.2.2) $L(\lambda)\,\psi = F_\phi(\emptyset,\lambda)\psi$

where $F_\phi(\emptyset,\lambda)$ is the derivative of $F(\phi,\lambda)$ with respect to $\phi$ at $\phi=\emptyset$.

Under the following two Hypotheses:

(H1) The linearization (3.2.2) has a simple purely imaginary characteristic root $z_\emptyset = iy_\emptyset \neq \emptyset$ and all characteristic roots $z_j \neq z_\emptyset$, $\bar{z}_\emptyset$ satisfy $z_j \neq mz_\emptyset$ for any integer m.

(H2) $\frac{d}{d\lambda} \operatorname{Re} z(\lambda) \neq \emptyset$,

we have

<u>Theorem</u> (3.2.3) ( <u>The Hopf Bifurcation Theorem for Retarded Functional Differential Equations</u>)

Hypotheses (H1) and (H2) imply there are nonconstant periodic solutions of equation (3.2.1) bifurcating from $(\emptyset,\lambda)$ which have period close to $2\pi/y_\emptyset$.

## 3.3 Examples of Delay Differential Equations

As mentioned in section 3.1, the equation (3.1.1) is a very general type of equation which includes ordinary differential equations, differential difference equations, delay differential equations as well as integro-differential equation. In this thesis, we are interested in delay differential equations. In this section, we give some examples of physical and biological systems in which the present rate of change of some unknown function depends upon past values of the same function.

### Mixing of Liquids

Consider a tank containing B litres of salt water brine. Fresh water flows in at the top of the tank at a rate of q litres per second. The brine in the tank is continually stirred, and the mixed solution flows out through a hole at the bottom, also at the rate of q litres per second.

Let x(t) be the amount(in kilograms) of salt in the brine in the tank at time t. If we assume continual, instantaneous, perfect mixing throughout the tank, then the brine leaving the tank contains $x(t)/B$ Kg of salt per litre, and hence

$$x'(t) = -q\, x(t) / B \tag{3.3.1}$$

But, more realistically, mixing cannot occur instantaneously throughout the tank. Thus the concentration

of the brine leaving the tank at time t will equal the average concentration at some earlier instant, say $t-\tau$, $\tau$ is a positive constant. The differential equation (3.3.1) then becomes a delay differential equation

$$(3.3.2) \qquad x'(t) = -q\, x(t-\tau) / B$$

or setting $c=q/B$, we have

$$(3.3.3) \qquad x'(t) = -c\, x(t-\tau)$$

where $\tau$ is the "delay" or "time lag".

Population Growth

If $N(t)$ is the population at time t of an isolated colony of animals, the most naive model for the growth of the population is

$$(3.3.4) \qquad N'(t) = k\, N(t),$$

where k is a positive constant. This implies exponential growth, $N(t) = N_0 e^{kt}$ where $N_0 = N(0)$.

A somewhat more realistic model is obtained if we allow the growth rate coefficient k not be constant but to diminish as $N(t)$ grows, because of overcrowding and shortage of food. This leads to the differential equation

$$(3.3.5) \qquad N'(t) = k\, [1-N(t)/P]\, N(t),$$

where k and P are both positive constants.

Now suppose that the biological self-regulatory reaction modelled by the factor $[1-N(t)/P]$ in (3.3.5) is not instantaneous, but responds only after some time lag $\tau>0$. Then instead of (3.3.5) we have the delay differential equation

$$N'(t) = k\,[1-N(t-\tau)/P]\,N(t). \tag{3.3.6}$$

By introducing $x(s) = N(\tau s)/P - 1$ and $c = k\tau$, equation (3.3.6) can be rewritten as follows:

$$x'(s) = \tau N'(\tau s)/P$$

$$x'(s) = \tau\, k\,[1-N(\tau(s-1))/P]\,N(\tau s)/P$$

$$x'(s) = \tau\, k\,[-x(s-1)]\,[1+x(s)]$$

$$x'(s) = -c\,x(s-1)[1+x(s)] \tag{3.3.7}$$

This equation (3.3.7) has been studied extensively by Wright [41],[42], Kakutani and Markus [17], Jones [16], Kaplan and Yorke [18], Hale[11], and others.

## Predator-Prey Population Models

Let $x(t)$ be the population at time $t$ of some species of animal (prey) and let $y(t)$ be the population of a predator species which lives off these prey. We assume that $x(t)$ would increase at a rate proportional to $x(t)$ if the prey were left alone, that is, we would have $x'(t) = a_1\,x(t)$, where $a_1>0$. However the predators are hungry, and the rate

at which each of them eats prey is limited only by his ability to find prey. Thus we shall assume that the activities of the predators reduce the growth rate of $x(t)$ by an amount proportional to the product $x(t)y(t)$, that is, $x'(t) = a_1x(t) - b_1x(t)y(t)$, where $b_1$ is another positive constant.

Now let us also assume that the predators are completely dependent on the prey as their food supply. If there were no prey, we assume $y'(t) = -a_2\, y(t)$, where $a_2 > 0$, that is, the predator species would die out exponentially. However, given food the predators breed at a rate proportional to their number and to the amount of food available to them. Thus we consider the pair of equations

$$\begin{aligned} x'(t) &= a_1\, x(t) - b_1\, x(t)\, y(t) \\ y'(t) &= -a_2\, y(t) + b_2\, x(t)\, y(t), \end{aligned} \tag{3.3.8}$$

where $a_1$, $a_2$, $b_1$ and $b_2$ are positive constants. This well known model was introduced and studied by Lotka[23,24] and Volterra[37,38].

Wangersky and Cunningham [40] proposed to modify system (3.3.8) so that the birth rate of prey has a further limitation as in equation (3.3.5), while the birth rate of predators responds to changes in the magnitudes of $x$ and $y$ only after a delay $\tau>0$. Thus they replaced system (3.3.8) with

$$x'(t) = a_1[1-x(t)/P]x(t) - b_1y(t)x(t)$$

(3.3.9)

$$y'(t) = -a_2y(t) + b_2x(t-\tau)y(t-\tau)$$

Note that (3.3.9) is a nonlinear autonomous system of delay differential equations which can be written as

$$u'(t) = f(u(t),u(t-\tau)) \quad t,\tau \in R, \quad u,f\epsilon R^2.$$

We have given some examples of physical and biological systems. In these examples, the present rate of change of some unknown function depends both upon its present and past values. This is how delay differential equations arise. As can be seen in these examples, parameters appear naturally in these equations. One is often interested in the change of behaviour of the solutions as these parameters vary. In general, a system of delay differential equations can be expressed as

(3.3.10) $$u'(t) = f(u(t),u(t-\tau),\lambda) \quad u,f\epsilon R^n$$

where $\tau$ is a delay or time lag and $\lambda$ is a parameter.

## 3.4 Hadeler's Technique for computing Bifurcation Diagram

As mentioned in Chapter 2 in this thesis, as $\lambda$ varies, bifurcations may occur. Numerical techniques to find the solutions and bifurcation diagram of (3.3.10) have been developed by [8,10,35].

In Hadeler's paper[10], the equation

$$(3.4.1) \qquad u'(t) = -\lambda f(u(t-1)) \qquad \lambda, u, f \in R.$$

is considered. In a series of paper, Nussbaum[31,32,33] has discussed Equation(3.4.1) and has proved theorems on the existence of periodic solutions. For Equation (3.4.1), we require that $f(u)u > 0$ for $u \neq 0$, that f be differentiable, $f'(0) = 1$ and that $f(u) \geq -k$ for all u where $k > 0$ is some constant. Then for $\lambda > \frac{\pi}{2}$, Equation (3.4.1) has a non-constant periodic solution. Nussbaum has also shown that at $\lambda = \frac{\pi}{2}$ a family of periodic solutions bifurcates from the zero solution. Hassard[13] and Wan[19] can explain this bifurcation as a Hopf bifurcation. Walther[39] has shown that if $f'(u) > 1$ for u close to 0, $u \neq 0$, then the bifurcation arc starts at $\frac{\pi}{2}$, bends backward with increasing amplitudes, then bends again to the right and tends to $\infty$.

From the behaviour of Hopf bifurcation in ordinary differential equations one can conjecture that the solutions on the backward branch are unstable, at least for small amplitudes. On the other hand for large values of $\lambda$ the periodic solutions appear very stable. At the turning point, there is a change of stability. The determination of the unstable arc has been an extremely difficult task. Hadeler, in his paper, presented a method to compute the bifurcation diagram for delay equations. His technique can also compute unstable periodic solutions and backward

bifurcations. He uses $\lambda$ as the continuation parameter everywhere except at turning points. At turning points, the amplitude of the solution is used instead. To detect secondary bifurcation points, he keeps track of the rate of convergence of his numerical method.

Let $C[0,1]$ be the set of all continuous functions mapping the closed interval $[0,1]$ into $R$. For $\phi \in C[0,1]$, let $u(t,\phi)$ be the solution with initial datum $\phi$ at time $t$. Let $u^t(\phi)$ denote a segment of that solution, namely,

$$u^t(\phi) = u(t+s\ ,\ \phi) \qquad 0 \leq s \leq 1$$

Then for every $t \geq 0$, by $\phi \to u^t(\phi)$ a differential mapping of $C[0,1]$ into itself is defined. Suppose Equation (3.4.1) has a non-constant periodic solution. Given an approximate segment $\phi \in C[0,1]$ and an approximate period $\rho$, if $u(.,\phi)$ is periodic with period $\rho$ then

$$(3.4.2) \qquad u^\rho(\phi) - \phi = 0.$$

Since a periodic solution can be freely translated, to achieve local uniqueness, Hadeler uses the condition

$$(3.4.3) \qquad \phi(0) = 0$$

Condition (3.4.3) can be used here because in the existence proofs for periodic solutions of (3.4.1), Hadeler shows that, for $\lambda > 1$, initial data with at most one sign change lead to slowly oscillating solutions, that is, to solutions

which have infinitely many zeros converging towards infinity such that the distance of two successive zeros is greater than one. This is also the reason why Hadeler can take a quadratic polynomial as an initial guess for $\phi$. To discretize the problem, a meshsize $h = \frac{1}{n}$ is chosen and $\phi \in C[0,1]$ is replaced by a vector of n+1 values $\phi_i = \phi((i-1)h)$, $i=1,\ldots,n+1$. To compute the approximation of $u^{\rho}(\phi)$, linear or cubic interpolation was used. Given initial data for $\lambda$, n, $\phi$ and $\rho$, the system is solved by Newton's method. Near turning points, the amplitude of the solution is chosen as continuation parameter while $\lambda$ is introduced as an unknown to be determined. Thus a condition

$$(3.4.4) \qquad \phi(1) - A = 0$$

is added where A is the fixed amplitude.

To detect secondary periodic bifurcations, Hadeler keeps track of the rate of convergence of the Newton's method. When the rate of convergence is getting slow, he expects there is a secondary periodic bifurcation. To switch branches at secondary bifurcation point, he chooses an appropriate initial function $\phi$ by trials. In the paper, the detection of period doubling bifurcations is not discussed.

In the following two chapters of this thesis, we present a different numerical technique to compute the bifurcation diagram for delay differential equation. Our technique can

also compute unstable solutions and backward bifurcation. By using Keller's continuation technique, we can compute solutions at turning points without any modification of the system. In this respect our method is more automatic than Hadeler's technique. By automatic, we mean, by given a solution point on a stationary solution branch and a given window for the diagram, our implementation can trace out the whole branch while stationary and Hopf bifurcation points can be detected. Then the implementation can also switch from one branch to another without any intervention of the user. Secondary bifurcation points including period doubling bifurcation points can be detected while computing a periodic solution branch.

# CHAPTER FOUR

## Computation of Steady State Solution Branches

### 4.1 Continuation of a solution branch

Consider a system of delay differential equations

$$(4.1.1) \qquad u'(t) = f(u(t), u(t-\tau), \lambda) \qquad \tau, \lambda \in R, \; u, f \in R^n$$

where $\tau$ is the delay and $\lambda$ is the parameter of the system. Steady state solutions or stationary solutions of equation (4.1.1) satisfy

$$u'(t) = f(u(t), u(t-\tau), \lambda) = 0, \text{ for all } t,$$

or equivalently

$$(4.1.2) \qquad g(u(t), \lambda) \equiv f(u(t), u(t), \lambda) = 0, \text{ for all } t.$$

To compute a steady state solution branch of (4.1.1) is equivalent to computing a solution branch of (4.1.2). To apply the pseudo arclength continuation technique as discussed in Section 2.1, let $(u_*, \lambda_*)$ be a point, but not a singular point, on the branch. Let $(\dot{u}_*, \dot{\lambda}_*)$ be a unit vector which spans the null space of $g'(u_*, \lambda_*)$. Then the inflated problem (2.1.2) can be written as

$$(4.1.3) \qquad \begin{aligned} g(u \,;\, \lambda) &= 0 \\ M(u, \lambda \,;\, \delta s) &\equiv (u-u_*)^T \dot{u}_* + (\lambda-\lambda_*)\, \dot{\lambda}_* - \delta s = 0. \end{aligned}$$

To determine the initial direction vector $(\dot{u}_0, \dot{\lambda}_0)$ at the starting point $(u_0, \lambda_0)$, we compute the null vector of $g'(u_0, \lambda_0)$ by Gauss elimination with complete pivoting, if necessary, and back substitution. An illustration is given below:

<u>Example</u>: To find the null vector of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix}$$

By Gauss elimination, we have

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -7 \end{bmatrix}$$

Now, let $(x,y,z)$ be the null vector. Set $z = 1$; by back substitution, from row 2, we have $y = -7$. From row 1, we have $x = 11$. Thus $(11,-7, 1)$ is the null vector desired.

The direction vector $(\dot{u}_k, \dot{\lambda}_k)$ at $(u_k, \lambda_k)$ for $k \geq 1$ can be approximated by

$$\dot{u}_k = (u_k - u_{k-1})/\delta s$$

and $$\dot{\lambda}_k = (\lambda_k - \lambda_{k-1})/\delta s.$$

To solve the k-th solution step on the branch, we solve (4.1.3) by Newton's method using the following procedure:

i) set $j=0$

ii) $u_k^{(j)} = u_{k-1} + \delta s\ \dot{u}_{k-1}$

$\lambda_k^{(j)} = \lambda_{k-1} + \delta s\ \dot{\lambda}_{k-1}$

iii) Solve for $\delta u$, $\delta\lambda$

$$\begin{bmatrix} g_u^{(j)} & g_\lambda^{(j)} \\ M_u^{(j)} & M_\lambda^{(j)} \end{bmatrix} \begin{bmatrix} \delta u \\ \delta\lambda \end{bmatrix} = \begin{bmatrix} -g^{(j)} \\ -M^{(j)} \end{bmatrix}$$

where $g_u^{(j)} \equiv \frac{\partial g}{\partial u}$, $g_\lambda^{(j)} \equiv \frac{\partial g}{\partial \lambda}$, $M_u^{(j)} \equiv \frac{\partial M}{\partial u} = \dot{u}^T$,

$M_\lambda^{(j)} \equiv \frac{\partial M}{\partial \lambda} = \dot{\lambda}$, $g^{(j)}$ and $M^{(j)}$ are evaluated

at $(u_k^{(j)}, \lambda_k^{(j)})$ respectively.

iv) set $j=j+1$

v) $u_k^{(j)} = u_k^{(j-1)} + \delta u$

$\lambda_k^{(j)} = \lambda_k^{(j-1)} + \delta\lambda$

vi) if $|\delta u|$ and $|\delta\lambda|$ are less than the prespecified tolerance or j is greater than the maximum number of iterations allowed then stop; go to step (iii).
else go to step (iii).

## 4.2 Detection of a steady state Bifurcation point

A steady state bifurcation point is an intersection point of two steady state solution branches. A bifurcation point is found if at such a point there are two or more mutually linearly independent direction vectors each of which would lead to a steady state solution. In other words, the n by n+1 matrix

$$g'(u,\lambda) \equiv (g_u(u(s),\lambda(s)) \mid g_\lambda(u(s),\lambda(s)))$$

has at least a 2-dimensional null space. If we use the inflated continuation method, the Jacobian of the left hand side of the equations (4.1.3)

$$J(u,\lambda) = \begin{bmatrix} g_u(u(s),\lambda(s) & g_\lambda(u(s),\lambda(s)) \\ M_u(u(s),\lambda(s) & M_\lambda(u(s),\lambda(s)) \end{bmatrix}$$

evaluated at a bifurcation point must be singular. Thus bifurcation points can be detected by monitoring the determinant of $J(u,\lambda)$. When a sign change of the determinant of $J(u,\lambda)$ is detected, we can locate the bifurcation point accurately by an iterative method, say the Secant method, as follows:

Let $D_k = \det(J(u_k,\lambda_k))$ be the determinant of the n+1 by n+1 Jacobian matrix $J(u,\lambda)$ evaluated at $(u_k,\lambda_k)$.

Let $\delta s_k$ be the stepsize used to find the k+1-st solution $(u_{k+1},\lambda_{k+1})$.

Suppose $D_{k-1} D_k < 0$, then the bifurcation point lies somewhere on the branch between the solution $(u_{k-1}, \lambda_{k-1})$ and $(u_k, \lambda_k)$. To locate the solution $(u_*, \lambda_*)$ such that $\det(g'(u_*, \lambda_*)) = 0$, we proceed

(i) $\delta s_k = \delta s_{k-1} D_k / (D_{k-1} - D_k)$

(ii) With $\delta s_k$, use Newton's method as stated in Section 4.1 to find a solution $(u_{k+1}, \lambda_{k+1})$ on the branch.

(iii) if $|D_{k+1}|$ is greater than the user specified tolerance, then set k=k+1; go to step (i), else stop and $(u_{k+1}, \lambda_{k+1})$ is the bifurcation point.

## 4.3 Branch Switching Technique for Stationary Solutions

Suppose we have located a bifurcation point, $(u_*, \lambda_*)$ along a branch of steady state solution. In order to switch from one branch to another, we need to know the directions of the bifurcation branches. To simplify notation, let $x(s) \equiv (u(s), \lambda(s))$ and $x(t) \equiv (u(t), \lambda(t))$ where s and t are parameters such that for all s and t, $x(s)$ and $x(t)$ are steady state solutions of two different branches. Let $x'$ and $\dot{x}$ be the derivative of x with respect to t and s respectively. Thus

$$g(x(s))=0 \text{ implies } g_x(x(s))\ \dot{x}(s) = 0$$

$$g(x(t))=0 \text{ implies } g_x(x(t))\ x'(t) = 0$$

where $g_x \equiv (g_u \mid g_\lambda)$.

In particular, at the bifurcation point $x_* = (u_*, \lambda_*)$

$$g_x^* \dot{x} = 0 \text{ and } g_x^* x' = 0.$$

So $g_x^*$ has at least a 2-D null space. Moreover,

$$g(x(t)) = 0$$

Taking the derivative with respect to x, we have

$$g_x(x(t))\ x'(t) = 0$$

Taking the second derivative with respect to x, we have

$$g_{xx}(x(t))\ x'(t)\ x'(t) + g_x(x(t))\ x''(t) = 0$$

or

$$g_x^* x_*'' = -g_{xx}^* x_*' x_*'$$

Thus

$$g_{xx}^* x_*' x_*' \in R(g_x^*),$$

where $R(g_x^*)$ denotes the range of $g_x^*$.

By elementary linear algebra, $R(g_x^*) = N(g_x^{*T})^\perp$.

Suppose $\psi$ is a null vector of $g_x^{*T}$.

Then $g_{xx}^* x_*' x_*' \perp \psi$, or

$$\psi^T g_{xx}^* x_*' x_*' = 0 \tag{4.3.1}$$

for all null vectors $\psi$ of $g_x^{*T}$.

Equation (4.3.1) is the <u>Algebraic Bifurcation Equation</u>.

Note that $g_x^*$ is a n by n+1 matrix,

$g_{xx}^*$ is a n by n+1 by n+1 bilinear form,

$x_*'$ is a n+1 by 1 vector, and

$\psi$ is a n by 1 vector.

Suppose the dimension of the null space of $g_x^*$

$$\dim N(g_x^*) = k, \qquad 2 \leq k \leq n+1,$$

then $$\dim N(g_x^{*T}) = k-1.$$

Let $N(g_x^*) = \text{span}\{\phi_1,\ldots,\phi_k\}$, $N(g_x^{*T}) = \text{span}\{\psi_1,\ldots,\psi_{k-1}\}$,

then $x_*' = c_1\phi_1+\ldots+c_k\phi_k$ and from (4.3.1) we have

$$\psi_1^T g_{xx}^* (c_1\phi_1+\ldots+c_k\phi_k)(c_1\phi_1+\ldots+c_k\phi_k) = 0$$

$$\vdots \qquad\qquad \vdots$$

(4.3.2)

$$\vdots \qquad\qquad \vdots$$

$$\psi_{k-1}^T g_{xx}^* (c_1\phi_1+\ldots+c_k\phi_k)(c_1\phi_1+\ldots+c_k\phi_k) = 0.$$

To solve for $x_*'$, we have to solve for the $c_i$'s $i=1,\ldots,k$. Note that the system (4.3.2) is not a linear system, it is a quadratic one rather. Let us consider the case when k=3. In this case we have the following system for $c_1$, $c_2$ and $c_3$.

$$a_{11}c_1^2 + a_{21}c_2^2 + a_{31}c_3^2 + a_{41}c_1c_2 + a_{51}c_2c_3 + a_{61}c_1c_3 = 0$$

(4.3.3)

$$a_{12}c_1^2 + a_{22}c_2^2 + a_{32}c_3^2 + a_{42}c_1c_2 + a_{52}c_2c_3 + a_{62}c_1c_3 = 0$$

where $a_{ij}$ $i=1,\ldots,6$; $j=1,2$ are known constants.
We can add one more constraint, say

(4.3.4) $$c_1^2 + c_2^2 + c_3^2 = 1$$

to (4.3.3) and try to solve it by some iterative method (say Newton's method). However (4.3.3) and (4.3.4) may have more than k solutions or even infinitely many solutions and an initial guess to start the iterative method is difficult to choose systematically.

We have discussed the difficulties in computing the bifurcating directions when dim $N(g_x^*)=k \geq 3$. However if $k=2$, we can compute the directions analytically. When $k=2$, the algebraic bifurcation equations consists of only one equation

(4.3.5) $$\psi^T g_{xx}^* (c_1\phi_1 + c_2\phi_2)(c_1\phi_1 + c_2\phi_2)=0.$$

Let $\psi = (\psi_1 \ldots \psi_n)^T$

$\phi_1 = (\phi_{11} \cdots \phi_{1\,n+1})^T$

$\phi_2 = (\phi_{21} \cdots \phi_{2\,n+1})^T$

$g_{xx}^* = (a_{ijk})$

where $i=1,..,n$; $j=1,..,n+1$; $k=1,..,n+1$

$$a_{ijk} = \frac{\partial}{\partial u_k} \frac{\partial}{\partial u_j} g_i \qquad \text{and} \qquad u_{n+1} \equiv \lambda$$

Then (4.3.5) can be written as

$$(4.3.6) \qquad \sum_{i=1}^{2} \sum_{k=1}^{2} \sum_{s=1}^{n} \sum_{j=1}^{n+1} \sum_{p=1}^{n+1} \psi_s \, a_{spj} \, \phi_{ij} \, \phi_{kp} \, c_k \, c_i = 0$$

or

$$(4.3.7) \qquad A \, c_1^2 + B \, c_1 c_2 + C \, c_2^2 = 0$$

where

$$A = \sum_{s=1}^{n} \sum_{p=1}^{n+1} \sum_{j=1}^{n+1} \psi_s \, a_{spj} \, \phi_{1j} \, \phi_{1p},$$

$$B = 2 \sum_{s=1}^{n} \sum_{p=1}^{n+1} \sum_{j=1}^{n+1} \psi_s \, a_{spj} \, \phi_{1j} \, \phi_{2p}, \text{ and}$$

$$C = \sum_{s=1}^{n} \sum_{p=1}^{n+1} \sum_{j=1}^{n+1} \psi_s \, a_{spj} \, \phi_{2j} \, \phi_{2p}.$$

To solve for $c_1, c_2$ from (4.3.7) we investigate the values of A,B and C. If A, B and C are nonzero, we check the discriminant $\delta = B^2 - 4AC$.

If $\delta < 0$ we do not have any real solutions.

If $\delta \geq 0$ we have $c_1 = \frac{-B+\sqrt{\delta}}{2A} c_2$.

If $A=0$ and $C \neq 0$ we have $c_2 = 0$ or $c_2 = \frac{-B}{C} c_1$.

If $C=0$ and $A \neq 0$ we have $c_1 = 0$ or $c_1 = \frac{-B}{A} c_2$.

If $A=C=\emptyset$ and $B\neq\emptyset$ we set either $c_1$ or $c_2=\emptyset$ but not both.

If $A=B=C=\emptyset$, we have a higher order singularity and higher derivatives must be considered.

## 4.4 Stability of Stationary Solutions

It is shown in [11] that a stationary solution of (4.1.1) is stable if the roots of the characteristic equation of the linearized system of (4.1.1) at the stationary solution, have negative real parts.

The linearized system of (4.1.1) at a stationary solution $u(t) \equiv u_*$ is

$$u'(t) = A(\lambda)\ u(t) + B(\lambda)\ u(t-\tau) \tag{4.4.1}$$

where $A(\lambda)$ and $B(\lambda)$ are $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ respectively, $v(t) \equiv u(t-\tau)$. Both are $n \times n$ matrices, evaluated at $u_*$.

The characteristic equation for a system of n homogeneous linear delay differential equations with constant coefficients is obtained from the equation by looking for nontrivial solutions $u(t)$ of the form $u(t) \equiv e^{zt}c$, where c is nonzero constant in $R^n$. To derive the characteristic equation of (4.4.1), substitute $u(t) = e^{zt}c$ in (4.4.1). This gives

$$z\ I\ e^{zt}c = A(\lambda)\ e^{zt}\ c + B(\lambda)\ e^{z(t-\tau)}\ c$$

where I is the $n \times n$ identity matrix.

Dividing both side by $e^{zt}$ and collecting terms, we have

$$( zI - A(\lambda) - B(\lambda)\ e^{-z\tau} )\ c = 0.$$

Since c must be nonzero, we have

$$\det\ [zI - A(\lambda) - B(\lambda)e^{-z\tau}] = 0 \tag{4.4.2}$$

which is the characteristic equation of (4.4.1).

To find a root, z, of the characteristic equation (4.4.2), let $z=x+iy$, $x,y\in R$. Substituting $z=x+iy$ into (4.4.2), we have

$$\det\ [(x+iy)I - A(\lambda) - B(\lambda)e^{-\tau x-i\tau y}] = 0.$$

Expressing the equation in its real and imaginary parts, we have

$$\det\ [(xI - A(\lambda) - B(\lambda)e^{-\tau x}\cos\ \tau y) + i(yI + B(\lambda)e^{-\tau x}\sin\ \tau y)] = 0. \tag{4.4.3}$$

Consider the case n=2, and let

$$A(\lambda) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B(\lambda) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Substituting $A(\lambda)$, $B(\lambda)$ into (4.4.3), we have

$$\begin{vmatrix} c_{11} + i\ d_{11} & (c_{12}-x) + i(d_{12}-y) \\ (c_{21}-x) + i(d_{21}-y) & c_{22} + i\ d_{22} \end{vmatrix} = 0 \tag{4.4.4}$$

where $c_{ij} = x - a_{ij} - b_{ij}e^{-\tau x}\cos \tau y$

and $d_{ij} = y + b_{ij}e^{-\tau x}\sin \tau y$ for $i,j=1,2$.

It is difficult, as we can see, to find all the roots for x and y of the system (4.4.4) analytically or numerically. These difficulties are present even more strongly for any n greater than 2. However, for the case n=1, the characteristic equation of (4.1.1) is

$$(4.4.5) \qquad z - A(\lambda) - B(\lambda)\, e^{-z\tau} = 0$$

where $A(\lambda)$ and $B(\lambda)$ are respectively $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ evaluated at a stationary solution $u_*$.

With the help of the following theorem which can be found in the appendix of Hale's book [11], we can determine the stability of a stationary solution easily:

<u>Theorem</u> (4.4.1):

All roots of the equation $(z+a)e^{z}+b=0$, where a and b are real, have negative real parts if and only if

$$a > -1$$

$$a + b > 0$$

$$b < x \sin(x) - a \cos(x)$$

where x is the root of $x = -a \tan(x)$,

$$0 < x < \frac{\pi}{2} \text{ if } a \neq 0,$$

$$x = \frac{\pi}{2} \text{ if } a = 0.$$

In order to apply this theorem, we have to scale (4.1.1) such that the delay term $\tau$ is equal to one. Scaling t to $\frac{t}{\tau}$ we have

$$(4.4.6) \qquad u'(t) = \tau f(u(t),u(t-1),\lambda).$$

Hence the characteristic equation of (4.4.6) is

$$(4.4.7) \qquad z - \tau A(\lambda) - \tau B(\lambda) e^{-z} = 0$$

or

$$(4.4.8) \qquad (z - \tau A(\lambda)) e^{z} - \tau B(\lambda) = 0.$$

To apply Theorem 4.4.1 for (4.4.8), take $a=-\tau A(\lambda)$ and $b=-\tau B(\lambda)$. Any stationary solution of (4.4.6), thus of (4.1.1), is stable if and only if the conditions in Theorem (4.4.1) are satisfied.

Due to the difficulties we have discussed above for analyzing the characteristic equation of the system of n delay differential equations for $n\geq 2$, we have restricted ourselves to a single delay differential equation (that is $n=1$) in this thesis. However, apart from the analysis of the characteristic equation, the methods in this thesis are applicable to general systems of delay differential equations.

The following is an example that ilustrates how Theorem (4.4.1) can be applied to determine the stability of a stationary solution of a delay differential equation. The equation considered will receive further numerical analysis

in Chapter 5.

<u>Example</u>: Consider

$$u'(t) = -\lambda\, u(t-1)\,\frac{1+u(t-1)^2}{1+u(t-1)^4}.$$

Denoting the derivatives of the right hand side with respect to $u(t)$ and $u(t-1)$ by $A(\lambda)$ and $B(\lambda)$ respectively, we have

$$A(\lambda)=0$$

$$B(\lambda)=\lambda(v^6+3v^4-3v^2-1)/(1+v^4)^2, \quad \text{where } v=u(t-1)$$

Note that as $u'(t)=0$, we have two stationary solution branches

$$u(t) = u(t-1) = 0 \quad \text{for all } \lambda\in R$$

and

$$\lambda=0 \text{ for all } u\in R$$

To investigate the stability of the solutions along the first branch, we evaluate $A(\lambda)$ and $B(\lambda)$ at $u(t)=u(t-1)=0$:

$$A(\lambda)=0$$

$$B(\lambda)=-\lambda$$

Taking $a=-A(\lambda)=0$ and $b=-B(\lambda)=\lambda$ in Theorem (4.4.1), we find that the solutions of the first branch are stable if and only if $0<\lambda<\frac{\pi}{2}$.

Similarly, for the second branch, we have

$$A(\lambda)=0$$

$$B(\lambda)=0,$$

therefore all solutions on the second branch are unstable.

CHAPTER FIVE

# Computation of Periodic Solution Branches

## 5.1 Detection of Hopf Bifurcation points

As mentioned in Section 2.3, the Hopf Bifurcation refers to the development of periodic orbits from a equilibrium point. To compute periodic solution branches, it is natural to start at a Hopf Bifurcation point. To detect the Hopf Bifurcation point, we can apply the Hopf Bifurcation Theorem. In Section 3.2, we have stated the Hopf Bifurcation Theorem for retarded functional differential equations. As we know from Section 3.1, a retarded functional differential equation is a general type of equation which includes delay differential equations. To apply the Hopf Bifurcation Theorem for Delay Differential Equations, consider a one parameter family of Delay Differential Equations of the form

$$(5.1.1)\quad u'(t) = f(u(t), u(t-\tau), \lambda) \qquad \lambda, \tau \in R, \quad f(.,.,.), \; u \in R^n$$

where f has continuous first and second derivatives in $\lambda$ and u; and $f(u_0, u_0, \lambda_0) = 0$. The linearization of (5.1.1) is

$$(5.1.2)\qquad u'(t) = A(\lambda)u(t) + B(\lambda)v(t), \qquad v(t) \equiv u(t-\tau)$$

where $A(\lambda)$ and $B(\lambda)$ are respectively $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$ evaluated at $u=u_0$.

Recall the two hypotheses (H1) and (H2) stated in Section 3.2:

(H1) The linearization (5.1.2) has a simple purely imaginary characteristic root $z_0 = iy_0 \neq 0$ and all characteristic roots $z_j \neq z_0$, $\bar{z}_0$ satisfy $z_j \neq mz_0$ for any integer m.

(H2) $\frac{d}{d\lambda} \operatorname{Re} z(\lambda) \neq 0$.

Hale[11] has shown that Hypotheses (H1) and (H2) imply there are nonconstant periodic solutions of equation (5.1.1) bifurcating from $(u_0, \lambda_0)$ which have period close to $2\pi/y_0$.

In Section 4.4, the characteristic equation of (5.1.2) has been derived as

$$\det [zI - A(\lambda) - B(\lambda)e^{-z\tau}] = 0 \tag{5.1.3}$$

where I, $A(\lambda)$ and $B(\lambda)$ are n x n matrix.
In the same section, we have discussed the difficulties to analyze the characteristic equation for $n \geq 2$. For n=1, the characteristic equation of (5.1.2) is

$$z - A(\lambda) - B(\lambda)e^{-z\tau} = 0 \tag{5.1.4}$$

where $z = x + iy$ $x, y \in R$.

To investigate the behaviour of the roots of (5.1.4) we express (5.1.4) into real and imaginary parts

(5.1.5a) $x - A(\lambda) - B(\lambda)e^{-x\tau}\cos(\tau y) = 0$

(5.1.5b) $y + B(\lambda)e^{-x\tau}\sin(\tau y) = 0.$

The following two lemma give conditions which imply the Hypotheses (H1) and (H2) respectively.

Lemma (5.1.1)

Suppose $|B(\lambda)| > |A(\lambda)|$ and there exists $y \neq 0$ satisfying

$$A(\lambda) + B(\lambda)\cos\tau y = 0$$

and

$$y + B(\lambda)\sin\tau y = 0.$$

Then the Hypothesis (H1) is satisfied and $z=iy$ is the characteristic root.

Proof:

Suppose there exist $y \neq 0$ satisfying

(5.1.6a) $A(\lambda) + B(\lambda)\cos(\tau y) = 0$

(5.1.6b) $y + B(\lambda)\sin(\tau y) = 0.$

Then by elementary trigonometry, from (5.1.6), we have

$$y^2 + A^2(\lambda) = B^2(\lambda)\cos^2\tau y + B^2(\lambda)\sin^2\tau y = B^2(\lambda)$$

or

(5.1.7) $y^2 = B^2(\lambda) - A^2(\lambda).$

Since $|B(\lambda)| > |A(\lambda)|$

it follows that $y \in R$ and $y \neq 0$.

Thus $(0, y)$ is a solution of the equations (5.1.5a) and (5.1.5b). Hence $z = 0 + iy$ is a solution of (5.1.4) which is the characteristic equation of (5.1.2).

Now it is sufficient to show that for any integer m with $|m| > 1$, $z_* = \pm iy_* = imy$ is not a root of (5.1.6) or equivalently not a root of (5.1.7). Since

$$y_*^2 = m^2(B^2(\lambda) - A^2(\lambda)) \neq B^2(\lambda) - A^2(\lambda),$$

hence $y_*$ is not a root of (5.1.7). □

<u>Lemma</u> (5.1.2)

Let $z=x+iy$, $x=0$, $y\neq 0$ be a solution of (5.1.5).

If $A'(\lambda) + B'(\lambda)\cos(\tau y) - \tau(A(\lambda)A'(\lambda) - B'(\lambda)B(\lambda)) \neq 0$

where $A'(\lambda)$ and $B'(\lambda)$ are the derivatives of $A(\lambda)$ and $B(\lambda)$ with respect to $\lambda$,

then Hypothesis (H2) is satisfied.

<u>Proof</u>:

Differentiate (5.1.5) with respect to $\lambda$, to get

$$\text{(5.1.8a)}\quad x' - A'(\lambda) - B'(\lambda)e^{-x\tau}\cos(\tau y) + \tau y'B(\lambda)e^{-x\tau}\sin(\tau y) + B(\lambda)\cos(\tau y)\tau x'e^{-x\tau} = 0$$

$$\text{(5.1.8b)}\quad y' + B'(\lambda)e^{-x\tau}\sin(\tau y) + \tau y'B(\lambda)e^{-x\tau}\cos(\tau y) - \tau x'e^{-x\tau}B(\lambda)\sin(\tau y) = 0$$

Evaluating (5.1.8) at $x=0$, we have

$$\text{(5.1.9a)}\quad (1-\tau A(\lambda))x' - \tau yy' = A'(\lambda) + B'(\lambda)\cos(\tau y)$$

(5.1.9b) $\tau y\, x' + (1-\tau A(\lambda))y' = -B'(\lambda)\sin(\tau y)$

Since $y \neq 0$, we have

$$\begin{vmatrix} 1-\tau A(\lambda) & -\tau y \\ \tau y & 1-\tau A(\lambda) \end{vmatrix} \neq 0$$

Hence, there is a unique solution $(x',y')$ of (5.1.9) and

$$x' = \frac{\begin{vmatrix} A'(\lambda)+B'(\lambda)\cos(\tau y) & -\tau y \\ -B'(\lambda)\sin(\tau y) & 1-\tau A(\lambda) \end{vmatrix}}{\begin{vmatrix} 1-\tau A(\lambda) & -\tau y \\ \tau y & 1-\tau A(\lambda) \end{vmatrix}}$$

Now $\frac{d}{d\lambda}$ Re $z(\lambda) = x' \neq 0$

iff

$$(A'(\lambda) + B'(\lambda)\cos(\tau y))(1-\tau A(\lambda)) - \tau y B'(\lambda)\sin(\tau y) \neq 0$$

iff

(5.1.10)

$$A'(\lambda) + B'(\lambda)\cos(\tau y) - \tau(A(\lambda)A'(\lambda) - B'(\lambda)B(\lambda)) \neq 0. \square$$

With Lemma (5.1.1) and (5.1.2), we can establish a Hopf Bifurcation Theorem for a single delay differential equation.

Theorem: (5.1.1)

Consider a one parameter delay differential equation of the form

(5.1.11) $u'(t) = f(u(t), u(t-\tau), \lambda) \quad \tau, \lambda, u, f \in R$

where f has continuous first and second derivative in $\lambda$ and u; and $f(u_0, u_0, \lambda_0) = 0$. Define the linearization of (5.1.11) as

(5.1.12) $u'(t) = A(\lambda)u(t) + B(\lambda)v(t) \quad v(t) \equiv u(t-\tau)$,

where $A(\lambda)$ and $B(\lambda)$ are respectively $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ evaluated at $u=u_0$ and $\lambda=\lambda_0$.

Define $A'(\lambda)$ and $B'(\lambda)$ as the derivative of $A(\lambda)$ and $B(\lambda)$ with respect to $\lambda$.

Suppose $|B(\lambda)| > |A(\lambda)|$ and there exists $y \neq 0$, $y \in R$ such that

(5.1.13a) $A(\lambda) - B(\lambda)\cos \tau y = 0$,

(5.1.13b) $y + B(\lambda)\sin \tau y = 0$

and

(5.1.14)

$$A'(\lambda) + B'(\lambda)\cos(\tau y) - \tau(A(\lambda)A'(\lambda) - B'(\lambda)B(\lambda)) \neq 0.$$

Then there are nonconstant periodic solutions of equation (5.1.11) bifurcating from $(u_0, \lambda_0)$ which have period close to $2\pi/y$.

Corollary:

Instead of finding a $y \neq 0$, $y \in R$ to satisfy the conditions (5.1.13) and (5.1.14), it is equivalent to find a pair $(u_0, \lambda_0)$ which satisfies the same conditions where y is replaced by $\sqrt{(B^2(\lambda) - A^2(\lambda))}$. Moreover, if such a pair $(u_0, \lambda_0)$ exists, then $(u_0, \lambda_0)$ is a Hopf bifurcation point.

Proof: It has already been shown in (5.1.7) that

$$y^2 = B^2(\lambda) - A^2(\lambda). \square$$

Remark: The Hopf Bifurcation Theorem gives sufficient conditions for a bifurcation to occur. Other Hopf Bifurcation points not satisfying all these conditions, are called "degenerate", (not generic, exceptional) and need more effort to deal with. In this thesis, we only consider the non-degenerate ones.

Example: Consider

(5.1.15) $u'(t) = -\lambda u(t-1)(1+u(t))$.

If $u'(t)=0$ then

(i) $u(t) = 0$, or

(ii) $u(t) \equiv -1$ or

(iii) $\lambda = 0$.

Thus equation (5.1.15) has three branches of stationary solutions. They are

(i) $u(t) \equiv 0$ for all $\lambda$

(ii) $u(t) \equiv -1$ for all $\lambda$ and

(iii) $\lambda = 0$ for all u.

We have

$$A(\lambda) = \frac{\partial f}{\partial u} = -\lambda u(t-1)$$

$$B(\lambda) = \frac{\partial f}{\partial v} = -\lambda(1+u(t))$$

$$A'(\lambda) = \frac{\partial A}{\partial \lambda} = -u(t-1)$$

$$B'(\lambda) = \frac{\partial B}{\partial \lambda} = -(1+u(t))$$

To detect a Hopf Bifurcation along any one of the stationary branches, we can apply the Hopf Bifurcation Theorem. Let $A_*$, $B_*$, $A_*'$ and $B_*'$ be $A(\lambda)$, $B(\lambda)$, $A'(\lambda)$ and $B'(\lambda)$ evaluated at a stationary solution, $(u_*, \lambda_*)$, respectively. Suppose we are on the first branch, that is $u(t) \equiv 0$ for all $\lambda$. Then, for any $\lambda$, $(0,\lambda)$ is a stationary solution and

(5.1.16) $A_* = 0$, $B_* = -\lambda$, $A_*' = 0$ and $B_*' = -1$.

Substituting (5.1.16) into (5.1.6) and (5.1.10) where $y = \sqrt{(B^2(\lambda) - A^2(\lambda))}$, we have

(5.1.17a) $\cos(\lambda) = 0$

(5.1.17b) $1 - \sin(\lambda) = 0$

(5.1.17c) $\cos(\lambda) - \lambda \neq 0$

Solving (5.1.17a) and (5.1.17b) for $\lambda$, we find $\lambda=(4k+1)\pi/2$ for any integer k and for such $\lambda$, (5.1.17c) is satisfied. Thus for any integer k, $(0,(4k+1)\pi/2)$ is a non-degenerate

Hopf bifurcation point with period $\frac{4}{4k+1}$.

To detect Hopf bifurcation along the other stationary solution branches, consider the second branch where $u(t) \equiv -1$ for all $\lambda$. We have $A_* = \lambda$, $B_* = 0$, $A_*' = -1$ and $B_*' = 0$. For the third branch where $\lambda \equiv 0$ for all u, we have $A_* = 0$, $B_* = 0$, $A_*' = 0$ and $B_*' = 0$. Note that in both cases, the condition $|B_*| > |A_*|$ is violated. Hence there is no non-degenerate Hopf bifurcation on these two branches.

## 5.2 Computation of periodic solution branches

This section is devoted to the computation of branches of periodic solutions of delay differential equations. First the use of initial value techniques for solving delay differential equations is discussed and we indicate why the initial value technique is not appropriate for computing branches of periodic solutions, especially not near Hopf bifurcation points and for asymptotically unstable solutions. Instead, we show how Keller's pseudo arclength continuation can be used to compute branches of periodic solutions of delay differential equations.

The initial condition for a delay differential equation of the form

$$(5.2.1) \qquad u'(t) = f(u(t), u(t-\tau), \lambda) \qquad \tau, \lambda, f, u \in R$$

is a function $h : [-\tau, 0] \to R$ such that $u(t) \equiv h(t) \quad t \in [-\tau, 0]$.

We can use initial value techniques, like Euler's method, Predictor-Corrector methods, Runge Kutta methods, etc. to solve equation (5.2.1) for a given initial condition.

Let $u_j \equiv u(t_j)$; $v_j \equiv u(t_j-\tau)$ and $\delta t \equiv t_{j+1}-t_j$ for all j. Assume $\delta t$ is taken such that $\tau$ is an integral multiple of $\delta t$. We have for example

(1) Euler's method:

$$u_{j+1} = u_j + \delta t\ f(u_j, v_j, \lambda)$$

(2) Predictor-Corrector method or Modified Euler's method:

$$\bar{u}_{j+1} = u_j + \delta t\ f(u_j, v_j, \lambda)$$

$$u_{j+1} = u_j + \frac{\delta t}{2}\ [f(u_j, v_j, \lambda) + f(\bar{u}_{j+1}, v_{j+1}, \lambda)]$$

(3) 4th order Runge Kutta Method:

$$K_1 = \delta t\ f(u_j, v_j, \lambda)$$

$$K_2 = \delta t\ f(u_j + \tfrac{1}{2}K_1,\ v_j + \tfrac{1}{2}K_1, \lambda)$$

$$K_3 = \delta t\ f(u_j + \tfrac{1}{2}K_2,\ v_j + \tfrac{1}{2}K_2, \lambda)$$

$$K_4 = \delta t\ f(u_j + \tfrac{1}{2}K_3,\ v_j + \tfrac{1}{2}K_3, \lambda)$$

$$u_{j+1} = u_j + \tfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).$$

There are two main reasons why these initial value techniques are not chosen to compute branches of periodic solutions of delay differential equations. First, initial

value techniques can only give us stable solutions. It is difficult to extract unstable solutions from results computed by initial value technique. Second, initial value techniques have difficulties near Hopf bifurcation points. In practice, near the Hopf bifurcation, it takes more time to settle down to a stable periodic solution than away from the Hopf bifurcation point. For these reasons, a framework is established in which Keller's pseudo arclength continuation technique can be used to compute branches of periodic solutions.

To apply the pseudo arclength technique for periodic solutions, it is convenient to scale the independent variable, t, of (5.2.1) by the factor $\frac{2\pi}{\rho}$, where $\rho$ is the unknown period of the solution. This transforms the equation (5.2.1) into

$$u'(t) = \frac{\rho}{2\pi} f(u(t), u(t - \frac{2\pi}{\rho}\tau), \lambda)$$

to which we now want to determine $2\pi$ periodic solutions. In ordinary differential equations, a given initial condition $u(0)=u_0$ determines a unique solution and the requirement $u(0) = u(2\pi)$ is enough to determine a periodic solution. However, in delay differential equation, the initial condition is a function in an interval of length $\tau$. Periodic solutions cannot be obtained by simply requiring $u(0) = u(2\pi)$. A counter example is given in Figure 5.1 where a solution satisfying the condition $u(0) = u(2\pi)$ is

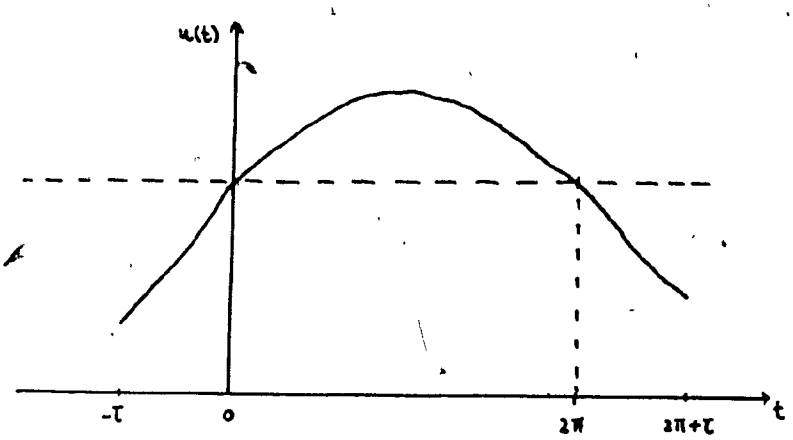



Fig. 5.1 $u(0)=u(2\pi)$ does not imply that u is $2\pi$-periodic solution to Delay Differential Equation.

not a $2\pi$-periodic solution of a delay differential equation.

Our approximate solutions, that are $2\pi$ periodic and continuously differentiable, are chosen from the space of truncated trigonometric expansions. The discrete system is obtained by collocation. More precisely, for fixed n, we seek

$$(5.2.2) \qquad u_n(t) = \sum_{k=-n}^{n} c_k e^{ikt}$$

$$= a_0 + \sum_{k=1}^{n} a_k \sin(kt) + \sum_{k=1}^{n} b_k \cos(kt)$$

that satisfies the differential equation (5.2.1) at 2n+1 equally spaced points $t_j = j\delta t$, $\delta t = \frac{2\pi}{2n+1}$, that is

$$(5.2.3) \qquad u_n'(t_j) = \frac{\rho}{2\pi} f(u_n(t_j),\ u_n(t_j - \frac{2\pi}{\rho}\tau),\ \lambda),$$

$$j=0,1,\ldots,2n.$$

A periodic solution can be translated freely in time; that is, if $u(t)=v(t)$ is a solution, then so is the solution $u(t)=v(t+r)$, for any $r$. A remaining difficulty is therefore the inherent non-uniqueness of $u$. The solution $u$ still must be "anchored". There are many possible choices for an additional equation to accomplish this. One is to simply fix one of the components of $u$ at $t=0$ to some constant $u_0$, where $\min u(t) < u_0 < \max u(t)$. (This is in fact what Hadeler uses [10]). However it requires knowledge of the bounds of the solution in advance. For the proof of the existence of solutions, a better choice is the orthogonality condition

$$(5.2.4) \qquad (u(0)-u_0(0))^T f(u_0(0), u_0(0), \lambda) = 0$$

which ensures that $u(0)$ on the orbit to be determined occupies a similar position as $u_0(0)$ on the known orbit. (see Figure 5.2). However, with the anchor equation (5.2.4), as it is shown in [7], the peaks of the solutions move as we go along the branch of periodic solutions. This motion becomes more pronounced as the front get steeper.

To derive an alternative for (5.2.4) that performs better on difficult problems, as suggested by [7], it is natural to seek a solution that minimizes the distance

$$(5.2.5) \qquad \int_0^{2\pi} (v(t+r)-u_0(t))^2\, dt$$

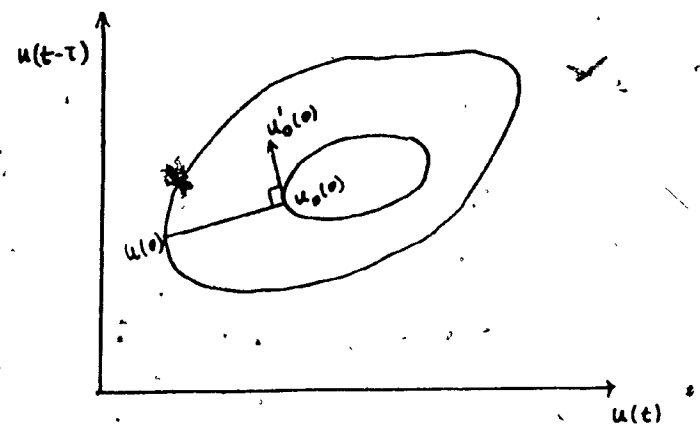



Fig. 5.2 Two "anchored" solutions with equation (5.2.4).

over r. This would force peaks to remain approximately in the same place. The minimizing $r^*$ is obtained by setting the derivative of (5.2.5) with respect to r equal to zero. That is

$$\int_0^{2\pi} (v(t+r^*)-u_0(t))\ v'(t+r^*)\ dt = 0$$

Letting $u(t) = v(t+r^*)$, we have

$$(5.2.6) \qquad \int_0^{2\pi} (u(t)-u_0(t))\ u'(t)\ dt = 0.$$

It is shown in the following lemma that with our discretization setting, the modified anchor equation (5.2.6) is equivalent to

$$(5.2.7) \qquad \int_0^{2\pi} (u(t)-u_0(t))\ u_0'(t)\,dt = 0.$$

Note that (5.2.7) is nothing but an integrated version of (5.2.4).

<u>Lemma</u> (5.2.1):

Let $u(t_j) = a_0 + \sum_{k=1}^{n} [a_k \sin kt_j + b_k \cos kt_j]$ and

$$u_0(t_j) = a_0^{(\ell-1)} + \sum_{k=1}^{n} [a_k^{(\ell-1)} \sin kt_j + b_k^{(\ell-1)} \cos kt_j]$$

Then $\int_0^{2\pi} [u(t)-u_0(t)]\, u'(t)\, dt$

$$= \int_0^{2\pi} [u(t)-u_0(t)]\, u_0'(t)\, dt$$

$$= \pi \sum_{k=1}^{n} k(a_k^{(\ell-1)} b_k - a_k b_k^{(\ell-1)}).$$

<u>Proof</u>:

Claim: $\int_0^{2\pi} u(t)\, u'(t)\, dt = \int_0^{2\pi} u_0(t)\, u_0'(t)\, dt = 0$.

Proof: Using integration by parts(IBP), we have

$$\int_0^{2\pi} u(t)\, u'(t)\, dt$$

$$= u^2(t)\Big|_0^{2\pi} - \int_0^{2\pi} u(t) u'(t)\, dt$$

On $u(t)$ is $2\pi$-periodic it follows that $u^2(t)\Big|_0^{2\pi} = 0$.

Hence $\int_0^{2\pi} u(t)\, u'(t)\, dt = 0$. Similarly, we have

$$\int_0^{2\pi} u_0(t)\, u_0'(t)\, dt = 0.$$

Now $\int_0^{2\pi} (u(t) - u_0(t))\, u'(t)\, dt$

$$= \int_0^{2\pi} u(t)\, u'(t)\, dt - \int_0^{2\pi} u_0(t)\, u'(t)\, dt$$

$$= - \int_0^{2\pi} u_0(t)\, u'(t)\, dt$$

$$= \int_0^{2\pi} u(t)\ u_0'(t)\ dt. \qquad \text{(using IBP again)}$$

And $\int_0^{2\pi} (u(t)-u_0(t))\ u_0'(t)\ dt$

$$= \int_0^{2\pi} u(t)\ u_0'(t)\ dt - \int_0^{2\pi} u_0(t)\ u_0'\ dt$$

$$= \int_0^{2\pi} u(t)\ u_0'\ dt.$$

Hence $\int_0^{2\pi} (u(t) - u_0(t))\ u'(t)\ dt$

$$= \int_0^{2\pi} (u(t) - u_0(t))\ u_0'(t)\ dt$$

$$= \int_0^{2\pi} u(t)\ u_0'(t)\ dt.$$

(5.2.8) $\int_0^{2\pi} u(t)\ u_0'(t)\ dt$

$$= \int_0^{2\pi} \{a_0 + \sum_{k=1}^{n} [a_k \sin kt + b_k \cos kt]\}$$

$$\sum_{k=1}^{n} k(a_k^{(\ell-1)} \cos kt - b_k^{(\ell-1)} \sin kt)\ dt$$

Since $\int_0^{2\pi} \sin mu \sin nu\ du = 0$ for $m \neq n$

$\int_0^{2\pi} \cos mu \cos nu\ du = 0$ for $m \neq n$

$\int_0^{2\pi} \sin mu \cos nu\ du = 0$ for all integer $m,n$

$\int_0^{2\pi} \sin^2 u\ du = \pi$ and

$\int_0^{2\pi} \cos^2 u\ du = \pi$

then (5.2.8) equals

$$\pi \sum_{k=1}^{n} k\,(a_k^{(\ell-1)} b_k - a_k b_k^{(\ell-1)}).\ \square$$

To fully specify a solution in Keller's general pseudo arclength continuation, we require the pseudo arclength between two consecutive solutions to equal a prespecified increment $\delta s$. Let $w=(u(t),\rho,\lambda)$ and $w_\emptyset=(u_\emptyset(t),\rho_\emptyset,\lambda_\emptyset)$. In the pseudo arclength equation(2.1.2b), we approximate $w_\emptyset'$ by

$$w_\emptyset' \approx \frac{w - w_\emptyset}{\delta s}.$$

Then (2.1.2b) becomes

$$(w-w_\emptyset)^T(w-w_\emptyset)-\delta s^2=\emptyset$$

i.e.

$$|w-w_\emptyset|^2 = \delta s^2$$

i.e.

$$|u-u_\emptyset|^2 + (\rho-\rho_\emptyset)^2 + (\lambda-\lambda_\emptyset)^2 = \delta s^2$$

or

$$\int_\emptyset^{2\pi} (u(t)-u_\emptyset(t))^2\,dt + (\rho-\rho_\emptyset)^2 + (\lambda-\lambda_\emptyset)^2 = \delta s^2. \tag{5.2.9}$$

Treating $\lambda$ as one of the unknowns as is done in [20] allows the computation to proceed past limit points in the Bifurcation diagram. Indeed, this capability to compute both stable and unstable solutions is difficult to achieve by initial value techniques.

In view of the form of the approximate solution the equations (5.2.3), (5.2.6) and (5.2.9) can be expressed in terms of the Fourier cofficients $a_k$ and $b_k$. Suppose we have computed the $(\ell-1)$th solution, and want to compute the next solution on the branch. Thus the system of $2n+3$ nonlinear algebraic equations that must be solved at the $\ell$-th step for the $a_k$'s, $b_k$'s, $\rho$ and $\lambda$ consists of the following

Let $u^{(\ell)}(t_j) = a_\emptyset + \sum_{k=1}^{n} [a_k \sin kt_j + b_k \cos kt_j]$ and

$$u^{(\ell-1)}(t_j) = a_\emptyset^{(\ell-1)} + \sum_{k=1}^{n} [a_k^{(\ell-1)} \sin kt_j + b_k^{(\ell-1)} \cos kt_j].$$

Then from (5.2.3) we have

$$(5.2.10) \quad \sum_{k=1}^{n} k\,(a_k \cos(kt_j) - b_k \sin(kt_j) =$$

$$\frac{\rho}{2\pi} f(a_\emptyset + \sum_{k=1}^{n} (a_k \sin(kt_j) + b_k \cos(kt_j)),$$

$$a_\emptyset + \sum_{k=1}^{n} (a_k \sin(k(t_j - \tfrac{2\pi}{\rho}\tau)) + b_k \cos(k(t_j - \tfrac{2\pi}{\rho}\tau))), \lambda),$$

$$j=\emptyset,1,\ldots,2n$$

From (5.2.6) and lemma (5.2.1) we have

$$(5.2.11) \quad \sum_{k=1}^{n} k\,(a_k^{(\ell-1)}\, b_k - a_k\, b_k^{(\ell-1)}) = \emptyset,$$

and from (5.2.9) we have

(5.2.12)

$$2\pi\ (a_0-a_0^{(\ell-1)})^2 + \pi \sum_{k=1}^{n} ((a_k-a_k^{(\ell-1)})^2 + (b_k-b_k^{(\ell-1)})^2)$$

$$+ (\rho-\rho^{(\ell-1)})^2 + (\lambda-\lambda^{(\ell-1)})^2 = \delta s^2.$$

To solve (5.2.10),(5.2.11) and (5.2.12), we use the Newton-Chord method. An accurate initial aproximation to the next solution $(u,\rho,\lambda)$ is obtained by extrapolation from the two preceeding solution points on the branch. (See Figure 5.3.)

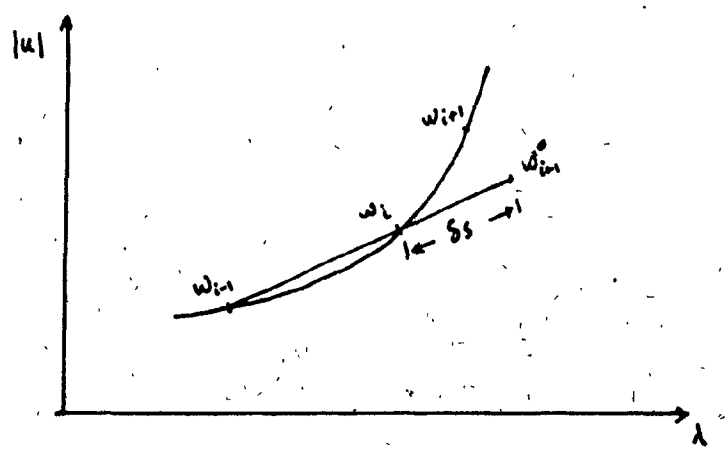



Fig. 5.3 Initial guess by extrapolation.

## 5.3 Branch Switching at a Hopf Bifurcation point

The general technique, we discussed in Section 5.2, to compute a branch of periodic solutions can be summarized as

follows. First, the pseudo-arc length between a new solution point and a previous solution point on the branch is required to equal a certain increment $\delta s$. Second, the new solution and the previous solution should satisfy the anchor equation, that is, the peaks of the solutions should line up as close as possible. Third, the new solution should satisfy the differential equations at every collocation point.

To compute a periodic solution branch, we start from the Hopf Bifurcation point, which is the intersection of a steady state solution branch and a periodic solution branch. Note that the solution at the Hopf Bifurcation point is a constant function, $u(t)=a_0$, for all t. In our sine-cosine representation of such a solution, $a_k=b_k=0$, for all $k \geq 1$. Thus the left hand side of the anchor equation (5.2.11) would be zero and this would make the Jacobian matrix of (5.2.10), (5.2.11) and (5.2.12) singular. To get started at the Hopf bifurcation on a periodic solution branch, the general procedure can be applied without change, except for the anchor equation. The modification is as follows. Instead of taking the constant steady state solution as the reference solution in the anchor equation we use an asymptotic estimate of the non-constant periodic solution near the Hopf bifurcation point.

The asymptotic estimate for the periodic solution in a neighbourhood of the Hopf bifurcation point follows from our

discussion in Section 5.1 for detecting Hopf bifurcation points. We find a solution $u(t)=e^{iyt}$, where iy is a pure imaginary eigenvalue of the linearized system (5.1.2). If iy is an eigenvalue of the system, then so is its conjugate, -iy, and thus $u(t)=e^{-iyt}$ is also a solution of the linearized system. Note that, any linear combination of two solutions of the linearized system is also a solution of the system. Hence sin(yt) and cos(yt) are solutions of the system. By scaling the independent variable t by the factor $\frac{2\pi}{\rho}$, where $\rho$ is the period of the solution, we may determine $2\pi$ periodic solutions. Hence sin(t) and cos(t) are solutions of the linearized system after scaling. Thus the asymptotic estimate for the periodic solution near the Hopf bifurcation point takes the form

$$(5.3.1) \qquad u^{(\varepsilon)}(t) = u^{(0)} + \varepsilon(c_1\cos t + c_2 \sin t) + O(\varepsilon^2)$$

The freedom of phase shift is still present in (5.3.1) and allows omitting, say, the sine term. Thus near the bifurcation point, the periodic solution can be represented asymptotically by

$$(5.3.2a) \qquad u^{(\varepsilon)}(t) = u^{(0)} + \varepsilon\cos t + O(\varepsilon^2)$$

It is also known [4] that

$$(5.3.2b) \qquad \lambda^{(\varepsilon)} = \lambda^{(0)} + O(\varepsilon^2),$$

and

$$(5.3.2c) \qquad \rho^{(\varepsilon)} = \rho^{(0)} + O(\varepsilon^2).$$

Further, for compatibility with the pseudo arclength condition, that is, to satisfy equation (5.2.9), we have

$$\int_0^{2\pi} (u^{(\varepsilon)}(t)-u^{(0)}(t))^2\,dt + (\rho^{(\varepsilon)}-\rho^{(0)})^2 + (\lambda^{(\varepsilon)}-\lambda^{(0)})^2=\delta s^2.$$

i.e. $\int_0^{2\pi} \varepsilon^2 \cos^2 t\,dt = \delta s^2,$

i.e. $\varepsilon = \frac{\delta s}{\sqrt{\pi}}.$

Now we use this asymptotic estimate,

$$\bar{u}^{(1)}(t) = u^{(0)} + \frac{\delta s}{\sqrt{\pi}}\cos t,$$

to align the first solution point $(u^{(1)},\rho^{(1)},\lambda^{(1)})$. Note that the Fourier representation of $\bar{u}^{(1)}(t)$ has $a_k = b_k = 0$ for all $k$, except $a_0 = u^{(0)}$ and $b_1 = \frac{\delta s}{\sqrt{\pi}}$. Thus, instead of the general anchor equation, at the starting point, we use

(5.3.3) $$\int_0^{2\pi} (u_n^{(1)}(t)-\bar{u}_n^{(1)}(t))\,u_n^{(1)\prime}(t)\,dt = 0$$

In terms of the Fourier coefficient, this simply becomes

(5.3.4) $$a_1 = 0.$$

Note that this starting procedure does not require much change in the general continuation procedure, thus keeping down the overall programming effort.

## 5.4 Detection of Secondary Periodic Bifurcation Points

So far we have discussed two types of bifurcation points. One is the steady state bifurcation point which is an intersection of two steady state solution branches. The other one is the Hopf bifurcation point which is an intersection of a steady state solution branch and a periodic solution branch. In this section, a third type of bifurcation, secondary periodic bifurcation, is discussed. A secondary periodic bifurcation point is an intersection of two periodic solution branches. This type of bifurcation point is of as great importance as the others because it is a state where periodic solutions change qualitatively and quantitatively. A special case of secondary periodic bifurcation is the period doubling bifurcation. In a neighbourhood of the period doubling bifurcation point, the period of the periodic solution on one branch is double the one on the other. To be more precise, (see Figure 5.4), let $(u(s), \rho(s), \lambda(s))$ and $(\bar{u}(s), \bar{\rho}(s), \bar{\lambda}(s))$ be solutions on branch A and B respectively, where s is a parameterization on the branches. Suppose

$$(\bar{u}(0), \bar{\rho}(0), \bar{\lambda}(0)) = (u(0), \rho(0), \lambda(0))$$

is a period doubling bifurcation point. Then as $s \to 0$, we have $\bar{\rho}(s) \to \rho$ and $\rho(s) \to 2\rho$.

To detect ordinary secondary periodic bifurcation and period doubling bifurcation, we can apply the Crandall and

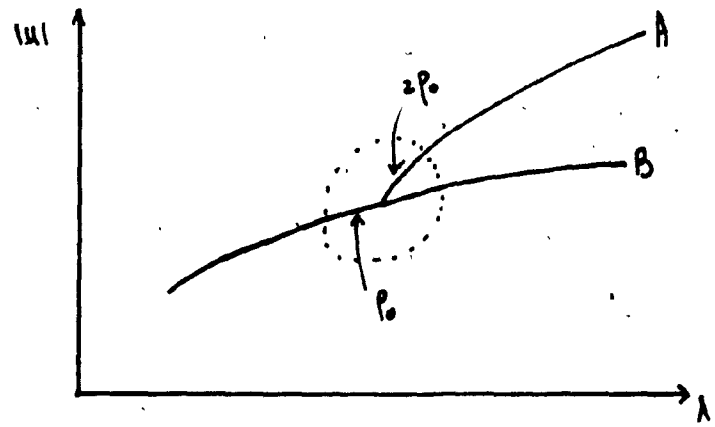



Fig. 5.4 A period doubling bifurcation.

Rabinowitz Bifurcation Theorem.

Let $G(w)=0$ represents the system of equation (5.2.3), (5.2.6) and (5.2.9) where $w=(u(t),\rho,\lambda)$ and $u(t)$ is a $\rho$-periodic solution. Suppose there is a periodic solution branch C, given by

$$C = \{w(s):w(s) = (u(t;s),\ \rho(s),\ \lambda(s)),\ s\in I\}$$

where I is an interval, such that $G(w)=0$ for $w\in C$. If every neighbourhood of $w(0)$ contains zeros of G not lying on C, then $w(0)$ is called a secondary periodic bifurcation point for the equation $G(w)=0$ with respect to the periodic solution branch C.

If $u(t)$ is $\rho$-periodic, then it is also $k\rho$-periodic, for any integer k. If we define the above periodic solution branch C by $C=\{w(s):\ w(s)=(u(2t;s),2\rho(s),\lambda(s)),\ s\in I\}$, then $G(w)=0$ for $w\in C$. In this setting, $w(s)$ is said to be a

period doubling bifurcation point if there is a number $s \in I$ such that every neighbourhood of $w(s)$ contain zeros of $G$ not lying on $C$.

With the Fourier expansion of the periodic solution, computing solutions along a periodic solution branch is equivalent to determining $a_k$'s, $b_k$'s, $\rho$ and $\lambda$ which satisfy (5.2.10), (5.2.11) and (5.2.12). Let $G(w) = 0$ denote the system containing equations (5.2.10),(5.2.11) and (5.2.12) where $w \equiv (\bar{a}, \bar{b}, \rho, \lambda)$, $\bar{a}=(a_0, a_1, \ldots, a_n)$, $\bar{b}=(b_1, \ldots, b_n)$. The Crandall and Rabinowitz Bifurcation Theorem can be used to determine bifurcation points of $G(w)=0$. Note that every solution $w=(\bar{a}, \bar{b}, \rho, \lambda)$ of $G(w)=0$ represents a $\rho$-periodic solution on the branch of periodic solutions. A bifurcation point of $G(w)=0$ is an ordinary secondary periodic bifurcation of the equation (5.2.10),(5.2.11) and (5.2.12).

To detect period doubling bifurcation points along a periodic solution branch, essentially the same technique can be applied except for a slight modification. The modification is as follows:

Suppose there is a $\rho$-periodic solution for some $\lambda$ on a branch of periodic solution.(see Figure 5.5). If we take the portion from $0$ to $\rho$ and scale it in the interval from $0$ to $2\pi$ and express it in trigonometric expansion, we have (see Figure 5.6)

$$u(t) = a_0 + \sum_{k=1}^{n} [a_k \sin(kt) + b_k \cos(kt)].$$

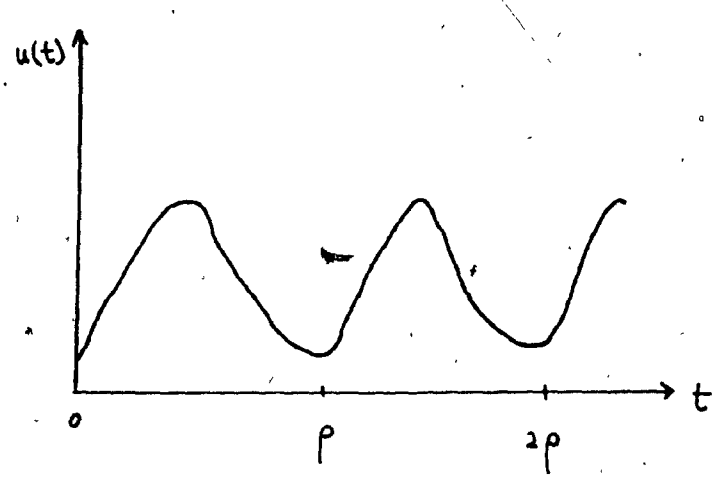



Fig. 5.5 A $\rho$-periodic solution.

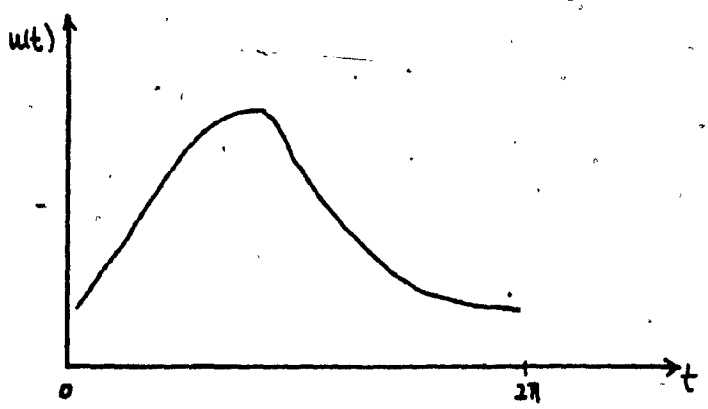



Fig. 5.6 $(u(t), \rho, \lambda)$

Then the triple $(u(t), \rho, \lambda)$ represents periodic solution on the branch. The same periodic solution can be represented as follows: if we take the portion from $0$ to $2\rho$, scale it from $0$ to $2\pi$ (see Figure 5.7) and express it in

trigonometric expansion as

$$(5.4.1) \qquad \bar{u}(t) = \bar{a}_0 + \sum_{k=1}^{m} [\bar{a}_k \sin(kt) + \bar{b}_k \cos(kt)].$$

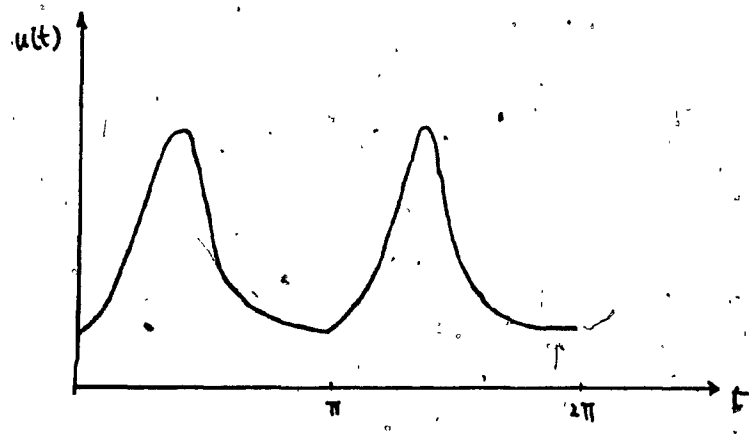



Fig. 5.7 $(\bar{u}(t), 2\rho, \lambda)$

Then $(\bar{u}(t), 2\rho, \lambda)$ represents the same periodic solution on the branch. Note that $\bar{u}(t)=u(2t)$ for all t. Thus we have

$$(5.4.2) \qquad u(2t) = a_0 + \sum_{k=1}^{n} [a_k \sin(2kt) + b_k \cos(2kt)]$$

$$= a_0 + \sum_{k=1}^{2n} [\hat{a}_k \sin(kt) + \hat{b}_k \cos(kt)].$$

where $\hat{a}_k = \hat{b}_k = 0$ for k is odd, and $\hat{a}_{2k} = a_k$ and $\hat{b}_{2k} = b_k$ for $k=1,\ldots,n$.

By comparing coefficients of $\bar{u}(t)$ with $u(t)$ in equation (5.4.1) and (5.4.2) we have $m=2n$; and $\bar{a}_k=\hat{a}_k$ and $\bar{b}_k=\hat{b}_k$ for

all k. To detect period doubling bifurcation points, we apply the same procedure as for detecting ordinary secondary periodic bifurcation with u(t) replaced by $\bar{u}(t)$, $\rho$ by $2\rho$ and n by 2n.

Difficulties may arise in the detection of ordinary secondary bifurcation. These are due to the fact that the ordinary secondary periodic bifurcation point is structurally unstable. This means that after a slight perturbation of the system, the bifurcation normally changes its structure (see Figure 5.8).

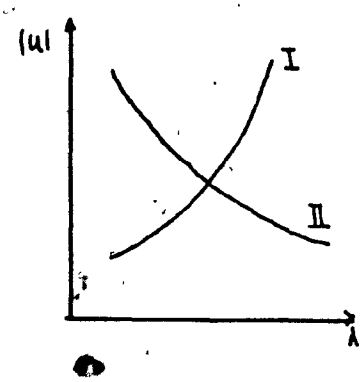



Fig. 5.8

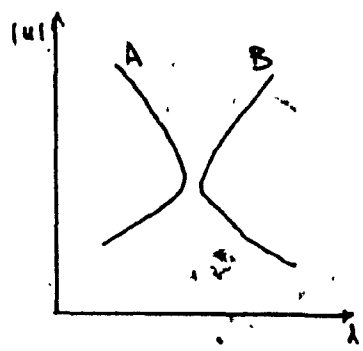



Fig. 5.9(a)

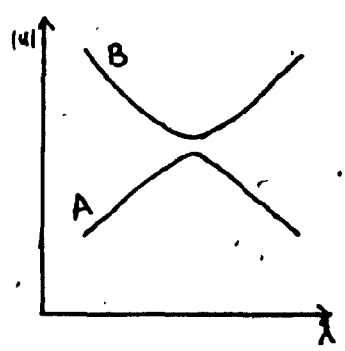



Fig. 5.9(b)

Theoretically the bifurcation is the intersection of the two periodic solution branches I and II. However, with a slight perturbation, like discretizing the system, the structure of the bifurcation is most likely to change to one of the two possibilities in Figure 5.9, where branches A and B are

composed of branches I and II. In this case, we have the following two possibilities. We either follow branch A completely without detecting the periodic bifurcation point, or, we step from branch A to branch B and detect a candidate secondary periodic bifurcation but fail to locate it accurately, since it does not exist in the discretized system. The situation for period doubling bifurcations is better. Once a periodic doubling bifurcation is detected, it can be located easily. Examples of both types of periodic bifurcations are presented in chapter six.

## 5.5 Branch Switching Technique at Secondary Periodic Bifurcation Points

In order to switch branches at a secondary periodic bifurcation point, we need to know the bifurcating directions. If we use the same notation as in Section 4.3 for stationary solutions, then $x(s)$ represents the triple $(u(s), \rho(s), \lambda(s))$ and $g(x(s))$ represents the system of equations (5.2.10) and (5.2.11). To find the bifurcation direction, it is in principle possible to apply the Algebraic Bifurcation Equation (4.3.1). This requires computation of the second derivative of the system $g$. For the case of secondary periodic bifurcation the computation of these derivatives of equation (5.2.10) and (5.2.11) is tedious.

An alternative is to approximate the bifurcation direction by a vector which is orthogonal to the known direction of the given branch. (see Figure 5.10).

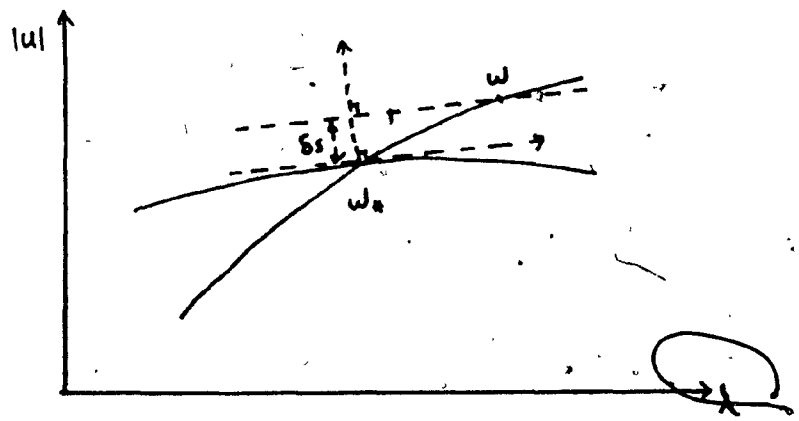



Fig. 5.10 Hyper Plan method.

Given this direction, $\eta$, we determine the next solution on the bifurcating branch such that it lies on the hyper plane at a distant $\delta s$ from the known direction $w_0'$. This technique has been introduced in [20] and is called the Hyper Plane method.

The remaining question is how to find the orthogonal direction, $\eta$. We know that at the secondary bifurcation point, the Jacobian matrix of the system of equations (5.2.10), (5.2.11) and (5.2.12) is singular. Let $g(w(s))$ represents equation (5.2.10) and (5.2.11). As discussed in Section 5.2, the pseudo arclength equation (5.2.12) originates from

$$w_0'^{*} (w - w_0) - \delta s = 0$$

Now the Jacobian matrix of (5.2.10), (5.2.11) and (5.2.12),

$$G_w(w(s)) = \begin{bmatrix} g_w(w(s)) \\ w_0' \end{bmatrix}$$

is singular at the bifurcation point. Hence there exists a vector $\eta$ such that

$$G_w(w(s))\ \eta = 0.$$

This implies

$$g_w(w(s))\ \eta = 0$$

and

$$w_0'\ \eta = 0.$$

Note that $\eta$ is a null vector of $G_w(w(s))$. $G_w(w(s))$ has been evaluated in Newton's method for solving the system of equations (5.2.10), (5.2.11) and (5.2.12).

Now,

$$g_w(w(s))\ \eta = 0 \text{ implies } \eta \in \text{span}\{w_0', w_0\}$$

where $w_0$ is the actual bifurcation direction, and

$$w_0'\ \eta = 0 \text{ implies } w_0' \perp \eta.$$

Hence $\eta$ is the direction orthogonal to the known direction vector $w_0'$. To compute the null vector $\eta$ of the singular matrix, $G_w(w(s))$, we can use Gauss elimination and back substitution. An illustration has been given in Section 4.1.

<u>Remark</u>: the Hyper Plane method can also be used to switch branches at steady state bifurcation points.

## Chapter Six

## A Model from Physiology and Other Examples

In order to demonstrate the accuracy of the method and to check the correctness of our implementation, we have recomputed the example given in Hadeler's paper [10]. The equation is

$$(6.1.1) \qquad u'(t) = -\lambda u(t-1)\,\frac{1+u(t-1)^2}{1+u(t-1)^4}\,.$$

For $\lambda$ positive and along the zero stationary solution branch (1), our program signals Hopf bifurcation points at $\lambda = \pi/2$, $5\pi/2$, $9\pi/2$, ... . The bifurcation diagram is shown in Figure 6.1. In the diagram branch (1) represents the zero stationary solution, while branches (2),(3) and (4) are the primary branches of periodic solutions. A secondary periodic bifurcation occurs along branch (2) at $\lambda$=4.67. The bifurcating branch of periodic solutions has been labelled (5) in Figure 6.1. This secondary bifurcation is an ordinary bifurcation and it is detected by our program DLAY as the sign changes in the determinant of the Jacobian of Equations (5.2.10),(5.2.11),(5.2.12), which is a necessary condition for a bifurcation to occur in the Crandall and Rabinowitz Bifurcation Theorem as discussed in Section 2.2. As is also noted in [10] and Section 5.4 in this thesis, such an ordinary secondary bifurcation is not structurally stable. Indeed, for insufficiently accurate discretization

the usual perturbed bifurcation is observed. (For a general discussion of the effect of discretization see [1,2,6].) For this reason, our program DLAY fails to detect the bifurcation point accurately. Instead, we obtain a bifurcation direction by trials. In this example the bifurcation diagram can be computed with n=12 or less in (5.2.10), (5.2.11) and (5.2.12). Near the secondary bifurcation we have used n=20 as in order to deal with the structural instability. Our program can handle Hopf bifurcation, secondary bifurcation, branch switching, backward bifurcation, turning points, etc. with little or no intervention by the user.

It is known [34] that the equation (6.1.1) has the property that there is a periodic solution with period $\rho = 4$ for all $\lambda > \frac{\pi}{2}$. This is rather unusual and allows us to compare the numerically obtained period to the actual period. Along the first bifurcating branch of periodic solutions (2) the period remains equal to 4. In the table below we list the numerically observed period on this branch at $\lambda$=3.0 for various choices of n in (5.2.10), (5.2.11) and (5.2.12). As is evident the convergence is indeed very rapid. (Along branches (3) and (4) in Figure 6.1 the period also remains constant and equals 4/5 and 4/9 respectively.)

| n | ρ |
|---|---|
| 2 | 4.724 |
| 4 | 4.135 |
| 6 | 3.975 |
| 8 | 3.9987 |
| 10 | 3.9985 |
| 12 | 3.99984 |
| 16 | 4.000000 |

In this problem, the zero stationary solution (1) is stable up to the first Hopf bifurcation point. The stability of stationary solutions is determined by Theorem (4.4.1). A procedure, based on the theorem, to determine the stability is also implemented in our DLAY program. To determine the stability properties of any particular periodic solution on the branch, we can use an initial value problem solver. If the solution resulting from the initial value problem solver agrees with the one from our DLAY program for sufficiently long time, then we assume that the solution is stable. In this way we find that periodic branch (2) is stable between the limit point and the secondary periodic bifurcation. The upper portion of branch (5) (past the limit point) is also stable. All other solutions indicated in the bifurcation diagram are asymptotically unstable. The actual solution u(t) at points 31 and 51 on branches (3) and (5) respectively is shown in Figure 6.2. Thus solution 31 is unstable and 51 is stable.

For both solution points $\lambda$ approximately equals 8.0. To illustrate the transitional behaviour of this equation we have also solved the differential equation for $\lambda$=8.0 using a simple initial value problem solver. As initial data on the time interval [-1,0] we have taken a sine function that approximates the unstable solution 31. The dynamic response of the differential equation to this starting condition is shown in Figure 6.3. Initially the solution oscillates near the unstable solution 31 which has period 0.8. Then a quick transition takes place from the unstable oscillation to the stable periodic solution 51 which has period approximately equal to 5.5. The period does not remain constant along the secondary periodic branch (5). On this periodic branch, the period of the solutions is increasing from the secondary bifurcation point along the upper branch while it is decreasing along the lower branch. (see Table 6.1 and 6.2).

| | | | | |
|---|---|---|---|---|
| $\lambda$ | 4.7653 | 5.5375 | 6.7581 | 8.0245 |
| $\rho$ | 5.1184 | 5.2828 | 5.4014 | 5.5182 |

Table 6.1 Upper branch (5)

| | | | | |
|---|---|---|---|---|
| $\lambda$ | 4.7677 | 5.5683 | 6.7472 | 8.8580 |
| $\rho$ | 3.9276 | 3.5627 | 3.3360 | 3.2223 |

Table 6.2 Lower branch (5)

Apart from Hadeler's paper [10] and our own work [8], the only other reference that we are aware of for computing bifurcation diagrams for delay equations is the recent Doctoral Thesis of Saupe [35].

## A Model from Physiology

Below, a simple model from physiology introduced in [26] is considered. As we shall see, complicated behavior of the solutions is possible, even though the model is quite simple.

Consider the ordinary differential equation

$$\frac{dx}{dt} = \lambda - \gamma x \tag{6.1.2}$$

where x is a variable of interest, $\lambda$ is the production rate for x, $\gamma$ is the destruction rate of x, and t is the time. For $\lambda$ and $\gamma$ constant, $x \to \lambda/\gamma$ in the limit $t \to \infty$. However, in many physiological systems $\lambda$ and $\gamma$ at t may depend on $x(t)$ and/or $x(t-\tau)$ where $\tau$ is a time delay.

A simple model for the control of peripheral blood cell numbers via a feedback mechanism was studied by Glass and Mackey [9,26]. Let $x(t)$ be the concentration of circulating cells (cells/kg) and assume that cells are randomly lost from the circulation at a rate $\gamma$/day proportional to their concentration. To reproduce the effects of feedback control from the circulating population of cells, the flux ($\lambda$ in

cells/kg/day) into the circulation from the system cell compartment presumably depends on x at time $t-\tau$, and thus the dynamics of $x(t)$ is governed by

$$(6.1.3) \qquad \frac{dx}{dt} = \lambda(x(t-\tau)) - \gamma x$$

A form of $\lambda(x)$

$$(6.1.4) \qquad \lambda(x) = \frac{\lambda\theta^n x}{\theta^n + x^n}$$

was suggested by Glass and Mackey, where n, $\theta$(cells/kg) and $\lambda$ (kg/day-cell) are parameters. Graphs of (6.1.4) with $\lambda(x)$ against x are given in Figure 6.4. As can be seen $\lambda(x)$ is a single hump function. The hump gets bigger as $\theta$ increases; and it gets sharper as n gets bigger. The sharper the hump, the more sudden the change of the production rate. Combining equation (6.1.3) with equation (6.1.4), we have

$$(6.1.5) \qquad \frac{dx}{dt} = \frac{\lambda\theta^n x(t-\tau)}{\theta^n + x(t-\tau)^n} - \gamma x,$$

with free parameters $\theta$ and n. Equation (6.1.4) is flexible enough to model many possible production rates $\lambda(x)$. The qualitative behaviour of (6.1.5) in response to parameter changes is of interest. To illustrate this behaviour, Glass and Mackey assume that $\gamma=1$, $\lambda=2$, $\theta=1$ and $\tau=2$. By using a predictor-corrector integration routine with a step size of 0.05 for various values of n, starting from an initial

condition $x(t) = 0.05$, $-\tau < t < 0$, they observed that as n is increased, oscillations occur and these undergo a sequence of bifurcations. N. MacDonald [25] has summarized the subsequent results in the following table. Here 3, for example, stands for a triple loop in the trajectory of $x(t-\tau)$ against $x(t)$. Such a sequence is analogous to the sequence of bifurcations observed in a class of finite-difference equations in 1-dimension[22,28,29,30].

| n | 7 | 7.75 | 8.50 | 8.79 | 9.65 | 9.6575 | 9.76 | 10.0 |
|---|---|---|---|---|---|---|---|---|
| trajectory | 1 | 2 | 2 | 4 | chaos | 3 | 6 | chaos |

Table 6.3 Results of Glass and Mackey(1978) for solutions of (6.1.5)

With the same set of parameter values, taking n as the bifurcation parameter, we have analyzed equation (6.1.5) with our DLAY program. We start at $x=1$, $n=1$ which is a steady state solution of equation (6.1.5) and we get a branch of steady state solutions where $x\equiv 1$ (see Figure 6.5). At $n = 5.04$, a Hopf bifurcation point is detected. Our DLAY program also shows that steady state solutions with $x = 1$ and $n < 5.04$ are stable and that those with $x = 1$ and $n > 5.04$ are unstable. Starting from the Hopf bifurcation point, a branch of periodic solutions, branch (2) is

computed. The period of the periodic solution at the Hopf bifurcation point is 5.49. At the solution point labelled 21 the period is 5.48939 and at the solution labelled 22, it is 5.3603. At n=7.40587 on this branch, a period doubling bifurcation point is detected. Starting from this period doubling bifurcation, another branch of periodic solutions, (branch (3)), is computed. The periods at solutions labelled 33 and 34 are respectively 11.731 and 11.8767. The period of solutions on this branch is increasing with n. At n=8.692 on this branch, another period doubling bifurcation is detected. Starting again from the latter period doubling bifurcation point, periodic solution branch (4) is computed. The period at solution labelled 45 is 22.86. At n=8.8473, another period doubling bifurcation is detected. We have not traced out the corresponding bifurcation branch.

Solutions with labels are plotted in two fashions in Figures 6.6 to 6.10. One with x against time and the other with $x(t-\tau)$ against $x(t)$. In Figures 6.7b and 6.8b, one can observe that a period doubling bifurcation has occurred: the number of loops in Figure 6.8b is twice of that in Figure 6.7b. Similary, one can observe that the same phenomenon occurs again in Figures 6.9b and 6.10b.

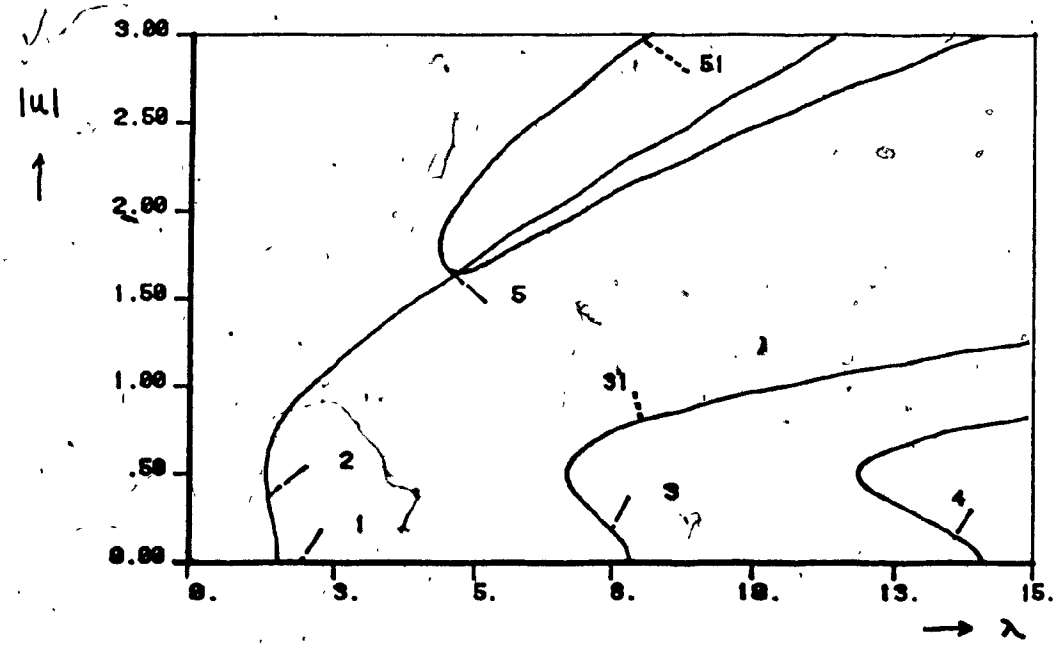



FIG. 6,1 BIFURCATION DIAGRAM OF (6.1.1)

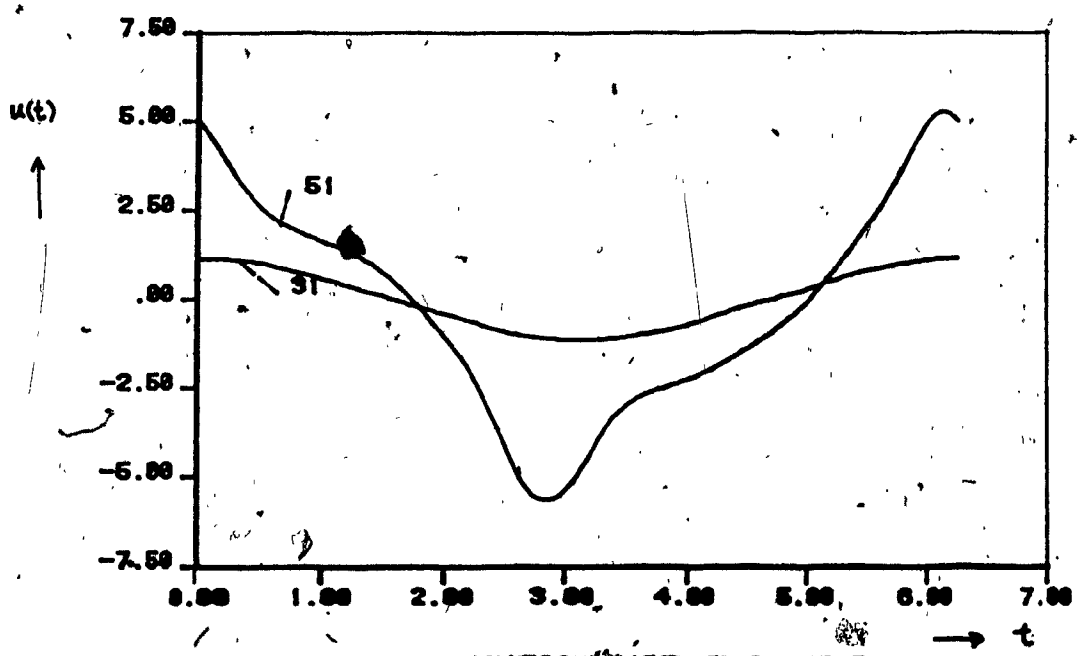



FIG. 6.2 SOLUTIONS LABELLED 31 AND 51

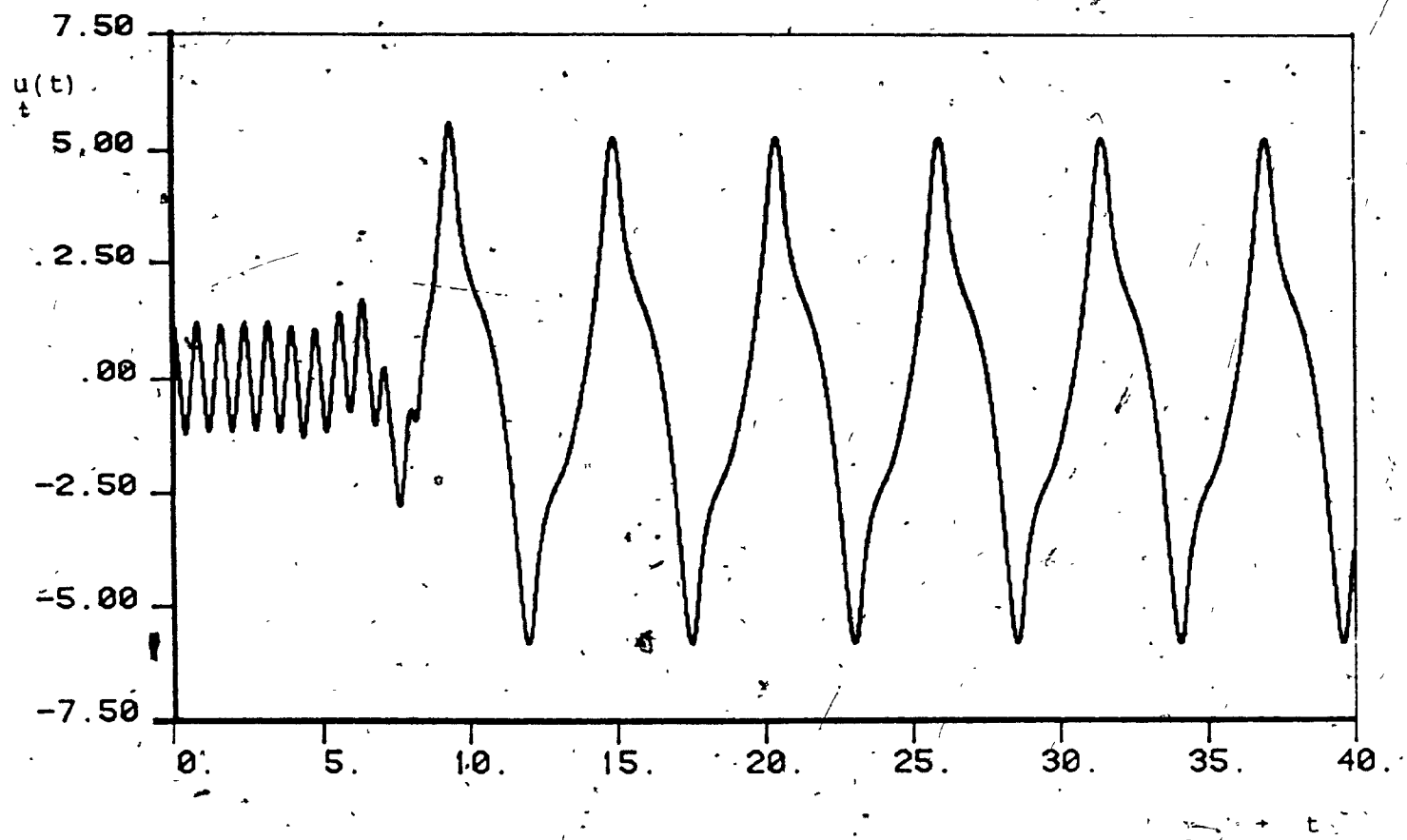



Fig. 6.3 Transitional behaviour between solutions 31 and 51.

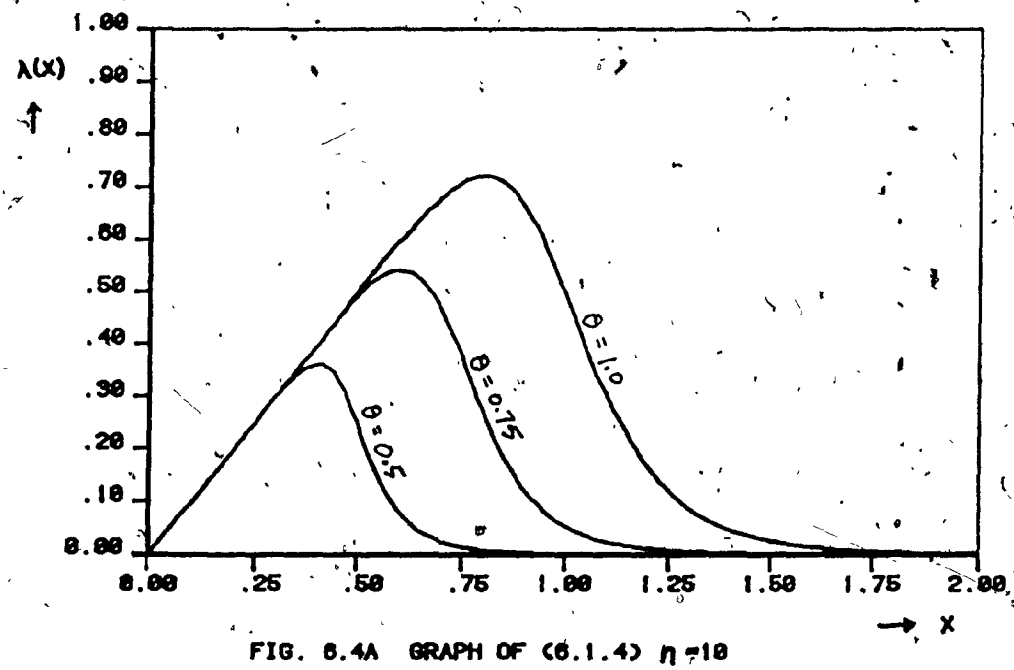



FIG. 6.4A GRAPH OF (6.1.4) $\eta = 10$

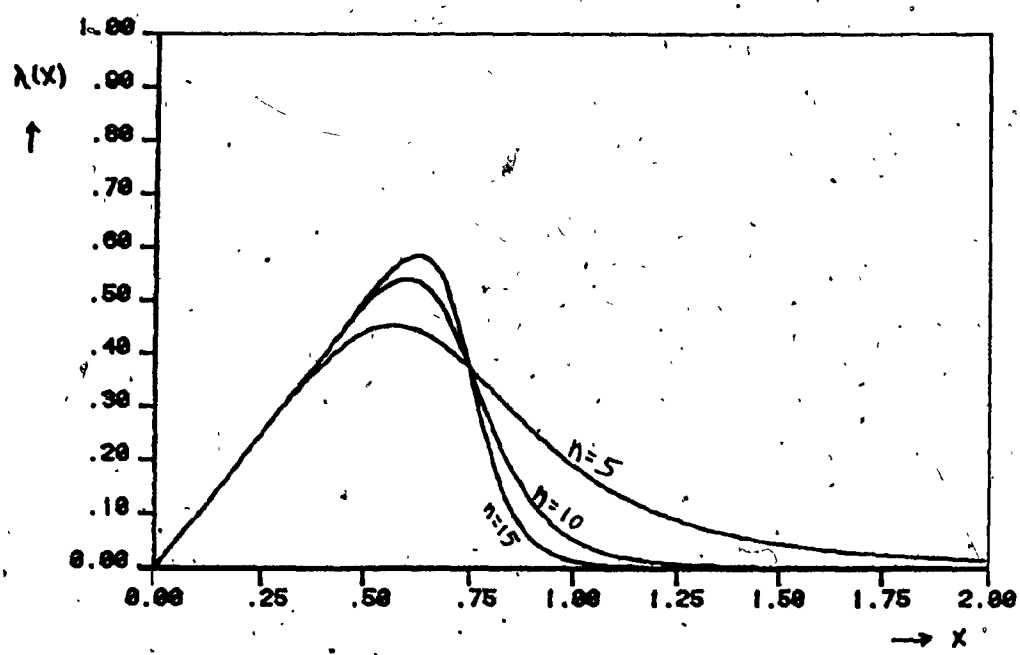



FIG. 6.4B GRAPH OF (6.1.4) $\theta = 0.75$

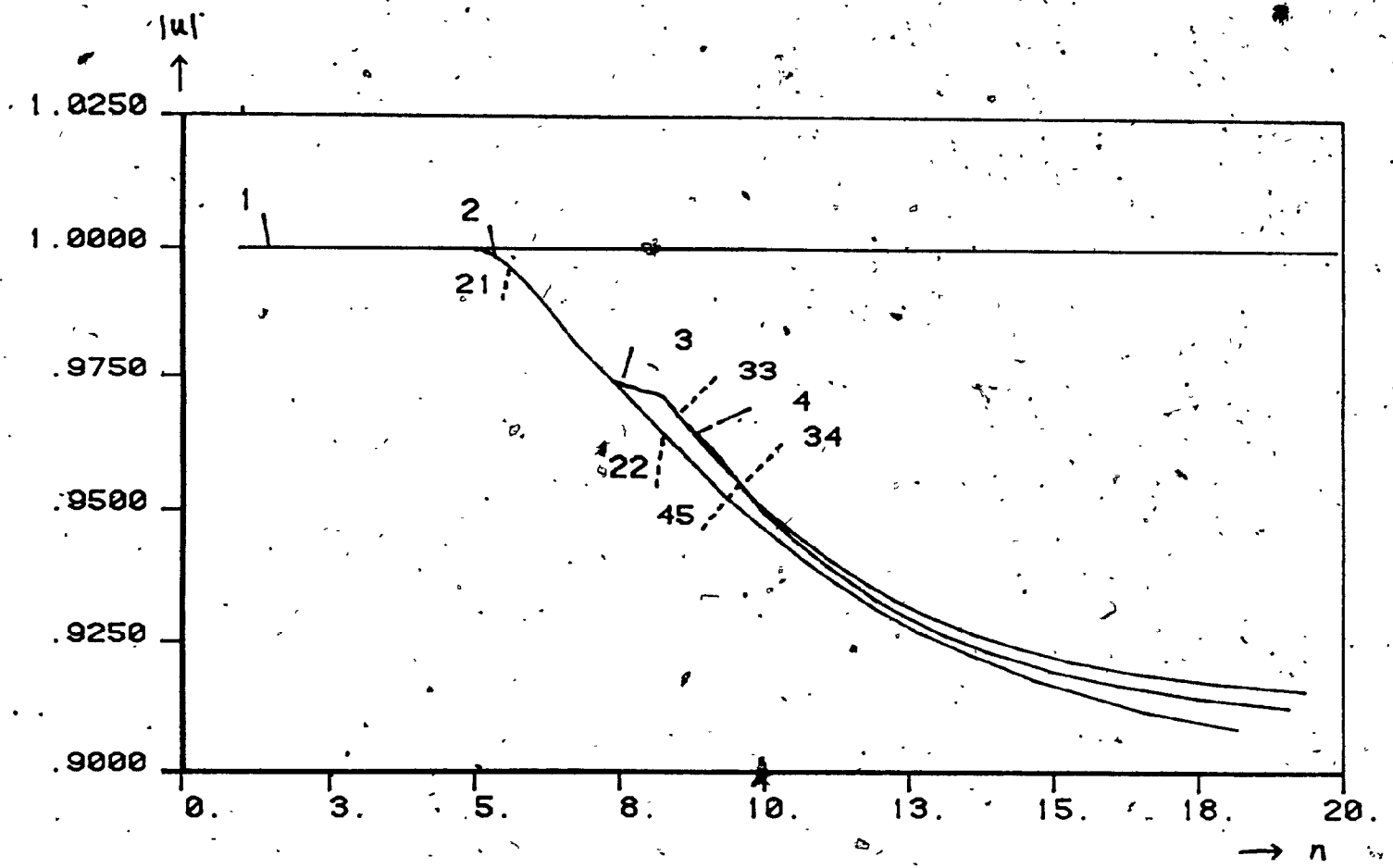



FIG.6.5 BIFURCATION DIAGRAM OF (6.1.5)

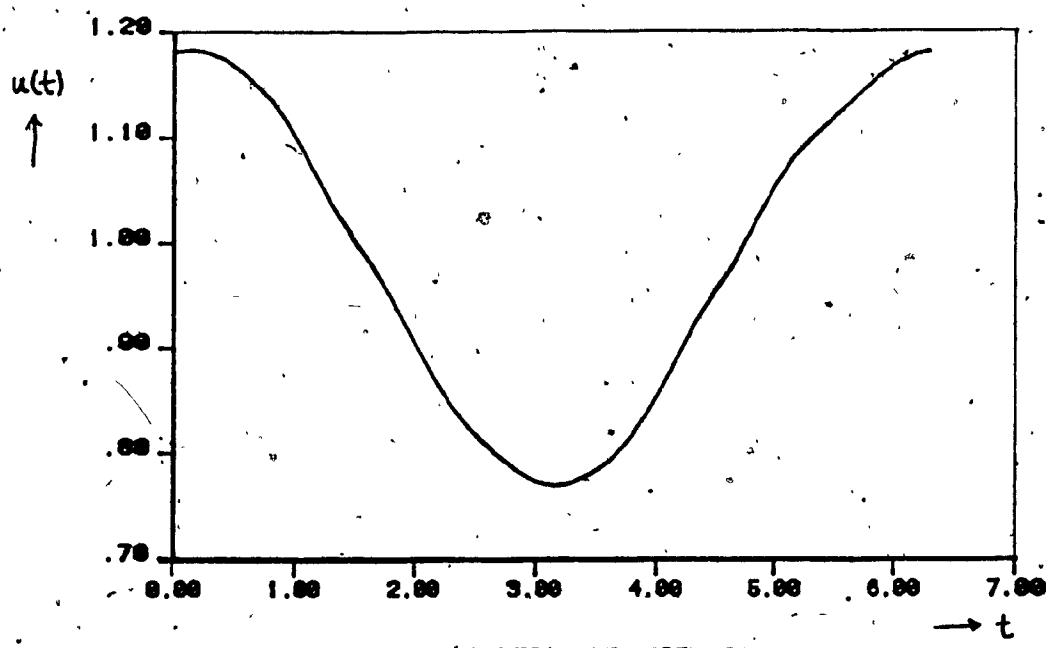



FIG. 6.6A SOLUTION OF LABEL 21

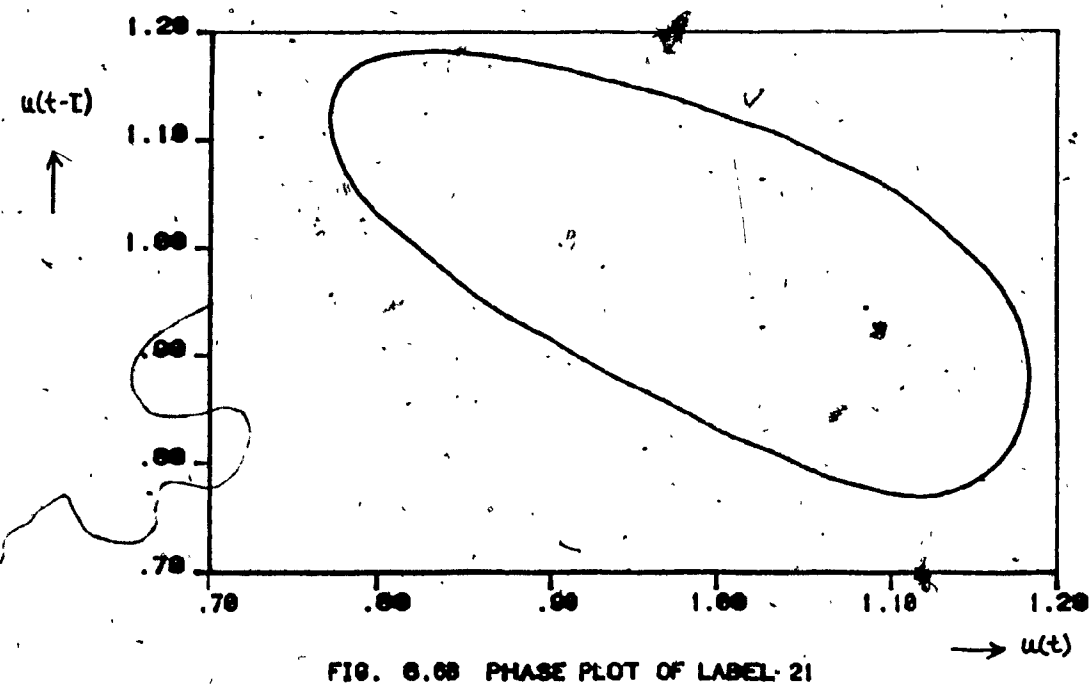



FIG. 6.6B PHASE PLOT OF LABEL 21

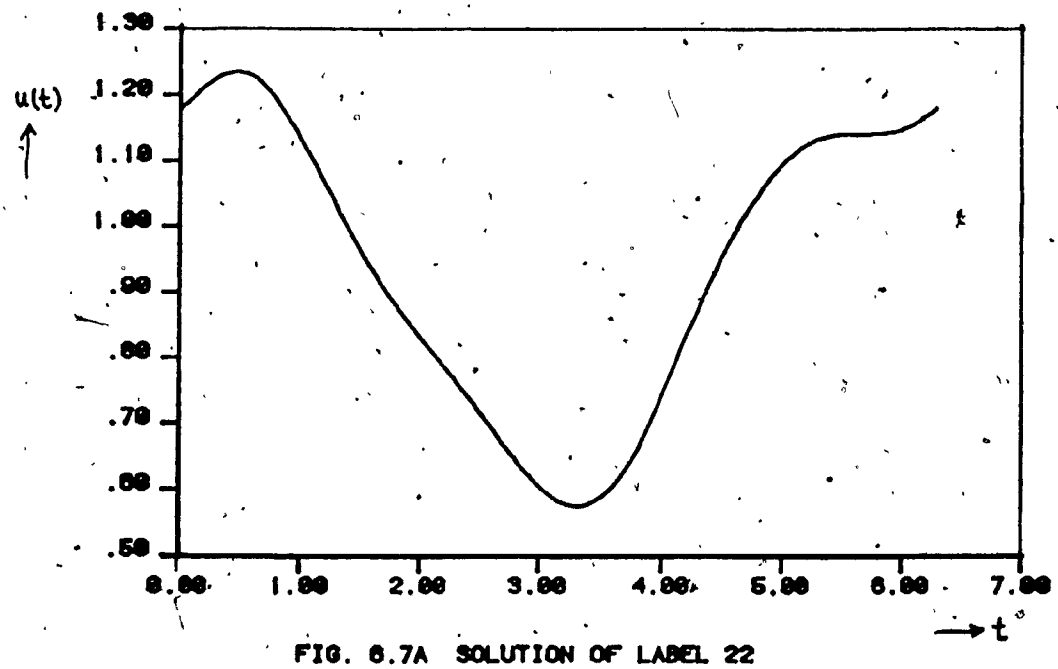



FIG. 6.7A SOLUTION OF LABEL 22

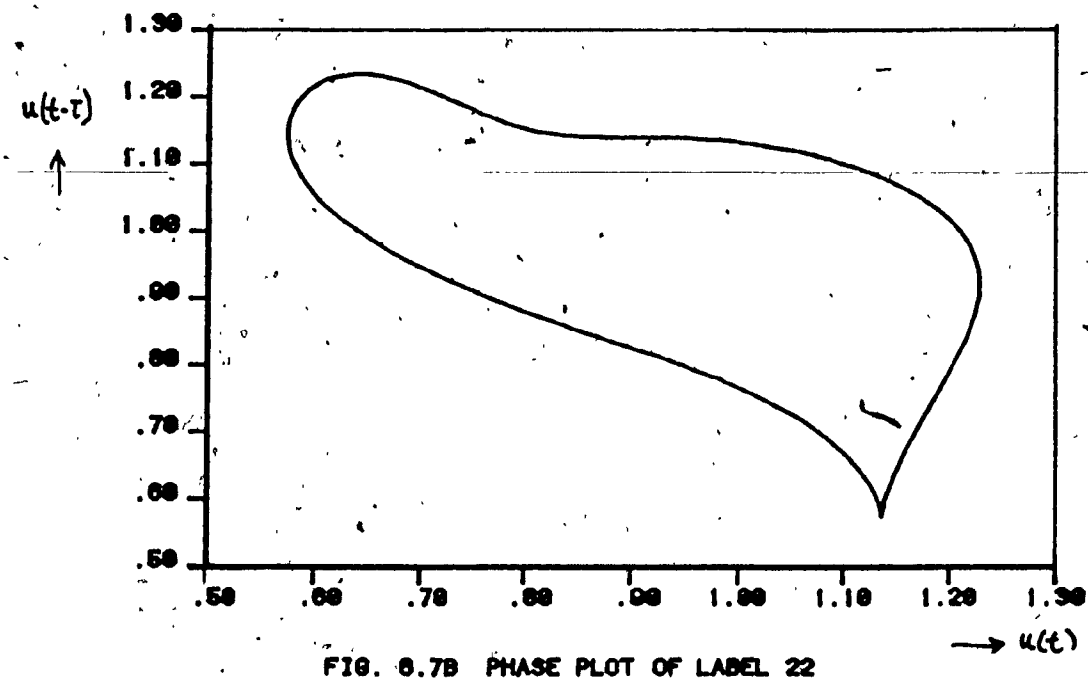



FIG. 6.7B PHASE PLOT OF LABEL 22

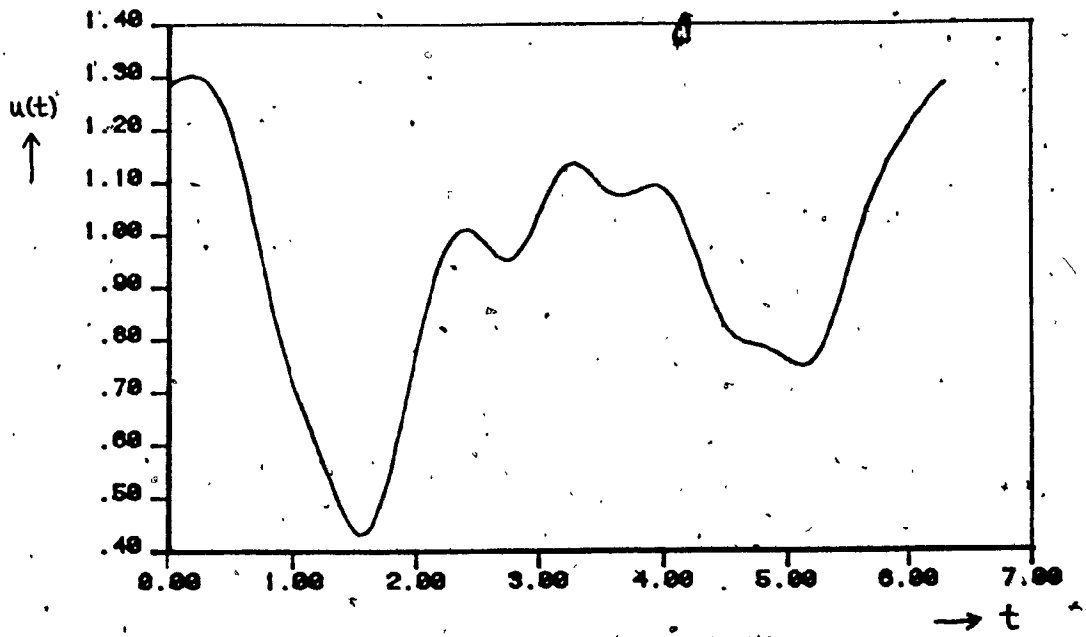



FIG. 6.8A SOLUTION OF LABEL 33

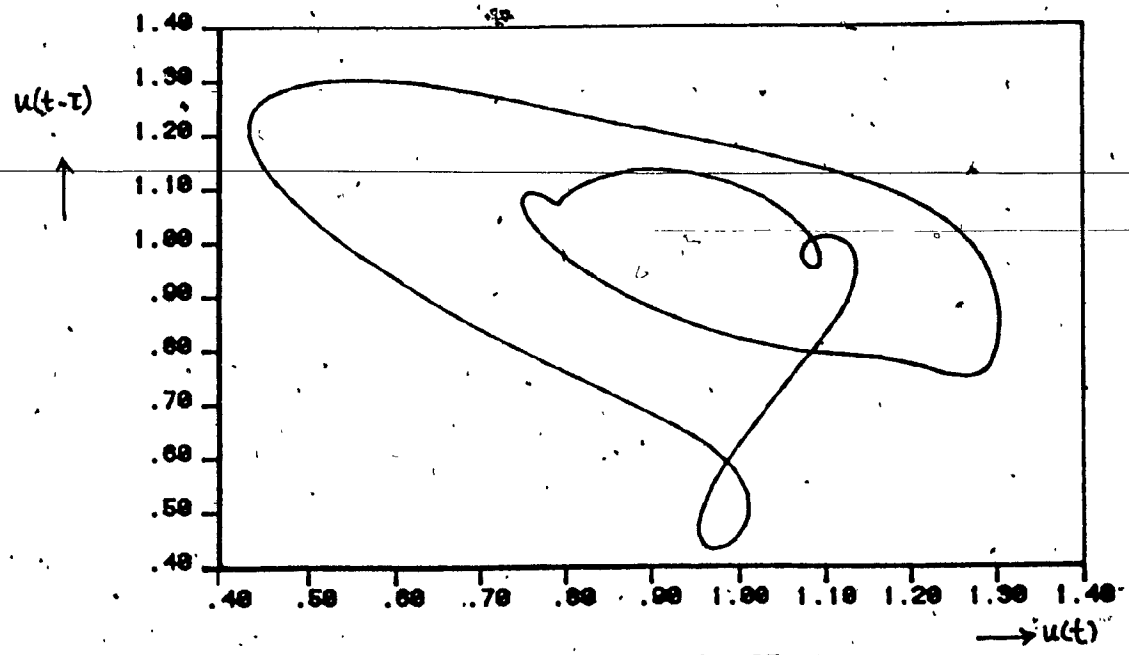



FIG. 6.8B PHASE PLOT OF LABEL 33

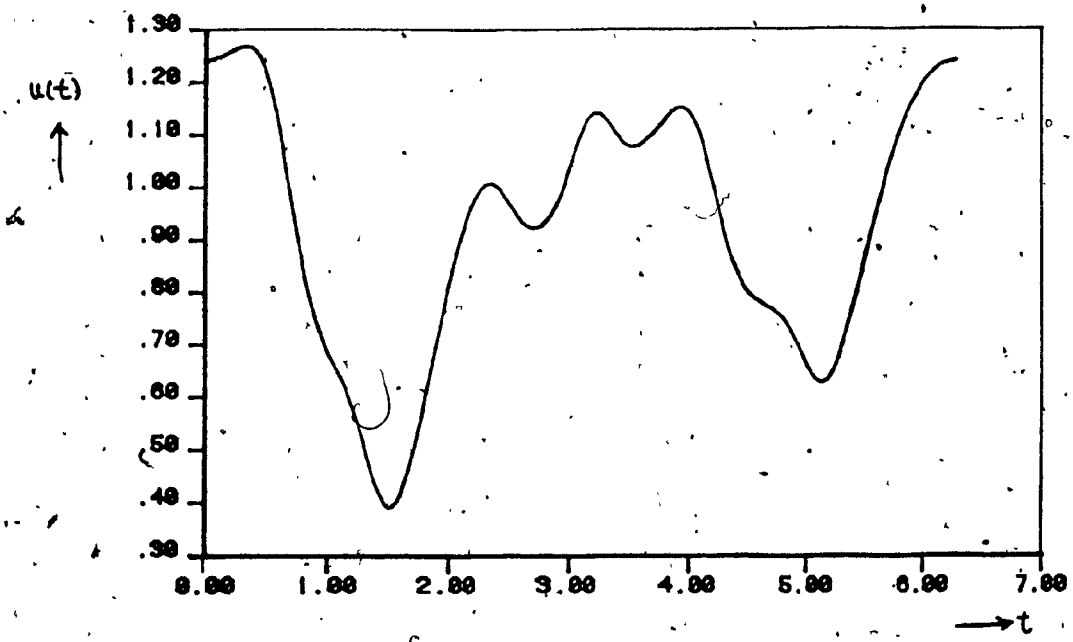



FIG. 6.9A SOLUTION OF LABEL 34

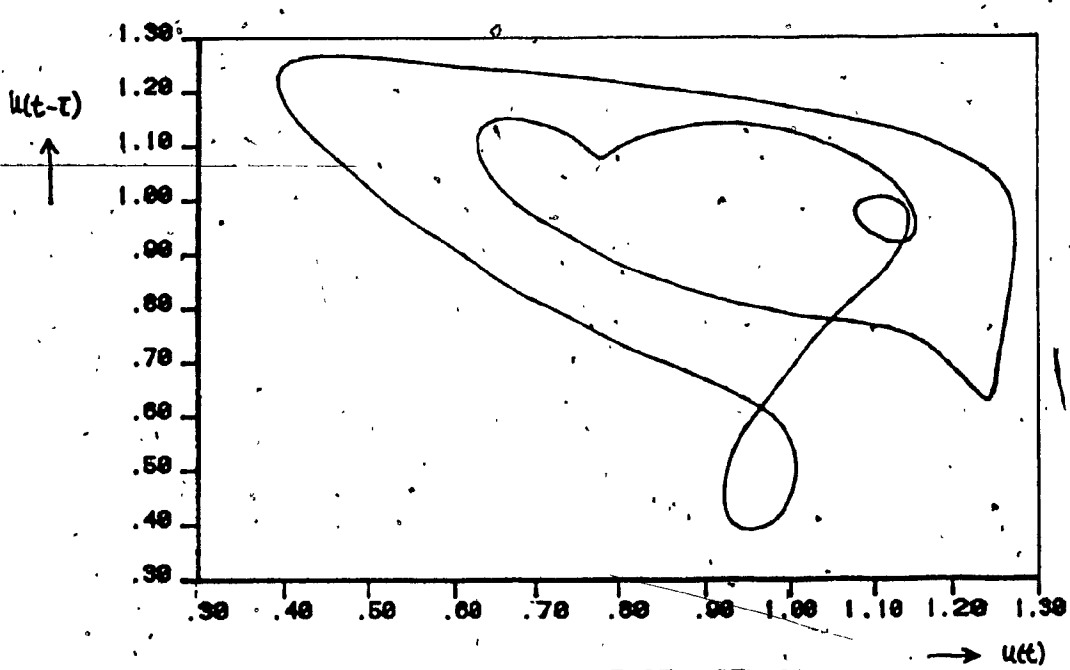



FIG. 6.9B PHASE PLOT OF LABEL 34

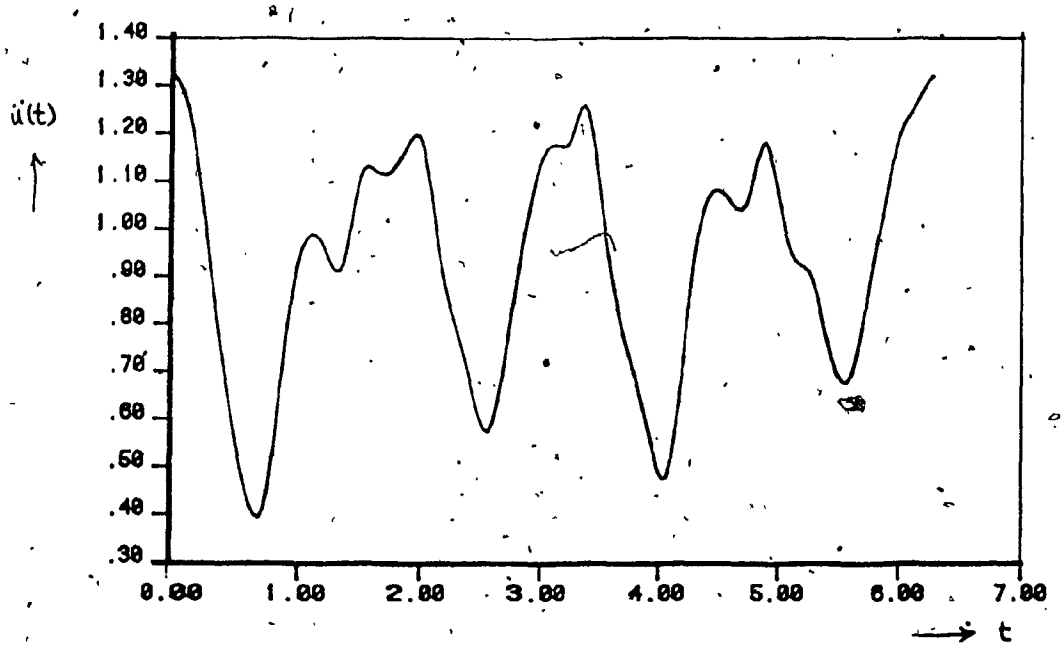



FIG. 6.10 SOLUTION OF LABEL 45

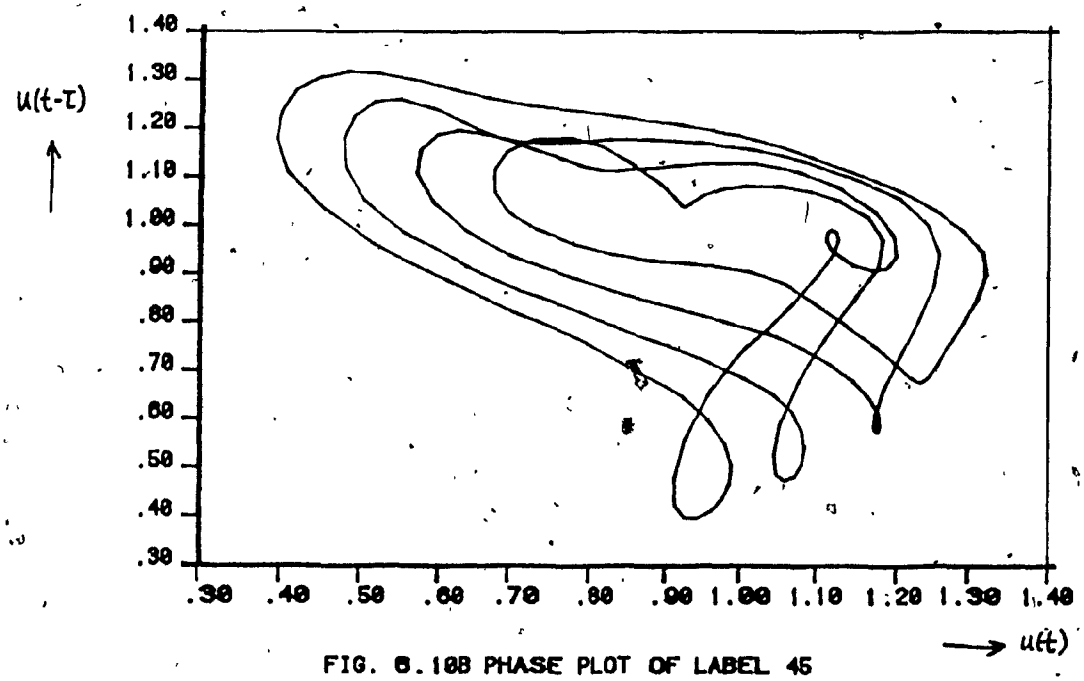



FIG. 6.10B PHASE PLOT OF LABEL 45

# CHAPTER SEVEN

## Conclusion

In this thesis, we have discussed the use of Keller's pseudo arclength technique. We use this technique for the continuation of steady state and periodic solution branches of delay differential equations. In the computation of steady state solution branches we have developed techniques to detect and locate steady state bifurcations and Hopf bifurcations. In the computation of periodic solution branches we have developed a method to detect and locate secondary periodic bifurcations including period doubling bifurcations. To switch branches at a Hopf bifurcation point, the bifurcation direction is obtained by making use of the symmetry of the bifurcating branches about the Hopf bifurcation point. An asymptotic estimate near the Hopf bifurcation point is used to anchor the next solution on the branch. To switch branches at other bifurcations mentioned in this thesis, we use the Hyper-plane method.

Further studies can be done on the generalization of the program DLAY to systems of differential equations with multiple delays, or with distributed lag. Generalization of the methods to delay differential equations with diffusion (partial differential equations) would be especially challenging. On the other hand, as far as Numerical Analysis is concerned, a complete convergence proof of our

method of computing is still lacking. Research on other schemes of computing and comparison of the efficiency of alterate computational schemes to that of our method would be of interest also.

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## APPENDIX

### User Guide for the DLAY PROGRAM

The DLAY program is a FORTRAN written package for computing the bifurcation diagram of a delay differential equation of the form

$$(*) \qquad u'(t) = f(u(t), u(t-\tau), \lambda)$$

where $\tau$ is the delay and $\lambda$ is the free parameter.

The package can be used in two situations. One is the first run of a specfic problem starting from the very beginning. The other one is the restart of the program at a certain labelled position in the bifurcation diagram generated from the previous run. A sample user program will be given at the end of the appendix, illustrating how this package is applied. A flow chart of the package is also given thereafter.

I. For the first run

For the first run of a specific problem, the user must prepare the following:

1. The main program which consists of only one call statement to the program driver subroutine, DLAY.

2. A subroutine with defining statement

```
SUBROUTINE FUNC(U,V,ICP,PAR,IJAC)
```

and two common blocks

```
COMMON/BLFCN/F,DFDU,DFDV,DFDP
COMMON/BLDDF/D2DU,D2DUV,D2DV,IDFFLAG
```

where

U, V  $U=u(t)$ and $V=u(t-\tau)$.

PAR  is a 1-dimensional array of size 10 which is used to store the values of parameters in the delay differential equation. Note that PAR(2) is reserved for the constant delay, $\tau$.

ICP  is the location of the free parameter in the parameter array PAR (ICP≠2).

IJAC  is a flag passed from the referencing routine to signify whether the derivatives are needed or not. The function value $f(u(t),u(t-\tau),\lambda)$ must be computed in any case. If IJAC=1, the first derivative must be computed. If IJAC=2 the second derivative must be computed.

F  is used to store the function value of the right hand side of the problem (*).

DFDU  is the partial derivative of f with respect to u.

DFDV is the partial derivative of f with respect to $u(t-\tau)$.

DFDP is the partial derivative of f with respect to the free parameter $\lambda$.

D2DU is the second partial derivative of f with respect to u.

D2DUV is the mixed second partial derivative of f with respect to u and $u(t-\tau)$.

D2DV is the second partial derivative of f with respect to $u(t-\tau)$.

IDFFLAG is a flag, which when set to 0 indicates that the second partial derivatives of f are provided. Otherwise it is set to 1. (See also Section 4 below.)

In this subroutine, the function itself and its first derivative have to be provided. However the second derivative is optional. If it is provided also, the steady state bifurcation directions will be computed exactly, otherwise a hyperplane method is used to approximate the directions.

3. A subroutine with the defining statement

```
SUBROUTINE STPNT(U,NPAR,ICP,PAR)
```

where U,ICP,PAR are as before. NPAR is the number of parameters used in the problem.

In this subroutine, the initial values of the parameters with a corresponding steady state solution are to be provided. Only the free parameter PAR(ICP) varies during execution. The other parameters remain fixed. The steady state solution provided should not be a bifurcation point or a limit point, otherwise the program may abort with an appropriate message.

4. A subroutine with the defining statement

```
      SUBROUTINE INIT
```

and common blocks

```
COMMON /PLOTU/ UT(203),VT(203),INPL,ITPLSC
COMMON /BLCSS/NDIM,ITMX,NPAR,ICP,IID,NMX,IPS,ITS,MTHCOD
COMMON /BLCPS/ NTST,NCOL,IANCH,NMXPS,IAD,NAD,NPR,NWTN
COMMON /BLDLS/ DS,DSMIN,DSMAX,IADS,NADS,DS0,DS1
COMMON /BLEPS/ EPSU,EPSL,EPSS,EPSR
COMMON /BLLIM/ RL0,RL1,A0,A1,PAR(10)
COMMON /BLDDF/ D2DU,D2DUV,D2DV,IDFFLAG
COMMON /BLGE/ DETGE,DETPS,IPRDBL,DETGE1
COMMON /ORDER/ M
```

is used to initialize the variables in the common blocks. These variables can be divided into two types: (i) Problem Dependent Data, which describe the problem

and should always be provided; (ii) Control Variables, which control the execution of the program. If any of the control values is not defined then a default value will be used.

i. Problem dependent data

NPAR number of parameters

ICP location of the free parameter

IDFFLAG =0 second derivative is provided
=1 otherwise

ii. Control variables

| NAME | DEFINITION | DEFAULT |
|---|---|---|
| M | the number of sine and cosine terms used in the truncated Fourier series to approximate the solution of the solution of the problem $u(t) = a_0 + \sum_{k=1}^{M} (a_k \sin kt + b_k \cos kt)$ where $1 \leq M \leq 100$. | 5 |
| INPL | the number of interpolation points for the solution between a pair of collocation points | 20 |
| EPSU | tolerance for the solution values | 1.0E-4 |

| | | |
|---|---|---|
| EPSL | tolerance for the free parameter | 1.0E-4 |
| EPSS | tolerance for the step size | 1.0E-4 |
| EPSR | tolerance for the period of the solution | 1.0E-4 |

Note that the tolerances above are relative.

| | | |
|---|---|---|
| RL0 | lower bound of the free parameter | -1.0E10 |
| RL1 | upper bound of the free parameter | 1.0E10 |
| A0 | lower bound of the solution norm | 0.0 |
| A1 | upper bound of the solution norm | 1.0E10 |
| DS | initial step size | 0.01 |
| IADS | =1 step size is adaptive<br>=0 step size is fixed | 1 |
| NADS | if number of iterations >NADS, then DS decreases, otherwise it increases. DS is fixed only if IADS=0. | 1 |
| ITMX | the maximum number of iteration allowed for each step | 10 |
| NMX | the maximum number of steps allowed for a steady state solution branch | 200 |
| NMXPS | the maximum number of steps allowed for a peroidic solution branch | 200 |

| | | |
|---|---|---|
| NPR | the number of steps between two labelled periodic solutions for which the complete orbit is written on unit 8. | 40 |
| MTHCOD | the number of full Newton iterations used before the Chord method starts | 0 |
| IID | =0 no debugging output on unit 9<br>=4 extensive debugging output is written on unit 9<br>=9 information on the Newton-chord iterations is written on unit 9 | 0 |
| IPS | =0 only steady state solutions are computed<br>=1 both steady state and periodic solutions are computed | 1 |
| IRS | =0 for the first run<br>otherwise, restart the program at the labelled point IRS | 0 |
| IPRDBL | =1 to compute period doubling bifurcation<br>=0 do not compute period doubling | 0 |

At the termination of the program, there will be four data files, namely, units 3, 7, 8 and 9. Unit 3 contains the restarting information for every labelled point in the bifurcation diagram. Unit 7 contains the plotting

information for the bifurcation diagram. Unit 8 contains the actual periodic solution with domain scaled to an interval of length $2\pi$. Unit 9 contains debugging information as requested.

II. To restart the program

The program can be restarted at any labelled solution point in the bifurcation diagram generated in the previous run. The restarting information of these labelled solutions is stored in file(unit 3). In a restarting run, the user should provide this file(unit 3). The user can retain the four routines mentioned above without change except the variable IRS. IRS must be set to the label number where the program is to restart. The control variables in Subroutine INIT can be changed if desired.

III. Example

An example is given below to illustrate how to get a bifurcation analysis of the delay differential equation

$$u'(t) = -\lambda\ u(t-\tau)\ (u(t) + 1)$$

using the DLAY program. In this example, $u \equiv 0$ is always a steady state solution for any $\lambda$. We take $u = 0$ and $\lambda = 1.0$ as a starting point. At the termination of the DLAY program, there will be four data files as described in section (I) above. They are units 3, 7, 8 and 9. Also a terminal output, unit 6, will appear. The terminal output

for this example is shown after this sample program.

```
      PROGRAM DLAYJH1
C     ------- -------
C
      IMPLICIT DOUBLE PRECISION     (A-H,O-Z)
C
      CALL DELAY
C
      STOP
      END

      SUBROUTINE FUNC(U,V,ICP,PAR,IJAC)
C     ---------- ----
C
      IMPLICIT DOUBLE PRECISION     (A-H,O-Z)
      COMMON /BLFCN/ F,DFDU,DFDV,DFDP
      COMMON /BLDDF/ D2DU,D2DUV,D2DV,IDFFLAG
      DIMENSION PAR(10)
C
      RL=PAR(1)
      RT=PAR(2)
C
      F(1) = -RL*V*(1+U)
C
      IF (IJAC.EQ.1) THEN
C
         DFDU(1,1) = -RL*V
         DFDV(1,1) = -RL*(1+U)
         DFDP(1,1) = -V*(1+U)
C
      ELSEIF (IJAC.EQ.2) THEN
C
C THE SECOND DERIVATIVES OF F ARE DEFINED HERE IF PROVIDED
C
      ENDIF
C
      RETURN
      END

      SUBROUTINE STPNT(U,NPAR,ICP,PAR)
C     ---------- -----


      IMPLICIT DOUBLE PRECISION  (A-H,O-Z)
      DIMENSION PAR(10)

      PAR(1)=1.0
      PAR(2)=1.0
C
      U = 0.0
C
```

```
      RETURN
      END

      SUBROUTINE INIT
C     ---------- ----
C
      IMPLICIT DOUBLE PRECISION     (A-H,O-Z)
C
      COMMON /PLOTU/ UT(203),VT(203),INPL,ITPLSC
      COMMON /BLCSS/ NDIM,ITMX,NPAR,ICP,IID,NMX,IPS,IRS,MTHCOD
      COMMON /BLCPS/ NTST,NCOL,IANCH,NMXPS,IAD,NAD,NPR,NWTN
      COMMON /BLDLS/ DS,DSMIN,DSMAX,IADS,NADS,DS0,DS1
      COMMON /BLEPS/ EPSU,EPSL,EPSS,EPSR
      COMMON /BLLIM/ RL0,RL1,A0,A1,PAR(10)
      COMMON /BLDDF/ D2DU,D2DUV,D2DV,IDFFLAG
      COMMON /BLGE/  DETGE,DETPS,IPRDBL,DETGE1
      COMMON /ORDER/ M
C
C     PROBLEM DEPENDENT DATA
C
      NDIM=1
      NPAR=2
      ICP=1
      IDFFLAG=1
C
C     CONTROL VARIABLES
C
      M = 10
      INPL = 10
C
      RL0=-1.0
      RL1=10.
      A0=0.
      A1=10.
C
      DS=0.01
      IADS=1
C
      ITMX=8
      NMX=100
      NMXPS=100
      NPR=20
      MTHCOD=2
      IID=9
C
      IPS=1
      IRS=0
C
      RETURN
      END
```

The terminal output(unit 6) consists of two parts. The first part is the information for the steady state solutions. The second part is the information for the periodic solutions. In the listing below,

```
LAB=  1 RLDA=  1.000E+00  AMP=  0.000E+00  N=   1        EP
LAB=  2 RLDA=  1.571E+00  AMP=  0.000E+00  N=  57        HB
LAB=  3 RLDA=  2.001E+00  AMP=  0.000E+00  N= 100        EP
*** START WITH A HOPF BIF.  0.000E+00  4.000E+00  1.571E+00
LAB=  4 RLDA=  0.158E+01 ROE=  0.401E+01 AMP=  0.179E+00
LAB=  5 RLDA=  0.164E+01 ROE=  0.405E+01 AMP=  0.415E+00
LAB=  6 RLDA=  0.174E+01 ROE=  0.413E+01 AMP=  0.633E+00
LAB=  7 RLDA=  0.187E+01 ROE=  0.425E+01 AMP=  0.830E+00
```

LAB is the label at that point. RLDA represents the free parameter. AMP is the norm of the solution, $|u|$. N is the step number on the branch. EP indicates that it is an end point of the branch and HB indicates it is a Hopf bifurcation point. The fourth line indicates that the following lines are information on a periodic branch starting at a Hopf bifurcation point with $|u|=0.00$, $\rho=4.00$ and $\lambda=1.571$. Here ROE represents the period of the solution, $\rho$.

Figures A1 and A2 are bifurcation diagram and solutions plots of the example respectively. File unit 7 contributes to the bifurcation diagram and file unit 8 contributes to the solution plots.

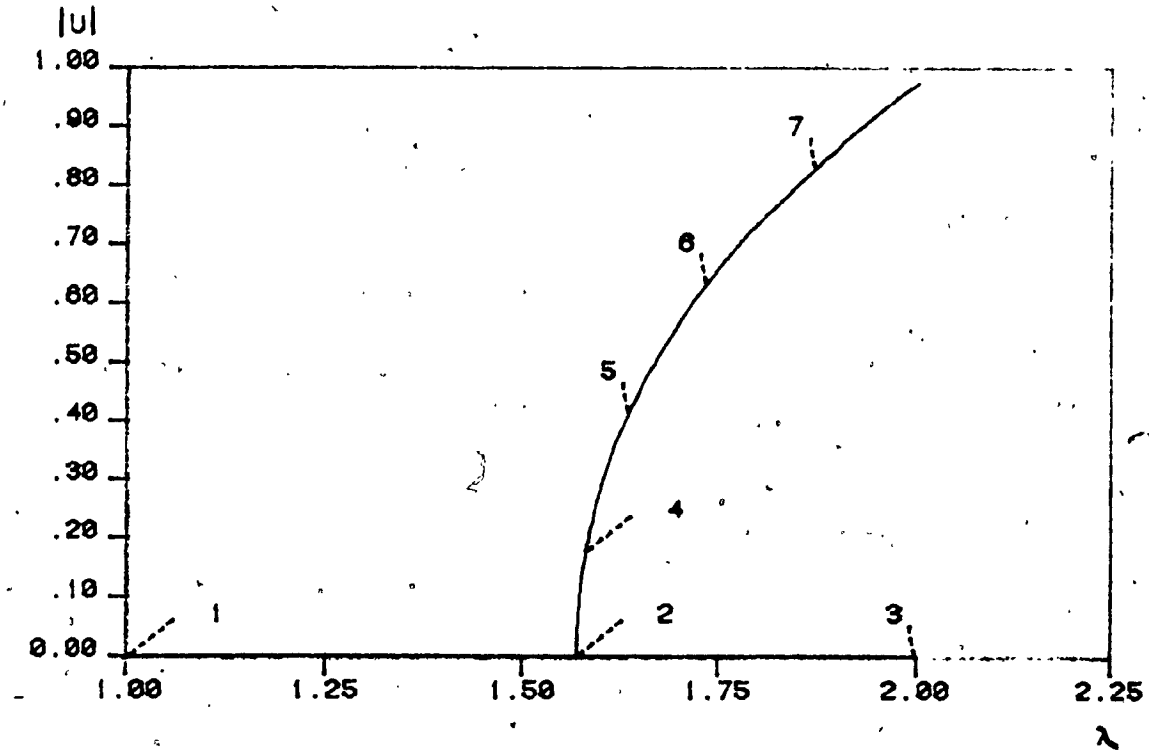



FIGURE A1 BIFURCATION DIAGRAM

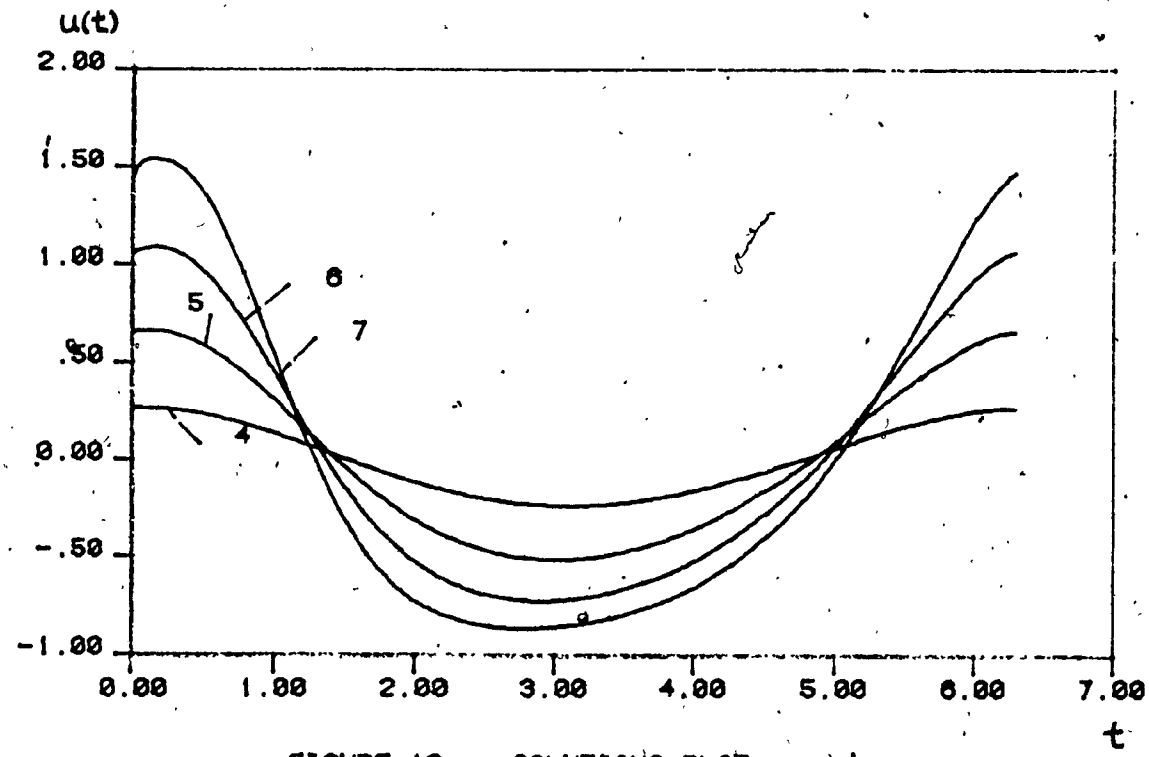



FIGURE A2 SOLUTIONS PLOT

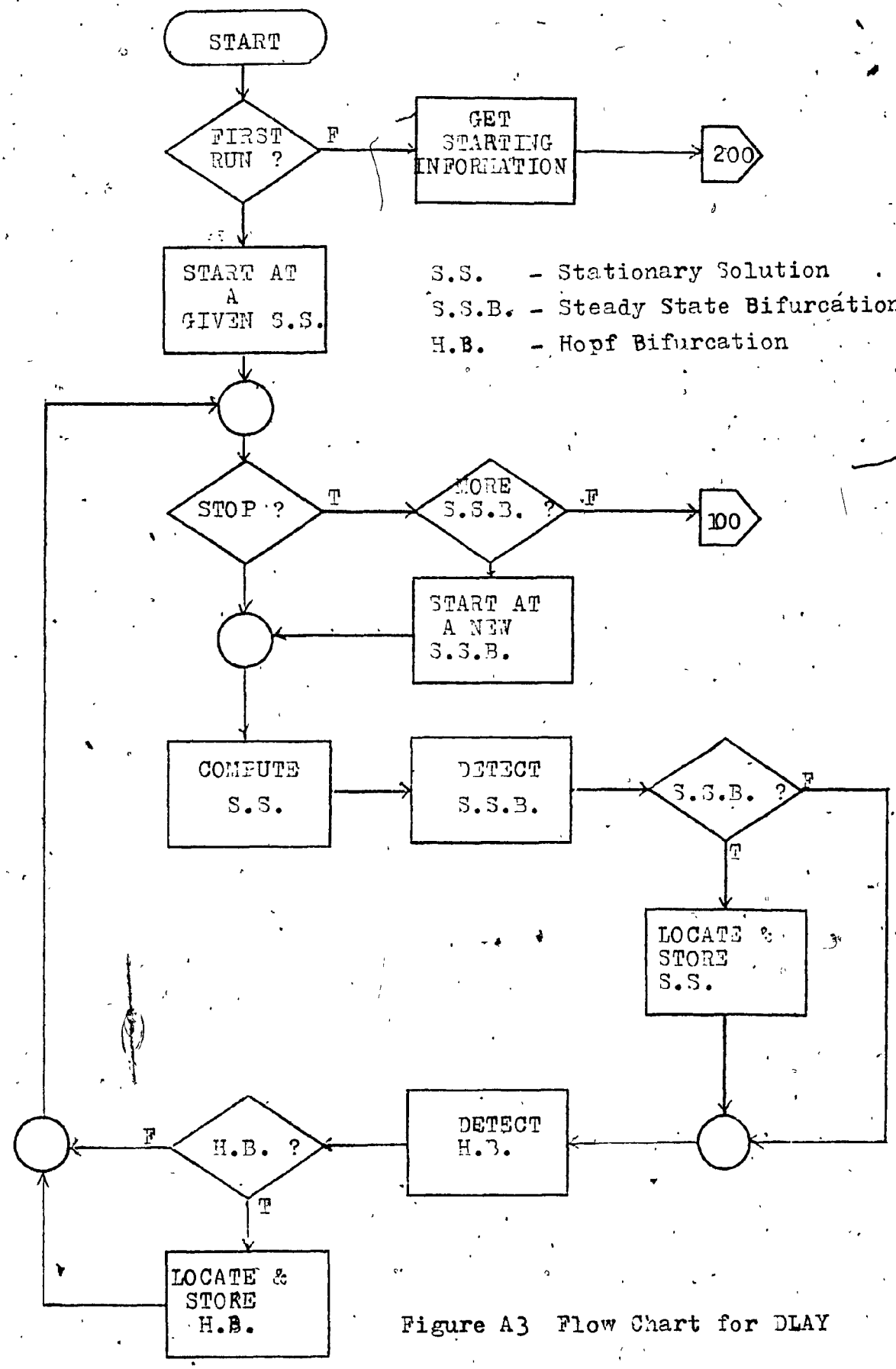
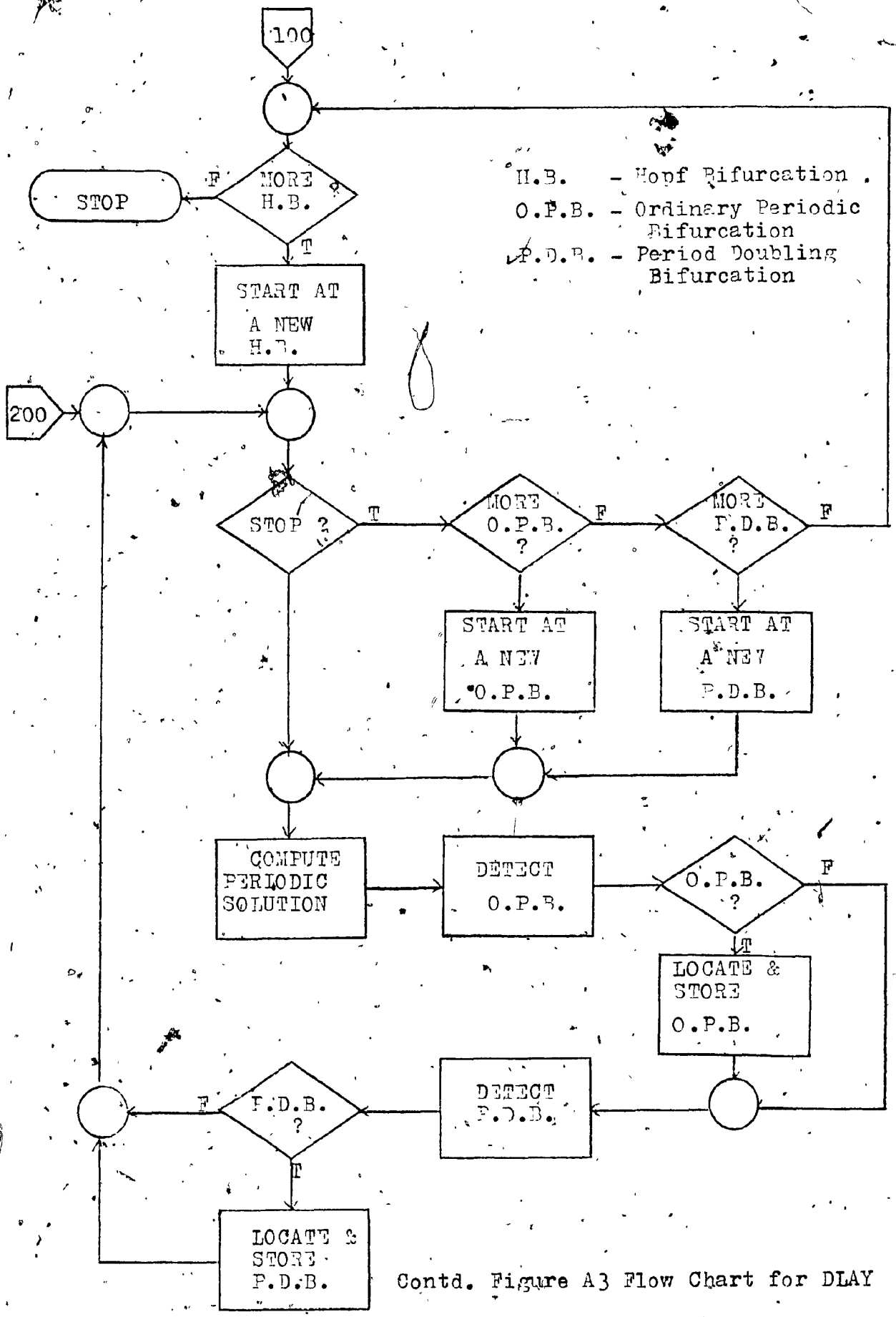



Figure A3 Flow Chart for DLAY




Contd. Figure A3 Flow Chart for DLAY