Schwarz's Surface and the Theory of Minimal Surfaces

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ABSTRACT

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and the
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We explore some general properties of minimal surfaces, and their historical origins. I am particularly interested in the Schwarz surface, which is spanned by a regular tetrahedral skew quadrilateral. We use the Weierstrass-Enneper representation formulas to derive the analytic function $R(\omega)$ obtained by Schwarz and use a representation in terms of elliptic integrals to investigate the relation to the hyperbolic paraboloid.

To Dr. Harald Proppe,

my mentor,

without whose help this thesis

would not have been possible,

thank you.

PREFACE

My facination with surfaces goes back to the late seventies during my studies at the McGill School of Architecture. One of my teachers, the late Stuart Wilson, was a Buckminster Fuller in architect's clothing. His interest in geometry rubbed off on me and it was while doing a project with him that I was first introduced to minimal surfaces.

Several years later, while working as a registered architect, and motivated by a desire to continue my studies, I initiated a thought process, occupying my evenings with experiments. I was intrigued by the fact that the octahedron fit inside the tetrahedron and the tetrahedron fit inside the cube. I reasoned that there must be a geometry linking these three polyhedra.

After a year of frustration, I gave up on lattices and tried using surfaces. The hyperbolic paraboloid solved my problem. It contained four of the six vertices of the octahedron and four of the six edges of the tetrahedron. By composing these 'hypars' contained in a cube, I found that they generated a continuous surface, simply connected, free of self-intersections and extending in all directions. In the fall of 1987 I constructed a model using piano wire, nylon stockings and fiberglass resin (see Fig. 1).

I decided that I would base my studies on this geometry. It soon became apparent, however, that architecture was not the best academic route. Mathematics seemed a better choice. So in the fall of 1991 I enroled as an independent at Concordia. It was not until August of 1995, while then a graduate student, that I finally saw a book that dealt with this kind of surface. It was Schwarz's surface discovered in 1865 in the context of minimal surfaces.

This thesis is an expository treatment of the theory of minimal surfaces as it relates to Schwarz's surface.

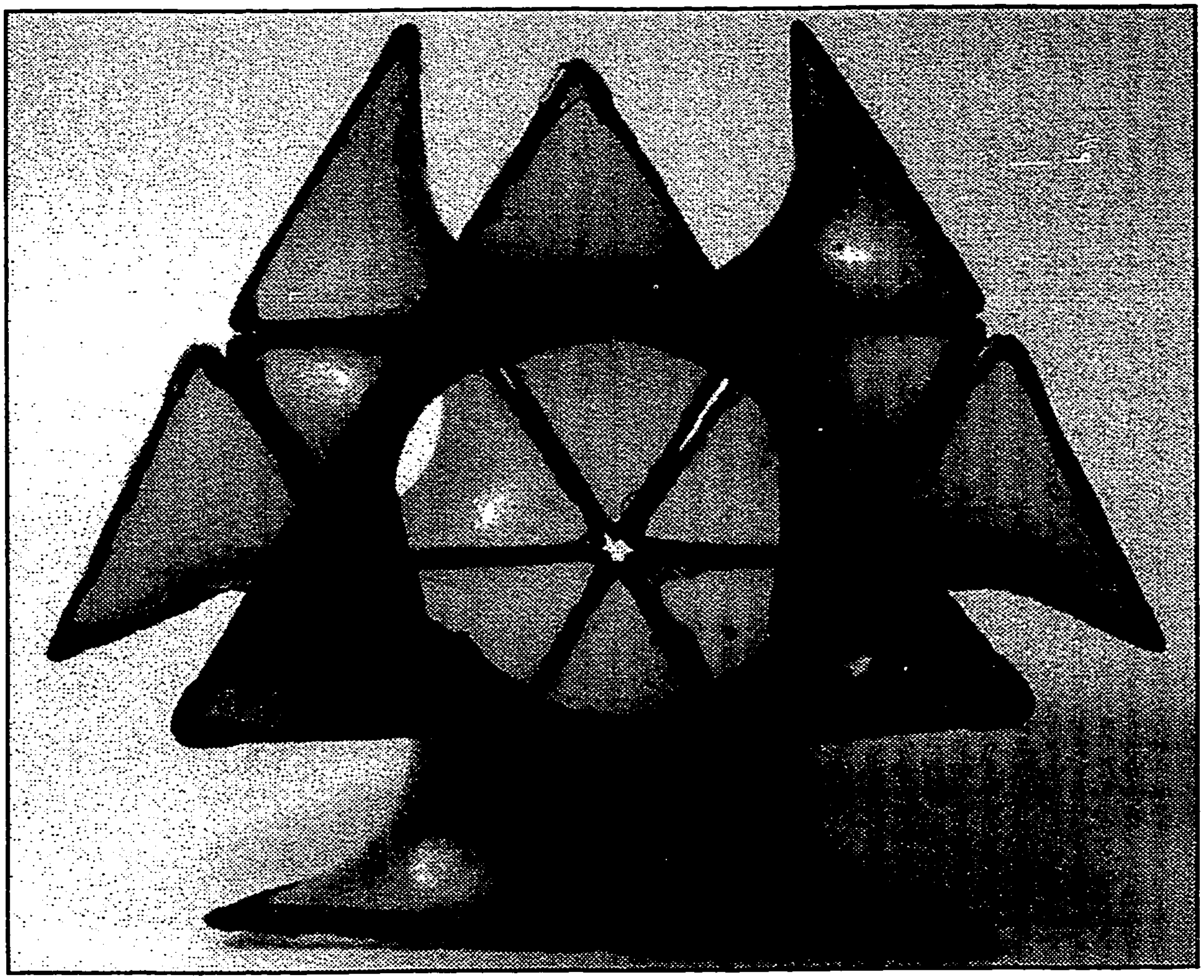

Figure 1

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I. Introduction

1. Historical Outline

The theory of minimal surfaces is one of the most prolific branches of modern mathematics. For centuries mathematicians have been intrigued by the inherent properties of the surface with minimum area spanning a given fixed wire frame in three-dimensional space. The physicist Joseph Antoine Ferdinand Plateau obtained physical models of such surfaces in his famous experiments with soap films during the nineteenth century.

The man whose work foreshadowed the study of minimal surfaces was Joseph Louis Lagrange. In 1762 he introduced a method for the calculus of variations to find the surface of smallest area bounded by a given closed curve.[1] That surface $z = z(x,y)$ was a solution to the Euler-Lagrange differential equation

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)=0. \tag{1}$$

An important contribution was then made by Jean Baptiste Marie Meusnier who in 1776 discovered that the right helicoid and the catenoid both satisfy the Euler-Lagrange equation. He also showed that the left-hand side of equation (1) is twice what today is known as the mean curvature H of the surface $z = z(x,y)$, i.e.

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1+z_x^2+z_y^2}}\right)+\frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1+z_x^2+z_y^2}}\right)$$
$$=\frac{(1+z_y^2)z_{xx}-2z_xz_yz_{xy}+(1+z_x^2)z_{yy}}{(1+z_x^2+z_y^2)^{\frac{3}{2}}}=2H. \tag{2}$$

Henceforth, it has been customary to use the term 'minimal surface' for *any* surface of vanishing mean curvature, even though it may not have the least area for the given contour.

[1] What since Henri Léon Lebesgue is referred to as the *Plateau problem*.

Equation (2) yields the minimal surface equation

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0. \tag{3}$$

In 1783 Gaspard Monge, Meusnier's teacher, integrated this equation to derive representation formulas—soon to be amended by Adrien Marie Legendre—in terms of analytic functions. Using these formulas, Heinrich Ferdinand Scherk, between 1830 and 1834, derived explicit equations for five additional real minimal surfaces.

Then came the golden age of the theory from 1855 to 1890. New minimal surfaces were discovered by Catalan, Enneper, Henneberg, and Schwarz. Important contributions were also made by Pierre Ossian Bonnet, Joseph Alfred Serret, Georg Friedrich Bernhard Riemann, Karl Theodor Wilhelm Weierstrass, Johannes Leonard Gottfried Julius Weingarten, and many others.

2. Lagrange

Given a domain D in $\mathbf{R}^2$ and a (sufficiently smooth) map $z = z(x, y)$ defined on ∂D, the boundary of D, we have a fixed closed curve C, the image of ∂D in $\mathbf{R}^3$. The problem is to extend $z(x, y)$ to all of D so that the resulting surface in $\mathbf{R}^3$ is minimal.

Lagrange used the calculus of variations developed by Leonhard Euler to minimize the area

$$A(S) = \int\!\!\int_D \sqrt{1 + |\nabla \bar{z}|^2}\, dxdy$$

of the surface $\bar{z} = \bar{z}(x, y) = z(x, y) + \varepsilon\zeta(x, y)$ near the solution surface $z = z(x, y)$, defined on D, with the common boundary C. Here $\zeta(x, y)$ is an arbitrary function satisfying suitable regularity conditions and vanishing on C, while ε can be viewed as a 'small' variable. Thus the integral

$$I(\varepsilon) = \int\!\!\int_D \sqrt{1 + \bar{z}_x^2 + \bar{z}_y^2}\, dxdy,$$

considered as a function of ε, has a minimum value when $I'(0) = 0$. In detail

$$I(\varepsilon) = \iint_D \sqrt{1 + z_x^2 + z_y^2 + 2\varepsilon z_x \zeta_x + 2\varepsilon z_y \zeta_y + \varepsilon^2 \zeta_x^2 + \varepsilon^2 \zeta_y^2}\, dx dy,$$

$$I'(\varepsilon) = \iint_D \frac{1}{2} \cdot \frac{2 z_x \zeta_x + 2 z_y \zeta_y + 2\varepsilon \zeta_x^2 + 2\varepsilon \zeta_y^2}{\sqrt{1 + z_x^2 + z_y^2 + 2\varepsilon z_x \zeta_x + 2\varepsilon z_y \zeta_y + \varepsilon^2 \zeta_x^2 + \varepsilon^2 \zeta_y^2}}\, dx dy,$$

$$I'(0) = \iint_D \frac{z_x \zeta_x + z_y \zeta_y}{\sqrt{1 + z_x^2 + z_y^2}}\, dx dy$$

$$= \iint_D \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) \bullet (\zeta_x, \zeta_y)\, dx dy.$$

Integrating by parts we get

$$I'(0) = \left[\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) \cdot \zeta(x, y) \right]$$

$$- \iint_D \zeta(x, y) \left\{ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \bullet \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) \right\} dx dy,$$

where the first term vanishes since $\zeta(x, y)$ vanishes on C. Hence,

$$I'(0) = - \iint_D \zeta(x, y) \left\{ \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \bullet \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}, \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) \right\} dx dy$$

$$= - \iint_D \left\{ \frac{\partial}{\partial x} \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) \right\} \zeta(x, y)\, dx dy = 0,$$

which implies that

$$\frac{\partial}{\partial x} \left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}} \right) + \frac{\partial}{\partial y} \left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \right) = 0.^2 \tag{1}$$

[2] This implies that $\frac{z_x dy - z_y dx}{\sqrt{1 + z_x^2 + z_y^2}}$ is an exact differential.

3. Monge

Monge first attempted to solve Lagrange's minimal surface equation

$$(1+q^2)r - 2pqs + (1+p^2)t = 0 \tag{3'}$$

in 1775, but only succeeded in doing so ten years later[3] (here we have used Monge's notation, replacing $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ by p, q, r, s, t respectively). In his paper entitled *Mémoire sur le calcul intégral des équations aux différences partielles*, presented to the Royal Academy of Sciences in Paris the first of February 1786, he outlines a method for solving a general linear equation of the form

$$Ar + Bs + Ct + D = 0.$$

Then, beginning in section XXIII, he solves Lagrange's minimal surface equation (see §II.1, equations (4')). His method was flawed,[4] however, and upon the objections of Pierre Simon Laplace and others he revised his work, creating at the same time his theory of characteristics and envelopes—which Legendre and others described as 'metaphysical principles'.

These methods, along with the revised solution to Lagrange's minimal surface equation, appear in *Feuilles d'Analyse appliquée à la Géométrie à l'usage de l'École Polytechnique* published in 1795.[5] In 'feuille XX' he gives the x, y, and z coordinate functions of a minimal surface as

$$\begin{aligned} x &= -\Phi'(a) + \Psi'(b), \\ y &= -\Phi(a) + a\Phi'(a) + \Psi(b) - b\Psi'(b), \\ z &= \int \Phi''(a)da\sqrt{-1-a^2} + \int \Psi''(b)db\sqrt{-1-b^2}, \end{aligned} \tag{4}$$

[3] René Taton, *L'œuvre Scientifique de Monge* (Paris, 1951), p. 292.
[4] By a simple change of variables, Legendre corrected the error. Taton, p. 293.
[5] Taton, p. 300.

where Φ, Ψ are constants of (partial) integration and

$$a = \frac{pq + \sqrt{-1 - p^2 - q^2}}{1 + q^2},$$
$$b = \frac{pq - \sqrt{-1 - p^2 - q^2}}{1 + q^2}.$$

4. Weierstrass

In 1830, Scherk used the representation formulas (4), obtained by Monge and Legendre, to discover two new explicit examples of minimal surfaces.[6] His prize-winning essay was published in 1832, at about the same time as Siméon Denis Poisson published a paper in which he qualified the Monge-Legendre formulas as practically unusable.[7] Soon after, in 1834, Scherk submitted another paper in which he derives equations for three further examples.

Meanwhile, great advances were being made in the theory of functions of a complex variable by Augustin Louis Cauchy, Carl Friedrich Gauss, Weierstrass, and others. The stage was set for the application of this new theory to minimal surfaces.

In 1866, Weierstrass succeeded in writing the Monge-Legendre formulas (4) in the very elegant form

$$\begin{aligned} x &= x_0 + \Re \int_0^\gamma (\Phi^2 - \Psi^2)\, d\gamma', \\ y &= y_0 + \Re \int_0^\gamma i(\Phi^2 + \Psi^2)\, d\gamma', \\ z &= z_0 + \Re \int_0^\gamma 2\Phi\Psi\, d\gamma', \end{aligned} \tag{5}$$

[6] Johannes Nitsche, *Lectures on Minimal Surfaces* (Cambridge, 1989), p. 10.
[7] Taton, p. 292.

where Φ and Ψ are single-valued analytic functions of the (complex) variable γ with no common zeros,[8] the integrals are path-independent line integrals, and (x_0, y_0, z_0) is the point on the surface corresponding to the origin in the γ-plane. The Monge-Legendre formulas were then further simplified by Enneper and Weierstrass to the form

$$x = x_0 + \Re \int_{\omega_0}^{\omega} (1 - \omega'^2) R(\omega')\, d\omega',$$

$$y = y_0 + \Re \int_{\omega_0}^{\omega} i(1 + \omega'^2) R(\omega')\, d\omega', \tag{6}$$

$$z = z_0 + \Re \int_{\omega_0}^{\omega} 2\omega' R(\omega')\, d\omega',$$

where $R(\omega)$ is a function which is analytic and $R(\omega) \neq 0$.

5. Schwarz

In 1865, a pupil of Weierstrass, named Hermann Amandus Schwarz, derived the first successful solution of Plateau's problem in a concrete case.[9] He began with the skew quadrilateral and, using the representation formulas developed by Weierstrass, determined the analytic function

$$R(\omega) = \frac{\kappa}{\sqrt{1 - 14\omega^4 + \omega^8}}, \quad \kappa > 0, \tag{7}$$

pertaining to this closed contour (see Figure 2).

This surface of Schwarz is remarkable in that it is extremely well approximated by the hyperbolic paraboloid with the same contour. The ratio of the areas of the two surfaces is 1.0012.[10] We will explore the difference between these two surfaces and determine their commonality in Chapter IV. By assembling eighteen copies

[8] Φ, Ψ not the same as Monge-Legendre formulas (4).

[9] Nitsche, p. 233.

[10] Nitsche, p. 241. Also Darboux, p. 440.

of Schwarz's surface, we obtain a new minimal surface bounded by four identical equilateral triangles (see Figure 1).

"Unlimited continuation leads to a complete minimal surface[...]which extends to infinity in all directions. This surface contains an infinite number of straight lines and its genus is infinite. Historically, this represents the first example of a periodic minimal surface. It has no self-intersections and effects a peculiar division of space into two congruent, disjoint, intertwined labyrinthic domains of infinite connectivity."[11]

To an architect these forms are especially pleasing. The structural properties of minimal surfaces are well documented,[12] and these natural forms, with their efficient intricate spacial configurations, hold much potential for the architecture of tomorrow. But more important are the clues, hidden in these shapes, to a four-dimensional geometry, a parting with the Cartesian tradition, and a new approach to space. Is there a way to express these periodic surfaces using four variables based on a tetrahedral coordinate system? Is the Schwarz surface an elementary shape in four-dimensional geometry, just as the square is in two-dimensional geometry? These are open questions which are certainly not in the scope of this thesis, but such questions were the motivation behind it.

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[11] Nitsche, p. 239.

[12] See, for example, Frei Otto (ed.), *Tensile Structures* (Cambridge,1969).

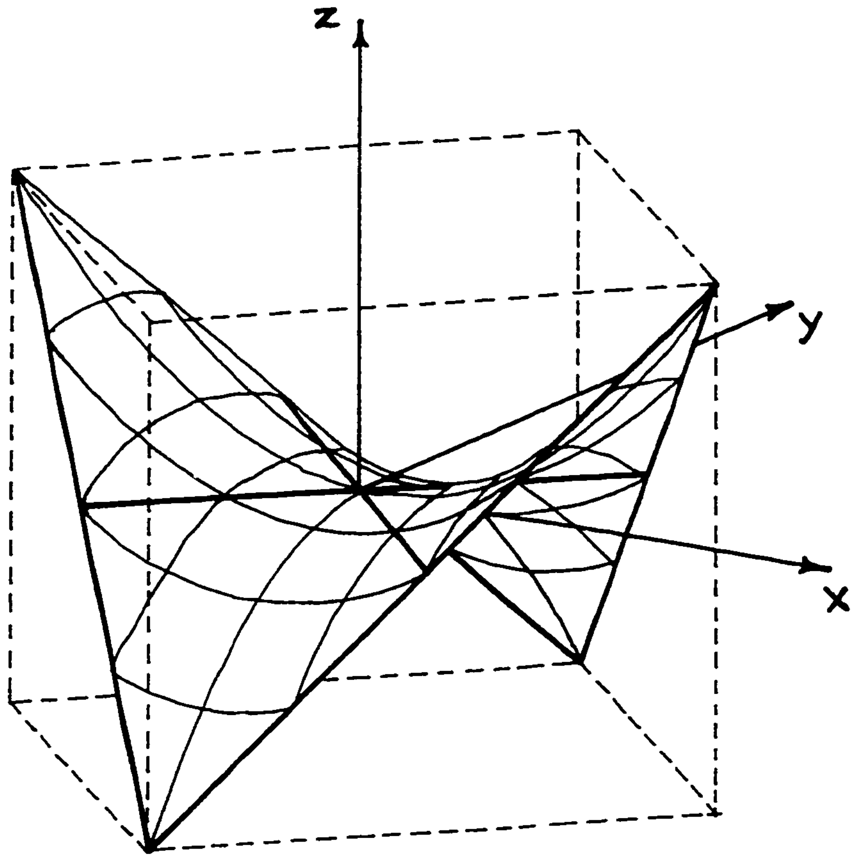

Figure 2

Schwarz's Surface

II. The Equations of Weierstrass

1. Overview

Monge was looking for the solution $z(x,y)$ to the differential equation

$$(1+q^2)r - pqs + (1+p^2)t = 0. \tag{3'}$$

His initial solution

$$\begin{aligned} x &= \int \frac{\psi'(b)db - \phi'(a)da}{a-b}, \\ y &= \int \frac{a\phi'(a)da - b\psi'(b)db}{a-b}, \\ z &= \int \frac{\psi'(b)db\sqrt{-1-b^2} + \phi'(a)da\sqrt{-1-a^2}}{a-b}, \end{aligned} \tag{4'}$$

where ϕ, ψ are constants of (partial) integration and

$$\begin{aligned} a &= \frac{pq + \sqrt{-1-p^2-q^2}}{1+q^2}, \\ b &= \frac{pq - \sqrt{-1-p^2-q^2}}{1+q^2}, \end{aligned}$$

was soon to be rewritten (see equations (4)), but it became clear that Monge was dealing with expressions of x, y, and z in terms of analytic functions.

In 1797, soon after his *Feuilles d'Analyse appliquée à la Géométrie* was published, Monge introduces the idea of considering the three coordinates x, y, z as functions of two parameters; u and v.[13] The minimal surface equation (3) then becomes

$$E(Xx_{vv}+Yy_{vv}+Zz_{vv}) - 2F(Xx_{uv}+Yy_{uv}+Zz_{uv}) + G(Xx_{uu}+Yy_{uu}+Zz_{uu}) = 0,$$

where E, F, G are the first fundamental coefficients (see Appendix A) and X, Y, Z are the coordinates of the unit normal. Monge notes that this equation is satisfied

[13] G. Darboux, *Leçons sur la Théorie Générale des Surfaces* (Paris, 1914), p. 274.

by $E=0$, $G=0$, and $x_{uv}=y_{uv}=z_{uv}=0$, which yields

$$\begin{aligned} x &= f_1(u)+g_1(v), \\ y &= f_2(u)+g_2(v), \\ z &= f_3(u)+g_3(v), \end{aligned} \tag{8}$$

where the f_i, g_j are complex-valued functions since they must satisfy

$$\begin{aligned} df_1^2+df_2^2+df_3^2 &= 0, \\ dg_1^2+dg_2^2+dg_3^2 &= 0. \end{aligned} \tag{9}$$

Enneper and Weierstrass rewrote equations (8) in the form

$$\begin{aligned} x &= x_0+\frac{1}{2}\int_{\omega_0}^{\omega}(1-\omega'^2)R(\omega')d\omega'+\frac{1}{2}\int_{\omega_1^0}^{\omega_1}(1-\omega_1'^2)R_1(\omega_1')d\omega_1', \\ y &= y_0+\frac{i}{2}\int_{\omega_0}^{\omega}(1+\omega'^2)R(\omega')d\omega'-\frac{i}{2}\int_{\omega_1^0}^{\omega_1}(1+\omega_1'^2)R_1(\omega_1')d\omega_1', \\ z &= z_0+\int_{\omega_0}^{\omega}\omega'R(\omega')d\omega'+\int_{\omega_1^0}^{\omega_1}\omega_1'R_1(\omega_1')d\omega_1', \end{aligned} \tag{10}$$

but it was Weierstrass who realized the full application of analytic function theory.[14] He showed that, by considering $\omega_1, \omega_1^0, R_1$ as the complex conjugates of ω, ω_0, R respectively, we can represent *all* real minimal surfaces in the form

$$\begin{aligned} x &= x_0+\Re\int_{\omega_0}^{\omega}(1-\omega'^2)R(\omega')d\omega', \\ y &= y_0+\Re\int_{\omega_0}^{\omega}i(1+\omega'^2)R(\omega')d\omega', \\ z &= z_0+\Re\int_{\omega_0}^{\omega}2\omega'R(\omega')d\omega', \end{aligned} \tag{6}$$

[14] Darboux, p. 288.

where the two parameters are the real and imaginary parts of the complex variable ω, and (x_0, y_0, z_0) is the point on the surface corresponding to ω_0 in the ω-plane.

2. A Classical Approach

Following the method of Darboux,[15] we show that the sum of the principal curvatures (see Appendix A) must be zero for a minimal surface and we derive Weierstrass's equations (see Appendix E).

Consider a continuous portion of the surface Σ bounded by a contour C, and let us find the variation in area as we pass to a surface Σ' which is 'infinitely close by'. Let x, y, z be the coordinates of a point of Σ and X, Y, Z the coordinates of the unit normal $\mathbf{X}$ at this point. We will assume a locally orthogonal parametrization and let the u- and v-parameter curves be the lines of curvature.

If we designate by R and R' the two radii of normal curvature at the point being considered, then the equations of Rodrigues (see Appendix A) give us

$$\mathbf{x}_u + R\mathbf{X}_u = 0, \qquad \text{and} \qquad \mathbf{x}_v + R'\mathbf{X}_v = 0.$$

Moreover, since $R\mathbf{X}_u = -\mathbf{x}_u$ and $R'\mathbf{x}_v = -\mathbf{x}_v$, then

$$R\mathbf{X}_u \bullet R'\mathbf{X}_v = \mathbf{x}_u \bullet \mathbf{x}_v = 0,$$

and so, $\mathbf{X}_u \bullet \mathbf{X}_v = 0$. To simplify, let

$$e = \mathbf{X}_u \bullet \mathbf{X}_u, \qquad E = \mathbf{x}_u \bullet \mathbf{x}_u,$$

$$g = \mathbf{X}_v \bullet \mathbf{X}_v, \qquad G = \mathbf{x}_v \bullet \mathbf{x}_v.$$

Now we consider a surface Σ' 'infinitely close to' Σ, where the normal at a point M of Σ passes through Σ' at M'. Designate by λ the distance MM'. The surface Σ' is defined if we give λ as a function of u and v, and the coordinates x', y', z' of a point of Σ' are determined by the equation

$$\mathbf{x}' = \mathbf{x} + \lambda\mathbf{X}.$$

[15] Darboux, p. 281.

Suppose further that the surface Σ' shares the same contour C, i.e. λ vanishes on all points of C.

The area of Σ' will be

$$A = \iint \sqrt{EG}\left(1 - \frac{\lambda}{R}\right)\left(1 - \frac{\lambda}{R'}\right)\sqrt{1 + \frac{\lambda_u^2}{e(\lambda - R)^2} + \frac{\lambda_v^2}{g(\lambda - R')^2}}\, dudv. \qquad (11)$$

We can approximate the area by using a Taylor series expansion,

$$\left[1 + \left(\frac{\lambda_u^2}{e(\lambda - R)^2} + \frac{\lambda_v^2}{g(\lambda - R')^2}\right)\right]^{\frac{1}{2}} \approx 1 + \frac{1}{2}\left(\frac{\lambda_u^2}{e(\lambda - R)^2} + \frac{\lambda_v^2}{g(\lambda - R')^2}\right),$$

and further approximate by

$$1 + \frac{1}{2}\left(\frac{\lambda_u^2}{e(\lambda - R)^2} + \frac{\lambda_v^2}{g(\lambda - R')^2}\right) \approx 1 + \frac{\lambda_u^2}{2eR^2} + \frac{\lambda_v^2}{2gR'^2}$$

for $\lambda \to 0$. From (11) the approximate area of Σ' is thus

$$\begin{aligned} A_\lambda = &\iint \sqrt{EG}\, dudv - \iint \sqrt{EG}\, \lambda\left(\frac{1}{R} + \frac{1}{R'}\right) dudv \\ &+ \iint \sqrt{EG}\left[\frac{\lambda^2}{RR'} + \frac{\lambda_u^2}{2eR^2} + \frac{\lambda_v^2}{2gR'^2}\right] dudv. \end{aligned}$$

If we let $\lambda = 0$, we get

$$A_0 = \iint \sqrt{EG}\, dudv,$$

the area of Σ. If Σ is a minimal surface, then

$$\begin{aligned} A_\lambda - A_0 = &-\iint \sqrt{EG}\, \lambda\left(\frac{1}{R} + \frac{1}{R'}\right) dudv \\ &+ \iint \sqrt{EG}\left[\frac{\lambda^2}{RR'} + \frac{\lambda_u^2}{2eR^2} + \frac{\lambda_v^2}{2gR'^2}\right] dudv > 0. \end{aligned}$$

This means that the negative term must vanish, since we can choose λ in such a way that the positive term vanishes. It follows that

$$\frac{1}{R} + \frac{1}{R'} = 0,$$

i.e.

$$R = -R'.$$

Thus, the first condition for the area of Σ to be as small as possible is that the sum of the principal curvatures is zero at every point of the surface.

We now proceed to derive Weierstrass's equations. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch on a surface of class 2. As did Monge a century earlier (see § II.1), Darboux[16] assumes that x, y, z are complex-valued coordinate functions satisfying $E{=}G{=}0$, $F{\neq}0$. Thus, for a minimal surface, it follows that

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = 0 \quad \Rightarrow \quad M = 0.$$

We have $M = \mathbf{x}_{uv} \bullet \mathbf{X} = 0$, and so $\mathbf{x}_{uv} = A\mathbf{x}_u + B\mathbf{x}_v$. Also,

$$E = \mathbf{x}_u \bullet \mathbf{x}_u = 0, \quad \text{and} \quad G = \mathbf{x}_v \bullet \mathbf{x}_v = 0.$$

Differentiating E with respect to v we get

$$\frac{\partial}{\partial v}(\mathbf{x}_u \bullet \mathbf{x}_u) = 2\mathbf{x}_u \bullet \mathbf{x}_{uv} = 2\mathbf{x}_u \bullet (A\mathbf{x}_u + B\mathbf{x}_v) = 0,$$

and so $B\mathbf{x}_u \bullet \mathbf{x}_v = 0$, which means that $B = 0$ since $F = \mathbf{x}_u \bullet \mathbf{x}_v \neq 0$. Similarly, by differentiating G with respect to u, we find that $A = 0$. Thus $\mathbf{x}_{uv} = \mathbf{0}$, i.e.

$$\frac{\partial^2 x}{\partial u \partial v} = 0, \quad \frac{\partial^2 y}{\partial u \partial v} = 0, \quad \frac{\partial^2 z}{\partial u \partial v} = 0,$$

and their solutions are

$$\begin{aligned} x &= f_1(u) + g_1(v), \\ y &= f_2(u) + g_2(v), \\ z &= f_3(u) + g_3(v). \end{aligned} \tag{8}$$

But since $E = G = 0$, we must have

$$\begin{aligned} f_1'^2(u) + f_2'^2(u) + f_3'^2(u) = 0, \\ g_1'^2(v) + g_2'^2(v) + g_3'^2(v) = 0. \end{aligned} \tag{9'}$$

16 Darboux, p. 284.

Now, if we let

$$\frac{f_1'(u) + if_2'(u)}{-f_3'(u)} = \omega,$$

then from (9′),

$$f_1'^2(u) + f_2'^2(u) = -f_3'^2(u),$$

$$[f_1'(u) - if_2'(u)][f_1'(u) + if_2'(u)] = -f_3'^2(u),$$

$$\left[\frac{f_1'(u) - if_2'(u)}{f_3'(u)}\right]\left[\frac{f_1'(u) + if_2'(u)}{-f_3'(u)}\right] = 1,$$

$$\frac{f_1'(u) - if_2'(u)}{f_3'(u)} \cdot \omega = 1,$$

i.e.

$$f_1'(u) - if_2'(u) = \frac{f_3'(u)}{\omega}.$$

The relationship between f_1', f_2', and f_3' is thus

$$\frac{f_1'(u)}{1-\omega^2} = \frac{f_2'(u)}{i(1+\omega^2)} = \frac{f_3'(u)}{2\omega}.$$

If these equations share the common value $\frac{1}{2}R(\omega)\frac{d\omega}{du}$, then we get

$$f_1(u) = \frac{1}{2}\int(1-\omega^2)R(\omega)d\omega,$$
$$f_2(u) = \frac{i}{2}\int(1+\omega^2)R(\omega)d\omega,$$
$$f_3(u) = \int \omega R(\omega)d\omega.$$

Similarly, if we let

$$\frac{g_1'(v) - ig_2'(v)}{-g_3'(v)} = \omega_1,$$

we will get

$$g_1(v) = \frac{1}{2}\int(1-\omega_1^2)R_1(\omega_1)d\omega_1,$$
$$g_2(v) = -\frac{i}{2}\int(1+\omega_1^2)R_1(\omega_1)d\omega_1,$$
$$g_3(v) = \int \omega_1 R_1(\omega_1)d\omega_1.$$

From equations (8) we now get the equations of Enneper,

$$
\begin{aligned}
x &= \frac{1}{2}\int(1-\omega^2)R(\omega)d\omega + \frac{1}{2}\int(1-\omega_1^2)R_1(\omega_1)d\omega_1,\\
y &= \frac{i}{2}\int(1+\omega^2)R(\omega)d\omega - \frac{i}{2}\int(1+\omega_1^2)R_1(\omega_1)d\omega_1,\\
z &= \int\omega R(\omega)d\omega + \int\omega_1 R_1(\omega_1)d\omega_1,
\end{aligned}
\tag{12}
$$

and by considering $R(\omega)$ and $R_1(\omega_1)$ as complex conjugates, as well as ω and ω_1, we get the equations of Weierstrass,

$$
\begin{aligned}
x &= \Re\int(1-\omega^2)R(\omega)d\omega,\\
y &= \Re\int i(1+\omega^2)R(\omega)d\omega,\\
z &= \Re\int 2\omega R(\omega)d\omega.
\end{aligned}
$$

3. Theorem of Weierstrass

Let $\mathcal{U} \subseteq \mathbf{R}^2$ be an open subset, $(\alpha, \beta) \in \mathcal{U}$. A patch $\mathbf{x} : \mathcal{U} \to \mathbf{R}^3$ of class C^2 is called *isothermal* provided there exists a differentiable function $\mu : \mathcal{U} \to \mathbf{R}$ such that $\mathbf{x}_\alpha \bullet \mathbf{x}_\alpha = \mathbf{x}_\beta \bullet \mathbf{x}_\beta = \mu^2$ and $\mathbf{x}_\alpha \bullet \mathbf{x}_\beta = 0$. We call μ the scaling function of the isothermal patch. A *regular patch* is a patch $\mathbf{x} : \mathcal{U} \to \mathbf{R}^3$ for which the Jacobian matrix $\mathcal{J}(\mathbf{x})(\alpha, \beta)$ has rank 2 for all $(\alpha, \beta) \in \mathcal{U}$. We say that $\mathbf{x}$ is *harmonic* provided each of its coordinate functions is harmonic, i.e. $\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta} = \mathbf{0}$.

LEMMA: Let $\mathbf{x}$ be a regular isothermal patch with scaling function μ and mean curvature H. Then

$$\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta} = 2\mu^2 H\mathbf{X}.$$

Proof: We have $\mathbf{x}_\alpha \bullet \mathbf{x}_\alpha = \mathbf{x}_\beta \bullet \mathbf{x}_\beta$ and $\mathbf{x}_\alpha \bullet \mathbf{x}_\beta = 0$. Differentiating we get

$$\mathbf{x}_{\alpha\alpha} \bullet \mathbf{x}_\alpha = \mathbf{x}_{\beta\alpha} \bullet \mathbf{x}_\beta \quad \text{and} \quad \mathbf{x}_{\beta\beta} \bullet \mathbf{x}_\alpha = -\mathbf{x}_{\alpha\beta} \bullet \mathbf{x}_\beta.$$

Therefore, $(\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta}) \bullet \mathbf{x}_\alpha = 0$. Similarly, $(\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta}) \bullet \mathbf{x}_\beta = 0$. It follows that $\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta}$ is collinear with $\mathbf{X}$. Thus,

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{N + L}{2\mu^2},$$

i.e.

$$\frac{(\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta}) \bullet \mathbf{X}}{2\mu^2} = H,$$

and

$$\mathbf{x}_{\alpha\alpha} + \mathbf{x}_{\beta\beta} = 2\mu^2 H\mathbf{X}.$$

The theorem of Weierstrass can now be stated as a corollary: *A regular isothermal patch $\mathbf{x} : \mathcal{U} \to \mathbf{R}^3$ is a minimal surface if and only if it is harmonic.* Consequently, the coordinates x, y, z as functions of α and β are harmonic (and therefore analytic, as will be shown in the next section). Hence, we can write:

$$x = \Re f_1(\gamma), \quad y = \Re f_2(\gamma), \quad z = \Re f_3(\gamma),$$

where f_1, f_2, f_3 are analytic functions of the complex variable $\gamma = \alpha + i\beta$.

Differentiating we get

$$f_1'(\gamma) = x_\alpha - ix_\beta = \phi_1,$$
$$f_2'(\gamma) = y_\alpha - iy_\beta = \phi_2,$$
$$f_3'(\gamma) = z_\alpha - iz_\beta = \phi_3,$$

where ϕ_1, ϕ_2, ϕ_3 are again analytic functions of γ. The sum of the squares of these three functions must vanish since α, β are isothermal parameters, i.e.

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = E - G - 2iF = 0,$$

when $E = G$ and $F = 0$. Also, the regularity condition is equivalent to the condition that ϕ_1, ϕ_2, ϕ_3 do not vanish simultaneously, i.e.

$$\mathcal{J}(\mathbf{x})(\alpha, \beta) = \begin{pmatrix} x_\alpha & y_\alpha & z_\alpha \\ x_\beta & y_\beta & z_\beta \end{pmatrix}$$

has rank 2, implies that

$$\begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2 = \|\mathbf{x}_\alpha \times \mathbf{x}_\beta\|^2 \neq 0,$$

which implies that

$$x_\alpha^2 + y_\alpha^2 + z_\alpha^2 \neq 0, \quad \text{and} \quad x_\beta^2 + y_\beta^2 + z_\beta^2 \neq 0,$$

and so ϕ_1, ϕ_2, ϕ_3 cannot all be zero.

Thus, every minimal surface can be represented locally by equations of the form

$$x = \Re \int_0^\gamma \phi_1 \, d\gamma', \quad y = \Re \int_0^\gamma \phi_2 \, d\gamma', \quad z = \Re \int_0^\gamma \phi_3 \, d\gamma', \tag{13}$$

where ϕ_1, ϕ_2, ϕ_3 satisfy (under certain domain restrictions) the two conditions:

1. $\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$.
2. ϕ_1, ϕ_2, ϕ_3 do not vanish simultaneously.

Conversely, if ϕ_1, ϕ_2, ϕ_3 satisfy these conditions, then the equations (13) define a minimal surface.[17]

In his paper[18] (see Appendix E) dated June 25, 1866, Weierstrass shows that every triple ϕ_1, ϕ_2, ϕ_3, satisfying the above conditions, can be represented locally by the formulas

$$\phi_1 = \Phi^2 - \Psi^2,$$
$$\phi_2 = i(\Phi^2 + \Psi^2),$$
$$\phi_3 = 2\Phi\Psi,$$

where in fact, $\Phi(\gamma) = \sqrt{\frac{1}{2}[\phi_1(\gamma) - i\phi_2(\gamma)]}$, and $\Psi(\gamma) = i\sqrt{\frac{1}{2}[\phi_1(\gamma) + i\phi_2(\gamma)]}$. It is easy to see that ϕ_1 and ϕ_2 cannot also vanish simultaneously. This means that

[17] Tibor Radó, *On the problem of Plateau* (Berlin, 1933), p. 28.
[18] Karl Weierstrass, *Mathematische Werke* (Berlin, 1903), III, p. 39.

Φ and Ψ can be defined to be single-valued functions with no roots in common.[19] Thus, every minimal surface can be represented locally by the equations

$$
\begin{aligned}
x &= x_0 + \Re \int_0^\gamma (\Phi^2 - \Psi^2)\, d\gamma', \\
y &= y_0 + \Re \int_0^\gamma i(\Phi^2 + \Psi^2)\, d\gamma', \\
z &= z_0 + \Re \int_0^\gamma 2\Phi\Psi\, d\gamma'.
\end{aligned}
\tag{5}
$$

He then defines a new variable ω by

$$\omega = \frac{\Psi(\gamma)}{\Phi(\gamma)} = \frac{\phi_1 + i\phi_2}{-\phi_3}, \tag{14}$$

and a function $R(\omega)$ by[20]

$$\Phi^2(\gamma)\, d\gamma = \frac{1}{2}(\phi_1 - i\phi_2)\, d\gamma = R(\omega)\, d\omega.$$

From equations (5), we now get

$$
\begin{aligned}
x &= x_0 + \Re \int_{\omega_0}^{\omega} (1 - \omega'^2) R(\omega')\, d\omega', \\
y &= y_0 + \Re \int_{\omega_0}^{\omega} i(1 + \omega'^2) R(\omega')\, d\omega', \\
z &= z_0 + \Re \int_{\omega_0}^{\omega} 2\omega' R(\omega')\, d\omega',
\end{aligned}
\tag{6}
$$

and Weierstrass notes that the mapping defined by (14) is equivalent to the stereographic projection of the Gauss map onto the ω-plane (see Appendix B).

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19 Nitsche, p. 146.

20 See Nitsche, p. 147. Weierstrass used s and $\mathcal{F}(s)$ (see Appendix E).

4. A Modern Approach

Following the method of Radó,[21] we show that a minimal surface locally admits isothermal parameters, and that every solution of the minimal surface equation is analytic.

Let S be a minimal surface given in a parametric representation $\mathbf{x} = \mathbf{x}(u, v)$, where (u, v) varies in a region Q, and let $\lambda(u, v)$ be a function which vanishes on the boundary of Q. Then the area of the surface

$$x = x(u, v) + \varepsilon\lambda(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

is a function $A(\varepsilon)$ which has a minimum when $\varepsilon = 0$. The condition $A'(0) = 0$ gives

$$\frac{\partial}{\partial u}\frac{Gx_u - Fx_v}{W} + \frac{\partial}{\partial v}\frac{Ex_v - Fx_u}{W} = 0,$$

where E, F, G are the first fundamental coefficients and $W = \sqrt{EG - F^2}$. Similar variations of the y and z coordinates give

$$\frac{\partial}{\partial u}\frac{Gy_u - Fy_v}{W} + \frac{\partial}{\partial v}\frac{Ey_v - Fy_u}{W} = 0,$$

$$\frac{\partial}{\partial u}\frac{Gz_u - Fz_v}{W} + \frac{\partial}{\partial v}\frac{Ez_v - Fz_u}{W} = 0.$$

It follows that

$$\frac{Ex_v - Fx_u}{W}du - \frac{Gx_u - Fx_v}{W}dv, \tag{15}$$

$$\frac{Ey_v - Fy_u}{W}du - \frac{Gy_u - Fy_v}{W}dv, \tag{16}$$

$$\frac{Ez_v - Fz_u}{W}du - \frac{Gz_u - Fz_v}{W}dv, \tag{17}$$

are exact differentials.

[21] Radó, p. 22.

Suppose now the minimal surface S is given in a nonparametric representation $z = z(x,y)$, where (x,y) varies in a region P. The expressions (15), (16), and (17) then reduce to

$$-\frac{pq}{W}dx - \frac{1+q^2}{W}dy, \quad \frac{1+p^2}{W}dx + \frac{pq}{W}dy, \quad \frac{q}{W}dx - \frac{p}{W}dy, \tag{18}$$

where $W = \sqrt{1+p^2+q^2}$. Choosing the unit normal $\mathbf{X} = (-\frac{p}{W}, -\frac{q}{W}, \frac{1}{W})$, we find that the expressions (18) are identical to

$$Ydz - Zdy, \quad Zdx - Xdz, \quad Xdy - Ydx; \tag{19}$$

the coordinates of $\mathbf{X} \times d\mathbf{x}$. Hence, expressions (18) and (19) are exact differentials, which means that the line integrals

$$\begin{aligned} \mu_1 &= -\int_{(0,0)}^{(x,y)} \frac{pq}{W}dx' + \frac{1+q^2}{W}dy', \\ \mu_2 &= \int_{(0,0)}^{(x,y)} \frac{1+p^2}{W}dx' + \frac{pq}{W}dy', \\ \mu_3 &= \int_{(0,0)}^{(x,y)} \frac{q}{W}dx' - \frac{p}{W}dy', \end{aligned} \tag{20}$$

are independent of path, i.e. $\int \mathbf{X} \times d\mathbf{x}$ is path-independent.

We now introduce new variables ξ, η defined by $\xi = x + \mu_2(x,y)$ and $\eta = y - \mu_1(x,y)$. It can be shown that the map $(x,y) \mapsto (\xi, \eta)$ is expanding and hence one-to-one.[22] Also, we have that

$$\xi_x = 1 + \frac{1+p^2}{W}, \quad \xi_y = \frac{pq}{W},$$

$$\eta_x = \frac{pq}{W}, \quad \eta_y = 1 + \frac{1+q^2}{W},$$

[22] Nitsche, p. 120.

so

$$d\xi^2 + d\eta^2 = (\xi_x^2 + \eta_x^2)dx^2 + 2(\xi_x\xi_y + \eta_x\eta_y)dxdy + (\xi_y^2 + \eta_y^2)dy^2$$
$$= \frac{(1+W)^2}{W^2}\left[(1+p^2)dx^2 + 2pqdxdy + (1+q^2)dy^2\right].$$

It follows that

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$= (1+p^2)dx^2 + 2pqdxdy + (1+q^2)dy^2$$
$$= \frac{W^2}{(1+W)^2}(d\xi^2 + d\eta^2).$$

Thus, ξ, η are isothermal parameters on S. Therefore, a minimal surface admits isothermal parameters. That is to say, *every* minimal surface can be represented, locally, by equations

$$x = x(\xi,\eta), \quad y = y(\xi,\eta), \quad z = z(\xi,\eta),$$

with $E = G$, and $F = 0$. According to the theorem of Weierstrass, $x(\xi,\eta)$,$y(\xi,\eta)$, $z(\xi,\eta)$ are harmonic functions. We can therefore write:

$$x = \Re f_1(\zeta), \quad y = \Re f_2(\zeta), \quad z = \Re f_3(\zeta),$$

where f_1, f_2, f_3 are analytic functions.

Furthermore, the formulas

$$dx = \frac{W+1+q^2}{(W+1)^2}\,d\xi - \frac{pq}{(W+1)^2}\,d\eta,$$
$$dy = -\frac{pq}{(W+1)^2}\,d\xi + \frac{W+1+p^2}{(W+1)^2}\,d\eta,$$
$$dz = \frac{p}{W+1}\,d\xi + \frac{q}{W+1}\,d\eta,$$
$$d\mu_1 = -\frac{pq}{(W+1)^2}\,d\xi - \frac{W+1+q^2}{(W+1)^2}\,d\eta,$$
$$d\mu_2 = \frac{W+1+p^2}{(W+1)^2}\,d\xi + \frac{pq}{(W+1)^2}\,d\eta,$$
$$d\mu_3 = \frac{q}{W+1}\,d\xi - \frac{p}{W+1}\,d\eta,$$

reveal that

$$\begin{aligned} x_\xi &= (-\mu_1)_\eta, & x_\eta &= -(-\mu_1)_\xi, \\ y_\xi &= (-\mu_2)_\eta, & y_\eta &= -(-\mu_2)_\xi, \\ z_\xi &= (-\mu_3)_\eta, & z_\eta &= -(-\mu_3)_\xi. \end{aligned} \tag{21}$$

In other words, x and $-\mu_1$, y and $-\mu_2$, z and $-\mu_3$, as functions of ξ and η, satisfy the Cauchy-Riemann equations. The formulas defining the functions in equations (21) have continuous partial derivatives of the first order, so $-\mu_1, -\mu_2, -\mu_3$ are the harmonic conjugates of x, y, z respectively. It follows that $x - i\mu_1$, $y - i\mu_2$, and $z - i\mu_3$ are analytic functions of the complex variable $\zeta = \xi + i\eta$, i.e.

$$f = (f_1, f_2, f_3) = \mathbf{x} - i \int_{(0,0)}^{(x,y)} \mathbf{X} \times d\mathbf{x}$$

is an analytic function of ζ, from (19) and (20).

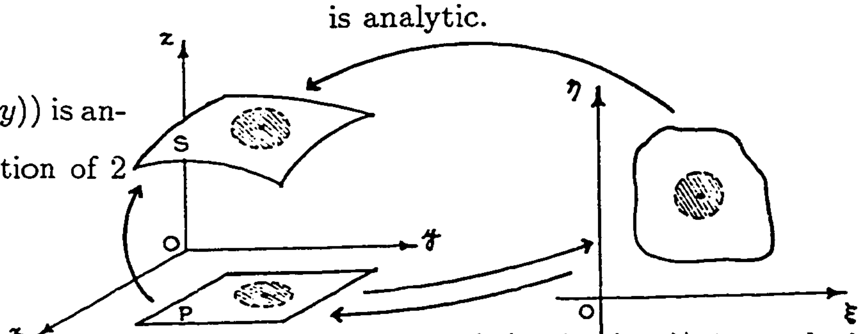
Moreover, since x, y, z are analytic functions of ξ and η, every solution $z(x,y)$, with continuous partial derivatives of the first and second order, of the equation

$$(1+q^2)r - 2pqs + (1+p^2)t = 0, \tag{3'}$$

is analytic (see Figure 3).

$(\xi,\eta) \mapsto (x(\xi,\eta), y(\xi,\eta), z(x(\xi,\eta), y(\xi,\eta)))$ is analytic.

$(x,y) \mapsto (x, y, z(x,y))$ is analytic, by composition of 2 analytic functions.



$(\xi,\eta) \mapsto (x(\xi,\eta), y(\xi,\eta))$ is analytic and locally invertible (see Radó, p. 26).

Figure 3

5. Global Conformal Mapping

Until now we have treated the minimal surface locally, where the minimal condition has been rendered by a vanishing mean curvature H. It was recognized early on in the literature, however, that *the area of a minimal surface, bounded by a given Jordan curve, is not necessarily a minimum.* To answer the Plateau problem (see Appendix C) as well as the problem of least area (together known as *the simultaneous problem*), we must treat the minimal surface globally.

Let the minimal surface S be represented nonparametrically by $\{z = z(x, y) : (x, y) \in P\}$, where P is an open convex domain containing the origin in the (x, y)-plane. Following the method of Nitsche,[23] we show the existence of global isothermal parameters on S.

As we have seen in § III.4, the mapping $(x, y) \mapsto (\xi, \eta)$ defined by

$$\begin{aligned} \xi = \xi(x,y) &= x + \int_{(0,0)}^{(x,y)} \frac{1}{W}[(1+p^2)\,dx' + pq\,dy'], \\ \eta = \eta(x,y) &= y + \int_{(0,0)}^{(x,y)} \frac{1}{W}[pq\,dx' + (1+q^2)\,dy'], \end{aligned} \tag{22}$$

provides us with local isothermal parameters on S. It can be shown[24] that the condition for the existence of isothermal parameters $\xi = \xi(x, y), \eta = \eta(x, y)$ is an elliptic system of two first order linear partial differential equations. This system generalizes the Cauchy-Riemann equations and is called the Beltrami system:

$$\xi_x = \frac{1}{W}[-F\eta_x + E\eta_y], \quad \xi_y = \frac{1}{W}[-G\eta_x + F\eta_y], \tag{23}$$

where E, F, G are the first fundamental coefficients and $W = \sqrt{EG - F^2}$.

[23] Nitsche, p. 126.

[24] Dubrovin, Fomenko, Novikov, *Modern Geometry* (New York, 1984), I, p. 110.

From equations (23), we get the following theorem: *For a pair of functions $\alpha(x,y)$ and $\beta(x,y)$ to be solutions of the Beltrami system, it is necessary and sufficient that the complex-valued function $\alpha(x(\xi,\eta),y(\xi,\eta)) + i\beta(x(\xi,\eta),y(\xi,\eta))$ is an analytic function of the complex variable $\zeta = \xi + i\eta$.*

Hence, the composite map T, defined by $T(\zeta) = \alpha + i\beta$, is conformal. Furthermore, since the image Π of P under (22) is simply connected and P is not the entire (x,y)-plane, the Riemann mapping theorem shows that the domain Π can be mapped bijectively and conformally onto the open unit disc in the complex γ-plane, where $\gamma = \alpha + i\beta$ (see Figure 4). Thus, α and β are admissible isothermal parameters on S.

We now endow the parameter domain P with the inner product induced by S, and P becomes a geometric surface (see Appendix A). Thus, the mapping $\mathbf{x} : P \to \mathbf{R}^3$ defined by $\mathbf{x}(x,y) = x\mathrm{e}_1 + y\mathrm{e}_2 + z(x,y)\mathrm{e}_3$ is an isometry, and $\alpha(x,y), \beta(x,y)$ are local isothermal parameters on S. In other words, every point of S has a neighbourhood (the image of an open subset U in the parameter domain) which is mapped bijectively and conformally (with respect to the metric on S) onto the interior of the unit circle in the complex γ-plane by the mapping $(x,y) \mapsto (\alpha,\beta)$.

It can be proven, using analytic function theory and the Koebe-Poincaré uniformization theorem, that there is a single mapping $(x,y) \mapsto (\alpha,\beta)$ of the entire surface S onto the open unit disc $|\gamma| < 1$ in the γ-plane which is bijective and conformal. In particular, α and β are global isothermal parameters on S.

6. Recapitulation

We assume that the minimal surface S is represented nonparametrically by $\{z = z(x,y) : (x,y) \in P\}$ where P is an open convex domain containing the origin in the (x,y)-plane. Again we replace the often recurring quantity $\sqrt{1+p^2+q^2}$ by W.

In general, the parameters x and y for a nonparametric minimal surface $z = z(x,y)$ are not isothermal. We can obtain isothermal parameters ξ and η by introducing the function $\zeta = \xi + i\eta$ where

$$\begin{aligned} \xi = \xi(x,y) &= x + \int_{(0,0)}^{(x,y)} \frac{1}{W}[(1+p^2)\,dx' + pq\,dy'], \\ \eta = \eta(x,y) &= y + \int_{(0,0)}^{(x,y)} \frac{1}{W}[pq\,dx' + (1+q^2)\,dy']. \end{aligned} \tag{22}$$

The mapping $(x,y) \mapsto (\xi,\eta)$ is one-to-one[25] and ξ,η are admissible parameters on S. Moreover, the inverse map $(\xi,\eta) \mapsto (x,y)$ followed by the map $(x,y) \mapsto \mathbf{x} - i\int \mathbf{X} \times d\mathbf{x}$ is an analytic (vector-valued) function of the complex variable $\zeta = \xi + i\eta$.

Using the Riemann mapping theorem, we now introduce new isothermal parameters α and β, where $\gamma = \alpha + i\beta$ (see Figure 4). There is a global conformal mapping[26] of S onto the normal domain $|\gamma| < 1$. The minimal surface S can then be represented parametrically as

$$\mathbf{x} = \mathbf{x}(\alpha,\beta) = \mathbf{x}_0 + \Re \int_0^{\gamma} \mathbf{F}(\gamma')d\gamma', \tag{24}$$

where

$$\mathbf{F}(\gamma) = \{\phi_1(\gamma), \phi_2(\gamma), \phi_3(\gamma)\},$$

[25] Nitsche, p. 120.
[26] Nitsche, p. 126.

and

$$\phi_1(\gamma) = x_\alpha - ix_\beta,$$

$$\phi_2(\gamma) = y_\alpha - iy_\beta,$$

$$\phi_3(\gamma) = z_\alpha - iz_\beta.$$

$\mathbf{F}(\gamma)$ is a nonvanishing analytic vector satisfying

$$\mathbf{F}^2(\gamma) = \phi_1^2(\gamma) + \phi_2^2(\gamma) + \phi_3^2(\gamma) = 0.$$

Conversely, every such vector generates a unique (up to translation) open, simply connected minimal surface.[27]

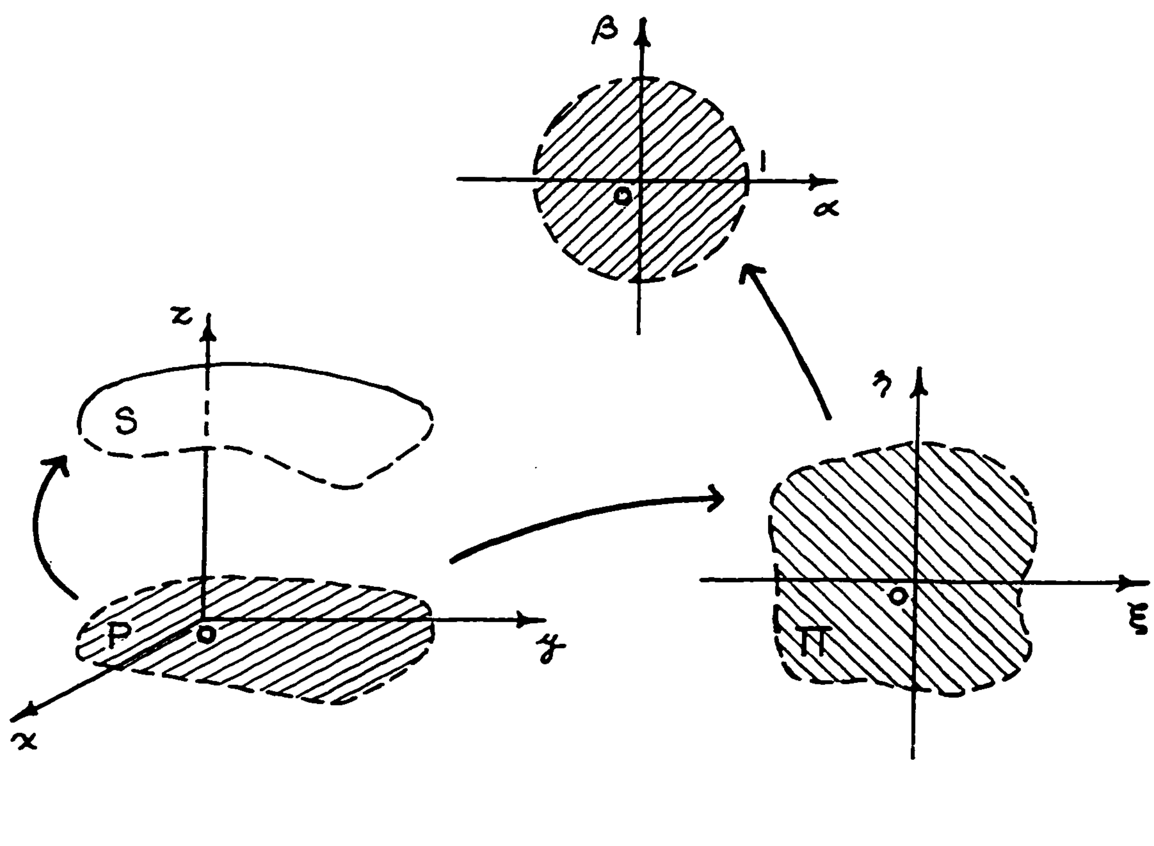


Figure 4

[27] Nitsche, p. 138.

Finally, we obtain the Weierstrass-Enneper representation formulas by introducing the functions

$$\Phi(\gamma) = \{\frac{1}{2}[\phi_1(\gamma) - i\phi_2(\gamma)]\}^{\frac{1}{2}},$$
$$\Psi(\gamma) = i\{\frac{1}{2}[\phi_1(\gamma) + i\phi_2(\gamma)]\}^{\frac{1}{2}}.$$

From (24) we get

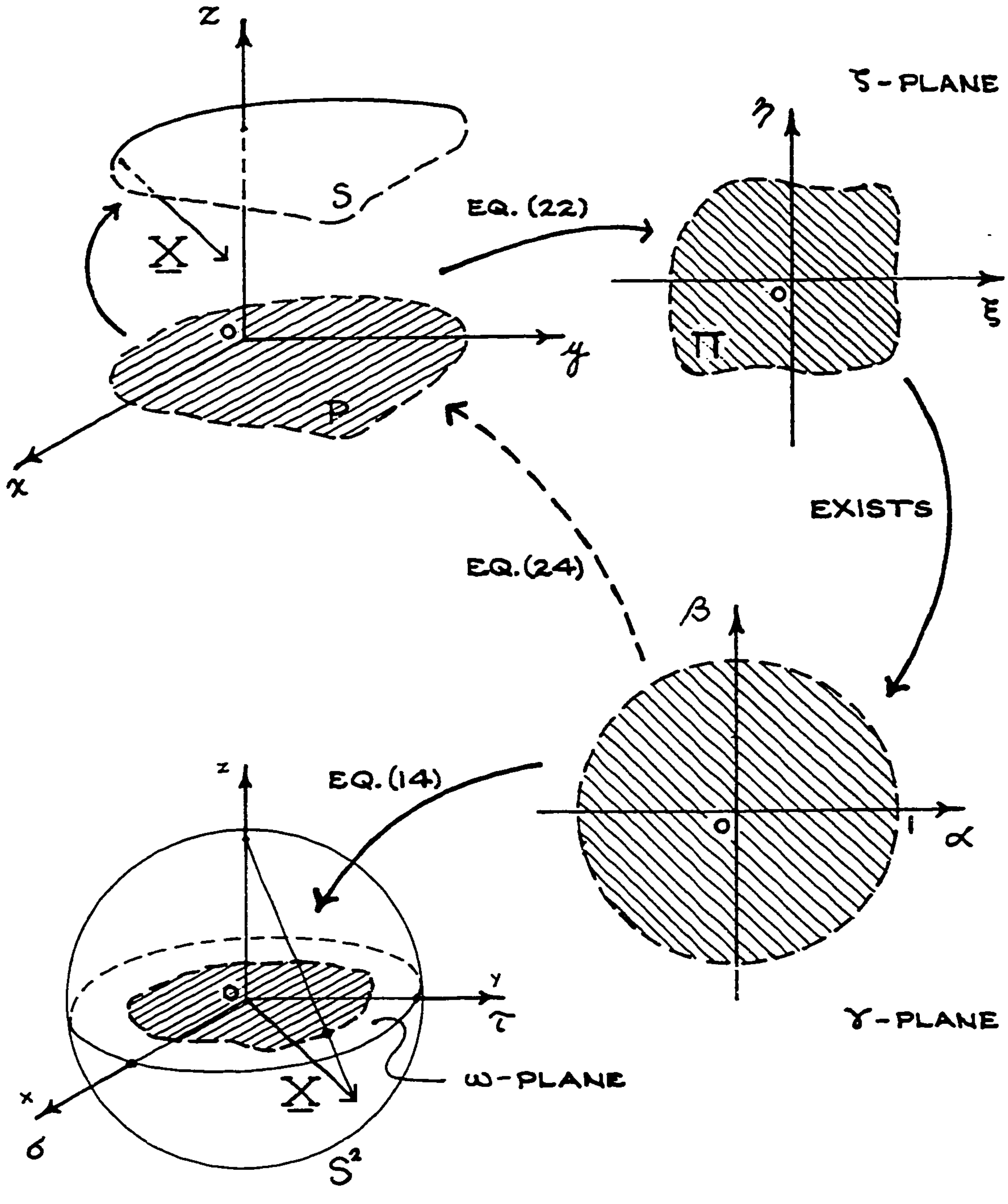
$$x = x_0 + \Re\int_0^\gamma (\Phi^2 - \Psi^2)d\gamma',$$
$$y = y_0 + \Re\int_0^\gamma i(\Phi^2 + \Psi^2)d\gamma', \tag{5}$$
$$z = z_0 + \Re\int_0^\gamma 2\Phi\Psi d\gamma'.$$

The mapping $\omega(\gamma) = \frac{\Psi(\gamma)}{\Phi(\gamma)}$ (which is equivalent to the stereographic projection of the Gauss image[28]) and the function $R(\omega)$ defined by $\Phi^2(\gamma)d\gamma = R(\omega)d\omega$ then yield

$$x = x_0 + \Re\int_{\omega_0}^{\omega} (1 - \omega'^2)R(\omega')d\omega',$$
$$y = y_0 + \Re\int_{\omega_0}^{\omega} i(1 + \omega'^2)R(\omega')d\omega', \tag{6}$$
$$z = z_0 + \Re\int_{\omega_0}^{\omega} 2\omega' R(\omega')d\omega',$$

where $\omega = \sigma + i\tau$, (see Figure 5).

28 Nitsche, p. 147. Also, see Appendix B.


z
S
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EQ. (22)
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EQ.(24)
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z
α
1
y
τ
γ - PLANE
ω-PLANE
x
σ
S²

Figure 5

7. Example: $R(\omega) = \omega$**.**

With the aid of MATHEMATICA, we now generate a minimal surface using equations (6) for the case $R(\omega) = \omega$.

$$x = \Re \int_0^\omega (1 - t^2) t \, dt = \Re \int_0^\omega (t - t^3) \, dt$$

$$= \Re \left[\frac{t^2}{2} - \frac{t^4}{4} \right]_0^\omega = \Re \left[\frac{\omega^2}{2} - \frac{\omega^4}{4} \right]$$

$$= \frac{\sigma^2 - \tau^2}{2} - \frac{\sigma^4 - 6\sigma^2\tau^2 + \tau^4}{4}$$

$$= \frac{1}{4}(2\sigma^2 - 2\tau^2 + 6\sigma^2\tau^2 - \sigma^4 - \tau^4),$$

$$y = \Re \int_0^\omega i(1 + t^2) t \, dt = \Re \, i \int_0^\omega (t + t^3) \, dt$$

$$= \Re \, i \left[\frac{t^2}{2} + \frac{t^4}{4} \right]_0^\omega = \Re \, i \left[\frac{\omega^2}{2} + \frac{\omega^4}{4} \right]$$

$$= \sigma\tau^3 - \sigma\tau - \sigma^3\tau,$$

$$z = \Re \int_0^\omega 2t \cdot t \, dt = \Re \, 2 \int_0^\omega t^2 \, dt$$

$$= \Re \, 2 \left[\frac{t^3}{3} \right]_0^\omega = 2\Re \left[\frac{\omega^3}{3} \right]$$

$$= 2 \left[\frac{\sigma^3}{3} - \sigma\tau^2 \right];$$

$$\mathbf{x}(\sigma, \tau) = \left(\frac{1}{4}[2\sigma^2 - 2\tau^2 + 6\sigma^2\tau^2 - \sigma^4 - \tau^4], -\sigma\tau[1 + \sigma^2 - \tau^2], 2\sigma \left[\frac{\sigma^2}{3} - \tau^2 \right] \right).$$

We verify that σ and τ are indeed isothermal parameters:

$$\mathbf{x}_\sigma = (\sigma[1 - \sigma^2 + 3\tau^2], -\tau[1 + 3\sigma^2 - \tau^2], 2[\sigma^2 - \tau^2]),$$

$$\mathbf{x}_\tau = (-\tau[1 - 3\sigma^2 + \tau^2], -\sigma[1 + \sigma^2 - 3\tau^2], -4\sigma\tau);$$

$$E = \mathbf{x}_\sigma \bullet \mathbf{x}_\sigma = \mathbf{x}_\tau \bullet \mathbf{x}_\tau = G$$

$$= \sigma^2 + 2\sigma^4 + 4\sigma^2\tau^2 + \sigma^6 + 3\sigma^4\tau^2 + 3\sigma^2\tau^4 + \tau^2 + 2\tau^4 + \tau^6, \quad \text{and}$$

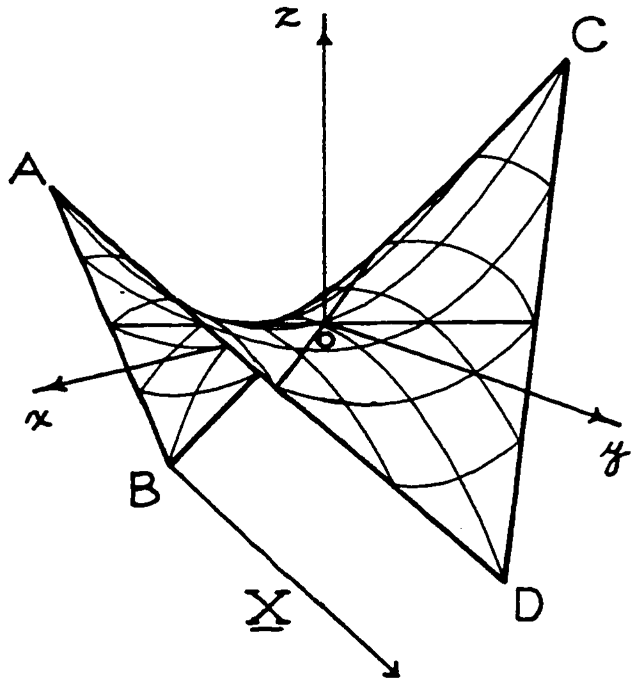
$$F = \mathbf{x}_\sigma \bullet \mathbf{x}_\tau = 0.$$

We can now plot this intricate self-intersecting minimal surface (see Appendix D).

III. Schwarz's Surface

1. Schwarz's Method

Schwarz capitalized on the fact that the complex variable ω, of equations (6), is equivalent to the stereographic projection of the Gauss map (see Figure 6).



The sides AB,BC,CD,DA are four edges of a regular tetrahedron (with side 1). Vertex B corresponds to the point b in the ω-plane.

The stereographic projection is defined by the formulas:[29]

$$\sigma = \frac{X}{1-Z},$$
$$\tau = \frac{Y}{1-Z},$$

where $\mathbf{X} = (X, Y, Z)$.

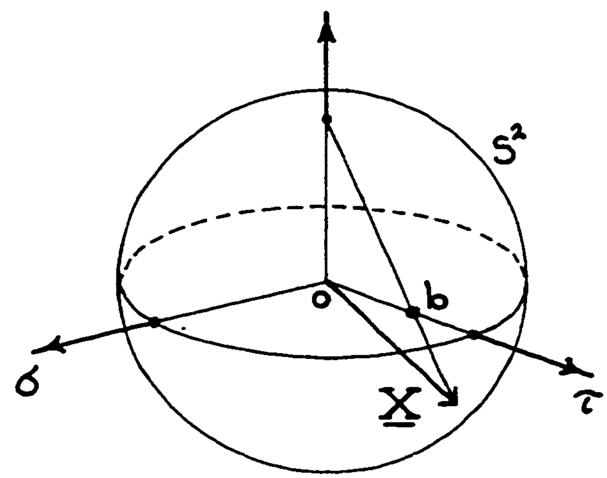


Figure 6

[29] Nitsche, p. 55.

He determined that the image of the skew quadrilateral in the ω-plane under this projection was the circular quadrilateral *abcd* (see Figure 7).

$$a = \frac{\sqrt{3}-1}{\sqrt{2}},$$
$$b = \frac{i(\sqrt{3}-1)}{\sqrt{2}},$$
$$c = \frac{1-\sqrt{3}}{\sqrt{2}},$$
$$d = \frac{i(1-\sqrt{3})}{\sqrt{2}}.$$

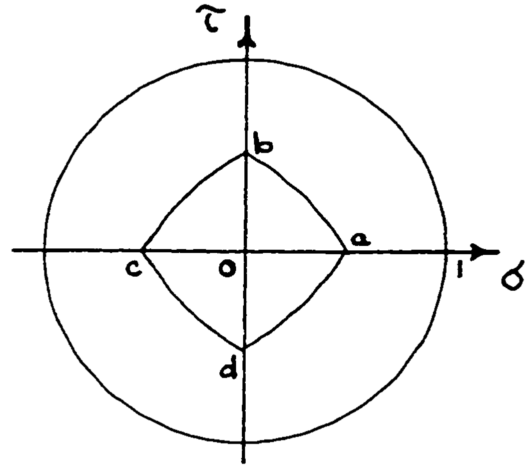


Figure 7

Using the equations of Weierstrass, he then wrote an expression for the quotient $\frac{\overline{R(\omega)}}{R(\omega)}$ for each circular arc $\widehat{ab}$, $\widehat{bc}$, $\widehat{cd}$, and $\widehat{da}$.

Constrained by these boundary conditions, Schwarz then obtained an expression for $R(\omega)$ based on the assumption that this function is analytic in the interior of the circular quadrilateral *abcd* and on the open boundary arcs, but singular at the vertices a, b, c, and d. However, this expression, namely

$$R(\omega) = f(\omega)[(\omega-a)(\omega-b)(\omega-c)(\omega-d)]^{-\frac{1}{2}},$$

did not satisfy the boundary conditions. To obtain four more points of singularity, he then extended the original surface across all four edges by reflection (see Figure 10). The stereographic projection of the unit normal at the eight disjoint outer vertices then yielded the four singular points a', b', c', and d' (see Figure 8). He could now rewrite the trial function as

$$R(\omega) = f(\omega)\prod_{j=1}^{8}(\omega-\omega_j)^{-\frac{1}{2}},$$

and, substituting the values for the points ω_j, he found that

$$R(\omega) = \frac{\kappa}{\sqrt{1 - 14\omega^4 + \omega^8}}, \quad \kappa > 0, \tag{7}$$

which did satisfy the boundary conditions.[30]

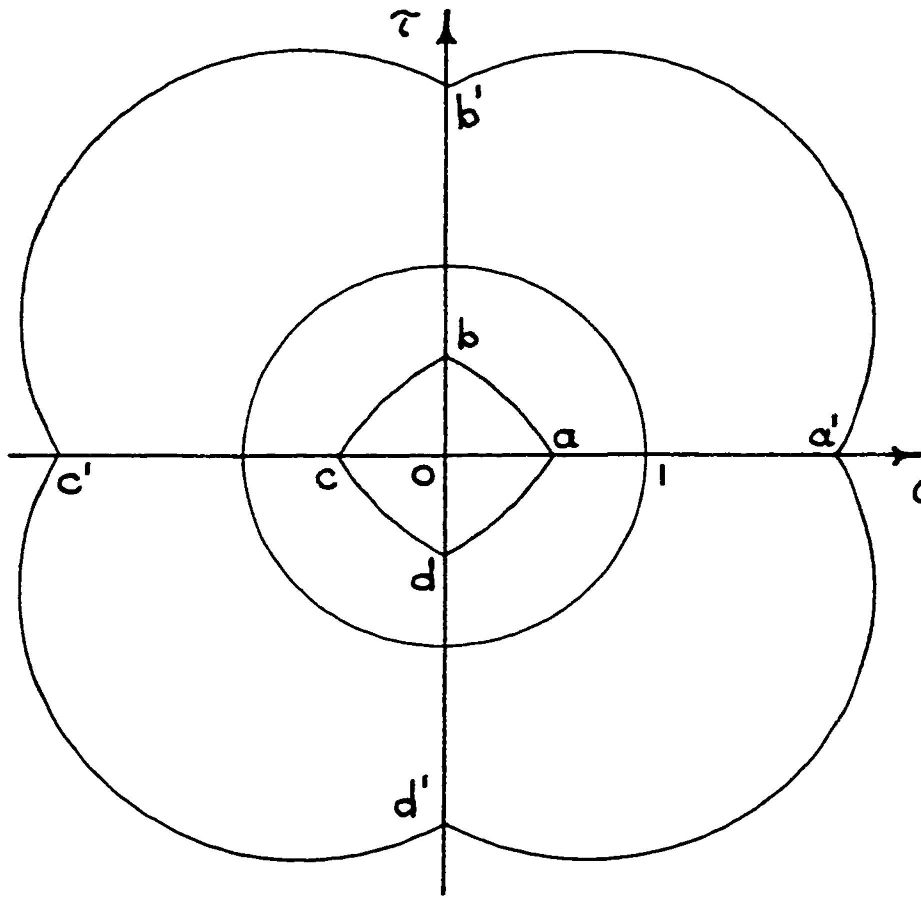


Figure 8

2. Derivation of R(ω)

The direction cosines of the unit normal **X** at vertex B (see Figure 6) are $\left(0, \frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$. We can calculate the point b by stereographic projection, i.e.

$$\omega = \frac{X}{1-Z} + i\frac{Y}{1-Z} = i\frac{\sqrt{3}-1}{\sqrt{2}}.$$

[30] Nitsche, p. 237. Here $\kappa \approx 0.84$ for side-length 1 (p. 238).

Similarly, a is the point $\frac{\sqrt{3}-1}{\sqrt{2}}$. A point following the straight-line segment from A to B then traces the circular arc

$$\left(\sigma + \frac{1}{\sqrt{2}}\right)^2 + \left(\tau + \frac{1}{\sqrt{2}}\right)^2 = 2,$$

from a to b.[31] The parametric representation of this arc is

$$\omega = -\frac{\sqrt{2}}{2}(1+i) + \sqrt{2}\, e^{i\theta}, \qquad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}.$$

The same point moving along the linear segment AB has direction cosines $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$, and so $dx = dy = -\frac{1}{2}ds$ (see Figure 9).

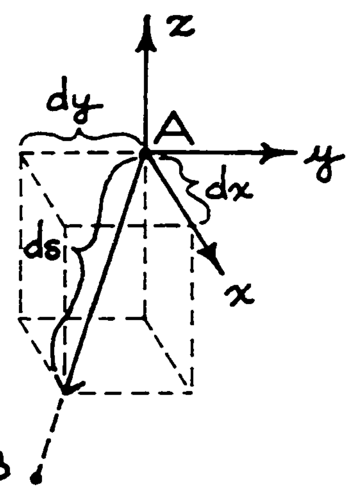


Figure 9

Using equations (6) we can write

$$dx + i\,dy = -\omega^2 R(\omega)\, d\omega + \overline{R(\omega)}\, d\overline{\omega} = -\frac{1+i}{2}\, ds, \tag{25}$$

and so

$$dx - i\,dy = R(\omega)\, d\omega - \overline{\omega}^2 \overline{R(\omega)}\, d\overline{\omega} = -\frac{1-i}{2}\, ds. \tag{26}$$

[31] Nitsche, p. 236.

Multiplying equation (25) by i and adding it to equation (26) we get

$$(1 - i\omega^2)R(\omega)\,d\omega + (i - \overline{\omega}^2)\overline{R(\omega)}\,d\overline{\omega} = 0.$$

Hence,

$$\frac{\overline{R(\omega)}}{R(\omega)} = \frac{(1 - i\omega^2)\,d\omega}{(\overline{\omega}^2 - i)\,d\overline{\omega}} = -e^{4i\theta},$$

when

$$\omega = -\frac{1+i}{\sqrt{2}} + \sqrt{2}\,e^{i\theta}, \qquad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}.$$

In the same way, we can derive similar relations for line segments BC, CD, and DA. These would then be our boundary conditions.

To determine the nature of the singularity at the point a, we expand $R(\omega)$ in a neighbourhood of a in the form

$$R(\omega) = (a - \omega)^k[1 + \ldots],$$

where the square brackets contain a power series in $(a - \omega)$ and k is determined[32] to be $-\frac{1}{2}$. By symmetry, the function $R(\omega)$ must have the same behavior near the points b, c, and d. However, the trial function

$$R(\omega) = f(\omega)[(\omega - a)(\omega - b)(\omega - c)(\omega - d)]^{-\frac{1}{2}}$$

does not satisfy our boundary conditions.

We obtain four additional singular points by reflecting Schwarz's surface across the four linear segments AB, BC, CD, and DA. The stereographic projection of the Gauss map at the vertices B' and B'' (see Figure 10) is the point $b' = \frac{i(\sqrt{3}+1)}{\sqrt{2}}$ in the ω-plane (see Figure 8). Similarly, $a'=\frac{1}{a}$, $c'=\frac{1}{c}$, and $d'=-\frac{1}{d}$. The new trial function can now be written

$$R(\omega) = f(\omega)[(\omega - a)(\omega - b)(\omega - c)(\omega - d)(\omega - a')(\omega - b')(\omega - c')(\omega - d')]^{-\frac{1}{2}}.$$

[32] Nitsche, p. 237.

For reasons of symmetry, $f(\omega)$ must be constant.[33] The resulting function is

$$R(\omega) = \frac{\kappa}{\sqrt{1 - 14\omega^4 + \omega^8}}, \qquad \kappa > 0, \tag{7}$$

where we choose that branch of the square root which takes the value 1 at the point $\omega = 0$. $R(\omega)$ now satisfies the boundary conditions; for example,

$$\begin{aligned} \frac{\overline{R(\omega)}}{R(\omega)} &= \frac{\frac{\kappa}{\sqrt{1-14\overline{\omega}^4+\overline{\omega}^8}}}{\frac{\kappa}{\sqrt{1-14\omega^4+\omega^8}}} = \frac{\sqrt{1-14\omega^4+\omega^8}}{\sqrt{1-14\overline{\omega}^4+\overline{\omega}^8}} \\ &= \frac{-4ie^{2i\theta}\sqrt{\Omega(\theta)}}{4ie^{-2i\theta}\sqrt{\Omega(\theta)}} = -e^{4i\theta}, \end{aligned}$$

for

$$\omega = -\frac{1+i}{\sqrt{2}} + \sqrt{2}e^{i\theta}, \qquad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}.$$

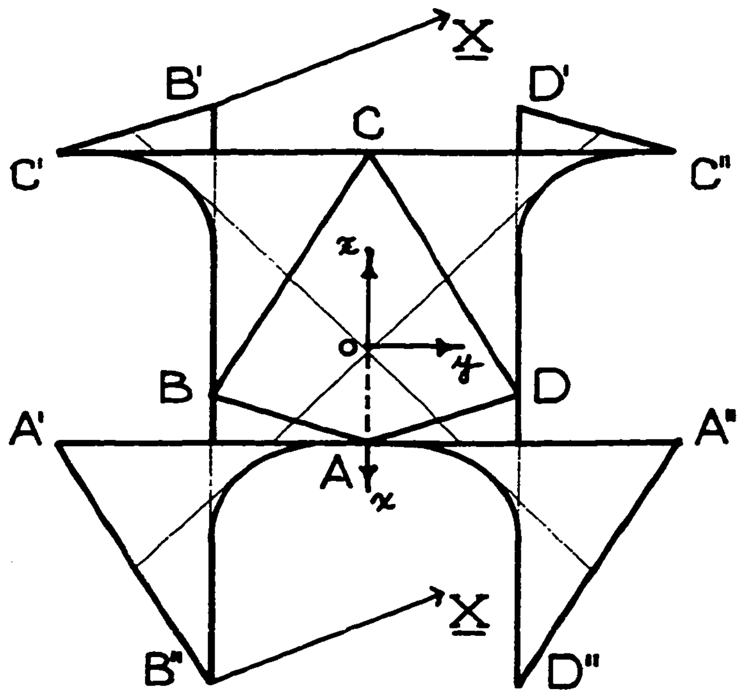


Figure 10

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[33] *Ibid.*

3. Representation

After attempting to plot the Schwarz surface using equations (6) and (7), one soon discovers that the task is a little overwhelming for the standard personal computer. Instead, we exploit the fortuitous fact[34] that this surface is expressible in the form $f(x) + g(y) + h(z) = 0$. The class of minimal surfaces representable in this form can be expressed in terms of elliptic functions which are the inverses of certain elliptic integrals of the first kind. We can determine all such minimal surfaces using a method[35] developed by Weingarten and René Maurice Fréchet. Following Nitsche, we now summarize this method as it applies to the Schwarz surface.

When dealing with questions of a local nature, surfaces with a regular parametrization may be assumed to be represented implicitly by $\Theta(x, y, z) = 0$, where $\Theta(x, y, z)$ is twice continuously differentiable and satisfies $\Theta_x^2 + \Theta_y^2 + \Theta_z^2 > 0$. The formula for the mean curvature is then

$$H = \frac{1}{2}\left\{\frac{\partial}{\partial x}\left(\frac{\Theta_x}{\sqrt{\Theta_x^2+\Theta_y^2+\Theta_z^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\Theta_y}{\sqrt{\Theta_x^2+\Theta_y^2+\Theta_z^2}}\right) + \frac{\partial}{\partial z}\left(\frac{\Theta_z}{\sqrt{\Theta_x^2+\Theta_y^2+\Theta_z^2}}\right)\right\},$$

and so the vanishing mean curvature is expressed by

$$\Theta_{xx}[\Theta_y^2 + \Theta_z^2] + \Theta_{yy}[\Theta_x^2 + \Theta_z^2] + \Theta_{zz}[\Theta_x^2 + \Theta_y^2] = 0,$$

i.e.

$$\begin{aligned} f''(x)[g'^2(y) + h'^2(z)] + g''(y)[f'^2(x) + h'^2(z)] + h''(z)[f'^2(x) + g'^2(y)] &= 0, \\ f(x) + g(y) + h(z) &= 0. \end{aligned} \tag{27}$$

We now introduce new variables $u{=}f(x), v{=}g(y), w{=}h(z)$ and the abbreviations $X(u){=}f'^2(x), Y(v){=}g'^2(y), Z(w){=}h'^2(z)$ to rewrite equations (27) as

$$\begin{aligned} X'(Y + Z) + Y'(X + Z) + Z'(X + Y) &= 0, \\ u + v + w &= 0. \end{aligned} \tag{28}$$

[34] Nitsche, p. 234.
[35] Nitsche, p. 71.

After some algebaic manipulation we eventually arrive at the system

$$\begin{aligned} \Delta X' + X'Y'Z'X''' &= 0, \\ \Delta Y' + X'Y'Z'Y''' &= 0, \\ \Delta Z' + X'Y'Z'Z''' &= 0, \end{aligned} \tag{29}$$

where $\Delta = X'Y''Z'' + Y'Z''X'' + Z'X''Y''$. If none of the functions X', Y', Z' vanishes identically, then we have

$$X''' = nX', \quad Y''' = nY', \quad Z''' = nZ',$$

for some real number n. If $n \neq 0$, then we get

$$\begin{aligned} X(u) &= a_1 + b_1 e^{u\sqrt{n}} + c_1 e^{-u\sqrt{n}}, \\ Y(v) &= a_2 + b_2 e^{v\sqrt{n}} + c_2 e^{-v\sqrt{n}}, \\ Z(w) &= a_3 + b_3 e^{w\sqrt{n}} + c_3 e^{-w\sqrt{n}}, \end{aligned} \tag{30}$$

where a_1, a_2, a_3 are real constants and the pairs $(b_1, c_1), (b_2, c_2), (b_3, c_3)$ are real for $n > 0$.

The functions $f(x), g(y), h(z)$ can now be calculated as inverses of the elliptic integrals

$$x = \pm \int_0^u \frac{du}{\sqrt{X(u)}}, \quad y = \pm \int_0^v \frac{dv}{\sqrt{Y(v)}}, \quad z = \pm \int_0^w \frac{dw}{\sqrt{Z(w)}}. \tag{31}$$

By choosing $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = c_1 = c_2 = c_3 = 1$ and setting $n = 4$, equations (30) and (31) give

$$\begin{aligned} X(u) &= 1 + e^{2u} + e^{-2u}, \\ Y(v) &= 1 + e^{2v} + e^{-2v}, \\ Z(w) &= 1 + e^{2w} + e^{-2w}, \end{aligned} \tag{32}$$

and

$$z_1 = \int_0^w \frac{dw}{\sqrt{1 + e^{2w} + e^{-2w}}} = \int_0^w \frac{e^w dw}{\sqrt{1 + e^{2w} + e^{4w}}} = \int_1^{e^w} \frac{d\tau}{\sqrt{1 + \tau^2 + \tau^4}}.$$

We obtain similar expressions for x_1 and y_1, and the equation of the minimal surface can be written in the form

$$\ln \mathcal{E}(x_1) + \ln \mathcal{E}(y_1) + \ln \mathcal{E}(z_1) = 0, \text{ or } \mathcal{E}(x_1)\mathcal{E}(y_1)\mathcal{E}(z_1) = 1. \tag{33}$$

However, for a different choice of the sign in equations (31) we would get

$$z_2 = \int_1^{e^{-u}} \frac{d\tau}{\sqrt{1+\tau^2+\tau^4}},$$

where $x_1 = x_2$, $y_1 = y_2$, and z_2 is merely the reflection of z_1 with respect to the plane halfway between them, i.e. $\mathcal{E}(x_2)\mathcal{E}(y_2) = \mathcal{E}(z_2) = \frac{1}{\mathcal{E}(z_1)}$.

The minimal surface contained in the cube

$$0 \leq x_2, y_2, z_2 \leq \int_0^{\infty} \frac{d\tau}{\sqrt{1+\tau^2+\tau^4}} = p_0 \approx 1.68575, \tag{34}$$

is that part of the extended Schwarz surface (Figure 1) where six copies of the Schwarz surface meet. This surface is bounded by the six edges emanating from two opposite corners of a cube with side length p_0 (see Figure 11).[36]

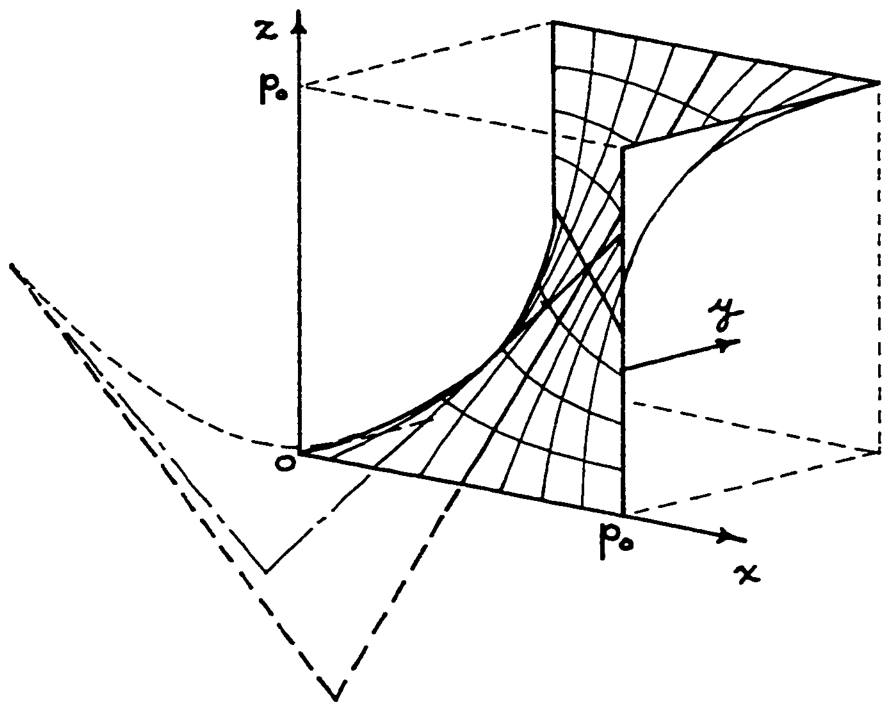

Figure 11

[36] *Ibid.*

We now substitute $\tau=\frac{\sigma-1}{\sigma+1}$ and $t=\frac{s-1}{s+1}$, to get

$$z_1 = \int_{t=0}^{t=1} \frac{d\tau}{\sqrt{1+\tau^2+\tau^4}} = \int_{s=1}^{\infty} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} = \frac{p_0}{2}.$$

If we denote the inverse of the second elliptic integral by $s = \mathcal{F}(z_1)$, then $\mathcal{E}(z_1) = \frac{\mathcal{F}(z_1)-1}{\mathcal{F}(z_1)+1}$, and equations (33) transform into

$$\mathcal{F}(y_1)\mathcal{F}(z_1) + \mathcal{F}(z_1)\mathcal{F}(x_1) + \mathcal{F}(x_1)\mathcal{F}(y_1) + 1 = 0. \tag{35}$$

Also, since

$$p_0 = \int_0^{\infty} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} = \int_{-\infty}^{0} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}},$$

then

$$\begin{aligned}\int_0^{s} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} &= \int_{-s}^{0} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} = \int_{-\infty}^{-\frac{1}{s}} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} \\ &= \int_{-\infty}^{0} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} - \int_{-\frac{1}{s}}^{0} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} \\ &= p_0 + \int_0^{-\frac{1}{s}} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}},\end{aligned}$$

i.e.

$$p_0 = \int_0^{s} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}} + \int_0^{\frac{1}{s}} \frac{d\sigma}{\sqrt{\frac{3}{4}+\frac{5}{2}\sigma^2+\frac{3}{4}\sigma^4}}. \tag{36}$$

We will need (35) and (36) in the next chapter.

Schwarz's surface is now defined by the inequalities

$$0 \le x_1, y_1 \le p_0, \text{ and } \quad -p_0 \le z_1 \le 0.$$

It lies inside the cube $0 \le x_1, y_1, z_1 + p_0 \le p_0$ and is bounded by the four edges of a regular tetrahedron (see Figure 12).

Using equations (31) and (32), we obtain a representation for our surface in the form

$$x = \int_0^{e^u} \frac{de^u}{\sqrt{1+e^{2u}+e^{4u}}},$$
$$y = \int_0^{e^v} \frac{de^v}{\sqrt{1+e^{2v}+e^{4v}}},$$
$$z = \int_0^{e^w} \frac{de^w}{\sqrt{1+e^{2w}+e^{4w}}}.$$

If we substitute $e^u = \frac{i(1+\omega^2)}{2\omega}$, $e^v = \frac{2\omega}{(1-\omega^2)}$, $e^w = \frac{1-\omega^2}{i(1+\omega^2)}$, then we get

$$\begin{aligned} x &= \int (1-\omega^2) R(\omega) d\omega, \\ y &= \int i(1+\omega^2) R(\omega) d\omega, \\ z &= \int 2\omega R(\omega) d\omega, \end{aligned} \tag{37}$$

where $R(\omega) = \frac{2}{i\sqrt{1+14\omega^4+\omega^8}}$. It is interesting that the minimal surfaces (37) are identical to those defined by equations (6), after a suitable choice of the constants of integration[37] and a rotation of the domain by $\frac{\pi}{4}$.

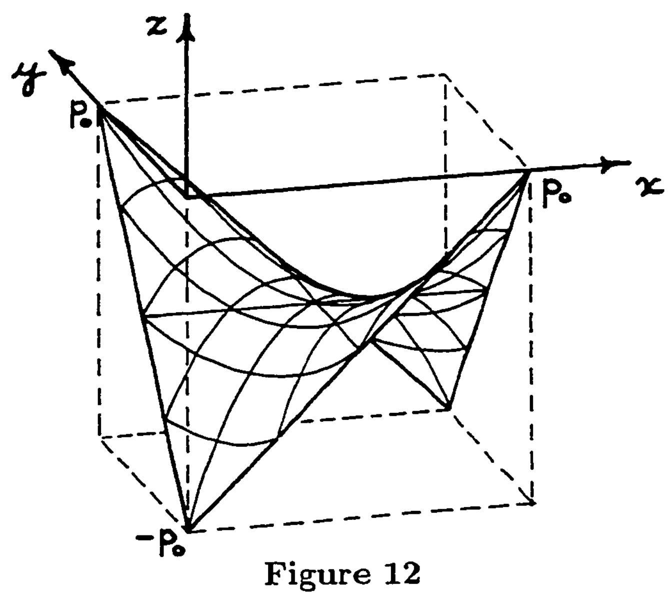

Figure 12

[37] Nitsche, p. 77.

IV. Comparison to the Hyperbolic Paraboloid

1. Outline

We now compare the Schwarz surface with the hyperbolic paraboloid when these two surfaces share a common contour. To simplify our computations, we place the coordinate axes at the upper corner of the cube containing both surfaces, as shown in Figure 12. In this way, the formula for the hyperbolic paraboloid becomes

$$z_H = -\frac{p_0}{2}\left[\left(\frac{2}{p_0}x - 1\right)\left(\frac{2}{p_0}y - 1\right) + 1\right]. \tag{38}$$

Using the method outlined in § III.3, we can represent the Schwarz surface as

$$\begin{aligned} x(s_1) &= \int_0^{s_1} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}}, \\ y(s_2) &= \int_0^{s_2} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}}, \\ z(s_3) &= \int_0^{s_3} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}}, \end{aligned} \tag{39}$$

where the $s_i \in [0, \infty)$. Equation (35) now becomes $s_2 s_3 + s_3 s_1 + s_1 s_2 + 1 = 0$. Solving for s_3 we get

$$s_3 = -\frac{s_1 s_2 + 1}{s_1 + s_2},$$

and the z-coordinate of equations (39) becomes

$$z_S(s_1, s_2) = \int_0^{-\frac{s_1 s_2 + 1}{s_1 + s_2}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}}.$$

Note that, $(x, y, z_S) = (0, 0, -p_0)$ when $s_1 = s_2 = 0$; $0 \le x, y \le \frac{p_0}{2}$ when $0 \le s_1, s_2 \le 1$; and $\frac{p_0}{2} \le x, y \le p_0$ when $1 \le s_1, s_2 \le \infty$.

To examine the vertical difference between these two surfaces, we define a function

$$D(s_1, s_2) = \left(x(s_1), y(s_2), z_S(s_1, s_2) - \left(-\frac{p_0}{2}\right)\left[\left(\frac{2}{p_0}x(s_1) - 1\right)\left(\frac{2}{p_0}y(s_2) - 1\right) + 1\right]\right).$$

With the aid of MATHEMATICA (see Appendix D), we can now plot the function D for $0 \leq x(s_1), y(s_2) \leq p_0$ and $-p_0 \leq z_S(s_1, s_2) \leq 0$. The resulting graph is shown in Figure 13. It shows that, for the regions $0 < x, y < \frac{p_0}{2}$ and $\frac{p_0}{2} < x, y < p_0$, the Schwarz surface is above the hyperbolic paraboloid; and, for the regions $x \in (\frac{p_0}{2}, p_0)$, $y \in (0, \frac{p_0}{2})$ and $x \in (0, \frac{p_0}{2})$, $y \in (\frac{p_0}{2}, p_0)$, the Schwarz surface is below.

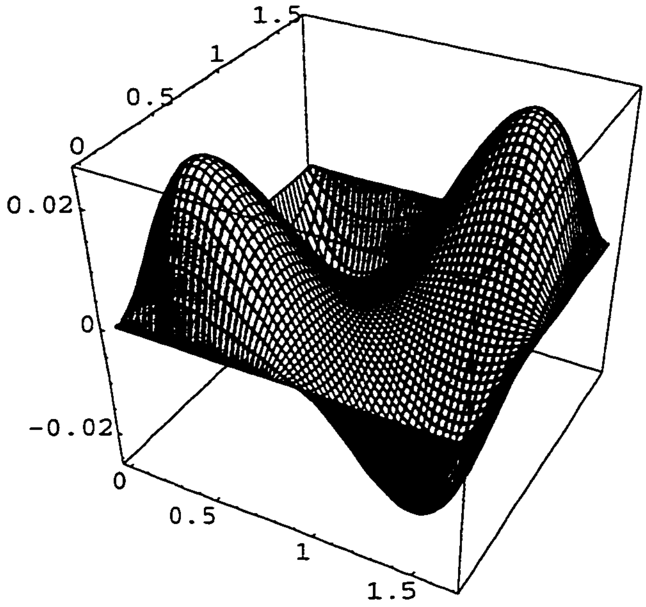

Figure 13

By superimposing the two surfaces, we see that this is indeed the case. For the lower half of the cube, the Schwarz surface is above the hyperbolic paraboloid; and, for the upper half, it is below (see Figure 14 and Appendix D). The two surfaces intersect along the contour and along the lines $x = \frac{p_0}{2}$, $z = -\frac{p_0}{2}$ and $y = \frac{p_0}{2}$, $z = -\frac{p_0}{2}$.

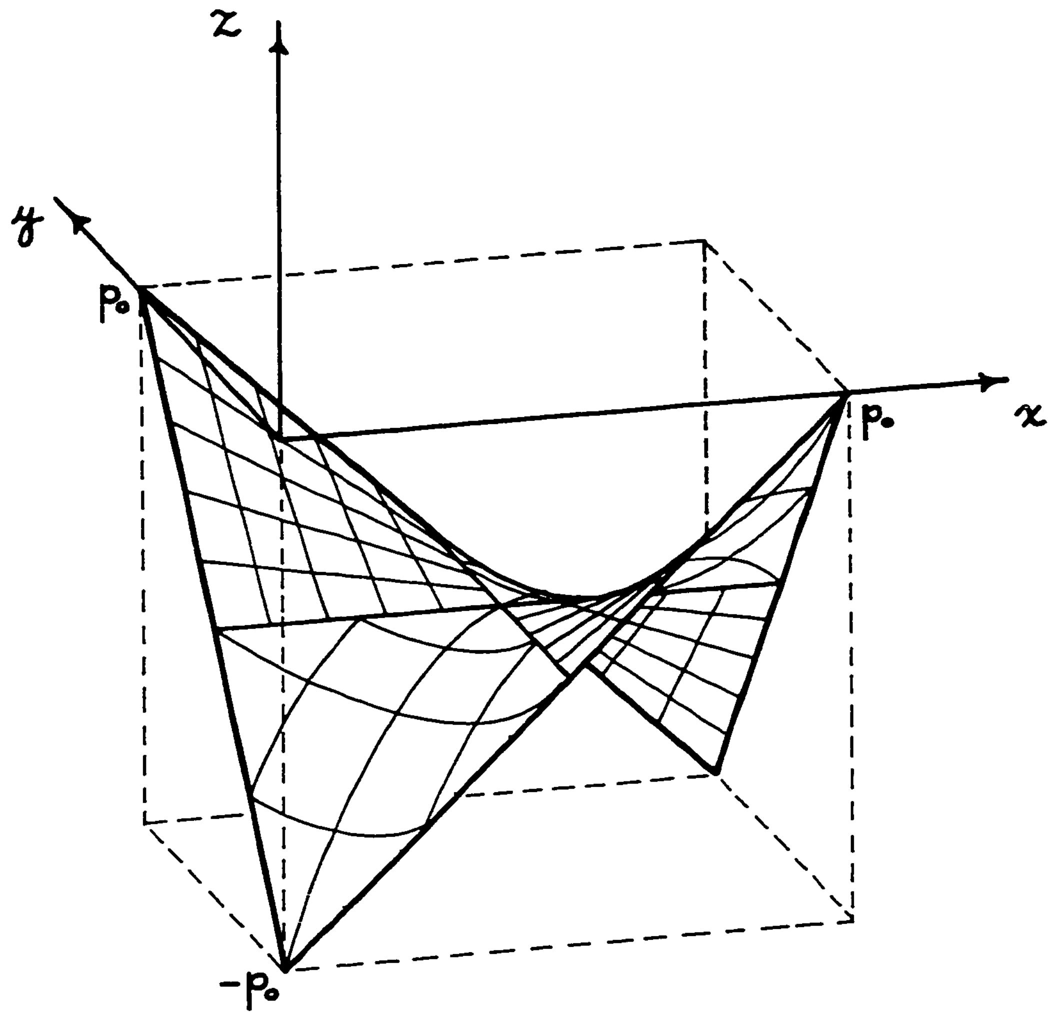

Figure 14

2. Graphing

In §IV.1, we mentioned three graphs; namely, the hyperbolic paraboloid, the Schwarz surface, and the difference function D. Plotting the hyperbolic paraboloid is straightforward, using equation (38) and the nonparametric representation

$$\mathbf{x}(x, y) = (x, y, z_H(x, y)),$$

where $x, y \in [0, p_0]$. But when we attempt to plot the latter two, we are immediately confronted with the problem of dividing the parameter interval $[0, \infty)$ into n equal parts. The solution is simple. We divide the interval $[0, 1]$ into $\frac{n}{2}$ equal parts to obtain the values $s_1, s_2 \in \{0, \frac{2}{n}, \frac{4}{n}, ..., \frac{n}{n}\}$ and use the reciprocals of these on the interval $[1, \infty)$. For example, if $n=10$, then

$$s_1, s_2 \in \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, \frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \infty\}.$$

In practice, however, it is convenient to divide the graphs of the Schwarz surface and the difference function into four regions. These correspond to:

(i) $0 \le s_1 \le 1$ and $0 \le s_2 \le 1$, where we use the formulas:

$$x = \int_0^{s_1} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$y = \int_0^{s_2} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$z = \int_0^{-\frac{s_1 s_2 + 1}{s_1 + s_2}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

i.e. $x \in [0, \frac{p_0}{2}]$, $y \in [0, \frac{p_0}{2}]$, and $z \in [-p_0, -\frac{p_0}{2}]$;

(ii) $0 \le s_1 \le 1$ and $0 \le \frac{1}{s_2} \le 1$, where we use the formulas:

$$x = \int_0^{s_1} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$y = p_0 - \int_0^{\frac{1}{s_2}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$z = -p_0 - \int_0^{-\frac{\frac{s_1}{s_2}+1}{s_1+\frac{1}{s_2}}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

i.e. $x \in [0, \frac{p_0}{2}]$, $y \in [\frac{p_0}{2}, p_0]$, and $z \in [-\frac{p_0}{2}, 0]$;

(iii) $0 \le \frac{1}{s_1} \le 1$ and $0 \le s_2 \le 1$, where we use the formulas:

$$x = p_0 - \int_0^{\frac{1}{s_1}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$y = \int_0^{s_2} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$z = -p_0 - \int_0^{-\frac{\frac{s_2}{s_1}+1}{\frac{1}{s_1}+s_2}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

i.e. $x \in [\frac{p_0}{2}, p_0]$, $y \in [0, \frac{p_0}{2}]$, and $z \in [-\frac{p_0}{2}, 0]$;

(iv) $0 \le \frac{1}{s_1} \le 1$ and $0 \le \frac{1}{s_2} \le 1$, where we use the formulas:

$$x = p_0 - \int_0^{\frac{1}{s_1}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$y = p_0 - \int_0^{\frac{1}{s_2}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

$$z = \int_0^{-\frac{\frac{1}{s_1 s_2}+1}{\frac{1}{s_1}+\frac{1}{s_2}}} \frac{d\sigma}{\sqrt{\frac{3}{4} + \frac{5}{2}\sigma^2 + \frac{3}{4}\sigma^4}},$$

i.e. $x \in [\frac{p_0}{2}, p_0]$, $y \in [\frac{p_0}{2}, p_0]$, and $z \in [-p_0, -\frac{p_0}{2}]$.

3. The Hyperbolic Paraboloid

We now examine the regular tetrahedral hyperbolic paraboloid. As mentioned in § IV.2, this surface can be represented nonparametrically by the Monge patch

$$\mathbf{x}(x,y) = \left(x, y, -\frac{p_0}{2}\left[\left(\frac{2}{p_0}x - 1\right)\left(\frac{2}{p_0}y - 1\right) + 1\right]\right),$$

where $x \in [0, p_0]$, $y \in [0, p_0]$ (see Figure 15).

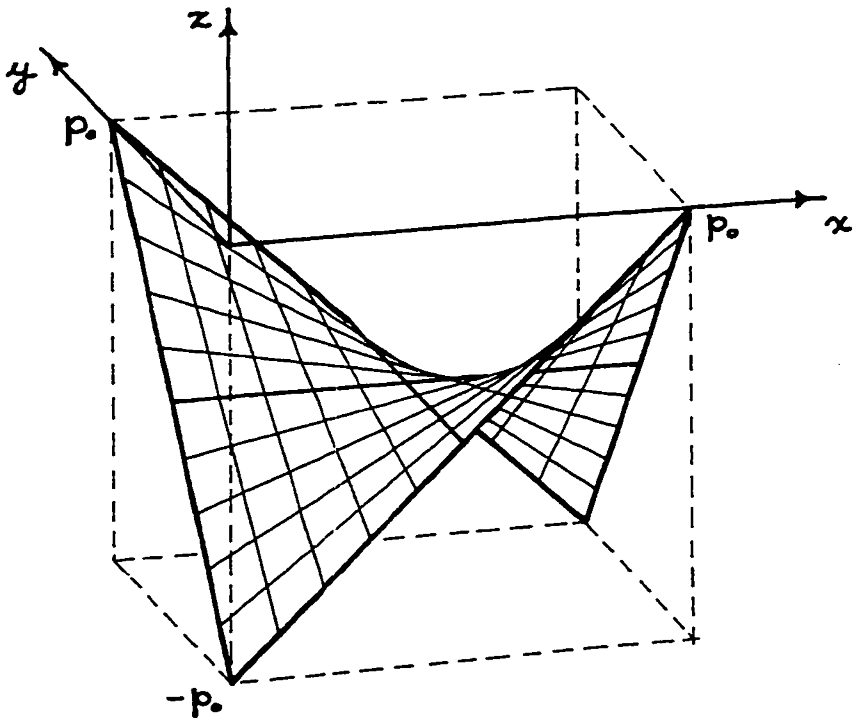

Figure 15

We first show that the mean curvature H vanishes along the lines $x = \frac{p_0}{2}$, $z = -\frac{p_0}{2}$ and $y = \frac{p_0}{2}$, $z = -\frac{p_0}{2}$, i.e along the directrixes.

$$\mathbf{x}_x = \left(1, 0, 1 - \frac{2}{p_0}y\right) \qquad E = \frac{4}{p_0^2}y^2 - \frac{4}{p_0}y + 2$$

$$\mathbf{x}_y = \left(0, 1, 1 - \frac{2}{p_0}x\right), \qquad F = \frac{4}{p_0^2}xy - \frac{2}{p_0}x - \frac{2}{p_0}y + 1$$

$$G = \frac{4}{p_0^2}x^2 - \frac{4}{p_0}x + 2,$$

$$W = \sqrt{EG - F^2} = \sqrt{\frac{4}{p_0^2}(x^2 + y^2) - \frac{4}{p_0}(x + y) + 3}\,,$$

$$\mathbf{X} = \frac{1}{W}\left(\frac{2}{p_0}y - 1, \frac{2}{p_0}x - 1, 1\right),$$

$$\begin{aligned} \mathbf{x}_{xx} &= (0,0,0) & L &= 0 \\ \mathbf{x}_{xy} &= (0,0,-\frac{2}{p_0}) & M &= -\frac{2}{p_0 W} \\ \mathbf{x}_{yy} &= (0,0,0), & N &= 0. \end{aligned}$$

Thus, the mean curvature

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{2\left(\frac{4}{p_0^2}xy - \frac{2}{p_0}x - \frac{2}{p_0}y + 1\right)}{p_0(EG - F^2)^{\frac{3}{2}}} = 0,$$

along the two directrixes.

If we now combine eighteen of these hyperbolic paraboloids in the same manner as the extended Schwarz surface (Figure 1), then we get the configuration drawn in Figure 16. At the center of this figure is what appears to be a circle of diameter equal to the side-length of one surface, and contained in a plane perpendicular to our point of view. But this is not a circle. In actual fact, this opening bows out hexangularly and attains its maximum diameter when centered on opposing pairs of the six constituent surfaces. Since this is where the Schwarz surface 'fills in' the hyperbolic paraboloid, it raises the obvious question as to whether or not the corresponding opening of the extended Schwarz surface is circular. The answer to this question is not in the scope of this thesis, but an affirmative result would support the idea that the Schwarz surface is an elementary building block of four-dimensional geometry.

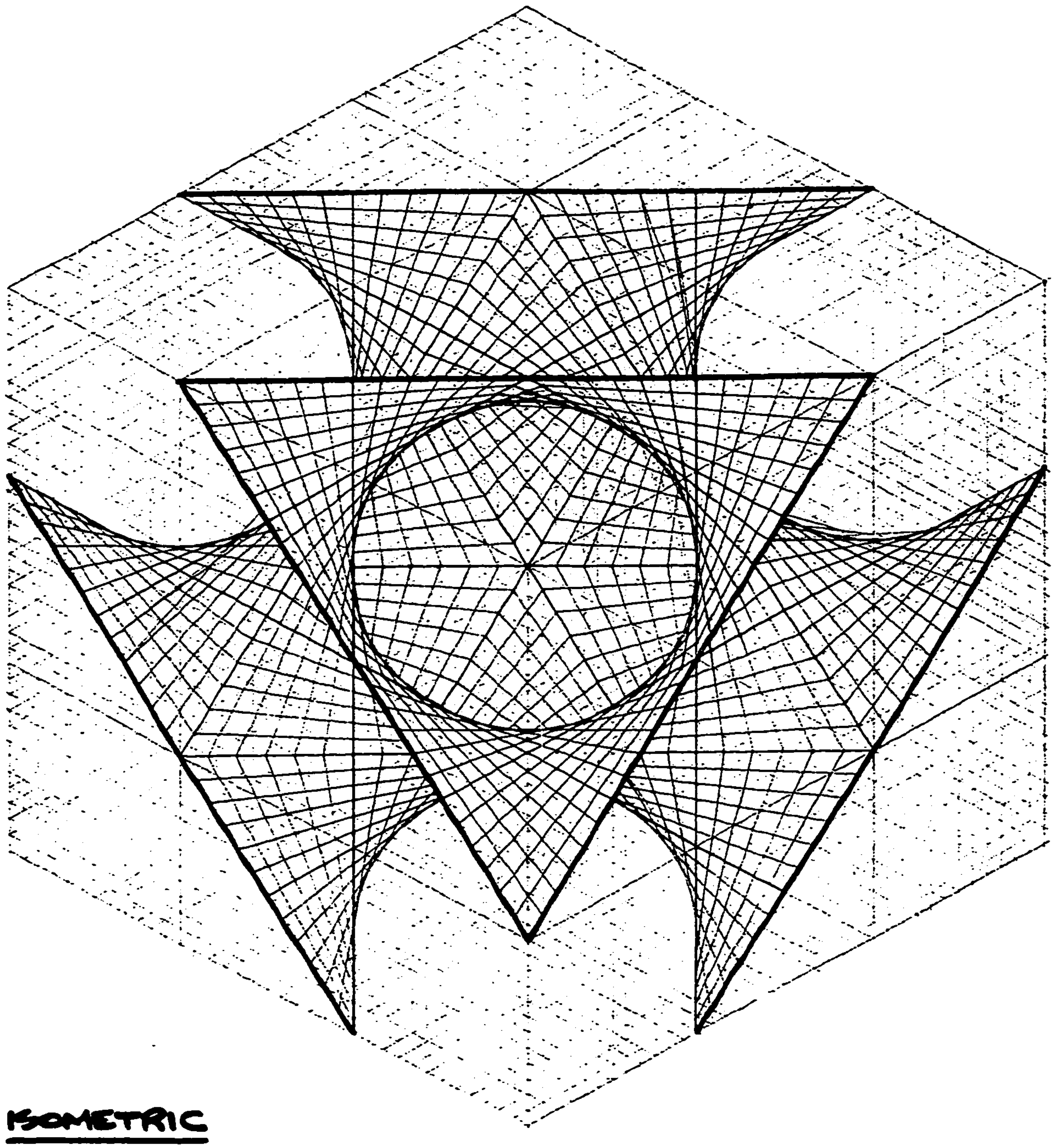

Figure 16

V. Conclusion

We have seen how the theory of minimal surfaces has blossomed from its modest beginnings in the mid-eighteenth century, with the likes of Lagrange and Monge, through to the end of the nineteenth century, in the hands of Weierstrass and Schwarz. Of course, the picture presented here is incomplete, since we have followed that thread of research linked to our main theme, namely, the surface of Schwarz. For this reason, we have focused on the work surrounding the minimal surface equation (3). In the beginning, the challenge was an exercise in pure analysis, in the sense that they were just seeking the solution to this partial differential equation. But later, after advances in complex analysis, these solutions became formulas, and using them one could create any, indeed, all minimal surfaces both real and imaginary.

The breakthrough came with the theorem of Weierstrass, which states that a necessary and sufficient condition for a surface given in isothermic representation to be a minimal surface is that the coordinate functions be harmonic. This means that we can consider each coordinate function as the real part of an analytic function of a complex variable. The integral equations (6) of Weierstrass follow, with which we can represent all real minimal surfaces. It is interesting to note that the imaginary part also represents a minimal surface, the conjugate minimal surface (see Gray, p. 462).

As we have shown, the value of the complex variable $\omega = \sigma + i\tau$ in the Weierstrass equations is equivalent to the stereographic projection of the Gauss map onto the equatorial plane, where the x, y axes of $\mathbf{R}^3$ are collinear with the σ, τ axes of the ω-plane, respectively. Schwarz used this fact to derive an expression for the analytic function $R(\omega)$— of equations (6)—particular to the minimal surface spanned by the regular tetrahedral skew quadrilateral. By substituting this function (7) back into equations (6), he thus had a parametric representation of his surface and the first explicit solution to the Plateau problem.

The representation formulas derived by Schwarz involve elliptic integrals over a complex domain, and we have mentioned that these are unwieldy in practice. We adopt, instead, the method outlined in § III.3, to represent Schwarz's surface using elliptic integrals over a real domain. Of particular interest is the difference between the hyperbolic paraboloid and our surface. Intuitively, since the former is a ruled surface and, hence, made up of straight lines, one would expect that it have minimal area. This is not the case, however, and a quick calculation shows that the mean curvature $H=0$ only along the two directrixes, where these asymptotic lines bisect the angle between the lines of curvature.

By superimposing the two surfaces, we have discovered that the two surfaces intersect along the two directrixes as well as on the contour. The Schwarz surface fills in slightly where the hyperbolic paraboloid is concave. This brings up an intriguing question. Is the near-circular opening at the center of Figure 16 now made circular when we replace these hyperbolic paraboloids with Schwarz surfaces? Perhaps this will be the topic of a future study, but it is just one question amongst many others. As in any scientific inquiry, by delving deeper into a subject we naturally expose new desiderata.

In closing, it remains to be stated that in this thesis we have focused exclusively on the events pertaining to the representation of the Schwarz surface. That is to say, we have concentrated on the period 1762-1890 in the annals of minimal surface theory when the equations of Weierstrass came into being, and we have selected those events which had an immediate impact on them. However, in doing so we have not given due mention to such great geometers as Courant, Douglas, McShane, Morrey, Morse, Shiffman, Tompkins, Tonelli, and others, all participants in a second golden age of the theory from about 1930 to 1940. Their work, in the first half of the twentieth century, brought strong results to the study of two-dimensional surfaces. The complete solution of the Plateau problem was obtained for surfaces of arbitrary topological type, spanning a fixed contour in Euclidean space. Concurrent with the advent of personal computers has been a third golden age in recent years. To the trend of ever increasing abstraction has been added a renewed interest in solutions with explicit representations and the ancient desire to study geometrical objects for their own sake.

Appendix A

Key Concepts of Differential Geometry

A1. We say that the mapping $\mathbf{x}$ is of class C^m in $\mathcal{U}$ if all partial derivatives of $\mathbf{x}$ of order m or less are continuous in $\mathcal{U}$. A *regular parametric representation of class* C^m $(m \geq 1)$ of a set of points S (S for surface) in $\mathbf{R}^3$ is a mapping $\mathbf{x} = \mathbf{x}(u, v)$ of an open set $\mathcal{U}$ in the (u, v)-plane onto S such that

(i) $\mathbf{x}$ is of class C^m in $\mathcal{U}$.

(ii) If $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a basis in $\mathbf{R}^3$ and $\mathbf{x}(u, v) = x(u, v)\mathbf{e}_1 + y(u, v)\mathbf{e}_2 + z(u, v)\mathbf{e}_3$, then for all (u, v) in $\mathcal{U}$, the Jacobian matrix $\mathcal{J}(\mathbf{x})(u, v)$ has rank 2.

A2. A regular parametric representation $\mathbf{x} = \mathbf{x}(u, v)$ is thus a continuous mapping of a set $\mathcal{U}$ (in $\mathbf{R}^2$) onto S (in $\mathbf{R}^3$). If $\mathbf{x}$ is one-to-one, there exists an inverse mapping $\mathbf{x}^{-1}$ of S back onto $\mathcal{U}$. If $\mathbf{x}^{-1}$ is continuous, then $\mathbf{x}$ is called a *topological mapping* or *homeomorphism* of $\mathcal{U}$ onto S. A *coordinate patch*, or *regular patch*, is a regular parametric representation of a part of S, where $\mathbf{x}$ is a homeomorphism of $\mathcal{T} \subset \mathcal{U}$ onto $R \subset S$.

A3. A function of the form $\mathbf{x} = u\mathbf{e}_1 + v\mathbf{e}_2 + f(u, v)\mathbf{e}_3$ defines a coordinate patch of class C^m if $f(u, v)$ is a function of class C^m. Such a patch is called a *Monge patch* or a *nonparametric representation.*

A4. If $\mathbf{x} = \mathbf{x}(u, v)$ is a regular parametric representation of S defined on $\mathcal{U}$, then the image of the coordinate line $v = v_0$ in $\mathcal{U}$ is the *u-parameter curve* $\mathbf{x} = \mathbf{x}(u, v_0)$ on S. Similarly, the image of the coordinate line $u = u_0$ is the *v-parameter curve* $\mathbf{x} = \mathbf{x}(u_0, v)$ on S.

A5. We write $\mathbf{x}_u(u_0, v_0)$ for the partial derivative of $\mathbf{x}$ at (u_0, v_0) in the direction of the u-axis. Hence, $\mathbf{x}_u(u_0, v_0)$ is a vector which is tangent to the u-parameter curve at $\mathbf{x}(u_0, v_0)$ in the direction of increasing u. Similarly, $\mathbf{x}_v(u_0, v_0)$ is a vector tangent to the v-parameter curve at $\mathbf{x}(u_0, v_0)$ in the direction of v.

A6. The vectors $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$ are linearly independent, and thus form a basis for the tangent plane at $\mathbf{x}(u_0, v_0)$. The normal vector to S at $\mathbf{x}(u_0, v_0)$ is $\mathbf{x}_u \times \mathbf{x}_v$, and so the *unit normal vector* is

$$\mathbf{X} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

A7. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a coordinate patch on a surface of class ≥ 1. The function $I = d\mathbf{x} \bullet d\mathbf{x} = Edu^2 + 2Fdudv + Gdv^2$ is called the *first fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$, where

$$E = \mathbf{x}_u \bullet \mathbf{x}_u, \quad F = \mathbf{x}_u \bullet \mathbf{x}_v, \quad G = \mathbf{x}_v \bullet \mathbf{x}_v,$$

and E, F, G are called the *first fundamental coefficients*. The area of a region R on $\mathbf{x} = \mathbf{x}(u, v)$ is the double integral

$$A(S) = \int\int_D \sqrt{EG - F^2}\, dudv,$$

where D is the set of points in the parameter plane which maps onto R. Note that $\|\mathbf{x}_u \times \mathbf{x}_v\|^2 = EG - F^2$.

A8. Suppose $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2. The function $II = -d\mathbf{x} \bullet d\mathbf{X} = Ldu^2 + 2Mdudv + Ndv^2$ is called the *second fundamental form* of $\mathbf{x} = \mathbf{x}(u, v)$, where

$$L = -\mathbf{x}_u \bullet \mathbf{X}_u, \quad M = -\frac{1}{2}(\mathbf{x}_u \bullet \mathbf{X}_v + \mathbf{x}_v \bullet \mathbf{X}_u), \quad N = -\mathbf{x}_v \bullet \mathbf{X}_v,$$

and L, M, N are called the *second fundamental coefficients*. Alternatively,

$$L = \mathbf{x}_{uu} \bullet \mathbf{X}, \quad M = \mathbf{x}_{uv} \bullet \mathbf{X}, \quad N = \mathbf{x}_{vv} \bullet \mathbf{X}.$$

A9. Let P be a point on a surface S and $\mathbf{X}$ the unit normal at P. Imagine a plane, containing $\mathbf{X}$, and intersecting the surface along a curve. The *normal curvature* at

P is simply the reciprocal of the radius of curvature R of that curve at P. If we now rotate the intersecting plane around $\mathbf{X}$ until we obtain a curve of maximum normal curvature at P, then this would be a *principal curvature* of S at P. We then rotate 90° to obtain the minimum normal curvature; the other principal curvature at P. When the intersecting curve bends toward $\mathbf{X}$, the normal curvature is positive. When it bends away from $\mathbf{X}$, the normal curvature is negative. On a minimal surface, the principal curvatures k_1 and k_2 are equal but opposite everywhere, i.e.

$$k_1 = -k_2.$$

Thus, the *mean curvature*, or average curvature,

$$H = \frac{1}{2}(k_1 + k_2) = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

is zero at all points on a minimal surface.

A10. Suppose $\mathbf{x} = \mathbf{x}(u, v)$ is a patch on a surface of class ≥ 2. A curve on the surface whose tangent at each point is along a principal direction (direction of principal curvature) is called a *line of curvature*. The *equations of Rodrigues*, namely

$$dx + \rho dX = 0, \quad dy + \rho dY = 0, \quad dz + \rho dZ = 0,$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{X} = (X, Y, Z)$, completely characterize the lines of curvature on the surface. The function ρ is the radius of normal curvature in the direction of the line of curvature.

A11. An *orthogonal* coordinate patch $\mathbf{x} = \mathbf{x}(u, v)$ is one for which $F = \mathbf{x}_u \bullet \mathbf{x}_v = 0$. If in addition $E{=}G{\neq}0$, then we have an *isothermal* coordinate patch and $\mathbf{x} : \mathcal{U} \rightarrow \mathbf{R}^3$ is a *conformal mapping*. An *isometry* is a conformal mapping with $E{=}G{=}1$.

A12. *An n-dimensional manifold* M is a set furnished with a collection $\mathcal{P}$ of *abstract patches* such that

(i) M is covered by the images of the patches in the collection $\mathcal{P}$.

(ii) For any two patches $\mathbf{x}$, $\mathbf{y}$ in the collection $\mathcal{P}$, the composite functions $\mathbf{y}^{-1}\mathbf{x}$ and $\mathbf{x}^{-1}\mathbf{y}$ are Euclidean-differentiable.

A *surface* is a two-dimensional manifold. A manifold M of arbitrary dimension furnished with a (differentiable) inner product on each of its tangent spaces is called a *Riemannian manifold*. A *geometric surface* is a two-dimensional Riemannian manifold.

A13. Let $S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in \mathcal{U}\}$ have a regular parametric representation of class C^m. A topological mapping $(u, v) \mapsto (u', v')$ from $\mathcal{U}$ onto a set $\mathcal{U}'$ in the (u', v')-plane is called an *admissible change of parameters*, if the Jacobian determinants $\partial(u, v)/\partial(u', v'), \partial(u', v')/\partial(u, v)$ exist and are nonzero, and if the functions $u(u', v'), v(u', v')$ and the inverse functions $u'(u, v), v'(u, v)$ are m-times differentiable on the interiors of $\mathcal{U}'$ and $\mathcal{U}$, respectively.

A14. A *Jordan curve* is a simple closed curve.

Appendix B

Geometric Interpretation of ω

B1. We now examine the relation between a point on the minimal surface and the unit normal at that point. Using the notation of Ahlfors,[38] we can rewrite equations (11) as:

$$\begin{aligned} x(\omega,\bar\omega) &= \frac{1}{2}\int(1-\omega^2)R(\omega)d\omega + \frac{1}{2}\int(1-\bar\omega^2)\bar R(\bar\omega)d\bar\omega, \\ y(\omega,\bar\omega) &= \frac{i}{2}\int(1+\omega^2)R(\omega)d\omega - \frac{i}{2}\int(1+\bar\omega^2)\bar R(\bar\omega)d\bar\omega, \\ z(\omega,\bar\omega) &= \int\omega R(\omega)d\omega + \int\bar\omega\bar R(\bar\omega)d\bar\omega. \end{aligned} \tag{11'}$$

The unit normal $\mathbf{X} = (X, Y, Z)$ is defined by the equations

$$\mathbf{X}\bullet\mathbf{x}_\omega = 0, \quad \mathbf{X}\bullet\mathbf{x}_{\bar\omega} = 0, \quad \mathbf{X}\bullet\mathbf{X} = 1,$$

i.e.

$$\begin{aligned} Xx_\omega + Yy_\omega + Zz_\omega &= 0, \\ Xx_{\bar\omega} + Yy_{\bar\omega} + Zz_{\bar\omega} &= 0, \\ X^2+Y^2+Z^2 &= 1. \end{aligned} \tag{40}$$

Using equations (11′), we can determine the partial derivatives in equations (40), and substituting we get:[39]

$$X = \frac{\omega+\bar\omega}{1+\omega\bar\omega}, \quad Y = i\frac{\bar\omega-\omega}{1+\omega\bar\omega}, \quad Z = \frac{\omega\bar\omega-1}{1+\omega\bar\omega}. \tag{41}$$

We now set $\omega = \sigma + i\tau$. We can then rewrite equations (41) as:

$$X = \frac{2\sigma}{\sigma^2+\tau^2+1}, \quad Y = \frac{2\tau}{\sigma^2+\tau^2+1}, \quad Z = \frac{\sigma^2+\tau^2-1}{\sigma^2+\tau^2+1}. \tag{42}$$

The inverse mapping

$$\sigma = \frac{X}{1-Z}, \qquad \tau = \frac{Y}{1-Z},$$

defines a stereographic projection from the north pole of the unit sphere onto its equatorial (σ,τ)-plane[40] (see Figure 5).

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38 Lars V. Ahlfors, *Complex Analysis* (New York, 1979), p. 27.
39 Darboux, p. 296.
40 Nitsche, p. 55.

B2. We now show that the mapping $\omega = \frac{\Psi(\gamma)}{\Phi(\gamma)}$ is equivalent to the stereographic projection of the Gauss map onto the ω-plane (see § II.3). It can be shown[41] that

$$\mathbf{X} = \left(\frac{\Phi\bar{\Psi} + \bar{\Phi}\Psi}{|\Phi|^2 + |\Psi|^2}, \frac{i(\Phi\bar{\Psi} - \bar{\Phi}\Psi)}{|\Phi|^2 + |\Psi|^2}, \frac{|\Psi|^2 - |\Phi|^2}{|\Phi|^2 + |\Psi|^2} \right).$$

From B.1,

$$\begin{aligned} \omega &= \frac{X}{1-Z} + i\frac{Y}{1-Z} \\ &= \frac{\frac{\Phi\bar{\Psi}+\bar{\Phi}\Psi}{|\Phi|^2+|\Psi|^2}}{1 - \frac{|\Psi|^2-|\Phi|^2}{|\Phi|^2+|\Psi|^2}} + i\frac{\frac{i(\Phi\bar{\Psi}-\bar{\Phi}\Psi)}{|\Phi|^2+|\Psi|^2}}{1 - \frac{|\Psi|^2-|\Phi|^2}{|\Phi|^2+|\Psi|^2}} \\ &= \frac{\Phi\bar{\Psi} + \bar{\Phi}\Psi}{2|\Phi|^2} - \frac{\Phi\bar{\Psi} - \bar{\Phi}\Psi}{2|\Phi|^2} \\ &= \frac{2\bar{\Phi}\Psi}{2|\Phi|^2} = \frac{\bar{\Phi}\Psi}{\Phi\bar{\Phi}} = \frac{\Psi}{\Phi}. \end{aligned}$$

41 Nitsche, p. 146.

Appendix C
The Plateau Problem

The Plateau Problem is a fruitful branch of modern mathematics which brings together many different fields of study to bear on the question of minimal surfaces. In its simplest form, the problem reads as follows: Find the surface of minimal area spanning a given fixed wire frame in three-dimentional space. It was Lagrange who first formulated the problem in 1762, but it gets its name from the famous physicist Plateau (1801-1883), who spent many years conducting soap-film experiments, solving it for several different contours. It was Lebesgue who first coined the name in his thesis of 1902. To demonstrate the depth of this problem, we cite Radó (p. 32) for a more recent version which reads: Given, in the xyz-space, a Jordan curve Γ^*, determine a minimal surface, of the type of the circular disc [i.e. orientable, with one contour and Euler characteristic 1], bounded by Γ^*, such that

(i) the solution admits of a representation $S : \mathbf{r} = \mathbf{r}(u,v)$, $u^2+v^2 \leq 1$, where the components $x(u,v)$, $y(u,v)$, $z(u,v)$ of $\mathbf{r}(u,v)$ are continuous for $u^2+v^2 \leq 1$, harmonic for $u^2+v^2 < 1$, and satisfy for $u^2+v^2 < 1$ the equations $E = G$, $F = 0$;

(ii) the equations $x{=}x(u,v)$, $y{=}y(u,v)$, $z{=}z(u,v)$ carry $u^2+v^2 = 1$ in a topological way into Γ^*;

(iii) the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ admit of a representation of the form

$$x = \Re \int^{w} (\Phi^2 - \Psi^2)\, dw,$$

$$y = \Re \int^{w} i(\Phi^2 + \Psi^2)\, dw,$$

$$z = \Re \int^{w} 2\Phi\Psi\, dw,$$

where Φ and Ψ denote single-valued analytic functions of $w = u + iv$ in $|w| < 1$;

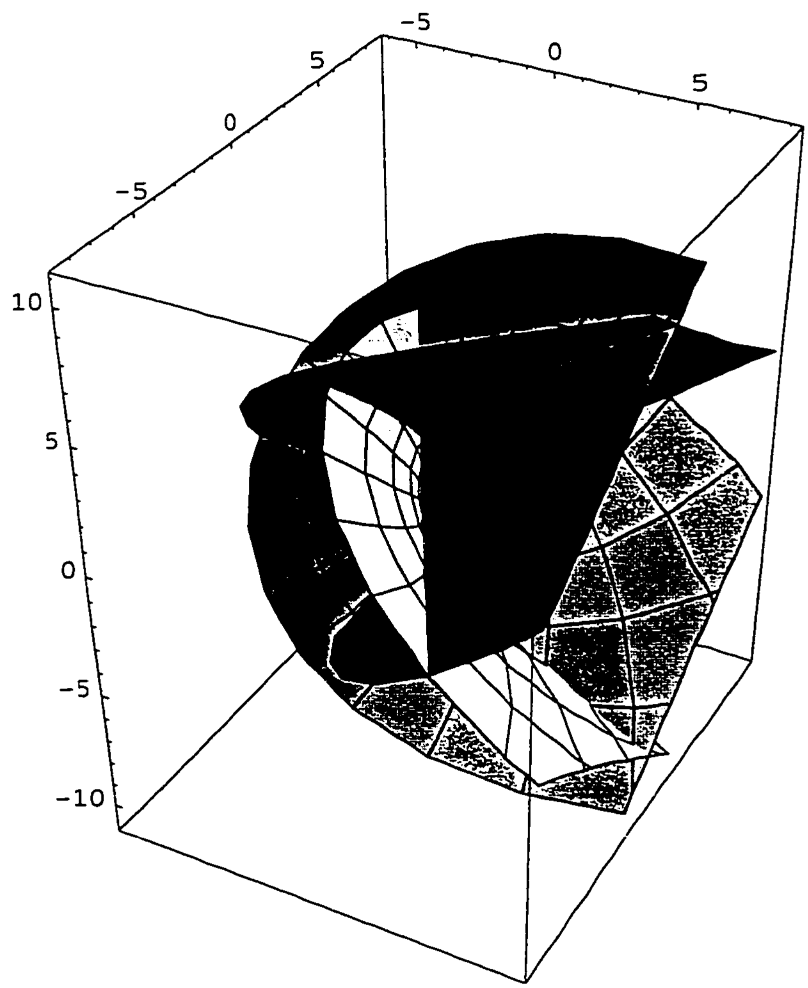
(iv) Φ and Ψ do not have any common zero in $|w| < 1$.

Appendix D

Computer Graphics

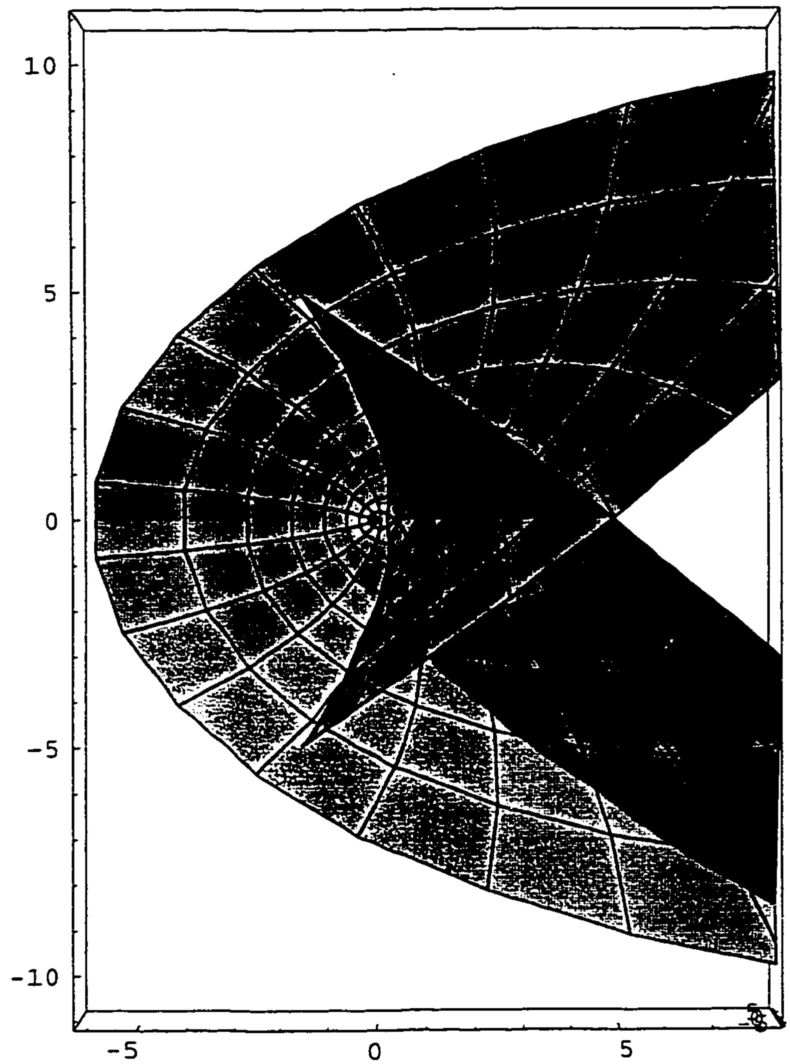
D1. Using MATHEMATICA, we can plot the parametric representation derived in § II.7. We first show a perspective and front view for $\sigma \in [-2,2]$, $\tau \in [-2,2]$. Following this, we restrict the parameter domain to $\sigma \in [-1,1]$ and $\tau \in [-1,1]$, to investigate the behavior of this minimal surface near the origin.

```
ParametricPlot3D[{(2 s^2 - 2 t^2 + 6 s^2 t^2 - s^4 - t^4) / 4, s t^3 - s t - s^3 t, 2 s^3 / 3 - 2 s t^2},
  {s, -2, 2}, {t, -2, 2}]
```



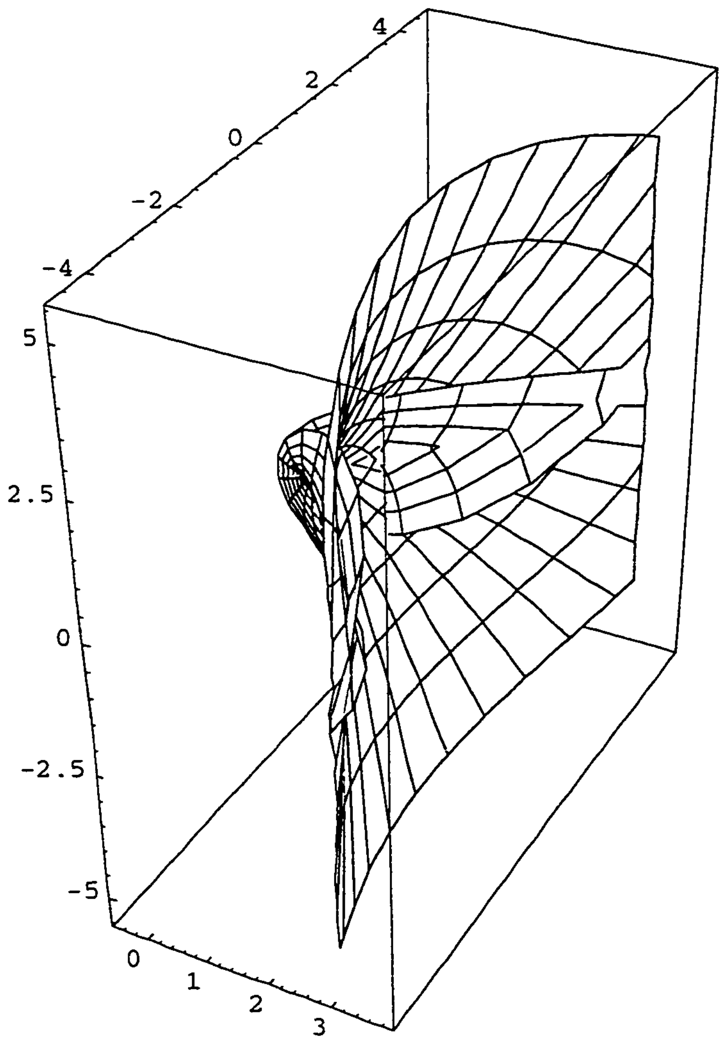

```
- Graphics3D -
```

```
Show[%, ViewPoint -> {0, -20, 0}]
```



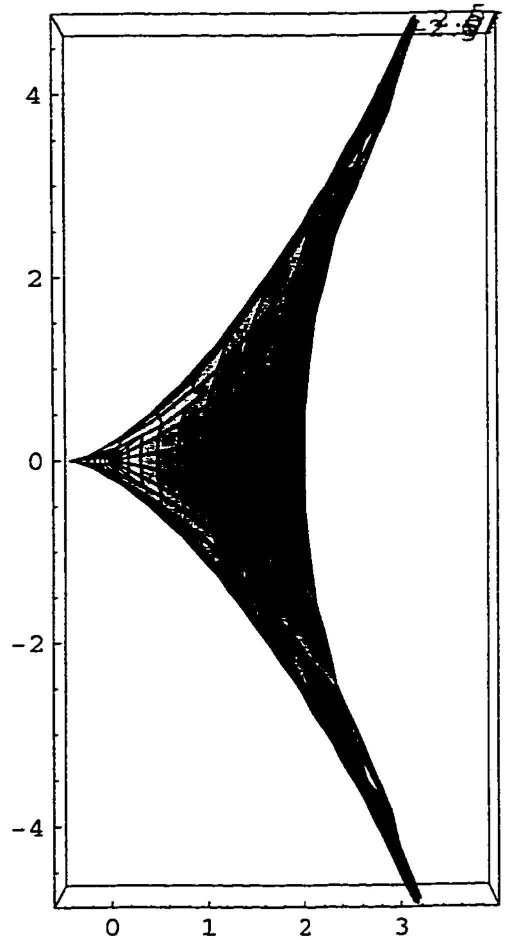

```
- Graphics3D -
```

Show[%1, Shading -> False]



- Graphics3D -

```
Show[%1, ViewPoint -> {0, 0, 20}]
```




```
- Graphics3D -
```

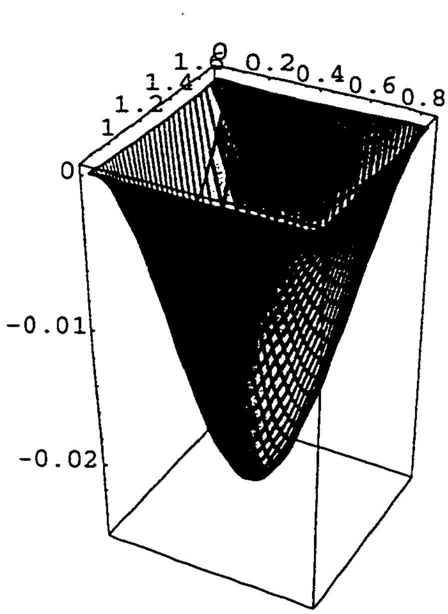
D2. Using MATHEMATICA, we can plot the difference function D defined in §IV.1. We graph the four regions; $0 \le x, y \le \frac{p_0}{2}$; $x \in [\frac{p_0}{2}, p_0]$, $y \in [0, \frac{p_0}{2}]$; $x \in [0, \frac{p_0}{2}]$, $y \in [\frac{p_0}{2}, p_0]$; and $\frac{p_0}{2} \le x, y \le p_0$.

$$\text{EL2[u\_] := } \int_0^u \frac{1}{\sqrt{\frac{3}{4} + 5 * \frac{s^2}{2} + 3 * \frac{s^4}{4}}} \, ds$$

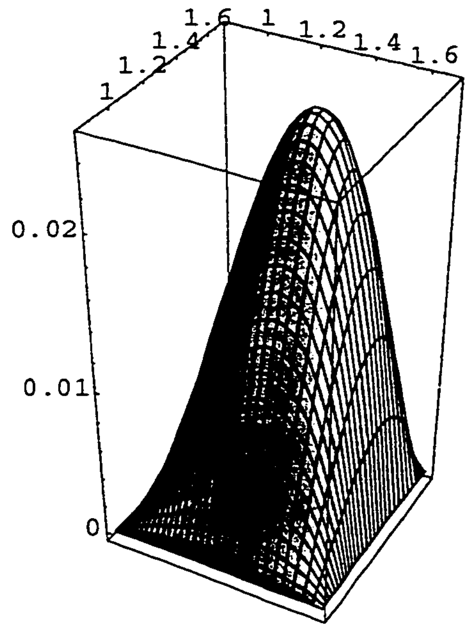
```
Hdiff[v_, w_] = { N[EL2[v]], N[EL2[w]], N[EL2[(-1 - v*w) / (v + w)]] +
   (1.68575 / 2) * (((2 / 1.68575) *N[EL2[v]] - 1) * ((2 / 1.68575) *N[EL2[w]] - 1) + 1)}
```

$$\{-(1.1547\, I \sqrt{1. + 0.333333\, v^2}\, \sqrt{1. + 3.\, v^2}\; \text{EllipticF}[1.\, I\, \text{ArcSinh}[1.73205\, v],\ 0.111111])\,/\,(\sqrt{3. + 10.\, v^2 + 3.\, v^4}),$$
$$-(1.1547\, I \sqrt{1. + 0.333333\, w^2}\, \sqrt{1. + 3.\, w^2}\; \text{EllipticF}[1.\, I\, \text{ArcSinh}[1.73205\, w],\ 0.111111])\,/\,(\sqrt{3. + 10.\, w^2 + 3.\, w^4}),$$
$$0.842875\, (1 + (-1 - (1.36995\, I \sqrt{1. + 0.333333\, v^2}\, \sqrt{1. + 3.\, v^2}\; \text{EllipticF}[1.\, I\, \text{ArcSinh}[1.73205\, v],\ 0.111111])\,/\,(\sqrt{3. + 10.\, v^2 + 3.\, v^4}))$$
$$(-1 - (1.36995\, I \sqrt{1. + 0.333333\, w^2}\, \sqrt{1. + 3.\, w^2}\; \text{EllipticF}[1.\, I\, \text{ArcSinh}[1.73205\, w],\ 0.111111])\,/\,(\sqrt{3. + 10.\, w^2 + 3.\, w^4}))) -$$
$$\left(0.666667\, I \sqrt{\frac{1. + 3.\, v^2 + 8.\, v w + 3.\, w^2 + v^2 w^2}{(v+w)^2}}\, \sqrt{1. + \frac{3.\,(-1. - 1.\, v w)^2}{(v+w)^2}}\right.$$
$$\left.\text{EllipticF}\left[1.\, I\, \text{ArcSinh}\left[\frac{1.73205\,(-1. - 1.\, v w)}{v+w}\right],\ 0.111111\right]\right) \Big/$$
$$\left(\sqrt{3. + \frac{10.\,(-1. - 1.\, v w)^2}{(v+w)^2} + \frac{3.\,(-1. - 1.\, v w)^4}{(v+w)^4}}\right)\}$$

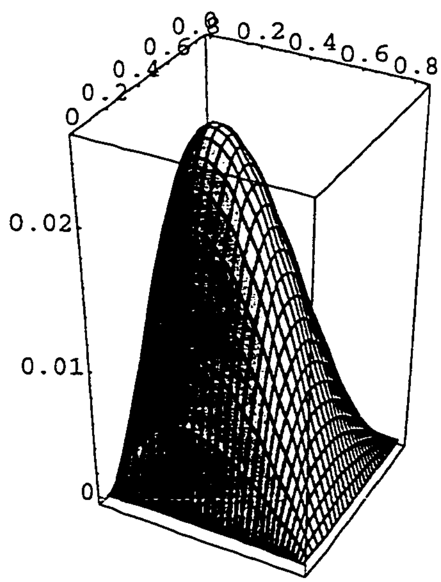
```
D1 = ParametricPlot3D[Hdiff[e, f],
  {e, .001, 1}, {f, .001, 1}, {PlotPoints -> 30}, BoxRatios -> {1, 1, 2}]
D2 = ParametricPlot3D[Hdiff[1 / e, f],
  {e, .001, 1}, {f, .001, 1}, {PlotPoints -> 30}, BoxRatios -> {1, 1, 2}]
D3 = ParametricPlot3D[Hdiff[e, 1 / f],
  {e, .001, 1}, {f, .001, 1}, {PlotPoints -> 30}, BoxRatios -> {1, 1, 2}]
D4 = ParametricPlot3D[Hdiff[1 / e, 1 / f],
  {e, .001, 1}, {f, .001, 1}, {PlotPoints -> 30}, BoxRatios -> {1, 1, 2}]
Show[D1, D2, D3, D4, BoxRatios -> {1, 1, 1}]
```



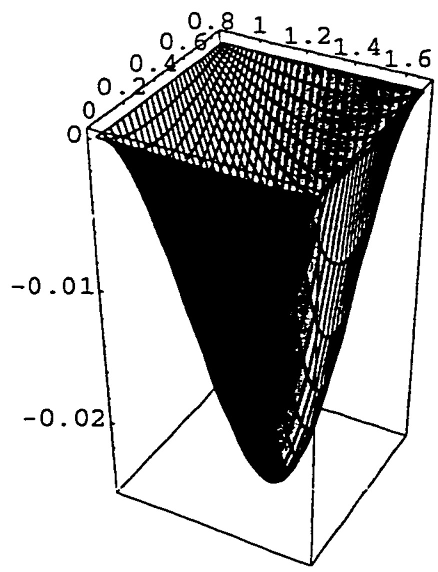
- Graphics3D -



- Graphics3D -



- Graphics3D -

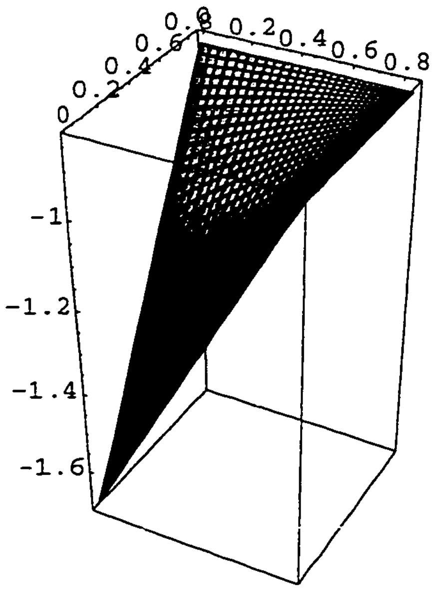


- Graphics3D -

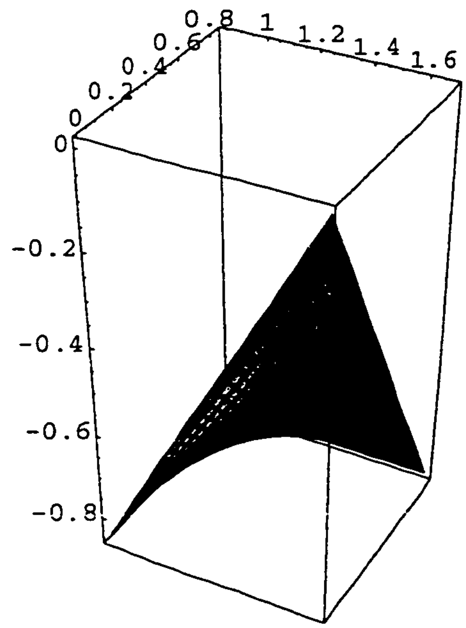
D3. Using MATHEMATICA, we plot pieces of the Schwarz surface under the four regions; $0 \le x, y \le \frac{p_0}{2}$; $x \in [\frac{p_0}{2}, p_0]$, $y \in [0, \frac{p_0}{2}]$; $x \in [0, \frac{p_0}{2}]$, $y \in [\frac{p_0}{2}, p_0]$; and $\frac{p_0}{2} \le x, y \le p_0$. We then compose these four parts to graph Schwarz's minimal surface. Finally, we plot the hyperbolic paraboloid for the same contour, and superimpose it with the Schwarz surface.

$$\text{EL2[u\_]} := \int_0^u \frac{1}{\sqrt{\frac{3}{4} + 5 * \frac{s^2}{2} + 3 * \frac{s^4}{4}}}\, ds$$

```
H[v_,w_]={ N[EL2[v]],N[EL2[w]],N[EL2[(-1-v*w)/(v+w)]]}

ParametricPlot3D[H[e,f],{e,.001,1},{f,.001,1},{PlotPoints->30},BoxRatios->{1,1,2}]

Bh2=ParametricPlot3D[H[1/e,f],{e,.001,1},{f,.001,1},{PlotPoints->30},BoxRatios->{1,1
,2}]
```

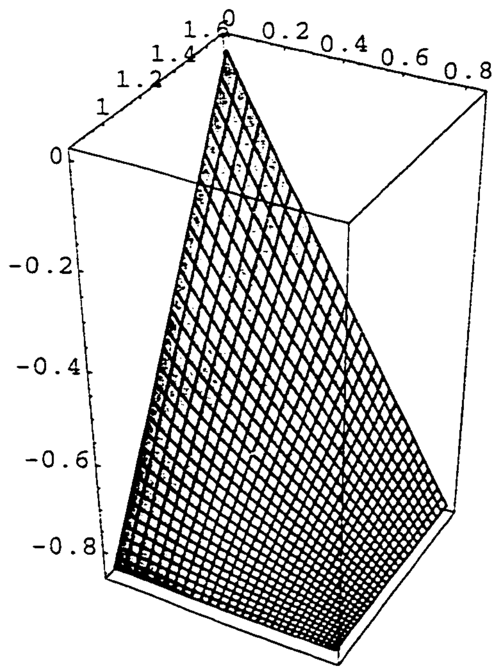



```
- Graphics3D -

Bh1 = %
```

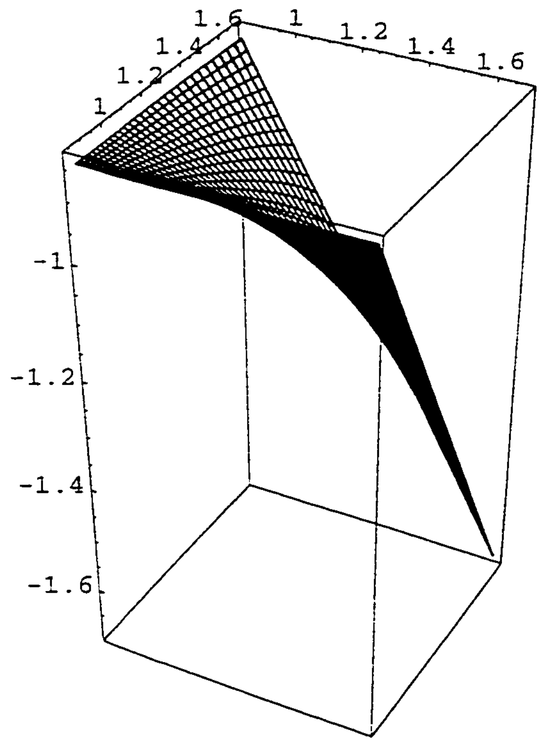



```
- Graphics3D -
```

```
Bh3=ParametricPlot3D[H[e,1/f],{e,.001,1},{f,.001,1},{PlotPoints->30},BoxRatios->{1,1
,2}]

Bh4 = ParametricPlot3D[H[1/e, 1/f], {e, 0.001, 1}, {f, 0.001, 1},
   {PlotPoints -> 30}, BoxRatios -> {1, 1, 2}]
```

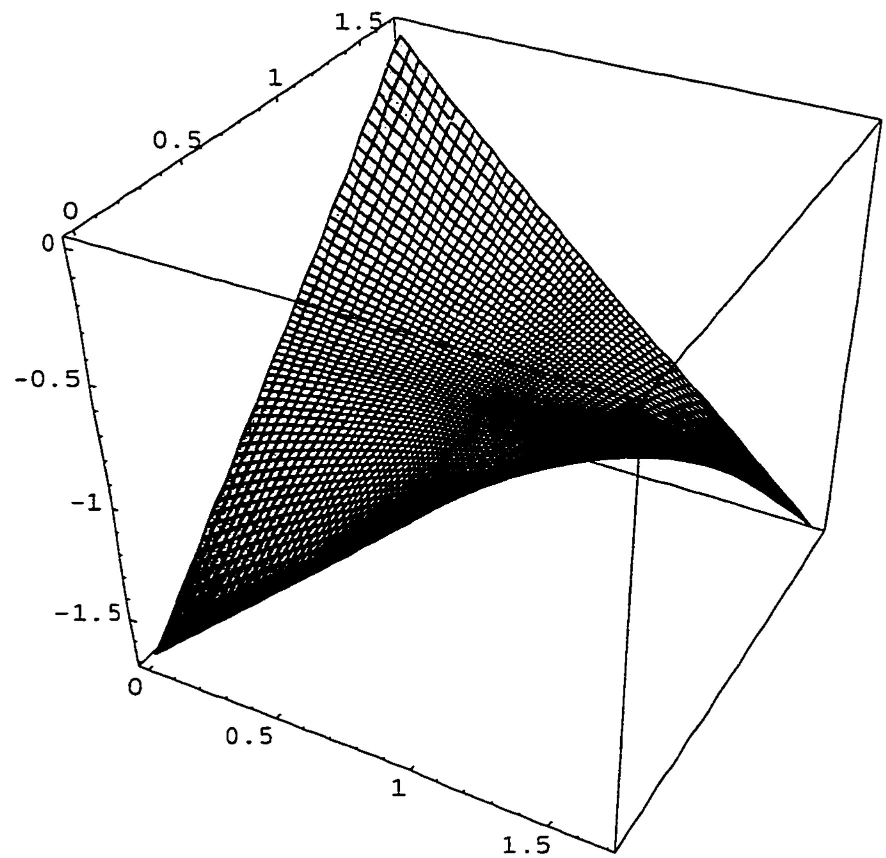


- Graphics3D -



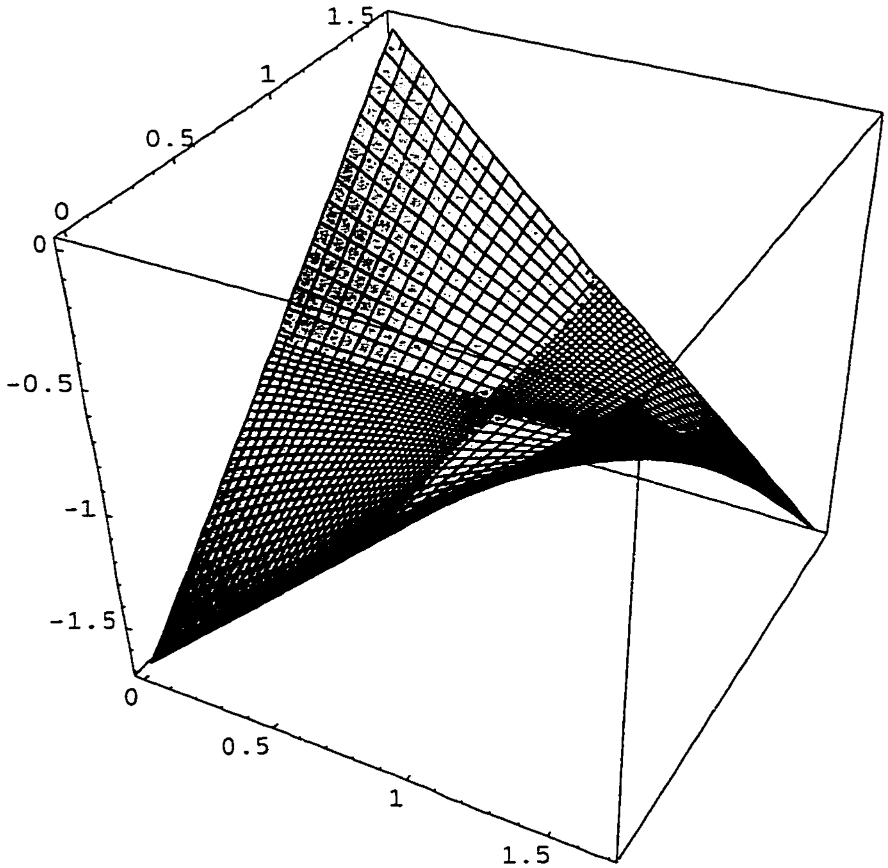
- Graphics3D -

```
Show[Bh1,Bh2,Bh3,Bh4,BoxRatios->{1,1,1}]
```




```
- Graphics3D -
```

```
Show[Bh1,Bh2,Bh3,Bh4,Bhp,BoxRatios->{1,1,1}]
```




```
- Graphics3D -
```

Appendix E

The Equations of Weierstrass

Darboux derived Weierstass's equations by using the notion of "curves of zero length". These are in fact asymptotic directions—directions along which the normal curvature, and hence also the second fundamental form, vanishes. By introducing an appropriate complex coordinate system one has $E{=}G{=}0$, $F{\neq}0$ along such curves. Darboux then proceeded to derive Weierstass's equations from this point of departure.

The following derivation, which was presented by Weierstrass in 1866 to the Akademie der Wissenschaften, is more straightforward. He begins by introducing local isothermal coordinates $x(p,q)$, $y(p,q)$, $z(p,q)$ satisfying

$$\left(\frac{\partial x}{\partial p}\right)^2+\left(\frac{\partial y}{\partial p}\right)^2+\left(\frac{\partial z}{\partial p}\right)^2=\left(\frac{\partial x}{\partial q}\right)^2+\left(\frac{\partial y}{\partial q}\right)^2+\left(\frac{\partial z}{\partial q}\right)^2=k,$$

$$\frac{\partial x}{\partial p}\frac{\partial x}{\partial q}+\frac{\partial y}{\partial p}\frac{\partial y}{\partial q}+\frac{\partial z}{\partial p}\frac{\partial z}{\partial q}=0.$$

Letting H be the mean curvature of the surface at (x,y,z) we have

$$H=\frac{1}{k^2}=\begin{vmatrix} \frac{\partial x}{\partial p} & \frac{\partial x}{\partial q} & \frac{\partial^2 x}{\partial p^2}+\frac{\partial^2 x}{\partial q^2} \\ \frac{\partial y}{\partial p} & \frac{\partial y}{\partial q} & \frac{\partial^2 y}{\partial p^2}+\frac{\partial^2 y}{\partial q^2} \\ \frac{\partial z}{\partial p} & \frac{\partial z}{\partial q} & \frac{\partial^2 z}{\partial p^2}+\frac{\partial^2 z}{\partial q^2} \end{vmatrix}.$$

The conditions $E{=}G{=}k$, $F{=}0$ imply that the three minors $\frac{\partial y}{\partial p}\frac{\partial z}{\partial q}-\frac{\partial z}{\partial p}\frac{\partial y}{\partial q}$, $\frac{\partial z}{\partial p}\frac{\partial x}{\partial q}-\frac{\partial x}{\partial p}\frac{\partial z}{\partial q}$, $\frac{\partial x}{\partial p}\frac{\partial y}{\partial q}-\frac{\partial y}{\partial p}\frac{\partial z}{\partial q}$ cannot vanish simultaneously, since the sum of the squares of these three minors is k^2. Thus the two column vectors on the left are linearly independent. Moreover, the third column vector is normal to the tangent plane. Therefore the condition $H{=}0$ implies

$$\frac{\partial^2 x}{\partial p^2}+\frac{\partial^2 x}{\partial q^2}=\frac{\partial^2 y}{\partial p^2}+\frac{\partial^2 y}{\partial q^2}=\frac{\partial^2 z}{\partial p^2}+\frac{\partial^2 z}{\partial q^2}=0.$$

This means the coordinate functions are harmonic. If we put $u{=}p + iq$, the above equations become

$$x = \Re f(u), \quad y = \Re g(u), \quad z = \Re h(u),$$

where f, g, and h are entire functions satisfying the equation

$$(f'(u))^2 + (g'(u))^2 + (h'(u))^2 = 0.$$

This last condition can be expressed as follows for arbitrary entire functions Φ, Ψ:

$$\begin{aligned} f'(u) &= \Phi^2 - \Psi^2, \\ g'(u) &= i(\Phi^2 + \Psi^2), \\ h'(u) &= 2\Phi\Psi. \end{aligned}$$

From the equation

$$\left(\frac{1}{2}f' - \frac{1}{2}ig'\right)\left(-\frac{1}{2}f' - \frac{1}{2}ig'\right) = \frac{1}{4}(h')^2.$$

it is clear that one of the factors on the left-hand side can vanish at a point a only when h' vanishes, and only that factor whose power series expansion in powers of $u - a$ begins with an even power. This is only possible if Φ, Ψ are functions that never vanish simultaneously. One can now write the above equations as follows (using the notation u^0 for a fixed value of u and x_0, y_0, z_0 the corresponding point of the surface):

$$\begin{aligned} x &= x_0 + \Re \int_{u^0} (\Phi^2(u) - \Psi^2(u))du, \\ y &= y_0 + \Re \int_{u^0} i(\Phi^2(u) + \Psi^2(u))du, \\ z &= z_0 + \Re \int_{u^0} 2\Phi(u)\Psi(u)du. \end{aligned}$$

Finally, in place of u we define

$$s = \frac{\Psi(u)}{\Phi(u)} = \frac{f' + ig'}{-h'}, \text{ which means that}$$

$$\frac{1}{s} = \frac{f' - ig'}{h'}.$$

By shrinking the domain, if necessary, so that s is a 1-1 function of u (and thus also u is a 1-1 function of s) we conclude that the function

$$\Phi^2(u)\frac{du}{ds} = \frac{1}{2}(f' - ig')\frac{du}{ds} = \mathcal{F}(s)$$

is also 1-1, and we have

$$\begin{aligned} x &= x_0 + \Re \int_{s^0} (1 - s^2)\mathcal{F}(s)ds, \\ y &= y_0 + \Re \int_{s^0} i(1 + s^2)\mathcal{F}(s)ds, \\ z &= z_0 + \Re \int_{s^0} 2s\mathcal{F}(s)ds. \end{aligned}$$

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