THE REAL PART OF A MULTIVARIABLE POSITIVE REAL FUNCTION AND THE SYNTHESIS OF MULTIVARIABLE LADDER NETWORKS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF IMPORTANT SYMBOLS AND NOTATIONS

MHPB	Multivariable Hurwitz Polynomial in Broad Sense
MHPN	Multivariable Hurwitz Polynomial in the Narrow Sense
MLPL	Multivariable Low-Pass Ladder Network
MPRF	Multivariable Positive Real Function
MRF	Multivariable Reactance Function
PRF	Positive Real Function
$s, p_1, p_2, \ldots, p_n$	Complex Frequency Variables
TLPL	Two-Variable Low-Pass Ladder Network.
UE	Unit element
τ	Time delay of a Uniform Lossless Transmission Line
D_{p_i}	$\dfrac{\partial D(p_1, p_2, \ldots, p_n)}{\partial p_i}$
$\hat{D}$	$\sum_{i=1}^{n} \gamma_i \cdot D_{p_i}$
N_{p_i}	$\dfrac{\partial N(p_1, p_2, \ldots, p_n)}{\partial p_i}$
$\hat{N}$	$\sum_{i=1}^{n} \gamma_i \cdot N_{p_i}$
Q_{p_i}	$\dfrac{\partial Q(p_1, p_2, \ldots, p_m)}{\partial p_i}$
$\hat{Q}$	$\sum_{i=1}^{m} \delta_i \cdot Q_{p_i}$

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ABSTRACT

This thesis is concerned with the study of the properties of multivariable positive real functions (MPRFs) and the realization of multivariable ladder networks.

Generation of MPRFs by the differential operator and from the given multivariable real part are studied. The differential operator method provides various means of generating an n-and (n-1)-variable positive real functions (PRFs) from the given n-variable PRF as well as a necessary test on the coefficients of an MPRF. The problem of generation of an MPRF from the prescribed multivariable non-negative real part is investigated. The numerator and the denominator polynomials of the real part have to satisfy certain conditions. Further, a method of testing a two-variable polynomial for non-negativeness is developed by extending the single-variable Sturm test.

A multivariable array is proposed by means of which the realizability conditions for the multivariable low-pass ladder network - consisting of p_i-type ($1 \leq i \leq n$) of inductors in series in the series arms and p_i-type of capacitors in parallel in the shunt arms - are given. By suitable transformations, various other ladder networks are derived. The realizability conditions for the resistively terminated lossless ladder networks are also established.

An equivalence relation is developed between a cascade of resistively terminated uniform lossless transmission lines separated by lumped series inductor on one side and lumped shunt capacitor on the other side and a two-variable low-pass ladder network. This enables the realizability conditions for such a mixed lumped-distributed structure to be derived

in terms of the two-variable low-pass ladder network.

The realization of a class of two-variable reactance networks in a form similar to the single-variable Foster form is studied. The equivalent ladder and unsymmetrical lattice structures are also developed. The realizability conditions for these networks are derived directly from the network functions rather than from the multivariable array.

CHAPTER I

INTRODUCTION

1.1 General:

Linear, lumped, finite, passive networks may be completely characterized by rational functions of the complex frequency variable s. A great deal of literature is available on the properties and realization of such network functions. However, if we allow the network to include distributed elements as well as lumped elements, the network functions are in general no longer rational in s. For example, if the network consists of lumped reactances in addition to commensurate lossless uniform transmission lines (unit elements, UEs), the network functions are irrational. Similarly, if the network consists of non-commensurate lengths of transmission lines described by $p_i = \tanh \tau_i s$, where the different τ_i's are not rationally related, the network functions once again are irrational in any one of the variables p_i. Such mixed lumped-distributed* structures are highly desirable as may be seen from the following discussion.

1.2 Mixed Lumped-Distributed Structures:

There are many useful circuits containing both lumped and distributed elements. Common examples are networks containing semiconductor elements and transmission lines or wave guides, or indeed networks of transmission lines alone where lumped discontinuities inevitably occur. And for a variety of reasons it is desirable to include the lumped lossless elements in the microwave networks. In particular the following points may be claimed as advantages for their inclusion:

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* In the literature, the terms "Mixed Lumped-Distributed" and "Lumped-Distributed" are used synonymously. However, the term "Mixed Lumped-Distributed" appears to be more commonly used. As such, this term is used throughout this thesis.

(i) In the case of mixed lumped-distributed structures, allowance may be made for the parasitics in the terminating impedances, that is, the terminations need not be purely resistive. Levy[1] has shown how one or two parasitic elements across a terminating resistor may be absorbed in the filter design.

(ii) A conventional quarter-wave transformer gives small or no attenuation at the higher harmonics. On the other hand, the mixed lumped-distributed impedance transformer can be designed to have the properties of an impedance transformer and a low-pass filter. This property is useful where combined filtering and impedance transformation is desirable.

(iii) In the case of comb-line filters, lumped capacitive coupling at the input and output reduces the filter size by eliminating the transmission line matching section[2].

(iv) In the case of cascaded UL filters, the number of UEs can be reduced from (2n+1) to n if the cascaded lines are separated by (n+1) lumped capacitors.

Thus, we see that there are certain definite advantages of the mixed lumped-distributed networks, and hence the study of such structures is essential. However, the realization techniques available in the lumped network theory are not directly applicable in such cases due to the transcendental nature of the network functions. In trying to solve the realization problem of these mixed lumped-distributed networks, basically two different approaches are followed. One of them directly deals with transcendental functions which we term as the single-variable approach. In the other approach, the transcendental functions of s are converted into polynomial functions of several variables s and p_i. For

obvious reasons, we term this as the multivariable approach. A brief review of both these approaches follows.

1.3 Realization by Single-Variable Approach:

For a class of distributed structures, viz: commensurate uniform transmission lines, by suitable transformations the network functions can be transformed into rational functions of the transformed variable. Hence, the realization methods of the linear, lumped, finite, passive networks can be applied for the synthesis of such a class of distributed networks. In particular, by introducing $p = \tanh \tau s$ (where τ is the time-delay of the transmission line), Richards[3] has shown that the input immittance of a network consisting of finite number of lumped resistors, transformers and UEs of commensurate lengths is a positive real function of p.

Kinariwala[4] has derived the realizability conditions for a resistively terminated cascade of commensurate or non-commensurate UEs. The input impedance function of such a structure is expressed as a ratio of sums of exponentials, the coefficients of the exponentials being real constants. A procedure for the realization of such a network is also given.

Using Kinariwala's algorithm as a guide, Riederer[5] deals with the realization of a resistively terminated cascade connection of commensurate UEs separated by passive lumped lossless two-port networks. The realizability conditions are obtained in terms of the specification on the ensignant (numerator of the even part of the function) and a matrix formed from the polynomial coefficients of the input impedance in addition to the positive real criterion of the input impedance function.

Carlin and Gupta[6] have proposed a method for realizing prescribed insertion loss characteristics either as a structure consisting of a cascade of UEs with frequency dependent characteristic impedances operating between pure resistances or as a structure consisting of a cascade of UEs with constant characteristic impedances operating between frequency dependent loads. The method utilizes scattering matrix normalization and is implemented by a computer program.

More recently, Gupta[7] has presented a numerical algorithm for the design of equiripple low-pass filters consisting of mixed lumped-distributed cascade structures. For the filter considered, the number of ripples in the pass band is equal to the number of distributed elements. The elemental values of the filter are arrived at by the solution of a set of simultaneous non-linear equations, obtained from the properties of the equiripple filter.

Levy[8] proposed a distributed prototype filter consisting of a cascade of shunt stubs of equal electrical lengths alternating with UEs, each having twice the stub length. Depending upon whether the stubs are open-circuited or short-circuited, they may be replaced by lumped capacitors or inductors to synthesize a mixed lumped-distributed filter.

The methods discussed thus far assume cascaded configurations for the mixed lumped-distributed networks. To the best of the author's knowledge, no attempt has been made to treat the arbitrary interconnection of the lumped and distributed elements by this approach.

1.4 Realization by Multivariable Approach:*

The other widely used approach is to treat the mixed lumped-distributed structures by the multivariable network functions. By this approach, the problem of arbitrary interconnection of lumped and distributed structures, as well as the particular case of cascaded UEs separated by lumped passive lossless two-ports are extensively studied. The available literature on this approach may be broadly classified as follows: (i) General Synthesis, (ii) Cascade Synthesis and (iii) Approximation. The general synthesis covers the properties and realization of a given positive real matrix of two-or n-variables. Considerable amount of work has been done in this topic. In cascade synthesis, attempts have been made to solve the problem of realizing a multivariable positive real function of arbitrary degree in each variable as a tandem connection of resistively terminated commensurate and/or non-commensurate lines separated by lumped lossless two-ports. However, the problem has been solved only for the case of two-variable functions where the degree of each variable is arbitrary. Not much work has been done in the topic of approximation.

Before proceeding with the review of the literature in this area, we will present certain accepted definitions and properties of such network functions.**

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* The theory of multivariable functions has been applied for the synthesis of variable networks[32,33] and in the design of two-dimensional digital filtering as well[34,35]. But, in this thesis no attempt is made to review the literature in these areas.

** It may be noted that these definitions are logical extensions of those for the corresponding single-variable functions in Brune sense.

Definition 1.1:

A rational function $F(p_1,p_2,\ldots,p_n)$ of n-complex variables $p_1,p_2,\ldots,p_n$ is called a Multivariable Positive Real Function (MPRF) when the following conditions are satisfied:

(i) $F(p_1,p_2,\ldots,p_n)$ is a real function of $p_1,p_2,\ldots,p_n$, and

(ii) $\mathrm{Re}\ F(p_1,p_2,\ldots,p_n) \geq 0$ in the polydomain $\mathrm{Re}\ p_i \geq 0$, $i = 1,2,\ldots,n$.

Definition 1.2:

A rational function $F(p_1,p_2,\ldots,p_n)$ is called a Multivariable Reactance Function (MRF) when the following conditions are satisfied:

(i) $F(p_1,p_2,\ldots,p_n)$ is an MPRF, and

(ii) $F(p_1,p_2,\ldots,p_n) = -F(-p_1,-p_2,\ldots,-p_n)$

Definition 1.3:

A polynomial of n-complex variables $p_1,p_2,\ldots,p_n$ is called a Multivariable Hurwitz Polynomial in Narrow Sense (MHPN) if it has no zeros in the regions $\mathrm{Re}\ p_1 > 0,\ldots,\mathrm{Re}\ p_{i-1} > 0$, $\mathrm{Re}\ p_i \geq 0$, $\mathrm{Re}\ p_{i+1} > 0,\ldots,$ $\mathrm{Re}\ p_n > 0$ for all i $(1 \leq i \leq n)$.

Definition 1.4:

A polynomial of n-complex variables $p_1,p_2,\ldots,p_n$ is called a Multivariable Hurwitz Polynomial in Broad Sense (MHPB) if it has no zeros in the open polydomain $\mathrm{Re}\ p_i > 0$, and those zeros for $\mathrm{Re}\ p_i = 0$ must be simple.

Property 1.1: [17]

The numerator and the denominator of an MPRF, prescribed ~~in the~~ irreducible form, are MHPBs.

Property 1.2:

The ratio of even to odd (odd to even) of an MHPB is an MRF.

1.4.1 Realization of General Mixed Lumped-Distributed Structures:

Ozaki and Kasami[9], motivated by the earlier work of Levenstein[10] on variable networks, initiated the study of network functions of several variables. In particular they reported some properties and realizations of such network functions.

Ansell[11] has shown that the driving point immittance of a finite network containing commensurate length UEs and lumped reactances is a two-variable reactance function in p_1, p_2 where $p_1 = s$ and $p_2 = \tanh \tau s$. In addition, he provided a method of testing for the reactance property of two-variable rational functions. He has also given a synthesis method for two-variable reactance functions based upon the single-variable Lunelli's[12] method. This synthesis method requires a two-variable polynomial decomposition with certain properties. As Ansell himself has pointed out, the procedure has one drawback: even though the decomposition is assured by network realizability conditions there is no algorithm available for the decomposition.

The next advance was made by Koga[13], who succeeded in synthesizing any rational $m \times m$ two-variable reactance matrix $Z(p_1, p_2)$ as the impedance matrix of a reactive m-port containing p_1-type and p_2-type reactances. Any such reactance matrix may always be realized by a passive two-variable m-port which is bilateral or non-bilateral depending upon whether $Z(p_1, p_2)$ is symmetric or non-symmetric. This method leans heavily on the theory of algebraic functions and the structure of para-unitary matrices and does not guarantee the use of minimum number of elements.

Youla[14] solved the problem of synthesizing a two-variable bounded real $m \times m$ scattering matrix by extending an algebraic theorem due to Kalman[15] concerning the decomposition of rational matrices of a single-variable. The bounded real scattering matrix $W(p_1,p_2)$ which is finite at $p_2 = \infty$ possesses a decomposition of the form

$$W(p_1,p_2) = J(p_1) + H(p_1)[p_2 \cdot I_k - F(p_1)]^{-1} G(p_1)$$

where $J(p_1)$, $H(p_1)$, $F(p_1)$ and $G(p_1)$ are real rational matrices of appropriate orders and are identified with the scattering parameters of a $(m + 2\ell)$-port passive lumped network. The 2ℓ-ports of the passive lumped network are terminated by ℓ equidelay UEs. This method demonstrates that a regular para-unitary scattering matrix may be realized with absolute minimum number of lumped reactances and UEs.

T.N. Rao[16], by defining $p_1 = \sqrt{s}$ and $p_2 = \tanh \frac{\sqrt{rcs}}{2} \ell$ (where r and c are respectively the per unit length resistance and capacitance of a uniform RC-line of length ℓ), has shown that the mixed lumped-distributed network consisting of lumped resistors, capacitors and commensurate uniform RC-lines is equivalent to a two-variable reactance network in p_1 and p_2. He has given a method of realizing a given $m \times m$ two-variable reactance matrix, utilizing a matrix decomposition similar to that of Youla. This method always yields the minimum number of elements in p_1 and p_2.

All the three methods discussed above deal with matrices of two-variables. In a later contribution, Koga[17] has shown that an $m \times m$

positive real matrix of several variables is realizable as the immittance matrix of a finite passive multivariable m-port network.* This method consists of the following steps:

(i) The given $m \times m$ positive real matrix $W(p_1, p_2, \ldots, p_n)$ is converted into an $m \times m$ reactance matrix $Z(\mu_1, \mu_2, \ldots, \mu_r)$, whose elements are bilinear in $\mu_i (1 \leq i \leq r)$.

(ii) From the derived matrix $Z(\mu_1, \mu_2, \ldots, \mu_r)$, an $m \times m$ polynomial matrix $F(\mu_1, \mu_2, \ldots, \mu_{r-1})$ which is real for real μ_i, quadratic in each μ_i and non-negative Hermitian for $\mathrm{Re}\, \mu_i = 0$, is obtained.

(iii) There exists a matrix decomposition

$$F(\mu_1, \mu_2, \ldots, \mu_{r-1}) = M(\mu_1, \mu_2, \ldots, \mu_{r-1}) \cdot M_t(-\mu_1, -\mu_2, \ldots, -\mu_{r-1})$$

where t indicates the transposition, and M's are of appropriate order.

(iv) From the matrices M's and Z, the immittance parameters of an (m + q)-port network of $\mu_1, \mu_2, \ldots, \mu_{r-1}$ are obtained, whose q-ports are terminated in μ_r-type inductors. The procecure is repeated until a network of μ_1-type is obtained, which can be synthesized by the single-variable methods.

(v) By replacing the μ_i-type $(1 \leq i \leq r)$ of elements with the appropriate p_j-type $(1 \leq j \leq n)$ of elements, the required network is obtained.

The foregoing methods require either ideal transformers or ideal gyrators or both. However, some work has been done to realize an MPRF without transformers. Soliman and Bose[19] have given sufficient conditions utilizing Saito's[20] generalization of Richards' transformation for

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* A brief review on the synthesis of rational multivariable positive real matrices is presented by Youla[18].

transformerless realization of MPRFs. They have followed the single-variable Bott-Duffin method. More recently, Kamp and Belevitch[21] presented the necessary and sufficient conditions under which an MPRF of first degree in all variables except one, can be realized by Bott-Duffin method. Both of the above mentioned methods point out that although the Bott-Duffin method can be used to realize any given PRF of a single-variable as a transformerless one-port, this is not the case for MPRFs.

The foregoing methods pertain to the realization of an arbitrary interconnection of the multivariable network elements. Extensive work has also been done on the problem of realizing multivariable network functions as a resistively terminated cascade of commensurate or non-commensurate, uniform, lossless, transmission lines and lumped, passive, lossless two-port networks.

1.4.2 Realization of Cascade of Mixed Lumped-Distributed Structures:

Cascades of mixed lumped-distributed structures are particularly important in the design of microwave filters using TEM mode, with or without lumped discontinuities, networks containing semiconductor elements and commensurate transmission lines, and acoustic filters using pipes etc. Various authors[22-31] have studied these cascaded structures using the multivariable approach. However, each author has considered a pre-assumed structure. A brief review follows:

Ansell[22] has given the realizability conditions for a symmetrical cascaded structure consisting of commensurate UEs with a lumped shunt capacitor at its center. Saito[20] has by extending Richards' transformation to the case of multivariables, studied the realization of the following structures:

(i) Cascaded commensurate UEs terminated by a lumped reactance

(ii) Cascaded non-commensurate UEs terminated by a resistor, and

(iii) Cascaded commensurate UEs terminated by a resistor and shunted by lumped reactances.

Scanlan and Rhodes[23] have presented the realizability conditions under which an n-variable positive real function bilinear in (n-1)-variables may be realized as a cascade of passive, lumped lossless two-port network separated by non-commensurate UEs and terminated in a positive resistor.

Shirikawa, Takahasi and Ozaki[24] discussed the problem of synthesizing some specific structures consisting of cascaded UEs and open-circuited stubs. These networks can generate more general class of network functions than those above mentioned contributions but still somewhat restrictive in nature.

Youla and Ott[25] established conditions for the realizability of a driving point impedance as a cascade containing at most two commensurate UEs and three shunt capacitors. However, it is difficult to extend this method to cascade structures having more number of elements. Uruski and Piekarski[26] have given realizability conditions for a resistively terminated cascade of commensurate UEs separated by shunt lumped capacitors or series lumped inductors. Even though they do not explicitly mention the insufficiency of the two conditions viz: Positive realness of the driving point function and the corresponding even part condition, they imply the requirement for some additional conditions on the driving point function for such cascaded structures.

Kamp[27] gave the realizability conditions for an MPRF of arbitrary degree in each variable to be realizable as a cascade of non-commensurate UEs terminated in a positive resistor. In fact, he has shown these conditions to be equivalent to Pinariwala's[4] conditions that were derived on the single-variable basis. Recently Koga[28] has presented the realizability conditions for a resistively terminated cascade of UEs, commensurate or non-commensurate and lumped passive lossless two-port networks in an arbitrary sequence. Subsequently Rhodes and Marston[29] have shown the insufficiency of Kamp's and Koga's conditions by a counter example following which, Youla, Rhodes and Marston[30,31] have presented the realizability conditions for a resistively terminated cascade of commensurate UEs separated by lumped passive lossless two-port networks.

There could be many other possibilities of cascaded structures, each of them requiring a different method of solution. It may be pointed out that all the cascade realization methods discussed above make use of Richards' transformation[20]. Also, the realizability conditions are given in terms of the even part of the MPRF and its polynomial coefficients. Finally, we note that some of the cascade realizations and most of the general synthesis techniques available in the literature require either ideal transformers or ideal gyrators or both.

1.5 Scope of the Thesis:

This thesis aims at the study of the even part of an MPRF and gives the realization of a class of cascaded mixed lumped-distributed structures without taking recourse to Richards' transformation. Also the realization of a class of network functions by the multivariable ladder networks is studied. These networks require neither the ideal transformers

nor the ideal gyrators.

In Chapter II, the problem of generating an MPRF by the differential operator and from the given multivariable real part is considered. The general conditions under which an MPRF can be generated from the given real part are studied and some particular cases are treated in greater detail. A procedure of testing a two-variable real polynomial for non-negativeness is given by extending the single-variable Sturm test.

Chapter III introduces a multivariable array which reduces to the Routh-Hurwitz array for the case of a single-variable. By means of this array, the realizability conditions of multivariable low-pass ladder networks (MLPL, consisting of series p_i-type inductors in the series arms and p_i-type of capacitors in parallel in the shunt arms) are obtained. By suitable transformations, other kinds of ladder networks are derived, the realizability conditions for them being given in terms of the MLPL. Realizability conditions for resistively terminated ladder networks are also derived.

In Chapter IV, some equivalence relations are derived between lumped, low-pass, lossless, ladder networks and mixed lumped-distributed networks. Incorporating these equivalence relations, the realizability conditions for a resistively terminated cascade of commensurate UEs separated by lumped shunt capacitor on one side and lumped series inductor on the other are given in terms of the two-variable low-pass ladder network (TLPL), the realizability conditions of which are already known from the two-variable array. Also a method of realizing a cascade of non-commensurate UEs separated by lumped lossless two-ports is given.

In Chapter V, we discuss the realization of a class of two-variable reactance functions as structures which are similar to the single-variable Foster form. From this the equivalent ladder and unsymmetrical lattice structures are developed, and it is shown that these networks are a class of TLPL. The partial polynomial derivatives of such network functions are also studied. Utilizing the above mentioned ladder networks, transformerless synthesis method for a class of two-variable PRFs similar to the single-variable Miyata method is proposed.

CHAPTER II

THE MULTIVARIABLE POSITIVE REAL FUNCTIONS AND SOME PROPERTIES

2.1 Introduction:

Several theorems concerning the generation and properties of single-variable and multivariable positive real functions are proved in the literature[9,17,22,36-40]. In particular, the following results closely connected with the present investigation have been proved[17] using the operation of differentiation:

(i) The partial logarithmic derivative $\frac{N_{p_i}}{N}$, $1 \leq i \leq n$, (where N is an MHPB*) of an MHPB in an MPRF.

(ii) A multivariable positive real matrix (MPRM) remains an MPRM under partial polynomial differentiation.

Such results do exist for the single-variable case also.

In the case of a single-variable, there are several methods of generating a positive real function from the given real part[41]. It is

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* The following mathematical notations have been used throughout the thesis:

$$N(p_1,p_2,\ldots,p_n) = N \qquad \frac{\partial N(p_1,p_2,\ldots,p_n)}{\partial p_i} = N_{p_i}$$

$$D(p_1,p_2,\ldots,p_n) = D \qquad \frac{\partial D(p_1,p_2,\ldots,p_n)}{\partial p_i} = D_{p_i} \qquad 1 \leq i \leq n$$

$$Q(p_1,p_2,\ldots,p_m) = Q$$

Multivariable Hurwitz Polynomial in Broad Sense: MHPB

$$\sum_{i=1}^{n} \gamma_i N_{p_i} = \hat{N} \; ; \quad \sum_{i=1}^{n} \beta_i D_{p_i} = \hat{D} \; ; \quad \sum_{i=1}^{m} \delta_i Q_{p_i} = \hat{Q}$$

and it is assumed that γ's, β's and δ's are positive constants.

known that, in the case of single-variable, if the given real part is non-negative for all real frequencies, a positive real function can always be generated an turm test can be applied for testing the non-negativeness of such functions.

In this chapter, several methods of generating MPRFs and a method of testing the real part of a two-variable function for its non-negativeness are presented. Section 2.2 deals with the generation of MPRFs by the differential operator. In section 2.3, the generation of an MPRF from the given real part is discussed. The conditions under which an MPRF can be generated from the given real part are studied. In section 2.4, a method of testing the real part of a two-variable function for its non-negativeness is presented.

2.2 Generation of MPRFs by Differential Operator[42]:

We present the key theorem to this section, which is the multivariable version of Talbot's theorem[38,39]. Utilising this theorem, several methods of generating MPRFs using the Differential Operator are studied.

Theorem 2.1:

The multivariable real rational function $\frac{N}{D}$ (or $-\frac{N}{D}$) is reduced and positive, if and only if $(N + j\alpha D)$ is an MHPB, where α is any real, non zero, finite constant and the term "reduced" here means that N and D have no common factors in the open polydomain Re $p_i > 0$ $(1 \leq i \leq n)$.

Proof: Necessity:

If $\frac{N}{D}$ (or $-\frac{N}{D}$) is a reduced MPRF, N and D have no zeros in Re $p_i > 0$ and hence $(N + j\alpha D) \neq 0$ in Re $p_i > 0$. Suppose $N + j\alpha D = 0$ in Re $p_i > 0$, then Re $\frac{N}{D} = 0$ there, which is not possible and hence the necessity follows.

Sufficiency:

Assume that $(N + j\,\alpha\,D)$ is an MHPB. Then $D \neq 0$ in $\mathrm{Re}\ p_i > 0$. If $D = 0$ in $\mathrm{Re}\ p_i > 0$, then for large values of α, $N + j\,\alpha\,D = 0$ contrary to the hypothesis. Hence $\frac{N}{D}$ is regular and not imaginary in $\mathrm{Re}\ p_i > 0$; that is $\mathrm{Re}\ \frac{N}{D} \neq 0$ in $\mathrm{Re}\ p_i > 0$. Since $\frac{N}{D}$ is a continuous function of the variables $p_1, p_2, \ldots, p_n$, and if the sign of $\mathrm{Re}\ \frac{N}{D}$ is opposite at two sets of points $(p_{10}, p_{20}, \ldots, p_{n0})$ and $(p'_{10}, p'_{20}, \ldots, p'_{n0})$ in $\mathrm{Re}\ p_i > 0$, then there must be at least one set of points in the open polydomain where $\mathrm{Re}\ \frac{N}{D}$ is zero, which is not possible. Hence, $\pm\ \mathrm{Re}\ \frac{N}{D} > 0$ in $\mathrm{Re}\ p_i > 0$. Thus the theorem follows.

Corollary 2.1.1:

If $(N + j\,\alpha\,D)$ is an MHPB, so is $(N - j\,\alpha\,D)$ and hence their product $(N + j\,\alpha\,D)\ (N - j\,\alpha\,D) = (N^2 + \alpha^2\,D^2)$ is also an MHPB.

This is the multivariable version of Brockett's theorem[43] for the case of a single-variable.

Theorem 2.2:

If N/D is a reduced MPRF, so are (provided they exist)

(i) the partial polynomial derivatives, i.e. $\frac{N_{p_i}}{D_{p_i}}$ $(1 \leq i \leq n)$

(ii) the iterated partial polynomial derivatives, i.e. $\frac{\partial^{m_i} N/\partial p_i^{m_i}}{\partial^{m_i} D/\partial p_i^{m_i}}$

and (iii) the mixed partial polynomial derivatives, i.e.

$$\frac{\partial^k N/\partial p_1^{a_1} . \partial p_2^{a_2} . \ldots . \partial p_n^{a_n}}{\partial^k D/\partial p_1^{a_1} . \partial p_2^{a_2} . \ldots . \partial p_n^{a_n}} \text{ where } a_1 + a_2 + \ldots + a_n = k$$

Proof:

(i) If N/D is an MPRF, by Theorem 2.1, we have $(N + j\,\alpha\, D)$ is an MHPB for any real α. Since the partial derivative of an MHPB is also an MHPB, $(N_{p_i} + j\,\alpha\, D_{p_i})$ is an MHPB and hence by Theorem 2.1 $\pm \frac{N_{p_i}}{D_{p_i}}$ is an MPRF. To determine the correct sign, let N and D be expressed as polynomials of p_i,

$$\text{i.e. let } \frac{N}{D} = \frac{c_0 + c_1 p_i + c_2 p_i^2 + \dots + c_r . p_i^r}{d_0 + d_1 p_i + d_2 p_i^2 + \dots + d_{r'} . p_i^{r'}}$$

where the c's and d's are polynomials in $p_1, p_2, \dots, p_{i-1}, p_{i+1}, \dots, p_n$. We know that $r - r' = \pm 1$ or 0 and $\frac{c_r}{d_{r'}}$ is an (n-1)-variable PRF or a positive constant[9]. Then

$$\pm \frac{N_{p_i}}{D_{p_i}} = \pm \frac{c_1 + 2c_2 . p_i + \dots \dots + r . c_r . p_i^{r-1}}{d_1 + 2d_2 . p_i + \dots \dots + r' . d_{r'} . p_i^{r'-1}}$$

If $\frac{c_r}{d_{r'}}$ has to remain positive, then $-\frac{N_{p_i}}{D_{p_i}}$ cannot be positive. Hence $+\frac{N_{p_i}}{D_{p_i}}$ is the MPRF.

This furnishes an alternate proof for Koga's[17] result for an MPRF.

(ii) If N/D is a reduced MPRF, then we have $\frac{N_{p_i}}{D_{p_i}}$ as a reduced MPRF by the above proposition. Then $(N_{p_i} + j\,\alpha\, D_{p_i})$ is an MHPB for any

real α. Hence $\left(\frac{\partial^2 N}{\partial p_i^2} + j\,\alpha\,\frac{\partial^2 D}{\partial p_i^2}\right)$ is also an MHPB. By a process similar to the one followed in the above proposition, we conclude that $\frac{\partial^2 N/\partial p_i^2}{\partial^2 D/\partial p_i^2}$ is an MPRF. Thus, by repeating this process m_i times, proposition (ii) is proved.

(iii) For a reduced MPRF N/D, we have $\left(\frac{\partial^{a_1} N}{\partial p_1^{a_1}} + j\,\alpha\,\frac{\partial^{a_1} D}{\partial p_1^{a_1}}\right)$ as an MHPB.

Then, the partial derivative w.r.t. p_2, $\left(\frac{\partial^{a_1+1} N}{\partial p_1^{a_1}.\partial p_2} + j\,\alpha\,\frac{\partial^{a_1+1} D}{\partial p_1^{a_1}.\partial p_2}\right)$ is also an MHPB. Similarly the polynomial

$$\left(\frac{\partial^k N}{\partial p_1^{a_1}.\partial p_2^{a_2}.\ \ldots\ .\partial p_n^{a_n}} + j\,\alpha\,\frac{\partial^k D}{\partial p_1^{a_1}.\partial p_2^{a_2}.\ \ldots\ .\partial p_n^{a_n}}\right) \text{ where } a_1+a_2+..+a_n=k$$

is also an MHPB. Hence by proposition (i), we can conclude that

$$+\ \frac{\partial^k N/\partial p_1^{a_1}.\partial p_2^{a_2}.\ \ldots\ .\partial p_n^{a_n}}{\partial^k D/\partial p_1^{a_1}.\partial p_2^{a_2}.\ \ldots\ .\partial p_n^{a_n}} \text{ is an MPRF.}$$

<u>Theorem 2.3:</u>

If $\frac{N}{D}$ is a reduced MPRF, so are

(i) $\frac{\hat{N}}{\hat{D}}$

and (ii) $\frac{N + \beta.\hat{N}}{D + \beta.\hat{D}}$

Proof:

Since N is an MHPB, $\frac{\gamma_1 N_{p_1}}{N} + \frac{\gamma_2 . N_{p_2}}{N} + \ldots + \frac{\gamma_n . N_{p_n}}{N} = \frac{\sum_{i=1}^{n} \gamma_i N_{p_i}}{N}$

$= \frac{\hat{N}}{N}$ is an MPRF. Hence, $\hat{N}$ and $(\hat{N} + \beta.\hat{N})$ are MHPBs for positive constant β. Similarly $\hat{D}$ and $(\hat{D} + \beta.\hat{D})$ are MHPBs. Since $(N + j\,\alpha\,D)$ is an MHPB, $(\hat{N} + j\,\alpha\,\hat{D})$ and $[(\hat{N} + \beta.\hat{N}) + j\,\alpha\,(\hat{D} + \beta.\hat{D})]$ are also MHPBs.

Therefore by Theorems 2.1 and 2.2

(i) $\hat{N}/\hat{D}$

and (ii) $\frac{(\hat{N} + \beta.\hat{N})}{(\hat{D} + \beta.\hat{D})}$

are MPRFs.

Corollary 2.3.1:

If $\frac{N}{D}$ is a reduced MRF, so is $\hat{N}/\hat{D}$.

Proof:

For N even (odd), N_{p_i} $1 \leq i \leq n$ are odd (even) and for D odd (even), D_{p_i} are even (odd) polynomials. Hence $\hat{N}$ and $\hat{D}$ are respectively the odd (even) and even (odd) polynomials. Thus $\hat{N}/\hat{D}$ which is an MPRF, is an odd function and hence an MRF.

Corollary 2.3.2:

If N/D is a reduced MPRF (MRF), so is

$$\frac{N + \beta_1.\hat{D}}{D + \beta_2.\hat{N}}$$

Proof:

We have $$\frac{N + \beta_1 . \hat{D}}{D + \beta_2 . \hat{N}} = \frac{1}{\frac{D}{N} + \beta_1 . \frac{\hat{N}}{N}} + \frac{1}{\frac{D}{\beta_1 . \hat{D}} + \frac{\beta_2}{\beta_1} . \frac{\hat{N}}{\hat{D}}} .$$

As the individual terms on the right hand side are MPRFs (MRFs) their sum is an MPRF (MRF).

Corollary 2.3.3:

If $\frac{N}{D}$ is a reduced MPRF (MRF), so is

$$\frac{N.\hat{D} + \beta_1 . D . \hat{N}}{D.\hat{D} + \beta_2 . N . \hat{N}}$$

Proof:

We have $$\frac{N.\hat{D} + \beta_1 . D . \hat{N}}{D.\hat{D} + \beta_2 . N . \hat{N}} =$$

$$\frac{1}{\frac{D}{N} + \beta_2 . \frac{\hat{N}}{\hat{D}}} + \frac{1}{\frac{\beta_2 . N}{\beta_1 . D} + \frac{\hat{D}}{\beta_1 . \hat{N}}}$$

Since the individual terms on the right hand side are MPRFs (MRFs), their sum is also an MPRF (MRF).

Theorem 2.4:

If $\frac{N}{D}$ is a reduced MPRF, so are

(i) $$\frac{\beta . N . \hat{Q} + Q . \hat{N}}{\beta . D . \hat{Q} + Q . \hat{D}} \qquad ..(2.1)$$

and (ii) $$\frac{\beta.N.Q + \hat{N}.\hat{Q}}{\beta.D.Q + \hat{D}.\hat{Q}} \qquad ..(2.2)$$

where Q is an MHPB.

Proof:

(i) Since N and Q are MHPBs, $[\frac{\hat{N}}{N} + \beta.\frac{\hat{Q}}{Q}]$ is an MPRF. Hence $[Q.\hat{N} + \beta.N.\hat{Q}]$ is an MHPB. Similarly $[Q.\hat{D} + \beta.D.\hat{Q}]$ is also an MHPB.

For an MPRF N/D and MHPB Q we have $[\frac{\hat{N} + j\,\alpha\,\hat{Q}}{N + j\,\alpha\,Q} + \beta.\frac{\hat{Q}}{Q}]$ is a multivariable positive function. Hence $(Q.\hat{N} + \beta.N.\hat{Q}) + j\,\alpha\,(Q.\hat{D} + \beta.D.\hat{Q})$ is an MHPB and by Theorems 2.1 and 2.2

$$\frac{\beta.N.\hat{Q} + Q.\hat{N}}{\beta.D.\hat{Q} + Q.\hat{D}} \text{ is an MPRF.}$$

(ii) We can prove that $(\beta.N.Q + \hat{N}.\hat{Q}) + j\,\alpha\,(\beta.D.Q + \hat{D}.\hat{Q})$ is an MHPB. Hence by Theorems 2.1 and 2.2

$$\frac{\beta.N.Q + \hat{N}.\hat{Q}}{\beta.D.Q + \hat{D}.\hat{Q}} \text{ is an MPRF.}$$

Corollary 2.4.1:

If $\frac{N}{D}$ is an MRF and Q is an m-variable odd or even Hurwitz polynomial in Broad sense, then (2.1) and (2.2) are MRFs.

Proof:

Let us consider that N is an even and D and Q are odd polynomials. We can prove the same result on the same lines for the other combinations as well.

For the case under consideration $(\beta.N.\hat{Q} + Q.\hat{N})$ and $(\beta.D.Q + \hat{D}.\hat{Q})$ are multivariable even polynomials whereas, $(\beta.D.\hat{Q} + Q.\hat{D})$ and $(\beta.N.Q + \hat{N}.\hat{Q})$ are multivariable odd polynomials. Thus, the MPRFs (2.1) and (2.2) are odd functions and hence are MRFs.

Theorems 2.1 to 2.4 provide some methods of generating an MPRF from a given MPRF. In Theorem 2.5, we show a method of generating an (n-1)-variable PRF from a given n-variable PRF.

It may be noted that if[9]:

$$F(p_1,p_2,\ldots,p_n) = \frac{N}{D} = \frac{c_0 + c_1.p_1 + \ldots + c_r.p_1^r}{d_0 + d_1.p_1 + \ldots + d_{r'}.p_1^{r'}} \qquad ..(2.3)$$

is an MPRF, where the c_i and d_i are polynomials of $p_2,\ldots,p_n$ and c_r, $d_{r'} \not\equiv 0$, then

$r - r' = \pm 1$ or 0, and

$\frac{c_r}{d_{r'}}$ is a positive constant or (n-1)-variable PRF respectively.

Using this result and Theorem 2.1 we prove the following theorem.

<u>Theorem 2.5</u>:

If $F = \frac{N}{D}$ is an MPRF, then $\frac{c_i}{d_i}$ $(0 \leq i \leq r-1)$ are (n-1)-variable PRFs of $p_2,p_3,\ldots,p_n$.

<u>Proof</u>: <u>Case (a)</u>:

Let $r = r'$. Making $p_1 \to \frac{1}{s_1}$ transformation and keeping the other variables unchanged in (2.3), we have

$$F(s_1,p_2,\ldots,p_n) = \frac{N(s_1,p_2,\ldots,p_n)}{D(s_1,p_2,\ldots,p_n)} = \frac{c_0.s_1^r + c_1.s_1^{r-1} +\ldots+ c_r}{d_0.s_1^r + d_1.s_1^{r-1} +\ldots+ d_r} \quad ..(2.4)$$

is an MPRF of $s_1,p_2,\ldots,p_n$ and c_0, $d_0 \neq 0$.

From (2.3), $\frac{c_0}{d_0}$ is an (n-1)-variable PRF of $p_2,p_3,\ldots,p_n$. By Theorem 2.2, the partial polynomial derivative of $F(s_1,p_2,\ldots,p_n)$ w.r.t. s_1, namely

$$F_1(s_1,p_2,\ldots,p_n) = \frac{N_{s_1}}{D_{s_1}} = \frac{r.c_0.s_1^{r-1} +(r-1)c_1.s_1^{r-2} + \ldots+ c_{r-1}}{r.d_0.s_1^{r-1} +(r-1)d_1.s_1^{r-2} + \ldots+ d_{r-1}} \quad ..(2.5)$$

is an MPRF of $s_1,p_2,\ldots,p_n$.

Now making $s_1 \to \frac{1}{p_1}$ transformation on (2.5), we obtain

$$F_1(p_1,p_2,\ldots,p_n) = \frac{r.c_0 + (r-1)c_1.p_1 +\ldots+ c_{r-1}.p_1^{r-1}}{r.d_0 + (r-1)d_1.p_1 +\ldots+ d_{r-1}.p_1^{r-1}} \quad ..(2.6)$$

From this, $\frac{c_{r-1}}{d_{r-1}}$ is an (n-1)-variable PRF of $p_2,p_3,\ldots,p_n$.

Thus, by the process of repeated transformation and partial polynomial differentiation, we arrive at the conclusion that

$\frac{c_0}{d_0}, \frac{c_1}{d_1}, \ldots \ldots \frac{c_i}{d_i}, \ldots, \frac{c_{r-1}}{d_{r-1}}$ are (n-1)-variable PRFs.

<u>Case (b)</u>:

Let $r = r'+1$. Then (2.3) becomes:

$$F(p_1,p_2,\ldots,p_n) = \frac{c_0 + c_1.p_1 + \ldots + c_r.p_1^r}{d_0 + d_1.p_1 + \ldots + d_{r-1}.p_1^{r-1}} \qquad ..(2.7)$$

we know that[9], $\frac{c_r}{d_{r-1}}$ is identically equal to a positive constant.

Making $p_1 \to \frac{1}{s_1}$ transformation in (2.7), we have

$$F(s_1,p_2,\ldots,p_n) = \frac{c_0.s_1^r + c_1.s_1^{r-1} + \ldots + c_{r-1}.s_1 + c_r}{d_0.s_1^r + d_1.s_1^{r-1} + \ldots + d_{r-2}.s_1^2 + d_{r-1}.s_1} \qquad ..(2.8)$$

$\frac{c_0}{d_0}$ is an (n-1)-variable PRF of $p_2,p_3,\ldots,p_n$. By taking the partial derivatives of the numerator and denominator w.r.t. s_1 and then making $s_1 \to \frac{1}{p_1}$ transformation in (2.8), we get

$$F_1(p_1,p_2,\ldots,p_n) = \frac{r.c_0 + (r-1).c_1.p_1 + \ldots + c_{r-1}.p_1^{r-1}}{r.d_0 + (r-1).d_1.p_1 + \ldots + d_{r-1}.p_1^{r-1}} \qquad ..(2.9)$$

By earlier reasoning, $\frac{c_{r-1}}{d_{r-1}}$ is an (n-1)-variable PRF of $p_2,p_3,\ldots,p_n$.

Thus once again by repeated transformation and partial polynomial differentiation, we can prove that $\frac{c_0}{d_0}, \frac{c_1}{d_1}, \ldots, \frac{c_{r-2}}{d_{r-2}}$ are (n-1)-variable PRFs of $p_2,p_3,\ldots,p_n$.

<u>Case (c)</u>:

Let $r = r'-1$. This is equivalent to considering that the p_1 degree of the denominator is one greater than the numerator degree.

Then

$$F(p_1,p_2,\ldots,p_n) = \frac{c_0 + c_1 \cdot p_1 + \ldots + c_{r-1} \cdot p_1^{r-1}}{d_0 + d_1 \cdot p_1 + \ldots + d_{r-1} \cdot p_1^{r-1} + d_r \cdot p_1^r} \qquad ..(2.10)$$

We know that, $\frac{c_{r-1}}{d_r}$ is identically equal to a positive constant. By following a procedure of repeated transformation and partial polynomial differentiation as in Cases (a) and (b), we can conclude that

$\frac{c_0}{d_0}, \frac{c_1}{d_1}, \ldots, \frac{c_{r-1}}{d_{r-1}}$ are PRFs of $p_2, p_3, \ldots, p_n$. Thus the proof of the theorem follows.

Instead of expressing N and D as polynomials in p_1, if we express them as polynomials of either p_2 or $p_3, \ldots,$ or p_n we can prove that the corresponding $\frac{c_i}{d_i}$ $(0 \leq i \leq r-1)$ are PRFs of the remaining (n-1)-variables.

<u>Corollary 2.5.1</u>:

If $F(p_1,p_2,\ldots,p_n)$ is an MRF of $p_1,p_2,\ldots,p_n$, then $\frac{c_i}{d_i}$ $(0 \leq i \leq r-1)$ are (n-1)-variable reactance functions.

<u>Proof</u>:

By applying Ozaki and Kasami's[9] condition in all the above cases, we arrive at the above conclusion.

Thus Theorems 2.1 to 2.5 provide various methods of generating MPRFs from the given MPRF using the operation of differentiation. It may be noted that Theorems 2.1 to 2.4 hold true for single-variable as well as for multivariable functions. Regarding Theorem 2.5, in the case of single-variable PRF, the ratio is always a positive constant for all

coefficients and in the case of MPRF, it is an (n-1)-variable PRF for the first (r-1) coefficient ratios and an (n-1)-variable PRF or a positive constant for the highest degree coefficient ratio depending upon the difference in the degrees. This theorem provides a necessary test on the coefficients of an MPRF.

2.3 Generation of an MPRF from the given Real Part:[44]

In the synthesis of single-variable network functions, the numerator of the even part of a positive real function (PRF) plays an important role. Methods are available for generation of a PRF from the given even part[41]. Also it is known that if the given real part is non-negative, a PRF can always be generated.

In the case of a multivariable positive real function (MPRF) also, $Z(p_1,p_2,\ldots,p_n) = \frac{M_1+N_1}{M_2+N_2}$ (where the M's and N's are respectively the even and odd polynomials in $p_1,p_2,\ldots,p_n$), the even part, $\frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2}$ occupies a dominant role in the cascade realization of mixed lumped-distributed networks. However, there does not seem to be any study made regarding the real part of an MPRF.

In this section, the generation of an MPRF from the given multivariable real part is studied. The conditions under which an MPRF can be generated from the given real part are discussed.

From the definition of an MPRF, the even part $Z_e(p_i) = \frac{1}{2}[Z(p_i) + Z(-p_i)]$ shall be non-negative for Re $p_i \geq 0$, $(1 \leq i \leq n)$. Given $Z(p_i) = \frac{M_1+N_1}{M_2+N_2}$, it is evident that $Z_e(p_i) = \frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2}$ can be determined

uniquely. It is also known that, in the case of a single-variable, a PRF can always be generated from the given non-negative real part. In this section we look at the question: whether it is possible to generate an MPRF $Z(p_i)$ from the given non-negative $Z_e(p_i)$, and if so, whether any conditions need be satisfied? For the sake of convenience, the case of two-variable functions will be considered. These results are applicable for the case of n-variables as well. Before going into the details of generation, some properties of the even part are studied.

Definition 2.1:

The kth degree* n-variable polynomial

$$Q(p_1,p_2,\ldots,p_n) = \sum_{\delta_j=0}^{k} \sum_{\substack{i_1,i_2,\ldots,i_n = 0 \\ i_1+i_2+\ldots+i_n = \delta_j}}^{\delta_j} a_{i_1,i_2,\ldots,i_n} \cdot p_1^{i_1} \cdot p_2^{i_2} \cdots p_n^{i_n}$$

is a polynomial in $p_1,p_2,\ldots,p_n$ with no missing terms if $a_{i_1,i_2,\ldots,i_n}$, the coefficient of $p_1^{i_1} \cdot p_2^{i_2} \cdots p_n^{i_n}$, is a non-zero, constant for all combinations of $i_1, i_2, \ldots, i_n$.

Lemma 2.1:

If $Z(p_1,p_2)$ is analytic in and on the boundary of the domain $\mathrm{Re}\, p_1 \geq 0$, $\mathrm{Re}\, p_2 \geq 0$, then the minimum value of $\mathrm{Re}\, Z(p_1,p_2)$ in this domain occurs along $p_i = j\omega_i$, $i = 1,2$.

Proof:

Let us consider

* The term "degree" here implies the total degree of the n-variables unless otherwise mentioned.

$$\left| e^{-Z(p_1,p_2)} \right| = \left| e^{-\mathrm{Re}\, Z(p_1,p_2)} \cdot e^{-\mathrm{Im}\, Z(p_1,p_2)} \right|$$

$$= e^{-\mathrm{Re}\, Z(p_1,p_2)}$$

Hence $$\left| e^{-Z(p_1,p_2)} \right|_{\mathrm{Max.}} = e^{-\mathrm{Re}\, Z_{\mathrm{min.}}(p_1,p_2)} \geq e^{-\mathrm{Re}\, Z(p_1,p_2)}$$

In the open polydomain $\mathrm{Re}\, p_i > 0$, $Z(p_1,p_2)$ is analytic and hence $e^{-Z(p_1,p_2)}$ is also analytic in that region. Then, by maximum modulus theorem, we have:

$$\left| e^{-Z(p_1,p_2)} \right|_{\substack{\mathrm{Max.} \\ \mathrm{Re}\, p_i = 0}} > \left| e^{-Z(p_1,p_2)} \right|_{\mathrm{Re}\, p_i > 0.}$$

Since $$\left| e^{-Z(p_1,p_2)} \right|_{\substack{\mathrm{Max.} \\ \mathrm{Re}\, p_i = 0}} = e^{-\mathrm{Re}\, Z_{\mathrm{min.}}(p_1,p_2)} \Big|_{\mathrm{Re}\, p_i = 0}$$

$$\geq e^{-\mathrm{Re}\, Z(p_1,p_2)} \Big|_{\mathrm{Re}\, p_i > 0,}$$

the lemma follows immediately.

Thus, if the numerator and denominator of a given two-variable function are Hurwitz polynomials in the narrow sense, it is sufficient to test the given function on $j\omega_i$-axes for the real part condition.

$$\text{Let Re } Z(j\omega_1, j\omega_2) = Z_e(p_1, p_2)\Big|_{p_i=j\omega_i} = \frac{M_1M_2 - N_1N_2}{M_2^2 - N_2^2}\Bigg|_{p_i=j\omega_i} = \frac{A(\omega_1,\omega_2)}{B(\omega_1,\omega_2)} \qquad ..(2.11)$$

be a real rational non-negative function of ω_1 and ω_2. $A(\omega_1,\omega_2)$ and $B(\omega_1,\omega_2)$ can be arranged as follows:*

$$\begin{aligned} A(p_1,p_2)\Big|_{p_i=j\omega_i} &= (A_{0,0} + A_{0,2}\cdot\omega_2^2 + \cdots + A_{0,2t}\cdot\omega_2^{2t}) \\ &+ \omega_1(A_{1,1}\cdot\omega_2 + A_{1,3}\cdot\omega_2^3 + \cdots + A_{1,2t-1}\cdot\omega_2^{2t-1}) \\ &+ \cdots \\ &\vdots \\ &+ \omega_1^{2\ell}(A_{2\ell,0} + A_{2\ell,2}\cdot\omega_2^2 + \cdots + A_{2\ell,2t}\cdot\omega_2^{2t}) \end{aligned} \qquad ..(2.12)$$

and

$$\begin{aligned} B(p_1,p_2)\Big|_{p_i=j\omega_i} &= (B_{0,0} + B_{0,2}\cdot\omega_2^2 + \cdots + B_{0,2r}\cdot\omega_2^{2r}) \\ &+ \omega_1(B_{1,1}\cdot\omega_2 + B_{1,3}\cdot\omega_2^3 + \cdots + B_{1,2r-1}\cdot\omega_2^{2r-1}) \\ &\vdots \\ &+ \omega_1^{2s}(B_{2s,0} + B_{2s,2}\cdot\omega_2^2 + \cdots + B_{2s,2r}\cdot\omega_2^{2r}) \end{aligned} \qquad ..(2.13)$$

where the coefficients A's and B's are real constants.

---

* $A_{i,j}$ and $B_{i,j}$ are coefficients of $\omega_1^i \cdot \omega_2^j$ term.

2.3.1 Properties of $B(p_1,p_2)$:

From (2.11), $B(p_1,p_2) = M_2^2 - N_2^2 = (M_2+N_2)(M_2 - N_2)$

Hence

(i) $B(p_1,p_2) = B(-p_1,-p_2)$

(ii) $B(j\omega_1,j\omega_2) > 0$ for all real ω_1 and ω_2.

(However, $B(j\omega_1,j\omega_2)$ may be equal to zero for $\omega_1 = \omega_2 = 0$)

(iii) $B(p_1,p_2)$ is a product of the factors $[\beta_i(p_2).p_1 + \alpha_i(p_2)].[-\beta_i(-p_2).p_1 + \alpha_i(-p_2)]$, where the β's and α's are polynomials of p_2 and $\mathrm{Re}\,\dfrac{\alpha_i(p_2)}{\beta_i(p_2)} \geq 0$ for $\mathrm{Re}\,p_2 \geq 0$.

It may be observed that, in the case of a single-variable, properties (i) and (iii) imply the same, whereas it is not so in the case of multivariable functions. Also it may be noted that property (i) is implied by property (iii), but not the converse. This is the main difference between the single-variable and multivariable Hurwitz polynomials factorization. Utilising these properties, we give, in the following theorem, the conditions under which a given $B(p_1,p_2)$ can be factorized as $Q(p_1,p_2).Q(p_1,-p_2)$, which is a product of two-variable Hurwitz and anti-Hurwitz polynomials. (A method of determining Q can be found on page 34). This solves the problem mentioned in [45].

Theorem 2.6:

A two-variable real polynomial $B(p_1,p_2)$ can be factorized as $B(p_1,p_2) = Q(p_1,p_2).\,Q(-p_1,-p_2)$ where $Q(p_1,p_2)$ and $Q(-p_1,-p_2)$ are respectively the two-variable real Hurwitz and anti-Hurwitz polynomials in narrow sense, if and only if $B(p_1,p_2)$ is a product of factors of the nature $[\beta_i(p_2).p_1 + \alpha_i(p_2)].[-\beta_i(-p_2).p_1 + \alpha_i(-p_2)]$ where for $\mathrm{Re}\,p_2 \geq 0$, $\mathrm{Re}\,\dfrac{\beta_i(p_2)}{\alpha_i(p_2)} \geq 0$.

Proof: Necessity:

It is known that a given two-variable polynomial can always be factorized into the product of irreducible factors. Hence, $Q(p_1,p_2)$ and $Q(-p_1,-p_2)$ can be factorized into their irreducible factors. Because $Q(p_1,p_2)$ is a two-variable Hurwitz polynomial in the narrow sense, it does not have any zeros in either Re $p_1 \geq 0$, Re $p_2 > 0$ or Re $p_1 > 0$, Re $p_2 \geq 0$. Immediatly we notice that the factor $[\beta_i(p_2)p_1 + \alpha_i(p_2)]$ contributes to the Hurwitz character in the narrow sense in two-variables, and hence $[-\beta_i(-p_2).p_1 + \alpha_i(-p_2)$ corresponds to the anti-Hurwitz portion. Hence $Q(p_1,p_2)$ is formed as a product of factors of the nature $[\beta_1(p_2)p_1+\alpha_1(p_2)]$ and $Q(-p_1,-p_2)$ that of $[-\beta_1(-p_2).p_1 + \alpha_1(-p_2)]$. Thus the necessity follows.

Sufficiency:

Since $B(p_1,p_2) = \Pi[\beta_i(p_2).p_1 + \alpha_1(p_2)].[-\beta_i(-p_2).p_1 + \alpha_1(-p_2)]$, the first factor on the right hand side contributes to the Hurwitz character in the narrow sense in two-variables and the second factor gives rise to the anti-Hurwitz portion. As $B(p_1,p_2)$ is a product of such factors, by associating $\Pi[\beta_i(p_2).p_1 + \alpha_1(p_2)]$ with $Q(p_1,p_2)$ and $\Pi[-\beta_i(-p_2).p_1 + \alpha_i(-p_2)]$ with $Q(-p_1,-p_2)$, we obtain the required result. Thus the theorem follows.

As an example, even though the polynomial

$$B(p_1,p_2) = (1 - 4\,p_1p_2 - p_1^2 - p_2^2 + p_1^2p_2^2) = (1 - p_1p_2)^2 - (p_1+p_2)^2$$

does satisfy properties (i) and (ii), it does not obey the property (iii) and hence it cannot be an eligible denominator polynomial of $Z_e\,(p_1,p_2)$.

2.3.2 Properties of $A(p_1,p_2)$:

We have from (2.11) that $A(p_1,p_2) = M_1M_2 - N_1N_2$.

Hence it satisfies the following properties:

(i) $A(p_1,p_2) = A(-p_1,-p_2)$

(ii) $A(j\omega_1,j\omega_2) \geq 0$ for all real values of ω_1 and ω_2, and

(iii) $t \leq r$, $\ell \leq s$ and hence $(t + \ell) \leq (r + s)$.

It may be noted that there is no necessity for $A(p_1,p_2)$ to satisfy a condition similar to property (iii) of $B(p_1,p_2)$. Thus, even though $(1 - 4\,p_1p_2 - p_1^2 - p_2^2 + p_1^2\,p_2^2)$ cannot be an eligible denominator polynomial of $Z_e\,(p_1,p_2)$, we see that it can be a numerator polynomial of $Z_e\,(p_1,p_2)$. However, there are certain other restrictions imposed upon $A(p_1,p_2)$, which we will be discussing in Section 2.3.4.

2.3.3 To calculate $(M_2 + N_2)$ from the given $B(p_1,p_2)$:

We know that $B(p_1,p_2)$ can be factorized as $B(p_1,p_2) = \Pi\psi_i(p_1,p_2)$, where $\psi_i(p_1,p_2)$ are the irreducible factors. If the given $B(p_1,p_2)$ satisfies the conditions of Theorem 2.6, by assigning the left half polydomain roots to $(M_2 + N_2)$, we obtain the required two-variable Hurwitz polynomial. Even though the factorization of the type $B(p_1,p_2) = (M_2 + N_2).(M_2 - N_2)$ is guaranteed, no systematic procedure is known for the factorization of two-variable polynomials. Hence, we present below a systematic method of calculating $(M_2 + N_2)$ from the given $B(p_1,p_2)$:

Let us assume that $(M_2 + N_2) = \sum_{i_1=0}^{s} \sum_{i_2=0}^{r} b_{i_1,i_2}.p_1^{i_1}.p_2^{i_2}$, where

the b_{i_1,i_2} is the non-negative coefficient of $p_1^{i_1}.p_2^{i_2}$. From this assumed $(M_2 + N_2)$, calculate $(M_2^2 - N_2^2)$ and equate the resulting even polynomial in p_1 and p_2 to the given $B(p_1,p_2)$. It may be seen immediately that we

get a set of non-linear equations in b's and the number of equations are more than the number of unknowns.

If the given $B(p_1,p_2)$ is of degree 2s in p_1 and 2r in p_2, the resulting equations and unknowns are respectively $[(r+1)(s+1)+r.s]$ and $(r+1)\times(s+1)$, providing an extra set of r + s equations. Instead, if the degree of $B(p_1,p_2)$ is 2k, the number of equations and the number of unknowns are respectively $(k+1)^2$ and $\frac{(k+1)(k+2)}{2}$ thus, providing an extra set of $\frac{k(k+1)}{2}$ equations.

Thus, in either case, there are some extra equations. If a solution has to exist, these equations must be consistent. Since by Theorem 2.6 the decomposition of $B(p_1,p_2)$ into $B(p_1,p_2) = (M_2+N_2).(M_2-N_2)$ is guaranteed, we are assured of the solution of the set of equations.

Step (1):

From the assumed value of $(M_2^2-N_2^2)$, putting $p_2 = 0$ and equating to $B(p_1,0)$, we can obtain the coefficients of the type $b_{0,0}$, $b_{1,0}$, $b_{2,0}$, ..., $b_{i,0}$ etc. Similarly, by putting $p_1 = 0$, the values of the coefficients of the type $b_{0,0}$, $b_{0,1}$, $b_{0,2}$,..., $b_{0,j}$,... etc. can be calculated immediately. To calculate the coefficients of the type $b_{i,j}$ where $i,j \neq 0$, we proceed to step (2).

Step (2):

Consider the equations whose p_1-degree is one and p_2-degree is $(2j-1)$, where $j = 1,2,3,\ldots,r$. From this set of equations, the values of $b_{1,j}$ can be calculated. Similarly, consider the set of equations whose p_1-degree is 2s and p_2-degree is 2j. From this set of equations, we can immediately get the values of $b_{s,j}$. Next, consider the equations, whose

p_1-degree is 2 and p_2-degree is 2j, by solving which, we can obtain the values of $b_{2,j}$. Similarly, by solving the equations, whose p_1-degree is (2s-1) and p_2-degree is (2j-1), the values of $b_{2s-1,j}$ can be calculated. Thus by proceeding in this manner we can calculate the values of $b_{i,j}$, $0 \leq i < s$, and $0 \leq j \leq r$ from (s+1).(r+1) equations.

Step (3):

By substituting the values of the b's in the remaining r.s equations, we can test for the consistency of the equations. Thus by this process we do not encounter redundant equations while calculating for the b's.

If, in Step 3, the equations are found to be inconsistent, the conclusion is that the given even polynomial does not satisfy the conditions of Theorem 2.6.

Example 2.1:

Let us consider the two-variable Hurwitz polynomial

$$M_2 + N_2 = (4 + p_1 + p_2 + p_1p_2 + p_2^2 + p_1p_2^2)$$

Calculating $B(p_1,p_2)$ we have

$$B(p_1,p_2) = 16 - p_1^2 + 6\,p_1p_2 + 7\,p_2^2 - p_1^2p_2^2 - p_1^2p_2^4 + p_2^4$$

Let us assume that we are given the above $B(p_1,p_2)$ and calculate (M_2+N_2) by the procedure discussed earlier.

Solution:

$$\text{Let } M_2 + N_2 = \sum_{i=0}^{1} \sum_{j=0}^{2} b_{i,j} \cdot p_1^i \cdot p_2^j$$

By calculating $M_2^2 - N_2^2$ from the assumed (M_2+N_2) and equating it to the given $B(p_1,p_2)$ the following set of equations are obtained:

$$b_{00}^2 = 16 \qquad ..(2.14a)$$

$$b_{10}^2 = 1 \qquad ..(2.14b)$$

$$2\, b_{00}.b_{02} - b_{01}^2 = 7 \qquad ..(2.14c)$$

$$b_{02}^2 = 1 \qquad ..(2.14d)$$

$$2\, b_{00}.b_{11} - 2\, b_{01}.b_{10} = 6 \qquad ..(2.14e)$$

$$2\, b_{11}.b_{02} - 2\, b_{01}.b_{12} = 0 \qquad ..(2.14f)$$

$$b_{11}^2 - 2\, b_{10}.b_{12} = -1 \qquad ..(2.14g)$$

$$b_{12}^2 = 1 \qquad ..(2.14h)$$

As expected, there are eight equations and six unknowns leaving two extra equations. By solving 2.14(a) and (b), we get the values of b_{00} and b_{10} as $b_{00} = 4$, $b_{10} = 1$. By solving 2.14(a), (c) and (d), the values of b_{00}, b_{01}, and b_{02} are obtained as $b_{00} = 1$, $b_{01} = 1$ and $b_{02} = 1$. The equations 2.14(d) and (e) give the values of b_{11} and b_{12} as $b_{11} = 1$, $b_{12} = 1$. The equations 2.14(g) and (h) are extra and when the above calculated values of b's are substituted in these equations, they must be satisfied; otherwise the equations will be inconsistent. As could be seen immediately when the values of b's are substituted in 2.14(g) and (h) they are satisfied, thus we obtain the original polynomial with which we have started.

2.3.4 To Calculate (M_1+N_1) from the given $A(p_1,p_2)$ and (M_2+N_2):

The Brune-Gewertz method of the single-variable theory is utilized for this purpose.

It must be pointed out that the other methods of generating PRFs from the given real part in the single-variable case may not be applicable

for the multivariable case, as the factorization of the multivariable Hurwitz polynomials similar to the single-variable Hurwitz polynomials is not known.

Let us assume

$$M_1 + N_1 = \sum_{i_1=0}^{s} \sum_{i_2=0}^{r} a_{i_1,i_2} \cdot p_1^{i_1} \cdot p_2^{i_2}$$

where s and r are respectively the p_1-and p_2-degrees of (M_2+N_2) and a_{i_1,i_2} are real non-negative constants. Calculating $M_1M_2 - N_1N_2$ from the assumed (M_1+N_1) and the given (M_2+N_2), and equating the coefficients with $A(p_1,p_2)$, a set of linear equations is obtained which may be put in a compact form as follows:

$$\sum_{\substack{i_1=0 \\ x_1=0 \\ i_1+x_1=u}}^{2s} \sum_{\substack{i_2=0 \\ x_2=0 \\ i_2+x_2=v}}^{2r} (-1)^{i_1+i_2} \cdot a_{i_1,i_2} \cdot b_{x_1,x_2} = A_{u,v}$$

such that $0 \leq u \leq 2s$ and $0 \leq v \leq 2r$.

Once again there are $(s+1).(r+1)$ unknowns and $[(s+1).(r+1)+s.r]$ equations resulting in an excess of $s \times r$ equations.

The above set of equations can be arranged in the form of the following matrix equation

$$[\mathfrak{b}].[\mathfrak{a}] = [A]$$

where $[\mathfrak{b}]$ is the coefficient matrix consisting of the given b's; $[\mathfrak{a}]$ is the column vector consisting of the unknown a's, and $[A]$ is the column vector consisting of the given A's. The order of the matrices depends

upon the type of denominator polynomial under consideration. If the given (M_2+N_2) is of degree s in p_1 and r in p_2 then

[Б] is of order $[(s+1)(r+1) + s.r] \times (s+1)(r+1)$

[a] is of order $(s+1)(r+1) \times 1$, and

[A] is of order $[(s+1)(r+1) + s.r] \times 1$

Instead, if the given $A(p_1,p_2)$ is of degree 2k at most and (M_2+N_2) is of degree k with no missing terms, the set of equations can be arranged in a compact form as follows:

$$\sum_{\substack{i_1=0 \\ x_1=0 \\ i_1+x_1=u}}^{2k} \sum_{\substack{i_2=0 \\ x_2=0 \\ i_2+x_2=v}}^{2k} (-1)^{i_1+i_2} \cdot a_{i_1,i_2} \cdot b_{x_1,x_2} = A_{u,v}$$

where $0 \leq u \leq 2k$, $0 \leq v \leq 2k$ and $0 \leq u+v \leq 2k$.

In this case there are $(k+1)^2$ equations and $\frac{(k+1)(k+2)}{2}$ unknowns and hence $\frac{k(k+1)}{2}$ extra equations; and the orders of the matrices are as follows:

[Б] is of order $(k+1)^2 \times \frac{(k+1).(k+2)}{2}$

[a] is of order $\frac{(k+1).(k+2)}{2} \times 1$, and

[A] is of order $(k+1)^2 \times 1$

For the existence of a solution in either case, the following conditions must hold:

(a) For the consistency of the equations:

Rank of [Б] = Rank of [Б A].

(b) For the solution to be unique:

Rank of [Б] = $\frac{(k+1).(k+2)}{2}$, if (M_2+N_2) is a kth degree

polynomial with no missing terms

$= (s+1).(r+1)$, if p_1-and p_2-degrees of (M_2+N_2) are respectively s and r.

Once we know that the equations are consistent, the evaluations of the a's can be done as outlined below. Here we give the method for the case where the p_1-and p_2-degrees are respectively s and r. For the case of the degree of M_2+N_2 being k, the same method can be applied.

Step (1):

From the calculated value of $(M_1.M_2-N_1.N_2)$, by putting $p_2 = 0$ and equating to $A(p_1, 0)$, we obtain the coefficients of the type $a_{0,0}$, $a_{1,0}, \ldots,$ $a_{i,0}$ etc. Similarly, by putting $p_1 = 0$, the values of the coefficients of the type $a_{0,0}$, $a_{0,1}$, $a_{0,2}, \ldots$ etc. can be calculated immediately. To determine the coefficients of the type $a_{i,j}$, where $i, j \neq 0$, we proceed to step (2).

Step (2):

Consider the equations whose p_1-degree is one and p_2-degree is $(2j-1)$, where $j = 1,2,\ldots, r$. From this set of equations, the values of $a_{1,j}$ can be calculated. Similarly, consider the set of equations whose p_1-degree is 2s and p_2-degree is 2j. From this set of equations, we can immediately get the values of $a_{s,j}$. Next, consider the equations whose p_1-degree is 2 and p_2-degree is 2j, by solving which, we can obtain the values of $a_{2,j}$. Similarly, by solving the equations whose p_1-degree is $(2s-1)$ and p_2-degree is $(2j-1)$, the values of $a_{(2s-1),j}$ can be calculated. Thus proceeding in this manner we can calculate the values of the coefficients $a_{i,j}$, $0 \leq i \leq s$ and $0 \leq j \leq r$ from $(s+1).(r+1)$ equations.

Step (3):

By substituting the values of the a's in the remaining s.r equations, we can test for the consistency of the equations. Thus by this process we do not encounter any redundant equations while calculating for the a's.

The same method is applicable for the functions of more than two-variables. It may be noted that the MPRF generated is unique within an additive reactance function.

Example 2.2:

Let us assume that

$$Z_e(p_1,p_2) = \frac{(1-4\,p_1^2).(2+p_2^2)^2}{16 - p_1^2 + 6\,p_1p_2 + 7\,p_2^2 - p_1^2p_2^2 - p_1^2p_2^4 + p_2^4}.$$

Calculate a two-variable positive real function by the above discussed method.

Solution:

From Example 2.1, we know that the given $B(p_1,p_2)$ satisfies the conditions of Theorem 2.6 and the calculated $(M_2+N_2) = 4+p_1+p_2+p_1p_2+p_2^2+p_1p_2^2$. From this (M_2+N_2) and the given non-negative $A(\omega_1,\omega_2)$ we will calculate for (M_1+N_1) as below:

$$\text{Let } M_1+N_1 = \sum_{i=0}^{1}\sum_{j=0}^{2} a_{i,j}.\,p_1^i.p_2^j$$

Calculating $(M_1M_2-N_1N_2)$ and equating to the given $A(p_1,p_2)$, the following set of equations is obtained.

$$4\,a_{00} = 4 \qquad ..(2.15a)$$

$$a_{10} = 16 \qquad ..(2.15b)$$

$$a_{00} + 4\,a_{02} - a_{01} = 4 \qquad ..(2.15c)$$

$$a_{02} = 1 \qquad ..(2.15d)$$

$$a_{00} + 4\,a_{11} - a_{10} - a_{01} = 0 \qquad ..(2.15e)$$

$$a_{02} + a_{11} - a_{01} - a_{12} = 0 \qquad ..(2.15f)$$

$$a_{11} - a_{10} - a_{12} = -16 \qquad ..(2.15g)$$

$$a_{12} = 4 \qquad ..(2.15h)$$

Thus, there are eight equations and six unknowns providing two extra equations as expected. By solving equations 2.15(a) to (f), the values of a's are obtained as below:

$$a_{00} = 1,\ a_{10} = 16,\ a_{02} = 1,\ a_{01} = 1,\ a_{11} = 4,\ a_{12} = 4$$

By substituting these values in 2.15(g) and (h), we find that the equations are consistent.

The two-variable positive real function that is generated from the given $Z_e(p_1, p_2)$ is:

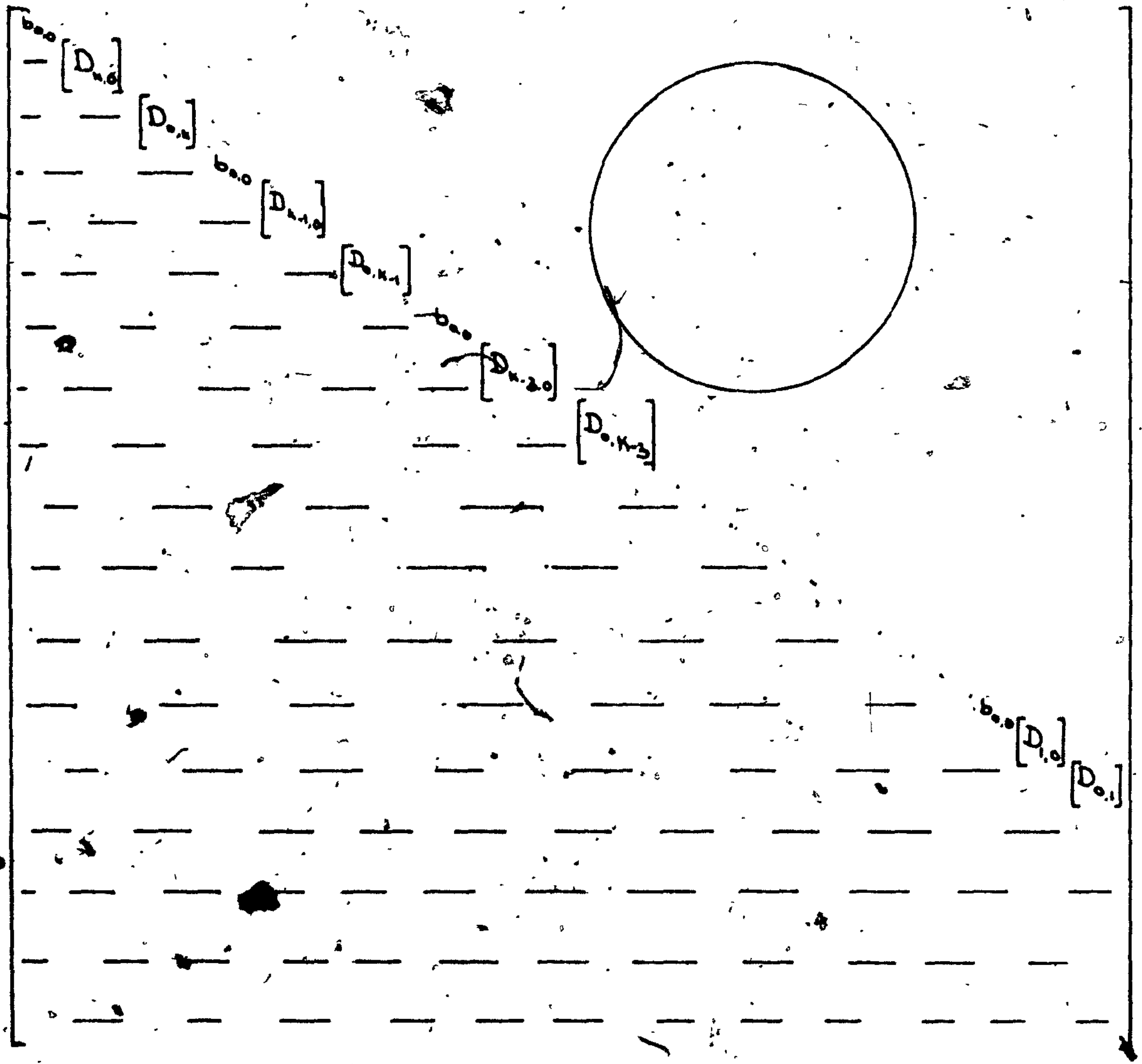
$$Z(p_1, p_2) = \frac{1 + 16\,p_1 + p_2 + 4\,p_1p_2 + p_2^2 + 4\,p_1p_2^2}{4 + p_1 + p_2 + p_1p_2 + p_2^2 + p_1p_2^2}$$

If (M_2+N_2) is a polynomial in p_1, p_2 with no missing terms of degree k, then the above matrix equation can be arranged as shown in equations (2.16) and (2.17) respectively for the cases k even and k odd.

Depending upon the terms that are present in [A] and [B], the above mentioned conditions (a) and (b) can be tested. It may be observed that, unlike the case of a single-variable, where a PRF can always be generated as long as the real part is non-negative, it is not necessary to have a two-variable PRF and hence an MPRF generated even if the real part is non-negative unless the condition (a) is satisfied.

We can see, from the matrix equations (2.16) and (2.17), that if the given (M_2+N_2) is a polynomial of kth degree with no missing terms, then all the submatrices $[D_{i,0}]$, $[D_{0,i}]$, $0 \leq i \leq k$ are non-singular. Thus for both k even and odd, the matrix $[\mathfrak{B}]$ is of rank $\frac{(k+1)(k+2)}{2}$. Hence, whatever the terms that are present in [A], for a solution to exist the rank of $[\mathfrak{B}A]$ must be $\frac{(k+1)(k+2)}{2}$.

Thus, there are some restrictions upon the given non-negative $Z_e(p_1,p_2)$ so that a two-variable PRF can be generated. The foregoing discussion deals with the general conditions under which an MPRF can be generated. Some particular cases of the $Z_e(p_1,p_2)$ are considered in Chapter III.



$$\begin{bmatrix} b_{0,0} & & & & & & & & & \\ & [D_{k,0}] & & & & & & & & \\ & & [D_{0,k}] & & & & & & & \\ & & & b_{0,0} & & & & & & \\ & & & & [D_{k-1,0}] & & & & & \\ & & & & & [D_{0,k-1}] & & & & \\ & & & & & & b_{0,0} & & & \\ & & & & & & & [D_{k-2,0}] & & \\ & & & & & & & & [D_{0,k-3}] & \\ & & & & & & & & & \ddots \\ & & & & & & & & & b_{0,0} \\ & & & & & & & & & [D_{1,0}] \\ & & & & & & & & & [D_{0,1}] \end{bmatrix} \begin{bmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{k,0} \\ a_{0,1} \\ \vdots \\ a_{0,k} \\ a_{1,1} \\ \vdots \\ a_{k-1,1} \\ a_{1,2} \\ \vdots \\ a_{1,k-1} \\ a_{2,2} \\ \vdots \\ a_{\frac{k}{2},\frac{k}{2}} \\ a_{\frac{k+2}{2},\frac{k}{2}} \\ a_{\frac{k}{2},\frac{k+2}{2}} \end{bmatrix} = \begin{bmatrix} A_{0,0} \\ \vdots \\ A_{2k,0} \\ A_{0,2} \\ \vdots \\ A_{0,2k} \\ A_{1,1} \\ \vdots \\ A_{2k-1,1} \\ A_{1,3} \\ \vdots \\ A_{1,2k-1} \\ A_{2,2} \\ \vdots \\ A_{k,k} \end{bmatrix} \quad ..(2.16)$$

$$\begin{bmatrix}
b_{0,0} & & & & & & & & & & & & & \\
- & [D_{k,0}] & & & & & & & & & & & & \\
- & - & [D_{0,k}] & & & & & & \bigcirc & & & & & \\
- & - & - & b_{0,0} & & & & & & & & & & \\
- & - & - & - & [D_{k-1,0}] & & & & & & & & & \\
- & - & - & - & - & [D_{0,k-1}] & & & & & & & & \\
- & - & - & - & - & - & b_{0,0} & & & & & & & \\
- & - & - & - & - & - & - & [D_{k-3,0}] & & & & & & \\
- & - & - & - & - & - & - & - & [D_{0,k-3}] & & & & & \\
- & - & - & - & - & - & - & - & - & \ddots & & & & \\
- & - & - & - & - & - & - & - & - & - & b_{0,0} & & & \\
- & - & - & - & - & - & - & - & - & - & - & [D_{2,0}] & & \\
- & - & - & - & - & - & - & - & - & - & - & - & [D_{0,2}] & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & b_{0,0} \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - & - & - & -
\end{bmatrix}
\begin{bmatrix}
a_{0,0} \\ \vdots \\ a_{k,0} \\ a_{0,1} \\ \vdots \\ a_{0,k} \\ a_{1,1} \\ \vdots \\ a_{k-1,1} \\ a_{1,2} \\ \vdots \\ a_{1,k-1} \\ a_{2,2} \\ \vdots \\ a_{\frac{k+1}{2},\frac{k+1}{2}}
\end{bmatrix}
=
\begin{bmatrix}
A_{0,0} \\ \vdots \\ A_{2k,0} \\ A_{0,2} \\ \vdots \\ A_{0,2k} \\ A_{1,1} \\ \vdots \\ A_{2k-1,1} \\ A_{1,3} \\ \vdots \\ A_{1,2k-1} \\ A_{2,2} \\ \vdots \\ A_{k,k}
\end{bmatrix}
\qquad ..(2.17)$$

where the dashes indicate appropriate b's.

where

$$[D_{k,0}] = \begin{bmatrix} b_{1,0} & b_{0,0} & 0 & 0 & 0 & \cdots & 0 \\ b_{3,0} & b_{2,0} & b_{1,0} & b_{0,0} & \cdot & \cdots & \cdot \\ b_{5,0} & b_{4,0} & b_{3,0} & b_{2,0} & \cdot & \cdots & \cdot \\ b_{7,0} & b_{6,0} & b_{5,0} & b_{4,0} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{k-3,0} \quad b_{k-4,0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{k-1,0} \quad b_{k-2,0} \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & b_{k,0} \end{bmatrix}_{(k\times k)}$$

and

$$[D_{0,k}] = \begin{bmatrix} b_{0,1} & b_{0,0} & 0 & 0 & \cdot & \cdots & 0 \\ b_{0,3} & b_{0,2} & b_{0,1} & b_{0,0} & 0 & \cdots & 0 \\ b_{0,5} & b_{0,4} & b_{0,3} & b_{0,2} & \cdot & \cdots & \cdot \\ b_{0,7} & b_{0,6} & b_{0,5} & b_{0,4} & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{0,k-3} \quad b_{0,k-4} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & b_{0,k-1} \quad b_{0,k-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots 0 \cdot 0 & b_{0,k} \end{bmatrix}_{(k\times k)}$$

$[D_{k-i,0}]$ $(1 \leq i \leq k)$ is matrix $[D_{k,0}]$ with the last i number of columns and i number of rows deleted, and similarly $[D_{0,k-i}]$.

2.4 Testing the Real Part of a Two-Variable Positive Real Function:

A two-variable rational function $Z(p_1,p_2)$ is positive real, if and only if:

(i) $Z(p_1,p_2)$ is real for real p_1 and p_2, and

(ii) $\text{Re } Z(p_1,p_2) \geq 0$ for $\text{Re } p_i \geq 0$, $i = 1,2$

Condition (i) can be tested by inspection, hence we concentrate on the testing of condition (ii). By Lemma 2.1 if, $Z(p_1,p_2)$ is analytic in the right half polydomain, then the minimum value of $\text{Re } Z(p_1,p_2)$ occurs along $j\omega_i$-axes. Thus, if the numerator and denominator of $Z(p_1,p_2)$ are Hurwitz polynomials in the narrow-sense, which can be ascertained by Ansell's method[11], the following procedure for testing the non-negativeness of $\text{Re } Z(p_1,p_2)$ along $j\omega_i$-axes can be applied.

Let $\text{Re } Z(j\omega_1,j\omega_2) = \dfrac{A(\omega_1,\omega_2)}{B(\omega_1,\omega_2)}$ be a real rational function of ω_1 and ω_2. For $Z(p_1,p_2)$ to be a PRF $\dfrac{A(\omega_1,\omega_2)}{B(\omega_1,\omega_2)} \geq 0$ for $0 \leq |\omega_i| \leq \infty, i=1,2$. Since $B(\omega_1,\omega_2) > 0$ for all real values of ω_1,ω_2, only $A(\omega_1,\omega_2)$ need be tested for non-negativeness.

If $A(\omega_1,\omega_2)$ can be factored easily, then the testing for non-negativeness is simple. Also, since the factors which are independent of the other variable can be found easily[46], we assume that the given $A(\omega_1,\omega_2)$ is not having any factors independent of the other variable. Such $A(\omega_1,\omega_2)$ can be arranged as below:

$$A(\omega_1,\omega_2) = A_{0,0} + (A_{2,0}\cdot\omega_1^2 + A_{1,1}\,\omega_1\cdot\omega_2 + A_{0,2}\cdot\omega_2^2)$$

$$+ (A_{4,0}\cdot\omega_1^4 + A_{3,1}\,\omega_1^3\cdot\omega_2 + A_{2,2}\,\omega_1^2\cdot\omega_2^2 + A_{1,3}\omega_1\cdot\omega_2^3 + A_{0,4}\omega_2^4)$$

$$\vdots$$

$$+ (A_{2q,0}\,\omega_1^{2q} + A_{2q-1,1}\,\omega_1^{2q-1}.\omega_2 + \ldots + A_{0,2q}.\omega_2^{2q})$$

$A(\omega_1,\omega_2)$ must be non-negative for $-\infty \leq \omega_1,\omega_2 \leq \infty$, the testing of which is divided into the following steps:

Step (1):

In this step, the non-negativeness of $A(\omega_1,\omega_2)$ is checked either for $0 \leq |\omega_1| \leq \infty$, $\omega_2 = 0, \pm\infty$ or for $\omega_1 = 0, \pm\infty$, $0 \leq |\omega_2| \leq \infty$ depending upon the convenience by means of the tests listed in Table 2.1.

Since the resulting $A(\omega_1,\omega_2)$ for Tests (1), (2), (7) and (8) is a single-variable even polynomial in ω_1 or ω_2, its non-negativeness can be determined by the known methods.

Tests (3) to (6) can be carried out by inspection. By means of Tests (1) to (5), $A(\omega_1,\omega_2)$ is tested for the range of $0 \leq |\omega_1| \leq \infty$ and $\omega_2 = 0, \pm\infty$. Similarly, by means of Tests (4) to (8), $A(\omega_1,\omega_2)$ is tested for the range of $\omega_1 = 0, \pm\infty$ and $0 \leq |\omega_2| \leq \infty$. If $A(\omega_1,\omega_2)$ is tested for non-negativeness either by Tests (1) to (5) or by Tests (4) to (8) we proceed to step (2). It may be noted that for testing the entire ranges of ω_1 and ω_2, either set of the above mentioned tests is sufficient in this step, since the remaining ranges will be covered in the steps to follow. Without loss of generality, it is assumed that Tests (1) to (5) are performed and $A(\omega_1,\omega_2) \geq 0$ during that range. Of course, if $A(\omega_1,\omega_2) < 0$ during that range, the given $Z(p_1,p_2)$ is not positive real.

Step (2):

For covering the remaining range of ω_1 and ω_2, we follow an extended Sturm test as described below.

TABLE 2.1

The real part at particular values of ω_1 and ω_2.

Test No.	Values of ω_1 and ω_2		The resulting $A(\omega_1,\omega_2)$
	ω_1	ω_2	
1	$0 \leq \lvert\omega_1\rvert \leq \infty$	0	$A_{0,0} + A_{2,0}\,\omega_1^2 + \ldots +$
2	$0 < \lvert\omega_1\rvert < \infty$	$\pm\infty$	$A_{0,2t} + A_{2,2t}\,\omega_1^2 + \ldots +$
3	0	$\pm\infty$	The highest degree coefficient in ω_2 alone
4	$\pm\infty$	$\pm\infty$	$A_{2q,0} + A_{2q-1,1} + A_{2q-2,2} + \ldots + A_{0,2q}$
5	$\pm\infty$	$\mp\infty$	$A_{2q,0} - A_{2q-1,1} + A_{2q-2,2} - + \ldots + A_{0,2q}$
6	$\pm\infty$	0	The highest degree coefficient in ω_1 alone
7	0	$0 \leq \lvert\omega_2\rvert \leq \infty$	$A_{0,0} + A_{0,2}\,\omega_2^2 + A_{0,4}\,\omega_2^4 + \ldots +$
8	$\pm\infty$	$0 < \lvert\omega_2\rvert < \infty$	$A_{2\ell,0} + A_{2\ell,2}\,\omega_2^2 + \ldots +$

The given $A(\omega_1,\omega_2)$ is arranged as a polynomial in ω_1, the coefficients being polynomials of ω_2 as in equation (2.12). Let $P_0(\omega_1,\omega_2)$, $P_1(\omega_1,\omega_2)\ldots$ etc. be the corresponding Sturm functions which are given as below:

$$P_0(\omega_1,\omega_2) = A(\omega_1,\omega_2) = \omega_1^{2\ell}.\ f_0(\omega_2) + \omega_1^{2\ell-1}.f_1(\omega_2)+\ldots+ f_{2\ell}(\omega_2)$$

$$P_1(\omega_1,\omega_2) = 2\ell.\omega_1^{2\ell-1}.\ f_0(\omega_2) + (2\ell-1).\omega_1^{2\ell-2}.f_1(\omega_2)+\ldots+f_{2\ell-1}(\omega_2)$$

$$P_2(\omega_1,\omega_2) = \omega_1^{2\ell-2}.\ g_0(\omega_2) + \omega_1^{2\ell-3}.\ g_1(\omega_2)+\ldots+ g_{2\ell-2}(\omega_2)$$

$$P_3(\omega_1,\omega_2) = \omega_1^{2\ell-3}.\ h_0(\omega_2) + \ldots+ h_{2\ell-3}(\omega_2)$$

$$\vdots$$

$$P_{2\ell}(\omega_1,\omega_2) = y_0(\omega_2).$$

The above Sturm functions are calculated by considering $P_0(\omega_1,\omega_2)$ as a polynomial in ω_1. Thus

$$P_1(\omega_1,\omega_2) = \frac{\partial P_0(\omega_1,\omega_2)}{\partial\omega_1}.$$

we have that $\dfrac{P_0(\omega_1,\omega_2)}{P_1(\omega_1,\omega_2)} = \dfrac{\omega_1}{2\ell} + \beta_1(\omega_2) + \dfrac{R_1(\omega_1,\omega_2)}{f_0(\omega_2).P_1(\omega_1,\omega_2)}$.

where $\dfrac{R_1(\omega_1,\omega_2)}{f_0(\omega_2)}$ is the remainder. Since $f_0(\omega_2)$ is non-negative for all real values of ω_2, we consider the next Sturm function $P_2(\omega_1,\omega_2) = -R_1(\omega_1,\omega_2)$. The cases when $f_0(\omega_2) = 0$ for real values of ω_2 will be considered in step (3).

Similarly when $P_1(\omega_1,\omega_2)$ is divided by $P_2(\omega_1,\omega_2)$ to give a two-term

quotient, let $\frac{R_2(\omega_1,\omega_2)}{g_0^2(\omega_2)}$ be the remainder. Since $g_0^2(\omega_2)$ is non-negative for all real values of ω_2, we consider $P_3(\omega_1,\omega_2) = - R_2(\omega_1,\omega_2)$. Again, the cases when $g_0(\omega_2) = 0$ for real values of ω_2 will be dealt in step (3).

Thus, on similar lines the Sturm functions are obtained. As a result of multiplication of the remainders by $f_0(\omega_2)$, $g_0^2(\omega_2)$, $h_0^2(\omega_2)$... etc, the Sturm functions are polynomials in ω_1,ω_2 rather than being ratios of polynomials. For finding the non-negativeness of $A(\omega_1,\omega_2)$ for the values of $0 \leq |\omega_1| \leq \infty$ and $0 < |\omega_2| < \infty$, the Sturm table is formed as shown in Table 2.2.

Since P's are polynomials of ω_2, instead of being constants for $\omega_1 = \pm \infty$, the evaluation of their signs is not straight forward. Hence, to obtain the signs of the P's, the entire range of ω_2 is divided such that during any interval under consideration none of the P's changes sign. Since the P's are polynomials, the number of such intervals are finite. Thus instead of having one Sturm table, we will be having a number of them covering the entire range of ω_2 and for each interval the sign of the P's is evaluated and from that the number of sign variations $\underline{v}_{-\infty,\infty}$ for ω_1 varying from $-\infty$ to $+\infty$ is calculated. Thus if

$\underline{v}_{-\infty,\infty} \neq 0$, for any interval of ω_2, $A(\omega_1,\omega_2)$ goes negative and hence $Z(p_1,p_2)$ is not positive real.

$\underline{v}_{-\infty,\infty} = 0$, for the entire range of ω_2 (except at the real zeros of f_0, g_0, h_0, .. y_0, which is going to be tested next), then proceed to step (3).

TABLE 2.2

A Sturm Table

	Sign of the Sturm functions:					
Value of ω_1	$P_0(\omega_1,\omega_2)$	$P_1(\omega_1,\omega_2)$	$P_2(\omega_1,\omega_2)$	....	$P_{2\ell}(\omega_1,\omega_2)$	$V_{-\infty,\infty}$
$+\infty$	$f_0(\omega_2)$	$f_0(\omega_2)$	$g_0(\omega_2)$	....	$y_0(\omega_2)$	
$-\infty$	$f_0(\omega_2)$	$-f_0(\omega_2)$	$g_0(\omega_2)$	....	$y_0(\omega_2)$	

Step (3):

$P_0(\omega_1,\omega_2)$ is tested for fixed values of ω_2, the range of ω_1 being varying from $-\infty$ to $+\infty$. The fixed values of ω_2 are the real roots of $f_0(\omega_2)$, $g_0(\omega_2)$, $h_0(\omega_2)$ ... $y_0(\omega_2)$. Thus, if $P_0(\omega_1,\omega_2)$ is non-negative for all these fixed values of ω_2 - which is again tested by Sturm test - the given $A(\omega_1,\omega_2)$ is non-negative for $0 \leq |\omega_i| \leq \infty$, $i = 1,2$.

Thus by dividing the range of ω_2, and covering the entire range of ω_1 at one stretch, we could cover the entire range of ω_1 and ω_2. Now the following comments are in order:

(i) If $f_0(\omega_2)$ is having any real root - which obviously must be even multiple - then $f_1(\omega_2)$ must also be having the same real root, if $A(\omega_1,\omega_2)$ is to be non-negative.

(ii) Similarly, if $f_{2\ell}(\omega_2)$ is having any real root, so must be $f_{2\ell-1}(\omega_2)$ at the same value of ω_2.

(iii) If $P_i(\omega_1,\omega_2) \equiv 0$, then $P_{i-1}(\omega_1,\omega_2)$ is a factor of second order in $P_0(\omega_1,\omega_2)$

(iv) If $P_i(\omega_1,\omega_2) = 0$ for some particular value of $\omega_2 = \omega_{20}$(say), then $P_{i-1}(\omega_1,\omega_{20})$ is a double root of $P_0(\omega_1,\omega_2)$.

Example 2.3:

Test the function

$$Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2} = \frac{2 + 2\,p_1+2\,p_2+ p_1^2+ 4\,p_1\,p_2+p_2^2+2\,p_1^2p_2+2\,p_1p_2^2+2\,p_1^2p_2^2}{1 + p_1 + p_2 + 3\,p_1p_2 + p_1p_2^2 + p_1^2p_2 + p_1^2p_2}$$

for the two-variable positive real property.

Solution:

To apply the above testing procedure for the real part, we test by means of Ansell's method[11] whether the numerator and the denominator are two-variable Hurwitz polynomials in the narrow sense.

Testing the numerator polynomial for Hurwitz character:

$$H(p_1,p_2) = (2 + 2\,p_2+p_2^2)+ p_1.(2 + 4\,p_2+ 2\,p_2^2)+ p_1^2(1+2\,p_2 + 2\,p_2^2)$$

we see that

$$H(p_1,1) = 5 + 8\,p_1 + 5\,p_1^2$$

is having its roots in the open L.H.S. plane of p_1.

$$H(j\omega_1,j\omega_2) = [(2-\omega_1^2) + \omega_2.(-4\,\omega_1) + \omega_2^2(-1 + 2\,\omega_1^2)]$$

$$+ \quad j[2\,\omega_1 + \omega_2.(2-2\omega_1^2) + \omega_2^2(-2\omega_1)]$$

The corresponding polynomial matrix is given by:

$$D(\omega_1) = 4 \times \begin{bmatrix} 1-\omega_1^2 + 2\,\omega_1^4 & -\omega_1 - \omega_1^3 \\ -\omega_1 - \omega_1^3 & 2 + \omega_1^2 + \omega_1^4 \end{bmatrix}$$

Its successive principal minors, $4(1-\omega_1^2+2\,\omega_1^4)$ and $32(1-\omega_1^2+\omega_1^4+\omega_1^8)$ are non-negative for all real values of ω_1. Hence the given numerator polynomial is a two-variable Hurwitz polynomial in the narrow sense.

Testing the Denominator polynomial for Hurwitz character:

$$H(p_1,p_2) = (1 + p_2) + p_1(1 + 3\,p_2 + p_2^2) + p_1^2.(p_2 + p_2^2)$$

Then we see that

$$H(p_1,1) = (2 + 5\,p_1 + 2\,p_1^2)$$

is having its roots in the open L.H.S. plane.

$$H(j\omega_1, j\omega_2) = [1+\omega_2.(-3\ \omega_1)+\omega_2^2.(-\omega_1^2)]+j[\omega_1+\omega_2(1-\omega_1^2)+\omega_2^2(-\omega_1)]$$

The corresponding polynomial matrix is given by

$$D(\omega_1) = \begin{bmatrix} 2\omega_1^2 + \omega_1^4 & -\omega_1 - \omega_1^3 \\ -\omega_1 - \omega_1^3 & 1 + 2\omega_1^2 \end{bmatrix}$$

The leading principal minors, $(2\ \omega_1^2 + \omega_1^4)$ and $\omega_1^2(1 + 3\ \omega_1^2 + \omega_1^4)$ are non-negative for all real values of ω_1. Hence the given denominator polynomial is a two-variable Hurwitz polynomial in the narrow sense.

Having verified that the numerator and the denominator are Hurwitz polynomials, it remains to test whether the even part on the $j\omega_i$-axes is non-negative before we can ascertain that the given function satisfies the two-variable positive real conditions.

<u>Testing the Real part for non-negativeness for all real values of ω_1 and ω_2:</u>

$$A(\omega_1, \omega_2) = \omega_1^4.(\omega_2^2+2\ \omega_2^4)+\omega_1^3.(-\omega_2-6\omega_2^3)+\omega_1^2(1+8\ \omega_2^2+\omega_2^4)$$
$$+\ \omega_1.(-6\ \omega_2-\omega_2^3)+(2+\omega_2^2)$$

<u>Step (1):</u>

Tests (1) to (5) are performed in the following manner:

$$A(\omega_1, 0) = 2 + \omega_1^2$$

$$A(\omega_1, \pm\infty) = \omega_1^2 + 2\ \omega_1^4 \quad \text{for } 0 < |\omega_1| < \infty$$

$$A(0, \pm\infty) = 0$$

$$A(\pm\infty, \pm\infty) = 2$$

$$A(\pm\infty, \mp\infty) = 2$$

Thus we see that $A(\omega_1,\omega_2) \geq 0$ for $0 \leq |\omega_1| \leq \infty$, $\omega_2 = 0, \pm\infty$.

Step (2):

$$P_0(\omega_1,\omega_2) = \omega_1^4 \cdot f_0 + \omega_1^3 \cdot f_1 + \omega_1^2 \cdot f_2 + \omega_1 \cdot f_3 + f_4$$

$$P_1(\omega_1,\omega_2) = 4\,\omega_1^3 \cdot f_0 + 3\omega_1^2 \cdot f_1 + 2\,\omega_1 \cdot f_2 + f_3$$

$$P_2(\omega_1,\omega_2) = \omega_1^2 \cdot g_0 + \omega_1 \cdot g_1 + g_2$$

$$P_3(\omega_1,\omega_2) = \omega_1 \cdot h_0 + h_1$$

$$P_4(\omega_1,\omega_2) = k_0$$

where

$$f_0 = \omega_2^2 + 2\,\omega_2^4 \;; \qquad f_1 = -\omega_2 - 6\,\omega_2^3$$

$$f_2 = 1 + 8\,\omega_2^2 + \omega_2^4 \;; f_3 = -6\omega_2 - \omega_2^3$$

$$f_4 = 2 + \omega_2^2$$

$$g_0 = -(5\,\omega_2^2 + 44\,\omega_2^4 + 28\,\omega_2^6 + 16\,\omega_2^8)$$

$$g_1 = -2\,\omega_2 + 44\,\omega_2^3 + 58\,\omega_2^5 + 12\,\omega_2^7$$

$$g_2 = -(26\,\omega_2^2 + 43\,\omega_2^4 + 26\,\omega_2^6)$$

$$h_0 = 16\,\omega_2^4 \cdot (-6 + 8\,\omega_2^2 - 95\,\omega_2^4 - 192\,\omega_2^6 + 82\,\omega_2^8 - 4\,\omega_2^{10} + 336\,\omega_2^{12} - 16\,\omega_2^{14} - 32\,\omega_2^{16})$$

$$h_1 = \omega_2^5(-448 + 64\,\omega_2^2 + 2880\,\omega_2^4 + 2752\,\omega_2^6 + 1472\,\omega_2^8$$

$$-1728\,\omega_2^{10} - 2560\,\omega_2^{12} + 256\,\omega_2^{14})$$

and $$k_0 = \omega_2^{10}(1 + 18.1\,\omega_2^2 + 96.16\,\omega_2^4 + 131.98\,\omega_2^6 + 139.47\,\omega_2^8$$

$$+ 491.93\,\omega_2^{10} + 453.71\,\omega_2^{12} + 710.42\,\omega_2^{14} + 1249.84\,\omega_2^{16}$$

$$+ 440.06\,\omega_2^{18} + 1621\,\omega_2^{20} + 109.68\,\omega_2^{22} + 1267.36\omega_2^{24} + 154.56\,\omega_2^{26}$$

$$+ 673.92\,\omega_2^{28} + 85.76\,\omega_2^{30} + 227.84\,\omega_2^{32} + 40.96\,\omega_2^{34} + 40.96\,\omega_2^{36})$$

From the above values it is seen that:

$f_0(\omega_2) \geq 0$ for all real ω_2

$g_0(\omega_2) \leq 0$ for all real ω_2

$h_0(\omega_2) \lesseqgtr 0$ for real values of ω_2.

The reason being that it is having real roots at $\omega_2 = 0, \pm 0.9546$ and ± 1.731.

$k_0(\omega_2) \geq 0$ for all real values of ω_2.

As mentioned earlier there will be a number of Sturm tables instead of one; in this case the number is three. Since, f_0, g_0, h_0 and k_0 are even polynomials of ω_2, it is sufficient to consider the range of ω_2 from 0 to $+\infty$. The Sturm tables are as shown in Table 2.3.

It may be noted in this case no matter what the sign of $h_0(\omega_2)$ is going to be for the specified ranges of ω_2, $v_{-\infty,\infty} = 0$ for all the ranges. Thus $P_0(\omega_1,\omega_2) > 0$ for all the three ranges of ω_2 and $0 \leq |\omega_1| \leq \infty$.

TABLE 2.3

Sturm Tables for Example 2.3

Range of ω_2	Value of ω_1	P_0	P_1	P_2	P_3	P_4	$v_{-\infty,\infty}$
$0<\omega_2<0.9546$	$+\infty$	+	+	−		+	
	$-\infty$	+	−	−		+	
$0.9546<\omega_2<1.731$	$+\infty$	+	+	−		+	
	$-\infty$	+	−	−		+	
$1.731<\omega_2<\infty$	$+\infty$	+	+	−		+	
	$-\infty$	+	−	−		+	

Step (3):

The given $P_0(\omega_1,\omega_2)$ has to be tested at the real roots of f_0, g_0, h_0 and k_0. The real roots of f_0, g_0 and k_0 are at $\omega_2 = 0$ and we know that

$$P_0(\omega_1,0) = 2 + \omega_1^2 > 0 \quad \text{for } 0 \le |\omega_1| \le \infty.$$

The only values of ω_2 for which $P_0(\omega_1,\omega_2)$ is yet to be tested are at $\omega_2 = \pm\, 0.9546$ and $\pm\, 1.731$.

$$P_0(\omega_1,0.9546) = 2.572\ (\omega_1^4-2.4004\ \omega_1^3+3.5461\ \omega_1^2-2.565\ \omega_1+1.1319)$$

$$P_0(\omega_1,-0.9546) = 2.572\ (\omega_1^4+2.4004\ \omega_1^3+3.5461\ \omega_1^2+2.505\ \omega_1+1.1319)$$

$$P_0(\omega_1,1.731) = 20.9528\ (\omega_1^4-1.5679\ \omega_1^3+1.6203\ \omega_1^2-0.7432\ \omega_1+0.2386)$$

$$P_0(\omega_1,-1.731) = 20.9528\ (\omega_1^4+1.5679\ \omega_1^3+1.6203\ \omega_1^2+0.7432\ \omega_1+0.2386)$$

For these fixed values of ω_2, $P_0(\omega_1,\omega_2) > 0$ since $P_0(\omega_1,\omega_2)$ is not having any real roots of ω_1.

Thus the given $A(\omega_1,\omega_2)$ has satisfied all the required tests for the positiveness. Hence, the given $Z(p_1,p_2)$ is a two-variable positive real function.

Special case (i):

The two-variable function $Z(p_1,p_2) = \dfrac{a_{10}p_1+a_{01}p_2+a_{00}}{b_{10}p_1+b_{01}p_2+b_{00}}$ is positive real if and only if,

(i) The a's and the b's are non-negative, and

(ii) $a_{10}.b_{01} = a_{01}.b_{10}$.

Proof: Necessity:

If $Z(p_1,p_2)$ is a PRF, then the numerator and the denominator are

two-variable Hurwitz polynomials and hence the necessity of condition (i) follows.

We have $A(\omega_1,\omega_2) = a_{10}.b_{10}\,\omega_1^2+[(a_{10}.b_{01}+a_{01}.b_{10})\omega_2]\omega_1 + (a_{00}.b_{00}+a_{01}.b_{01}\omega_2^2)$

For $A(\omega_1,\omega_2)$ to be non-negative for all real values of ω_1 and ω_2 we must have:

$$(a_{10}b_{01}+a_{01}b_{10})^2.\omega_2^2 - 4\,a_{10}.b_{10}(a_{00}.b_{00}+a_{01}.b_{01}\omega_2^2) \leq 0.$$

which upon simplification results in

$$(a_{10}.b_{01}-a_{01}.b_{10})^2.\omega_2^2 - 4\,a_{10}.a_{00}.b_{10}.b_{00} \leq 0 \text{ for all real } \omega_2$$

or $a_{10}.b_{01} = a_{01}.b_{10}$

Thus the necessity follows.

<u>Sufficiency:</u>

The given $Z(p_1,p_2)$ can be arranged as below:

$$Z(p_1,p_2) = \frac{a_{10}}{b_{10}} \cdot \frac{p_1 + \frac{a_{01}}{a_{10}}.p_2 + \frac{a_{00}}{a_{10}}}{p_1 + \frac{b_{01}}{b_{10}}.p_2 + \frac{b_{00}}{b_{10}}} = \frac{a_{10}}{b_{10}} \cdot \frac{p_1 + k_1p_2 + \frac{a_{00}}{a_{10}}}{p_1 + k_1p_2 + \frac{b_{00}}{b_{10}}}$$

where $k_1 = \frac{a_{01}}{a_{10}} = \frac{b_{01}}{b_{10}}$

$$\text{Therefore } Z(p_1,p_2) = \frac{a_{10}}{b_{10}} \cdot \left[\frac{1}{1 + \frac{b_{00}}{b_{10}} \cdot \frac{1}{p_1 + k_1p_2}} + \frac{a_{00}/a_{10}}{p_1 + k_1p_2 + \frac{b_{00}}{b_{10}}} \right]$$

which is synthesized by the network shown in Fig. 2.1. Thus the validity of above conclusion follows.

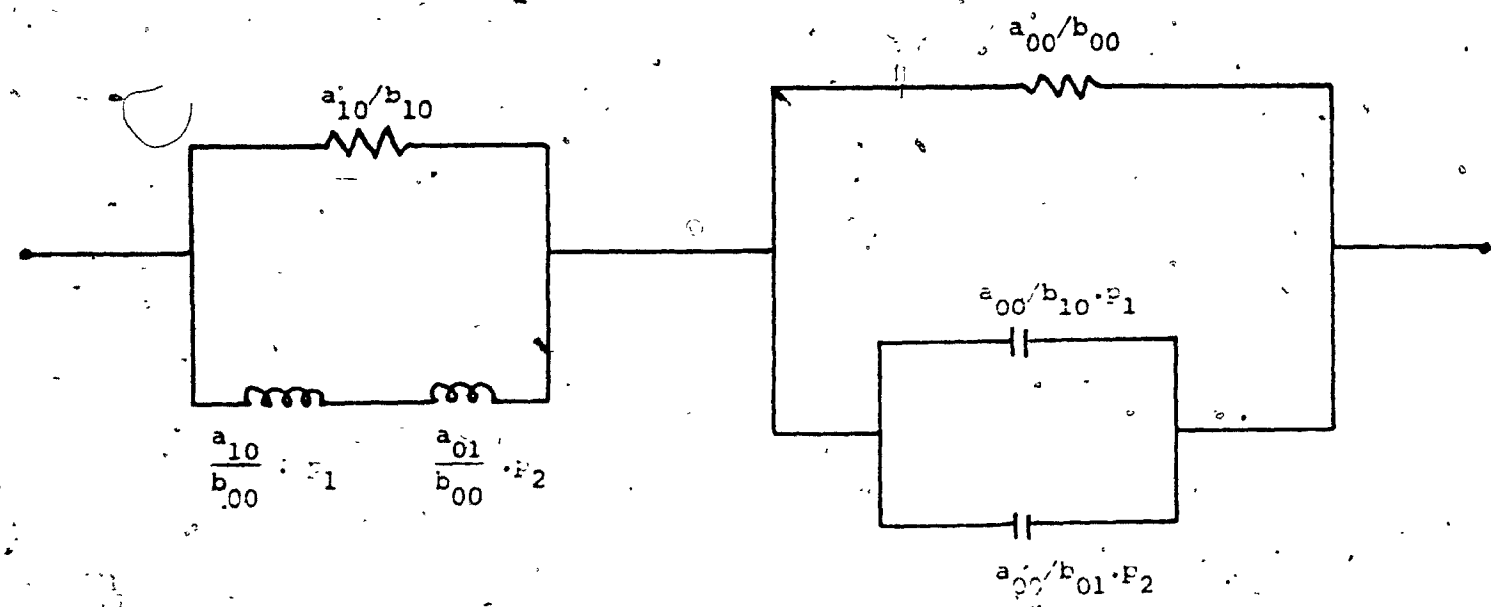


Fig. 2.1

Realization of two-variable first degree function

Special case (ii):

When the given two-variable even polynomial is of fourth degree in ω_1 and ω_2, a simple testing procedure which avoids the above lengthy method is given below:

$$\text{Let } A(\omega_1,\omega_2) = A_{0,0} + (A_{2,0}\omega_1^2 + A_{1,1}\omega_1.\omega_2 + A_{0,2}\omega_2^2)$$
$$+ (A_{4,0}\omega_1^4 + A_{3,1}\omega_1^3.\omega_2 + A_{2,2}\omega_1^2.\omega_2^2 + A_{1,3}\omega_1.\omega_2^3$$
$$+ A_{0,4}\omega_2^4)$$

By defining new variables x and y, the above polynomial is transformed into another polynomial, with the degree of one of the variables being two as below:

$$A(x,y) = A(\omega_1,\omega_2) = A_{0,0} + y(A_{2,0} + A_{1,1}.x + A_{0,2}.x^2)$$
$$+ y^2(A_{4,0} + A_{3,1}.x + A_{2,2}.x^2 + A_{1,3}.x^3 + A_{0,4}.x^4)$$

where $y = \omega_1^2$ and $x = \omega_2/\omega_1$.

$A(\omega_1,\omega_2)$ is non-negative for all real ω_1 and ω_2 if and only if,

(i) $A_{4,0}, A_{0,0}, A_{0,4} \geq 0$

(ii) $(A_{4,0} - A_{3,1} + A_{2,2} - A_{1,3} + A_{0,4}) \geq 0$

(iii) $(A_{4,0} + A_{3,1} + A_{2,2} + A_{1,3} + A_{0,4}) \geq 0$, and

(iv) $(4\,A_{00}.A_{40} - A_{20}^2) + x(4.A_{00}.A_{31} - 2\,A_{20}.A_{11})$

$+ x^2(4A_{00}.A_{22} - A_{11}^2 - 2A_{20}.A_{02}) + x^3(4A_{00}.A_{13} - 2A_{11}.A_{02})$

$+ x^4(4A_{00}.A_{04} - A_{02}^2) \geq 0$ for $-\infty \leq x \leq \infty$.

Proof of the above conclusion is quite simple and follows on the

same lines as that of the special case (i), and hence it is omitted.

2.5 Discussion:

In this chapter, methods of generating an MPRF from the given MPRF and the multivariable real part are considered. The generation of MPRFs by the differential operator makes use of the generalization of Talbot's theorem. By means of these theorems, a necessary coefficient test for an MPRF is provided.

The generation of an MPRF from the given multivariable real part is considered next. The necessary and sufficient conditions on the denominator and the numerator polynomials of the real part are obtained so that an MPRF can be generated. It is pointed out that, in addition to the non-negativeness of the given real part, there need be some extra conditions satisfied so that an MPRF can be generated. While solving for the numerator and the denominator polynomials of an MPRF, it is found out that the number of equations are more than the number of unknowns, and in fact these extra equations impose the conditions on the real part. This is contrary to the single-variable case where, the number of equations are equal to the number of unknowns and the non-negativeness of the real part alone guarantees the generation of a PRF.

The generation of an MPRF from the given imaginary part may be carried out by a method similar to the one followed when the real part is prescribed. Here also the number of equations are more than the number of unknowns. However, the solution seems to be non-unique as may be seen from the following example:

Example 2.4:

Let us consider a two-variable PRF $Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2} =$

$$\frac{1+16p_1+p_2+p_2^2+4p_1p_2+4p_1\ p_2^2}{4+p_1+p_2+p_2^2+p_1p_2+p_1\ p_2^2}$$

The odd part of the above PRF is:

$$Z_0(p_1,p_2) = \frac{N_1M_2-M_1N_2}{M_2^2-N_2^2} = \frac{12\ p_1^2p_2 + 27\ p_1p_2^2 + 63\ p_1 + 3\ p_2 + 3\ p_1p_2^4}{16 - p_1^2 + 6\ p_1p_2 + 7\ p_2^2 - p_1^2p_2^2 - p_1^2p_2^4 + p_2^4}$$

Now let us assume that the above $Z_0(p_1,p_2)$ is given and try to calculate a two-variable positive real function by a method similar to the one followed for the real part case.

The calculation of the denominator polynomial of $Z(p_1,p_2)$ is the same as before and hence

$$M_2 + N_2 = 4 + p_1 + p_2 + p_2^2 + p_1p_2 + p_1\ p_2^2.$$

To calculate the numerator polynomial, we will assume that

$$M_1 + N_1 = a_{00} + a_{10}\cdot p_1 + a_{01}\cdot p_2 + a_{11}\cdot p_1\cdot p_2 + a_{20}\cdot p_1^2 + a_{12}p_1\cdot p_2^2$$

where the a's are non-negative constants. Calculating $(N_1M_2-M_1N_2)$ from the assumed (M_1+N_1) and the calculated (M_2+N_2) and equating the corresponding coefficients the following set of equations are obtained.

$$4\ a_{10} - a_{00} = 63 \qquad ..(2.18a)$$

$$a_{10} + a_{01} + 4\ a_{12} - a_{11} - a_{20} - a_{00} = 27 \qquad ..(2.18b)$$

$$a_{12} - a_{20} = 3 \qquad ..(2.18c)$$

$$4\ a_{01} - a_{00} = 3 \qquad ..(2.18d)$$

$$a_{10} - a_{11} = 12 \qquad ..(2.18e)$$

$$a_{01} - a_{20} = 0 \qquad ..(2.18f)$$

$$a_{12} - a_{11} = 0 \qquad ..(2.18g)$$

It may be seen immediately that even though the number of equations which are seven in number are more than the number of unknowns which are six in number, still the evaluation of the a's is not unique. All the a's may be expressed in terms of any one of the a's, but we will express them in terms of a_{00} as below:

$$a_{10} = \frac{a_{00} + 63}{4}$$

$$a_{01} = \frac{a_{00} + 3}{4}$$

$$a_{20} = \frac{3 + a_{00}}{4}$$

$$a_{11} = \frac{a_{00} + 15}{4}, \text{ and}$$

$$a_{12} = \frac{a_{00} + 15}{4}$$

Hence, out of the seven equations, two are linearly dependent upon the remaining five. Now by assuming a proper value for a_{00}, the coefficients of the numerator polynomial may be obtained. Thus the solution is not unique.

When an MPRF is to be generated from the given real part there are certain conditions imposed in order that a solution exists. Also when the imaginary part is prescribed, there seems that such conditions do exist and the solution appears to be non-unique. However, a conclusive

proof for the non-uniqueness of the solution is lacking.

The method of testing the two-variable polynomial for non-negativeness is the extended version of the Sturm test for the single-variable. But it seems to be too cumbersome to extend the same idea for polynomials of more than two variables, and probably an entirely new approach may have to be adopted for testing such polynomials for non-negativeness. The method of testing the two-variable polynomials for non-negativeness may be useful in the testing of Hurwitz polynomials of three-variables.

CHAPTER III

A MULTIVARIABLE ARRAY AND ITS APPLICATIONS TO LADDER NETWORKS

3.1 Introduction:

The single-variable reactance function can always be realized as a low-pass ladder network by a continued-fraction expansion. The relation between the ladder network elements and the Routh-Hurwitz array is well established. It is also known that not every single-variable PRF can be realized by a continued-fraction expansion. Hence, it is natural to predict that not all two-variable reactance functions - being generalizations of single-variable PRFs[9] - and hence, not all the multivariable reactance functions are realizable by continued-fraction expansion as ladder networks, which is corroborated by different synthesis procedures[11,13,14,16]. Furthermore, for multivariable network functions, the conditions for the continued-fraction expansion and those for the realizability of ladder networks were unavailable in the literature[47].

In this chapter[47], we propose a multivariable array from which the realizability conditions for the multivariable low-pass ladder networks (MLPLs) consisting of series inductors and shunt capacitors are obtained. (Any series or shunt branch contains exactly one p_i-type ($1 \leq i \leq n$) element.) This array, for a single-variable, reduces to the Routh-Hurwitz array. By suitable transformations, several other types of multivariable ladder networks and their realizability conditions are derived starting from the MLPL.

3.2 Two-Variable Array and its Applications:

For the sake of convenience, we shall first discuss the two-variable array and show later how the same principle can be extended to the multi-variable array. In addition it is shown how the realizability conditions of the ladder networks are derived from these arrays.

3.2.1 The Two-Variable Array:

Let $Z(p_1,p_2) = \dfrac{M(p_1,p_2)}{N(p_1,p_2)}$ or $\dfrac{N(p_1,p_2)}{M(p_1,p_2)}$ be the two-variable reactance function* of kth degree in the variables p_1,p_2, and let

$$H(p_1,p_2) = M(p_1,p_2) + N(p_1,p_2) = (a_{k,0}\,p_1^k + a_{k-1,1}\,p_1^{k-1}.p_2 + \ldots + a_{0,k}\,p_2^k) + \ldots + (a_{1,0}\,p_1 + a_{0,1}\,p_2) + a_{0,0} \qquad ..(3.1)$$

be the sum of net numerator and denominator (after cancellation of all non-constant polynomial factors common to numerator and denominator) of $Z(p_1,p_2)$. From this two-variable Hurwitz polynomial $H(p_1,p_2)$, we form the following three-rowed array:

1st row:	$a_{k,0}$	$a_{0,k}$	$a_{k-1,1}\cdots a_{1,k-1}$	$a_{k-2,0}$	$a_{k-3,1}\cdots a_{0,k-2}\cdots$	
2nd row:	$a_{k-1,0}$	0	$a_{k-2,1}\cdots a_{0,k-1}$	$a_{k-3,0}$	$a_{k-4,1}\cdots 0\cdots$	
3rd row:	0	$a_{0,k-1}$	$a_{k-1,0}\cdots a_{1,k-2}$	0	$a_{k-3,0}\cdots a_{0,k-3}\cdots$	

..(3.2)

The rules for writing the array are as follows:

(i) The first row contains terms of degrees $k,(k-2),(k-4),\ldots$ etc. For convenience, the kth degree terms are arranged

---

* Where the $N(p_1,p_2)$ and $M(p_1,p_2)$ are odd and even polynomials in p_1,p_2 respectively. At places, for convenience, the arguments in the brackets are omitted.

first followed by (k-2), (k-4), (k-6)...etc. degree terms.

(ii) The second and third rows contain (k-1), (k-3), (k-5).. etc. degree terms in the following manner:

If $a_{x,y}$ is a particular term in the first row, then $a_{x-1,y}$ and $a_{x,y-1}$ are the terms respectively in the same column of the second and third rows. If (x-1) or (y-1) is less than zero, then the corresponding term is zero.

From the above array, we form the following 3 x 3 determinants:

$$\Delta_{(k,0),(0,k),(x,y)} = \begin{vmatrix} a_{k,0} & a_{0,k} & a_{x,y} \\ a_{k-1,0} & 0 & a_{x-1,y} \\ 0 & a_{0,k-1} & a_{x,y-1} \end{vmatrix} \qquad ..(3.3)$$

where $0 \leq x \leq k$ and $0 \leq y \leq k$

$$\text{and } b_{x,y} = \frac{\Delta_{(k,0),(0,k),(x,y)}}{a_{k-1,0} \cdot a_{0,k-1}} \qquad ..(3.4)$$

From the calculated values of $b_{x,y}$ and the earlier second and third row terms, the following new array is formed:

1st row:	$a_{k-1,0}$	$a_{0,k-1}$	$a_{k-2,1} \cdots a_{1,k-2}$	$a_{k-3,0}$	$a_{k-4,1} \cdots a_{0,k-3} \cdots$	
2nd row:	$b_{k-2,0}$	0	$b_{k-3,1} \cdots b_{0,k-2}$	$b_{k-4,0}$	$b_{k-5,1} \cdots$ 0	
3rd row:	0	$b_{0,k-2}$	$b_{k-2,0} \cdots b_{1,k-3}$	0	$b_{k-4,0} \cdots b_{0,k-4} \cdots$	

..(3.5)

In this array, the first row contains terms of degrees (k-1), (k-3), (k-5)....etc. and second and third rows contain terms of degrees (k-2), (k-4)....etc. The rules of forming this array are the same as

those for the earlier ones. From the array (3.5), we then calculate the following determinants:

$$\Delta_{(k-1,0),(0,k-1),(x_1,y_1)} = \begin{vmatrix} a_{k-1,0} & a_{0,k-1} & a_{x_1,y_1} \\ b_{k-2,0} & 0 & b_{(x_1-1),y_1} \\ 0 & b_{0,k-2} & b_{x_1,(y_1-1)} \end{vmatrix} \quad ..(3.6)$$

where $0 \leq x_1 \leq (k-1)$ and $0 \leq y_1 \leq (k-1)$

and $$c_{x_1,y_1} = \frac{\Delta_{(k-1,0),(0,k-1),(x_1,y_1)}}{b_{k-2,0} \cdot b_{0,k-2}} \quad ..(3.7)$$

From the above calculated values of c_{x_1,y_1} and $b_{x,y}$ we form the new arrays similar to the above and calculate d_{x_2,y_2}, e_{x_3,y_3},....etc.

It may be noted that for the case of a single-variable, the above arrays reduce to the Routh-Hurwitz array.

3.2.2 Two-Variable Low-Pass Ladder Networks(TLPL):

Using the above developed arrays, the realizability conditions for the TLPL are given by the following theorem:

Theorem 3.1:

The given two-variable reactance function

$Z(p_1,p_2) = \frac{M(p_1,p_2)}{N(p_1,p_2)}$ [or $\frac{N(p_1,p_2)}{M(p_1,p_2)}$] can be realized by the TLPL of Fig. 3.1 , if and only if

(i) $H(p_1,p_2)$, the sum of net numerator and denominator of $Z(p_1,p_2)$ is a polynomial in p_1,p_2 with no missing terms,

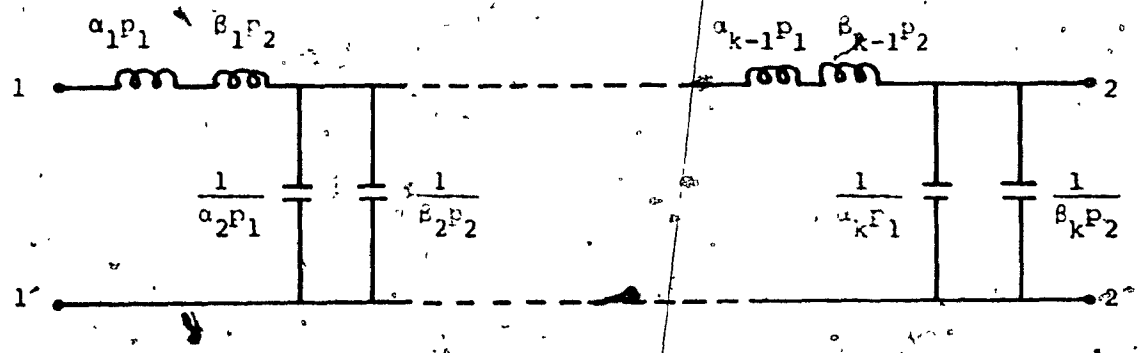


Fig. 3.1

Two-variable Low-Pass Ladder Network

and

(ii) when $H(p_1,p_2)$ is arranged in the form of the two-variable arrays, the following conditions hold:

(a) $b_{x,y} = 0$ if $x+y = k$

> 0 if $x+y < k$

(b) $c_{x_1,y_1} = 0$ if $x_1+y_1 = (k-1)$

> 0 if $(x_1+y_1) < (k-1)$

(c) and similar conditions hold for d_{x_2,y_2}, e_{x_3,y_3}, ...etc.

Proof: Necessity:

The transmission matrix of the two-port TLPL of Fig. 3.1 is the product of the following matrices:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & \alpha_1 p_1+\beta_1 p_2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \alpha_2 p_1+\beta_2 p_2 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ \alpha_k p_1+\beta_k p_2 & 1 \end{bmatrix}.$$

By direct matrix multiplication or otherwise[48], the necessity of condition (i) can be seen. If the given $Z(p_1,p_2)$ is realizable as TLPL, then the kth degree terms of $H(p_1,p_2)$ must be divisible by the $(k-1)^{th}$ degree terms with a quotient of $(\alpha_1 p_1+\beta_1 p_2)$ where α_1, $\beta_1 > 0$, with the remaining terms being positive. In fact this is what condition (ii)(a) implies. Thus, by repeating the same argument, the necessity of condition (ii) is proved.

Sufficiency:

For this part of the proof, we shall consider

$$Z(p_1,p_2) = \frac{N(p_1,p_2)}{M(p_1,p_2)}$$

and the same arguments apply for the case

$$Z(p_1,p_2) = \frac{M(p_1,p_2)}{N(p_1,p_2)}$$

Condition (ii)(a) implies that the kth degree terms of $H(p_1,p_2)$ are divisible by $(k-1)^{th}$ degree terms with a quotient of $(\alpha_1 p_1+\beta_1 p_2)$ with $\alpha_1 > 0$ and $\beta_1 > 0$ and all remaining terms being positive. Thus

$$Z(p_1,p_2) = (\alpha_1 p_1+\beta_1 p_2) + \frac{N'(p_1,p_2)}{M(p_1,p_2)}$$

and N'/M is of $(k-1)^{th}$ degree in p_1,p_2. By applying condition (ii)(b) we infer that

$$\frac{M(p_1,p_2)}{N'(p_1,p_2)} = (\alpha_2 p_1+\beta_2 p_2) + \frac{M'(p_1,p_2)}{N'(p_1,p_2)}$$

where $\alpha_2,\beta_2 > 0$ and M'/N' is of degree $(k-2)$. Condition (i) implies that no determinant is trivially zero, that is, no column in the array has only zeros. Thus, by repeating the above argument, the sufficiency follows.

3.2.3 Two-Variable Low-Pass Ladder with Resistive Termination:

If the TLPL is resistively terminated at 2-2', the resulting functions have some interesting features, which can be represented by mixed lumped-distributed structures. We shall give the realizability conditions for this type of ladder networks in Theorem 3.2, the proof of which is based on the following lemma:

Lemma 3.1:

If the

$$Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2}$$

is a two-variable PRF of kth degree with

(i) (M_2+N_2) being a polynomial in p_1, p_2 with no missing terms, and

(ii) $M_1M_2-N_1N_2 = R > 0$,

then (M_1+N_1) is also a polynomial in p_1, p_2 with no missing terms.

Proof:

If condition (ii) has to be satisfied, there must be a difference of one degree between those of the numerator and the denominator. Without loss of generality, let us consider that (M_2+N_2) is one degree higher. The same result can be proved by considering (M_1+N_1) to be one degree higher. Let us express $Z(p_1,p_2)$ in the following form:

$$Z(p_1,p_2) = \frac{g_0(p_2)+g_1(p_2).p_1+\ldots+g_{k-2}(p_2).p_1^{k-2}+g_{k-1}(p_2).p_1^{k-1}}{h_0(p_2)+h_1(p_2).p_1+\ldots+h_{k-2}(p_2).p_1^{k-2}+h_{k-1}(p_2).p_1^{k-1}+h_k(p_2)p_1^k} \quad ..(3.8)$$

where the g's and h's are polynomials in p_2.

Since (M_2+N_2) is a polynomial with no missing terms, condition (ii) assures the presence of the constant term in (M_1+N_1). Since (M_1+N_1) is one degree less than (M_2+N_2), $g_{k-1}(p_2)$ is a positive constant. As condition (ii) holds even when $p_1 = 0$, we can conclude that $g_0(p_2)$ is a Hurwitz polynomial in p_2 of the form $(g_{0,0}+g_{0,1}\,p_2+g_{0,2}\,p_2^2+\ldots+g_{0,k-1}.p_2^{k-1})$. Similarly by putting $p_2 = 0$, we can infer that the corresponding numerator polynomial will be Hurwitz of the form $(g_{0,0}+g_{1,0}\,p_1+g_{2,0}\,p_1^2+\ldots+g_{k-1,0}\,p_1^{k-1})$.

We have $\frac{g_{k-2}(p_2)}{h_{k-2}(p_2)}$ is a single-variable PRF [42], because $Z(p_1,p_2)$

is positive real. By condition (i), $h_{k-2}(p_2) = (h_{k-2,0}+h_{k-1,1}\,p_2+h_{k-2,2}\,p_2^2)$ and from the earlier reasoning, we know that the term $g_{k-2,0}$ is present in $g_{k-2}(p_2)$. Hence, for $\frac{g_{k-2}(p_2)}{h_{k-2}(p_2)}$ to be a PRF, it is necessary that $g_{k-2}(p_2)$ must have the term $g_{k-2,1}\cdot p_2$. Also, the p_2-degree of $g_{k-2}(p_2)$ cannot be more than one, as the total degree of (M_1+N_1) is at most $(k-1)$. Thus $g_{k-2}(p_2)$ must be of the form $g_{k-2}(p_2) = (g_{k-2,0}+g_{k-2,1}p_2)$.

By condition (i), we have

$h_{k-3}(p_2) = (h_{k-3,0}+h_{k-3,1}\,p_2+h_{k-3,2}\,p_2^2+h_{k-3,3}\,p_2^3)$. Then, for $\frac{g_{k-3}(p_2)}{h_{k-3}(p_2)}$ to be a PRF in p_2, the degree of $g_{k-3}(p_2)$ has to be at least two. Because the total degree of (M_1+N_1) is assumed to be $(k-1)$, we can have the degree of $g_{k-3}(p_2)$ at most two. Hence $g_{k-3}(p_2)$ is of degree two.

For $\frac{g_{k-3}(p_2)}{g_{k-2}(p_2)} = \frac{g_{k-3}(p_2)}{g_{k-2,0}+g_{k-2,1}p_2}$ to be a PRF, the first degree term in $g_{k-3}(p_2)$ must be present. We also know that $g_{k-3,0} > 0$ and hence $g_{k-3}(p_2)$ must be of the form $g_{k-3}(p_2) = (g_{k-3,0}+g_{k-3,1}\,p_2+g_{k-3,2}\,p_2^2)$.

Thus, by similar repeated arguments, we arrive at the conclusion that

$$g_1(p_2) = (g_{1,0}+g_{1,1}\,p_2+g_{1,2}\,p_2^2+\ldots+g_{1,k-2}\,p_2^{k-2}).$$

When these values of $g_i(p_2)$'s are substituted back in (M_1+N_1), we see that (M_1+N_1) is a polynomial with no missing terms of degree $(k-1)$ in p_1, p_2 and thus the lemma is proved.

Theorem 3.2:

The two-variable PRF $Z(p_1,p_2) = \dfrac{M_1+N_1}{M_2+N_2}$, where (M_1+N_1) and (M_2+N_2) are the polynomials with no missing terms, can be realized as TLPL with resistive termination, if and only if

$$M_1M_2-N_1N_2 = R > 0.$$

[It follows that $\dfrac{M_1}{N_1}$, $\dfrac{M_2}{N_2}$, $\dfrac{M_1}{N_2}$ and $\dfrac{M_2}{N_1}$ obey the conditions of Theorem 3.1].

Proof:

The necessity follows immediately since, for such a structure, all the transmission zeros are at $p_1 = \infty$ and $p_2 = \infty$ independent of the other variable. Hence, we give the proof of sufficiency.

Sufficiency:

Without loss of generality, let us assume that the degree of (M_1+N_1) is greater than that of (M_2+N_2). Since (M_1+N_1) and (M_2+N_2) are polynomials with no missing terms, it is obvious that $Z(p_1,p_2)$ has poles at $p_1 = \infty$ and $p_2 = \infty$ independent of the other variable which can be extracted as series inductors leaving a positive real function[9]

$$Z'(p_1,p_2) = \frac{M_1'+N_1'}{M_2+N_2}$$

Now $M_1'M_2-N_1'N_2 = R > 0$ and $Z'(p_1,p_2)$ is of one degree less than $Z(p_1,p_2)$. Hence, by lemma (3.1), $(M_1'+N_1')$ is also a polynomial with no missing terms and $Z'(p_1,p_2)$ has zeros at $p_1 = \infty$ and $p_2 = \infty$ independent of the other variable, which can be removed as shunt capacitors. The degree of the resulting positive real function has been reduced by two from that of

$Z(p_1,p_2)$ and the real part remains unchanged. Thus, by repeated extractions of poles and zeros, we arrive at a zero degree function, which can be realized as a resistor. Thus the sufficiency is established.

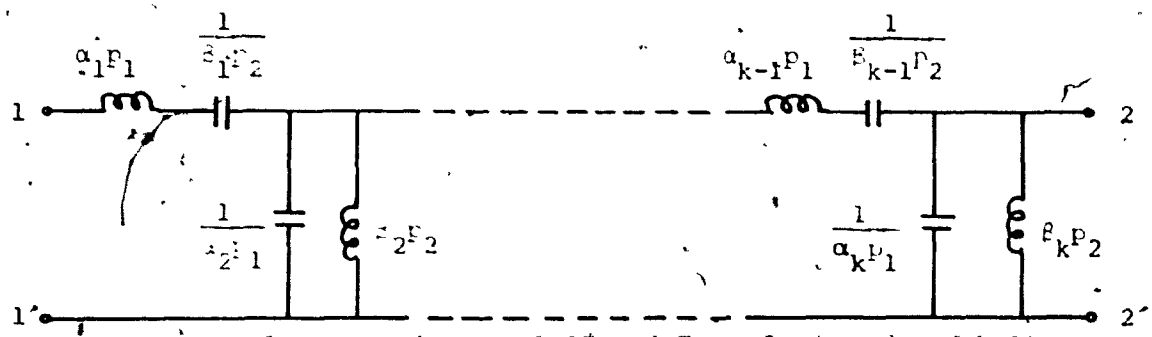
For the two-variable PRF $Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2}$, we can interpret M_1/N_1, N_2/M_2, N_1/M_2 and M_1/N_2 as the open-and short-circuit driving point functions of a lossless two-port network. Since these are the driving point functions of a TLPL, they satisfy the conditions of Theorem 3.1.

3.2.4 Other Types of Two-Variable Ladder Networks:

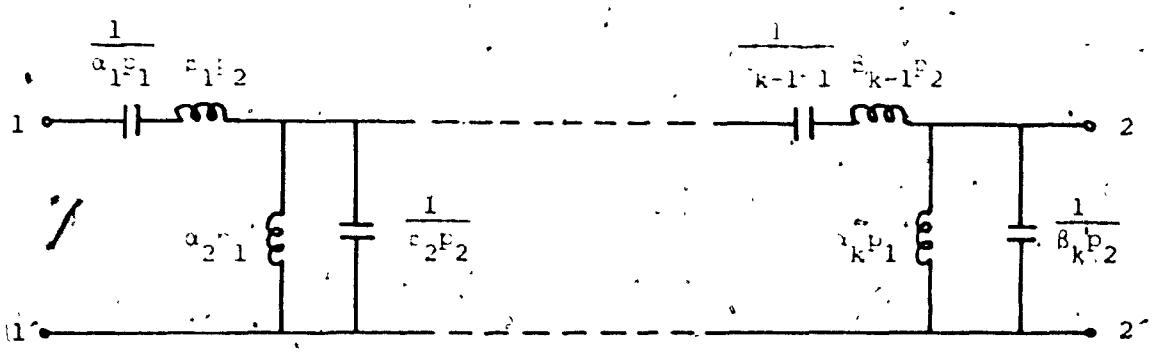
We propose some more types of two-variable ladder networks as shown in Fig. 3.2 and Fig. 3.3. The realizability conditions of Fig. 3.2(a), (b) and (c) ladder networks are derivable from the low-pass network as follows: By making $p_1 \to \frac{1}{p_1}$ keeping the other variable unchanged, $p_2 \to \frac{1}{p_2}$ while p_1 is unchanged, and $p_1 \to \frac{1}{p_1}$ & $p_2 \to \frac{1}{p_2}$ transformations on the network functions of Fig. 3.2(a), (b), and (c) respectively, we obtain the low-pass ladder of Fig. 3.1. Hence, the low-pass ladder network realizability conditions hold for these networks also, after making the proper transformations on the given network function.

Specifically, $Z(p_1,p_2)$, the given two-variable reactance function, can be realized by the ladder network of Fig. 3.2(a), if and only if, after making the transformation $p_1 \to \frac{1}{p_1}$ on $Z(p_1,p_2)$ the resulting network function satisfies the conditions of Theorem 3.1.

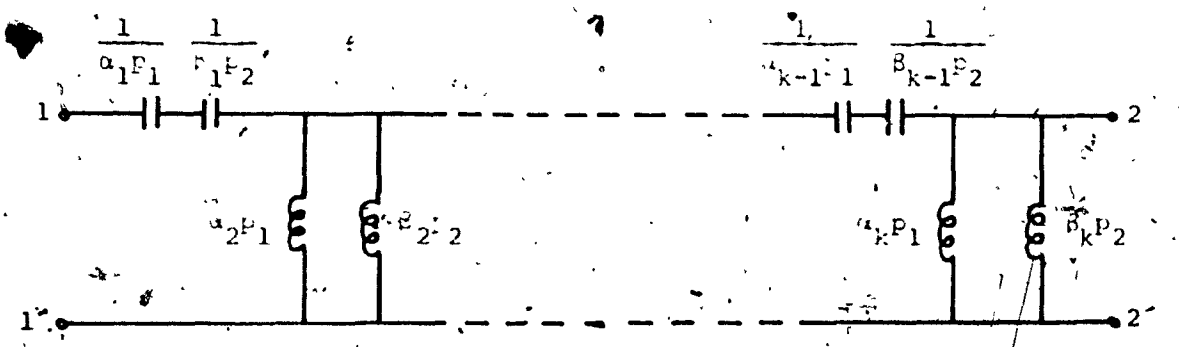
Also it may be noted that the driving point functions of these resistively terminated networks satisfy some necessary conditions, which can be used as inspection tests. For example, the input impedance



(a)



(b)



(c)

Fig. 3.2

Two-Variable Ladder Networks consisting of p_1-and p_2-type elements in series in the series arms and in parellel in the shunt arms.

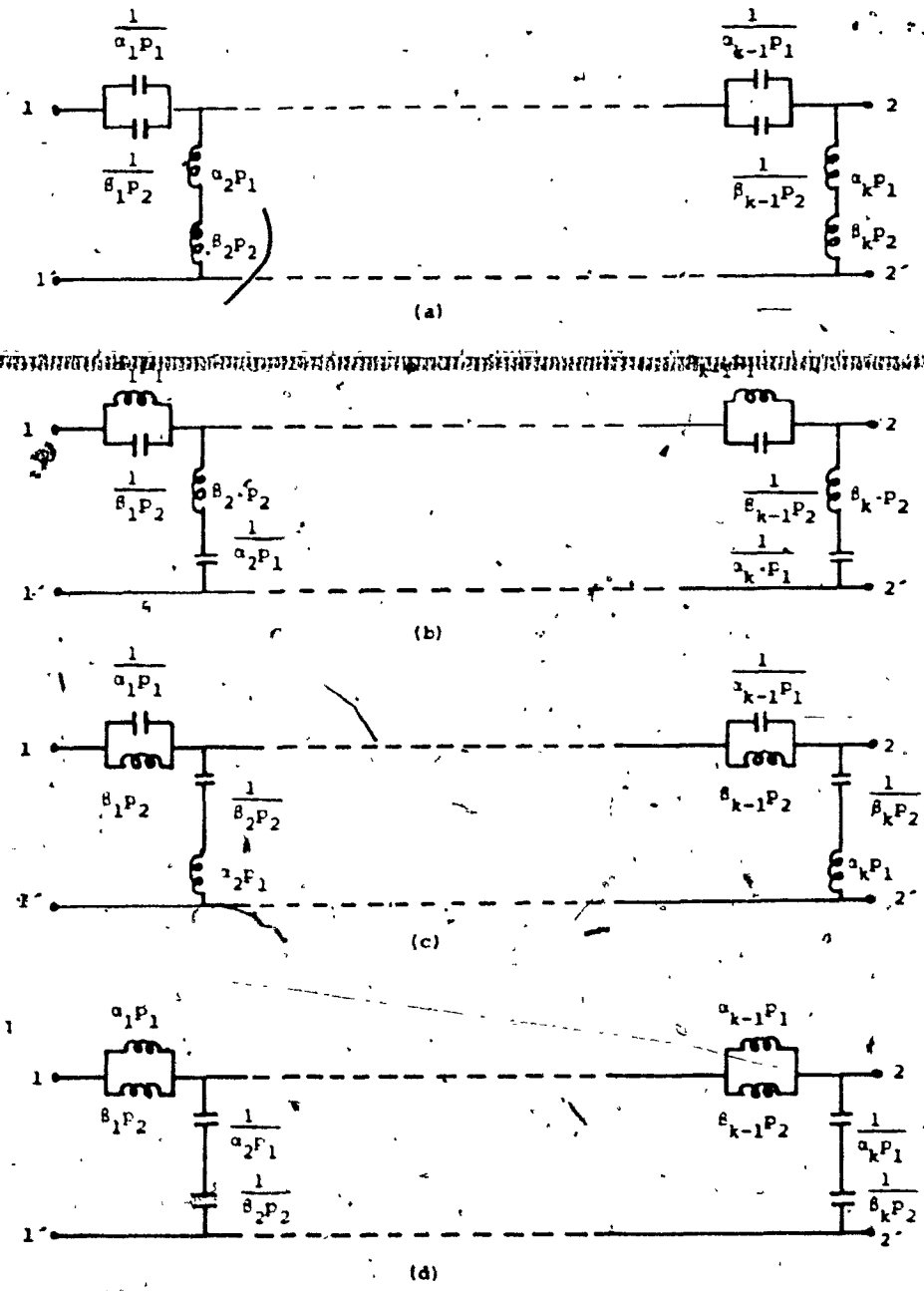


Fig. 3.3

Two-Variable ladder Networks consisting of p_1-and p_2-type elements in parallel in the series arms and in series in the shunt arms:

$Z(p_1,p_2)$ of a resistively terminated structure of Fig. 3.2(a) shall be of the following form:

$$Z(p_1,\lambda) = \frac{f(p_1,\lambda)}{g(p_1,\lambda)} \cdot p_1^{\varepsilon}$$

where $\lambda = p_1 p_2$

$f(p_1,\lambda)$ and $g(p_1,\lambda)$ are polynomials in p_1,λ with no missing terms, and $\varepsilon = +1$ if $g(p_1,\lambda)$ is of higher degree

$= -1$ if $g(p_1,\lambda)$ is of lower degree.

Conditions similar to these exist for the other networks also and they are all tabulated in Table 3.1.

In Fig. 3.3 are proposed another type of two-variable lossless ladder networks, where the series arms contain the p_1-and p_2-type elements in parallel and in the shunt arms they are in series. The realizability conditions of such networks are given by Theorems 3.3 and 3.4. Before that, we present a lemma which is used in proving the Theorems 3.3 and 3.4.

Lemma 3.2:

If $Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2}$, the two-variable positive real function, is the input impedance of the ladder structure of Fig. 3.3(a) with resistive termination, then the kth degree polynomials (M_1+N_1) and (M_2+N_2) satisfy the following conditions:

(i) They do not have any missing terms, except one of them not having the constant term.

(ii) The jth $(1 \le j \le k)$ degree terms are factorizable as product or sum of product of terms of the type $(\alpha_i p_1+\beta_i p_2)$, α_i, $\beta_i \ge 0$.

TABLE 3.1

Some properties of two-variable ladder networks consisting of p_1-and p_2-type elements in series in the series arms and in parallel in the shunt arms

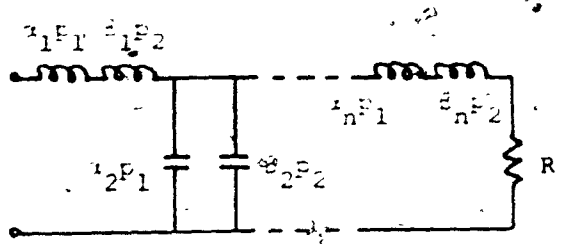
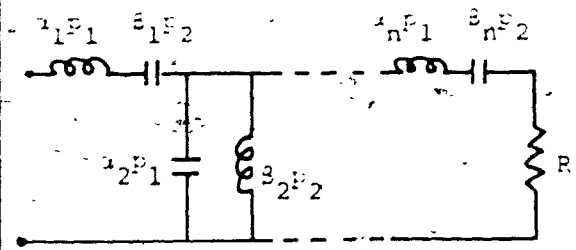
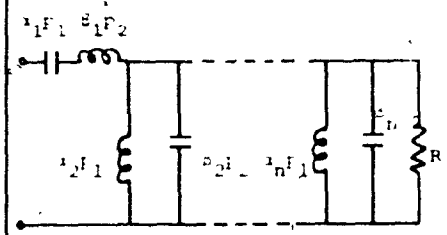
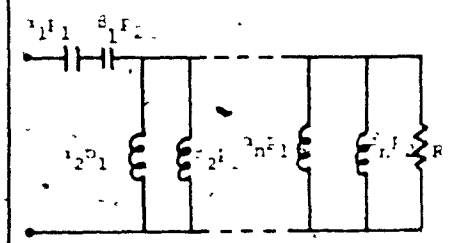
Type of Network	Nature of the polynomials	$M_1(p_1,p_2)\cdot M_2(p_1,p_2) - N_1(p_1,p_2)\cdot N_2(p_1,p_2)$
	$Z(p_1,p_2) = \dfrac{f(p_1,p_2)}{g(p_1,p_2)}$ f and g are polynomials with no missing terms	$R > 0$
	$Z(p_1,p_2) = Z(p_2,\lambda) = \dfrac{f(p_2,\lambda)}{g(p_2,\lambda)}\cdot p_2^{\varepsilon}$ where $\lambda = p_1 p_2$, f and g are polynomials with no missing terms, and $\varepsilon = +1$, if the degree of f is lower $= -1$, if the degree of f is higher	$\pm\, p_2^{2k}$

TABLE 3.1 - cont'd

Type of Network	Nature of the polynomials	$M_1(P_1,P_2)\cdot M_2(P_1,P_2) - N_1(P_1,P_2)\cdot N_2(P_1,P_2)$
	Same as above when P_2 is replaced by P_1	$\pm P_1^{2k}$
	$Z(P_1,P_2) = \dfrac{P_1^k\cdot f_k(P_2)+\ldots+P_1\cdot f_1(P_2)+f_0(P_2)}{P_1P_2\cdot[P_1^{k-1}\cdot g_{k-1}(P_2)+\ldots+g_0(P_2)]}$ where $f_k(P_2)$ is a polynomial in P_2 degree k with no missing terms $f_{k-1}(P_2)$ is a polynomial in P_2 degree with no missing terms except P_2^0 coefficient missing. $f_0(P_2)$ is a polynomial in P_2 degree k with no missing terms except P_2^0 to P_2^{k-1} coefficients missing $g_{k-1}(P_2)$ is a polynomial in P_2 degree (k-1) with no missing terms $g_{k-2}(P_2)$ is a polynomial in P_2 degree (k-1) with no missing terms except P_2^0 terms missing	$\pm P_1^{2k}\cdot P_2^{2k}$

(iii) The highest degree terms of (M_1+N_1) and (M_2+N_2) are the same and products of the type $(\alpha_i p_1+\beta_i p_2)$, and it is these factors that are present in the other degree terms also.

Proof:

The proof of this lemma will be given by mathematical induction.

We can calculate the input impedance of the network [illegible] transmission parameters as follows. Let

$$[a_k] = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} = [a]_1 \cdot [a]_2 \cdot \ldots [a]_i \ldots [a]_k$$

be the transmission matrix of the ladder network without the resistive load; where $[a]_i$ is the transmission matrix of the ith arm.

Thus

$$[a]_i = \begin{bmatrix} 1 & \frac{1}{\alpha_i p_1+\beta_i p_2} \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ \frac{1}{\alpha_i p_1+\beta_i p_2} & 1 \end{bmatrix}$$

depending upon the ith section is series arm or shunt arm respectively. Let us prove the lemma on the assumption that the first section is a series arm and it can be proved for a shunt arm being the first one on the same lines.

For the first section, the transmission matrix is given by:

$$[a_1] = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\alpha_1 p_1+\beta_1 p_2} \\ 0 & 1 \end{bmatrix}$$

and the input impedance for resistive termination is as follows:

$$Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2} = \frac{A_1R+B_1}{C_1R+D_1} = \frac{R.(\alpha_1p_1+\beta_1p_2)+1}{(\alpha_1p_1+\beta_1p_2)}, \text{ where } R > 0.$$

Thus for one section the lemma follows.

For two sections, the transmission matrix and the input impedance are as below:

$$[a_2] = \begin{bmatrix} 1 & \frac{1}{\alpha_1p_1+\beta_1p_2} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \frac{1}{\alpha_2p_1+\beta_2p_2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+(\alpha_1p_1+\beta_1p_2)(\alpha_2p_1+\beta_2p_2)}{(\alpha_1p_1+\beta_1p_2)(\alpha_2p_1+\beta_2p_2)} & \frac{1}{\alpha_1p_1+\beta_1p_2} \\ \frac{1}{\alpha_2p_1+\beta_2p_2} & 1 \end{bmatrix}$$

$$Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2} = \frac{R[1+(\alpha_1p_1+\beta_1p_2)(\alpha_2p_1+\beta_2p_2)] + (\alpha_2p_1+\beta_2p_2)}{R(\alpha_1p_1+\beta_1p_2) + (\alpha_1p_1+\beta_1p_2)(\alpha_2p_1+\beta_2p_2)}$$

where $R > 0$.

Thus, for resistive termination the conditions of the lemma are satisfied.

Now let us assume that the lemma is true for r sections and prove its validity for (r+1) sections. Thus for (r+1) sections, the transmission matrix is:

$$[a_{r+1}] = \begin{bmatrix} A_{r+1} & B_{r+1} \\ C_{r+1} & D_{r+1} \end{bmatrix} = \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2} \\ 0 & 1 \end{bmatrix}$$

or

$$= \begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ \dfrac{1}{\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2} & 1 \end{bmatrix}$$

depending upon whether the $(r+1)^{th}$ section is a series arm or a shunt arm respectively. Let us prove the lemma for the case of the $(r+1)^{th}$ section being a series arm, since the proof for the shunt arm case follows on similar lines. Then the transmission matrix for the $(r+1)$ sections is:

$$[a_{r+1}] = \begin{bmatrix} A_{r+1} & B_{r+1} \\ C_{r+1} & D_{r+1} \end{bmatrix}$$

$$= \begin{bmatrix} A_r & \dfrac{A_r+B_r\cdot(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)}{(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)} \\ C_r & \dfrac{C_r+D_r(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)}{(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)} \end{bmatrix}$$

and the input impedance is:

$$Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2} = \frac{R.A_{r+1}+B_{r+1}}{R.C_{r+1}+D_{r+1}}$$

$$= \frac{R.A_r\cdot(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)+[A_r+B_r\cdot(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)]}{R.C_r\cdot(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)+[C_r+D_r\cdot(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)]}$$

If A_r and D_r had the highest degree terms, then B_{r+1} and C_{r+1} are having the highest degree terms. We can see that the previous factors are multiplied by $(\alpha_{r+1}\cdot p_1+\beta_{r+1}\cdot p_2)$ thus satisfying the conditions (ii) and (iii). Since it is assumed that the first section to be the series arms, it is seen that (M_1+N_1) is having the constant term whereas (M_2+N_2) does

not have the constant term. Thus the lemma is true for (r+1) sections and hence true for k sections.

Using Lemma 3.2 and Theorems 3.1 and 3.2, we give the realizability conditions for the ladder network of Fig. 3.3(a) without and with resistive termination in the following two theorems.

Theorem 3.3:

The given two-variable reactance function $Z(p_1,p_2)$ can be realized by the ladder network of Fig. 3.3(a), if and only if after making the transformation $(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{(\alpha_i p_1+\beta_i p_2)}$ in the given function, ($(\alpha_i p_1+\beta_i p_2)$ known from Lemma 3.2) the resulting function satisfies the conditions of Theorem 3.1.

When the transformation $(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{(\alpha_i p_1+\beta_i p_2)}$ is applied on the network of Fig. 3.3(a), we obtain the TLPL. This establishes the above theorem.

Theorem 3.4:

The two-variable PRF $Z(p_1,p_2) = \frac{M_1+N_1}{M_2+N_2}$ can be realized by a ladder network of Fig. 3.3(a) with resistive termination, if and only if

(i) $M_1M_2-N_1N_2 = \pm R.\prod_{i=1}^{k} (\alpha_i p_1+\beta_i p_2)^2$, where α_i, $\beta_i > 0$, k is the degree of the given function, and, the positive sign applies if the degree of the right-hand side is an even multiple of two and the negative sign applies, if it is an odd multiple of two.

(ii) The kth degree polynomials (M_1+N_1) and (M_2+N_2) satisfy the conditions of Lemma 3.2.

The above conditions (i) and (ii), can be replaced by an equivalent statement as follows: In the given function $Z(p_1,p_2)$, if the transformation

$(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{\alpha_i p_1+\beta_i p_2}$ is made, the resulting function should satisfy the conditions of Theorem 3.2.

Proof: Necessity:

In Lemma 3.2, we have shown the necessity of condition (ii). Hence it is required to show the necessity of condition (i) only. We know, that for a ladder network the zero of transmission occurs, when the series arm becomes an open-circuit or the shunt arm becomes a short-circuit. Thus, in our case, the zeros of transmission occur whenever $(\alpha_i p_1+\beta_i p_2)= 0$. Hence $(M_1M_2-N_1N_2)$ contains the factors of the type $(\alpha_i p_1+\beta_i p_2)^2$ and thus the necessity of condition (i) follows.

Sufficiency:

By making $(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{\alpha_i p_1+\beta_i p_2}$ transformation on $(M_1M_2-N_1N_2)$, we get the resulting polynomial to be a constant. Since (M_2+N_2) is degree k, $(M_2^2-N_2^2)$ will be at most of degree 2k. Hence, when the above transformation is made on $\frac{M_1M_2-N_1N_2}{M_2^2-N_2^2}$, the factor $\prod_{i=1}^{k} (\alpha_i p_1+\beta_i p_2)^2$ cancels out between the numerator and denominator and finally we are left with a positive constant for $M_1M_2-N_1N_2$.

We can see that by making $(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{(\alpha_i p_1+\beta_i p_2)}$ transformation on the given $Z(p_1,p_2)$ the resulting numerator and denominator are polynomials with no missing terms.

Thus, we see that the given function satisfies the conditions of Theorem 3.2 after the transformation is made. Hence, the resulting function can be realized by TLPL with resistive termination. After the

realization of the resulting function as TLPL, by making the retransformation $(\alpha_i p_1+\beta_i p_2) \rightarrow \frac{1}{(\alpha_i p_1+\beta_i p_2)}$ on it, we obtain the structure of Fig. 3.3(a) with resistive termination. Thus the theorem is proved.

From the network of Fig. 3.3(a) by making $p_1 \rightarrow \frac{1}{p_1}$ keeping p_2 unchanged, $p_2 \rightarrow \frac{1}{p_2}$ while p_1 is unchanged, and $p_1 \rightarrow \frac{1}{p_1}$ & $p_2 \rightarrow \frac{1}{p_2}$ transformations, we can get the ladder networks of Fig. 3.3(b),(c) and (d) respectively. Hence, the realizability conditions stated in Theorems 3.3 and 3.4 hold for these networks also after making the proper transformation upon the given network function.

These network functions have some simple properties which can be utilized as inspection tests and these are tabulated in Table 3.2.

Thus far we are discussing two-variable ladder networks where each arm contains both p_1-type and p_2-type elements. It is also possible to have two-variable ladder networks for which some of the series elements or the shunt elements may be absent. A class of such ladder networks will be discussed in Chapter V. Depending upon the configuration of the networks, the realizability conditions for those networks can be derived from the TLPL. In the next section, it is shown how this array can be extended to multivariables also.

3.3 Multivariable Array and its Applications:

The above results can be extended to n-variable ladder networks. We first show how the n-variable array can be written, from which the realizability conditions for the n-variable low-pass ladder containing p_1-to p_n-types of inductors in the series arms and p_1-to p_n-types of

TABLE 3.2

Some properties of two-variable ladder networks consisting of p_1- and p_2-type elements in parallel in the series arms and in series in the shunt arms.

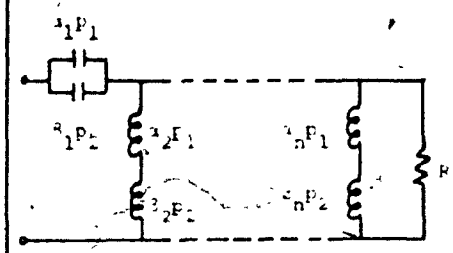
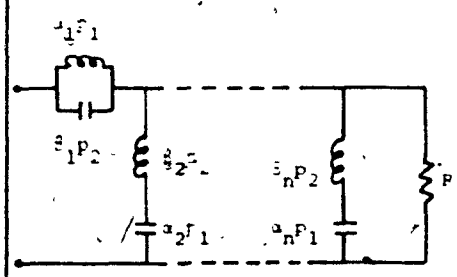
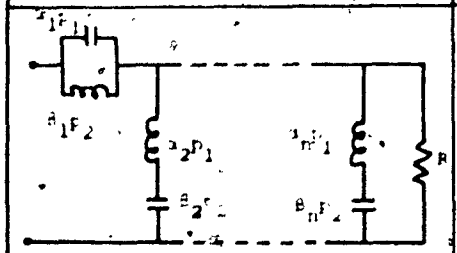
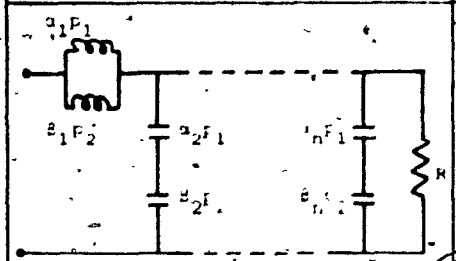
Type of Network	Nature of the polynomials	$M_1(p_1,p_2)\cdot M_2(p_1,p_2)+$ $N_1(p_1,p_2)\cdot N_2(p_1,p_2)$
	$Z(p_1,p_2)=\dfrac{f(p_1,p_2)}{g(p_1,p_2)}$ f and g are polynomials in p_1,p_2 with no missing terms except one of them not having the constant term	$\pm\Sigma\,(\alpha_1 p_1+\beta_1 p_2)^2$
	$Z(p_1,p_2)=Z(p_2')=\dfrac{f(p_2')}{g(p_2')}$ where $p_2'=p_1p_2$ f and g are polynomials in p_2' with no missing terms, except that one of them not having the highest degree coefficient of p_2'.	$-(\alpha_1+\beta_1 p_2)^2$

TABLE 3.2 - cont'd

Type of Network	Nature of the polynomials	$M_1(P_1,P_2)\cdot M_2(P_1,P_2) - N_1(P_1,P_2)\cdot N_2(P_1,P_2)$
	Same as above when P_2 is replaced by P_1.	$\Pi\,(\alpha_1+\beta_1 P_2)^2$
	$Z(P_1,P_2) = \dfrac{P_1^k\cdot f_k(P_2)+P_1^{k-1}\cdot f_{k-1}(P_2)+\dots+f_0(P_2)}{P_1^k\cdot g_k(P_2)+P_1^{k-1}\cdot g_{k-1}(P_2)+\dots+g_0(P_2)}$ $f_k(P_2)$, k^{th} degree polynomial with no missing terms $f_{k-1}(P_2)$, k^{th} degree polynomial with no missing terms except P_2^0 missing : $f_0(P_2)$, k^{th} degree polynomial with no missing terms except P_2^0 to P_2^{k-1} missing. The conditions on g_0 to g_{k-1} are the same as f_0 to f_{k-1}. $g_k(P_2)$ $(k-1)^{th}$ degree polynomial with no missing terms.	$\Pi\,(\alpha_1 P_1+\beta_1 P_2)^2$

capacitors in the shunt arms can be derived. The rules for writing the n-variable array are similar to those for the two-variable arrays. The n-variable array contains (n+1) rows and the first row contains the terms of degrees k,(k-2),(k-4),..etc., where k is the degree of the given function. The 2nd, 3rd, ...$(n+1)^{th}$ rows contain terms of degrees (k-1), (k-3), (k-5),...etc. according to the following rule: if $a_{\delta_1,\delta_2,\ldots\delta_n}$ is the term in the first row, then $a_{\delta_1-1,\delta_2,\ldots\delta_n}$, $a_{\delta_1,\delta_2-1,\delta_3\ldots\delta_n}$... and $a_{\delta_1,\ldots\delta_{n-1},\delta_n-1}$ are the terms respectively in the 2nd, 3rd,... and $(n+1)^{th}$ rows, but in the same column. If (δ_{i-1}) is less than zero, then the corresponding term is zero.

However, there is a slight deviation about the value of the determinants. Thus if,

$$\Delta_{(k,0..0),..(0,..0,k)(\delta_1..\delta_n)} = \begin{vmatrix} a_{k,0\ldots0}\cdots a_{0,\ldots0,k} & a_{\delta_1,\ldots\delta_n} \\ a_{k-1,0,\ldots0} \quad 0 & a_{\delta_1-1,\,\delta_2,\ldots\delta_n} \\ 0 & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0\ldots\ldots\ldots0 & \cdot \\ 0\ldots\ldots a_{0,\ldots0,k-1} & a_{\delta_1,\ldots\delta_{n-1},\delta_n-1} \end{vmatrix}$$

is a determinant of order (n+1), then the following conditions shall hold in order that the given n-variable reactance function is expressible as a continued-fraction:

1. $\Delta_{(k,0..0),..(0,..0,k),(\delta_1,..\delta_n)} = 0$, if $\delta_1+\ldots+\delta_n = k$

   > 0, if $(\delta_1+\ldots+\delta_n) < k$ and the

order of the determinant is odd

< 0, if $(\delta_1+\ldots+\delta_n) < k$ and the

order of the determinant is even.

2. Other degree terms also satisfy similar conditions.

The values of the b's are calculated as follows:

$$b_{\delta_1,\ldots\delta_n} = \pm \frac{\Delta_{(k,0\ldots0),\ldots(0,\ldots0,k),(\delta_1,\ldots\delta_n)}}{a_{k-1,0\ldots0} \cdot a_{0,k-1,0\ldots0} \cdot a_{0,\ldots0,k-1}}$$

The positive sign is to be taken if the order of the determinant is odd and negative sign, if it is even.

Thus, the realizability conditions for the n-variable ladder networks of different kinds can be given starting from the MLPL similar to the two-variable ladder networks.

Thus far we have been treating the variables as independent. However, by properly defining them, it is possible to derive various kinds of mixed lumped-distributed filters. For example, by defining $p_1 = s$ and $p_i = \tanh \tau_i s$, $(2 \leq i \leq n)$, the p_i-type inductors and capacitors can respectively be replaced by non-commensurate short-circuited and open-circuited stubs in the above derived ladder networks. The uses of such networks are discussed elsewhere[51].

Similarly, by defining $p_1 = \sinh s\tau$ and $p_2 = s \cosh s\tau$ different kind of mixed lumped-distributed filters can be developed starting from the TLPL. Networks of this kind are studied in Chapter IV.

3.4 Conclusions:

In this chapter, a multivariable array is proposed by means of which the realizability conditions for MLPL are derived. Starting from the TLPL, by means of various transformations, different ladder networks are obtained. Also the realizability conditions for the resistively terminated lossless ladder networks are derived. This brings out the important point that, for cascaded networks in addition to the real-part criterion, there need be some extra conditions satisfied. The extra conditions for the resistively terminated TLPL are that the numerator and the denominator of the given function must be polynomials with no missing terms. By suitable transformations, several other types of ladder networks are also derived, whose realizability conditions can be obtained from those of TLPL.

CHAPTER IV

CASCADE SYNTHESIS

4.1 Introduction:

The synthesis of resistively terminated cascade connection of UEs separated by lumped lossless elements has received considerable attention and has been discussed by several authors by the multivariable approach[22,31]. Such networks are important for the microwave filters using cascaded coaxial lines where lumped discontinuities inevitably occur or networks containing semiconductor elements and UEs, etc.

It has been shown that the synthesis of cascaded commensurate UEs can be carried out by means of two-element kind ladder networks[52,53]. In this chapter, By defining $p_1 = \sinh s\tau$ and $p_2 = s \cosh s\tau$, an equivalent relation between the cascade of commensurate UEs separated by lumped series inductors on one side and lumped shunt capacitors on the other side, and the TLPL is shown to exist. This enables the synthesis of such cascaded networks to be carried out by the continued-fraction expansion instead of the Richards' transformation. Also, a simple method of synthesizing $Z(p_i,s)$, bilinear in each p_i, by a cascade of non-commensurate UEs separated by lumped lossless two-ports terminated in a resistance is presented.

4.2 Cascade of Commensurate UEs and Lumped Lossless Elements[47]:

Consider the networks of Fig. 4.1(b) and (c). Both the circuits have an ideal transformer of turns ratio $1/\sqrt{1+p_1^2}$, which is frequency dependent. This is created for the mathematical convenience only and care should be taken to see that it must be eliminated from the realized network. The chain matrix for these networks is given by:

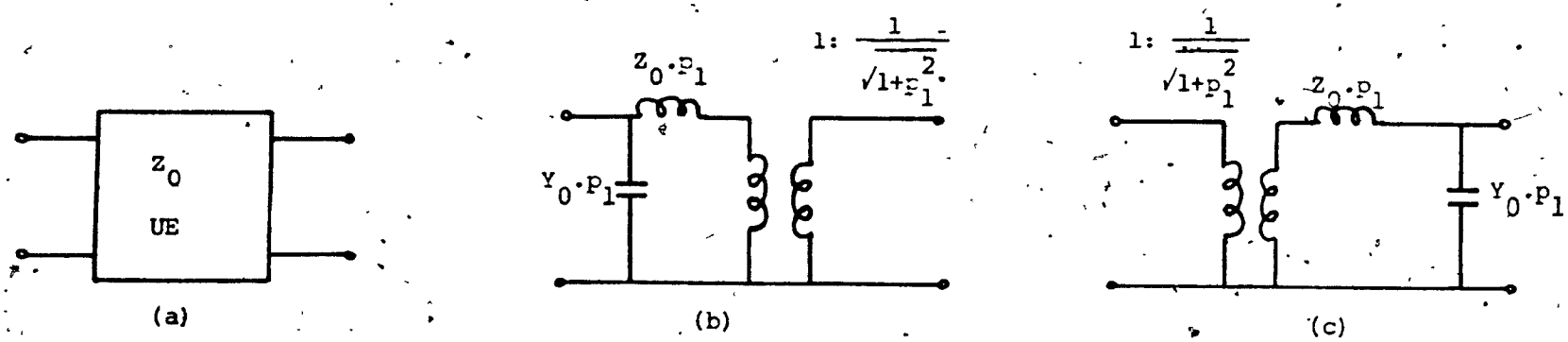


Fig. 4.1

ÜE and its Π-equivalent networks

$$a(p_1) = \begin{bmatrix} \sqrt{1+p_1^2} & \dfrac{Z_0 p_1}{\sqrt{1+p_1^2}} \\ Y_0 . p_1 . \sqrt{1+p_1^2} & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A(p_1) & B(p_1) \\ C(p_1) & D(p_1) \end{bmatrix}$$

where $Y_0 = 1/Z_0$.

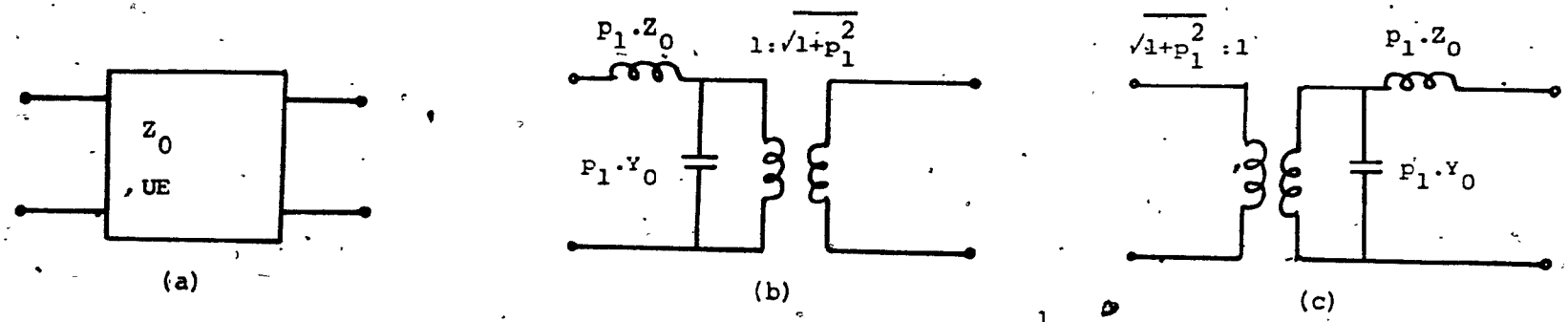
The transmission matrix of a UE in the p_1-plane is given by:

$$\bar{a}(p_1) = \begin{bmatrix} \sqrt{1+p_1^2} & Z_0 p_1 \\ Y_0 p_1 & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A(p_1) & B(p_1).F(s) \\ \dfrac{C(p_1)}{F(s)} & D(p_1) \end{bmatrix}$$

where $F(s) = \cosh s\tau$.

In this form $a(p_1)$ is equivalent to $\bar{a}(p_1)$, but for the term $F(s)$ in the off-diagonal elements. Following Pang[52], we shall term the networks of Fig. 4.1(b) and (c) as the left hand (L.H) and right hand (R.H), Π-equivalent LC circuits of the UE respectively. This equivalence is called the Π-equivalent since a cascade of two UEs is equivalent to a Π-configuration of the LC network in the p_1-plane; whenever the first UE is replaced by the L.H equivalent and the second UE by the R.H equivalent LC network.

Similarly, the networks of Fig. 4.2(b) and (c) consist of frequency dependent transformers of turns ratio $\sqrt{1+p_1^2}$. The chain matrix for these networks is given by:



$$Z_0 = \frac{1}{Y_0}$$

Fig. 4.2

UE and its T-equivalent networks

$$a_1(p_1) = \begin{bmatrix} \sqrt{1+p_1^2} & Z_0 p_1 \cdot \sqrt{1+p_1^2} \\ \dfrac{Y_0 \cdot p_1}{\sqrt{1+p_1^2}} & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A_1(p_1) & B_1(p_1) \\ C_1(p_1) & D_1(p_1) \end{bmatrix}$$

By expressing the chain matrix of UE in terms of $A_1(p_1)$, $B_1(p_1)$, $C_1(p_1)$ and $D_1(p_1)$ we have:

$$\bar{a}_1(p_1) = \begin{bmatrix} \sqrt{1+p_1^2} & Z_0 p_1 \\ Y_0 \cdot p_1 & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A_1(p_1) & \dfrac{B_1(p_1)}{F(s)} \\ C_1(p_1) \cdot F(s) & D_1(p_1) \end{bmatrix}$$

In this form, $a_1(p_1)$ is equivalent to $\bar{a}_1(p_1)$ but for the factor $F(s)$ in the off-diagonal elements. Once again following Pang[52], the networks of Fig. 4.2(b) and (c) shall be termed as the left hand (L.H) and right hand (R.H) T-equivalent LC networks of the UE respectively. As may be seen the cascade of two UEs is also equivalent to a T-configuration of LC network in the p_1-plane.

It may be noted that, by the transformation $p_1 = \sinh st$, the lossless uniform transmission line is transformed into an equivalent LC network in the p_1-plane, keeping the lossless nature in both the planes; whereas by the transformation[52] $q = \sinh^2 st$, the UE is transformed into an equivalent RC network in the q-plane. Also, to deal with the mixed lumped-distributed networks, the former transformation seems to be more convenient. By replacing the UEs alternately by their L.H and R.H, LC equivalents (Π or T), we obtain an LC ladder network in the p_1-plane. The ideal transformers of the L.H and R.H equivalents cancel each

other, if the number of UEs is even; otherwise, there will be a transformer at the far end. The equivalent relation remains invariant for the cascade of UEs. Thus, the synthesis can be carried out either by the Π-equivalent or the T-equivalent LC ladder network through the continued-fraction expansion in p_1-plane. The input immittances of these networks are given below, which apply both for the even number and odd number of UEs.

For the Π-equivalent:

$$y_{11}(p_1) = \frac{D(p_1)}{B(p_1)} = \bar{y}_{11}(p_1) \cdot F(s)$$

$$z_{11}(p_1) = A(p_1)/C(p_1) = \bar{z}_{11}(p_1) \; /F(s)$$

For the T-equivalent:

$$y_{11}(p_1) = \frac{D_1(p_1)}{B_1(p_1)} = \bar{y}_{11}(p_1)/F(s)$$

$$z_{11}(p_1) = \frac{A_1(p_1)}{C_1(p_1)} = \bar{z}_{11}(p_1) \cdot F(s)$$

where $y_{11}(p_1)$ and $z_{11}(p_1)$ are respectively the short-circuit and open-circuit driving-point functions of the LC equivalent networks and $\bar{y}_{11}(p_1)$ and $\bar{z}_{11}(p_1)$ are the corresponding quantities of the cascaded UEs.

Utilizing the above derived equivalent relations, we give here the realizability conditions of a resistively terminated cascade of UEs in terms of the Π(T)-equivalent LC ladder network in the following theorem:

Theorem 4.1:

The input impedance of a resistively terminated low-pass single-variable LC ladder network, $z(p_1)$, is equivalent to that of a cascade of UEs with resistive termination, if and only if

(a) <u>For T-equivalent*</u>

$$z(p_1) = \frac{m_1(p_1)/\sqrt{1+p_1^2} + n_1(p_1)\sqrt{1+p_1^2}}{m_2(p_1)\cdot\sqrt{1+p_1^2} + n_2(p_1)/\sqrt{1+p_1^2}}$$

(i) The zeros of the polynomials $m_1(p_1)$, $n_1(p_1)$, $m_2(p_1)$ and $n_2(p_1)$ are restricted to the imaginary axis of p_1 in the interval of $\pm j$.

(ii) $m_1(\pm j) = 0$, if the p_1 degree is even in $\frac{m_1+n_1}{m_2+n_2}$.

$n_1(\pm j) = 0$, if the p_1 degree is odd in $\frac{m_1+n_1}{m_2+n_2}$.

(b) <u>For Π-equivalent*</u>

$$z(p_1) = \frac{m_1(p_1)\cdot\sqrt{1+p_1^2} + n_1(p_1)/\sqrt{1+p_1^2}}{m_2(p_1)\cdot/\sqrt{1+p_1^2} + n_2(p_1)\sqrt{1+p_1^2}}$$

(i) The zeros of the polynomials $m_1(p_1)$, $n_1(p_1)$, $m_2(p_1)$ and $n_2(p_1)$ are restricted to the imaginary axis of p_1 in the interval of $\pm j$.

(ii) $m_2(\pm j) = 0$, if the p_1 degree is even in $\frac{m_1+n_1}{m_2+n_2}$.

$n_2(\pm j) = 0$, if the p_1 degree is odd in $\frac{m_1+n_1}{m_2+n_2}$.

<u>Proof:</u>

The proof of this theorem follows on similar lines that is given in [52].

---

* The factor $\sqrt{1+p_1^2}$ occurs only if the p_1 degree is even in $\frac{m_1+n_1}{m_2+n_2}$

Necessity:

For the cascaded UE network, the zeros of the polynomials

$$(1-\mu^2)^{\frac{m}{2}} \cdot m_1(p_1)\Big|_{p_1 = \frac{\mu}{\sqrt{1-\mu^2}}} \qquad (1-\mu^2)^{\frac{m}{2}} \cdot n_1(p_1)\Big|_{p_1 = \frac{\mu}{\sqrt{1-\mu^2}}} \qquad (1-\mu^2)^{\frac{m}{2}} \cdot m_2(p_1)\Big|_{p_1 = \frac{\mu}{\sqrt{1-\mu^2}}}$$

and $(1-\mu^2)^{\frac{m}{2}} \cdot m_2(p_1)\Big|_{p_1 = \frac{\mu}{\sqrt{1-\mu^2}}}$ where $\mu = \tanh s\tau$ and m is the degree of p_1 in $\frac{m_1+n_1}{m_2+n_2}$, must be restricted to the imaginary axis in the μ-plane. But the transformation $p_1 = \frac{\mu}{\sqrt{1-\mu^2}}$ maps the entire imaginary axis of μ-plane onto the interval $-j \leq p_1 \leq j$ in the p_1-plane. Thus condition (i) follows. By direct multiplication of the matrices we can see the necessity of condition (ii).

Sufficiency:

We give the proof for the T-equivalent network and it follows on the same lines for the Π-equivalent network as well.

Assume that m is odd, and the ladder LC network has been obtained by continued-fraction expansion. To obtain the cascaded UE network, the LC network is first decomposed in a manner indicated in Fig. 4.3. This process may involve two possible difficulties.

(i) Compatibility: Starting the decomposition process from the right hand end, we let $Z_i = 1/Y_i$ $(1 \leq i \leq m)$. The separation of the last shunt arm produces Y_0. But Y_0 may not be equal to $1/Z_0$, a pre-requisite that must be fulfilled in order to convert

the ladder network successfully into cascaded UE network.

The following is a proof that the LC network is always "compatible".

From Fig. 4.3 it is readily seen that the driving-point impedance $z(\pm j)$ at any mid-series position is always zero, and similarly the driving-point admittance at any mid-shunt position, $y(\pm j)$ is also zero. That is

$$y(\pm j) = z(\pm j) = 0 \qquad ..(4.1)$$

Progressing from the far end, we obtain

$$\frac{1}{y_{11}(\pm j)} = \pm j\, Z_0 + \frac{1}{\pm j\, Y_0 + y(\pm j)} = \pm j\, Z_0 + \frac{1}{\pm j\, Y_0}$$

But condition (ii) above yields

$$\frac{1}{y_{11}(\pm j)} = \frac{n_1(\pm j)}{m_2(\pm j)} = 0 \qquad ..(4.2)$$

Hence $Z_0 = \frac{1}{Y_0}$.

Thus the condition (ii) ensures that the LC network is always compatible.

(ii) Positive realness: The decomposition process does not preclude that one of the decomposed elements from being negative. The following is a proof that Z_i $(0 \leq i \leq m)$ are always positive.

Figs. 4.4(a) and (b) show the typical LC impedance and admittance functions respectively. From Fig. 4.5 $1/y_1(p_1)$ is the impedance function after a capacitance has been removed in the course of continued-fraction expansion, and $y_3(p_1)$ is the admittance function after a series inductance has just been removed. The dotted lines in Fig. 4.4 show the new zero positions when a series inductance or a shunt capacitance is removed from $1/y_1$ or y_3 respectively. It follows from condition (i) that the poles and zeros of $y_{11}(p_1)$ [zeros of $n_1(p_1)$ and $m_2(p_1)$] are initially within

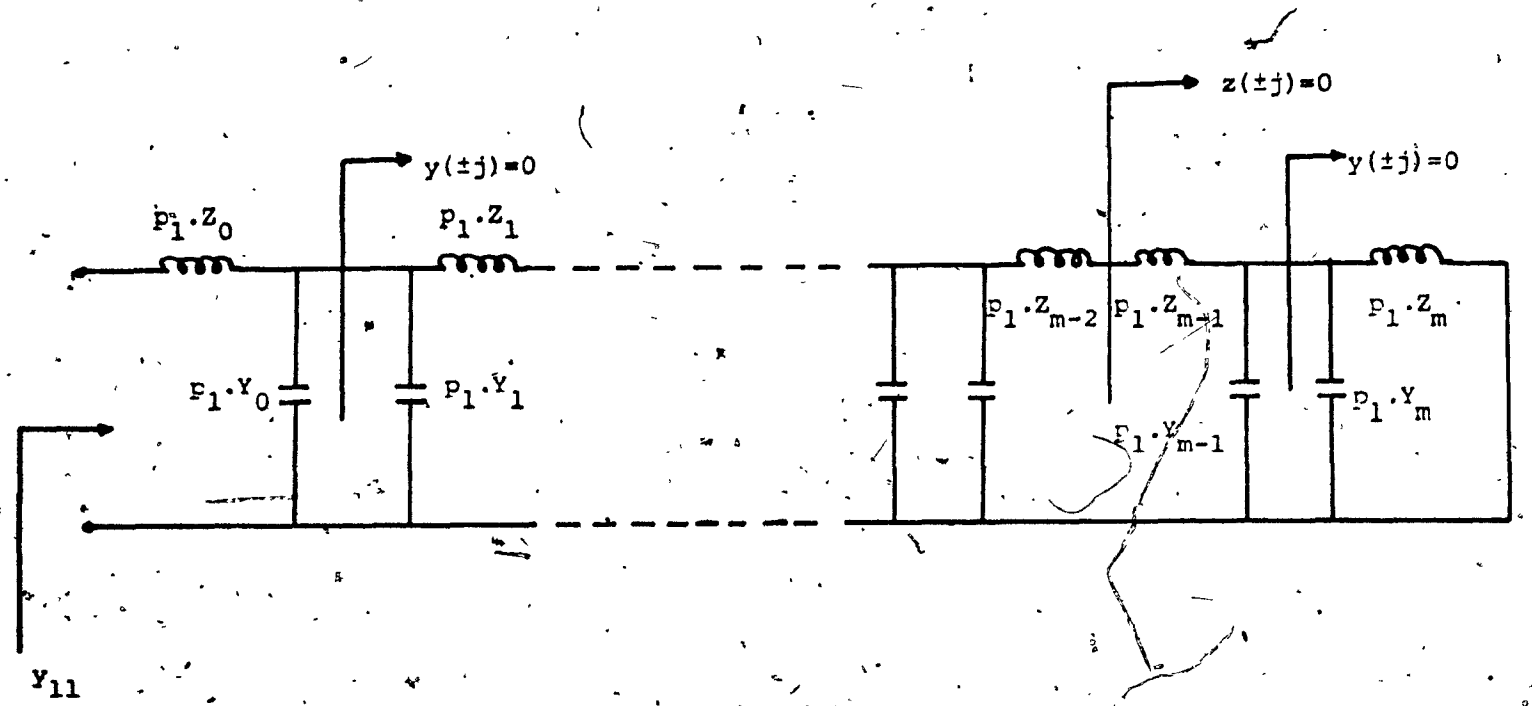
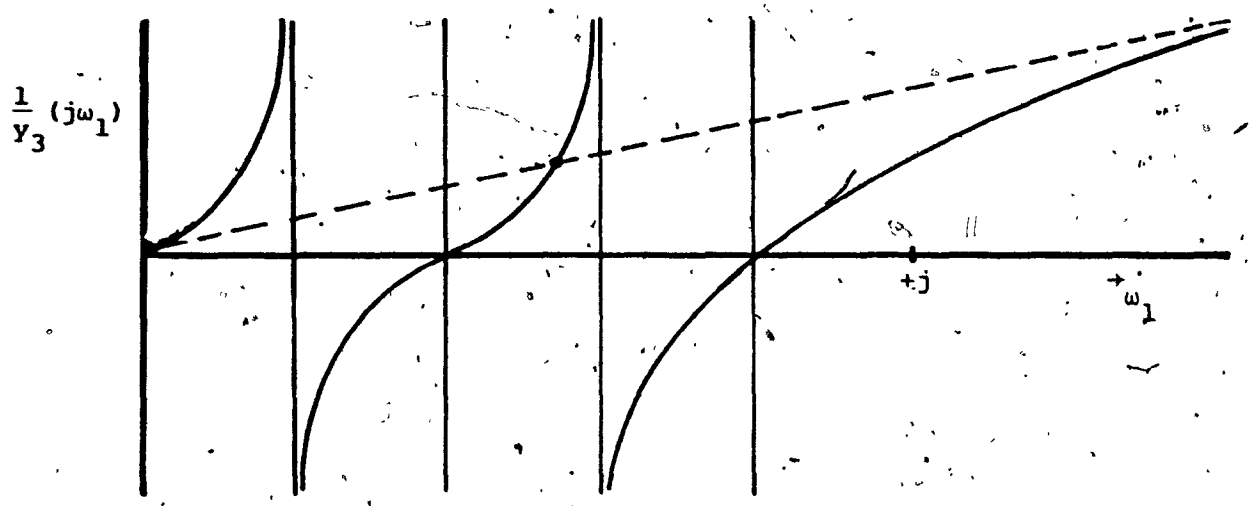


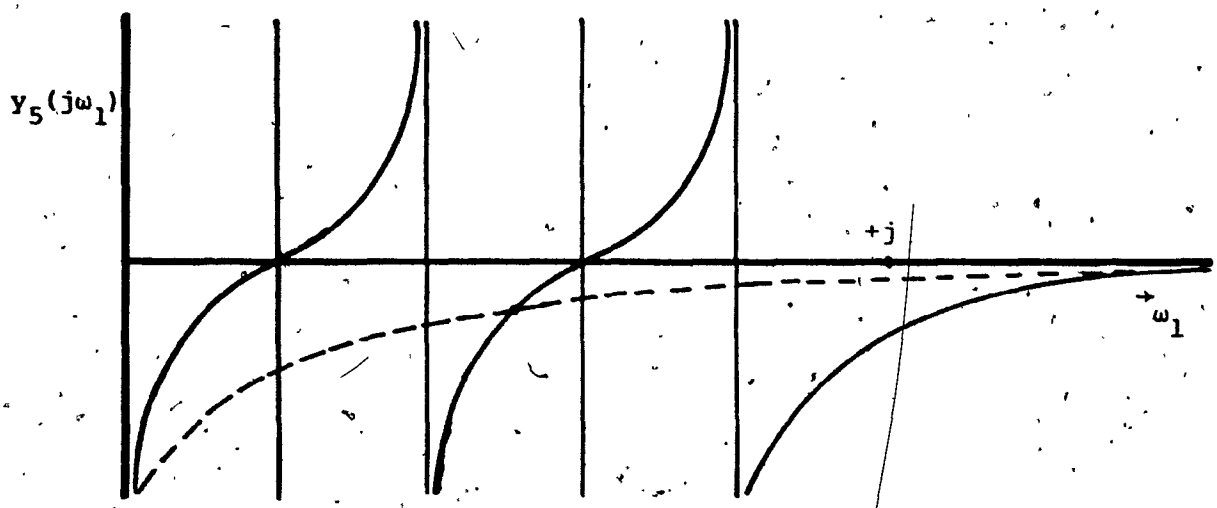
Fig. 4.3

The Decomposition Process



$$\frac{1}{Y_3(j\omega_1)} = \frac{1}{Y_1(j\omega_1)} - j\omega_1(Z_i+Z_{i+1})$$

(a)



$$Y_5(j\omega_1) = Y_3(j\omega_1) - j\omega_1(Y_{i+1}+Y_{i+2})$$

(b)

Fig. 4.4

Typical LC impedance and admittance functions

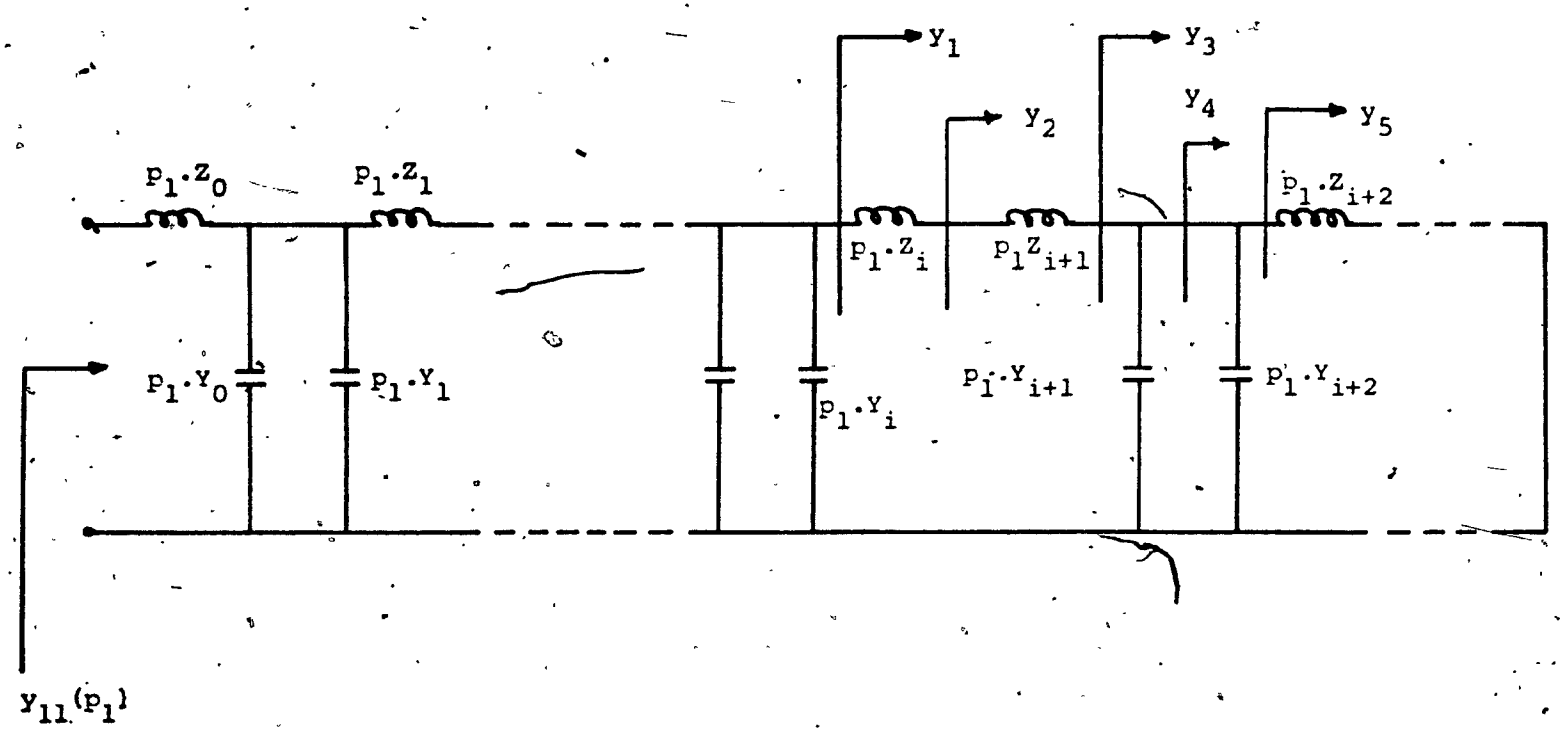


Fig. 4.5

Admittance Functions at Various points of the Network

$+j \leq \omega_1 \leq -j$. The sketches in Fig. 4.4 show that removal of a series inductance or a shunt capacitance would not shift the zeros and poles of the immittance function beyond $\omega_1 = \pm j$, except for a single pole at infinity in the case of $y_3(p_1)$. Hence we conclude that

$$\left.\begin{aligned} &\frac{\pm j}{y_1(\pm j)} > 0 \\ &\pm j . y_3(\pm j) < 0 \end{aligned}\right\} \qquad (4.3)$$

From Fig. 4.5, we have

$$\left.\begin{aligned} &\frac{1}{y_1(p_1)} = p_1 . Z_i + \frac{1}{y_2(p_1)} \\ &y_3(p_1) = p_1 . Y_{i+1} + y_4(p_1) \end{aligned}\right\} \qquad ..(4.4)$$

But it follows on the same reasoning as in (4.1) that

$$\frac{1}{y_2(\pm j)} = y_4(\pm j) = 0$$

Substituting this in (4.4) and with the help of (4.3), we have

$$\left.\begin{aligned} &Z_i = \frac{\pm j}{y_1(\pm j)} > 0, \text{ and} \\ &Y_{i+1} = -(\pm j) . y_3(\pm j) > 0 \end{aligned}\right\} \qquad ..(4.5)$$

We have thus shown that the values of the decomposed elements must be positive.

Thus by converting the decomposed LC network into the equivalent cascaded UE network the sufficiency follows.

Now, we derive some more equivalent relations, using which the realizability conditions for a cascade of UEs separated by lumped shunt/series elements can be given. The thain matrix of the two-variable ladder network with transformer as shown in Fig. 4.6(b) is:

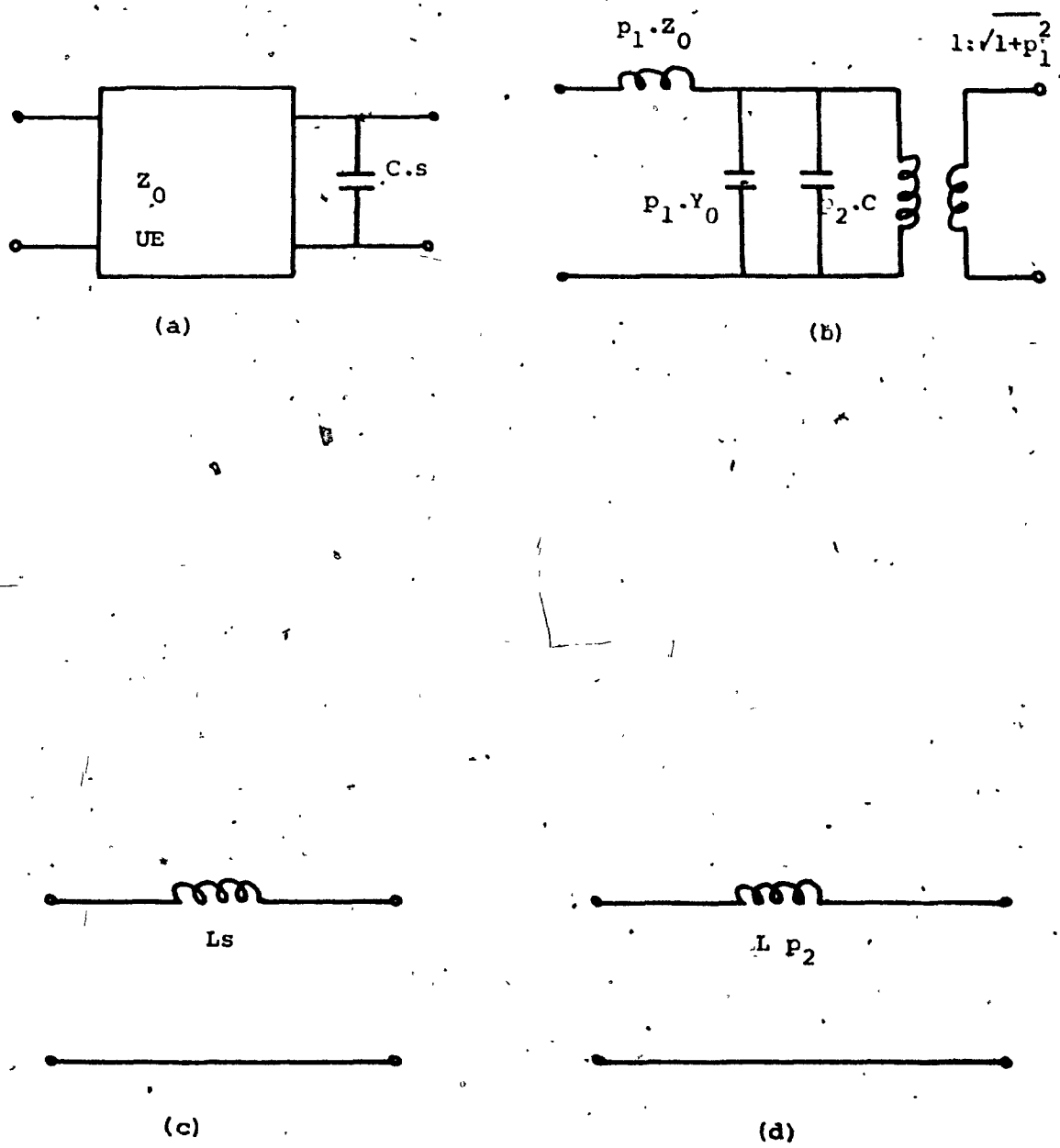


Fig. 4.6

(a) UE terminated in a capacitor, (b) Equivalent network of (a)

(c) and (d) Series inductor equivalents.

$$a(p_1,p_2)=\begin{bmatrix} \dfrac{1+p_1^2+Z_0\,C\,p_1p_2}{\sqrt{1+p_1^2}} & Z_0\cdot p_1\cdot\sqrt{1+p_1^2} \\ \dfrac{Y_0\cdot p_1+C\,p_2}{\sqrt{1+p_1^2}} & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A(p_1,p_2) & B(p_1,p_2) \\ C(p_1,p_2) & D(p_1,p_2) \end{bmatrix}$$

and the chain matrix of a UE in cascade with a capacitor as shown in Fig. 4.6(a) is:

$$\bar{a}(p_1,p_2)=\begin{bmatrix} \dfrac{1+p_1^2+Z_0\,C\,p_1p_2}{\sqrt{1+p_1^2}} & Z_0p_1 \\ Y_0\cdot p_1+C\,p_2 & \sqrt{1+p_1^2} \end{bmatrix} = \begin{bmatrix} A(p_1,p_2) & \dfrac{B(p_1,p_2}{F(s)} \\ C(p_1,p_2).F(s) & D(p_1,p_2) \end{bmatrix}$$

Thus, the transmission matrices are equal, but for the factor F(s) in the off-diagonal elements and these two networks shall be termed as equivalent. Similarly, we can see that the series inductors of Figs. 4.6(c) and (d) are equivalent; their respective chain matrices being

$$\bar{a}_1(p_2) = \begin{bmatrix} 1 & \dfrac{L.p_2}{F(s)} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_1(p_2) & \dfrac{B_1(p_2)}{F(s)} \\ C_1(p_2).F(s) & D_1(p_2) \end{bmatrix}$$

and $$a_1(p_2) = \begin{bmatrix} 1 & L\,p_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_1(p_2) & B_1(p_2) \\ C_1(p_2) & D_1(p_2) \end{bmatrix}$$

Utilizing the above derived equivalent relations, the realizability conditions for the cascade of UEs separated by lumped series inductor on

one side and shunt lumped capacitor on the other side with resistive termination as shown in Fig. 4.7(a) are derived in terms of the TLPL with resistive termination. In the cascaded structure of Fig. 4.7(a) replacing the UE and the right side capacitor by the equivalent network of Fig. 4.6(b), series inductors by the network of Fig. 4.6(d), and UE by the network of Fig. 4.2(c) we obtain the TLPL with resistive termination of Fig. 4.7(b) as its equivalent network. As its single-variable counterpart, there will be a transformer at the far end of the ladder, if the number of UEs is odd. When the chain matrices of the equivalent networks are multiplied, we see that the equivalent relation remains invariant for this network also. The relation between the input immittances of these two networks are given below:

$$y_{11}(p_1,p_2) = \frac{D(p_1,p_2)}{B(p_1,p_2)} = \bar{y}_{11}(p_1,p_2)/F(s)$$

$$z_{11}(p_1,p_2) = \frac{A(p_1,p_2)}{C(p_1,p_2)} = \bar{z}_{11}(p_1,p_2).F(s)$$

where $y_{11}(p_1,p_2)$ and $z_{11}(p_1,p_2)$ are the short-circuit and open-circuit driving-point functions of the TLPL of Fig. 4.7(b), where as $\bar{y}_{11}(p_1,p_2)$ and $\bar{z}_{11}(p_1,p_2)$ are the corresponding quantities of the lossless two-port of Fig. 4.7(a).

Now utilizing Theorems 3.2 and 4.1, the realizability conditions for the network of Fig. 4.7(a) are given in terms of the equivalent two-variable ladder of Fig. 4.7(b) in the following theorem.

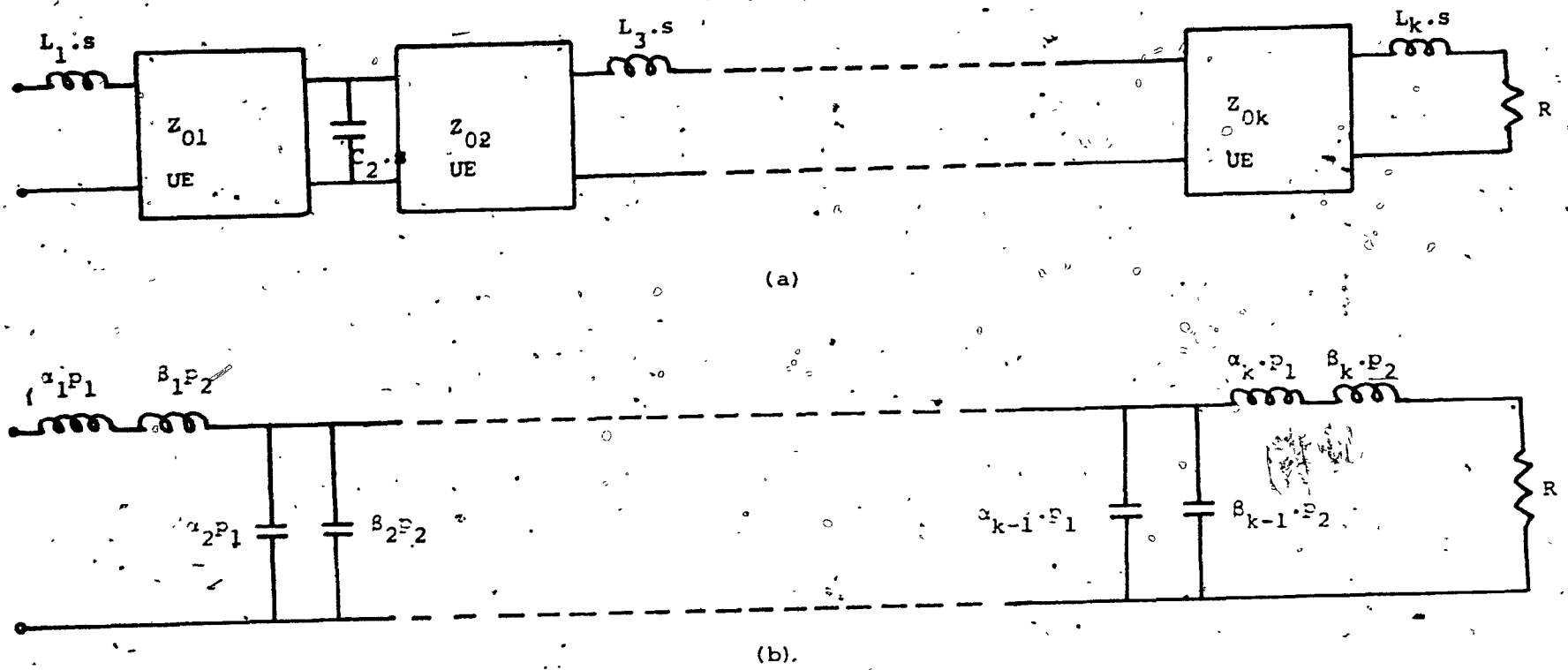


Fig. 4.7

(a) Cascade of UEs separated by series lumped inductor on one side and shunt lumped capacitor on the other side with resistive termination.

(b) Equivalent TLPL Structure.

Theorem 4.2:

The two variable function*

$$Z(p_1,p_2) = \frac{M_1(p_1,p_2)/\sqrt{1+p_1^2} + N_1(p_1,p_2)\sqrt{1+p_1^2}}{M_2(p_1,p_2)\sqrt{1+p_1^2} + N_2(p_1,p_2)/\sqrt{1+p_1^2}}$$

can be realized as the equivalent input impedance of a resistively terminated cascade of UEs separated by series inductors and shunt capacitors as shown in Fig. 4.7(a), if and only if

(i) $Z_1(p_1,p_2) = \dfrac{M_1(p_1,p_2) + N_1(p_1,p_2)}{M_2(p_1,p_2) + N_2(p_1,p_2)}$ obeys conditions of Theorem 3.2.

(ii) The zeros of the polynomials $m_1(p_1)$, $n_1(p_1)$, $m_2(p_1)$ and $n_2(p_1)$ are restricted to the imaginary axis of p_1 in the interval of $\pm j$, where $m_1(p_1) = M_1(p_1,0)$, $m_2(p_1) = M_2(p_1,0)$, $n_1(p_1) = N_1(p_1,0)$ and $n_2(p_1) = N_2(p_1,0)$.

and (iii) $m_1(\pm j) = 0$, if the p_1 degree is even in $\dfrac{m_1+n_1}{m_2+n_2}$

$n_1(\pm j) = 0$, if the p_1 degree is odd in $\dfrac{m_1+n_1}{m_2+n_2}$

Proof:

Condition (i) guarantees that the given two-variable function can be realized as a resistively terminated TLPL. Once this is done for $p_2 = 0$, the TLPL reduces to the single-variable low-pass ladder and then

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* The factor $\sqrt{1+p_1^2}$ appears only if the p_1 degree is even in $\dfrac{m_1+n_1}{m_2+n_2}$.

conditions (ii) and (iii) by Theorem 4.1 assure the decomposition of the single-variable ladder into an equivalent cascade of UEs with resistive termination. Thus the theorem follows.

Thus by defining $p_1 = \sinh s\tau$ and $p_2 = s.\cosh s\tau$ we have shown that cascade of resistively terminated UEs separated by lumped series inductors and shunt capacitors is equivalent to TLPL. Thus the cascade synthesis can be carried out by the continued-fraction expansion, the conditions for which are developed in Chapter III.

4.3 Cascade of Non-commensurate UEs and Lumped Lossless Networks[54]:

Premoli[55] has given a simple method of synthesizing $Z(p_1)$, where $p_i = \tanh \tau_i s$, bilinear in each variable, as a cascade connection of non-commensurate UEs terminated in a resistance. Here, we give a method of synthesizing $Z(p_1,s)$, bilinear in each p_i by a cascade of non-commensurate UEs separated by lumped lossless 2-ports, terminated in a resistance. The synthesis method is based upon the following theorem, which gives the necessary and sufficient conditions for an MPRF, $Z(p_i,s) = \frac{m_1+n_1}{m_2+n_2}$ of arbitrary degree in each variable to be realizable by a cascade of UEs separated by lumped lossless 2-ports terminated in 1Ω resistance, in terms of the functions m_1/n_2, m_2/n_2, n_1/m_2 and n_1/m_1.

Theorem 4.3:

The necessary and sufficient conditions for a multivariable positive real function $Z(p_i,s) = \frac{m_1+n_1}{m_2+n_2}$ to be realizable by cascade of non-commensurate UEs separated by passive lumped lossless 2-ports terminated in 1Ω resistor are

(i) $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$ ($\frac{n_1}{m_1}$ and $\frac{m_2}{n_2}$), the multivariable reactance functions must be synthesizable by cascade of UEs and passive lumped lossless 2-ports terminated in a capacitor and inductor respectively (inductor and capacitor) in one of the p_i variables, and

(ii) The values of the terminating capacitor and inductor are reciprocals of each other.

Proof: Necessity:

Let the realization of $Z(p_i,s) = \frac{m_1+n_1}{m_2+n_2}$ be as shown in Fig. 4.8.

Then we have that $\frac{m_1}{n_2} = Z_{11}$ and $\frac{n_1}{m_2} = \frac{1}{y_{11}}$ ($\frac{n_1}{m_1} = \frac{1}{y_{22}}$, $\frac{m_2}{n_2} = Z_{22}$) of the 2-port N. Since $Z(p_i,s)$ is cascade synthesizable, Z_{11}, $\frac{1}{y_{11}}$, Z_{22} and $\frac{1}{y_{22}}$ are also cascade synthesizable. In case of $\frac{m_1}{n_2}$ ($=Z_{11}$) the last UE is open-circuited with input impedance of $\frac{R_n}{p_n}$ (assuming that p_n-type is the last UE of characteristic impedance R_n), where as for $\frac{n_1}{m_2}$ ($= \frac{1}{y_{11}}$), the last UE is short-circuited with input impedance of $R_n \cdot p_n$. Thus the conditions of the theorem are satisfied. Similarly for the case of $\frac{n_1}{m_1}$ and $\frac{m_2}{n_2}$, the terminating inductors and capacitors will be of reciprocal values. Thus the proof of necessity follows.

Sufficiency:

We shall prove the sufficiency for $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$. It can be proved

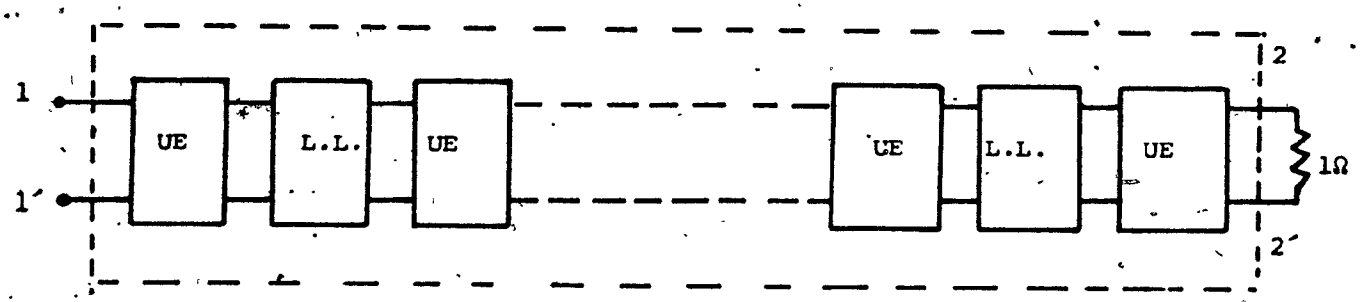

Fig. 4.8

Cascade of UEs separated by lumped lossless 2-ports.

on similar lines for $\frac{n_1}{m_1}$ and $\frac{m_2}{n_2}$.

If $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$ are cascade synthesizable with UEs and lumped 2-ports with inverse terminations, then $m_1m_2 - n_1n_2$ has

(a) $\prod (1 - p_i^2)^{\delta_i}$ corresponding to the UEs

(b) $\prod_{j=1}^{r} Q_j(s)\, Q_j(-s)$ corresponding to the lumped 2-ports, and

(c) $(1 - \lambda^2)$ corresponding to the inverse termination, where λ is the one of the p_i variables.

Thus $m_1m_2 - n_1n_2 \geq 0$, for Re p_i = Re $s = 0$.

We can easily infer that $F(p_i,s) = \frac{m_1+n_2}{n_1+m_2}$ and $Z(p_i,s) = \frac{m_1+n_1}{m_2+n_2}$ are multivariable positive real functions.

If we have extracted from $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$ a UE of say p_1-type with characteristic impedance R_1, then

$$\frac{m_1}{n_2} = \frac{n_1}{m_2} = R_1 = \frac{m_1+n_1}{m_2+n_2} \quad \text{for } p_1 = 1$$

Similarly if we can extract a lumped 2-port from $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$, the same can be extracted from $\frac{m_1+n_1}{m_2+n_2}$ also. And corresponding to the inverse terminations in $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$, $\frac{m_1+n_1}{m_2+n_2}$ will have a UE with 1Ω termination. Thus we have proved the sufficiency.

It may be noted that if we are proceeding with $\frac{n_1}{m_1}$ and $\frac{m_2}{n_2}$, the extraction of the 2-ports will be in the reverse sequence to that of $Z(p_i,s)$.

As a consequence of the above theorem, testing and synthesis of the given function are done simultaneously. The values of the 2-port parameters in $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$ realizations correspond to the 2-port parameters of $Z(p_i,s)$, but for the last 2-port. The one-port terminations in $\frac{m_1}{n_2}$ and $\frac{n_1}{m_2}$ provide respectively A/C and B/D; where A,B,C,D are the chain parameters of the last UE of $Z(p_i,s)$. Similarly, the one-port terminations in $\frac{n_1}{m_1}$ and $\frac{m_2}{n_2}$ correspond respectively to B_1/A_1 and D_1/C_1, where A_1,B_1,C_1,D_1 are the chain parameters of the first UE in $Z(p_i,s)$ realization.

If one is interested only in the synthesis part, this can be accomplished with any one of the above mentioned reactance functions, provided the first and the last 2-ports are UEs which is very easy to verify. But, only if the last 2-port is a UE and the first one is a lumped 2-port, then we can obtain all the necessary information about the cascaded structure from $\frac{m_1}{n_2}$ or $\frac{n_1}{m_2}$ and similarly if the first 2-port is a UE and the last is a lumped lossless one, we can get all the specifications from $\frac{n_1}{m_1}$ or $\frac{m_2}{n_2}$. Even if both the first and the last 2-ports are lumped lossless networks, we can synthesize $Z(p_i,s)$ from $\frac{n_1}{m_1}$ or $\frac{m_2}{n_2}$, after extracting the first lumped lossless 2-port from $Z(p_i,s)$ and then applying the

above procedure. Thus, as could be seen there is considerable saving in labour in the synthesis.

4.3.1 Synthesis method:

Now we present a method of synthesizing $Z(p_i,s)$, bilinear in each p_i by a cascade of non-commensurate UEs separated by lumped lossless 2-ports terminated in a resistance.

Without loss of generality, we shall assume the cascaded structure to be as shown in Fig. 4.9, where UE_i indicates a UE of p_i-type and L_1 to L_{n-1} being the lumped lossless 2-ports. Then $\frac{m_1}{n_2}$ and $\frac{m_2}{n_2}$ correspond respectively to Z_{11} and Z_{22} of the (2n-1) sections. Then

$$\left.\frac{m_2}{n_2}\right|_{p_n=1} = R_n$$ where R_n is the characteristic impedance of the nth UE

and $$\left.\frac{m_1}{n_2}\right|_{p_n=1} = \frac{m_1'R_n+n_1'}{n_2'R_n+m_2'}$$ corresponds to the input impedance of the first (2n-1) sections with R_n termination.

Now m_1'/n_2' and m_2'/n_2' are respectively the Z_{11} and Z_{22} of the 2(n-1) sections. The chain parameters of L_{n-1} are obtained from m_2'/n_2' by putting $p_1 = \ldots p_{n-1} = 1$. By extracting L_{n-1} from m_2'/n_2', we get the Z_{22} of the remaining (2n-3) sections.

$$\left.\frac{m_1}{n_2}\right|_{p_n=p_{n-1}=1} = \frac{m_1''R_{n-1}+n_1''}{n_2''R_{n-1}+m_2''}$$ in the irreducible form is the input impedance of the 2(n-2) sections with R_{n-1} termination. Now applying the above procedure, the UE and lumped lossless 2-port can be successively extracted.

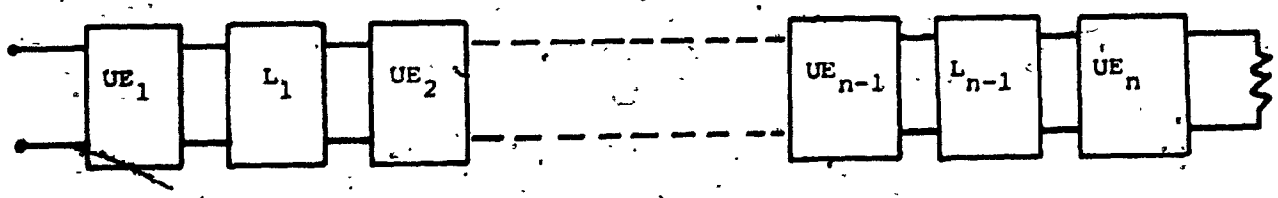

Fig. 4.9

The network for the synthesis method

The elements in the forward direction can be extracted, if we keep m_2/n_2 as the reference and the extraction being done using m_1/n_2. The whole synthesis can also be done with the other pair of functions n_1/m_1 and n_1/m_2 as well.

Thus, utilizing Theorem 4.3, a simple method of synthesizing $Z(p_i,s)$, bilinear in each p_i as cascade of non-commensurate UEs separated by lumped lossless 2-ports terminated in a resistor has been developed.

4.4 Conclusions:

In this chapter, some equivalent relations are developed between the UE and lossless lumped network in the p_1-plane. By means of these equivalent relations, a cascade of commensurate UEs is transformed into a low-pass lossless network in the p_1-plane. There exists a frequency dependent transformer at the far end, if the number of UEs is odd. Thus synthesizing a cascaded UE network is performed by continued-fraction expansion rather than taking recourse to the Richards' transformation. Utilizing some more equivalent relations developed, it is shown that a cascade of commensurate UEs separated by lumped series inductor on one side and lumped shunt capacitor on the other side is equivalent to TLPL. Hence, the realizability conditions for such a mixed lumped-distributed network are derived in terms of the TLPL.

Further the realizability conditions for the driving-point function $Z(p_i,s) = \frac{m_1+n_1}{m_2+n_2}$ by a cascade of non-commensurate UEs separated by lumped lossless 2-ports terminated in 1Ω resistor are derived in terms of the multivariable reactance functions m_1/n_2, m_2/n_2, n_1/m_2 and n_1/m_1. Based upon these conditions a simple method of realizing $Z(p_i,s)$, bilinear in

in each p_i, by a resistively terminated cascade of non-commensurate UEs separated by lumped lossless 2-port networks is developed.

CHAPTER V

REALIZATION OF SOME TWO-VARIABLE FUNCTIONS

5.1 Introduction:

Several synthesis procedures are available in the literature for the realization of two-variable reactance functions[11]. The realization methods for the two-variable reactance matrices given in [13,14,16] require either ideal gyrators or ideal transformers. However, the class of network functions considered in Chapter III require neither ideal gyrators nor ideal transformers and are realizable by ladder networks.

In this chapter, we propose a new type of two-variable reactance network similar to the single-variable Foster form*. The equivalent ladder and unsymmetrical lattice networks for the Foster forms are also derived. It is shown that this ladder network is a special case of the two-variable ladder network considered in Chapter III, in the sense that the series arms contain only p_1-(or p_2-) type of reactances and the shunt arms contain only p_2-(or p_1-) type of reactance elements. The realizability conditions for this class of reactance networks are directly derived from the nature of the reactance functions rather than taking recourse to the two-variable array. The partial polynomial derivatives of the above network functions are studied in detail.

By making use of these networks and the results of Chapter III, we derive the conditions on a two-variable PRF, so that it can be realized without transformers similar to the single-variable Miyata method.

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* In this thesis, for the sake of brevity these are called the Foster forms.

5.2 Realization of a class of Two-Variable Reactance Functions[49]:

Here we propose a two-variable structure similar to the single-variable Foster form and its realizability conditions are studied. We deal with two different classes of structures.

Class A: Inductors having reactances $p_1.L$ and capacitors having reactances $1/p_2.C$

Class B: Inductors having reactances $p_2.L$ and capacitors having reactances $1/p_1.C$

Table 5.1 gives Class A and Class B Foster forms and their equivalent ladder and unsymmetrical lattice networks. It may be noted, from the ladder equivalent networks, that these constitute a special case of two-variable ladders that were discussed in Chapter III. The realizability conditions for these two classes of structures are given in the following two theorems:

Theorem 5.1:

The necessary and sufficient conditions for the two-variable reactance function $Z(p_1,p_2)$ to be realizable by the Class A structures are:

(i) $Z(p_1,1)$ is an RL impedance function of the variable p_1

(ii) $Z(1,p_2)$ is an RC impedance function of the variable p_2, and

(iii) $Z(1,p_2)$ and $Z(p_1,1)$ possess the same internal critical frequencies.

Proof: Necessity:

The input impedance of the Foster first form is:

$$Z(p_1,p_2) = L_\infty.p_1 + \frac{1}{C_0.p_2} + \Sigma \frac{L_i p_1}{1 + L_i C_i p_1.p_2}$$

TABLE 5.1

Two-Variable Foster forms and their equivalent networks

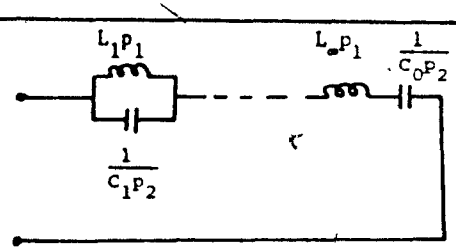
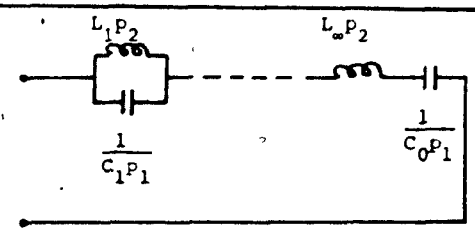
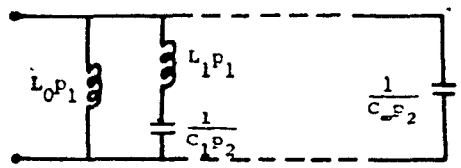
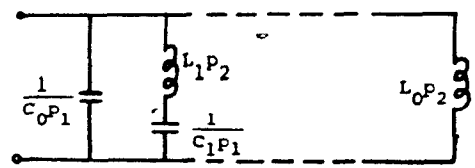
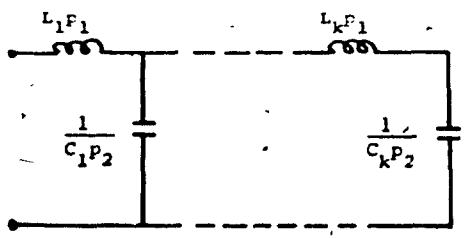
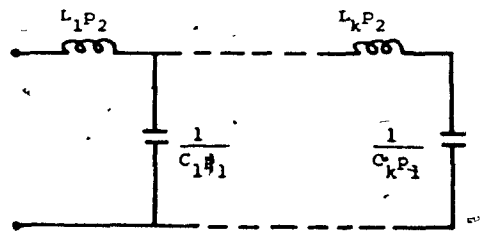
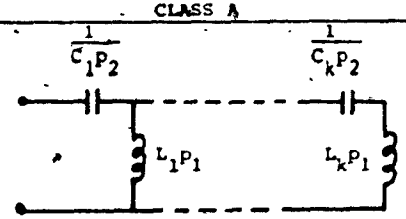
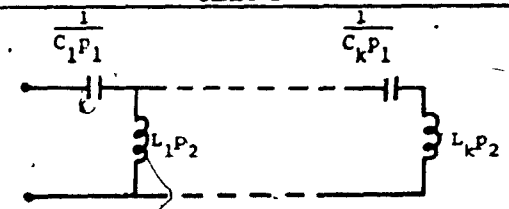
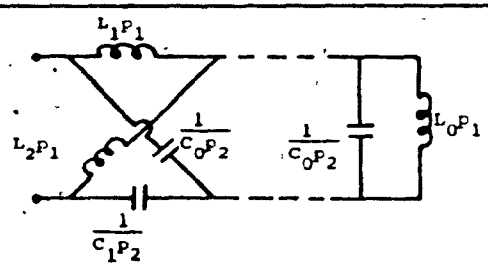
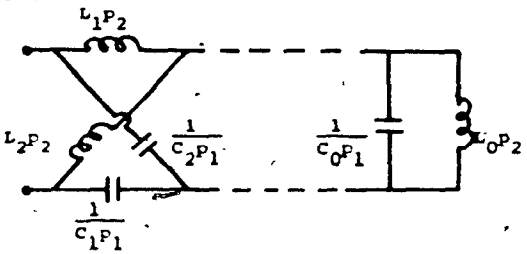
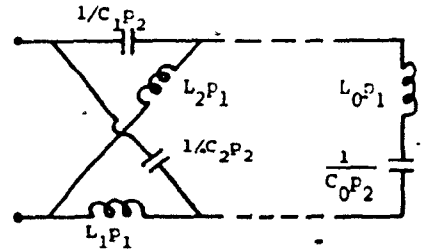
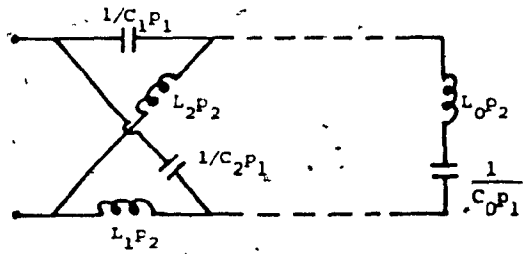
NETWORK FORM	CLASS A	CLASS B
Foster first form		
Foster second form		
Equivalent first ladder form		

TABLE 5.1 (cont'd)

NETWORK FORM	CLASS A	CLASS B
Equivalent second ladder form		
Equivalent first unsymmetrical form	 $Z(P_1,P_2) = \frac{P_1\,\Pi(P_1P_2+\alpha_i)}{\Pi(P_1P_2+\beta_i)}$	 $Z(P_1,P_2) = \frac{P_2\,\Pi(P_1P_2+z_i)}{\Pi(P_1P_2+s_i)}$
Equivalent second unsymmetrical form	 $Z(P_1,P_2) = \frac{\Pi(P_1P_2+\delta_i)}{P_2\,\Pi(P_1P_2+\lambda_i)}$	 $Z(P_1,P_2) = \frac{\Pi(P_1P_2+\mu_i)}{P_1\,\Pi(P_1P_2+\lambda_i)}$

Therefore

$$Z(p_1,1) = L_\infty \cdot p_1 + \frac{1}{C_0} + \Sigma \frac{L_i p_1}{1 + L_i \cdot C_i \cdot p_1}$$

which is an RL impedance function in p_1.

Similarly

$$Z(1,p_2) = L_\infty + \frac{1}{C_0 \cdot p_2} + \Sigma \frac{L_i}{1 + L_i \cdot C_i \cdot p_2}$$

is an RC impedance function in p_2.

It could be seen that condition (iii) is automatically satisfied for this network function and thus the proof of necessity follows.

<u>Sufficiency</u>:

The impedance function that satisfies conditions (i) and (ii) simultaneously could be only of the form

$$Z(p_1,p_2) = \frac{Hp_1 \, \Pi(p_1p_2+z_i)}{\Pi(p_1p_2+s_i)} \qquad ..(5.1)$$

or $$Z(p_1,p_2) = \frac{H\Pi(p_1p_2+\alpha_i)}{p_2 \, \Pi(p_1p_2+\beta_i)} \qquad ..(5.2)$$

where z_i, s_i, α_i and β_i are all real, positive constants,

$$0 < s_1 < z_1 < s_2 < \ldots,$$

and $$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots$$

First consider

$$Z(p_1,p_2) = \frac{Hp_1 \, \Pi(p_1p_2+z_i)}{\Pi(p_1p_2+s_i)}$$

From this we have

$$\frac{Z(p_1,p_2)}{p_1} = \frac{\Pi(p_1p_2+z_i)}{\Pi(p_1p_2+s_i)} = \frac{\Pi(p+z_i)}{\Pi(p+s_i)} = Z(p) \qquad ..(5.3)$$

where $p = p_1p_2$.

From the above conditions $Z(p)$ is an RC impedance function in the p-plane and can be realized canonically and its partial fraction expansion is

$$Z(p) = a_0 + \frac{a_1}{p} + \Sigma \frac{a_i}{p+s_i} \qquad ..(5.4)$$

From (5.3) and (5.4)

$$Z(p_1,p_2) = a_0p_1 + \frac{a_1}{p_2} + \Sigma \frac{a_ip_1}{p_1p_2+s_i} \qquad ..(5.5)$$

which can be realized by p_1-type inductors and p_2-type capacitors in the Foster first form.

Similarly, the function $Z(p_1,p_2) = \dfrac{\Pi(p_1p_2+\alpha_i)}{p_2\ \Pi(p_1p_2+\beta_i)}$ can be realized by p_1-type inductors and p_2-type capacitors. Thus the proof of the theorem follows.

The realizability conditions for Class B structures are given in Theorem 5.2, the proof of which is similar to Theorem 5.1 and hence it is omitted.

<u>Theorem 5.2</u>:

The necessary and sufficient conditions for the two variable reactance function $Z(p_1,p_2)$ to be realizable by the Class B structures are:

(i) $Z(p_1,1)$ is an RC impedance function

(ii) $Z(1,p_2)$ is an RL impedance function, and

(iii) $Z(p_1,1)$ and $Z(1,p_2)$ possess the same internal critical frequencies.

Theorems 5.1 and 5.2 show clearly that if the Foster first form exists, then the corresponding Foster second form, the equivalent ladder forms and the equivalent unsymmetrical lattice forms also exist.

Example 5.1:

Let $Z(p_1,p_2) = \dfrac{p_1(1 + 17\, p_1p_2 + 24\, p_1^2p_2^2)}{(1 + 21\, p_1p_2 + 77\, p_1^2p_2^2 + 72\, p_1^3p_2^3)}$ be the given reactance function. It can be seen that $Z(p_1,1)$ is an RL impedance function of p_1, $Z(1,p_2)$ is an RC impedance function of p_2 and both of the above functions have the same internal critical frequencies. Thus, the given reactance function satisfies the conditions of Theorem 5.1 and hence must be realizable by the Class A structures. In fact, the realized networks of various forms are as shown in Fig. 5.1.

5.3 Partial Polynomial Derivatives of this type of Network Functions:

Thus far, we have been considering the realizability conditions of Class A and Class B structures. The network functions obtained by partial polynomial differentiation of the above network functions are considered in the following theorem.

Theorem 5.3:

If $Z(p_1,p_2) = \dfrac{N(p_1,p_2)}{D(p_1,p_2)}$ is realizable by Class A or Class B structures, then the functions

(i) $\dfrac{\partial N(p_1,p_2)/\partial p_i}{\partial D(p_1,p_2)/\partial p_i}$, $i = 1,2$ and (ii) $\dfrac{\partial^2 N(p_1,p_2)/\partial p_1 \cdot \partial p_2}{\partial^2 D(p_1,p_2)/\partial p_1 \cdot \partial p_2}$

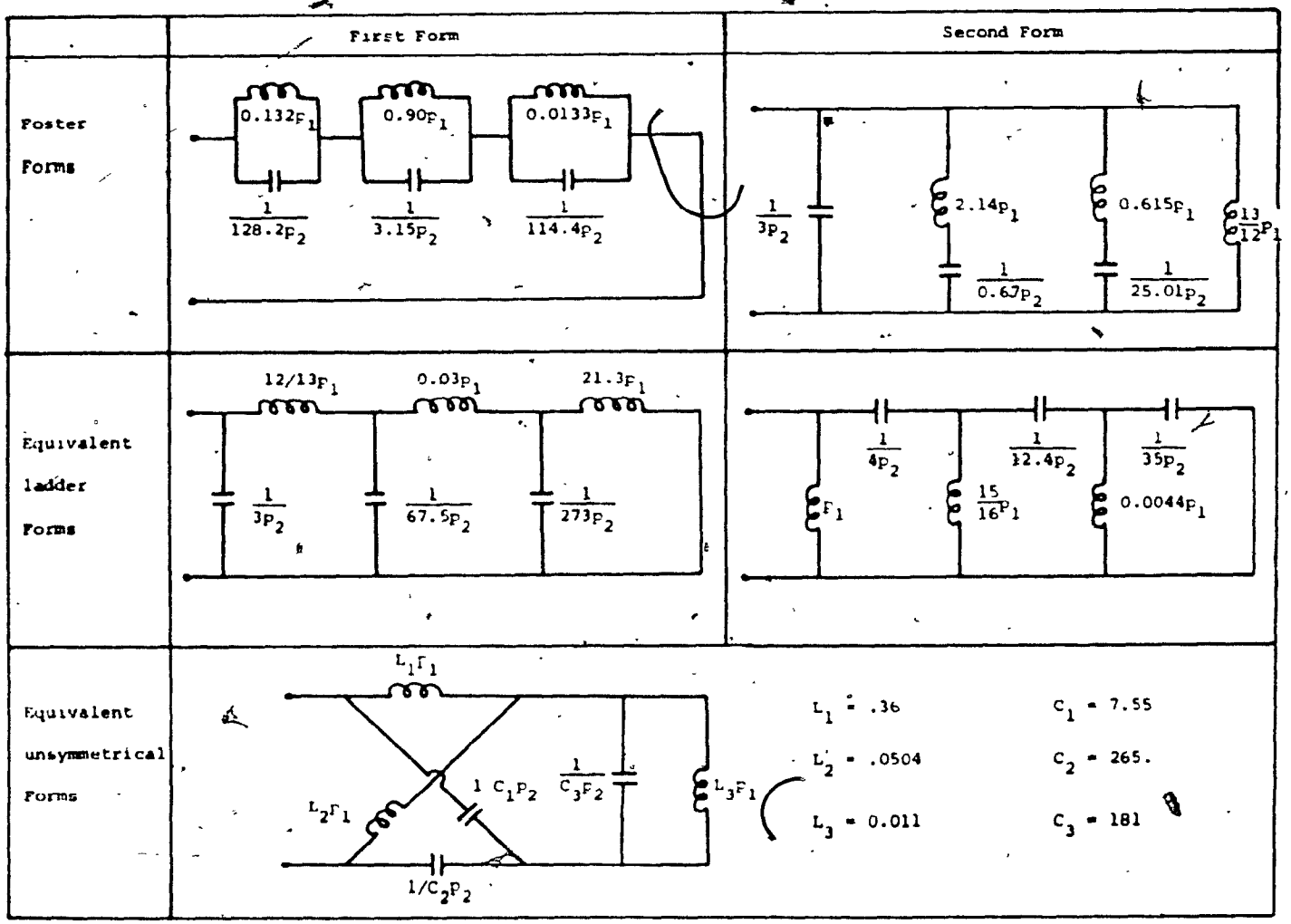


Fig. 5.1
Networks for Example 5.1

are also realizable by similar structures.

Proof:

The proof is given for a particular type of function, and similar proof holds for other types also.

Let

$$Z(p_1,p_2) = \frac{N(p_1,p_2)}{D(p_1,p_2)} = \frac{p_1 \prod(p_1p_2 + \alpha_i)}{\prod(p_1p_2 + \beta_i)} \qquad ..(5.6)$$

be the network function under consideration.

From Theorem 5.1, we have for constant positive value of

$p_2=p_{20}$(say), $Z(p_1,p_{20}) = \dfrac{N(p_1,p_{20})}{D(p_1,p_{20})}$ is an RC admittance function in p_1.

Hence,

$$\frac{\partial N(p_1,p_{20})/\partial p_1}{\partial D(p_1,p_{20})/\partial p_1} = \frac{\prod(p_1p_{20} + \alpha_i')}{p_{20}\prod(p_1p_{20} + \beta_i')} \qquad ..(5.7)$$

is again RC admittance function in p_1. Thus it is seen that (5.7) satisfies the conditions of Theorem 5.1 and hence can be realized by the Foster first form.

The proof can be given on similar lines for $\dfrac{\partial N(p_1,p_2)/\partial p_2}{\partial D(p_1,p_2)/\partial p_2}$ also.

By extension of the proof of proposition (i) with respect to p_2, proposition (ii) can be proved.

The network functions obtained by partial polynomial differentiation of Class A and Class B functions w.r.t. p_1 and p_2 are tabulated in Tables 5.2 and 5.3 or equivalently in Fig. 5.2. In Chapter II, we have dealt with generation of MRFs from a given MRF. Tables 5.2 and 5.3 provide us with some more methods of generating two-variable reactance functions.

TABLE 5.2

Network functions obtained under partial polynomial differentiation

Network function under Consideration	Operation Done	Resulting Network Function	Operation Done	Resulting Network Function
Class A type 1	N_{p1}/D_{p1}	Class A type 2	$p_2 \cdot N_{p1}/p_1 \cdot D_{p1}$	Class B type 2
Class A type 1	N_{p2}/D_{p2}	Class A type 1	$p_2 \cdot N_{p2}/p_1 \cdot D_{p2}$	Class B type 1
Class A type 2	N_{p1}/D_{p1}	Class A type 2	$p_2 \cdot N_{p1}/p_1 \cdot D_{p1}$	Class B type 2
Class A type 2	N_{p2}/D_{p2}	Class A type 1	$p_2 \cdot N_{p2}/p_1 \cdot D_{p2}$	Class B type 1
Class B type 1	N_{p1}/D_{p1}	Class B type 1	$p_1 \cdot N_{p1}/p_2 \cdot D_{p1}$	Class A type 1
Class B type 1	N_{p2}/D_{p2}	Class B type 2	$p_1 \cdot N_{p2}/p_2 \cdot D_{p2}$	Class A type 2
Class B type 2	N_{p1}/D_{p1}	Class B type 1	$p_1 \cdot N_{p1}/p_2 \cdot D_{p1}$	Class A type 1
Class B type 2	N_{p2}/D_{p2}	Class B type 2	$p_1 \cdot N_{p2}/p_2 \cdot D_{p2}$	Class A type 2

Class A

Type 1. $Z(p_1,p_2) = \dfrac{p_1 \, \Pi(p_1p_2+\alpha_1)}{\Pi(p_1p_2+\beta_1)}$

Type 2. $Z(p_1,p_2) = \dfrac{\Pi(p_1p_2+\delta_1)}{p_2 \, \Pi(p_1p_2+\lambda_1)}$

Class B

Type 1. $Z(p_1,p_2) = \dfrac{p_2 \, \Pi(p_1p_2+z_1)}{\Pi(p_1p_2+s_1)}$

Type 2. $Z(p_1,p_2) = \dfrac{\Pi(p_1p_2+\mu_1)}{p_1 \, \Pi(p_1p_2+\lambda_1)}$

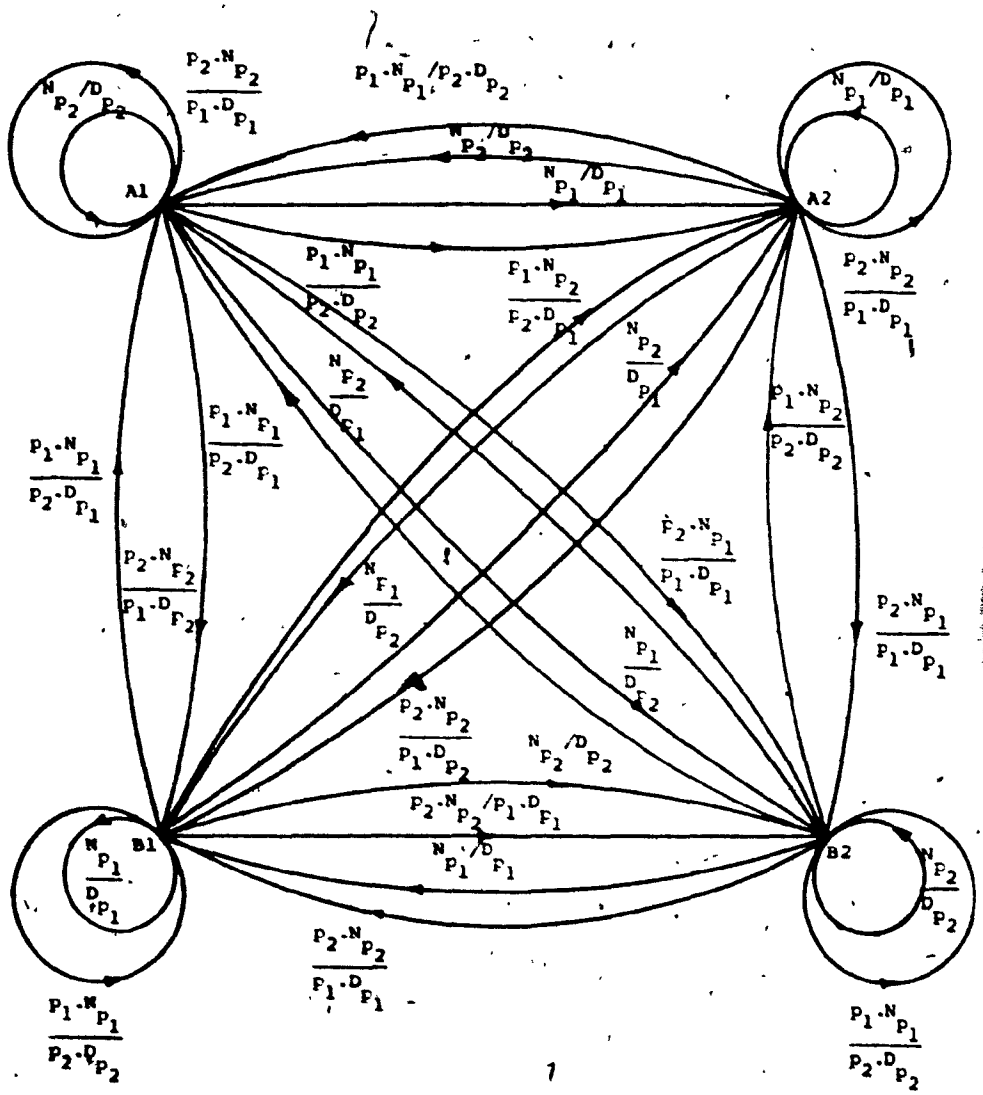
Notations used: $\dfrac{\partial N(p_1,p_2)}{\partial p_i} = N_{pi}$; $\dfrac{\partial D(p_1,p_2)}{\partial p_i} = D_{pi}$

TABLE 5.3

Network functions obtained under mixed partial polynomial differentiation

Network function under Consideration	Operation Done	Resulting Network Function	Operation Done	Resulting Network Function
Class A type 1	N_{p1}/D_{p2}	Class B type 2	$p_1 \cdot N_{p1}/p_2 \cdot D_{p2}$	Class A type 2
Class A type 1	N_{p2}/D_{p1}	Not TRF	$p_2 \cdot N_{p2}/p_1 \cdot D_{p1}$	Class A type 1
Class A type 2	N_{p1}/D_{p2}	Class B type 1	$p_1 \cdot N_{p1}/p_2 \cdot D_{p2}$	Class A type 1
Class A type 2	N_{p2}/D_{p1}	Not TRF	$p_2 \cdot N_{p2}/p_1 \cdot D_{p1}$	Class A type 2
Class B type 1	N_{p1}/D_{p2}	Not TRF	$p_1 \cdot N_{p1}/p_2 \cdot D_{p2}$	Class B type 1
Class B type 1	N_{p2}/D_{p1}	Class A type 2	$p_2 \cdot N_{p2}/p_1 \cdot D_{p1}$	Class B type 2
Class B type 2	N_{p1}/D_{p2}	Not TRF	$p_1 \cdot N_{p1}/p_2 \cdot D_{p2}$	Class B type 2
Class B type 2	N_{p2}/D_{p1}	Class A type 1	$p_2 \cdot N_{p2}/p_1 \cdot D_{p1}$	Class B type 1

NOTE: $\frac{p_i \, N_{pj}}{p_j \, D_{pi}}$ (for $i \neq j$) is not TRF (two-variable reactance function), the trivial cases of $\frac{N_{pi}}{D_{pj}} = 0, \infty$ are excluded.



$$A1: \quad Z(p_1,p_2) = \frac{p_1 \prod (p_1p_2 + \alpha_i)}{\prod (p_1p_2 + \beta_i)} \; ; \qquad B1: \quad Z(p_1,p_2) = \frac{p_2 \prod (p_1p_2 + z_i)}{\prod (p_1p_2 + s_i)}$$

$$A2: \quad Z(p_1,p_2) = \frac{\prod (p_1p_2 + \delta_i)}{p_2 \prod (p_1p_2 + \lambda_i)} \; ; \qquad B2: \quad Z(p_1,p_2) = \frac{\prod (p_1p_2 + \mu_i)}{p_1 \prod (p_1p_2 + \lambda_i)}$$

Fig. 5.2

The relation between various network functions under partial polynomial differentiation.

However, It may be noted that if we consider the network functions realizable by the ladder networks of Chapter III, the resulting functions after partial polynomial differentiation may not be realizable by such networks. The following example clarifies the above point:

Example 5.2:

Let us consider

$$Z(p_1,p_2) = \frac{N(p_1,p_2)}{D(p_1,p_2)} = \frac{1+4\ p_1^2+9\ p_1p_2+5\ p_2^2+2\ p_1^4+11\ p_1^3p_2+19\ p_1^2p_2^2+13p_1p_2^3+3p_2^4}{3\ p_1+2\ p_2+\ 2\ p_1^3+\ 7\ p_1^2p_2+8\ p_1p_2^2+\ 3\ p_2^3}$$

which is realizable by TLPL. But the functions $\frac{\partial N/\partial p_1}{\partial D/\partial p_1}$, $\frac{\partial N/\partial p_2}{\partial D/\partial p_2}$ and $\frac{\partial^2 N/\partial p_1.\partial p_2}{\partial^2 D/\partial p_1.\partial p_2}$ which are given as below are not realizable by TLPL.

$$\frac{\partial N/\partial p_1}{\partial D/\partial p_1} = \frac{8\ p_1+\ 10\ p_2+\ 11\ p_1^3+\ 38\ p_1^2p_2+\ 39\ p_1p_2^2+\ 12\ p_2^3}{2\ +\ 7\ p_1^2\ +\ 16\ p_1p_2\ +\ 9\ p_2^2}$$

$$\frac{\partial N/\partial p_2}{\partial D/\partial p_2} = \frac{9\ p_1+\ 10\ p_2+\ 11\ p_1^3+\ 38\ p_1^2p_2+\ 39\ p_1p_2^2+\ 12\ p_2^3}{2\ +\ 7\ p_1^2\ +\ 16\ p_1p_2\ +\ 9\ p_2^2}$$

$$\frac{\partial^2 N/\partial p_1.\partial p_2}{\partial^2 D/\partial p_1.\partial p_2} = \frac{\partial^2 N/\partial p_2.\partial p_1}{\partial^2 D/\partial p_2.\partial p_1} = \frac{9\ +\ 33\ p_1^2+\ 76\ p_1p_2+\ 39\ p_2^2}{14\ p_1\ +\ 16\ p_2}$$

5.4 Two-Variable Miyata Method:

Thus far, we are discussing about the Foster forms and their equivalent networks, which constitute a special class of two-variable ladder networks studied in Chapter III. By utilizing these results, we shall discuss in this section, a transformerless synthesis technique for two-

variable positive real functions, similar to the single-variable Miyata method[50].

The following theorem gives the conditions under which a given function can be realized both by TLPL and the network of Fig. 5.3, based upon which, a transformerless synthesis method for two-variable PRFs is proposed.

Theorem 5.4:

If a given two-variable reactance function $Z(p_1,p_2)$ is realizable as a TLPL and if the same function has to be realizable by a network of Fig. 5.3, then the necessary and sufficient conditions to be satisfied by the TLPL network are:

(i) The values of the series arms impedances are constant multiples of one another, and

(ii) The values of the shunt arms admittances are constant multiples of one another.

Proof: Sufficiency:

If the given $Z(p_1,p_2)$ is realizable as TLPL satisfying conditions (i) and (ii), we see that this is equivalent to another ladder network in the variables λ_1,λ_2, with λ_1-type inductors in the series arms and λ_2-type capacitors in the shunt arms. This network is nothing but the ladder equivalent of the first Foster form discussed above. We also know that if the first ladder equivalent exists, so does the second form. That is, the $Z(\lambda_1,\lambda_2)$ can be realized with λ_2-type series capacitors and λ_1-type shunt inductors. By replacing λ_1,λ_2 with the corresponding values of p_1,p_2 we see that the sufficiency follows.

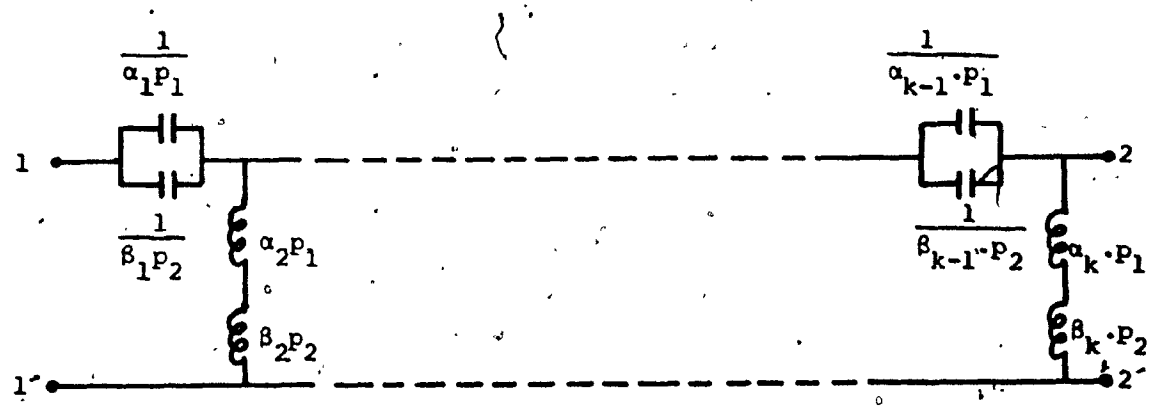


Fig. 5.3

Two-variable ladder network consisting of capacitors in parallel in the series arms and inductors in series in the shunt arms.

Necessity:

By analysis we can see that the chain parameter C of the TLPL satisfies the following conditions:*

a) The first degree term of $C = \sum_{j=1} y_j$

b) The highest degree term of $C = (\Pi\, Z_i) \cdot (\Pi\, y_j)$

Then A/C of TLPL is realizable as the network of Fig. 5.3, only if the first degree term of C is a common factor of all other degree terms of C. This can happen only if the shunt arm admittances are constant multiples of one another.

Similarly by proceeding with D/B of TLPL we arrive at the conclusion that the impedances of the series arms must be constant multiples of one another, and thus the theorem follows.

Utilizing the ideas developed thus far we present a transformerless synthesis method for a class of two-variable PRFs, similar to the single-variable Miyata method[50].

If $\frac{M_1+N_1}{M_2+N_2}$ is a two-variable PRF with $M_1M_2 - N_1N_2 = \sum_{i=0}^{k} (\alpha_1 p_1+\beta_1 p_2)^{2i}$ such that each degree term is positive for all ω_1, ω_2; and M_2/N_2 is realizable as TLPL with the values of the series arms impedances and shunt arms admittances being constant multiples of one another, then the given $\frac{M_1+N_1}{M_2+N_2}$ can be realized as a series connection of two-variable ladder networks.

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* y_j and Z_i are respectively the admittances and impedance of the shunt and series arms of a TLPL.

The validity of above conclusion can be explained as follows: If the values of series arms impedances and shunt arms admittances are constant multiples of each other, then the TLPL can be considered as a low-pass ladder of λ $[=(\alpha_1 p_1 + \beta_1 p_2)]$ and we know from [50], that if

$M_1M_2 - N_1N_2 = \sum_{i=0}^{k} a_i \cdot (-\lambda^2)^{2i}$, $a_i > 0$, then the corresponding immittance function can be realized as a series connection of resistively terminated ladder networks in λ-plane. Hence, by replacing λ by $(\alpha_1 p_1 + \beta_1 p_2)$ we obtain the required two-variable network.

From this result we see the restrictive nature of Miyata method - which is restrictive even to the single-variable case - when extended to two-variable functions. We have considered here a more general class of functions than were considered in [9].

5.5 Conclusions:

Necessary and sufficient conditions for the realization of a class of two-variable reactance functions in a form similar to the single-variable Foster form are derived. It is shown that if Foster first form exists, then the Foster second form, the equivalent ladder and unsymmetrical lattice networks also exist. It is also shown that these networks constitute a special class of two-variable ladder networks considered in Chapter III.

It may be noted that, if the two-variable reactance function is realizable by TLPL, it may not have the corresponding Foster form of the type discussed above, since, the even and odd polynomials of the TLPL functions may not necessarily be factorizable as required by the Foster forms. It has to be inferred that in the case of single-variable

reactance structures, if a given function is realizable by the Cauer ladder or the Foster form (and Lee's unsymmetrical lattice), the other structures are always realizable, whereas in the two-variable case this is not necessarily true.

It has been pointed that the partial polynomial derivatives of these network functions are again realizable by such networks, whereas the network functions considered in Chapter III are not.

It is pointed out that a given network function which is realizable by TLPL need not necessarily be realized by the network of Fig. 5.3 and the required conditions are established. Based upon this and the Foster forms, a transformerless realization for a class of two-variable PRFs is presented by extending the single-variable Miyata method. The restrictive nature of Miyata method for two-variable PRFs is mentioned.

CHAPTER VI

CONCLUSIONS AND SOME SUGGESTED PROBLEMS

6.1 Conclusions:

This thesis has studied the properties of MPRFs and the realization of multivariable ladder networks, the main topics of investigation being:

(i) Generation of MPRFs by the Differential Operator and from the given multivariable real part,

(ii) Realization of multivariable ladder networks, and

(iii) Synthesis of cascaded structures consisting of commensurate UEs and lumped elements.

Generation of MPRFs by the differential operator is considered first. Methods are given for the generation of MPRFs of n-and (n-1)-variables from the prescribed n-variable PRF by the differential operator. By these a necessary coefficient test is also provided for an MPRF.

A method of generating an MPRF from the prescribed multivariable real part is considered next. In order that an MPRF can be generated from the prescribed multivariable real part, in addition to its non-negativeness some more conditions need be satisfied by the numerator and the denominator polynomials of the real part. These conditions are caused by the fact that the number of equations to be solved are more than the number of unknowns. This is contrary to the single-variable case, where the number of equations is equal to the number of unknowns and non-negativeness of the real part alone suffices for the generation of a PRF.

By extending the Sturm test, a method of testing a two-variable polynomial for non-negativeness is developed. To cover the entire range

of both the variables, it is required to have a number of Sturm-tables as opposed to one table required in the single-variable case. For each such table, the range of one of the variables (say ω_2) is divided such that the sign of the corresponding Sturm functions does not change during that interval of ω_2 and the value of the other variable varies from $-\infty$ to $+\infty$. However, an extension for the case of more than two-variables appears to be tedious, and hence a new technique may have to be evolved.

A multivariable array is developed by means of which the realizability conditions for the MLPL are derived. This array reduces to the Routh-Hurwitz array for the case of a single-variable. By means of various transformations, other types of ladder networks are also derived from the original MLPL. The realizability conditions of the resistively terminated lossless ladder networks are established. It is also pointed out that, for the multivariable case, not all reactance functions are realizable as ladder networks. This is contrary to the case of a single-variable, where the reactance nature of the given function is both necessary and sufficient for it to be realizable by ladder networks.

The realizability conditions of a class of two-variable reactance networks similar to the single-variable Foster form are studied. It is shown that these networks are a class of the two-variable ladder networks considered earlier. The realizability conditions for these networks are derived directly from the given network functions rather than by the array discussed earlier. The equivalent ladder and non-symmetric lattice forms are also developed.

However, it is noted that if a ladder network of the form TLPL is

available, the equivalent non-symmetrical lattice structure or the structure similar to the single-variable Foster form may not exist. The properties of these functions under the operation of partial polynomial differentiation are studied. By making use of the two-variable ladder networks discussed earlier, a transformerless synthesis is proposed for a class of two-variable PRFs similar to the single-variable Miyata method. This synthesis is possible only (a) when the values of the series impedances are constant multiples of one another, and (b) when the values of the shunt admittances are constant multiples of one another, in a TLPL.

In the ladder networks discussed, by properly defining the variables p_i, it is shown that some interesting mixed lumped-distributed filters can be developed. Thus, when $p_i = \tanh \tau_i s$, the p_i-type inductors and capacitors can be respectively replaced by non-commensurate short-circuit and open-circuit stubs. By means of the equivalent relations derived, it is shown that for $p_1 = \sinh s\tau$ and $p_2 = s \cdot \cosh s\tau$, a cascade of commensurate UEs separated by series lumped inductor on one side and shunt lumped capacitor on the other side is equivalent to a TLPL. This enables the synthesis of such mixed lumped-distributed networks to be performed by continued-fraction expansion rather than taking recourse to the Richards' transformation[20], which has been the widely adopted technique in the realization of cascaded UE structures.

6.2 Some Suggested Investigations:

The foregoing discussion leads to the following problems which are suggested for further investigation:

(i) It has been discussed in section 2.5, that the generation of an MPRF from the prescribed imaginary part may not be unique. In the

single-variable case, as the number of equations is one less than the number of unknowns, it is concluded that the solution is non-unique. But, in the multivariable case, it appears that even though the number of equations may be more than the number of unknowns, the solution could still be non-unique. It is worthwhile to pursue this investigation further.

(ii) The testing of a polynomial of more than two-variables for non-negativeness by the method given in the thesis appears to be tedious. However, it is hoped that this method may prove useful in the testing of 3-variable Hurwitz polynomials similar to the Ansell's method. Also, in order to test the non-negativeness of more than two-variable polynomials, an entirely new approach may have to be developed.

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