SOME PROPERTIES OF INTEGRALS OF NETWORK FUNCTIONS

by

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ABSTRACT

This thesis initiates the study of some properties of integrals of network functions and makes an attempt to investigate their use in network theory.

The integrals of the even and odd parts of Hurwitz polynomials, M and N, have been considered first. Conditions have been established such that these are suitable as the odd and even parts of other Hurwitz polynomials respectively. Testing procedures including simple inspection tests have been enunciated. The polynomials M and N satisfying these conditions have been defined as being "LC-integrable".

The approach used to derive integrability conditions for M and N are then employed to examine the integrability of a polynomial F having only simple negative real axis zeros. Conditions have been found such that the integral of F (a) is Hurwitz (Hurwitz Integrability) or (b) has only simple negative real axis zeros (RC-Integrability).

Special cases of M, N or F, termed the medial M, the medial N or the medial F, arise when their integrals assume the value zero at the alternate zeros of the given functions. The properties of the medial M, N or F provide additional criteria which supplement the testing procedures described earlier.

The real and the imaginary parts of positive real functions are subjected to the operations of differentiation and integration. Conditions have been established such that the resulting function is suitable as the corresponding part of another positive real function.

As a possible application, it is shown that higher order positive real functions may be generated using the properties of integrals of network functions. Further, two new decompositions of a Hurwitz polynomial employing differentials and/or integrals of Hurwitz polynomials are considered and their use in network synthesis discussed.

CHAPTER I

INTRODUCTION

1.1 General:

Most problems dealing with the stability or the physical realizability of a linear network are stated in terms of the restrictions on the location of the zeros of polynomials in the complex frequency plane 's'. One well known result is that the numerator and the denominator polynomials of driving point functions realizable by a lumped, linear, finite, passive, bilateral and time-invariant network shall have no zeros in the right-half plane. Considerable amount of literature[1,2] exists on the subject of polynomials and this is being utilized at a rapidly increasing rate in the solutions of network theory problems. There is little doubt that this practice will continue in the years to come as a large amount of literature on the mathematics of polynomials is yet to be applied in the solutions of such problems. The mutual interaction between the mathematics of polynomials and network theory has profited both disciplines. In several instances, modern network theory has given considerable insight into the behavior of polynomials. The generation of polynomials with negative real axis zeros and an alternate proof of the Laguerre's theorem[3] can be quoted as examples in point. Recently, some links between theory of equations and realizability conditions for networks[4] have also been established.

One of the topics in the mathematics of polynomials closely connected to the subject of present investigation is concerned with relationships between a polynomial and its derivative. The application of these relationships in network theory has been relatively recent. The following section summarizes some results concerning the derivatives of polynomials.

1.2 Some known Results concerning the Derivatives of Polynomials:

A useful theorem in network theory concerning the properties of Hurwitz polynomials* states that

(a) "If F(s) is a Hurwitz Polynomial(HP), then so is $F'(s)$, where $F'(s) = \frac{d\,F(s)}{ds}$".

While a proof of this theorem has been furnished by Weinberg[5] utilizing the properties of a positive real function(PRF), it also follows as a direct consequence of a powerful theorem by Lucas[6], which states that:

(b) "The zeros of the derivative polynomial F'(s) lie within or on the perimeter of the smallest polygon which includes within itself, or on its boundary, all the zeros of the polynomial F(s)".

This theorem has been applied by Reza[7] to generalize in

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* A Hurwitz polynomial is one whose zeros do not lie in the right half plane. A Hurwitz polynomial may be either strictly Hurwitz (containing no zeros on the imaginary axis) or pseudo Hurwitz (if simple imaginary axis zeros are included). It is assumed that the even and odd parts do not contain any common factor.

a certain sense Foster's and Cauer's theorems. Specifically, it has been proved that:

(c) "If $P(s)/Q(s)$ is a rational two-element kind impedance function, then so is $P'(s)/Q'(s)$, thereby establishing the invariance of the nature of an impedance function under such an operation".

Another result of some importance concerning the derivative of a HP is that:

(d) "The logarithmic derivative $Z(s)$ of a HP $F(s)$ defined as $Z(s) = \frac{d\{\text{Log } F(s)\}}{ds}$ is a PRF".

Three important corollaries to (d) are:

(i) If $F(s)$ is a HP, then

$$Z_n(s) = \frac{\frac{d^n F(s)}{ds^n}}{\frac{d^{n-1} F(s)}{ds^{n-1}}}$$

is a PRF.

(ii) The logarithmic derivative of any polynomial $F(s)$ with zeros restricted to the negative real axis, represents the driving point impedance of an RC network.

(iii) The logarithmic derivative of any polynomial $F(s)$ with simple zeros restricted to the imaginary axis represents the driving point impedance of an LC network.

Talbot[8] has shown that the result (c) with respect to

the generalized Foster's and Cauer's theorems is, in fact, of a more general nature and holds for any impedance function. Specifically, it is established that:

(e) "If $\frac{F(s)}{G(s)}$ is a PRF, then so is $\frac{F'(s)}{G'(s)}$".

Further, using the known properties of the derivative of a polynomial, necessary coefficient conditions for the realizability of two-element kind networks have recently been given[9].

1.3 Derivatives of Network Functions:

Several uses of the derivatives of complete network functions have been reported in the literature. A summary of some important results follows in the next section.

An early reference is that of Van der Pol[10], who showed that the difference between the electric and the magnetic energy in an electric network is related to the derivative of its driving point function. This theorem states that:

"The excess of stored magnetic energy M over stored electric energy E, in any general RLC passive network fed by a DC current source I or a DC voltage source V, is given by

$$M - E = \frac{I^2}{2} \left.\frac{dZ(s)}{ds}\right|_{s=0}$$

$$M - E = -\frac{V^2}{2} \left.\frac{dY(s)}{ds}\right|_{s=0} \qquad ..(1.1)$$

respectively".

Using this theorem, driving point function synthesis

procedures have been formulated for two-element kind networks wherein the total R,C or L has been minimized[11].

Necessary coefficient relationships between the numerator and the denominator polynomials of driving point functions have resulted by taking derivatives of network functions and using the known properties of such derivatives[12,13,14,15,16]. These relationships provide simple inspection tests for positive realness of driving point functions.

In one of the above papers[16], it has been shown that the derivative of an RC driving point immittance function is positive real under certain coefficient conditions. Considering an RC driving point admittance function of the following form:

$$Y_{RC}(s) = K \frac{s^{n+1} + a_n s^n + \ldots + a_o}{s^n + b_{n-1} s^{n-1} + \ldots + b_o}$$

where either $a_o \neq 0$ and $b_o \neq 0$ or $a_o = 0$ and $b_o \neq 0$ and the inequality

$$1 < \frac{a_1 b_o - a_o b_1}{b_o^{\,2}} \leq 9 \qquad ..(1.2)$$

holds, it has been proved that $\frac{d\{Y_{RC}(s)\}}{ds}$ is positive real.*

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* What can be considered as an omission in the above result is the fact that the authors have not pointed out that (1.2) is a sufficient condition though not necessary. This can be demonstrated by the following example. Consider

$$Y_{RC}(s) = \frac{s^3 + 15.1s^2 + 51.5s + 5}{s^2 + 6.8s + 4.8} \qquad ..(1.3)$$

The same paper[16] goes on further to establish the conditions under which a rational function F(s) is the derivative of an RC admittance function.

The foregoing discussion shows that the derivatives of polynomials and network functions have been widely used in Circuit Theory. It seems reasonable to speculate that the integrals of such functions would be of at least equal importance[17]. Unfortunately, even though several relationships are known between a polynomial and its derivative, little is known at the present time about such relationships with respect to its integral. This thesis initiates the study of the properties of the integrals of network functions.

1.4 Scope of the Thesis:

The main aim of Chapter II is to establish the conditions under which the integral of a HP is another HP. Poly-

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The inequality (1.2) does not hold for expression (1.3). Differentiating (1.3), we get

$$\frac{dY_{RC}(s)}{ds} = \frac{s^4 + 13.6s^3 + 51.18s^2 + 144.96s + 247.20}{(s + 0.8)^2(s + 6)^2} \qquad ..(1.4)$$

The numerator of the real part of (1.4) is

$$s^8 - 77.94s^6 + 350.46s^4 - 2178.29s^2 + 5695.48$$

which is always positive when $s = j\omega$, $0 \leq \omega < \infty$. Hence, it follows that (1.2) is not a necessary condition for the positive realness of $dY_{RC}(s)/ds$.

nomials for which this holds are referred to as being 'HP-integrable', otherwise, they are termed 'HP-unintegrable'.* The suitability of the integrals of the even and the odd parts of the given HP as being the odd or the even part of another HP is first studied. This procedure has been referred to in the thesis as testing the 'LC-integrability'* of the even and the odd parts. If the two parts are separately integrable, then they are tested further to see if they can form the odd and the even parts of the same HP. If either part is unintegrable, the given HP is classed as being unintegrable.

In Chapter III, the integrability of polynomials containing only simple negative real axis zeros has been studied. Two aspects are considered; the conditions under which the integral of the given polynomial is HP and the conditions under which the integral is a polynomial containing only simple negative real axis zeros. These two cases are referred to respectively as 'Hurwitz integrability' and 'RC integrability'.

In Chapter IV, the real and imaginary parts of a PRF are subjected to the operations of differentiation and integration. Apart from the conventional operations, two more operations on a rational function $R(s) = \frac{P(s)}{Q(s)}$ are considered. These have been termed 'polynomial differentiation' defined by $\frac{P'(s)}{Q'(s)}$ and 'polynomial integration' defined by $\frac{\int P(s)ds}{\int Q(s)ds}$. After each of these operations on the real and imaginary

*In this thesis, they are referred to as 'integrable or 'unintegrable' respectively.

parts of a positive real function, the suitability of the new functions as the corresponding parts of another positive real function are examined.

Chapter V discusses the decomposition of a given HP into the sum of a HP and its integral which also is a HP. This is applied in the synthesis of a transfer function as a symmetrical lattice structure where all the inductances are lossy. Also, an integro-differential type of decomposition of a HP where each constituent polynomial is a HP is shown to be possible. A possible use of this decomposition is discussed.

CHAPTER II

INTEGRABILITY OF HURWITZ POLYNOMIALS

2.1 Introduction:

The determination of the character of a HP under integration can be considered as a logical starting point for the investigation of the properties of integrals of network functions. However, no precise mathematical relationship exists between the zeros of a polynomial and those of its integral. Therefore, recourse is taken to network techniques in the formulation of the integrability conditions of a HP.

The main objective of this chapter is to establish necessary and sufficient conditions under which the even and odd parts of a HP are integrable and hence obtain the necessary conditions for the integrability of a HP.

2.2 The Integral of a Hurwitz Polynomial:

More than one proof[5,6] exists to show that the derivative of a HP F(s) is Hurwitz. However, the integral of $F = M + N$, where M and N are its even and odd parts respectively, as given by

$$\int F ds = \int M ds + \int N ds + K \qquad ..(2.1)$$

need not be a HP*. This may occur due to either or both of

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* K, the constant of integration has to be associated with $\int N ds$ only, as this will form the even part of $\int F ds$. In addition, K has to be non-negative, otherwise $\int F ds$ will not be a HP.

the following reasons:

(i) $\int M ds$ and/or $\int N ds + K$ may not possess simple imaginary axis zeros with any choice of K.

(ii) Even if $\int M ds$ and $\int N ds + K$ possess such zeros, these two sets of zeros may not interlace. Examples (2.1) and (2.2) illustrate these two occurrences respectively.

Example 2.1:

Consider

$$F = (s^2+1)(s^2+3) + s(s^2+2) \qquad ..(2.2)$$

which is a HP. Then

$$\int F ds = (\frac{s^4}{4} + s^2 + K) + \frac{s}{15}(3s^4 + 20s^2 + 45). \qquad ..(2.3)$$

It is seen that the odd part of $\int F ds$ has complex conjugate zeros and therefore, (2.3) is not a HP. This holds irrespective of the value of K.

Example 2.2:

Let

$$F = (s^2 + 1)(s^2 + 5.01) + s(s^2 + 1.1)(s^2 + 5.5) \qquad ..(2.4)$$

where

$$M = s^4 + 6.01s^2 + 5.01 \qquad ..(2.5)$$

and

$$N = s^5 + 6.6s^3 + 6.05s \qquad ..(2.6)$$

Now,

$$\int F ds = \int M ds + \int N ds + K$$

$$= \frac{1}{5}s(s^2 + 4.825569)(s^2 + 5.191097) + \frac{1}{6}(s^6 + 9.9s^4 + 18.15s^2) + K \qquad ..(2.7)$$

It can be shown that for the range of values of K, $K \leq 1.59561$, $\int N ds + K$ can have only imaginary axis zeros and it can be written as,

$$\int N ds + K = \frac{1}{6}(s^2 + a)(s^2 + b)(s^2 + c) \qquad ..(2.8)$$

$$a,b,c > 0$$

If $\int F ds$ is to be a HP, the inequality,

$$0 < a < 4.825569 < b < 5.191097 < c \qquad ..(2.9)$$

shall hold.

Since $a + b + c = 9.9$ from (2.7) and (2.8), it follows that (2.9) cannot hold. Hence $\int F ds$ cannot be a HP.

The above examples have illustrated the need of knowing how to choose K, such that, the resulting polynomial is Hurwitz. The following section deals with the influence of the constant term in determining the nature of the polynomial with which it is associated.

2.3 The Constant Term of a Polynomial:

A given non-HP, $F = M + N$, may be made Hurwitz by associating with it a real constant C, provided $M + C$ and N contain only simple imaginary axis zeros and these zeros interlace with each other. In certain simple cases, it may be possible to get the range of C by inspection. Failing this, one may expand $\frac{M + C}{N}$ into partial fractions and choose a value of C, if one exists, such that the residues are all positive. If no such value exists, then $(F + C)$ can never be Hurwitz.

Starting with F, a HP, it is possible to determine the range over which its constant term may be allowed to vary such that F remains Hurwitz. The procedure is contained in the following theorem.

Theorem 2.1:

If $F = (M + N)$ is a HP, where

$$N = s \prod_{1}^{n} (s^2 + p_i), \qquad 0 < p_1 < p_2 < \ldots\ldots < p_n$$

then $(F + C)$ will be a HP if and only if,

$$\text{(i) for } C > 0 \;,\quad M\Big|_{s^2=-p_i} + C \le 0 \;;\quad p_i = p_1, p_3, p_5, \ldots$$

$$\text{or} \quad \text{(ii) for } C < 0 \;,\quad M\Big|_{s^2=-p_i} + C \ge 0 \;;\quad p_i = 0, p_2, p_4, \ldots.$$

Proof:

Since F is HP, $\frac{M}{N}$ is a reactance function and can be written as

$$\frac{M}{N} = \frac{A_o}{s} + \Sigma \frac{2A_i s}{s^2 + p_i} + A_\infty s \qquad ..(2.10)$$

As the A_i's are positive, we have,

$$M\Big|_{s^2=-p_i} < 0, \qquad \text{for } p_i = p_1, p_3, p_5, \ldots.$$

$$\text{and} \quad M\Big|_{s^2=-p_i} > 0, \qquad \text{for } p_i = 0, p_2, p_4, \ldots.$$

It follows that, for $(F + C)$ to be a HP, we must have,

$$\text{(i) for } C > 0 \;,\quad M\Big|_{s^2=-p_i} + C \le 0 \;;\quad p_i = p_1, p_3, p_5, \ldots$$

$$\text{or} \quad \text{(ii) for } C < 0 \;,\quad M\Big|_{s^2=-p_i} + C \ge 0 \;;\quad p_i = 0, p_2, p_4, \ldots.$$

Hence the theorem is proved.

Example 2.3:

Given $F = (M + N)$

$$= (s^2+1)(s^2+3)(s^2+5) + s(s^2+2)(s^2+4)(s^2+6) \qquad ..(2.13)$$

it is required to find the range of C such that (F + C) is a HP. We have

$$\begin{aligned} M + C\Big|_{s = 0} &= 15 + C \\ M + C\Big|_{s^2=-2} &= -3 + C \\ M + C\Big|_{s^2=-4} &= 3 + C \\ M + C\Big|_{s^2=-6} &= -15 + C \end{aligned} \qquad ..(2.14)$$

Hence, the permissible range of C is given as,

$$-3 \leq C \leq 3 \qquad ..(2.15)$$

With any integration process, a constant term is associated. The importance of Theorem 2.1 lies in the fact that it gives a range of the constant term over which the Hurwitz character of the polynomial remains invariant.

2.4 Conditions for the Integrability of Hurwitz Polynomials:

For the integrability of F, it is essential to establish the following:

Step (1): that $\int M ds$ contains only simple imaginary axis zeros and is, therefore, suitable as the odd part of a HP, or equivalently that $\frac{\int M ds}{M}$ is a reactance function.

Step 2: that $\int N ds + K$ contains only simple imaginary axis zeros and is, therefore, suitable as the odd part of a HP or equivalently that $\frac{(\int N ds + K)}{N}$ is also a reactance function.

Step 3: that $\frac{(\int N ds + K)}{\int M ds}$ is also a reactance function.

In effect, step (3) imposes the constraint that the imaginary axis zeros of $\int M ds$ and $(\int N ds + K)$ interlace with each other.

Necessary and sufficient conditions towards the fulfilment of steps (1) through (3) above are given in the following sections.

2.5 The Integrability of M:

If M is given as a polynomial, then it will be integrable provided the continued fraction expansion of $\frac{\int M ds}{M}$ has only real and positive coefficients. Alternatively, if M is given in the factored form containing m factors as,

$$M_m = (s^2 + X_1)(s^2 + X_2) \;\ldots\; (s^2 + X_m) \qquad ..(2.16)$$

$$X_1 < X_2 < \ldots\ldots < X_m$$

then the residues of the partial fraction expansion of $\frac{\int M_m ds}{M_m}$ must be real and positive. The same is also equivalent to the following inequalities,

$$\frac{1}{j} \int M_m ds > 0 \quad \text{at} \quad s = j\sqrt{X_{2i-1}}$$

$$\text{and} \quad \frac{1}{j} \int M_m ds < 0 \quad \text{at} \quad s = j\sqrt{X_{2i}} \qquad ..(2.17)$$

$$i = 1,2,3,\ldots.$$

Thus the problem is to formulate the conditions under which inequalities (2.17) hold, given M_m as in (2.16).

It will be necessary at many places to depict M_m or $\int M_m\, ds$ graphically. The next sub-section deals with the conventions and the geometrical interpretations of M_m and $\int M_m ds$.

2.5.1 Geometrical Interpretation of M_m and $\int M_m ds$:

Since M_m is known to have only conjugate pairs of simple zeros on the imaginary axis, it is enough to depict the configuration of $M_m(j\omega)$ vs. $j\omega$ for $\omega \geq 0$. If M_m is integrable, then its zeros will interlace with those of $\int M_m ds$ on the $j\omega$-axis. This condition can also be established by noting the signs of $\int M_m ds$ at the consecutive zeros of M_m. The same must alternate in order that integrability may hold.

It will be apparent that M_m is real for any value of ω while $\int M_m ds$ is always imaginary. However, in order to show the interlacing of the zeros of M_m and $\int M_m ds$, it has been found convenient to plot both on the same graph. In such plots, it has been assumed that the ordinate of M_m is real while that of $\int M_m ds$ is imaginary. No confusion is anticipated on this account because the ultimate concern is only with respect to the signs of $\int M_m ds$ at the zeros of M_m. Similarly, whenever $\int M_m ds$ is compared with a real quantity, it is tacitly assumed that it has been made real by simply removing its j multiplier.

The M_m shown in Fig. 2.1 is integrable since the plot $\int M_m ds$ is seen to change sign at consecutive zeros of M_m.

M_m will be unintegrable if either

$$\begin{aligned} &\int M_m ds < 0 \quad \text{at } s = j\sqrt{X_{2i-1}} \\ \text{or} \quad &\int M_m ds > 0 \quad \text{at } s = j\sqrt{X_{2i}} \\ &i = 1,2,3,\ldots \end{aligned} \qquad \cdot\cdot(2.18)$$

Figs. 2.2a and 2.2b illustrate these two situations.

$\int M_m ds$ for a given value of $s = j\omega$ can also be viewed as the area enclosed by M_m from the origin up to the given point. Symbolically,

$$\int M_m ds \Big|_{s = j\sqrt{X_k}} = \int_0^{j\sqrt{X_k}} M_m ds \qquad ..(2.19)$$

The validity of this interpretation is guaranteed by the fact that the constant of integration is assumed to be zero. As an immediate consequence of this and in view of the form of M_m as given in (2.16), it follows that $\int M_m ds$ is unconditionally positive at $s = j\sqrt{X_1}$ (or $s^2 = - X_1$), the first zero of M_m. In other words, the inequality,

$$\int_0^{j\sqrt{X_1}} M_m ds > 0 \qquad ..(2.20)$$

is always true.

The testing of the integrability of M_m is based on three theorems discussed in the following sub-section.

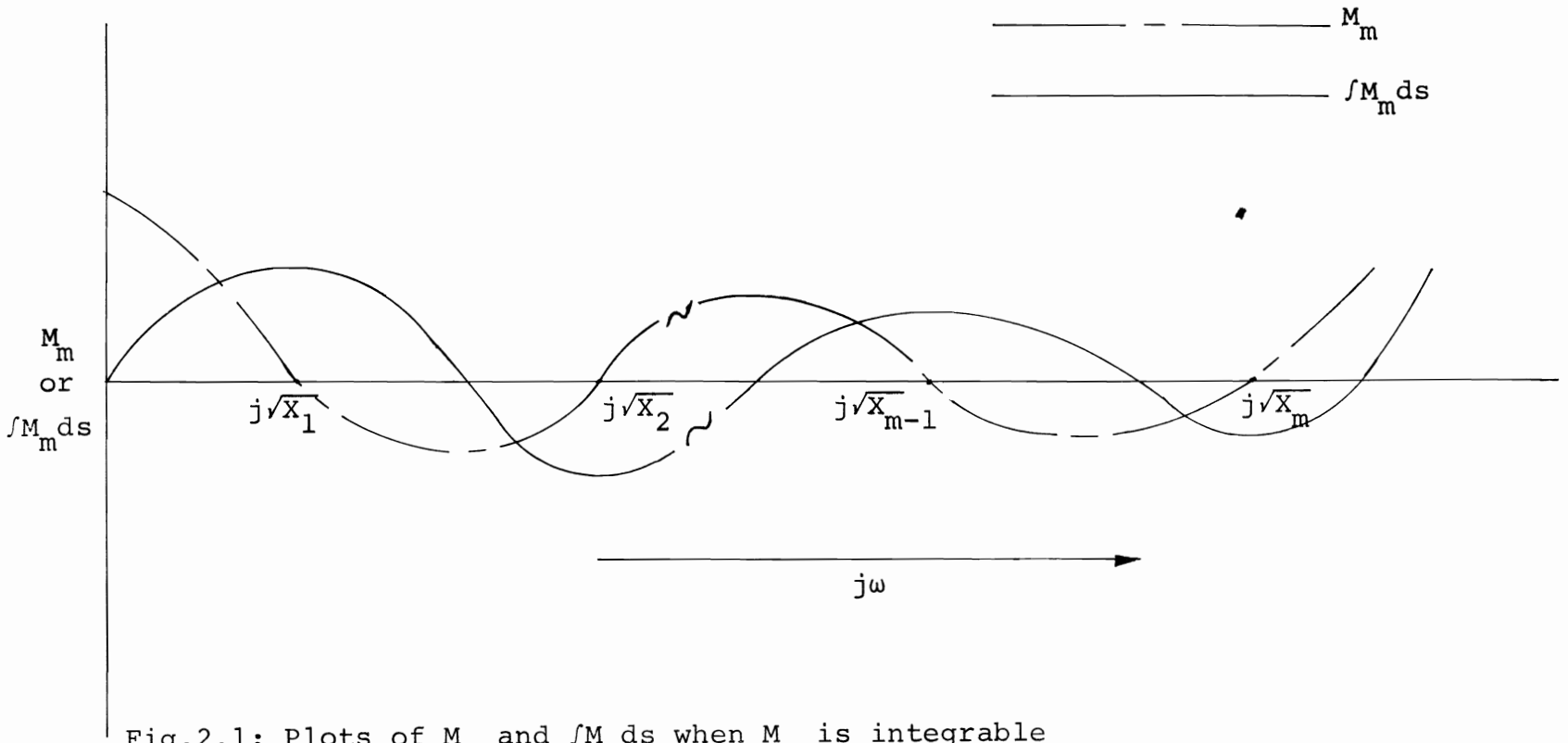

Fig.2.1: Plots of M_m and $\int M_m ds$ when M_m is integrable

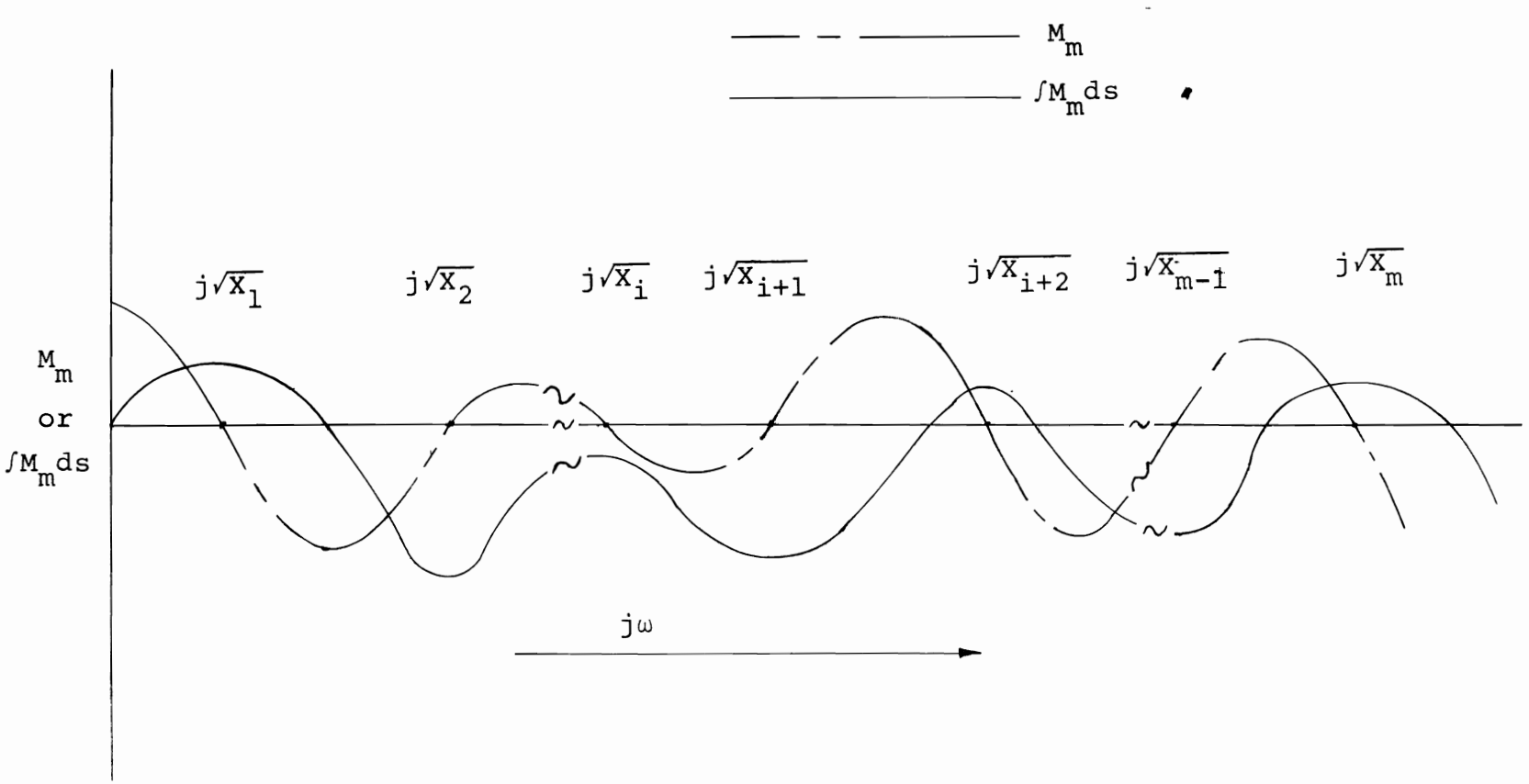

Fig.2.2(a): Plots of M_m and $\int M_m ds$ when M_m is unintegrable at an odd numbered zero

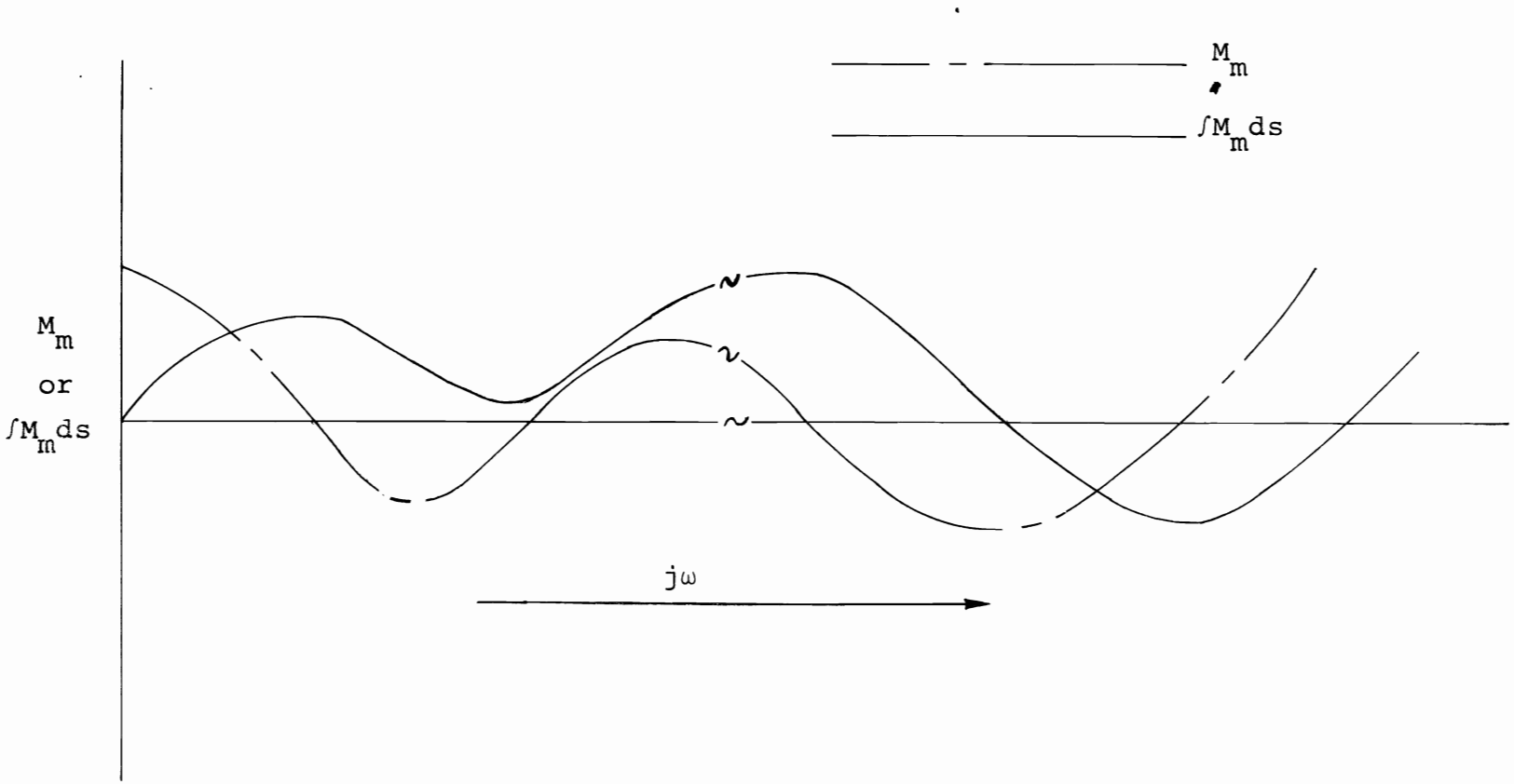

Fig.2.2(b): Plots of M_m and $\int M_m ds$ when M_m is unintegrable at an even numbered zero.

2.5.2 Some Theorems on the Integrability of M_m:

Theorem 2.2: If

$$\int_0^{j\sqrt{X_2}} M_m ds > 0, \text{ then}$$

$$\int_0^{j\sqrt{X_2}} M_m(s^2 + X_{m+1})ds > 0 \ , \ X_{m+1} > X_m.$$

Proof:

Integrating by parts, we have

$$\int_0^{j\sqrt{X_2}} M_m(s^2+X_{m+1})ds = (X_{m+1}-X_2)\int_0^{j\sqrt{X_2}} M_m ds - 2\int_0^{j\sqrt{X_2}} s(\int M_m ds)ds \qquad ..(2.21)$$

The term $(X_{m+1} - X_2)\int_0^{j\sqrt{X_2}} M_m ds$ is always positive. From Fig. 2.2b, it is seen that the plot of $\int s(\int M_m ds)ds$ will be as shown in Fig. 2.3 and hence will be negative at $s = j\sqrt{X_2}$. The theorem now follows.

Theorem 2.3:

A necessary condition for the integrability of a given M_m is that $(X_2/X_1) > 5$.

Proof: Consider M_2 given by,

$$M_2 = (s^2 + X_1)(s^2 + X_2) \qquad ..(2.22)$$

$$\int M_2 ds = \frac{s}{5}\{s^4 + \frac{5}{3}(X_1 + X_2)s^2 + 5\ X_1X_2\} \qquad ..(2.23)$$

The inequality (2.17) will be satisfied if $\int_0^{j\sqrt{X_2}} M_2 ds$ is negative. The same holds if $(X_2/X_1) > 5$.

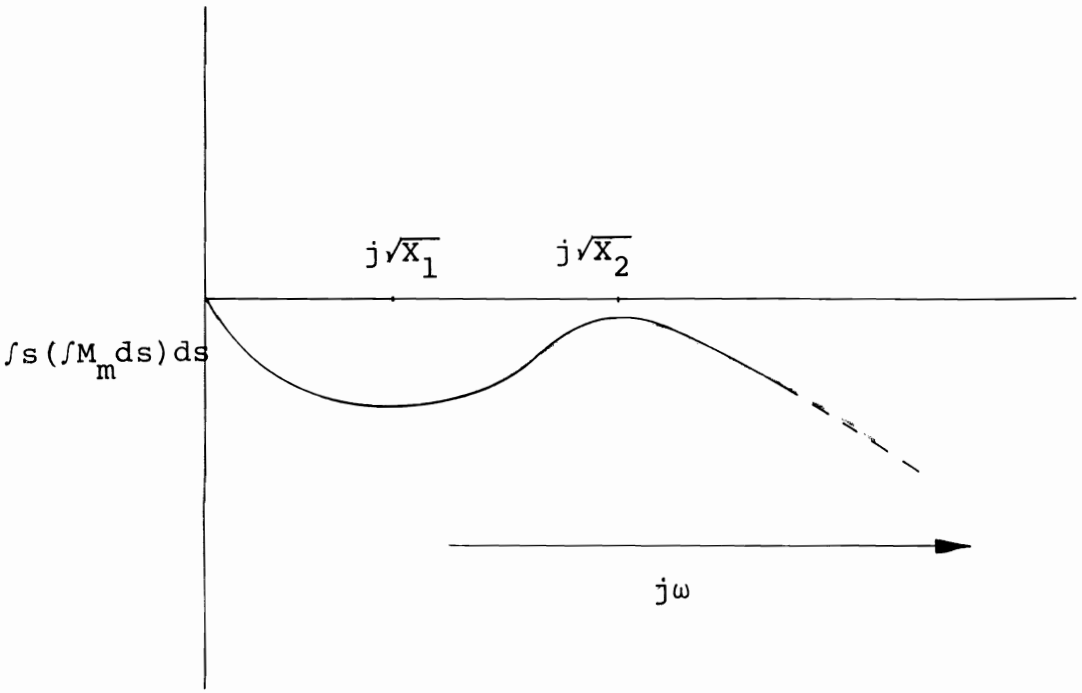

Fig.2.3: Plot of $\int s(\int M_m ds)ds$ when integrability fails at $s = j\sqrt{X_2}$

It may be noted that for M_2, this condition is both necessary and sufficient. From Theorem 2.2, it follows that a necessary condition for the integrability of M_m is that $M_2, \ldots, M_{m-2}, M_{m-1}$ meet the integrability condition at $s = j\sqrt{X_2}$. Combining this with the result obtained above, Theorem 2.3 follows.

Sufficient conditions of integrability are obtained only when the inequalities (2.17) for M_m are satisfied.

Let Δ_{m,X_i} be defined by

$$\Delta_{m,X_i} = \frac{1.3.5\ldots\ldots(2m+1)\left[\frac{1}{s}\int M_m ds\right]_0^{j\sqrt{X_i}}}{2X_i} \qquad ..(2.24)$$

Clearly, it is enough to test the signs of the Δ_{m,X_i}'s in order to ascertain the integrability of M_m. Δ_{m,X_i}'s are obtained successively from a knowledge of Δ_{m-1,X_i}'s using the following theorem.

<u>Theorem 2.4</u>:

(a)
$$\begin{aligned}\Delta_{m,X_1} = & -\beta(m,1,2,3)\ P(3,m-1,m-3)\begin{vmatrix}7 & 3\\ X_1 & X_2\end{vmatrix} X_1 \\ & +\beta(m,2,3,4)\ P(3,m-1,m-4)\begin{vmatrix}9 & 5\\ X_1 & X_2\end{vmatrix} X_1^{\,2} \\ & - \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +(-1)^{m-4}\ \beta(m,m-4,m-3,m-2)\ P(3,m-1,2)\begin{vmatrix}2m-3 & 2m-7\\ X_1 & X_2\end{vmatrix} X_1^{\,m-4}\end{aligned}$$

$$+ (-1)^{m-3} \beta(m,m-3,m-2,m-1)\, P(3,m-1,1) \begin{vmatrix} 2m-1 & 2m-5 \\ X_1 & X_2 \end{vmatrix} X_1^{m-3}$$

$$+ (-1)^{m-2} \beta(m,m-2,m-1,m) \begin{vmatrix} 2m+1 & 2m-3 \\ X_1 & X_2 \end{vmatrix} X_1^{m-2}$$

$$+ (2m+1) X_m \Delta_{m-1,X_1} \qquad ..(2.25)$$

where $\beta(m) = 1.3.5.7. \ldots\ldots\ldots (2m+1)$

$$\beta(m,i) = \frac{\beta(m)}{(2i+1)}$$

$$\beta(m,i,j,k) = \frac{\beta(m)}{(2i+1)(2j+1)(2k+1)}$$

$P(i,j,k)$ = the sum of the products of the X_i's lying within the subscript range i and j taken k at a time.*

---

* $$P(3,m-1,m-3) = \prod_{3}^{m-1} X_i$$

$$P(3,m-1,m-4) = \sum_{a=3}^{m-1} \sum_{b}^{m-1} \ldots \sum_{c}^{m-1} X_a X_b \ldots X_c$$

$$a < b < \ldots < c$$

$$c-a = m-5$$

and so on.

As an example, for m = 7,

$$P(3,6,3) = \sum_{a=3}^{6} \sum_{b}^{6} \sum_{c}^{6} X_a X_b X_c = X_3X_4(X_5+X_6) + X_3X_5X_6 + X_4X_5X_6$$

$$a < b < c$$

$$c-a = 2$$

(b) $\Delta_{m,X_{k+1}}$ $(k \leq m)$ is obtained by the following substitutions in Δ_{m,X_1}:

for $i + k \leq m$; replace X_i by X_{i+k}

or for $i + k > m$; replace X_i by X_{i+k-m}

<u>Proof</u>: (a) We have,

$$M_m = (s^2 + X_1)(s^2 + X_2) \ldots\ldots\ldots (s^2 + X_m)$$

$$= P(1,m,m) + P(1,m,m-1)s^2 + P(1,m,m-2)s^4 + \ldots\ldots\ldots +$$

$$P(1,m,2)s^{2m-4} + P(1,m,1)s^{2m-2} + s^{2m} \qquad ..(2.26)$$

Integrating (2.26) with respect to s, we obtain,

$$\int M_m ds = s\{P(1,m,m) + P(1,m,m-1)\frac{s^2}{3} + P(1,m,m-2)\frac{s^4}{5} + \ldots. +$$

$$P(1,m,2)\frac{s^{2m-4}}{2m-3} + P(1,m.1)\frac{s^{2m-2}}{2m-1} + \frac{s^{2m}}{2m+1}\} \qquad ..(2.27)$$

Multiplying each term of (2.27) by $\beta(m)$ and leaving out s, we obtain,

$$\beta(m)\ P(1,m,m) + \beta(m,1)\ P(1,m,m-1)s^2 + \beta(m,2)\ P(1,m,m-2)s^4 + .. +$$

$$\beta(m,m-2)\ P(1,m,2)s^{2m-4} + \beta(m,m-1)\ P(1,m,1)s^{2m-2} + \beta(m,m)s^{2m}$$

$$= G \text{ (say)} \qquad ..(2.28)$$

Evaluating (2.28) at $s^2 = -X_1$, we get

$$G\Big|_{s^2=-X_1} = \beta(m)\ P(1,m,m) - \beta(m,1)\ P(1,m,m-1)X_1 +$$

$$\beta(m,2)\ P(1,m,m-2)X_1^{\ 2} - \ldots\ldots\ldots\ldots\ldots +$$

$$(-1)^{m-2}\ \beta(m,m-2)\ P(1,m,2)X_1^{\ m-2} +$$

$$(-1)^{m-1}\ \beta(m,m-1)\ P(1,m,1)X_1^{\ m-1} + (-1)^m\ \beta(m,m)X_1^{\ m} \qquad ..(2.29)$$

$$= X_1 \{\beta(m) - (m,1)\} P(2,m,m-1) - \{\beta(m,1) - (m,2)\}$$
$$P(2,m,m-2)X_1 + \{\beta(m,2)-\beta(m,3)\} P(2,m,m-3)X_1^{\,2} - \ldots\ldots$$
$$(-1)^{m-2}\{\beta(m,m-2)-\beta(m,m-1)\} P(2,m,1)X_1^{\,m-2} +$$
$$(-1)^{m-1} \{\beta(m,m-1)-\beta(m,m)\}X_1^{\,m-1}$$
$$= 2X_1 \Delta_{m,X_1} \qquad ..(2.30)$$

Since X_1 is positive, the sign of (2.30) will be the same as that of Δ_{m,X_1}, which is the expression to be evaluated. From (2.30), we have,

$$\Delta_{m,X_1} = \frac{1}{2} X_m(2m+1) A + \frac{1}{2} B \qquad ..(2.31)$$

where

$$A = \{\beta(m-1)-\beta(m-1)\} P(2,m-1,m-2) - \{\beta(m-1,1)-\beta(m-1,2)\}$$
$$P(2,m-1,m-3)X_1 + \{\beta(m-1,2)-\beta(m-1,3)\} P(2,m-1,m-4)X_1^{\,2} - \ldots.$$
$$(-1)^{m-3} \{\beta(m-1,m-3)-\beta(m-1,m-2)\} P(2,m-1,1)X_1^{\,m-3} +$$
$$(-1)^{m-2} \{\beta(m-1,m-2)-\beta(m-1,m-1)\}X_1^{\,m-2}$$
$$= \Delta_{m-1,X_1} \qquad ..(2.32)$$

and

$$B = (2m+1) - \{\beta(m-1,1)-\beta(m-1,2)\} P(2,m-1,m-2)X_1 +$$
$$\{\beta(m-1,2)-\beta(m-1,3)\} P(2,m-1,m-3)X_1^{\,2} - \ldots\ldots\ldots$$
$$(-1)^{m-2} \{\beta(m-1,m-2)-\beta(m-1,m-1)\} P(2,m-1,1)X_1^{\,m-2} +$$
$$(-1)^{m-1} \{\beta(m,m-1)-\beta(m,m)\} X_1^{\,m-1} \qquad ..(2.33)$$

We can now write

$$\Delta_{m,X_1} = X_m(2m+1)\Delta_{m-1,X_1} + -\beta(m,1,2,3)P(3,m-1,m-3)(7X_2-3X_1)X_1$$
$$+ \beta(m,2,3,4)P(3,m-1,m-4)(9X_2-5X_1)X_1^{\,2} - \ldots\ldots\ldots\ldots +$$
$$(-1)^{m-4}\beta(m,m-4,m-3,m-2)P(3,m-1,2)\{(2m-3)X_2-(2m-7)X_1\}X_1^{\,m-4}$$
$$+ (-1)^{m-3}\beta(m,m-3,m-2,m-1)P(3,m-1,1)\{(2m-1)X_2-(2m-5)X_1\}X_1^{\,m-3}$$
$$+ (-1)^{m-2}\beta(m,m-2,m-1,m)\{(2m+1)X_2-(2m-3)X_1\}X_1^{\,m-2} \quad ..(2.34)$$

(a) follows from a rearrangement of (2.34)

(b) The (m-1) different expressions, i.e., $\Delta_{m,X_2}, \Delta_{m,X_3}, \ldots\ldots$ Δ_{m,X_m}, to be tested are obtained from (2.25) by the substitution of $s = j\sqrt{X_2}$, $j\sqrt{X_3}$, ... $j\sqrt{X_m}$. All the coefficients of the different powers of s involve all the X_i's in the same fashion. In other words, interchanging an X_i and X_j does not affect the coefficient. Keeping this in mind, it follows that if we obtain the expression corresponding to $s = j\sqrt{X_1}$, the remaining ones can be derived by a cyclic permutation of the X_i's.

The applications of Theorem 2.4 (a) in growing Δ_{m,X_1} from Δ_{m-1,X_1} are now given. We have

$$\Delta_{2,X_1} = \begin{vmatrix} 5 & 1 \\ X_1 & X_2 \end{vmatrix} \quad ..(2.35)$$

$$\Delta_{3,X_1} = -\begin{vmatrix} 7 & 3 \\ X_1 & X_2 \end{vmatrix} X_1 + 7X_3\Delta_{2,X_1} \quad ..(2.36)$$

$$\Delta_{4,X_1} = -9X_3\begin{vmatrix} 7 & 3 \\ X_1 & X_2 \end{vmatrix} X_1 + 1.3\begin{vmatrix} 9 & 5 \\ X_1 & X_2 \end{vmatrix} X_1^{\,2} + 9X_4\Delta_{3,X_1} \qquad ..(2.37)$$

$$\Delta_{5,X_1} = -99X_3X_4\begin{vmatrix} 7 & 3 \\ X_1 & X_2 \end{vmatrix} X_1 + 33(X_3+X_4)\begin{vmatrix} 4 & 5 \\ X_1 & X_2 \end{vmatrix} X_1^{\,2}$$

$$- 15\begin{vmatrix} 11 & 7 \\ X_1 & X_2 \end{vmatrix} X_1^{\,3} + 11X_5\Delta_{4,X_1} \qquad ..(2.38)$$

$$\Delta_{6,X_1} = - 1287X_3X_4X_5\begin{vmatrix} 7 & 3 \\ X_1 & X_2 \end{vmatrix} X_1 + 429(X_3X_4+X_3X_5+X_4X_5)\begin{vmatrix} 9 & 5 \\ X_1 & X_2 \end{vmatrix} X_1^{\,2}$$

$$- 195(X_3 + X_4 + X_5)\begin{vmatrix} 11 & 7 \\ X_1 & X_2 \end{vmatrix} X_1^{\,3}$$

$$+ 105\begin{vmatrix} 13 & 9 \\ X_1 & X_2 \end{vmatrix} X_1^{\,4} + 13X_6\Delta_{5,X_1} \qquad ..(2.39)$$

The application of Theorem 2.4 (b) gives Δ_{6,X_2}, Δ_{6,X_3} and so on.

2.5.3 Summary of the Testing Procedure for the Integrability of M_m:

Theorems 2.2, 2.3 and 2.4 enable us to outline the following testing procedure for determining the integrability of a given M_m as in (2.16):

(1) If $\frac{X_2}{X_1} \leq 5$, the given M_m is unintegrable. This will be referred to as a case where the second zero condition is violated.

(2) For $M_3, M_4, \ldots, M_m$, the corresponding Δ's at $s = j\sqrt{X_2}$ namely, Δ_{3,X_2}, Δ_{4,X_2}, ... , Δ_{m,X_2} are constructed in

the above order by means of Theorem 2.4. If at any stage of development, Δ_{i,X_2} $(i \leq m)$ is non-negative, the given M_m is unintegrable.

(3) Δ_{m,X_2}, Δ_{m,X_3}, ... Δ_{m,X_m} are computed by means of Theorem 2.4 in the order indicated. If at any stage Δ_{m,X_i} is non-negative for i even or non-positive for i odd, the given M_m is unintegrable.

(4) If the given M_m does not violate the integrability conditions referred to above, it is integrable.

<u>Example 2.3</u>: To find if

$$M_6 = (s^2 + 1)(s^2 + 6)(s^2 + 9)(s^2 + 18)(s^2 + 30)(s^2 + 100) \quad ..(2.40)$$

is integrable.

The second zero condition is satisfied. We have,

$$M_3 = (s^2 + 1)(s^2 + 6)(s^2 + 9) \quad ..(2.41)$$

From Theorem 2.4,

$$\Delta_{3,X_2} = -X_2 \begin{vmatrix} 7 & 3 \\ X_2 & X_3 \end{vmatrix} + 7X_1 \begin{vmatrix} 5 & 1 \\ X_2 & X_3 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 7 & 3 \\ 6 & 9 \end{vmatrix} + 7 \begin{vmatrix} 5 & 1 \\ 6 & 9 \end{vmatrix}$$

$$= 3 \quad ..(2.42)$$

Hence, condition (2) is not satisfied and the given M_m is, therefore, unintegrable.

Example 2.4: To find if

$$M_4 = (s^2 + 1)(s^2 + 6)(s^2 + 14)(s^2 + 25) \qquad ..(2.43)$$

is integrable.

The second zero condition is satisfied. Since there are only four factors in the given M_m, we may straightaway go to test 3 in the summary of the testing procedure. This test incorporates both the necessary and sufficient conditions.

For the given M_4,

$$\Delta_{4,X_2} = -9X_4 \begin{vmatrix} 7 & 3 \\ X_2 & X_3 \end{vmatrix} X_2 + 3 \begin{vmatrix} 9 & 5 \\ X_2 & X_3 \end{vmatrix} X_2^{\,2}$$

$$+ 9X_1 \left[- \begin{vmatrix} 7 & 3 \\ X_2 & X_3 \end{vmatrix} X_2 + 7X_4 \begin{vmatrix} 5 & 1 \\ X_2 & X_3 \end{vmatrix} \right]$$

$$= -145152 \qquad ..(2.44)$$

Applying cyclic permutations of the X's in the expression for Δ_{4,X_2} above, we similarly obtain expressions for Δ_{4,X_3} and Δ_{4,X_4}. Evaluating these, we get,

$$\Delta_{4,X_3} = 15792 \qquad ..(2.45)$$

$$\Delta_{4,X_4} = -17340 \qquad ..(2.46)$$

Hence the given M_m is integrable.

2.6 The Medial M_m:

If the zeros of a given M_m are such that $\int M_m ds$ assumes the value zero at all the even numbered roots of M_m, namely,

$j\sqrt{X_2}$, $j\sqrt{X_4}$, ... , then M_m is termed the "medial M_m". Such M_m provides an example of a limiting case of integrability, since even a slight perturbation of one or more zeros of M_m may result in the inequality,

$$\int_0^{j\sqrt{X_{2i}}} M_m ds > 0 \ , \quad i = 1,2,3,... \qquad ..(2.47)$$

being satisfied for one or more integral values of i. A question which may arise is whether the medial M_m itself can be considered integrable. It is true that $(\int M_m ds/M_m)$, where M_m is medial, is a reactance function as the common factors of $\int M_m ds$ and M_m, namely, $(s^2 + X_2),(s^2 + X_4)$, ... cancel out. However, in general, the double zeros contained in $\int M_m ds$ will render it unsuitable as the odd part of any HP. An exception occurs when $\int N_n ds + K$ has simple zeros corresponding to each of the double zeros of $\int M_m ds$, in which case, the medial M_m may be considered integrable. Depending on m being even or odd, two cases arise. These are discussed in Theorems 2.5 and 2.6.

<u>Theorem 2.5</u>:

A given M_m (m even) will be medial, if and only if

$$\sum_{i=1}^{\frac{m}{2}} \frac{X_j}{X_{2i} - X_j} = \frac{1}{4} \quad ; \quad j = 1,3,...., m-1 \qquad ..(2.48)$$

<u>Proof</u>:

(a) Necessity: If M_m is medial, we have, by definition,

$$\int M_m ds = \frac{s}{2m+1}(s^2 + X_2)^2(s^2 + X_4)^2 \ldots (s^2 + X_m)^2 \qquad ..(2.49)$$

Letting $\quad T = (s^2 + X_2)(s^2 + X_4) \ldots . (s^2 + X_m) \qquad ..(2.50)$

we have,

$$\int M_m ds = \frac{s}{2m+1} T^2 \qquad ..(2.51)$$

Differentiating (2.51) with respect to s, we get,

$$M_m = \frac{T^2}{2m+1}\left(1 + 2s\frac{T'}{T}\right) \qquad ..(2.52)$$

$X_1, X_3, X_5, \ldots, X_{m-1}$ are zeros of M and since T does not contain these zeros, we have,

$$1 + 2s\frac{T'}{T} = 0 \quad \text{for} \quad s^2 = -X_1, -X_3, \ldots, -X_{m-1} \qquad ..(2.53)$$

But $$\frac{T'}{T} = 2s \sum_{i=1}^{\frac{m}{2}} \frac{1}{s^2 + X_{2i}} \qquad ..(2.54)$$

Therefore, using (2.54) in (2.53), we obtain

$$1 + \sum_{i=1}^{\frac{m}{2}} \frac{4s^2}{s^2 + X_{2i}} = 0, \quad s^2 = -X_1, -X_3, \ldots, -X_{m-1} \qquad ..(2.55)$$

and hence (2.48) follows.

(b) Sufficiency: Starting with any of the given equalities, say at $-X_1$, and proceeding backwards from (2.55), we observe that $(s^2 + X_1)$ is contained as a factor in $(T + 2sT')$. Similarly, $(s^2 + X_3)$, $(s^2 + X_5)$, ... ,$(s^2 + X_{m-1})$ are also shown to be contained in $(T + 2sT')$. Since T contains the rest of the factors, M_m contains all the required factors.

Hence the theorem follows.

Theorem 2.6:

A given M_m (m odd, m > 1) will be medial if and only if, the expression

$$Y = \frac{3X_j - 4X_j^2 \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{X_{2i} - X_j}}{1 - 4X_j \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{X_{2i} - X_j}} \qquad ..(2.56)$$

remains invariant for j = 1,3,...,m and is a zero of $\int M_m ds$. (Hence, all the zeros of $\int M_m ds$ are known)

Proof:

(a) Necessity: If M_m is medial, we must have, by definition,

$$\int M_m ds = \frac{s}{2m+1} (s^2 + X_2)^2 (s^2 + X_4)^2 \ldots\ldots (s^2 + X_{m-1})^2 (s^2 + Y) \qquad ..(2.57)$$

Letting

$$T = (s^2 + X_2)(s^2 + X_4) \ldots (s^2 + X_{m-1}) \qquad ..(2.58)$$

we have,

$$\int M_m ds = \frac{s}{2m+1} T^2 (s^2 + Y) \qquad ..(2.59)$$

Differentiating (2.59), we get,

$$M_m = \frac{T^2}{2m+1} \{3s^2 + Y + \frac{2T'}{T} s(s^2 + Y)\} \qquad ..(2.60)$$

Since $\frac{T'}{T} = \sum_{i=1}^{\frac{m-1}{2}} \frac{2s}{s^2 + X_{2i}}$, we have,

$$M_m = \frac{T^2}{2m+1}\{Y(1 + 4s^2 \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{s^2 + X_{2i}}) + 3s^2 + 4s^4 \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{s^2 + X_{2i}}\} \quad ..(2.61)$$

$X_1, X_3, X_5, \ldots X_m$ are zeros of M_m and since T does not contain these zeros, we must have from (2.61),

$$Y = \frac{3X_j - 4X_j^2 \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{X_{2i} - X_j}}{1 - 4X_j \sum_{i=1}^{\frac{m-1}{2}} \frac{1}{X_{2i} - X_j}} \; ; \; j = 1,3,5,..,m \quad ..(2.62)$$

Hence the necessity follows.

(b) Sufficiency: Starting with the given expression Y at $s^2 = -X_1$ (say), and proceeding backwards from (2.61), we observe that $(s^2 + X_1)$ is contained as a factor in

$$3s^2T + YT + 2T's(s^2 + Y) \quad ..(2.63)$$

Similarly, $(s^2 + X_3), (s^2 + X_5), \ldots, (s^2 + X_m)$ are also contained as factors in (2.63). Since T contains the rest of the factors, it follows from (2.60) that M_m contains all the required factors. This proves Theorem 2.6.

Fig.2.4(a) and (b) give the plot of medial M_m and its integral for both the even and odd cases.

2.6.1 Some Theorems on the Medial M_m:

The following theorems hold for both the even and odd

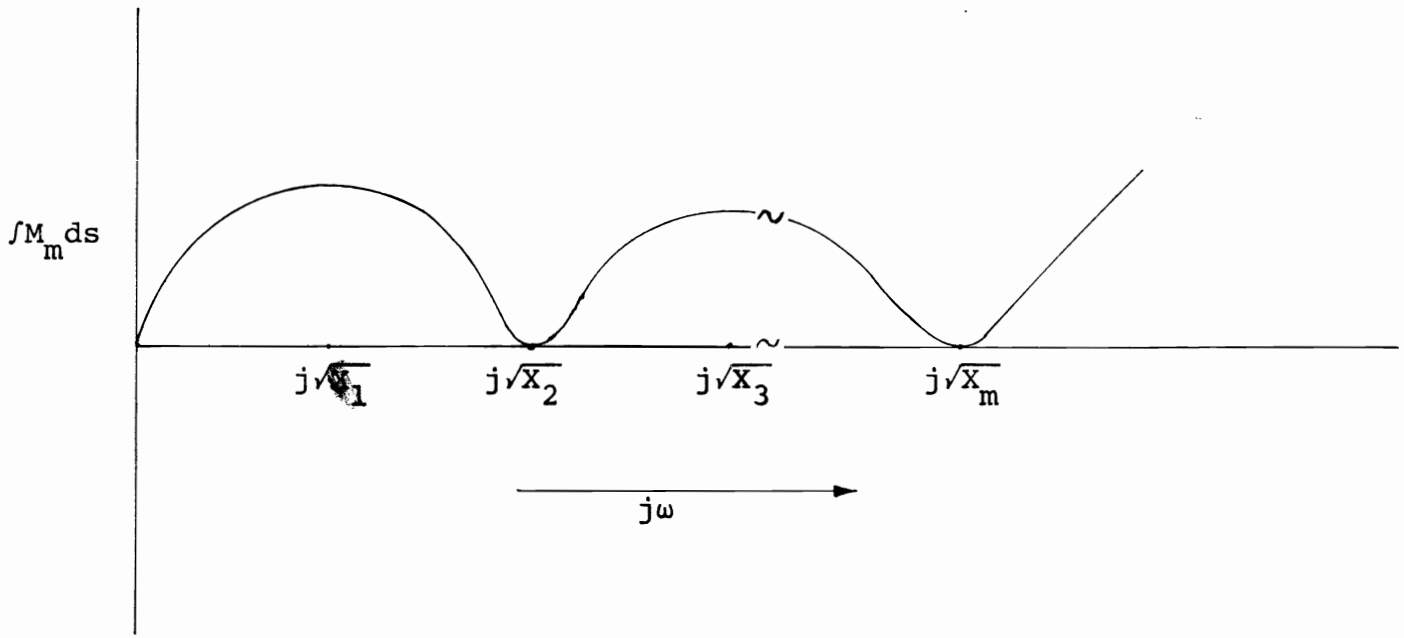

Fig.2.4(a): The integral of medial M_m for m even

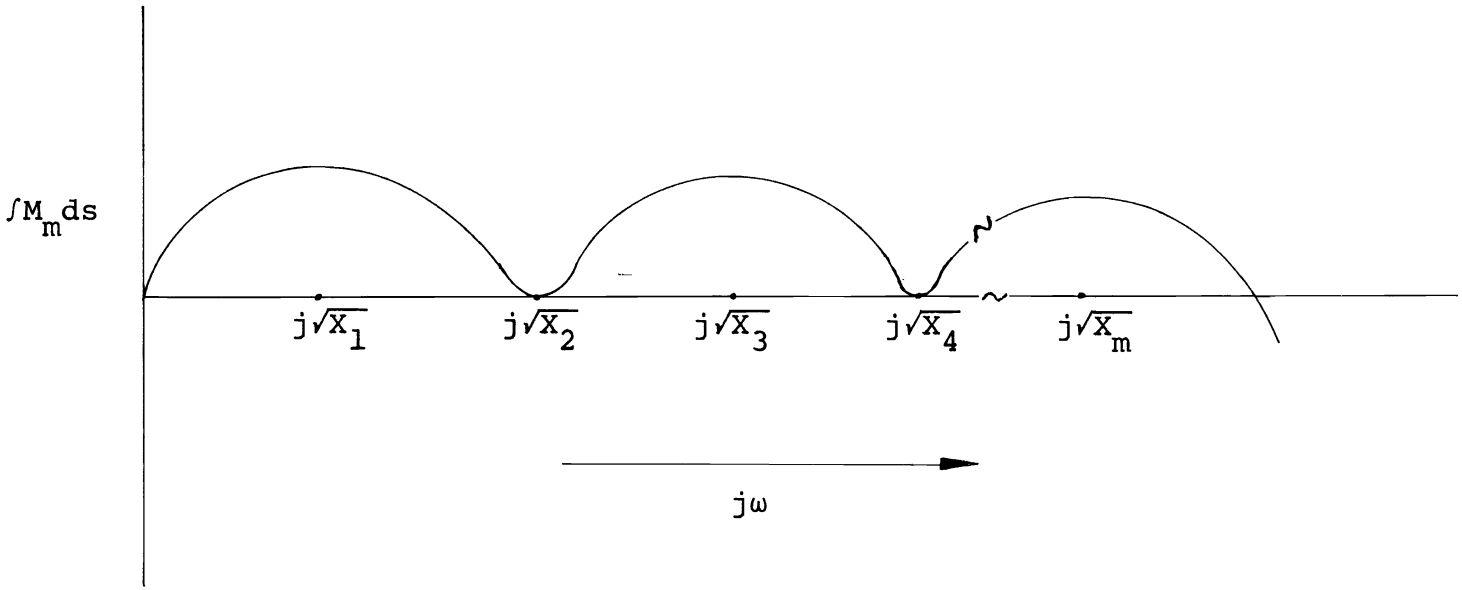

Fig.2.4(b): The integral of medial M_m for m odd

values of m.

Theorem 2.7:

If M_m is medial, then,

$$\int_0^{j\sqrt{X_{2i}}} M_{m-1}ds < 0 \quad , \quad 2i \leq m-1, \quad i = 1,2,3,\ldots$$

Proof:

The proof is by the method of induction. Since M_m is medial, we have,

$$\int_0^{j\sqrt{X_2}} M_m ds = 0$$

or

$$X_m \int_0^{j\sqrt{X_2}} M_{m-1}ds + \int_0^{j\sqrt{X_1}} s^2 M_{m-1}ds + \int_{j\sqrt{X_1}}^{j\sqrt{X_2}} s^2 M_{m-1}ds = 0 \qquad ..(2.64)$$

A plot of $s^2 M_{m-1}$ and M_{m-1} is as shown in Fig. 2.5, from which it follows that,

$$\int_0^{j\sqrt{X_1}} s^2 M_{m-1}ds > -X_1 \int_0^{j\sqrt{X_1}} M_{m-1}ds$$

$$..(2.65)$$

and $$\int_{j\sqrt{X_1}}^{j\sqrt{X_2}} s^2 M_{m-1}ds > -X_1 \int_{j\sqrt{X_1}}^{j\sqrt{X_2}} M_{m-1}ds$$

Hence, using (2.65) in (2.64), we get

$$(X_m - X_1) \int_0^{j\sqrt{X_2}} M_{m-1}ds < 0$$

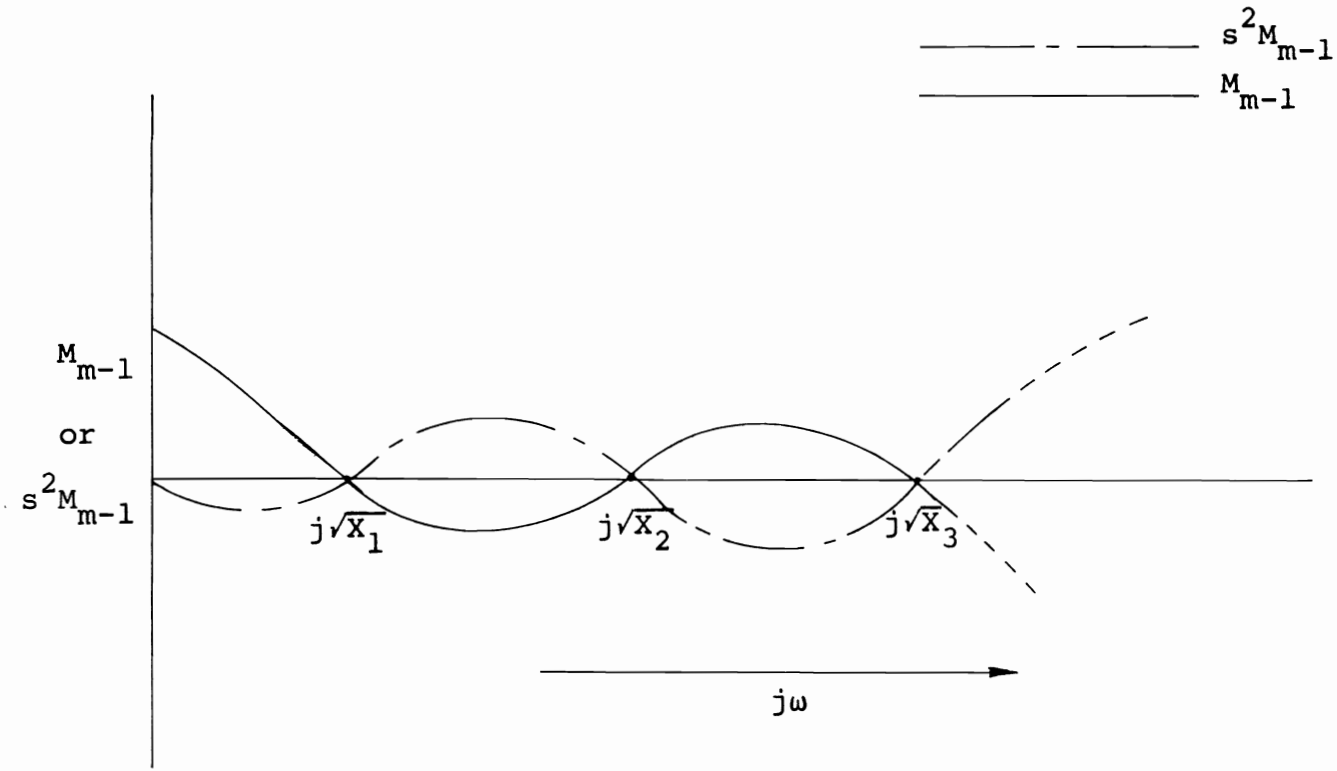

Fig.2.5: Plots of M_{m-1} and s^2M_{m-1} illustrating their behaviour between $s = 0$ and $s = j\sqrt{X_2}$

$$\text{or} \quad \int_0^{j\sqrt{X_2}} M_{m-1} ds < 0 \qquad ..(2.66)$$

Now, let

$$\int_0^{j\sqrt{X_{2i-2}}} M_{m-1} ds < 0 \qquad ..(2.67)$$

Since M_m is medial, we have,

$$X_m \int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i}}} M_{m-1} ds + \int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i-1}}} s^2 M_{m-1} ds + \int_{j\sqrt{X_{2i-1}}}^{j\sqrt{X_{2i}}} s^2 M_{m-1} ds = 0 \qquad ..(2.68)$$

where $\qquad 2i \leq m \ , \ i = 1,2,3,.....$

From Fig.2.6, which illustrates the behavior of $s^2 M_{m-1}$ and M_{m-1} between $j\sqrt{X_{2i-2}}$ and $j\sqrt{X_{2i}}$, we have as before,

$$\int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i-1}}} s^2 M_{m-1} ds > -X_{2i-1} \int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i-1}}} M_{m-1} ds \qquad ..(2.69)$$

$$\text{and} \quad \int_{j\sqrt{X_{2i-1}}}^{j\sqrt{X_{2i}}} s^2 M_{m-1} ds > -X_{2i-1} \int_{j\sqrt{X_{2i-1}}}^{j\sqrt{X_{2i}}} M_{m-1} ds$$

Using (2.69) in (2.68), we get,

$$(X_m - X_{2i-1}) \int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i}}} M_{m-1} ds < 0$$

$$\text{or} \quad \int_{j\sqrt{X_{2i-2}}}^{j\sqrt{X_{2i}}} M_{m-1} ds < 0 \qquad ..(2.70)$$

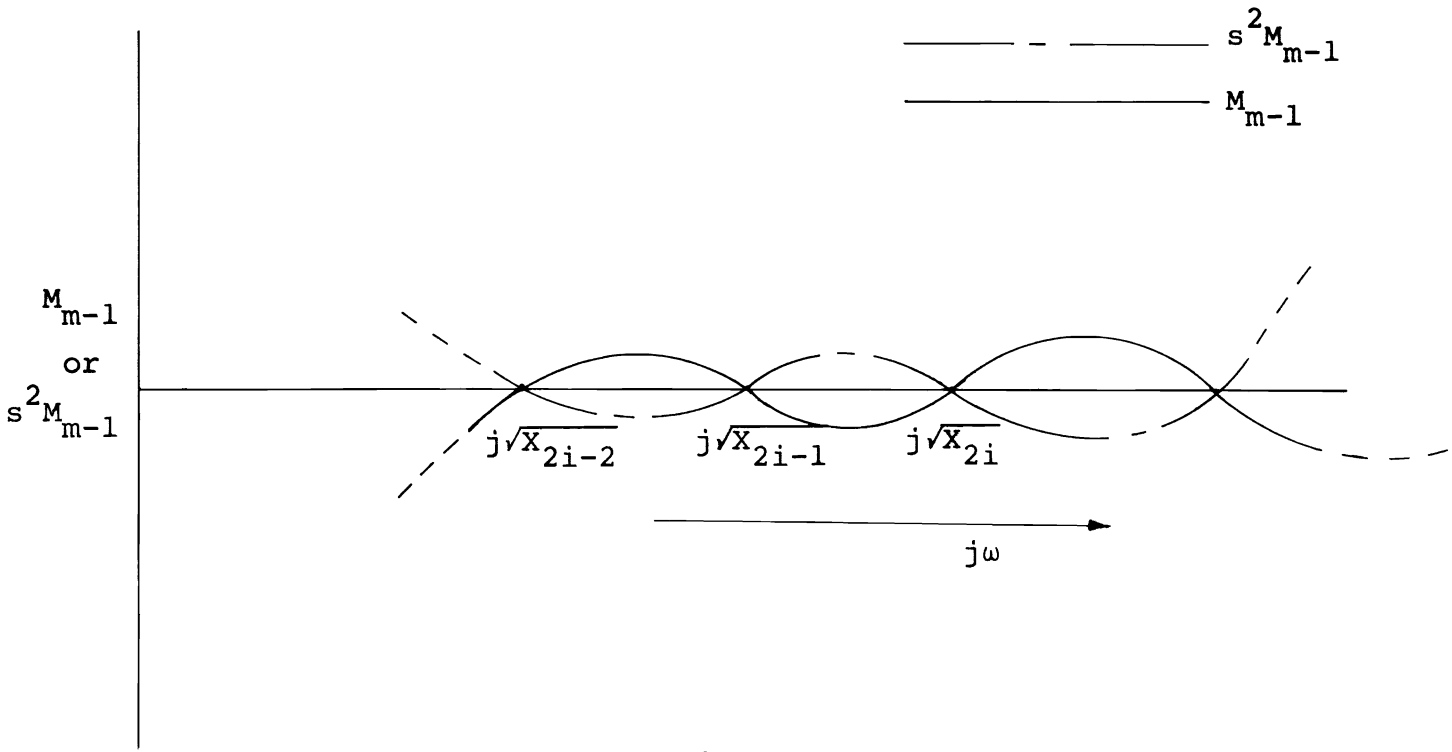

Fig.2.6: Plots of M_{m-1} and s^2M_{m-1} illustrating their behaviour between $s = j\sqrt{X_{2i-2}}$ and $s = j\sqrt{X_{2i}}$

Hence if (2.67) is true, we have,

$$\int_0^{j\sqrt{X_{2i}}} M_{m-1}ds < 0 \qquad ..(2.71)$$

The theorem now follows.

This theorem implies that if the introduction of an additional zeros in M_{m-1} makes the resulting M_m medial, then, the integrability conditions at all the even numbered zeros of M_{m-1} are satisfied.

<u>Theorem 2.8</u>:

If M_m is medial, then,

$$\int_0^{j\sqrt{X_2}} M_{m+1}ds > 0$$

<u>Proof</u>:

We have

$$\int_0^{j\sqrt{X_2}} M_{m+1}ds = \int_0^{j\sqrt{X_2}} s^2 M_m ds \qquad ..(2.72)$$

since $$X_{m+1} \int_0^{j\sqrt{X_2}} M_m ds = 0$$

From Fig.2.5, we obtain,

$$\int_0^{j\sqrt{X_2}} s^2 M_m ds > -X_1 \int_0^{j\sqrt{X_2}} M_m ds \qquad ..(2.73)$$

But $$\int_0^{j\sqrt{X_2}} M_m ds = 0 \qquad ..(2.74)$$

Hence the theorem follows.

This theorem leads to the following conclusions:

(i) For a given M_2 if $\frac{X_2}{X_1} = 5$, then M_2 is medial. Hence the second zero condition derived in Theorem 2.3 also follows from Theorem 2.8.

(ii) If $M_m = M_i M_j$, where M_i is known to be medial, then, M_m is unintegrable.

<u>Theorem 2.9</u>:

If any zero X_i of a medial M_m, other than the first zero X_1, is perturbed to $X_i^{(p)}$ such that

$$X_i^{(p)} = X_i - \varepsilon \ , \ \varepsilon > 0$$

then the resulting function $M_m^{(p)}$ is such that $\int_0^{j\sqrt{X_2}} M_m^{(p)} ds > 0$.

<u>Proof</u>:

Since M_m is medial,

$$\int_0^{j\sqrt{X_2}} M_m ds = 0$$

or $$X_i \int_0^{j\sqrt{X_2}} \{M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ \ldots \ (s^2 + X_m)\}ds$$
$$+ \int_0^{j\sqrt{X_2}} s^2\{M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ .. \ (s^2 + X_m)\}ds = 0 \qquad ..(2.75)$$

As before, it can be shown that,

$$\int_0^{j\sqrt{X_2}} s^2\{M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ldots (s^2 + X_m)\}ds >$$

$$-X_1 \int_0^{j\sqrt{X_2}} M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ldots (s^2 + X_m)ds \qquad ..(2.76)$$

Hence, using (2.76) in (2.75), we get,

$$\int_0^{j\sqrt{X_2}} \{M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ldots (s^2 + X_m)\}ds < 0 \qquad ..(2.77)$$

Now

$$\int_0^{j\sqrt{X_2}} M_m(p)ds = \int_0^{j\sqrt{X_2}} M_m ds - \varepsilon \int_0^{j\sqrt{X_2}} \{M_{i-1}(s^2 + X_{i+1})(s^2 + X_{i+2}) \ldots (s^2 + X_m)\}ds \qquad ..(2.78)$$

Hence using (2.77) in (2.78), the result follows.

Theorem 2.9 provides a comparison test for the integrability of a given M_m, in that, if the given M_m can be identified as being obtained by the perturbation of one or more zeros (except the first zero) of a medial M_m towards the origin, then the given M_m is unintegrable.

<u>Example 2.5</u>:

Let

$$M_4 = (s^2 + 1)(s^2 + 6)(s^2 + 13)(s^2 + 20) \qquad ..(2.79)$$

be given.

Knowing that

$$(s^2 + 1)(s^2 + 6)(s^2 + 14)(s^2 + 21) \qquad ..(2.80)$$

is medial, it can be concluded that the given M_4 is unintegrable.

Theorem 2.10:

If M_m is medial and X_m is perturbed to $X_m + \varepsilon$, $\varepsilon > 0$, then the perturbed function $M_m^{(p)}$ is such that

$$\int_0^{j\sqrt{X_2}} M_m^{(p)} ds < 0 \ , \ 2i \leq m, \ i = 1,2,... \qquad ..(2.81)$$

Proof:

We have

$$\int_0^{j\sqrt{X_{2i}}} M_m^{(p)} ds = \int_0^{j\sqrt{X_{2i}}} M_m ds + \varepsilon \int_0^{j\sqrt{X_{2i}}} M_{m-1} ds$$

$$= \varepsilon \int_0^{j\sqrt{X_{2i}}} M_{m-1} ds \qquad ..(2.82)$$

Hence the result follows from Theorem 2.7.

2.6.2 Summary of the Properties of the Medial M_m:

(1) A special case of M_m, called the medial M_m, results when its zeros satisfy either the equalities (2.48) or (2.56) depending on whether m is even or odd. Under this condition, $\int M_m ds$ assumes the value zero at all its even numbered zeros.

(2) If the introduction of one or more zeros in a given M_m leads to either (2.48) or (2.56) being satisfied, then the integral of the given M_m will be negative at all its even numbered zeros.

(3) If $M_m = M_i M_j$, where the zeros of M_i satisfy (2.48) or (2.56), then the given M_m is unintegrable.

(4) In a given medial M_m, if one or more zeros, except the first zero, are perturbed towards the origin, then the resulting M_m is unintegrable.

(5) If the last zero of a given medial M_m is perturbed away from the origin, then the resulting M_m meets the integrability criterion at all its even numbered zeros.

2.7 The Integrability of N:

N will be assumed to be given in the factored form, namely,

$$N_n = s(s^2 + X_1)(s^2 + X_2) \ldots (s^2 + X_n) \qquad ..(2.83)$$

$$X_1 < X_2 < \ldots < X_n$$

The difference between the integrability conditions of M and N results from the positive constant of integration, K, which is associated with $\int N ds$. Wherever this is done, the

integral of N is written as ($\int N ds + K$). The effect of this constant is equivalent to a shift of the origin along the vertical axis by the amount -K.

The plots of a typical N_n and its integral are shown in Fig. 2.7. Considering $\int_0^{j\sqrt{X_i}} N_n ds$ at a point as the area enclosed by N_n from the origin up to that point, it can be concluded that the first minimum of $\int N_n ds$ is unconditionally negative at $s^2 = -X_1$. That is,

$$\int_0^{j\sqrt{X_1}} N_n ds < 0 \qquad ..(2.84)$$

2.7.1 <u>Conditions for the Integrability of N_n</u>:

A given N_n will be integrable if

$$\int N_n ds + K < 0 \quad \text{at} \quad s = j\sqrt{X_{2i-1}}$$

$$\text{and} \quad \int N_n ds + K > 0 \quad \text{at} \quad s = j\sqrt{X_{2i}} \qquad ..(2.85)$$

$$i = 1,2,3,...$$

Since K is positive, (2.85) is equivalent to the following two conditions:

(i) $\int N_n ds$ is negative at all the odd numbered zeros of N_n, namely, $j\sqrt{X_1}$, $j\sqrt{X_3}$, ... etc., and

(ii) the minimum of the maxima of $\int N_n ds$ is greater than the maximum of the minima.

We now consider two cases: (a) when $n \leq 3$ and (b) when $n > 3$.

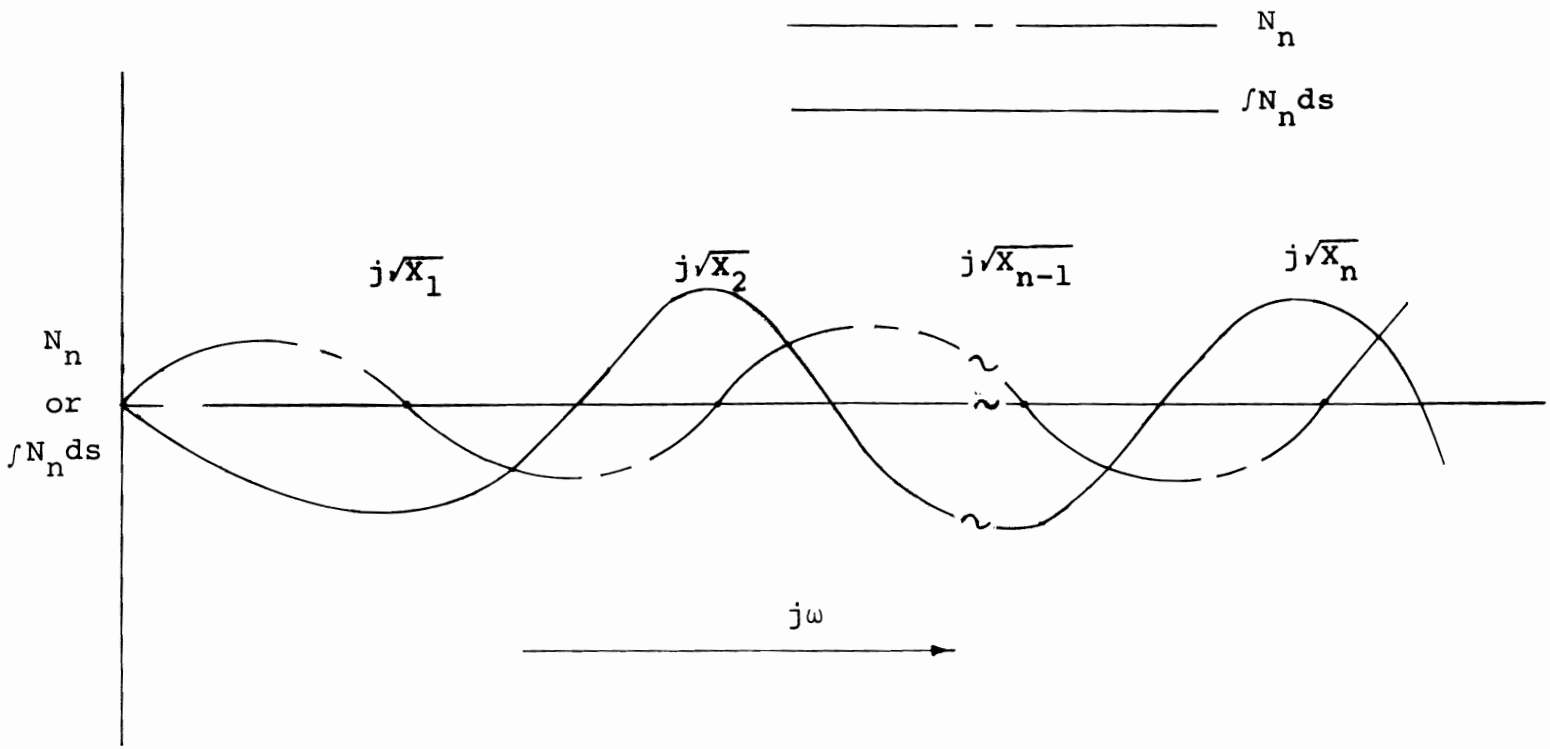

Fig.2.7: Plots of N_n and $\int N_n ds$, when N_n is integrable

2.7.2 Integrability of N_n when $n \leq 3$:

With $n \leq 3$, the integrability of N_n can be established almost by inspection.

Consider

$$N_2 = s(s^2 + X_1)(s^2 + X_2) \qquad ..(2.86)$$

then

$$\int N_2 ds = \frac{s^6}{6} + \frac{s^4}{4}(X_1 + X_2) + \frac{s^2}{2} X_1 X_2 \qquad ..(2.87)$$

Evaluating (2.87) at $s = j\sqrt{X_1}$ and $s = j\sqrt{X_2}$, we have,

$$\int_0^{j\sqrt{X_1}} N_2 ds = \frac{1}{12} X_1^{\,3} - \frac{X_1^{\,2} X_2}{4} \qquad ..(2.88)$$

$$\int_0^{j\sqrt{X_2}} N_2 ds = \frac{1}{12} X_2^{\,3} - \frac{X_1 X_2^{\,2}}{4} \qquad ..(2.89)$$

Since from (2.88) and (2.89),

$$\int_0^{j\sqrt{X_2}} N_2 ds > \int_0^{j\sqrt{X_1}} N_2 ds$$

there exists a positive value of K such that

$$\int_0^{j\sqrt{X_1}} N_2 ds + K < 0$$

$$..(2.90)$$

and

$$\int_0^{j\sqrt{X_2}} N_2 ds + K > 0$$

where

$$0 \leq K < \left| \int_0^{j\sqrt{X_1}} N_2 ds \right| \quad \text{for} \quad \int_0^{j\sqrt{X_2}} N_2 ds > 0$$

$$\text{or} \quad \left| \int_0^{j\sqrt{X_2}} N_2 ds \right| < K < \left| \int_0^{j\sqrt{X_1}} N_2 ds \right| \quad \text{for} \quad \int_0^{j\sqrt{X_2}} N_2 ds < 0 \qquad ..(2.91)$$

These conditions are illustrated in Figs. 2.8(a) and 2.8(b). It follows that N_2 is always integrable.

N_3 will be integrable, provided

$$\int_0^{j\sqrt{X_3}} N_3 ds < 0 \qquad ..(2.92)$$

Let

$$N_3 = s(s^2 + X_1)(s^2 + X_2)(s^2 + X_3) \qquad ..(2.93)$$

$\int N_3 ds$ has a maximum at $s = j\sqrt{X_2}$ flanked by the two minima at $s = j\sqrt{X_1}$ and $s = j\sqrt{X_3}$. It now follows that, given

$$\int_0^{j\sqrt{X_3}} N_3 ds < 0$$

there exists a K such that the maximum and minima of $(\int N_3 ds + K)$ are respectively positive and negative. K must satisfy either of the following inequalities:

$$0 \leq K < \min\left(\left| \int_0^{j\sqrt{X_1}} N_3 ds \right| , \left| \int_0^{j\sqrt{X_3}} N_3 ds \right| \right) \text{ for } \int_0^{j\sqrt{X_2}} N_3 ds > 0$$

$$\text{or} \quad \left| \int_0^{j\sqrt{X_2}} N_3 ds \right| < K < \min\left(\left| \int_0^{j\sqrt{X_1}} N_3 ds \right| , \left| \int_0^{j\sqrt{X_3}} N_3 ds \right| \right) \text{ for } \int_0^{j\sqrt{X_2}} N_3 ds < 0 \qquad ..(2.94)$$

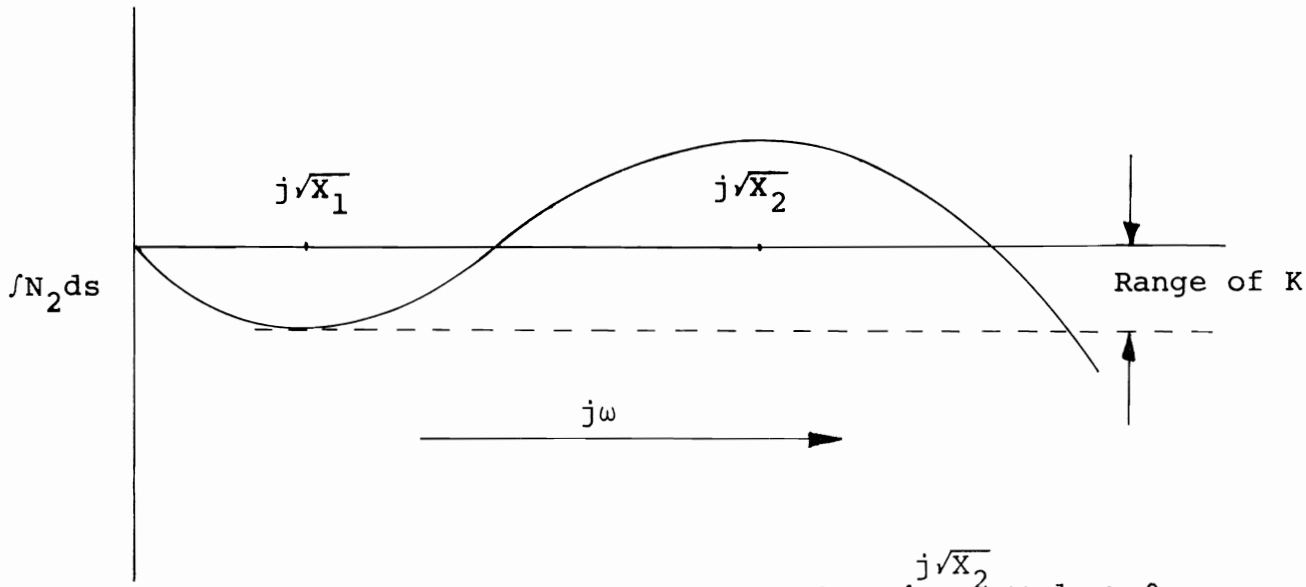

Fig.2.8(a): Choice of K for $\int N_2 ds$ when $\int_0^{j\sqrt{X_2}} N_2 ds > 0$

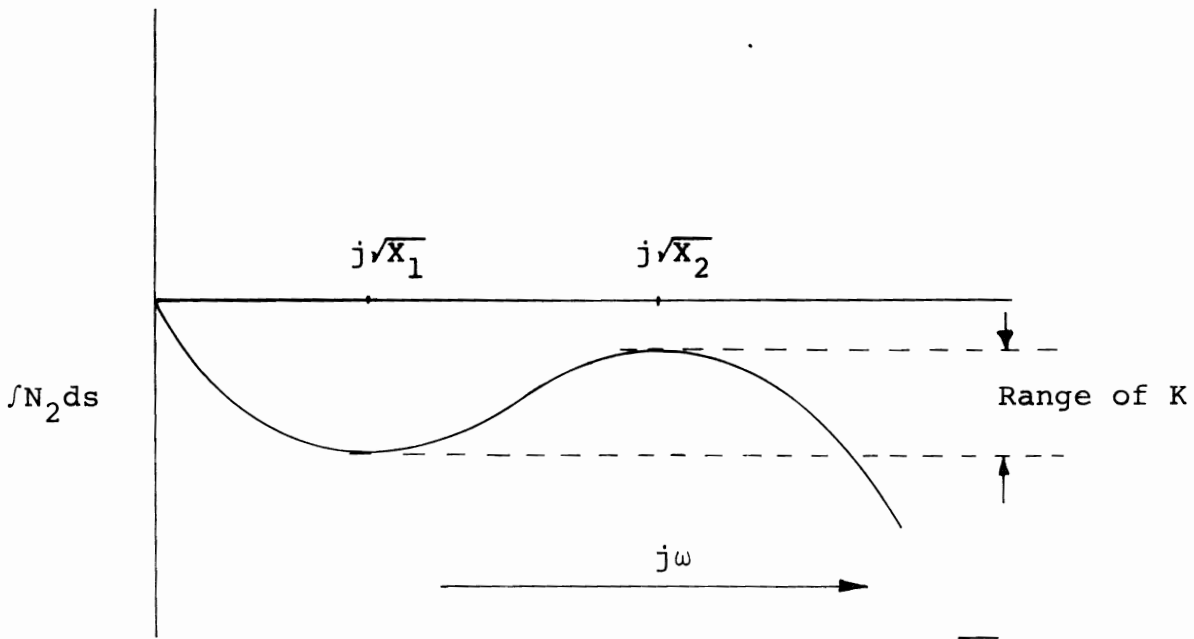

Fig.2.8(b): Choice of K for $\int N_2 ds$ when $\int_0^{j\sqrt{X_2}} N_2 ds < 0$

It is possible to derive a simple relationship between the zeros of N_3 which is equivalent to (2.92). Evaluating $\int N_3 ds$ at $s = j\sqrt{X_3}$, we have,

$$\int_0^{j\sqrt{X_3}} N_3 ds = \frac{1}{24} X_3^2(-X_3^2 + 2X_1X_3 + 2X_2X_3 - 6X_1X_2) \qquad ..(2.95)$$

Thus, in order that N_3 is integrable, the relationship to be satisfied is

$$X_3^2 - 2X_3(X_1 + X_2) + 6X_1X_2 > 0 \qquad ..(2.96)$$

<u>Example 2.6</u>:

It is required to investigate the integrability of

$$N_3 = s(s^2 + 1)(s^2 + 5)(s^2 + 8) \qquad ..(2.97)$$

Substituting the values of X_1, X_2 and X_3 in (2.96), we obtain,

$$X_3^2 - 2X_3(X_1 + X_2) + 6X_1X_2 = -2 \qquad ..(2.98)$$

Hence the given N_3 is unintegrable.

Inequality (2.96) also provides a necessary condition for the integrability of a HP of the seventh or the eighth order.

2.7.3 <u>Integrability of</u> N_n <u>when</u> $n > 3$:

In order to test for the integrability of higher order N_n, the evaluation of $\int N_n ds$ at the zeros of N_n becomes essential. The procedure for this will now be given.

Since a necessary condition for the integrability of N_n is

$$\int_0^{j\sqrt{X_{2i-1}}} N_n ds < 0, \quad i = 1,2,3,\ldots \qquad ..(2.99)$$

$\int N_n ds$ should be evaluated at the odd numbered zeros of N_n first. This will permit the immediate rejection of an unintegrable N_n. As noted in Theorem 2.4, the n different expressions required to be tested are obtainable from any one expression by a cyclic permutation of the X_i's. For this reason, only the expression for $\int_0^{j\sqrt{X_1}} N_n ds$ will be given.

<u>Theorem 2.11</u>:

$$\begin{aligned}\int_0^{j\sqrt{X_1}} N_n ds = {} & \frac{1}{12} P(2,n-1,n-2)X_1^{\,3} - \frac{1}{24} P(2,n-1,n-3)X_1^{\,4} \\ & + \frac{1}{40} P(2,n-1,n-4)X_1^{\,5} - \ldots\ldots\ldots\ldots\ldots \\ & + (-1)^{n-1} \frac{1}{2n(n-1)} P(2,n-1,,)X_1^{\,n} \\ & + (-1)^n \frac{1}{2n(n+1)} X_1^{\,n+1} + X_n \int_0^{j\sqrt{X_1}} N_{n-1} ds \qquad ..(2.100)\end{aligned}$$

The proof of this Theorem is similar to that of Theorem 2.4 and is omitted for the sake of brevity.

Repeated use of (2.100) will enable one to grow $\int_0^{j\sqrt{X_1}} N_n ds$ by a recursive process. For example:

$$\int_0^{j\sqrt{X_1}} s(s^2 + X_1) ds = \frac{-X_1^{\,2}}{4} \qquad ..(2.101)$$

$$\int_0^{j\sqrt{X_1}} N_2 ds = \frac{1}{12} X_1^3 + X_2 \int_0^{j\sqrt{X_1}} s(s^2 + X_1) ds \qquad ..(2.102)$$

$$\int_0^{j\sqrt{X_1}} N_3 ds = \frac{1}{12} X_1^3 X_2 - \frac{1}{24} X_1^4 + X_3 \int_0^{j\sqrt{X_1}} N_2 ds \qquad ..(2.103)$$

$$\int_0^{j\sqrt{X_1}} N_4 ds = \frac{1}{12} X_1^3 X_2 X_3 - \frac{1}{24} X_1^4 (X_2 + X_3) + \frac{1}{40} X_1^5 + X_4 \int_0^{j\sqrt{X_1}} N_3 ds \qquad ..(2.104)$$

.....

2.8 Summary of the Testing Procedure for the Integrability of N_n:

(1) A given N_2 is always integrable.

(2) N_3 is integrable provided its zeros satisfy the inequality (2.96).

(3) For $n > 3$, N_n is integrable if

(a) $$\int_0^{j\sqrt{X_{2i-1}}} N_n ds < 0, \; i = 2,3,4,... \qquad ..(2.105)$$

$\int_0^{j\sqrt{X_1}} N_n ds$ is computed using (2.100). The expression for the integral at other zeros of N_n are obtained by a cyclic permutation of the X_i's in (2.100).

and (b) the minimum of the maxima of $\int N_n ds$ is greater than the maximum of its minima.

The constant K, $K \geq 0$, must be chosen to lie in between the absolute values of the maximum of the minima and the minimum of the maxima.

Example 2.7:

We wish to investigate the integrability of

$$N_4 = s(s^2 + 1)(s^2 + 5)(s^2 + 10)(s^2 + 15) \qquad ..(2.106)$$

From Theorem 2.11,

$$\int_0^{j\sqrt{X_1}} N_4 ds = \frac{1}{12} X_1^3 X_2 X_3 - \frac{1}{24} X_1^4 (X_2 + X_3) + \frac{1}{40} X_1^5 +$$

$$+ X_4 \left[\frac{1}{12} X_1^3 X_2 - \frac{1}{24} X_1^4 + X_3 \{ \frac{1}{12} X_1^3 - \frac{1}{4} X_1^2 X_2 \} \right]$$

$$= -165.8083 \qquad ..(2.107)$$

By a cyclic permutation of the X_i's in (2.107), we similarly obtain expressions for the other zeros of N_4. Evaluating these, we obtain,

$$\int_0^{j\sqrt{X_3}} N_4 ds = -208.30 \qquad ..(2.108)$$

$$\int_0^{j\sqrt{X_2}} N_4 ds = 286.4625 \qquad ..(2.109)$$

$$\int_0^{j\sqrt{X_4}} N_4 ds = 703.2375 \qquad ..(2.110)$$

Hence N_4 is integrable for

$$0 \leq K < 165.8083$$

2.9 The Medial N_n:

If the zeros of a given N_n are such that $\int N_n ds$ assumes the value zero at all the even numbered roots of N_n, namely, $j\sqrt{X_2}$, $j\sqrt{X_4}$, ... , then N_n is termed the 'medial N_n'. ($\int N_n ds$ corresponding to the medial N_n contains s^2 as a factor). It is to be noted that the medial N_n is not a limiting curve in the same sense as the medial M_m. Unlike the medial M_m, the the medial N_n is always integrable, since there exists a range of K for which $\int N_n ds + K$ contains only simple imaginary axis zeros. The conditions which determine whether a given N_n is medial or not depend on whether n is even or odd. These are discussed in Theorems 2.12 and 2.13.

Theorem 2.12:

A given N_n (n even) will be medial, if and only if

$$\sum_{i=1}^{\frac{n}{2}} \frac{X_j}{X_{2i} - X_j} = \frac{1}{2}, \quad j = 1,3, \ldots, n-1 \qquad ..(2.111)$$

Proof:

(a) Necessity: If N_n is medial, we have, by definition,

$$\int N_n ds = \frac{s^2}{2n+2} (s^2 + X_2)^2 (s^2 + X_4)^2 \ldots (s^2 + X_n)^2 \qquad ..(2.112)$$

Letting $$T = (s^2 + X_2)(s^2 + X_4) \ldots (s^2 + X_n) \qquad ..(2.113)$$

we have,

$$\int N_n ds = \frac{s^2}{2n+2} T^2 \qquad ..(2.114)$$

Differentiating (2.114) with respect to s, we get,

$$N_n = \frac{2sT^2}{2n + 2} \left(1 + \frac{sT'}{T}\right) \qquad ..(2.115)$$

$j\sqrt{X_1}$, $j\sqrt{X_3}$, ... ,$j\sqrt{X_{n-1}}$ are zeros of N_n and since T does not contain these zeros, we have,

$$1 + \frac{sT'}{T} = 0 \quad \text{for} \quad s^2 = -X_1, -X_3, \ldots , -X_{n-1} \qquad ..(2.116)$$

Writing $\frac{T'}{T}$ in the partial fraction form and substituting in (2.116), we get,

$$1 + \sum_{i=1}^{\frac{n}{2}} \frac{2s^2}{s^2 + X_i} = 0, \quad s^2 = -X_1, -X_3, \ldots , -X_{n-1} \qquad ..(2.117)$$

from which the necessity follows.

(b) Sufficiency: Starting with any of the given inequalities, say at $s^2 = -X_1$, and proceeding backwards from (2.117) we observe that $(s^2 + X_1)$ is contained as a factor in $(T + sT')$. Similarly, $(s^2 + X_3), (s^2 + X_5), \ldots , (s^2 + X_{n-1})$ are also shown to be contained in $(T + sT')$. Since T contains the rest of the factors, N_n contains all the required factors.

Hence the Theorem follows.

<u>Theorem 2.13</u>:

A given N_n (n odd, n > 1) will be medial if and only if, the expression

$$Y = \frac{2X_j - 2X_j^2 \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}}{1 - 2X_j \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}} \qquad ..(2.118)$$

remains invariant for $j = 1,3, \ldots ,n$ and is a zero of $\int N_n ds$. (Hence, all the zeros of $\int N_n ds$ are known)

<u>Proof</u>:

(a) Necessity: If N_n is medial, we must have, by definition,

$$\int N_n ds = \frac{s^2}{2n + 2} (s^2 + X_2)^2 (s^2 + X_4)^2 \ldots (s^2 + X_{n-1})^2 (s^2 + Y) \qquad ..(2.119)$$

$$= \frac{s^2}{2n + 2} T^2 (s^2 + Y) \qquad ..(2.120)$$

Differentiating (2.120), we get,

$$N_n = \frac{2sT^2}{2n + 2} \{2s^2 + Y + \frac{sT'}{T} (s^2 + Y)\} \qquad ..(2.121)$$

$$= \frac{sT^2}{n + 1}\{Y(1 + \sum_{i=1}^{\frac{n-1}{2}} \frac{2s^2}{s^2 + X_{2i}}) + 2s^2 + \sum_{i=1}^{\frac{n-1}{2}} \frac{2s^4}{s^2 + X_{2i}}\} \qquad ..(2.122)$$

N_n vanishes for $s^2 = -X_1, -X_3, \ldots, -X_n$ and since T is not zero for these values, the necessity follows.

(b) Sufficiency: Starting with the given expression Y at $s^2 = -X_1$ (say) and proceeding backwards from (2.122), we observe that $(s^2 + X_1)$ is contained as a factor in

$$2s^2T + YT + sT'(s^2 + Y) \qquad ..(2.123)$$

Similarly, $(s^2 + X_3), (s^2 + X_5), \ldots, (s^2 + X_n)$ are also contained as factors in (2.123). Since T contains the rest of the factors, it follows from (2.121) that N_n contains all the required factors. This proves Theorem 2.13.

Figs. 2.9(a) and (b) give the plot of a medial N_n and its integral for both the even and odd cases.

As an aid in determining the integrability of a given N_n, it can be noted that if its zeros satisfy either (2.111) or (2.118), then N_n is integrable and no further tests need be made.

2.10 Integrability of Hurwitz Polynomials:

If a given M or N have been found integrable, then $\frac{\int M ds}{M}$ or $\frac{(\int N ds + K)}{N}$ will each be a reactance function. The integrability conditions of M and N have been summarized in 2.5.3 and 2.8. If M and N are the even and odd parts of the same HP, F, then integrability conditions of M and N are necessary conditions for F to be integrable. Sufficiency can be established only by a continued fraction expansion of $\frac{\int M ds}{(\int N ds + K)}$ if each of the coefficients is positive. This will be equivalent to ensuring that $\frac{\int M ds}{(\int N ds + K)}$ is a reactance function.

2.11 Generation of Higher Order Positive Real Functions using the Integrability Criteria of M and N:

Theorem 2.14:

If M is integrable, then,

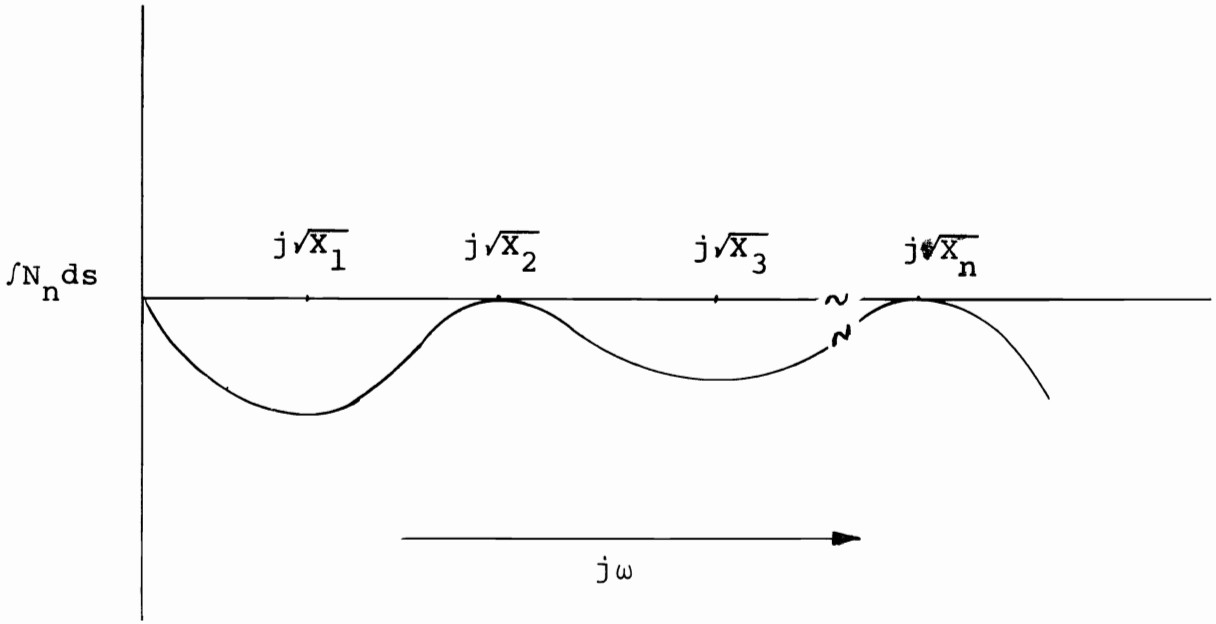

Fig.2.9(a): The integral of Medial N_n for n even

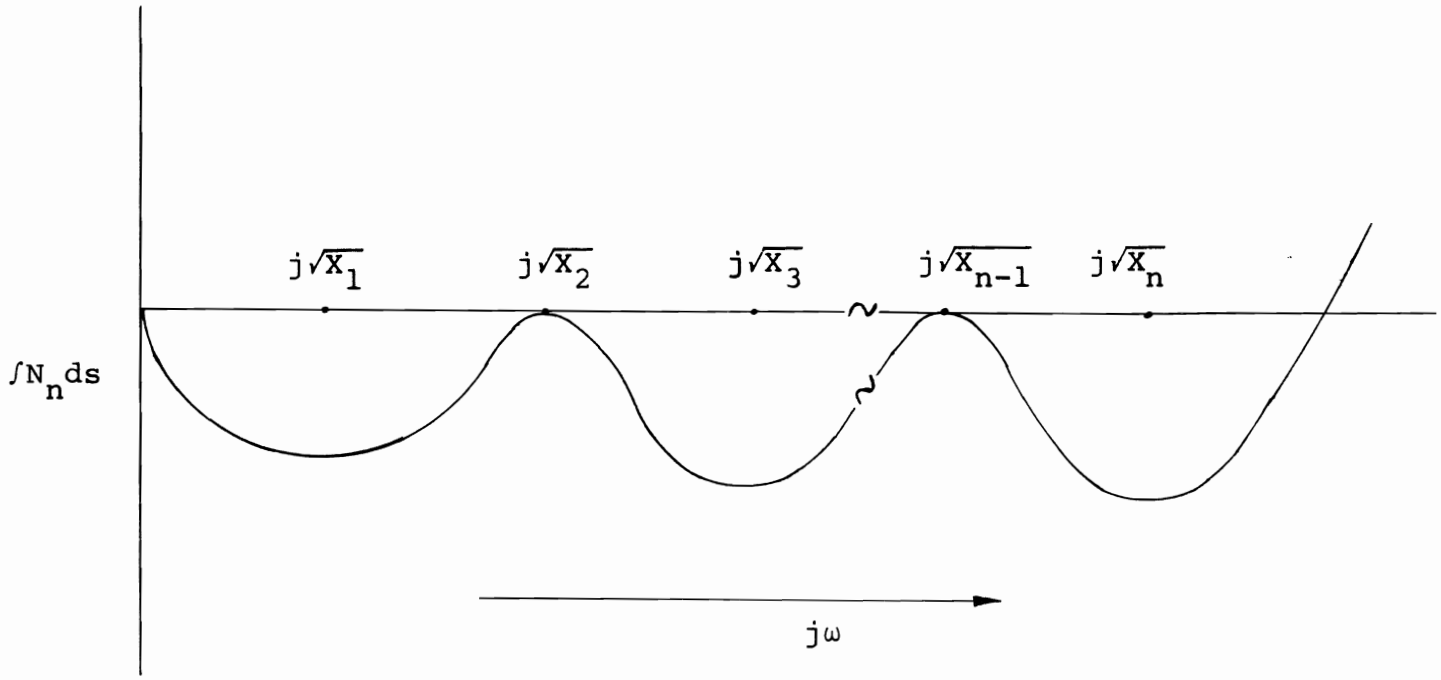

Fig.2.9(b): The integral of Medial N_n for n odd.

$$Z(s) = \frac{M + \int M ds}{M + M'} \qquad ..(2.124)$$

is a PRF.

Proof:

We have

$$\text{Ev } Z(s) = \frac{M^2 - M' \int M ds}{M^2 - M'^2} \qquad ..(2.125)$$

From (2.125), Z(s) will be positive real if,

$$(M^2 - M' \int M ds)\Big|_{s = j\omega} \geq 0 \quad \text{for } 0 \leq \omega < \infty \qquad ..(2.126)$$

Since $\frac{\int M ds}{M}$ is a reactance function,

$$\frac{d}{ds} \frac{\int M ds}{M}\Big|_{s = j\omega} > 0 \text{ for } 0 \leq \omega < \infty \qquad ..(2.127)$$

Hence (2.126) follows from (2.127). The Theorem is thus proved.

Theorem 2.15:

If N is integrable, then,

$$Z(s) = \frac{N + \int N ds + K}{N + N'} \qquad ..(2.128)$$

is a PRF.

The proof of this Theorem is similar to that of Theorem 2.14 and is omitted for the sake of brevity.

Theorems 2.14 and 2.15 enable one to generate higher order PRF's given M or N in factored form.

2.12 Conclusions:

In this Chapter, given the zeros of the even and odd parts of HP's, necessary and sufficient conditions have been established such that $\frac{\int M_m ds}{M_m}$ and $\frac{\int N_n ds + K}{N_n}$ are reactance functions.

Regarding the integrability of M_m, an inspection test is provided by the second zero condition which states that the ratio $\frac{X_2}{X_1}$ shall be greater than 5. Further, the integrals of all the lower order M_{m-k}'s, $k = 1,2, \ldots ,m-3$ obtained from the given M_m shall be negative at $s = j\sqrt{X_2}$ and this is facilitated by the application of Theorem 2.4(a), which enables one to write $\int_0^{j\sqrt{X_2}} M_{m-k} ds$ from a knowledge of $\int_0^{j\sqrt{X_2}} M_{m-k-1} ds$. These tests permit the immediate rejection of an unintegrable M_m.

Sufficiency tests for the integrability of M_m are obtained when the different $\int_0^{j\sqrt{X_i}} M_m ds$, constructed using Theorem 2.4(b), alternate in their sign.

A special case of M_m, called the medial M_m, arises when $\int M_m ds$ assumes the value zero at all the even numbered zeros of M_m. Certain prescribed relationships between the zeros of M_m are shown to exist in order that M_m is medial. Some properties of the medial M_m have been discussed. In particular, if the given M_m can be obtained by the perturbation of one or more zeros (except the first zero) of the corresponding medial

M_m towards the origin, then it is unintegrable. This provides a comparison test for the integrability of M_m. Also if M_m comprises the product of medial M_i and M_j, the given M_m is unintegrable.

Considering the integrability conditions of N_n, it has been established that a given N_n, $n \leq 2$, is always integrable. N_3 is integrable provided its zeros satisfy the given inequality (2.96). The Integrability of N_n's is established by requiring that $\int N_n ds$ at all the odd numbered zeros of N_n shall be negative and that the minimum of the maxima of $\int N_n ds$ shall be greater than the maximum of the minima. The latter permits the selection of a positive value of the constant of integration K. A special case of N_n, called the medial N_n, arises when $\int N_n ds$ assumes the value zero at all the even numbered zeros of N_n.

The medial N_n is always integrable since there exists a positive constant K such that $(\int N_n ds + K)$ has only simple imaginary axis zeros. If a given N_n can be identified as being medial, its integrability can be established without any further tests.

If M and N are the even and the odd parts of the same HP, F, then, F will be integrable provided

(i) M and N are integrable,

and (ii) $\frac{\int M ds}{\int N ds + K}$ is a reactance function.

The latter can be ensured by a continued fraction expansion.

The integrability criteria of M and N have been used to generate higher order positive real functions.

CHAPTER III

THE INTEGRABILITY OF POLYNOMIALS CONTAINING ONLY NEGATIVE REAL AXIS ZEROS

3.1 Introduction:

Network functions in which the zeros of the numerator and the denominator are restricted to the negative real axis and also interlace with each other are realizable by lossy two-element-kind networks. Of these, the RC networks are attractive whenever weight and size must be minimized. Besides, such networks have assumed even greater importance with the advent of integrated circuits, where generally speaking, the realization of inductances is not practicable. Polynomials with distinct zeros on the negative real axis are often referred to as RC polynomials[3] in network theory.

The integrability of RC polynomials can be studied in two ways. One way is to find the conditions under which an RC polynomial yields a HP on integration. The other is to study the conditions under which an RC polynomial will result when another RC polynomial is integrated. These two cases will be referred to as 'Hurwitz integrability' and 'RC integrability' of RC polynomials respectively. The subject matter of this Chapter deals with both these cases separately.

The Hurwitz integrability of an RC polynomial F(s)

can be investigated using the methods of Chapter II which require a knowledge of the zeros of its even and odd parts separately. However, considerable simplification in testing results if the zeros of F(s) are known. The testing procedures for Hurwitz integrability, established in this Chapter, assume this knowledge. On the other hand, the RC integrability of F(s) can be established whether its zeros are known or not employing two different techniques.

3.2 Conditions under which the Integral of an RC Polynomial is Hurwitz:

Let
$$F_n(s) = (s + X_1)(s + X_2) \ldots (s + X_n) \qquad ..(3.1)$$
$$0 < X_1 < X_2 < \ldots < X_n \quad *$$

Then,

$$\frac{\int F_n(s)ds + K}{F_n(s)} = \frac{s}{n+1} + \sum_{i=1}^{n} \frac{A_i}{s + X_i} + C_1 \qquad ..(3.2)$$

where K is the constant of integration and,

$$A_i = \left. \frac{(s + X_i)\{\int F_n(s)ds + K\}}{F_n(s)} \right|_{s = -X_i} \qquad ..(3.3)$$

---

* The case when $F_n(s)$ contains a zero at the origin can be readily excluded since in this case, $(\int F_n(s)ds + K)$ will have no term containing the first power in s, and hence, it will not be Hurwitz. If cancellation of the factor 's' is allowed, $\int F_n(s)ds/F_n(s)$ will still be an RC function for K = 0. This special case is not considered.

$$C_1 = \lim_{s \to -\infty} \left\{ \frac{\int F_n(s)ds + K}{F_n(s)} - \frac{s}{n+1} \right\}$$

$$= \frac{1}{n(n+1)} \sum_{i=1}^{n} X_i \qquad ..(3.4)$$

As can be seen, C_1 is positive, while the A_i's may be positive or negative. The following equality between C_1, K and the A_i's holds:

$$\frac{K}{\prod_{i=1}^{n} X_i} = \sum_{i=1}^{n} \frac{A_i}{X_i} + C_1 \qquad ..(3.5)$$

In (3.2), separating those A_i's which are negative and using the identity,

$$\frac{A_i}{s + X_i} = \frac{-\frac{A_i}{X_i} s}{s + X_i} + \frac{A_i}{X_i} \qquad ..(3.6)$$

for all such residues, (3.2) can be written as,

$$\frac{\int F_n(s)ds + K}{F_n(s)} = \frac{s}{n+1} + \sum_{1}^{m} \frac{A_i}{s + X_1} + \sum_{1}^{n-m} \frac{sB_j}{s + X_j} + C_2 \qquad ..(3.7)$$

where

$$B_j = \frac{-A_i}{X_i} \quad , \quad B_j > 0 \quad ,$$

$$A_i > 0,$$

and C_2 may be positive or negative.

Since $\frac{G'(s)}{G(s)}$ is a PRF if G(s) is a HP[5], the necessary and sufficient condition for $\int F(s)ds + K$ to be Hurwitz is that,

$$\mathrm{Re}\left.\frac{\int F_n(s)ds + K}{F_n(s)}\right|_{s = j\omega} \geq 0 \quad \text{for all } \omega \qquad ..(3.8)$$

The above condition will hold when

$$\sum_{1}^{m} \frac{A_i X_i}{\omega^2 + X_i^2} + \sum_{1}^{n-m} \frac{B_j \omega^2}{\omega^2 + X_j^2} + C_2 \geq 0 \text{ , for } 0 \leq \omega < \infty \qquad ..(3.9)$$

The first summation in (3.9), being the sum of monotonically decreasing functions of ω, is itself monotonically decreasing. In the same way, the second summation is a monotonically increasing function of ω. In Fig. 3.1, these two summations are plotted against ω. The sum of the ordinates of curves I and II can exhibit only one minimum at ω_{min}, which can be found as the real solution of,

$$\sum_{1}^{m} \frac{A_i X_i}{(X_i^2 + \omega^2_{min})^2} = \sum_{1}^{n-m} \frac{B_j X_j^2}{(X^2_j + \omega^2_{min})^2} \qquad ..(3.10)$$

Having obtained ω_{min}, we can rewrite inequality (3.9) as,

$$\sum_{1}^{m} \frac{A_i X_i}{\omega^2_{min} + X_i^2} + \Sigma \frac{B_j \omega^2_{min}}{\omega^2_{min} + X_j^2} + C_2 \geq 0 \qquad ..(3.11)$$

which is the condition for $\int F_n(s)ds + K$ to be Hurwitz. The entire range of K satisfying (3.11) will be available as the suitable choice for the constant of integration. From (3.7), it also follows that if for any range of K, C_2 is positive, then $\frac{\int F_n(s)ds + K}{F_n(s)}$ is realizable as a driving point immitance

$$I - \sum_{1}^{m} \frac{A_i X_i}{\omega^2 + X_i^2}$$

$$II - \sum_{1}^{n-m} \frac{B_j \omega^2}{\omega^2 + X_j^2}$$

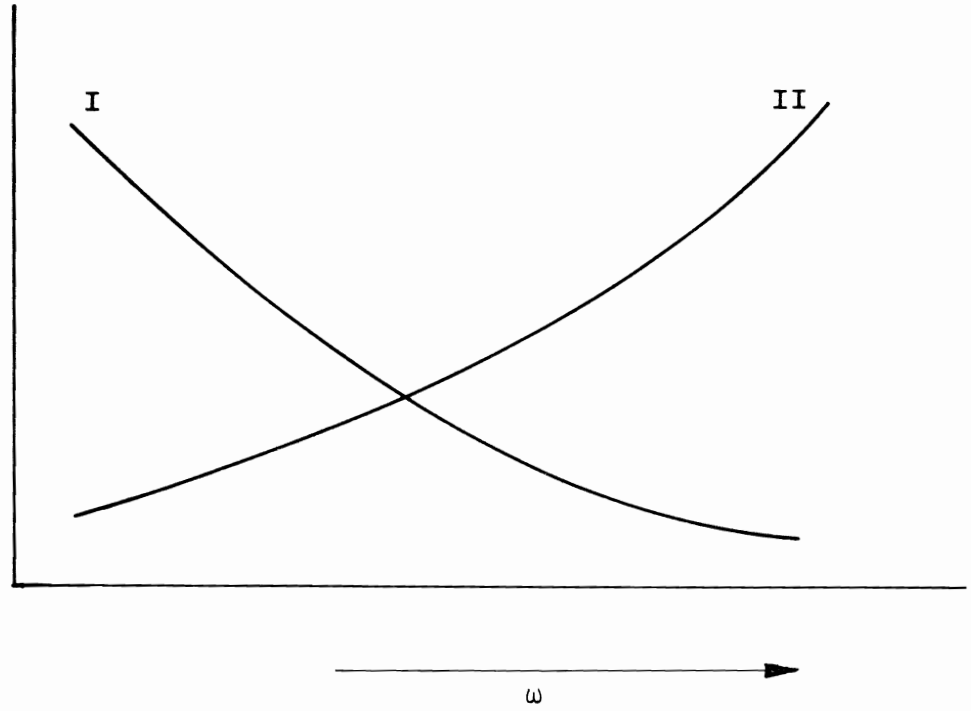

Fig.3.1: Plots of the real parts of RL and RC type terms of Equation 3.9.

function by a combination of RC and RL networks in series or parallel. Since for $K = 0$, $C_2 < 0$, $\frac{\int F_n(s)ds}{F_n(s)}$ is not realizable this way.

Example 3.1:

Let $$F(s) = (s + 1)(s + 5)(s + 8) \qquad ..(3.12)$$

then,

$$\frac{\int F(s)ds+K}{F(s)} = \frac{s}{4} + \frac{14}{3} + \frac{\frac{301.6665-4K}{28}}{s+1} - \frac{\frac{514.9998-4K}{12}}{s+5} + \frac{\frac{416-4K}{21}}{s+8} \qquad ..(3.13)$$

For an RC-RL decomposition to exist, we must have

(i) $K < 75.416$ ..(3.14)

and (ii) $\frac{\frac{514.9998-4K}{12}}{s+5}$ expandable as $\frac{As}{s+5} - B$, where $B < \frac{14}{3}$.

The latter condition gives

$$K > 58.75 \qquad ..(3.15)$$

Hence, choosing a value of K in between 58.75 and 75.416, F(s) is RC-RL decomposable. Letting $K = 60$, we obtain

$$\frac{\int F(s)ds + K}{F(s)} = \frac{s}{4} + \frac{2.203}{s+1} + \frac{4.583s}{s+5} + \frac{8.3809}{s+8} + 0.08366 \qquad ..(3.16)$$

3.3 Conditions for RC Integrability of a Polynomial:

In virtue of Lucas' theorem[6], a necessary condition for $(\int F_n(s)ds + K)$ to be an RC polynomial is that F(s) also be RC

in character.

Therefore, given F(s) as in (3.1), we wish to know the conditions under which its integral can be expressed as:

$$\int F_n(s)ds+K = (s + Y_1)(s + Y_2)(s + Y_3) \ldots (s + Y_{n+1}) \qquad ..(3.17)$$

$$0 \le Y_1 < X_1 < Y_2 < X_2 < \ldots < X_n < Y_{n+1}$$

$$K \ge 0$$

From a graphical point of view, (3.17) will hold if $\int F_n(s)ds+K$ changes its sign at the consecutive zeros of F(s). This is contained in the following inequalities:

$$\int_0^{-X_{2i}} F(s)ds + K > 0 \quad , \quad i = 1,2,3,\ldots$$

$$..(3.18)$$

and $$\int_0^{-X_{2i-1}} F(s)ds + K < 0 \quad , \quad i = 1,2,3,\ldots$$

Since only positive values of K are permitted, (3.18) is equivalent to the following two conditions:

(i) $\int F(s)ds$ is negative at all the odd numbered zeros of F(s) namely, $-X_1, -X_3, -X_5, \ldots$, and

(ii) the minimum of the maxima of $\int F(s)ds$ is greater than the maximum of its minima.

Fig. 3.2 shows the plots of a given F(s) and its integral. It can be observed that the area enclosed by $F_n(s)$ will be always negative between s = 0 and s = $-X_1$, since the ordinate

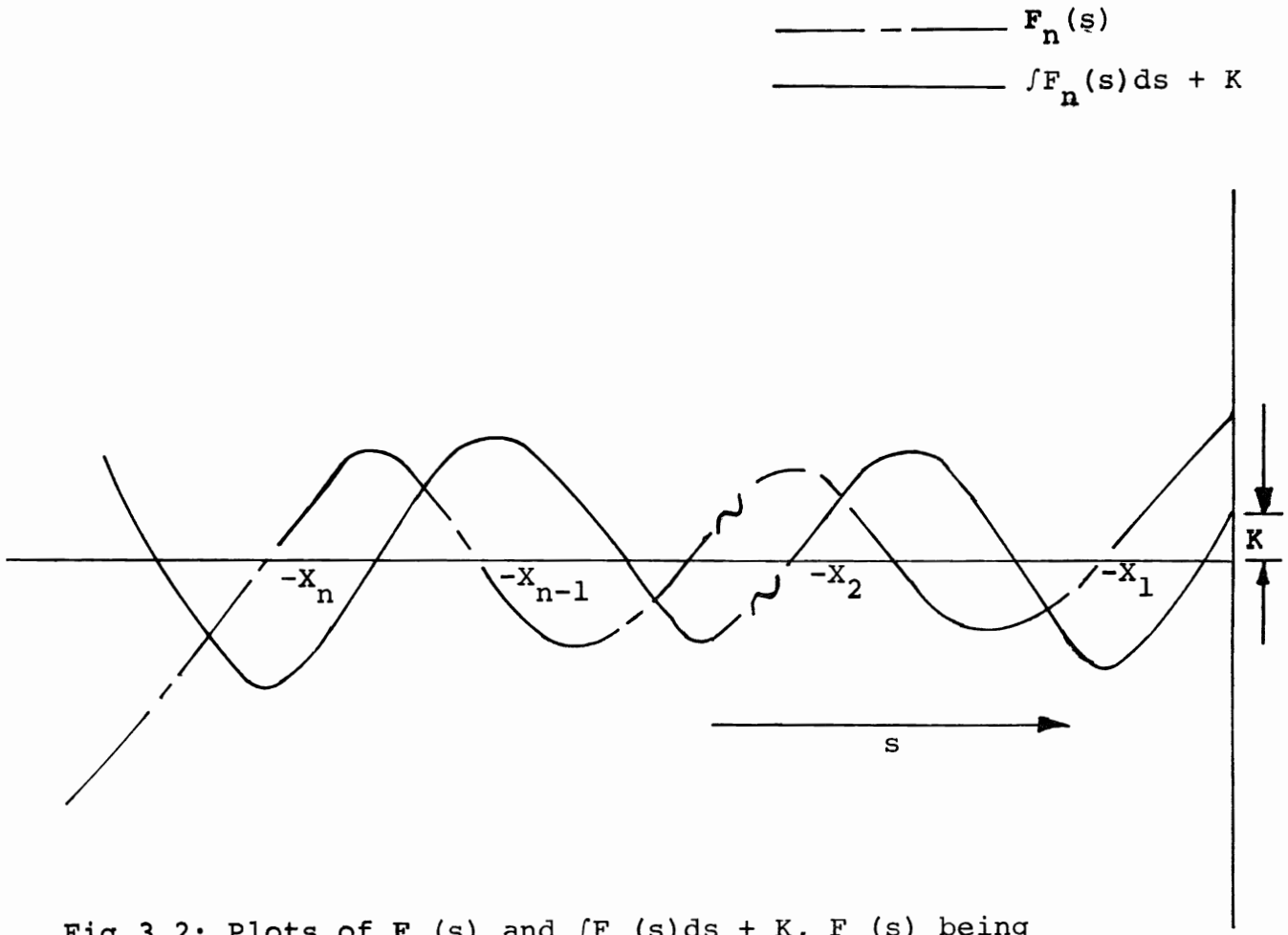

Fig.3.2: Plots of $F_n(s)$ and $\int F_n(s)ds + K$, $F_n(s)$ being integrable.

is positive while the abscissa is negative between these two points. The first minimum of $\int F_n(s)ds$ is, therefore, unconditionally negative. Two cases are considered: (a) when $n \leq 3$ and (b) when $n > 3$.

3.3.1 RC Integrability of $F_n(s)$ when $n \leq 3$:

The RC integrability of polynomials up to the third degree can be established almost by inspection.

Taking

$$F_2(s) = (s + X_1)(s + X_2) \qquad ..(3.19)$$

we have,

$$\int F_2(s)ds = \frac{s^3}{3} + \frac{1}{2}(X_1 + X_2)s^2 + X_1X_2s \qquad ..(3.20)$$

Since

$$\int_0^{-X_1} F_2(s)ds < 0$$

and

$$\int_0^{-X_2} F_2(s)ds > \int_0^{-X_1} F_2(s)ds, \qquad ..(3.21)$$

the necessary and sufficient conditions for the integrability of $F_2(s)$ are always satisfied. K can be chosen to have any value such that:

$$0 \leq K < \left|\int_0^{-X_1} F_2(s)ds\right| \quad \text{for} \quad \int_0^{-X_2} F_2(s)ds > 0 \qquad ..(3.22a)$$

or

$$\left|\int_0^{-X_2} F_2(s)ds\right| < K < \left|\int_0^{-X_1} F_2(s)ds\right| \quad \text{for} \quad \int_0^{-X_2} F_2(s)ds < 0 \qquad ..(3.22b)$$

These conditions are illustrated in Figs. 3.3(a) and 3.3(b).

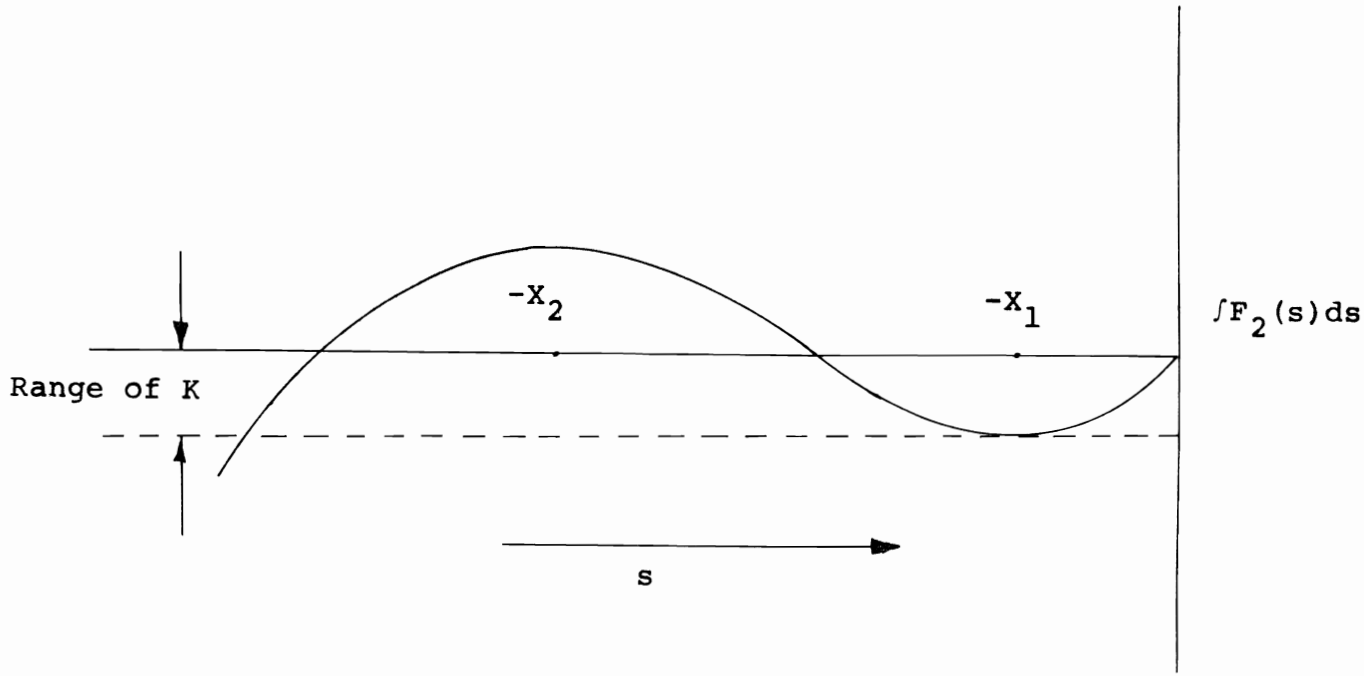

Fig.3.3(a): Choice of K for $\int F_2(s)ds$ when

$$\int_0^{-X_2} F_2(s)ds < 0$$

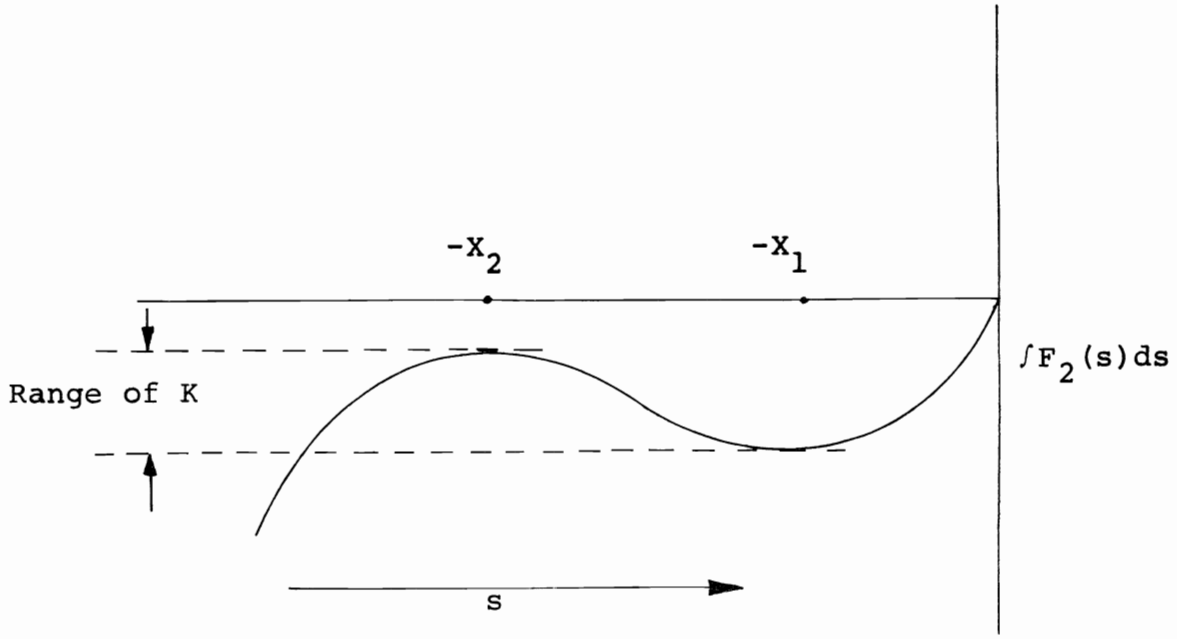

Fig.3.3(b): Choice of K for $\int F_2(s)ds$ when

$$\int_0^{-X_2} F_2(s)ds < 0$$

A polynomial $F_3(s)$ containing three factors is also RC integrable provided,

$$\int_0^{-X_3} F_3(s)ds < 0 \qquad ..(3.23)$$

(3.23) can be expressed as an inequality involving the zeros of $F_3(s)$. To obtain the same, let,

$$F_3(s) = (s + X_1)(s + X_2)(s + X_3) \qquad ..(3.24)$$

From (3.24), we get,

$$\int_0^{-X_3} F_3(s)ds = \frac{1}{12} X_3^2\{-X_3^2 + 2X_3(X_1 + X_2) - 6X_1X_2\} \qquad ..(3.25)$$

(3.23) will hold if,

$$X_3^2 - 2X_3(X_1 + X_2) + 6X_1X_2 > 0 \qquad ..(3.26)$$

K has the following range:

$$0 \leq K < \min\left(\left|\int_0^{-X_1} F(s)ds\right| , \left|\int_0^{-X_3} F(s)ds\right|\right) \text{ for } \int_0^{-X_2} F(s)ds > 0$$

$$\text{or} \quad \int_0^{-X_2} F(s)ds < K < \min\left(\left|\int_0^{-X_1} F(s)ds\right| , \left|\int_0^{-X_3} F(s)ds\right|\right) \text{ for } \int_0^{-X_2} F(s)ds < 0 \qquad ..(3.27)$$

<u>Example 3.2</u>:

To investigate the RC integrability of $F_3(s)$ given as:

$$F_3(s) = (s + 1)(s + 5)(s + 9) \qquad ..(3.28)$$

Substituting the values of X_1, X_2 and X_3 in (3.26), we obtain

$$X_3^2 - 2X_3(X_1 + X_2) + 6X_1X_2 = 3 \qquad ..(3.29)$$

Hence, $F_3(s)$ is RC integrable.

3.3.2 RC Integrability of $F_n(s)$ when $n > 3$:

In order to test the integrability of polynomials containing more than three factors, the evaluation of $\int F_n(s)ds$ at all the zeros of $F_n(s)$ becomes essential. The formalized approach, described below to achieve this, has the facility of a recursive relation. The evaluation of $\int F_n(s)ds$ is performed at the odd numbered zeros of $F_n(s)$ first so that a $F(s)$ which is not RC integrable could be rejected with less effort. As noted in connection with the integrability of M and N in Chapter II, the n different expressions required to be evaluated are obtainable from any one expression by a cyclic permutation of the different X_i's. For this reason, the expression for only $\int_0^{-X_1} F_n(s)ds$ will be given. The same is obtained using the following theorem:

Theorem 3.1:

$$\begin{aligned}\int_0^{-X_1} F_n(s)ds = {} & \frac{1}{6} P(2,n-1,n-2)X_1^3 - \frac{1}{12} P(2,n-1,n-3)X_1^4 \\ & + \frac{1}{20} P(2,n-1,n-4)X_1^5 - \ldots\ldots\ldots\ldots\ldots \\ & + (-1)^{n-1} \frac{1}{n(n-1)} P(2,n-1,1)X_1^n \\ & + (-1)^n \frac{1}{n(n+1)} X_1^{n+1} + X_n \int_0^{-X_1} F_{n-1}(s)ds \qquad ..(3.30)\end{aligned}$$

A proof of Theorem 3.1 is similar to that given for **M** in Theorem 2.4 and is omitted for the sake of brevity. Repeated use of (3.30) will allow one to grow $\int_0^{-X_1} F_n(s)ds$ by a recursive technique. For example:

$$\int_0^{-X_1} F_1(s)ds = \frac{-X_1^{\,2}}{2} \qquad ..(3.31)$$

$$\int_0^{-X_1} F_2(s)ds = \frac{1}{6} X_1^{\,3} + X_2 \int_0^{-X_1} F_1(s)ds \qquad ..(3.32)$$

$$\int_0^{-X_1} F_3(s)ds = \frac{1}{6} X_1^{\,3}X_2 - \frac{1}{12} X_1^{\,4} + X_3 \int_0^{-X_1} F_2(s)ds \qquad ..(3.33)$$

$$\int_0^{-X_1} F_4(s)ds = \frac{1}{6} X_1^{\,3}X_2X_3 - \frac{1}{12} X_1^{\,4}(X_2 + X_3) + \frac{1}{20} X_1^{\,5} + X_4 \int_0^{-X_1} F_3(s)ds \qquad ..(3.34)$$

.....

3.4 Summary of the Testing Procedure:

(i) An RC polynomial $F_n(s)$ is Hurwitz integrable if the inequality (3.9) is satisfied. Further, if in (3.11) $C_2 \geq 0$, then $\dfrac{\int F_n(s)ds + K}{F_n(s)}$ is realizable as a driving point immittance function by a combination of RC and RL networks in series or parallel.

(ii) A second degree RC polynomial is always RC integrable.

A third order RC polynomial is RC integrable provided its zeros satisfy the inequality (3.26).

Higher order RC polynomials $F_n(s)$, $n > 3$, are RC integrable if:

$$\text{(a)} \quad \int_0^{-X_{2i-1}} F_n(s)\,ds < 0 \;, \; i = 2,3,4,.. \qquad ..(3.35)$$

and (b) the minimum of the maxima of $\int F_n(s)\,ds$ is greater than the maximum of its minima.

$\int_0^{-X_1} F_n(s)\,ds$ is computed using Theorem 3.1. The expressions for the integral at other zeros of $F_n(s)$ are obtained by a cyclic permutation of the X_i's in (3.30).

The constant K can be chosen to lie in between the absolute values of the minimum of the maxima and the maximum of the minima.

<u>Example 3.3</u>:

To investigate the RC integrability of $F_4(s)$ given as:

$$F_4(s) = (s + 1)(s + 5)(s + 10)(s + 16) \qquad ..(3.36)$$

We have from Theorem 3.1,

$$\int_0^{-X_1} F_4(s)ds = \frac{1}{6} X_1^3 X_2 X_3 - \frac{1}{12} X_1^4 (X_2 + X_3) + \frac{1}{20} X_1^5$$

$$+ X_4 \left[- \frac{1}{12} X_1^4 + \frac{1}{6} X_1^3 X_2 + X_3 \{ \frac{1}{6} X_1^3 - \frac{1}{2} X_1^2 X_2 \} \right]$$

$$= - 354.2 \qquad ..(3.37)$$

By a cyclic permutation of the X_i's in the expression for $\int_0^{-X_1} F_4(s)ds$, we similarly obtain expressions for the other zeros of $F_4(s)$. Evaluating these, we obtain,

$$\int_0^{-X_3} F_4(s)ds = -500$$

$$\int_0^{-X_2} F_4(s)ds = 625 \qquad ..(3.38)$$

and $$\int_0^{-X_4} F_4(s)ds = 3020.8$$

Hence $F_4(s)$ is RC integrable for

$$0 \leq K < 354.2 \qquad ..(3.39)$$

<u>Example 3.4</u>:

To investigate the RC integrability of $F_4(s)$ given as:

$$F_4(s) = (s + 1)(s + 7)(s + 9)(s + 15) \qquad ..(3.40)$$

As in the previous example we obtain,

$$\int_0^{-X_1} F_4(s)ds = -424.5333$$

$$\int_0^{-X_3} F_4(s)ds = 1555.2000 \qquad ..(3.41)$$

Since $\int_0^{-X_3} F_4(s) \geq 0$, it follows that the necessary condition for the RC integrability of $F_4(s)$ is not satisfied.

3.5 The Medial $F_n(s)$:

If the zeros of a given $F_n(s)$ are such that $\int F_n(s)ds$ assumes the value zero at all the even numbered roots of M_m, namely, $-X_2, -X_4, \ldots$, then $F_n(s)$ is termed the medial $F_n(s)$. A medial $F_n(s)$ is RC integrable since there exists a K such that $\int F_n(s)ds + K$ is a RC polynomial. Depending upon n being even or odd, two different relationships exist between the zeros of $F_n(s)$ such that it is medial. These are discussed in Theorems 3.2 and 3.3 respectively.

Theorem 3.2:

$F_n(s)$, n even, will be medial if and only if

$$\sum_{i=1}^{\frac{n}{2}} \frac{X_j}{X_{2i} - X_j} = \frac{1}{2}, \; j = 1,3, \ldots , n-1 \qquad ..(3.42)$$

Proof:

(a) Necessity: If $F_n(s)$ is medial, we have,

$$\int F_n(s)ds = \frac{s}{n+1} (s + X_2)^2 (s + X_4)^2 \ldots\ldots (s + X_n)^2 \qquad ..(3.43)$$

by definition.

Letting $T = (s + X_2)(s + X_4) \ldots (s + X_n)$ ..(3.44)

$$\int F_n(s)ds = \frac{s}{n+1} T^2 \qquad ..(3.45)$$

Differentiating (3.45) with respect to s, we have,

$$F_n(s) = \frac{T^2}{n+1} \left(1 + 2s \frac{T'}{T}\right)$$

$$= \frac{T^2}{n+1} \left(1 + \sum_{i=1}^{n/2} \frac{2s}{s + X_{2i}}\right) \qquad ..(3.46)$$

The right hand side of (3.46) must vanish for $s = -X_1, -X_3, \ldots, -X_{n-1}$ and since T does not vanish for these values, we must have,

$$1 + \sum_{i=1}^{\frac{n}{2}} \frac{2s}{s + X_{2i}} = 0 \quad \text{for} \quad s = -X_1, -X_3, \ldots, -X_{n-1} \qquad ..(3.47)$$

(b) Sufficiency: Starting with any of the given equalities, say at S_1, and proceeding backwards from (3.47), we observe that $(s + X_1)$ is contained as a factor in $(T + 2sT')$. Similarly, $(s + X_3)$, $(s + X_5)$, ... , $(s + X_{n-1})$ are also shown to be contained in $(T + 2sT')$. Since it contains the rest of the factors, M_m has all the required factors.

Hence the Theorem follows.

<u>Theorem 3.3</u>:

A given $F_n(s)$, (n odd, $n > 1$) will be medial if and only if, the expression,

$$Y = \frac{2X_j - 2X_j^2 \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}}{1 - 2X_j \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}} \qquad ..(3.48)$$

remains invariant for $j = 1,3, \ldots n$ and is a zero of $\int F_n(s)ds$. (Hence all the zeros of $\int F_n(s)ds$ are known)

<u>Proof</u>:

If $F_n(s)$ is medial, we must have,

$$\int F_n(s)ds = \frac{s}{n+1} (s + X_2)^2 (s + X_4)^2 \ldots (s + X_{n-1})^2 (s + Y) \qquad ..(3.49)$$

$$= \frac{s}{n+1} T^2 (s + Y) \qquad ..(3.50)$$

where $T = (s + X_2)(s + X_4) \ldots\ldots\ldots (s + X_{n-1})$ ..(3.51)

Differentiating (3.51) with respect to s, we get,

$$F_n(s) = \frac{T^2}{n+1} \{Y(1 + 2s \frac{T'}{T}) + 2s + 2s^2 \frac{T'}{T}\}$$

$$= \frac{T^2}{n+1} \{Y(1 + 2s \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{s + X_{2i}}) + 2s + 2s^2 \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{s + X_{2i}}\} \qquad ..(3.52)$$

Since (3.52) must vanish for $s = -X_1, -X_3, \ldots, -X_n$, we must have,

$$Y = \frac{2X_j - 2X_j^2 \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}}{1 - 2X_j \sum_{i=1}^{\frac{n-1}{2}} \frac{1}{X_{2i} - X_j}}, \quad j = 1,3,\ldots\ldots,n \qquad ..(3.53)$$

Hence the necessity follows.

(b) Sufficiency: Starting with the given expression at $s = -X_1$ (say) and proceeding backwards from (3.53), we observe that $(s + X_1)$ is contained as a factor in,

$$YT + 2YsT' + 2sT + 2s^2T' \qquad ..(3.54)$$

Similarly, $(s + X_3), (s + X_5), \ldots, (s + X_n)$ are also contained as factors in (3.54). Since T contains the rest of the factors, it follows from (3.52) that $F_n(s)$ contains all the required factors. This proves Theorem 3.3.

Figs. 3.4(a) and 3.4(b) illustrate these two situations.

3.6 RC Integrability of a Polynomial whose Zeros are not known:

When the zeros of $F_n(s)$ are not known, an alternative method has to be employed in order to establish its RC integrability. The same is provided by a modification of the Routh-Hurwitz criterion.

The Routh-Hurwitz criterion[18] and other methods[19,20] have been used in the past to determine the negative real axis zeros of a polynomial. However, all these methods entail a fair amount of calculation. A very simple algorithm whereby the number of negative real axis zeros in a given polynomial having real coefficients can be determined exists[21], and is described below.

Consider the polynomial

$$F_n(s) = a_o + a_1 s + a_2 s^2 + \ldots + a_n s^n \qquad ..(3.55)$$

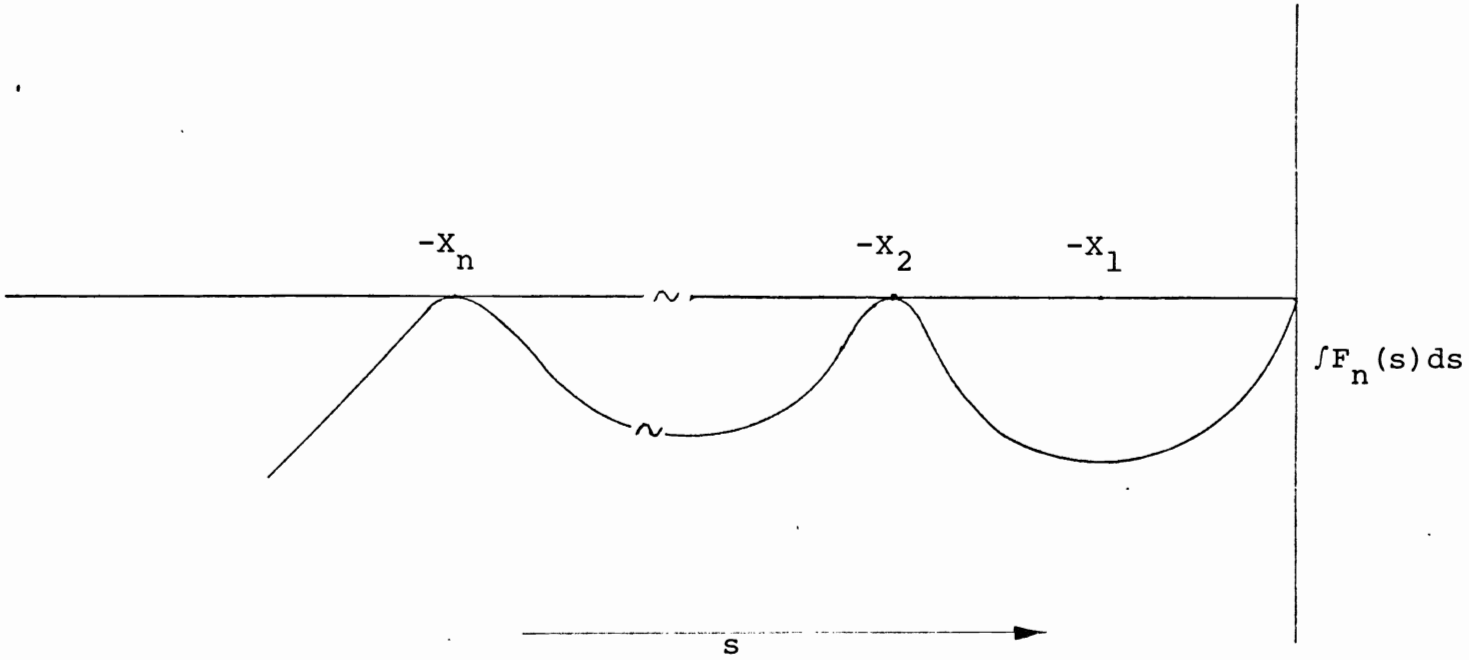

Fig.3.4(a): The Medial $F_n(s)$ for n even

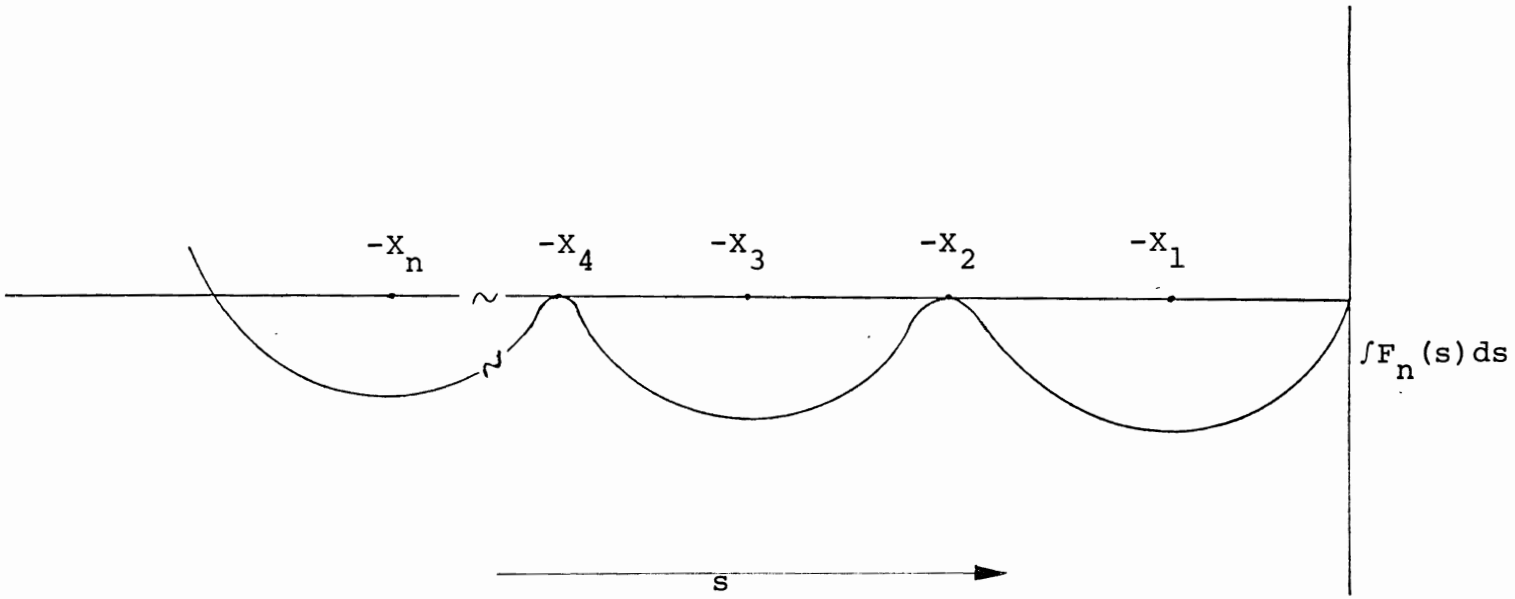

Fig.3.4(b): The Medial $F_n(s)$ for n odd

Now we write the first two rows of an array as,

$$\begin{array}{cccccc} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_o \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \cdots & a_1 & a_o \end{array}$$

where the second row consists of the coefficients of $F'_n(s)$. If the above array is treated as the first two rows of the Routh-Hurwitz array, then, we have the following theorem.

<u>Theorem 3.4</u>:

The number of zeros of $F_n(s)$ not on the negative real axis is equal to the number of sign changes in the first column of the Routh-Hurwitz array.

<u>Proof</u>:

Let $F_n(s)$ contain only negative real axis zeros. Then $\frac{F'_n(s)}{F(s)}$ will be an RC impedance function. Therefore, $\frac{sF'_n(s^2)}{F_n(s^2)}$ will be a reactance function and $F_n(s^2) + sF'_n(s^2)$ will be a HP. Applying the Routh-Hurwitz algorithm to $F_n(s^2) + sF'_n(s^2)$, we will have the first column without any change of sign. Suppose $F_n(s)$ has k zeros which do not lie on the negative real axis. Then the $s \to s^2$ transformation will result in the polynomial $F_n(s^2) + sF'_n(s^2)$ having k zeros in the right half plane. This will lead to k changes of sign in the first column of the array (3.44). Hence the theorem follows.

Example 3.4:

To investigate the RC integrability of $F_3(s)$ given as:

$$F_3(s) = s^3 + 15s^2 + 59s + 45 \qquad ..(3.57)$$

$F_3(s)$ is first integrated to obtain,

$$\int F_3(s)ds + K = 0.25s^4 + 5s^3 + 29.5s^2 + 45s + K \qquad ..(3.58)$$

The modified Routh-Hurwitz array can now be written as:

0.25	5	29.5	45	K	
1.00	15	59.00	45	0	
1.25	14.75	33.75	K	0	
3.20	3.20	$\frac{56.25-K}{1.25}$	0	0	
2.25	$\frac{51.75+K}{3.2}$	K	0	0	..(3.59)
$\frac{20.25-K}{2.25}$	$\frac{101.25-5K}{2.25}$	0	0	0	
$\frac{35.75+K}{3.2}$	K	0	0	0	
$\frac{1}{2.25}\frac{20.25-K}{35.75+K}(178.75-1.8K)$	0	0	0	0	

It follows that $\int F(s)ds + K$ will be a RC polynomial for $0 \leq K < 20.25$. Hence, $F(s)$ is RC integrable for this range of K.

3.7 Some Properties of the RC Integrable Polynomials[22]:

Theorem 3.3:

Let $B_1(s)$ and $B_2(s)$ be RC integrable such that,

$$A_1(s) = \int B_1(s)ds + K_1$$
$$\text{and} \quad A_2(s) = \int B_2(s)ds + K_2 \qquad ..(3.60)$$

contain only negative real axis zeros.

If

$$\frac{A_1(s^2) + sB_1(s^2)}{A_2(s^2) + sB_2(s^2)} \qquad ..(3.61)$$

is a PRF, the functions

$$\text{(a)} \quad \frac{A_1(s) + B_1(s)}{A_2(s) + B_2(s)}, \frac{A_1(s) + B_2(s)}{A_2(s) + B_1(s)} \qquad ..(3.62)$$

$$\text{and} \quad \text{(b)} \quad \frac{A_1^2(s) + sB_1^2(s)}{A_2^2(s) + sB_2^2(s)}, \frac{A_1^2(s) + sB_2^2(s)}{A_2^2(s) + sB_1^2(s)} \qquad ..(3.63)$$

are positive real, while the function

$$\text{(c)} \quad \frac{A_1^2(s) + B_1^2(s)}{A_2^2(s) + B_2^2(s)} \qquad ..(3.64)$$

may not necessarily be positive real.

<u>Proof</u>:

(a) Since $\frac{A_1(s)}{B_1(s)}$ is an RC admittance function, the zeros of $A_1(s) + B_1(s)$ designated as $-\alpha_i$'s are simple and lie on the negative real axis. The same holds for the zeros of $A_2(s) + B_2(s)$ which are designated as $-\beta_i$'s. Now let,

$$\begin{aligned} A_1(s) &= \pi(s + a_{1i}) \\ A_2(s) &= \pi(s + a_{2i}) \\ B_1(s) &= \pi(s + b_{1i}) \\ B_2(s) &= \pi(s + b_{2i}) \end{aligned} \qquad ..(3.65)$$

where

$$a_{1i} < b_{1i}$$
$$a_{2i} < b_{2i} \qquad ..(3.66)$$
$$a_{1i} < b_{2i}$$
$$a_{2i} < b_{1i}$$

Let us now examine the zeros of $A_1(s)$, $A_2(s)$ and $B_2(s)$. From inequality (3.66), it follows that the zeros of $A_1(s)$ and $A_2(s)$ as a block interlace with the zeros of $B_1(s)$ and $B_2(s)$ as a block. The zeros of $A_1(s) + B_1(s)$ or $-\alpha_i$'s can, therefore, interlace at the most in pairs with the zeros of $A_2(s) + B_2(s)$ or $-\beta_i$'s. We now write the overall function as,

$$\frac{A_1(s) + B_1(s)}{A_2(s) + B_2(s)} = \frac{(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)\ldots\ldots}{(s+\beta_1)(s+\beta_2)(s+\beta_3)\ldots\ldots} \qquad ..(3.67)$$

This can always be arranged as,

$$\frac{\dfrac{(s+\alpha_1)\ \ldots}{(s+\beta_1)\ \ldots}}{\dfrac{(s+\beta_2)\ \ldots}{(s+\alpha_2)\ \ldots}} \qquad ..(3.68)$$

which is of the form $\frac{Y_{RC1}}{Y_{RC2}}$ and the latter is a positive real function[23].

(b) We have

$$\frac{A_1^2(s) + sB_1^2(s)}{A_1(s)B_1(s)} = \frac{A_1(s)}{B_1(s)} + \frac{sB_1(s)}{A_1(s)} \qquad ..(3.69)$$

which is the sum of two RC admittance function. Therefore,

the negative real axis zeros of $A_1^2(s) + sB_1^2(s)$, designated as $-\gamma_i$'s, alternate with those of $A_1(s)B_1(s)$. For the same reason, the zeros of $A_2^2(s) + sB_2^2(s)$, designated as $-\delta_i$'s, alternate with those of $A_2(s)B_2(s)$. From the inequality (3.66), the zeros of $A_1(s)$ and $A_2(s)$ as a block interlace with those of $B_1(s)$ and $B_2(s)$ as a block. Therefore, the zeros of $A_1(s)B_1(s)$ interlace at the most in pairs with those of $A_2(s)B_2(s)$. This leads to the fact that γ_i's and δ_i's themselves alternate at the most in pairs. Proceeding as in (a) above, we can similarly show that

$$\frac{A_1^2(s) + sB_1^2(s)}{A_2^2(s) + sB_2^2(s)}$$

is a positive real function.

(c) This will be illustrated by means of a counter-example:

Let
$$\frac{A_1(s^2) + sB_1(s^2)}{A_2(s^2) + sB_2(s^2)} = \frac{s^2 + a + sb}{s^2 + c + sd} \qquad ..(3.70)$$

Then,
$$\frac{A_1^2(s) + B_1^2(s)}{A_2^2(s) + B_2^2(s)} = \frac{(s + a)^2 + b^2}{(s + c)^2 + d^2} \qquad ..(3.71)$$

If we now choose $a = 1$, $b = 10$, $c = 1$ and $d = 1$, then the condition for (3.70) to be positive real[24], namely

$$bd \geq (\sqrt{a} - \sqrt{c})^2$$

is satisfied, while the same for the positive realness of (3.71), namely,

$$4ac \geq (\sqrt{a^2 + b^2} - \sqrt{c^2 + d^2})^{\frac{1}{2}} \qquad ..(3.71)$$

does not hold good.

Hence, the theorem is proved.

It may be noted that $A_1(s)$, $B_1(s)$ and $A_2(s)$, $B_2(s)$ could, in general, be the two constituent parts of the Calahan decomposition[25] or the Horowitz decomposition[26].

Theorem 3.4:

If $F_n(s)$ is RC integrable, then and only then,

$$sF_n(s^2) + \int sF_n(s^2)ds + K \qquad ..(3.72)$$

is a HP.

Proof:

(a) Necessity: We have,

$$I = \int_0^{j\sqrt{X_i}} sF_n(s^2)ds = \int_0^{j\sqrt{X_i}} s(s^2+X_1)(s^2+X_2) \ldots (s^2+X_n)ds \qquad ..(3.73)$$

Substituting $s^2 = p$, we have,

$$I = \frac{1}{2} \int_0^{-X_i} (p + X_1)(p + X_2) \ldots (p + X_n)dp \qquad ..(3.74)$$

Hence, (3.74) can be rewritten as:

$$\int_0^{j\sqrt{X_i}} sF_n(s^2)ds = \frac{1}{2} \int_0^{-X_i} F_n(s)ds \qquad ..(3.75)$$

It follows from (3.75) that, if $F_n(s)$ is RC integrable, $sF_n(s^2)$ is integrable. Therefore, $\dfrac{\int sF_n(s^2)ds + K}{sF_n(s^2)}$ is a reactance function.

Hence, the theorem follows.

(b) Sufficiency: If $sF_n(s^2) + \int sF_n(s^2)ds + K$ is a HP, then $\dfrac{\int sF_n(s^2)ds + K}{sF_n(s^2)}$ is a reactance function and hence $sF_n(s^2)$ is integrable. From (3.75), it now follows that $F_n(s)$ is RC integrable.

Hence, the theorem follows.

An important implication of Theorem 3.4 lies in the fact that it indicates the basic similarity between the integrability tests of N_n and the RC integrability tests of $F_n(s)$. From an integrable N_n, a RC integrable $F_n(s)$ can be readily constructed as:

$$F_n(s) = \left.\frac{N_n}{s}\right|_{s^2 \to s} \qquad ..(3.76)$$

Similarly, from a RC integrable $F_n(s)$, we could obtain an integrable N_n as:

$$N_n = s\ F_n(s^2) \qquad ..(3.77)$$

It follows from Section 3.6 that should a given N_n be not available in the factored form, the corresponding $F_n(s)$

obtained from (3.76) could be tested for its integrability using Theorem 3.4 in order to ascertain the integrability of N_n.

3.8 Conclusions:

This Chapter has dealt with the integrability of RC polynomials $F_n(s)$, whose zeros are known. It is found that on integration, an RC polynomial may yield a HP under certain conditions or under more stringent conditions, it may yield another RC polynomial. The conditions under which either of these cases occur have been discussed. If $F_n(s)$ is Hurwitz integrable, then under certain conditions, it is possible to express $\frac{\int F_n(s)ds + K}{F_n(s)}$ as the sum of RC and RL driving point immitance functions.

If the zeros of $F_n(s)$ are not known, a modified Routh-Hurwitz criterion has been employed to establish its RC integrability. Also some properties of RC integrable polynomials have been discussed.

CHAPTER IV

THE DIFFERENTIALS AND INTEGRALS OF THE REAL AND IMAGINARY PARTS OF POSITIVE REAL FUNCTIONS

4.1 Introduction:

It has been observed earlier that some properties of network functions are invariant under polynomial differentiation while the same is not the case under polynomial integration. The investigation of such invariances in the properties of the real and imaginary parts of a PRF under the operations of differentiation and integration appear to be of interest and is carried out in this Chapter[27].

4.2 Real and Imaginary Parts of a Positive Real Function:

Let $Z(s) = \dfrac{m_1 + n_1}{m_2 + n_2}$ be a PRF. Then the real and imaginary parts of $Z(s)$ are:

$$\text{Re } Z(j\omega) = \left.\frac{m_1 m_2 - n_1 n_2}{m_2^{\,2} - n_2^{\,2}}\right|_{s = j\omega} = \frac{A(\omega^2)}{B(\omega^2)} = R(\omega) \qquad ..(4.1)$$

$$j\text{Im } Z(j\omega) = \left.\frac{m_2 n_1 - m_1 n_2}{m_2^{\,2} - n_2^{\,2}}\right|_{s = j\omega} = j\,\frac{\omega C(\omega^2)}{B(\omega^2)} = jX(\omega) \qquad ..(4.2)$$

It is known that $R(\omega)$ is positive and bounded for all positive values of ω.

$$\text{Let} \quad R(\omega) = \frac{a_o + a_2\omega^2 + a_4\omega^4 + \ldots + a_{2r}\omega^{2r}}{b_o + b_2\omega^2 + b_4\omega^4 + \ldots + b_{2t}\omega^{2t}}, \; r \leq t \qquad ..(4.3)$$

The coefficients a's and b's can be positive, negative or zero

with the restriction that a_o, b_o and the coefficients of ω^{2r} and ω^{2t} must each be non-negative. The numerator $A(\omega^2)$ may contain factors of the type:

(i) $\omega^2 + \delta_1$, δ_1 real and positive

(ii) $(\omega^2 + \delta_2)^{2n}$, $n = 1,2,\ldots,$ and δ_2 real and negative

(iii) $(\omega^2 + \delta_3)(\omega^2 + \bar{\delta}_3)$, δ_3 complex

The denominator may contain factors of the type (i) and (iii) only as it cannot vanish for any real and positive value of ω. A polynomial with factors of the type (i) will have all its terms positive. If factors of the type (ii) are contained in a polynomial, some terms will be definitely negative. No augmentation may help to get all the terms positive in this case[28,29]. The factors of the type (iii) which may be contained in the numerator or the denominator, may yield terms with negative coefficients if the real part of δ_3 is negative. However, any negativeness caused by factors of this type can be removed by suitable augmentation of the numerator and the denominator by $\pi(\omega^2 + \delta_i)$, $\delta_i > 0$. Thus, it follows that all the coefficients of $B(\omega^2)$ can always be made positive and those of $A(\omega^2)$ can also be made positive except when it contains factors of the type (ii).

4.3 Operations on the Real Part of a Positive Real Function:

(i) New function obtained as the polynomial differential of the real part:

This may be obtained with respect to ω or ω^2. However, it can be shown that:

$$\frac{\frac{dA(\omega^2)}{d\omega}}{\frac{dB(\omega^2)}{d\omega}} = \frac{\frac{dA(\omega^2)}{d\omega^2}}{\frac{dB(\omega^2)}{d\omega^2}} = \frac{E(\omega^2)}{F(\omega^2)} \qquad ..(4.4)$$

If $A(\omega^2)$ contained a factor of the type (ii), that is of the form $(\omega^2 + \delta_2)^{2n}$, then $E(\omega^2)$ will be negative in the immediate neighbourhood of $\omega^2 = |\delta_2|$. In this case, (4.4) would not be in an acceptable form as the real part of another PRF. In all the other cases, $A(\omega^2)$ and $B(\omega^2)$ could be suitably augmented, if necessary, to have all their terms positive, in which case, $E(\omega^2)$ and $F(\omega^2)$ will contain only positive terms, making the derived function (4.4) suitable as the real part of a PRF. Miyata's procedure[30] can therefore be always employed in order to synthesize the PRF obtained from the real part.

<u>Example 4.1</u>:

$$\text{Consider } R_1(\omega) = \text{Re } Z(j\omega) = \frac{1 + \omega^4}{(1 + \omega^2)^2} \qquad ..(4.5)$$

(4.5) corresponds to a driving point function

$$Z_1(s) = \frac{s^2 + s + 1}{s^2 + 2s + 1} \qquad ..(4.6)$$

A realization of (4.6) has the form given in Fig.4.1.

The polynomial differential of (4.5) with respect to ω or ω^2 yields

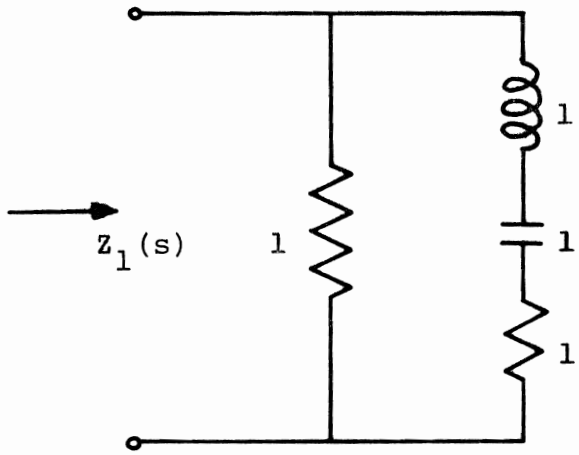

Fig.4.1: A realization of $Z_1(s)$

$$R_2(\omega) = \frac{\omega^2}{1 + \omega^2} \qquad ..(4.7)$$

(4.7) corresponds to a driving point function

$$Z_2(s) = \frac{s}{1 + s} \qquad ..(4.8)$$

A realization of (4.8) has been given in Fig.4.2.

(ii) New function obtained as the total differential of the real part:

(a) Differentiation carried out with respect to ω:

We have,

$$\frac{d}{d\omega}\{\frac{A(\omega^2)}{B(\omega^2)}\} = \frac{\{\frac{d}{d\omega}A(\omega^2)\}B(\omega^2) - \{\frac{d}{d\omega}B(\omega^2)\}A(\omega^2)}{B^2(\omega^2)} \qquad ..(4.9)$$

Since the numerator is no longer an even function of ω, the function is not suitable for being the real part of a PRF. This can, however, form the odd part of a PRF.

(b) Differentiation carried out with respect to ω^2:

This gives

$$\frac{d}{d\omega^2}\{\frac{A(\omega^2)}{B(\omega^2)}\} = \frac{\{\frac{d}{d\omega^2}A(\omega^2)\}B(\omega^2) - \{\frac{d}{d\omega^2}B(\omega^2)\}A(\omega^2)}{B^2(\omega^2)} \qquad ..(4.10)$$

While the numerator in this case is an even function of ω, it must satisfy the additional condition of being non-negative for all ω such that $0 \leq \omega < \infty$. The condition required to be fulfilled is,

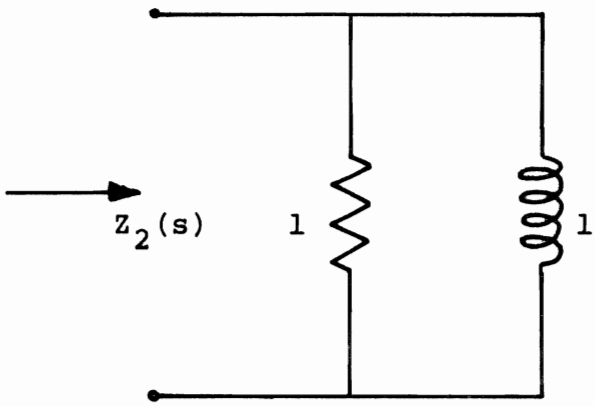

Fig.4.2: A realization of $Z_2(s)$

$$\frac{\frac{d}{d\omega^2} A(\omega^2)}{A(\omega^2)} \geq \frac{\frac{d}{d\omega^2} B(\omega^2)}{B(\omega^2)} \quad \text{for } 0 \leq \omega < \infty \qquad ..(4.11)$$

when the new function will be suitable as the real part of a PRF.

Example 4.2:

Considering (4.5), the operation carried out with respect to ω gives an odd function. On the other hand, since the condition expressed by (4.11) is not satisfied, $\frac{d}{d\omega^2}\{R_1(\omega)\}$ is also unsuitable as the real part of another PRF.

(iii) New function obtained as the polynomial integral of the real part:

(a) Integration carried out with respect to ω:

Integrating the numerator and the denominator of (4.3) term by term,

$$\frac{\int A(\omega^2)\, d\omega}{\int B(\omega^2)\, d\omega} \qquad ..(4.12)$$

can be put in the form $\frac{G(\omega^2)}{H(\omega^2)}$, provided the constants associated with integration are assumed to be zero so that ω in the numerator and the denominator are cancelled. $G(\omega^2)$ is seen to be non-negative for $0 \leq \omega < \infty$. This follows because $\{\omega G(\omega^2)\}$ can be viewed as the area under the curve $A(\omega^2)$ against ω from $\omega = 0$ upto ω. Since $A(\omega^2)$ is non-negative over this range, $\omega G(\omega^2)$ is also non-negative over the same. Hence the non-negativeness of $G(\omega^2)$ follows. Similar argument is applicable

to $H(\omega^2)$ which comes out to be always positive and finite for ω in the same range. Hence (4.12) corresponds to the real part of a PRF.

(b) Integration carried out with respect to ω^2:

This gives,

$$\frac{\int A(\omega^2)\,d\omega^2}{\int B(\omega^2)\,d\omega^2} = \frac{I(\omega^2)}{J(\omega^2)} \qquad ..(4.13)$$

In this case, arbitrary but non-negative constants of integration may be assumed. Associating $I(\omega^2)$ and $J(\omega^2)$ with the areas under the curves $A(\omega^2)$ and $B(\omega^2)$ against ω^2 respectively, we can, at once, see that the new function $I(\omega^2)$ is non-negative for $0 \leq \omega < \infty$, while $J(\omega^2)$ is always positive over the same range. All the required conditions for $\frac{I(\omega^2)}{J(\omega^2)}$ to form the real part of a positive real function are, therefore, met. Augmentation may be employed, if necessary, to render all the terms of the numerator and the denominator positive. Miyata's procedure can, therefore, be employed in order to synthesize the PRF obtained from the real part.

<u>Example 4.3</u>:

The polynomial integral of $R_1(\omega)$ in (4.5), carried out with respect to ω yields,

$$R_3(\omega) = \frac{\omega^4 + 5}{\omega^4 + 3.33333\omega^2 + 5} = \frac{G(\omega^2)}{H(\omega^2)} \qquad ..(4.14)$$

$R_3(\omega)$ corresponds to a driving point function

$$Z_3(s) = \frac{s^2 + 1.60073s + 2.23610}{s^2 + 2.79382s + 2.23604} \qquad ..(4.15)$$

A realization of $Z_3(s)$ has been furnished in Fig. 4.3.

The same operation carried out with respect to ω^2 gives,

$$R_4(\omega) = \frac{\omega^4 + 3}{\omega^4 + 3\omega^2 + 3} = \frac{I(\omega^2)}{J(\omega^2)} \qquad ..(4.16)$$

where the constants of integration have been assumed to be zero.

A convenient method of synthesizing $Z_4(s)$ of which $R_4(\omega)$ forms the real part, is furnished by the Miyata's procedure. Decomposing (4.16) as,

$$R_4(\omega) = \frac{\omega^4}{J(\omega^2)} + \frac{3}{J(\omega^2)} \qquad ..(4.17)$$

we obtain the realization for each term separately. The realization of $Z_4(s)$ is shown in Fig. 4.4.

(iv) New function obtained as the total integral of the real part:

(a) Integration carried out with respect to ω:

Consider

$$\int \frac{A(\omega^2)}{B(\omega^2)}\, d\omega. \qquad ..(4.18)$$

This integral will not be an even function of ω, and therefore, it will be unsuitable for being the real part of any PRF. In general, the integral will be a transcendental or an inverse

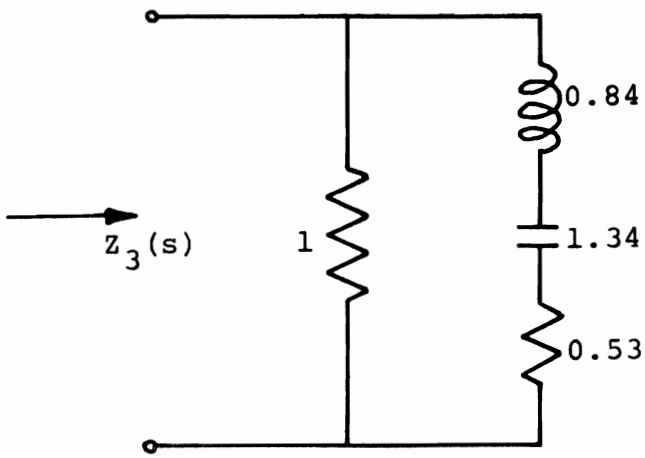

Fig.4.3: A realization of $Z_3(s)$

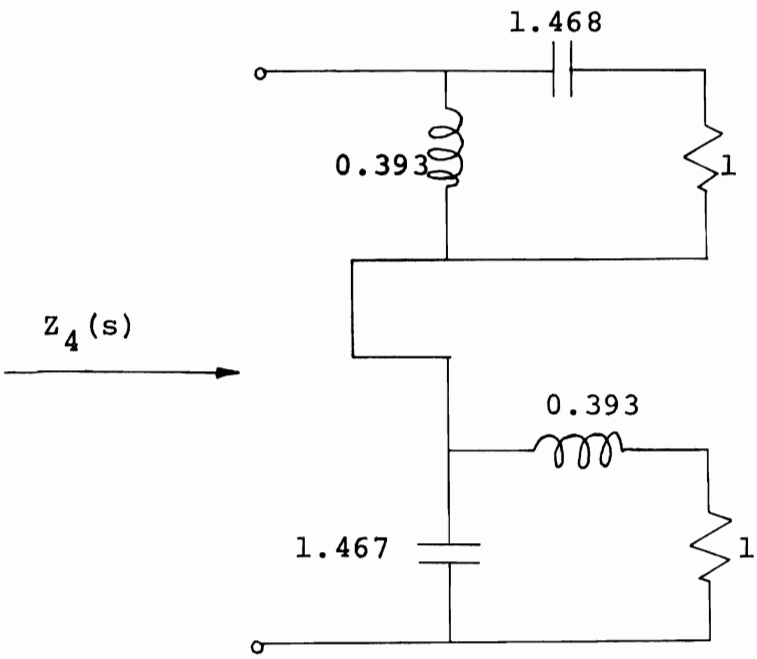

Fig.4.4: A Realization of $Z_4(s)$

trigonometric function.

(b) Integration carried out with respect to ω^2:

In general,

$$\int \frac{A(\omega^2)}{B(\omega^2)} d\omega^2 \qquad ..(4.19)$$

cannot be in the form $\frac{K(\omega^2)}{L(\omega^2)}$, where $K(\omega^2)$ and $L(\omega^2)$ are of the same form as the numerator and the denominator of (4.3) respectively. The new function will not be therefore suitable as the real part of a PRF. An exception occurs when the numerator is a constant and the denominator a perfect square or a perfect cube etc. In this case, using the absolute value of the integral as the real part, a PRF can be constructed.

4.4 Operations on the Imaginary Part of a Positive Real Function:

The numerator of Im $Z(j\omega)$ is an odd polynomial in ω, while the denominator is even. For this reason, the operation of differentiation or integration performed on them will yield new functions which do not maintain this relationship even after any cancellation that might take place. We, therefore, consider $\frac{\text{Im } Z(j\omega)}{\omega}$ and after desired operations on this function are made, investigate the possibility of the resultant function multiplied by ω being the imaginary part of another PRF. New functions are obtained by suitable operations on

$$\frac{\text{Im } Z(j\omega)}{\omega} = \frac{C(\omega^2)}{B(\omega^2)} \qquad ..(4.20)$$

(i) New function obtained as the polynomial differential of (4.20):

As before,

$$\frac{\frac{dC(\omega^2)}{d\omega}}{\frac{dB(\omega^2)}{d\omega}} = \frac{\frac{dC(\omega^2)}{d\omega^2}}{\frac{dB(\omega^2)}{d\omega^2}} \qquad ..(4.21)$$

If the numerator of (4.21) is multiplied by ω, the resulting function will be suitable as the odd part of a PRF.

(ii) New function obtained as total differential:

(a) Differentiation carried out with respect to ω:

This gives

$$\frac{d}{d\omega}\frac{A(\omega^2)}{B(\omega^2)} = \frac{\{\frac{d}{d\omega}A(\omega^2)\}\,B(\omega^2) - \{\frac{d}{d\omega}B(\omega^2)\}\,A(\omega^2)}{B^2(\omega^2)} \qquad ..(4.22)$$

The numerator of (4.22) is an odd function and further multiplication by ω makes it even, thereby rendering it unsuitable as the imaginary part of another PRF. However, this will be suitable as the real part of a PRF (containing a zero at the origin) provided

$$\frac{B(\omega^2)}{A(\omega^2)} \geq \frac{\frac{d}{d\omega}B(\omega^2)}{\frac{d}{d\omega}A(\omega^2)} \qquad \text{for } 0 \leq \omega < \infty \qquad ..(4.23)$$

(b) Differentiation carried out with respect to ω^2:

This gives

$$\frac{d}{d\omega^2}\left\{\frac{A(\omega^2)}{B(\omega^2)}\right\} = \frac{\left\{\frac{d}{d\omega^2} A(\omega^2)\right\} B(\omega^2) - \left\{\frac{d}{d\omega^2} B(\omega^2)\right\} A(\omega^2)}{B^2(\omega^2)} \qquad ..(4.24)$$

Multiplication of the numerator of (4.24) by ω makes it an odd function. A PRF therefore can be constructed by using the new function as its imaginary part.

(iii) New function obtained as the polynomial integral of (4.20):

The two cases arising out of the integration being carried out with respect to ω or ω^2 are corresponding to the two cases of the real part respectively, as discussed in Section 4.3. Similar arguments will show that the new functions are suitable as the imaginary parts of PRF's.

(iv) New function obtained as total integral:

Refering to cases (iv)a and (iv)b of the real part in Section 4.3, it can be observed that whether the integration is carried out with respect to ω or ω^2, the resultant function will not be, in general, suitable for the construction of a PRF. An exception occurs when the integration is being performed with respect to ω^2, the numerator is a constant and the denominator a perfect square, a perfect cube etc. In this case, the integral multiplied by ω will form the imaginary part of some PRF.

4.5 Conclusions:

The properties of eight new functions generated after the

different operations of differentiation and integration from the real or imaginary part of a PRF have been examined. It is found that the polynomial differentials of the real part of a PRF are conditionally suitable as the real part of another PRF. However, if polynomial integrations are employed to get the new function, it will always form the real part of another PRF. Total differentials and total integrals of the real part do not, in general, yield functions suitable as the real part of some PRF. In conclusion, it can be said that the properties of the real part are in a sense invariant with respect to polynomial integration but not with respect to polynomial differentiation. This furnishes a sharp contrast to the PRF itself, which retains invariance of positive realness under polynomial differentiation rather than under polynomial integration.

Almost analogous conclusions have been reached for the imaginary part of a PRF. In this case, $\frac{\text{Im } Z(j\omega)}{\omega}$ has been considered instead of Im $Z(j\omega)$ and the new function has been multiplied by ω before considering its suitability as the imaginary part of a PRF. The only important difference lies in the fact that both polynomial differentiation and polynomial integration yield invariant functions, i.e. functions which are suitable as the imaginary part of a PRF.

The accompanying Tables 4.1 and 4.2 summarize these results.

TABLE 4.1

FUNCTIONS GENERATED FROM THE REAL PART

Starting Function	Derived Function	Operation carried out with respect to ω or ω^2	Whether the new function is suitable as being the real part of another positive real function
Real part of a positive real function, General form is $\frac{A(\omega^2)}{B(\omega^2)} = R(\omega^2)$	Polynomial Differential of $R(\omega^2)$	ω	Conditionally suitable
		ω^2	Conditionally suitable
	Total Differential of $R(\omega^2)$	ω	Not suitable
		ω^2	Conditionally suitable
	Polynomial Integral of $R(\omega^2)$	ω	Always suitable
		ω^2	Always suitable
	Total Integral of $R(\omega^2)$	ω	Not suitable
		ω^2	Not suitable

TABLE 4.2

FUNCTIONS GENERATED FROM THE IMAGINARY PART

Starting Function	Derived Function	Operation carried out with respect to ω or ω^2	Whether the new function multiplied by ω will be suitable as the imaginary part of another positive real function
General form is $\frac{C(\omega^2)}{B(\omega^2)} = \frac{x(\omega^2)}{\omega}$ where $x(\omega^2)$ is is the imaginary part of a positive real function.	Polynomial Differential of $x(\omega^2)/\omega$	ω	Always suitable
		ω^2	Always suitable
	Total Differential of $x(\omega^2)/\omega$	ω	Not suitable
		ω^2	Always suitable
	Polynomial Integral of $x(\omega^2)/\omega$	ω	Always suitable
		ω^2	Always suitable
	Total Integral of $x(\omega^2)/\omega$	ω	Not suitable
		ω^2	Not suitable

CHAPTER V

POLYNOMIAL DECOMPOSITIONS USING INTEGRALS OF HURWITZ POLYNOMIALS

5.1 Introduction:

So far, the integrability conditions of HP's and RC polynomials have been discussed. In this Chapter, two new decompositions of a HP are presented and possible applications discussed. The first of these, the "Integral Decomposition", expresses a HP as the sum of another integrable HP and its integral. The Integral Decomposition is used to provide an alternative to Weinberg's[31,32,33] procedure for synthesizing transfer functions by symmetrical lattice structures containing lossy coils. The second decomposition, termed the "Integro-differential Decomposition" expresses a HP Q(s) as the sum of another integrable HP F(s), its derivative and its integral. A possible use of this decomposition is also indicated.

5.2 The Integral Decomposition of a HP:

The decomposition of a HP into the sum of another HP and its derivative has been shown to exist[32]. This has been used to realize transfer functions or immittances by symmetrical lattice structures with lossy coils. The Integral Decomposition and its use in the synthesis of symmetrical lattices will now be discussed.

Theorem 5.1:

Given a HP, Q(s), which does not contain only imaginary

axis zeros, it is possible to decompose Q(s) as:

$$Q(s) = A\,F(s) + G\int F(s)ds \qquad ..(5.1)$$

where $\int F(s)ds$ is a HP and includes a constant of integration, A and G being positive real constants.

Proof:

The existence of the above decomposition is always guaranteed by a theorem in algebra which states that the roots of a polynomial are continuous functions of the coefficients[2]. If A is made equal to zero, then $\int F(s)ds = \frac{Q(s)}{G}$ which is certainly Hurwitz. If A is increased slightly, $\int F(s)ds$ will still be a HP. However, beyond a certain maximum value of A, the roots of $\int F(s)ds$ may move to the right half of the s-plane, in which case, $\int F(s)ds$ is no longer Hurwitz. The procedure for obtaining the decomposition is now described:

$$\text{Let} \quad Q(s) = s^m + C_{m-1}s^{m-1} + C_{m-2}s^{m-2} + .. + C_1 s + C_o \qquad ..(5.2)$$

Assume that

$$F(s) = s^{m-1} + d_{m-2}s^{m-2} + d_{m-3}s^{m-3} + ... + d_2 s^2 + d_1 s + d_o \qquad ..(5.3)$$

where the d_i's are to be determined. Then

$$AF(s) + G\int F(s)ds = \frac{G}{m}s^m + (G\frac{d_{m-2}}{m-1} + A)s^{m-1} + (G\frac{d_{m-3}}{m-2} + Ad_{m-2})s^{m-2}$$
$$+ ... + (\frac{Gd_1}{2} + Ad_2)s^2 + (Gd_o + Ad_1)s$$
$$+ Ad_o + GK \qquad ..(5.4)$$

There are (m + 2) unknowns on the right hand side of (5.4) and once a suitable choice for the constant A has been made, the remaining (m + 1) unknowns can be determined one by one using the following set of equations:

$$\frac{G}{m} = 1$$

$$G\,\frac{d_{m-2}}{m-1} + A = C_{m-1}$$

$$G\,\frac{d_{m-3}}{m-2} + Ad_{m-2} = C_{m-2}$$

$$\cdots\cdots\cdots\cdots\cdots \qquad ..(5.5)$$

$$\frac{Gd_1}{2} + Ad_2 = C_2$$

$$Gd_o + Ad_1 = C_1$$

$$GK + Ad_o = C_o$$

Alternatively, it is possible to solve equations (5.5) simultaneously after making an initial choice of K. Thus, it is seen that there exists a definite implicit relationship between A and K. Depending on the choice of K, A will vary and vice versa*.

Since the initial choice of A (or K) is somewhat arbitrary, it is essential to confirm that $\int F(s)ds$ is a HP. This may be accomplished by using the integrability conditions of F(s) developed earlier. Should F(s) be unintegrable, it follows that the initial choice of A was too high. The d_i's are then

*An upper bound for A is the minimum of $\{C_i/d_i\}$, i = 0,1,2,..m-1.

computed again with a lower value of A and the operation repeated until the new F(s) comes out to be integrable.

It follows that the integrability conditions of F(s) help in the maximization of A by rejecting an unintegrable F(s).

The Integral Decomposition described above can be used as an alternative to the Differential Decomposition of Weinberg. The following example provides an application of this procedure in the synthesis of a voltage transfer function by an open-circuited symmetrical lattice.

Example 5.1:

It is required to realize the following voltage transfer function:

$$\frac{E_2}{E_1} = \frac{s-3}{H(s^4 + 12s^3 + 54s^2 + 108s + 80)} \qquad ..(5.6)$$

The maximization of the gain constant $\frac{1}{H}$ can be achieved as described above. It is equal to 6.94. Suppose for the purpose of this example that A is chosen to be 5. Then from (5.5),

$$\begin{aligned} G &= 4.0 \\ d_2 &= 5.25 \\ d_1 &= 13.875 \\ d_o &= 9.6562 \\ K &= 7.93 \end{aligned} \qquad ..(5.7)$$

Hence, the decomposition can be written as:

$$Q(s) = AF(s) + G\int F(s)ds$$

$$= 5\{s^3 + 5.25s^2 + 13.875s + 9.6562\}$$

$$+ 4\{0.25s^4 + 1.75s^3 + 6.9375s^2 + 9.6562s + 7.93\} \quad ..(5.8)$$

$\int F(s)ds$ can be seen to be a HP.

From (5.6),

$$\frac{E_2}{E_1} = \frac{\dfrac{s-3}{s^4+7s^3+27.75s^2+38.6248s+31.72}}{H\{1 + 1.25\,\dfrac{s^3+5.25s^2+13.875s+9.6562}{0.25s^4+1.75s^3+6.9375s^2+9.6572s+7.93}\}}$$

$$= \frac{N_r}{D_r} \quad ..(5.9)$$

The partial fraction expansion of the numerator and the denominator of (5.9) gives,

$$N_r = \frac{-0.085 - j0.0039}{s + 2.6301 + j3.1027} + \frac{-0.085 + j0.0039}{s + 2.6301 - j3.1027}$$

$$+ \frac{0.0848 - j0.1282}{s + 0.87 + j1.0711} + \frac{0.0848 + j0.1282}{s + 0.87 - j1.0711} \quad ..(5.10)$$

and $$D_r = H\{1 + 1.25(\frac{1}{s + 2.6301 + j3.1027} + \frac{1}{s + 2.6301 - j3.1027}$$

$$+ \frac{1}{s + 0.87 + j1.0711} + \frac{1}{s + 0.87 - j1.0711})\}$$

It remains now to compute the permissible range of H for each pole and choose a value which satisfies all the conditions[34] for the realization of $\frac{E_2}{E_1}$ as $\frac{Z_B - Z_A}{Z_B + Z_A}$, where Z_A and Z_B are positive real. The stronger condition yields,

$$H > 0.144.$$

Choosing H = 1, we obtain,

$$Z_A = 0.5 + \cfrac{1}{0.7507s + \cfrac{1}{0.3381 + \cfrac{1}{0.1174s + 0.1552}}}$$

$$Z_B = 0.5 + \cfrac{1}{0.8582s + \cfrac{1}{1.5491 + \cfrac{1}{1.0035s + 0.9913}}}$$

Therefore, the realization of the lattice is as shown in Fig. 5.1.

It is well known that under certain conditions, a lattice can be equivalently realized by an unbalanced network. Hence the unbalanced form of realization can be obtained using the above technique under those conditions.

5.3 An Integro-differential Decomposition of a HP:

In the previous section, it was seen that it is possible to obtain Differential and Integral Decompositions of a HP. These two aspects may be combined to obtain an Integro-differential type of a decomposition where a HP is expressed as the sum of another HP, its derivative and its integral, which is also a HP.

Theorem 5.2:

Given a HP, Q(s), which does not contain only imaginary axis zeros, it is possible to decompose Q(s) as:

$$Q(s) = AF(s) + BF'(s) + G\int F(s)ds \qquad ..(5.11)$$

where $\int F(s)ds$ is a HP and includes a suitable constant of integration, A,B and G being positive real constants.

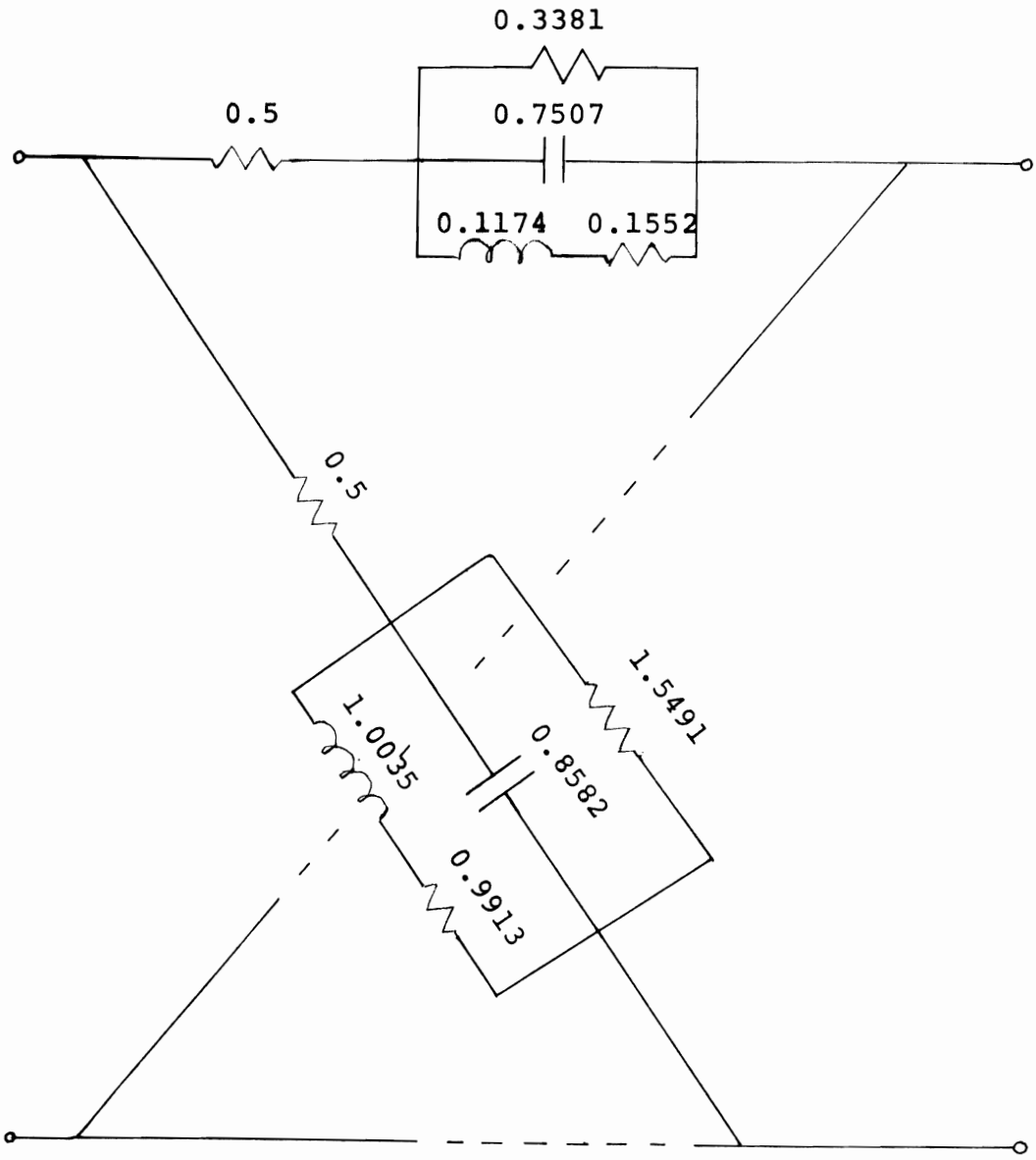

Fig.5.1: Symmetrical Lattice Realization of Voltage Transfer Function using Integral Decomposition

Proof: Let the form of Q(s) be as in (5.2) and let F(s) be as in (5.3). The d_i's are to be determined.

We have,

$$AF(s) + BF'(s) + G\int F(s)ds = \frac{G}{m} s^m + \{G \frac{d_{m-2}}{m-1} + A\}s^{m-1}$$

$$+ \{G \frac{d_{m-3}}{m-2} + Ad_{m-2} + B(m-1)\}s^{m-2}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ (Gd_o+Ad_1+2Bd_2)s + (Ad_o+GK+Bd_1) \qquad ..((5.12)$$

This decomposition is always possible for values of A and B upto a certain maximum. Above these values, $\int F(s)ds$ will no longer be Hurwitz. There are (m + 3) unknowns on the right hand side of (5.12) and we need to choose two constants A and B in this case. Once this has been done, the d_i's can be computed one by one using the following set of equations:

$$\frac{G}{m} = 1$$

$$G \frac{d_{m-2}}{m-1} + A = C_{m-1}$$

$$G \frac{d_{m-3}}{m-2} + Ad_{m-2} + B(m - 1) = C_{m-2} \qquad ..(5.13)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Gd_o + Ad_1 + 2Bd_2 = C_1$$

$$GK + Ad_o + Bd_1 = C_o$$

With F(s) obtained, it now remains to test its integrability. If F(s) is found unintegrable, the given Q(s) must be decomposed again with lower values of A and B until it becomes integrable.

5.4 A Possible Application of the Integro-differential Decomposition in the Synthesis of Transfer Immittances:

Consider two networks N_a and N_b in tandem, as shown in Fig. 5.2. We have,

$$y_{12} = \frac{y_{12a}y_{12b}}{y_{22a} + y_{11b}} = \frac{N(s)}{D(s)} \qquad ..(5.14)$$

and

$$z_{12} = \frac{z_{12a}z_{12b}}{z_{22a} + z_{11b}} \qquad ..(5.15)$$

The synthesis procedure described below for y_{12} may also be used for realizing a given z_{12}.

Decomposing D(s) by the Integro-differential technique given in Section 5.3, we obtain,

$$D(s) = AF(s) + BF'(s) + G\int F(s)ds \qquad ..(5.16)$$

Hence (5.14) can be rewritten as

$$y_{12} = \frac{\frac{N(s)}{AF(s)}}{1 + \frac{B}{A}\frac{F'(s)}{F(s)} + \frac{G}{A}\frac{\int F(s)ds}{F(s)}} \qquad ..(5.17)$$

Since each of the terms in the denominator of (5.16) is a PRF, there are four different ways in which y_{22a} and y_{11b} can be identified. These different alternatives are shown in

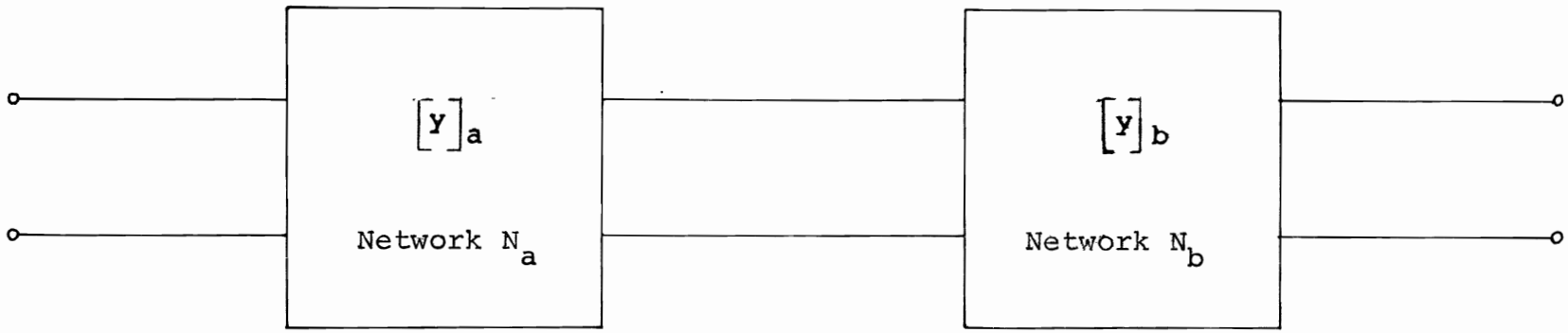

Fig.5.2: Two Networks connected in tandem

Table 5.1. Corresponding to each of these identifications, y_{12a} and y_{12b} may be chosen, writing the numerator of (5.17) as:

$$\frac{N(s)}{F(s)} = \frac{N_1(s)N_2(s)}{F_1(s)F_2(s)} \qquad ..(5.18)$$

and identifying,

$$y_{12a} = \frac{N_1(s)}{F_1(s)}$$

$$y_{12b} = \frac{N_2(s)}{F_2(s)} \qquad ..(5.19)$$

The additional poles contained in y_{22a} or y_{11b} may be realized as private poles. The networks can now be synthesised using either Weinberg's procedure[31] wherein each inductance is lossy or by the procedure of Fialkow et el[35] if N(s) does not have zeros on the positive real axis. The latter may not result in all inductances being lossy.

5.5 Conclusions:

In this Chapter, two new polynomial decompositions, namely, the Integral Decomposition and the Integro-differential Decomposition have been discussed. These two decompositions along with the Differential Decomposition obtained by Weinberg possess the following property: if a given HP Q(s) is decomposed using any of these techniques into a HP F(s), its derivative, and/or its integral (also a HP), then $\frac{Q(s)}{F(s)}$ is a sum of PRF's. This property is used in the synthesis of transfer functions

TABLE 5.1

FOUR WAYS OF IDENTIFYING y_{22a} AND y_{11b}

y_{22a}	y_{11b}
$1 + \frac{B}{A}\frac{F'(s)}{F(s)}$	$\frac{G}{A}\frac{\int F(s)ds}{F(s)}$
$1 + \frac{G}{A}\frac{\int F(s)ds}{F(s)}$	$\frac{B}{A}\frac{F'(s)}{F(s)}$
$\frac{G}{A}\frac{\int F(s)ds}{F(s)}$	$1 + \frac{B}{A}\frac{F'(s)}{F(s)}$
$\frac{B}{A}\frac{F'(s)}{F(s)}$	$1 + \frac{G}{A}\frac{\int F(s)ds}{F(s)}$

or transfer admittances using symmetrical lattices with lossy coils or as grounded RLC transformerless networks.

CHAPTER VI

CONCLUSIONS AND SOME SUGGESTED INVESTIGATIONS

6.1 Conclusions:

This thesis has initiated a study concerning the integrals of network functions. The main topics of investigation have been:

(i) the integrability criteria of polynomials containing simple imaginary axis zeros or simple negative real axis zeros.

(ii) the differentials and integrals of the real and imaginary parts of PRF's and,

(iii) polynomial decompositions using integrals of HP's.

The driving point immittances of lossless networks were considered first. Given the zeros of M and N, conditions have been established such that $\int M ds$ and $\int N ds + K$ contain only simple imaginary axis zeros. Suitable testing procedures including simple inspection tests have been enunciated. These inspection tests facilitate the immediate rejection of an unintegrable M, or an unintegrable N containing three factors. Once the inspection tests are complied with, sufficiency tests can be performed in the prescribed manner. The integrability conditions of M and N are necessary condi-

tions for $\frac{\int Mds}{\int Nds + K}$ to represent the driving point immittance function of a lossless network.

Two special cases of M and N, called the medial M and the medial N, arise when their zeros satisfy prescribed relationships. The medial M is generally unintegrable, while the medial N is always integrable since a real positive constant is associated with its integral. The properties of the integrals of the medial M or the medial N provide additional criteria which aid in the acceptance or rejection of the given function as being integrable.

The driving point immittance functions of lossy two-element-kind networks have also been considered. The conditions for the following two possibilities have been enunciated:

(i) the integral of a RC polynomial being another RC polynomial and,

(ii) the integral of a RC polynomial being a HP.

It has been shown that the RC integrability conditions of a polynomial F(s) whose zeros are given, are the same as the integrability conditions of the odd part of a HP obtained as $N = sF(s^2)$. Thus from an integrable odd part, it is possible to obtain an integrable RC polynomial and vice versa. If the zeros of F(s) are not known, a modified Routh-Hurwitz criterion has been employed to test its RC integrability. It follows that a given N, whose zeros are not known, can be tested for

its integrability by ascertaining the RC integrability of the polynomial obtained as $F(s) = \frac{N}{s}\Big|_{s^2 \to s}$. The Hurwitz integrability conditions of RC polynomials are less strict than those required for their RC integrability. It is also shown that under certain conditions $\frac{\int F(s)ds + K}{F(s)}$ can be expressed as the sum of RC and RL immittances.

It follows that neither the two-element-kind property nor the positive realness of a driving point immittance function remains invariant under polynomial integration. This is in contrast to the operation of polynomial differentiation where this invariance is always guaranteed.

That the above contrast is not general is proved when the real and imaginary parts of a PRF are considered. It is found that under polynomial integration, the real part of a PRF always yields a function which is suitable as the real part of another PRF. This is not necessarily true under polynomial differentiation. However, the modified form of the imaginary part considered here, yields a similar function under both the operations of polynomial differentiation and polynomial integration.

Generation of higher order PRF's can be achieved using the criteria reported in this thesis. Higher order PRF's may be constructed using the integrability criteria of M or N or by the polynomial integration of the real or (modified)

imaginary part.

Two new decompositions of HP's, the Integral Decomposition and the Integro-differential Decomposition, have been described. The advantage of these decompositions in yielding functions which can be broken up into a sum, such that each constituent part is a PRF, has been discussed and their possible use in network synthesis using lossy coils indicated.

6.2 Some Suggested Investigations:

The foregoing discussions lead to the following problems which are suggested for further investigations:

(a) It has been proved in Theorem 2.6 that if a given M_m is unintegrable at $s^2 = -X_2$, then so is M_{m+1} at the same point. It appears reasonable to conjecture that a necessary condition for the integrability of M_{m+1} is that M_m be integrable. In other words, if $\int M_m ds$ has one or more pairs of complex zeros, $\int M_{m+1} ds$ also has at least one pair of complex zeros.

Similarly, one may conjecture that if a given N_n is unintegrable, then so is N_{n+1}. Both the above conjectures have been found to be true for several numerical examples, but a rigorous proof is essential before their validity can be established.

(b) Considering the medial M_m, if any one or more of its even numbered zeros are moved towards the origin, then from

Theorem 2.9, the integrability is violated. For the case when m is even, all the equalities (2.48) become inequalities, that is

$$\sum_{i=1}^{\frac{m}{2}} \frac{X_j}{X_{2i} - X_j} > \frac{1}{4}, \quad j = 1,3,5,\ldots\ldots\ldots\ldots, m-1$$

However, this inequality does not imply unintegrability. This is obvious when one considers the perturbation of the odd numbered zeros. This does not necessarily mean that all the terms in the summation decrease or increase simultaneously. However, it is reasonable to conjecture that there exists a function γ, such that

$$\sum_{i=1}^{\frac{m}{2}} \frac{X_j}{X_{2i} - X_j} < \gamma, \quad j = 1,3,5,\ldots\ldots\ldots\ldots, m-1$$

is a ncessary condition for integrability. A similar result may be anticipated for the case when m is odd.

(c) Necessary coefficient relationships for two-element kind network functions are available using the properties of differentials[9,15]. Since the polynomial integration imposes some restrictions on the locations of the zeros, it is expected that some additional constraints regarding the coefficient relationships may result using the integrability criteria.

(d) In Chapter V, two new decompositions, namely the Integral and the Integro-differential decompositions have been discussed. It may be possible to extend this to a decomposition

containing higher order differentials and integrals, and to examine their use in network theory.

It is hoped that this thesis will stimulate the investigation of further properties and applications of integrals of network functions.

REFERENCES

1. M. Marden, The Geometry of Polynomials, American Mathematical society, New York, 1965.

2. J.L. Walsh, The Location of Critical Points of Analytic and Harmonic Functions, American Mathematical Society, New York, 1950.

3. F.M. Reza, Use of the Derivative in Electrical Network Problems, Research Lab. of Electronics, M.I.T., Cambridge, Mass., Quarterly Research Report, Pt. 3, pp. 78-82, July 15, 1953.

4. F.M. Reza and N.K. Bose, Some Links between Theory of Equations and Realizability Theory, 12th Midwest Symposium on Circuit Theory, Purdue University, pp. V-2-1 to V-2-11, May 1967.

5. L. Weinberg, Network Analysis and Synthesis, McGraw-Hill Book Co., Inc., New York, pp. 254, 1962.

6. M. Marden, The Geometry of Polynomials, American Mathematical Society, New York, pp. 22, 1965.

7. F.M. Reza, A Generalization of Foster's and Cauer's Theorems, IRE Convention Record, Vol. 3, Pt. 2, pp. 22-26, 1955.

8. A. Talbot, Some Theorems on Positive Real Functions, IEEE Trans. on Circuit Theory, Vol. CT-12, No. 4, pp. 607-608, December, 1965.

9. F.M. Reza and N.K. Bose, Derived Positive Real Functions and Matrices and Consequent Coefficient Relations, IEEE Trans. on Circuit Theory, Vol. CT-15, No.1, pp. 76-78, March, 1968.

10. B. Van der Pol, A New Theorem on Electrical Networks, Physica, Vol. IV, pp. 585-589, 1937.

11. A. Bar-Lev, Minimum Reactance Realization of Networks Comprising Two Types of Elements, Proc. of the NEC, Vol. 21, pp. 749-752, 1965.

12. A.J. Schneider, A Necessary Condition on the Coefficients of a Positive Real Rational Function, PIEEE, Vol. 53, No. 10, pp. 1659, October, 1965.

13. R.J. Krueger, Synthesis by Partial Fraction Expansion, Electronics Letters, Vol. 4, No. 10, pp. 192-193, 17th May, 1968.

14. G. Martinelli, Relationship among Coefficients of a P.R. Functions due to the Presence of an Imaginary Pole, Electronics Letters, Vol. 4, No. 17, pp. 349-350, 23rd August, 1968.

15. S.K. Mitra and D.O. Larimer, Generation of RC Impedance Functions and Applications, Second Asilomar Conference on Circuits and Systems, Naval Postgraduate School, Univ. of Santa Clara, pp. 353-356, November, 1968.

16. R.J. Krueger and D.P. Brown, Positive Real Derivatives of Driving Point Functions, Journal of Franklin Institute, Vol. 287, No. 1, pp. 51-60, January, 1969.

17. L. Weinberg, A General RLC Synthesis Procedure, IRE Convention Record, Pt. 5, pp. 2-16, 1953.

18. A. Lepschy, A New Criterion for Evaluating the Number of Complex Roots of an Algebraic Equation, PIEEE, Vol. 50, No. 9, pp. 1981, September, 1962.

19. N.K. Bose, On Real Eigenvalues of Real Non-Symmetric Matrices, PIEEE, Vol. 56, No. 8, pp. 1380, August, 1968.

20. N.K. Bose, Comments on 'On Real Eigenvalues of Real Non-Symmetric Matrices', PIEEE, Vol. 57, No. 4, pp. 705, April, 1969.

21. J.C. Giguere, V. Ramachandran and P.K. Verma, Use of Routh-Hurwitz Algorithm for Determining the Number of Negative Real Axis Zeros in a given Polynomial having Real Coefficients, Electronics Letters, Vol. 5, No. 22, pp. 546, 30th October, 1969.

22. P.K. Verma and V. Ramachandran, Some Functions Generated by the Two Parts of Calahan or Horowitz Decomposition, Twelfth Midwest Symposium on Circuit Theory, The Univ. of Texas at Austin, pp. XI.8.1 to XI.8.6, April, 1969.

23. N. Balbanian, Network Synthesis, Prentice Hall, Inc., pp. 134, Prob. 3-16, 1958.

24. M.E. Van Valkenburg, Introduction to Modern Network Synthesis, John Wiley and Sons, Inc., pp. 111, Prob. 4-18, 1965.

25. D.A. Calahan, Restrictions on the Natural Frequencies of an RC-RL Network, Journal of Franklin Institute, Vol. 272, pp. 112-133, August, 1961.

26. I.M. Horowitz, Optimization of Negative Impedance Methods of Active RC Synthesis, IRE Trans. on Circuit Theory, Vol. CT-6, No. 3, pp. 296-303, September, 1959.

27. P.K. Verma and V. Ramachandran, On the Differentials and Integrals of the Real and Imaginary Parts of Positive Real Functions, Proc. of 13th Midwest Symposium on Circuit Theory, University of Minnesota, pp. XVI.6.1. - XVI.6.7, May, 1970.

28. D. Hazony, Elements of Network Synthesis, Reinhold Publishing Corporation, New York, pp. 235, 1963.

29. A.D. Fialkow and I. Gerst, The Transfer Function of General Two Terminal-pair RC Networks, Quarterly of Applied Mathematics, Vol. 10, pp. 113-127, April, 1952.

30. L. Weinberg, Network Analysis and Synthesis, McGraw-Hill Book Co., New York, Chapter 10, 1962.

31. L. Weinberg, RLC Lattice Networks, PIRE, Vol. 41, No. 9, pp. 1139-1144, September, 1953.

32. L. Weinberg, Synthesis of Unbalanced RLC Networks, Jour. Applied Physics, Vol. 24, No. 3, pp. 300-306, March, 1953.

33. L. Weinberg, Unbalanced RLC Networks containing only one Resistance and one Real Transformer, PIRE, Vol. 42, No. 2, pp. 467-475, February, 1954.

34. W.C. Yengst, Procedures of Modern Network Synthesis, The Macmillan Company, New York, Chapter 9, 1964.

35. A. Fialkow, D. Hazony, P. Kodali and T. Zeren, Transfer-Function Realizability of Multiport RLC Transformerless Grounded Networks, IEEE Trans. on Circuit Theory, Vol. CT-11, No. 1, pp. 73-79, March, 1964.