Technical Report No. 1/04, May 2004

PRICING OF EQUITY-LINKED LIFE INSURANCE CONTRACTS WITH FLEXIBLE GUARANTEES

A Melnikov and V. Skornyakova

Pricing of equity - linked life insurance contracts with flexible guarantees

Alexander Melnikov[1]

Department of Mathematical and Statistical Sciences
University of Alberta, Edmonton, Canada T6G2G1.
(e-mail: melnikov@ualberta.ca).

Victoria Skornyakova

Russian Academy of Economics,
Moscow, Russia.
(e-mail: vskorn@shaw.ca).

Abstract. The paper deals with a particular class of equity-linked life insurance contracts called "pure endowment with guarantee". In our setting, these contracts are based on two risky assets in a two-factor jump-diffusion market. The first asset is responsible for the maximal size of future profits, while the second one provides a flexible guarantee to the insured. Quantile hedging methodology and Margrabe's formula are exploited to price such contracts.

Key words: Equity-linked life insurance, pure endowment, flexible guarantee, quantile hedging, jump-diffusion model.

JEL Classification: G10, G12, D81

Subject Categories: IM10, IE43, IB10.

[1]This work was partially supported by discovery grants of NSERC and SSHERC.
Prof. A. Melnikov, Department of Math. and Stat. Sciences
University of Alberta, Edmonton, Canada T6G2G1.
Ph: (780) 492-0568. Fax (780) 492-6826,
e-mail: melnikov@ualberta.ca))

1 Introduction

Since the middle of the 1970s, equity-linked life insurance contracts have been studied as an innovative insurance instrument that combines both financial and insurance risks. This type of mixed contracts allows insurance companies to be competitive in the modern financial system. Most papers on this topic were devoted to the pricing of "pure endowment life insurance contracts with guarantee" (see, for instance, Brennan and Schwartz (1976,1979), Boyle and Schwartz (1977), Delbaen (1986), Bacinello and Ortu (1993), Aase and Persson (1994), Boyle and Hardy (1997), Moeller (1998, 2002), Bacinello (2001)). Such contracts held some deterministic guarantees to the insured, and they were priced by means of perfect and mean variance hedging. A general feature of all papers was a reduction of a given mixed contract to a call (put) option with the strike price as the corresponding guarantee. Thus, if the underlying risky asset follows a geometric Brownian motion, some variants of the Black-Scholes formula naturally arise in the process of pricing.

We study the contracts in the framework of a two-factor jump-diffusion model with two risky assets. The first risky asset, S^1, is responsible for the size of possible future gains of the holder of the contract. The second one, S^2, is more reliable. We identify the second asset with a flexible guarantee for the insured. Then the contract under consideration should have the payoff of the form $\max\{S_T^1, S_T^2\}$, where T is the maturity time. We show how this contract can be naturally reduced to the option to exchange S_T^1 for S_T^2. This explains why the formula of Margrabe (1978) and its generalization appears in this paper. Our approach here is using quantile hedging to price equity-linked life insurance contracts with flexible guarantees (see Föllmer and Leukert (1999,2000) and also the books of Föllmer and Schied (2002), Melnikov et al (2002), and Melnikov (2003))in a framework of a jump-diffusion market. We describe an actuarial analysis of such contracts for a simplified (diffusion) model to illustrate our theoretical results. Numerical calculations are given with the help of financial indices such as the Dow Jones Industrial Average and the Russell 2000.

2 Auxiliary notions and results

The model of the financial market is given by two linear stochastic differential equations with respect to a Wiener process W_t and a Poisson process Π (with intensity $\lambda > 0$):

$$dS_t^i = S_{t-}^i(\mu^i dt + \sigma^i dW_t - \nu^i d\Pi_t), i = 1, 2\,, \tag{2.1}$$

where $\mu^i \in \mathbb{R},\ \sigma^i > 0,\ \nu^i < 1$.

We suppose that all processes are given on a standard stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ and are adapted to the filtration $\mathbb{F}$, generated by the independent processes W and Π, whose paths are right-continuous with finite left limits at each $t > 0$ (see, for example, Shiryaev (1999) or Melnikov et al (2002)).

The risky asset S^i is defined by its price process $S_t^i,\ t \geq 0,\ i = 1, 2$. For the

sake of simplicity, we assume that the non-risky asset $B_t \equiv 1$, hence the interest rate $r = 0$.
Every predictable process $\pi = (\pi_t)_{t\geq 0} = (\beta_t, \gamma_t^1, \gamma_t^2)_{t\geq 0}$ is called a *trading strategy*, or a *portfolio*. The value (capital) of π at time t equal to

$$X_t^\pi = \beta_t + \sum_{i=1}^{2} \gamma_t^i S_t^i. \tag{2.2}$$

The class of portfolios π with a value evolution

$$X_t^\pi = X_0^\pi + \sum_{i=1}^{2} \int_0^t \gamma_u^i dS_u^i. \tag{2.3}$$

is denoted *SF*. We call π *self financing* if $\pi \in SF$. We consider *admissible* only those portfolios $\pi \in SF$ whose capital is nonnegative.

Recall that the market (2.1)-(2.2) is complete if

$$\frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_2\nu_1 - \sigma_1\nu_2} > 0, \qquad \sigma_2\nu_1 - \sigma_1\nu_2 \neq 0. \tag{2.4}$$

Denote P^* a unique martingale measure which has a local density

$$Z_t = \left.\frac{d\mathrm{P}^*}{d\mathrm{P}}\right|_{\mathcal{F}_t} = \exp\left(\alpha^* W_t - \frac{{\alpha^*}^2}{2} t + (\lambda - \lambda^*)t + (\ln\lambda^* - \ln\lambda)\Pi_t\right),$$

and we find (α^*, λ^*) from the unique solution of the system

$$\begin{cases} \mu_1 = -\sigma_1\alpha^* + \nu_1\lambda^* \\ \mu_2 = -\sigma_2\alpha^* + \nu_2\lambda^* \end{cases} \tag{2.5}$$

therefore

$$\alpha^* = \frac{\mu_2\nu_1 - \mu_1\nu_2}{\sigma_2\nu_1 - \sigma_1\nu_2}, \qquad \lambda^* = \frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_2\nu_1 - \sigma_1\nu_2}.$$

The processes $W_t^* = W_t - \alpha^* t$ and Π_t are independent Wiener and Poisson processes (with another intensity $\lambda^* > 0$) under the measure P^*.

Let us fix a time horizon T. Any nonnegative $\mathcal{F}_T$-measurable random variable H will be called a *contingent claim.*
Let us take an admissible strategy π and form its value starting from an initial capital $x = X_0^\pi$, bounded by X_0. We call $A(x,\pi)$ the *set of successful hedging* if

$$A(x,\pi) = \{\omega : X_T^\pi \geq H\}.$$

Remark 2.1 It follows from option pricing theory for complete markets that there exists a strategy π^* with the property

$$P\left(A(E^*[H], \pi^*)\right) = 1, \tag{2.6}$$

where $X_0^\pi = X_0 = E^*[H]$, and π^* is a perfect hedge.

However, very often $X_0 < E^*[H]$, so we cannot provide appropriate financing for the perfect hedge in the sense of (2.6). In this case, the following criteria should be used to find an appropriate strategy π^*:

$$P\{A(x,\pi)\} \longrightarrow \max_\pi, \tag{2.7}$$

under the restriction $x \le X_0 < E^*[H]$.

According to Föllmer and Leukert (1999) (see also Melnikov et al (2002)), the set $A^* = A(X_0, \pi^*)$ is called a *maximal set of successful hedging.* The structure of this set is

$$A^* = \{Z_T^{-1} \ge a \cdot H\}, \tag{2.8}$$

and optimal strategy π^* becomes a perfect hedge for the modified option

$$H_{A^*} = H I_{A^*}. \tag{2.9}$$

The maximization problem in (2.7) and the structure of A^* in (2.8) have a statistical flavor connected with the fundamental Neuman-Pearson lemma. Therefore, this hedging methodology is called quantile hedging.

It is obvious from previous considerations that the bound on the initial capital X_0 has an important role. We are interested in determining this value in connection with the contingent claim exercised under some condition. We introduce such a condition through an insurance factor – mortality of the client.

Following actuarial traditions, we use a random variable $T(x)$ on some probability space $((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}))$ to represent the remaining lifetime of an insured of age x. Let us consider a pure endowment contract with the payoff function (see Bowers et al (1997))

$$H(T(x)) = H \cdot I\{T(x) > T\} \tag{2.10}$$

To find a natural value of X_0 we take the expected value in (2.10) w.r.t. $P^* \times \tilde{P}$:

$$\begin{aligned} E^* \times \tilde{E}[H(T(x))] &= E^*[H] \cdot \tilde{E} I_{\{T(x)>T\}} \\ &= E^*[H] \cdot {}_T p_x, \end{aligned} \tag{2.11}$$

where ${}_T p_x = \tilde{P}(T(x) > T)$. In view of (2.11), for this contingent claim a natural value of the initial capital of the hedging portfolio should be

$$X_0 = E^*[H] \cdot_T p_x < E^*[H]. \tag{2.12}$$

The condition (2.12) shows us that in order to provide a hedge with maximal probability, we should use quantile hedging methodology (2.7)-(2.9).

We finish the section with the generalizaton of the Margrabe formula that later will be used to price pure endowment life insurance contracts with flexible guarantees.

Let us represent $S_t^i,\ i = 1, 2$ in an exponential form:

$$\begin{aligned} S_t^i &= S_0^i \exp(\sigma^i W_t + [\mu^i - \frac{1}{2}(\sigma^i)^2]t + \Pi_t \ln(1 - \nu^i)] \\ &= S_0^i \exp(\sigma^i W_t^* + [\nu^i \lambda^* - \frac{1}{2}(\sigma^i)^2]t + \Pi_t \ln(1 - \nu^i)). \end{aligned} \tag{2.13}$$

Let's consider $H = (S_T^1 - S_T^2)^+$ and find its expected value w.r.t. P^*. Using (2.13) and the independence of W^* and Π under P^*, we have

$$\begin{aligned} E^*(S_T^1 - S_T^2)^+ &= E^*[E^*\left((S_T^1 - S_T^2)^+|\Pi_T\right)] \\ &= \sum_{n=0}^{\infty} E^*[(S_T^1 - S_T^2)^+|\Pi_T = n]p^*_{n,T}\,, \end{aligned} \tag{2.14}$$

where $p^*_{n,T} = \exp(-\lambda^* T)\frac{(\lambda^* T)^n}{n!}$ are components of a Poisson distribution with intensity λ^*.

Note that

$$E^*[(S_T^1 - S_T^2)^+|\Pi_T = n] = E^*[(s^1_{n,T} - s^2_{n,T})^+]\,, \tag{2.15}$$

where $s^i_{n,T},\ i = 1, 2$, are lognormally distributed random variables under P^*:

$$\ln s^i_{n,T} \sim N\left(\ln S_0^i(1-\nu^i)^n + [\nu^i\lambda^* - \frac{1}{2}(\sigma^i)^2]T, (\sigma^i)^2 T\right), \qquad i = 1, 2.$$

Assuming $\sigma_1 > \sigma_2$, we can apply the Margrabe formula in this partial situation and from (2.14)-(2.15) find, that

$$E^*(S_T^1 - S_T^2)^+ = \sum_{n=0}^{\infty} \mathbb{C}^{Mar}(S_0^1\vartheta^1_{n,T}, S_0^2\vartheta^2_{n,T}, T)p^*_{n,T}, \tag{2.16}$$

where $\mathbb{C}^{Mar}$ denotes a variant of the Margrabe formula: $n = 0, 1, 2, \ldots$

Equality (2.16) gives the price of the option to exchange S^1 for S^2 (from the model (2.1)) at time T in through Poisson weighting of the Margrabe formula.

3 Pricing formulas for the pure endowment contract with flexible guarantee

Let us define H as $\max\{S_T^1, S_T^2\}$, keeping in mind that the first asset, S^1, is more risky than the second one, S^2. Therefore we assume that $\sigma_1 > \sigma_2$, and S^2 will play the role of the flexible guarantee of the pure endowment life insurance contract with the payoff

$$\max\{S_T^1, , S_T^2\} \cdot I_{\{T(x)>T\}}\,, \tag{3.1}$$

where $T(x)$ is the remaining life of the insured of age x, as defined in (2.10).
We observe that

$$\max\{S_T^1, S_T^2\} = S_T^2 + (S_T^1 - S_T^2)^+\,,$$

and under the martingale measure P^* we obtain

$$E^*\max\{S_T^1, S_T^2\} = E^* S_T^2 + E^*(S_T^1 - S_T^2)^+ = S_0^2 + E^*(S_T^1 - S_T^2)^+. \tag{3.2}$$

Using (3.1)-(3.2) and (2.11) we arrive to the natural initial price ${}_TU_x$ of the contract (3.1):

$$ {}_TU_x = S_0^2 {}_Tp_x + E^*(S_T^1 - S_T^2)^+ \cdot {}_Tp_x. \tag{3.3} $$

The difference ${}_TU_x - S_0^2 {}_Tp_x$ can be viewed as an upper bound for the initial value of a hedging strategy for the option $(S_T^1 - S_T^2)^+$.

Taking into account (2.8)-(2.9), from the described above quantile hedging methodology and from (2.9) and (3.3) we obtain

$$ {}_Tp_x = \frac{E^*(S_T^1 - S_T^2)^+ \cdot I_{A^*}}{E^*(S_T^1 - S_T^2)^+}\,, \tag{3.4} $$

where A^* is the maximal set of successful hedging for $(S_T^1 - S_T^2)^+$.

In our actuarial analysis of the contract (3.1), the equality (3.4) plays a key role. Let us build A^* and a hedging strategy π^* such that the maximization of the successful hedging is fulfilled for $(S_T^1 - S_T^2)^+$.
To do this, we rewrite the key representation (2.8) for A^* as follows:

$$ \begin{aligned} A^* &= \left\{ Z_T^{-1} \geq a \cdot (S_T^1 - S_T^2)^+ \right\} \\ &= \left\{ Z_T^{-1} \geq a \cdot S_T^2 \left(\frac{S_T^1}{S_T^2} - 1 \right)^+ \right\} \\ &= \left\{ (Z_T S_T^2)^{-1} \geq a \cdot \left(\frac{S_T^1}{S_T^2} - 1 \right)^+ \right\}, \end{aligned} \tag{3.5} $$

where a is some appropriate constant.
The representation (3.5) shows that we should work with the ratio $Y_T = \frac{S_T^1}{S_T^2}$.

Using (2.13), we obtain the next exponential form for Y_T:

$$ \begin{aligned} Y_T &= \frac{S_0^1}{S_0^2} \left(\frac{1-\nu_1}{1-\nu_2} \right)^{\Pi_T} \exp\left\{ (\sigma_1 - \sigma_2) W_T + \left[(\mu_1 - \mu_2) - (\sigma_1^2 - \sigma_2^2)/2 \right] \cdot T \right\} \\ &= \frac{S_0^1}{S_0^2} \left(\frac{1-\nu_1}{1-\nu_2} \right)^{\Pi_T} \exp\{ (\sigma_1 - \sigma_2) W_T^* + \left[(\nu_1 - \nu_2)\lambda^* - (\sigma_1^2 - \sigma_2^2)/2 \right] \cdot T \}. \end{aligned} \tag{3.6} $$

Taking into account the formula for Z_T and (3.5)-(3.6), we want to express A^* in terms of Y_T. To do this, let us rewrite W_T in the following way:

$$ \begin{aligned} W_T &= 2W_T - W_T = 2\sigma_1^{-1}(\sigma_1 W_T) - \sigma_2^{-1}(\sigma_2 W_T) \\ &= 2\sigma_1^{-1}\left[\sigma_1 W_T + (\mu_1 - \sigma_1^2/2)T \right] - \sigma_2^{-1}\left[\sigma_2 W_T + (\mu_2 - \sigma_2^2/2)T \right] \\ &\quad -2\sigma_1^{-1}(\mu_1 - \sigma_1^2/2)T + \sigma_2^{-1}(\mu_2 - \sigma_2^2/2)T\,. \end{aligned} \tag{3.7} $$

Using (3.7), we obtain

$$
\begin{aligned}
Z_T S_T^2 &= \exp\left\{\alpha^* W_T - \frac{(\alpha^*)^2}{2}T + (\lambda-\lambda^*)T + \Pi_T \ln\frac{\lambda}{\lambda^*}\right\} \cdot S_T^2 \\
&= (S_0^1)^{\frac{2\alpha^*}{\sigma_1}} \exp\left\{\frac{2\alpha^*}{\sigma_1}\left[\sigma_1 W_T + (\mu_1 - \sigma_1^2/2)T + \Pi_T \ln(1-\nu_1)\right]\right\} \\
&\quad \times (S_0^2)^{\frac{-\alpha^*}{\sigma_2}} \exp\left\{\frac{-\alpha^*}{\sigma_2}\left[\sigma_2 W_T + (\mu_2 - \sigma_2^2/2)T + \Pi_T \ln(1-\nu_2)\right]\right\} \\
&\quad \times S_T^2 (S_0^1)^{-\frac{2\alpha^*}{\sigma_1}} (S_0^2)^{\frac{\alpha^*}{\sigma_2}} \exp\left\{\Pi_T \ln\left[(1-\nu_1)^{\frac{-2\alpha^*}{\sigma_1}} \cdot (1-\nu_2)^{\frac{\alpha^*}{\sigma_2}} \frac{\lambda^*}{\lambda}\right]\right\} \\
&\quad \times \exp\left\{\frac{-2\alpha^*}{\sigma_1}(\mu_1 - \sigma_1^2/2)T + \frac{\alpha^*}{\sigma_2}(\mu_2 - \sigma_2^2/2)T - \frac{(\alpha^*)^2}{2}T + (\lambda-\lambda^*)T\right\} \\
&= (S_T^1)^{\frac{2\alpha^*}{\sigma_1}} (S_T^2)^{1-\frac{\alpha^*}{\sigma_2}} \cdot b^{\Pi_T} \cdot g\,,
\end{aligned}
\tag{3.8}
$$

where

$$
\begin{aligned}
b &= (1-\nu_1)^{-\frac{2\alpha^*}{\sigma_1}} (1-\nu_2)^{\frac{\alpha^*}{\sigma_2}} \cdot \frac{\lambda^*}{\lambda}, \\
g &= (S_0^1)^{-\frac{2\alpha^*}{\sigma_1}} (S_0^2)^{\frac{\alpha^*}{\sigma_2}} \exp\{\frac{-2\alpha^*}{\sigma_1}(\mu_1 - \sigma_1^2/2)T + \frac{\alpha^*}{\sigma_2}(\mu_2 - \sigma_2^2/2)T - \frac{(\alpha^*)^2}{\sigma_2}T + (\lambda-\lambda^*)T\}.
\end{aligned}
$$

Let us represent (3.8) in the form

$$
Z_T S_T^2 = (Y_T)^{\alpha} \cdot b^{\Pi_T} \cdot g\,, \tag{3.9}
$$

where α should be chosen as

$$
\frac{2\alpha^*}{\sigma_1} = \alpha = \frac{\alpha^*}{\sigma_2} - 1.
$$

Hence,

$$
\alpha^* = \frac{\sigma_1\sigma_2}{\sigma_1 - 2\sigma_2}\,, \qquad \sigma_1 \neq 2\sigma_2. \tag{3.10}
$$

Taking into account (2.5) and (3.10) we arrive to the following condition on parameters of the model (2.1) to provide (3.9):

$$
\frac{\mu_2\nu_1 - \mu_1\nu_2}{\sigma_2\nu_1 - \sigma_1\nu_2} = \frac{\sigma_1\sigma_2}{\sigma_1 - 2\sigma_2}, \tag{3.11}
$$

where $\sigma_1 > \sigma_2, \quad \sigma_1 \neq 2\sigma_2,$ and $\quad \sigma_2\nu_1 \neq \sigma_1\nu_2.$

Relations (3.5) and (3.9) give us

$$
A^* = \{Y_T^{-\frac{2\alpha^*}{\sigma_1}} \geq b^{\Pi_T} \cdot g \cdot a(Y_T - 1)^+\}. \tag{3.12}
$$

To analyze A^* in the form of (3.12), we consider the set $\{\Pi_t = n\}, n = 0, 1, 2, \ldots,$ and the following equation

$$
x^{-\frac{2\alpha^*}{\sigma_1}} = b^n \cdot g \cdot a(x-1)^+. \tag{3.13}
$$

Assuming (3.11), we distinguish two cases in connection with the equation (3.13):

case 1: $\sigma_1 > 2\sigma_2$ (or $-\frac{2\alpha^*}{\sigma_1} \leq 1$) and *case 2*: $\sigma_2 < \sigma_1 < 2\sigma_2$ (or $-\frac{2\alpha^*}{\sigma_1} > 1$). Using (3.10), we can easily check that (3.13) has the only solution $c(n,a)$ in the first case and two solutions $c_1(n,a) < c_1(n,a)$ in the second case.

In the first case we shall use $c(n,a)$ to construct the set of successful hedging in the form $\{Y_T \leq c(n,a)\}$ on each set $\{\Pi_T = n\}, n = 0, 1, \ldots$. This form will depend on a parameter a that can be identified from (3.4), if the corresponding survival probability ${}_Tp_x$ is given.

We shall calculate the numerator $E^*(S_T^1 - S_T^2)^+ I_{A^*}$ of (3.4), since the denominator $E^*(S_T^1 - S_T^2)^+$ is given by (2.16). It is sufficient to compute

$$E^*\big[(S_T^1 - S_T^2)^+ I_{A^*}|\Pi_T = n\big] = E^*\big[(S_T^1 - S_T^2)^+ I_{\{Y_T \leq c(a,n)\}}|\Pi_T = n\big]$$

with further averaging of the Poisson distribution $p^*_{n,T}$.

As $c(n,a) > 1$, it is easy to see that

$$\begin{aligned} E^*\big[(S_T^1 - S_T^2)^+ I_{\{Y_T \leq c(a,n)\}}|\Pi_T = n\big] &= E^*\big[(S_T^1 - S_T^2)^+|\Pi_T = n\big] - E^*\big[S_T^1 I_{\{Y_T > c(a,n)\}}|\Pi_T = n\big] \\ &+ E^*\big[S_T^2 I_{\{Y_T > c(a,n)\}}|\Pi_T = n\big] \\ &= E^*\big[(S_T^1 - S_T^2)^+|\Pi_T = n\big] - E^*\big[(S_T^1 - S_T^2)|\Pi_T = n\big] \\ &- E^*\big[S_T^1 I_{\{Y_T \leq c(a,n)\}}|\Pi_T = n\big] - E^*\big[S_T^2 I_{\{Y_T \leq c(a,n)\}}|\Pi_T = n\big] \end{aligned} \tag{3.14}$$

The first term in the last equality of (3.14) is given by (2.15). To find the second term, we use (2.13) and obtain

$$\begin{aligned} E^*\big[(S_T^1 - S_T^2)|\Pi_T = n\big] &= \exp\{\ln S_0^1(1-\nu_1)^n + \nu_1\lambda^* T\} - \exp\{\ln S_0^2(1-\nu_2)^n + \nu_2\lambda^* T\} \\ &= \tilde{s}^1_{0,n} - \tilde{s}^2_{0,n}. \end{aligned} \tag{3.15}$$

To simulate

$$E^*\big[S_T^i I_{\{Y_T \leq c(n,a)|\Pi_T = n\}}\big], \qquad i = 1, 2, \tag{3.16}$$

we rewrite

$$\{Y_T \leq c(n,a)\} = \{\ln Y_T \leq \ln c(n,a)\}$$

and denote $\zeta = \ln Y_T$, $\quad S_T^i = \exp\{-\eta_i\}$, where the gaussian random variables

$$\eta_i = -\left[\ln\left(S_0^i(1-\nu_i)^n\right) + \sigma_i W_T^* + (\nu_i\lambda^* - \sigma_i^2/2)T\right], \qquad i = 1, 2, \tag{3.17}$$

are defined by (2.13). Under the condition $\{\Pi_T = n\}$, the pairs (ζ, η_1) and (ζ, η_2) are two systems of Gaussian random variables. According to the Lemma on p. 797 of Shiryaev (1999) (see also Lemma 2.4 in Melnikov (2003)), for $i = 1, 2$ we get

$$E^*[\exp -\eta_i I_{\{\zeta \leq \ln c\}}|\Pi_T = n] = \exp\{\frac{\sigma^2\eta_i}{2} - \mu_{\eta_i}\}\Phi(\frac{\ln(c) - (\mu_\zeta - cov(\zeta, \eta_i)}{\sigma_\zeta}).$$

The parameters, mean μ and variance σ^2, can be easily recognized from (2.13) and (3.6):

$$\begin{aligned}
\mu_\zeta &= \mu_{lnY_T} = E^*[\ln(Y_T)|\Pi_T = n] \\
&= \ln\{\frac{S_0^1}{S_0^2}(\frac{1-\nu_1}{1-\nu_2})^n\} + \big[(\nu_1-\nu_2)\lambda^* - \frac{\sigma_1^2-\sigma_2^2}{2}\big]T\,, \\
\sigma_\zeta^2 &= (\sigma_1-\sigma_2)^2 T\,, \\
\mu_{\eta_i} &= E^*[\eta_i|\Pi_T = n] = -\ln S_0^1(1-\nu_1)^n - [\nu_i\lambda^* - \frac{\sigma_i^2}{2}]T\,, \\
\sigma_{\eta_i}^2 &= \sigma_i^2 T\,, \\
cov(\zeta,\eta_i) &= -\sigma_i(\sigma_1-\sigma_2)T\,, \qquad i=1,2.
\end{aligned}$$

Putting the values of these parameters into the formula (3.17), we find the value of (3.16) and also the difference of the last terms of (3.14):

$$\begin{aligned}
& E^*[(S_T^1 - S_T^2)I_{\{Y_T\le c(a,n)|\Pi_T=n\}}] = \exp\sigma_1^2/2T + \ln S_0^1(1-\nu_1)^n + \big[\nu_1\lambda^* - \sigma_1^2/2\big]T\} \\
\times\ & \Phi\left((\frac{\ln(c(n,a)) - \big[\ln\{\frac{S_0^1}{S_0^2}(\frac{1-\nu_1}{1-\nu_2})^n\} + \big[(\nu_1-\nu_2)\lambda^* - \frac{\sigma_1^2-\sigma_2^2}{2}\big]T + \sigma_1(\sigma_1-\sigma_2)T\big]}{(\sigma_1-\sigma_2)\sqrt{T}}\right) \\
& -\exp\{\sigma_2^2/2\cdot T + \ln S_0^2(1-\nu_2)^n + \big[\nu_2\lambda^* - \sigma_2^2/2\big]T\} \\
\times\ & \Phi\left(\frac{\ln(c(n,a)) - \big[\ln\{\frac{S_0^1}{S_0^2}(\frac{1-\nu_1}{1-\nu_2})^n\} + \big[(\nu_1-\nu_2)\lambda^* - \frac{\sigma_1^2-\sigma_2^2}{2}\big]T + \sigma_2(\sigma_1-\sigma_2)T\big]}{(\sigma_1-\sigma_2)\sqrt{T}}\right) \\
& = \tilde S_{0,n}^1\Phi\left(\frac{\ln c(n,a)}{(\sigma_1-\sigma_2)\sqrt{T}} - b_+(\tilde S_{0,n}^1, \tilde S_{0,n}^2, T)\right) - \tilde S_{0,n}^2\Phi\left(\frac{\ln c(n,a)}{(\sigma_1-\sigma_2)\sqrt{T}} - b_-(\tilde S_{0,n}^1, \tilde S_{0,n}^2, T)\right) \\
& = \tilde S_{0,n}^1\Phi\big(-b_+(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big) - \tilde S_{0,n}^2\Phi\big(-b_-(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big) \\
& = (\tilde S_{0,n}^1 - \tilde S_{0,n}^2) - \big[\tilde S_{0,n}^1\Phi\big(b_+(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big) - \tilde S_{0,n}^2\Phi\big(b_-(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big)\big].
\end{aligned} \tag{3.18}$$

Combining (3.14), (3.15) and (3.18) with (3.4), we finally derive

$${}_Tp_x = 1 - \frac{\sum_0^\infty p_{n,T}^*\big[\tilde S_{0,n}^1\Phi\big(b_+(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big) - \tilde S_{0,n}^2\Phi\big(b_-(\tilde S_{0,n}^1, c\tilde S_{0,n}^2, T)\big)\big]}{\sum_0^\infty p_{n,T}^*\big[\tilde S_{0,n}^1\Phi\big(b_+(\tilde S_{0,n}^1, \tilde S_{0,n}^2, T)\big) - \tilde S_{0,n}^2\Phi\big(b_-(\tilde S_{0,n}^1, \tilde S_{0,n}^2, T)\big)\big]}. \tag{3.19}$$

The second case, where $\sigma_2 < \sigma_1 < 2\sigma_2$ or $-\frac{2\alpha^*}{\sigma_1} > 1$, can be treated in a similar fashion. The set of successful hedging A^* (again on $\{\Pi_t = n\}; n = 0,1,\ldots$) consists of two parts: $\{Y_T \le c_1(a,n)\}$ and $\{Y_T > c_2(a,n)\}$, where the parameter a should be identified from the condition (3.4) for a given survival probability ${}_Tp_x$. Hence we have

$$I_{A^*} = I_{\{Y_T\le c_1(a,n)\}\cup\{Y_T>c_2(a,n)\}}. \tag{3.20}$$

Using the inequalities $1 \le c_1 \le c_2$ and (3.19), we obtain

$$
\begin{aligned}
E^*[I_{A^*}(S_T^1 - S_T^2)^+ | \Pi_T = n] &= E^*[(S_T^1 - S_T^2)|\Pi_T = n] + E^*[I_{Y_T \le c_1}(S_T^1 - S_T^2)|\Pi_T = n] \\
&- E^*[I_{Y_T \le c_2}(S_T^1 - S_T^2)|\Pi_T = n].
\end{aligned}
\tag{3.21}
$$

The first term in the right hand side of (3.22) is simulated with the help of (2.15). The other two terms in (3.22) are calculated as in (3.16)-(3.17) with evident changes. All these manipulations lead us to a concrete form of (3.4) for the second case:

$$
\begin{aligned}
{}_Tp_x = 1 &- \frac{\sum_0^\infty p_{n,T}^* \big[\tilde{S}_{0,n}^1 \Phi\big(b_+(\tilde{S}_{0,n}^1, c_1 \tilde{S}_{0,n}^2, T)\big) - \tilde{S}_{0,n}^2 \Phi\big(b_-(\tilde{S}_{0,n}^1, c_1 \tilde{S}_{0,n}^2, T)\big)\big]}{\sum_0^\infty p_{n,T}^* \big[\tilde{S}_{0,n}^1 \Phi\big(b_+(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T)\big) - \tilde{S}_{0,n}^2 \Phi\big(b_-(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T)\big)\big]} \\
&+ \frac{\sum_0^\infty p_{n,T}^* \big[\tilde{S}_{0,n}^1 \Phi\big(b_+(\tilde{S}_{0,n}^1, c_2 \tilde{S}_{0,n}^2, T)\big) - \tilde{S}_{0,n}^2 \Phi\big(b_-(\tilde{S}_{0,n}^1, c_2 \tilde{S}_{0,n}^2, T)\big)\big]}{\sum_0^\infty p_{n,T}^* \big[\tilde{S}_{0,n}^1 \Phi\big(b_+(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T)\big) - \tilde{S}_{0,n}^2 \Phi\big(b_-(\tilde{S}_{0,n}^1, \tilde{S}_{0,n}^2, T)\big)\big]}
\end{aligned}
\tag{3.22}
$$

In conclusion, we give the following remarks regarding our actuarial analysis.

Remark 3.1 Under known probabilities ${}_Tp_x$, formulas (3.18), (**??**) give us a possibility to determine a^* of the maximal set of successful hedging A^*. The corresponding hedge π^* will be a perfect hedge for the modified claim (2.9). The capital $\mathbb{C}(t, S_t^1, S_t^2)$ of π^* can be computed in a similar way as the initial price of the option. The components $(\beta^*, \gamma_1^*, \gamma_2^*)$ of π^* satisfy a system of stochastic differential equations (see for instance Krutchenko and Melnikov (2001)).

Remark 3.2 We can fix the probability of the set of successful hedging as $1 - \epsilon,\ \epsilon \in (0,1)$, and determine the optimal value a_ϵ^* from this condition. Then, applying (3.18), we can find the survival probabilities ${}_Tp_x$ and use Life Tables (see Bowers et al (1997)) to choose an appropriate group of insured for a given contract.

Further for the group of size l_x of insureds of age x, one can consider the following cumulative claim $l_{x+T} \cdot H$. Let us define a constant n_α by the condition that the event $\{l_{x+T} \le n_\alpha\}$ has a probability $1 - \alpha$, where $\alpha \in (0,1)$ can be interpreted as a mortality risk level. This probability has binomial distribution with probability of success ${}_Tp_x$. If a strategy with initial price $H_0(\epsilon)$ hedges H with risk level ϵ then, the same strategy hedges with the same risk level the claim $\frac{n_\alpha}{l_x} H$ starting at $\frac{n_\alpha}{l_x} H_0(\epsilon)$. All these considerations lead to the conclusion that the above cumulative claim can be hedged with probability greater or equal to $1 - (\alpha + \epsilon)$.

4 Actuarial analysis in a simplified model and numerical example

The goal of this section is to present a package of actuarial calculations regarding a concrete life insurance contract with a flexible guarantee. A simplified model will be considered, since the corresponding simulations in the framework of the model (2.1) are much more complicated and may demand a special numerical technique. Let us concentrate on a limiting variant of (2.1) without its jump component:

$$dS^i_t = S^i_{t-}(\mu^i dt + \sigma^i dW_t), \ i = 1, 2, \ t \le T. \tag{4.1}$$

Remark 4.1 The case S^1 and S^2 generated by different Wiener processes demands some special considerations. It is realized in a forthcoming paper.
The model (4.1) can be viewed as a complete, one risky asset (for example S^1) model of such a market. The second risky asset, S^2, can be expressed through S^1 with the help of (2.13):

$$\begin{aligned}
S^2_T &= S^2_0 \exp\{\sigma_2 W_T + [\mu_2 - \frac{1}{2}(\sigma^i)^2]T\}. \\
S^2_T &= S^2_0 \exp\{\frac{\sigma_2}{\sigma_1}\left(\sigma_1 W_T + [\mu_1 - \frac{1}{2}(\sigma_1)^2]T\right) - \frac{\sigma_2}{\sigma_1}[\mu_1 - \frac{1}{2}(\sigma_1)^2]T + [\mu_2 - \frac{1}{2}(\sigma_2)^2]T\} \\
&= (S^1_0)^{\frac{\sigma_2}{\sigma_1}}(S^1_T)^{\frac{\sigma_2}{\sigma_1}}(S^2_0)\exp\{-\frac{\sigma_2}{\sigma_1}[\mu_1 - \frac{1}{2}(\sigma_1)^2]T + [\mu_2 - \frac{1}{2}(\sigma_2)^2]T\} \\
&= (S^1_0)^{-\frac{\sigma_2}{\sigma_1}}(S^2_0)\exp\{[\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1 + \frac{\sigma_2}{2}(\sigma_1 - \sigma_2)]T\}(S^1_T)^{\frac{\sigma_2}{\sigma_1}}.
\end{aligned} \tag{4.2}$$

It is well known (see Shiryaev (1999) or Melnikov et al (2002)) that the unique martingale measure P^* is given here by the density

$$Z_T = \exp\{-\frac{\mu_1}{\sigma_1}W_T - \frac{1}{2}\left(\frac{\mu_1}{\sigma_1}\right)^2 T\}. \tag{4.3}$$

Consider the contract (3.1). According to (3.2)-(3.5), it is reduced to the pricing of another contract:

$$(S^1_T - S^2_T)^+ \cdot I_{\{T(x)>T\}}$$

with a key equality (3.4) depending on a maximal set of successful hedging of the option $(S^1_T - S^2_T)^+$.

Our leading idea lies in determining the set A^* in terms of the ratio $Y_T = \frac{S^1_T}{S^2_T}$ based on (3.5). Let us reproduce the same analysis as in Section 3, using another representation of W_T in place of (3.7): for some positive γ,

$$\begin{aligned}
W_T &= (1+\gamma)W_T - \gamma W_T \\
&= \frac{1+\gamma}{\sigma_1}[\sigma_1 W_T + (\mu_1 - \sigma_1^2/2)T] - \frac{\gamma}{\sigma_2}[\sigma_2 W_T + (\mu_2 - \sigma_2^2/2)T] \\
&- \frac{1+\gamma}{\sigma_1}[\mu_1 - \frac{\sigma_1^2}{2}T] + \frac{\gamma}{\sigma_2}[\mu_2 - \sigma_2^2/2] \cdot T.
\end{aligned} \tag{4.4}$$

Using (4.3)-(4.4), we rewrite $Z_T S^2_T$ in (3.4) as follows:

$$\begin{aligned}
Z_T \cdot S^2_T &= (S^1_T)^{-\frac{(1+\gamma)\mu_1}{(\sigma_1)^2}}(S^2_T)^{1+\frac{\gamma\mu_1}{\sigma_1\sigma_2}} \cdot G \\
&= Y^\alpha_T \cdot G,
\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}
G &\equiv G(\gamma) \\
&= (S_0^1)^{\frac{(1+\gamma)\mu_1}{(\sigma_1)^2}} (S_0^2)^{\frac{\gamma\mu_1}{\sigma_1\sigma_2}} \exp\{-\frac{(1+\gamma)}{\sigma_1}[\mu_1 - \frac{1}{2}(\sigma_1)^2]T - \frac{\gamma\mu_1}{\sigma_1\sigma_2}[\mu_2 - \frac{1}{2}(\sigma_2)^2]T - \frac{1}{2}(\frac{\mu_1}{\sigma_1})^2 T\}, \\
\alpha &= -\frac{(1+\gamma)\mu_1}{(\sigma_1)^2} = -\frac{\gamma\mu_1}{\sigma_1\sigma_2} - 1.
\end{aligned}$$

The last equality for α can be utilized to find additional conditions on the parameters of the model (4.1).

Assuming $0 < \sigma_1 - \sigma_2 << \sigma_1$ and σ_2, we take $\gamma = (\sigma_1 - \sigma_2)^n,\ n \geq 1$ and obtain

$$\mu_1 = \frac{\sigma_1^2\sigma_2}{\sigma_2} - (\sigma_1 - \sigma_2)^{n+1}. \tag{4.6}$$

Now we consider the equation

$$x^{-\alpha} = G \cdot a \cdot (x-1)^+, \tag{4.7}$$

where $G = G((\sigma_1 - \sigma_2)^n)$ and a is an unknown parameter in (3.5). This is similar to the equation (3.13).

In section 3 we distinguished two cases for (3.13), $-\alpha \leq 1$ and $-\alpha > 1$, to reconstruct $A^* = A_a^*$. But here, due to (4.6), the parameter $-\alpha$ is close to 1:

$$-\alpha = \frac{1 + (\sigma_1 - \sigma_2)^n}{\sigma_1^2} \cdot \frac{\sigma_1^2\sigma_2}{\sigma_2 - (\sigma_1 - \sigma_2)^{n+1}} = \frac{\sigma_2 + (\sigma_1 - \sigma_2)^n \cdot \sigma_2}{\sigma_2 - (\sigma_1 - \sigma_2)^{n+1} \cdot \sigma_2} \simeq 1.$$

Hence we can replace (4.7) by its approximation

$$x = G \cdot a \cdot (x-1)^+. \tag{4.8}$$

Denote $c(a) = \frac{-G \cdot a}{1 - G \cdot a}$ a solution to (4.8) and consider the set $A_a^* = \{Y_T \leq c(a)\}$ as an approximation for A^*. To identify the parameter a, we can fix $P(A^*)$ to be $1 - \epsilon,\ \epsilon > 0$. It follows from (3.6) that in the case of the model (4.1), the structure of Y_T is

$$\begin{aligned}
Y_T &= \frac{S_0^1}{S_0^2} \exp\{(\sigma_1 - \sigma_2)W_T + \left[(\mu_1 - \mu_2) - \frac{\sigma_1^2 - \sigma_2^2}{2}\right] T\} \\
&= \frac{S_0^1}{S_0^2} \exp\{(\sigma_1 - \sigma_2)W_T^* - \frac{(\sigma_1 - \sigma_2)^2 T}{2} + \frac{\mu_1\sigma_2 - \mu_2\sigma_1 - \sigma_1\sigma_2(\sigma_1 - \sigma_2)}{\sigma_1} T\},
\end{aligned} \tag{4.9}$$

where $W_t^* = W_t + \frac{\mu_1}{\sigma_1} t$ is a new Wiener process with respect to P^*.

Looking at the representations of Y_T in (4.9), we conclude that $\ln(Y_T)$ is given by

$$N\left(\ln \frac{S_0^1}{S_0^2} - \frac{(\sigma_1 - \sigma_2)^2 T}{2} + \frac{\mu_1\sigma_2 - \mu_2\sigma_1 - \sigma_1\sigma_2(\sigma_1 - \sigma_2)}{\sigma_1} T,\ (\sigma_1 - \sigma_2)^2 T\right) =$$
$$N\left(\mu_{\ln Y_T}, \sigma^2_{\ln Y_T}\right)$$

with respect to P^* ,, and

$$N\left(\ln \frac{S_0^1}{S_0^2} + \left(\mu_1 - \mu_2 - \frac{\sigma_1^2 - \sigma_2^2}{2}\right) T,\ (\sigma_1 - \sigma_2)^2 T\right)$$

with respect to P.

So we have an equation to identify $a = a_\epsilon^*$ and $c = c(a_\epsilon^*)$:

$$1 - \epsilon = P(A^*_{a_\epsilon^*}) = P\{\ln Y_T \le \ln c(a_\epsilon^*)\} = \Phi_{\mu,\sigma^2}(\ln c(a_\epsilon^*)) \tag{4.10}$$

Now we consider the equality (3.4). The denominator of (3.4) is given by the Margrabe formula

$$\mathbb{C}^{Mar}(S_0^1, S_0^2, T) = S_0^1 \Phi(b_+(S_0^1, S_0^2, T)) - S_0^2 \Phi(b_-(S_0^1, S_0^2, T)). \tag{4.11}$$

We shall determine the denominator of (3.4) as

$$E^*[(S_T^1 - S_T^2)^+ I_{A^*_{a_\epsilon^*}}].$$

Using the same reasoning as in (3.4), we get

$$\begin{aligned} E^*[(S_T^1 - S_T^2)^+ I_{A^*_{a_\epsilon^*}}] &= E^*[(S_T^1 - S_T^2)^+ I_{\{Y_T \le c(a_\epsilon^*)\}}] \\ &= E^*[(S_T^1 - S_T^2)^+] - E^*[S_T^1 I_{\{Y_T > c(a_\epsilon^*)\}}] + E^*[S_T^2 I_{\{Y_T > c(a_\epsilon^*)\}}] \\ &= E^*[(S_T^1 - S_T^2)^+] - E^*[S_T^1] - E^*[S_T^2] + E^*[(S_T^1 - S_T^2) I_{\{Y_T \le c(a_\epsilon^*)\}}]. \end{aligned} \tag{4.12}$$

Due to the martingale property of S^1 w.r.t. P^*, we find that $E^*[S_T^1] = S_0^1$. Another expected value $E^* S_T^2$ is calculated simply as

$$E^*[S_T^2] = E Z_T S_T^2 = S_0^2 \exp\{\frac{\sigma_1 \mu_2 - \sigma_2 \mu_1}{\sigma_1} T\} = S_0^2 \exp\{\mu_2 T - \mu_1 \frac{\sigma_2}{\sigma_1} T\}. \tag{4.13}$$

To determine $E^* S_T^i I_{\{Y_T \le c\}},\ i = 1, 2$, in (4.12), we use the same approach as in (3.16)-(3.18). Applying (4.11)-(4.13) and the new denotations

$$\tilde{S}_0^1 = S_0^1 \exp\{\mu_1 \frac{\sigma_2}{\sigma_1} T\} \text{ and } \tilde{S}_0^2 = S_0^2 \exp\{\mu_2 T\}$$

we find that

$$\begin{aligned} E^*[(S_T^1 - S_T^2)^+ I_{\{Y_T \le c(a_\epsilon^*)\}}] &= S_0^1 \Phi(-b_+(\tilde{S}_0^1, c\tilde{S}_0^2, T)) - \\ &\quad - \tilde{S}_0^2 \exp\{-\mu_1 \frac{\sigma_2}{\sigma_1} T\} \Phi(-b_-(\tilde{S}_0^1, c\tilde{S}_0^2, T)) \\ &= \exp\{\mu_1 \frac{\sigma_2}{\sigma_1} T\}[(\tilde{S}_0^1 - \tilde{S}_0^2) - \\ &\quad - (\tilde{S}_0^1 \Phi(b_+(\tilde{S}_0^1, c\tilde{S}_0^2, T)) - \tilde{S}_0^2 \Phi(b_-(\tilde{S}_0^1, c\tilde{S}_0^2, T)))], \end{aligned}$$

and finally

$$
{}_Tp_x = 1 - \frac{\exp\{-\mu_1 \frac{\sigma_2}{\sigma_1} T\}[\tilde{S}_0^1 \Phi(b_+(\tilde{S}_0^1, c\tilde{S}_0^2, T)) - \tilde{S}_0^2 \Phi(b_-(\tilde{S}_0^1, c\tilde{S}_0^2, T))]}{[S_0^1 \Phi(b_+(S_0^1, S_0^2, T)) - S_0^2 \Phi(b_-(S_0^1, S_0^2, T))]}.
$$

Now we give a numerical example to illustrate this methodology.

Consider the financial indices Russell 2000 (RUT-I) and Dow Jones Industrial Average (DJIA) as risky assets S^1 and S^2. Russell 2000 is the index of small US companies' stocks, whereas Dow Jones is based on the portfolio consisting of 30 blue-chip stocks in the USA. The first index, RUT-I, is supposed to be more risky than DJIA.

Example 4.1. Using daily observations of prices from August 1, 1997, until July 31, 2003. We estimate (μ_1, σ_1) and (μ_2, σ_2), the rate of return, and volatility for RUT-I and DJIA empirically, We get the following numbers:

$$
\begin{aligned}
\mu_1 &= 0.0481, \qquad \sigma_1 = 0.2232, \\
\mu_2 &= 0.0417, \qquad \sigma_2 = 0.2089.
\end{aligned}
$$

We observe that the condition (4.6) with $\gamma = \sigma_1 - \sigma_2$ is approximately fulfilled, as the right-hand side of (4.6) equals 0.0499.

The initial prices of these indices are 414.21 and 8194.04. Therefore, we use $\frac{8194.04}{414.21} \cdot S_t^1$ as the value of the first asset to make initial values of both assets the same.

Utilizing the formulas (4.10) and (4.14) with $T = 1, 3, 5, 10$ and $\epsilon = 0.01$, 0.025, 0.05, we obtain the values of the corresponding survival probabilities ${}_Tp_x$ (see Table 1).

Now we can find an age of the insured using Life Tables (see, for instance, Bowers et al (1997)). The data is displayed in Table 2.

When the level of financial risk ϵ (the probability that the insurance company cannot hedge $(S_T^1 - S_T^2)^+$ or, equivalently, $\max\{S_T^1, S_T^2\}$) increases, the company should restrict the group of insured by attracting older clients. As a result, the company diminishes the insurance component of risk to compensate for the increasing financial risk.

Issuing contracts for a longer term T allows the insurance company to diminish insurance risk with fixed ϵ. Therefore, the company can afford to work with younger groups of clients.

We also do the same for the contract with fixed guarantee, taking into account that it is a particular case of the contract with flexible guarantee $(\mu_2 = \sigma_2 = 0)$.
Taking $K = 1.1$, 0.8194 as the fixed guarantee, we calculate survival probability

and ages of the insured using the same procedure (see Table 3 and Table 4). Comparing the ages in Table 2 and Table 4, we conclude that the company should attract older clients for the contract with flexible guarantee to compensate for the riskier characteristic of this contract relatively to the contract with a fixed guarantee.

Let us pay more attention in our methodology to mortality risk. We consider the cumulative claim $l_{x+T}(S_T^1 - S_T^2)^+$, where l_{x+T} is the number of insureds at the end of the contract from the group of size l_x (see Remark 3.2).

Denote $\pi = \pi_\epsilon$ a quantile hedge of the risk level ϵ with initial (quantile) price C_ϵ and terminal value X_T^π so that

$$P\left(X_T^\pi \geq (S_T^1 - S_T^2)^+\right) = 1 - \epsilon\,. \tag{4.14}$$

The maximal set of successful hedging is invariant with respect to multiplication by a positive constant δ. Hence, the claim $\delta(S_T^1 - S_T^2)^+$ can be hedged at the same risk level ϵ with the initial price δC_ϵ. Take $\delta = \frac{n_\alpha}{l_x}$, where the number n_α is determined from the equality

$$P(l_{x+T} \leq n_\alpha) = 1 - \alpha\,. \tag{4.15}$$

The parameter $\alpha \in (0,1)$ characterizes the level of mortality risk of the company, and the probability in (4.15) can be computed with the help of the binomial distribution with parameter ${}_Tp_x$.

Using independence of l_{x+T} and the market we derive that

$$\begin{aligned}
P\left(l_x X_T^\pi \geq l_{x+T}(S_T^1 - S_T^2)^+\right) &\geq P\left(X_T^\pi \geq \frac{l_{x+T}}{l_x}(S_T^1 - S_T^2)^+\right) \\
&\geq P\left(X_T^\pi \geq \frac{n_\alpha}{l_x}(S_T^1 - S_T^2)^+\right) P(l_{x+T} \leq n_\alpha) \\
&\geq (1-\epsilon)(1-\alpha) \\
&\geq 1 - (\epsilon + \alpha)\,.
\end{aligned} \tag{4.16}$$

Inequalities in (4.16) give the following. Let us take $T = 1,\ 5\ ,\ 10$, fix both risk levels $\epsilon = \alpha = 0.025$, and consider the contract for the group of size $l_x = 100$. We find $n_\alpha = 94;\ 95;\ 96$, and the modified quantile prices $C_{\epsilon,\alpha} = 50.44;\ 156.56;\ 274.44$.

These results show us that under the combined risk level $\epsilon + \alpha = 5\%$ the initial prices can be reduced by $12 - 18\%$ in comparison with the fair prices $C^{mar} = 61.15;\ 183.33;\ 313.84$. At the same time the corresponding quantile prices $C_\epsilon = C_{0.025}$ reduce C^{mar} by $9 - 12\%$

5 Tables

	$\epsilon = 0.01$	$\epsilon = 0.025$	$\epsilon = 0.05$
$T = 1$	0.9447	0.8774	0.7811
$T = 3$	0.9511	0.8910	0.8041
$T = 5$	0.9549	0.8989	0.8174
$T = 10$	0.9605	0.9108	0.8378

Table 1: Survival probabilities (Flexible Guarantee)

	$\epsilon = 0.01$	$\epsilon = 0.025$	$\epsilon = 0.05$
$T = 1$	78	87	94
$T = 3$	61	71	79
$T = 5$	53	63	71
$T = 10$	41	50	58

Table 2: Age of Insured (Flexible Guarantee)

	$\epsilon = 0.01$	$\epsilon = 0.025$	$\epsilon = 0.05$
$T = 1$	0.9733	0.9306	0.8585
$T = 3$	0.9700	0.9247	0.8510
$T = 5$	0.9706	0.9266	0.8553
$T = 10$	0.9732	0.9332	0.8679

Table 3: Survival probabilities (Fixed Guarantee)

	$\epsilon = 0.01$	$\epsilon = 0.025$	$\epsilon = 0.05$
$T = 1$	68	80	88
$T = 3$	55	67	76
$T = 5$	48	59	68
$T = 10$	36	47	56

Table 4: Age of Insured (Fixed Guarantee)

References

Aase, K. K., and S. A. Persson (1994): Pricing of unit-linked life insurance policies. *Scand. Actuar. J.* 1, 26-52.

Bacinello, A. R., Ortu F.(1993): Pricing of unit-linked life insurance with endegeneous minimum guarantees. *Insurance: Math and Economics* 12, 245-257.

Bacinello, A. R.(2001): Fiar pricing of life insurance participating policies with a minimum interest rate guaranteed? *Astin bulletin,* V.31, 2, 275-297.

Bowers, N. L., Gerber, G. C. Hickman, H. U., Jones, D. A. and Nesbit, C. J. (1997): Actuarial Mathematics. *Schaumburg, Illinois: The Society of Actuaries.*

Boyle, P. P., and E. S. Schwartz (1977): Equilibrium prices of guarantees under equity- linked contracts. *Journal of Risk and Insurance* 44, 639-680.

Boyle, P. P., and M. R. Hardy (1997): Reserving for maturity guarantees: two approaches. *Insurance: Mathematics and Economics* 21, 113-127.

Brennan, M. J., and E. S. Schwartz (1976): The pricing of equity-linked life insurance policies with an asset value guarantee. *J. Financial Economics* 3, 195-213.

Brennan, M. J., and E. S. Schwartz (1979): Alternative investment strategies for the issuers of equity-linked life insurance with an asset value guarantee. *Journal of Business* 52, 63-93.

Delbaen, F. (1986): Equity-linked policies. Bulletin Association Royal Actuaries Belges,80 33-52.

Föllmer, H., and P. Leukert (1999): Quantile hedging. *Finance Stochast.* 3, 251-273.

FÖllmer, H., and P. Leukert (2000): Efficient hedging: cost versus shortfall risk. *Finance Stochast.* 4, 117-146.

Föllmer, H., and A. Schied (2002): *Stochastic Finance: An introduction in discrete time.* Berlin – N. Y: Walter de Gruyter.

Krutchenko, R. N. and A., Melnikov (2001): Quantile hedging for a jump-diffusion financial market, *Trends in Mathematics, Ed M. Kohlmann, Birkhäuser Verlag Basel/ Switzerland,* 215-229.

Margrabe, W. (1978): The value of an option to exchange one asset for another, *Journal of finance* 33, 177-186.

Melnikov, A. V., S. N. Volkov, and M. L. Nechaev (2002): *Mathematics of Financial Obligations*. Providence, RI: American Math. Soc.

Melnikov, A. V. (2003): *Risk Analysis in Finance and Insurance*. Chapman & Hall/CRC.

Moeller, T. (1998): Risk-minimizing hedging strategies for unit-linked life insurance contracts. ASTIN Bulletin 28, 17-47.

Moeller, T. (2002): Hedging equity-linked life insurance contracts. *North American Actuarial Journal* 5, 79-95.

Shiryaev, A., N. (1999): Essentials of stochastic Finance, World Scientific.

46. Eduardo Rodriguez, *Scoring Methods for Risk Classification*, April 2000

47. R. Rodríguez-Ramos, L. Cortés-Carrasquero, J. Bravo-Castillero and R. Guinovart-Díaz, *Homogeneization in Elasto-plastic Composites*, June 2000

48. A. A. Alvarez-Mesquida, R. Rodríguez-Ramos, F. Comas, R. Esquivel and G. Monsivais, *Scattering of Shear Horizontal Elastic Waves in a Heterogeneous Piezoelectric Medium*, June 2000

49. M. L. Filshtinsky, R. Rodríguez-Ramos and O. Sanchez-Casals, *Fracture Mechanic in Piezoceramic Composite Plate*, June 2000

50. F. Lebon, R. Rodriguez-Ramos and A. Mesejo, *Homogenization and Wavelet-Galerkin Method for a Nonlinear One-dimensional Problem*, June 2000

51. L. Yang, *The Impact of Mortality Improvement on Social Security*, August 2000

52. Rodrigo Arias López and José Garrido, *Bounds and Other Properties of the Inverse, Moments of a Positive Binomial Variate*, September 2000

53. B. N. Dimitrov, Z. Khalil, M. E. Ghitany and V. V. Rykov, *Likelihood Ratio Test for Almost Lack of Memory Distributions*, November 2001

54. Yogendra P. Chaubey and Anthony Crisalli, *The Generalized Smoothing Estimator*, April 2002

55. Yogendra P. Chaubey and Pranab K. Sen, *Smooth Isotonic Estimation of Density, Hazard and MRL Functions*, April 2002

56. Pablo Olivares, *Maximum Likelihood Estimators for a Branching-Diffusion Process*, August 2002

57. Shuanming Li and José Garrido, *On Ruin for the Erlang*(n) *Risk Process*, June 2003

58. G. Jogesh Babu and Yogendra P. Chaubey, *Smooth Estimation of a Distribution and Density function on a Hypercube Using Bernstein Polynomials for Dependent Random Vectors* , August 2003

59. Shuanming Li and José Garrido, *On the Time Value of Ruin for a Sparre Anderson Risk Process Perturbed by Diffusion*, November 2003

60. Yogendra P. Chaubey, Cynthia M. DeSouza and Fassil Nebebe, *Bayesian Inference for Small Area Estimation under the Inverse Gaussian Model via Cibbs Sampling*, December 2003

61. Alexander Melnikov and Victoria Skornyakova, *Pricing of Equity–Linked Life Insurance Contracts with Flexible Guarantees*, May 2004

Copies of technical reports can be requested from:

Prof. Xiaowen Zhou
Department of Mathematics and Statistics
Concordia University
7141, Sherbrooke Street West
Montréal (QC) H4B 1R6 CANADA