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ON CHARACTERIZING DISTRIBUTIONS BY CONDITIONAL EXPECTATIONS OF FUNCTIONS OF GENERALIZED ORDER STATISTICS

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Let $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ be n generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) distribution. We give characterizations of distributions by means of $E\{\psi(X(s,n,m,k))|X(r,n,m,k) = x\} = g(x)$ and $E\{\psi(X(r,n,m,k))|X(s,n,m,k) = x\} = g(x), s > r$ under some mild conditions on $\psi(.)$ and $g(.)$. It is shown that most of the known characterization results based on conditional expectations are special cases of the results of this paper.

1. Introduction. Let $X_1, X_2, \ldots X_n$ be a random sample of size n from an absolutely continuous (with respect to Lebesgue measure) distribution function (df) $F(x)$ and the corresponding probability distribution function (pdf) $f(x)$. We will take the support of $F(x) = (\alpha, \beta)$, where $\alpha = inf\{x \in \mathbb{R},\ F(x) > 0\}$ and $\beta = sup\{x \in \mathbb{R},\ F(x) < 1\}$. Ferguson (1967) introduced the characterization of distributions based on the linearity of regression of adjacent order statistics $E(X_{r+1,n}|X_{r,n} = x)$ and its dual $E(X_{r,n}|X_{r+1,n} = x)$, where $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$ denote the corresponding order statistics. The characterization of distributions by the regression of non-adjacent order statistics was obtained by Wesolowski and Ahsanullah (1997). They gave the characterization of distributions by the following relation

$$E(X_{r+2,n}|X_{r,n} = x) = ax + b.$$

Dembinska and Wesolowski (1998) gave further generalization by characterizing distributions by means of the equation

$$E(X_{r+j,n}|X_{r,n} = x) = ax + b.$$

They used a result of Rao and Shanbhag (1994) dealing with extended version

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of the integrated Cauchy functional equation. The same result was proved earlier by Lopez-Blazquez and Moreno-Rebollo (1997) by using solution of a polynomial equation, see also Franco and Ruiz (1997). It may be remarked that Rao and Shanbhag's (1994) result is applicable only when the conditional expectation is a linear function.

As for the reverse order, Ferguson (1967) raised the question whether the linearity of $E(X_{r,n}|X_{r+2,n} = x)$ characterize distributions. Pudeg (1990) solved what Ferguson (1967) pointed out. Dembinska and Wesolowski (2000) characterized distributions using the relation

$$E(X_{r,n}|X_{r+j,n} = x) = ax + b.$$

Suppose that $\{X_i, i \geq 1\}$ is a sequence of independent and identically distributed (iid) random variables with absolutely continuous (with respect to Lebesgue measure) df $F(x)$ and pdf $f(x)$. Let $R_1 = X_1, R_2, \ldots$ be the upper record values of the sequence. Nagaraja (1977) characterized continuous distributions by using the relation

$$E(R_{r+1}|R_r = x) = ax + b.$$

Wesolowski and Ahsanullah (1997) extended the result of Nagaraja (1977) to non-adjacent record values. They characterized distributions by using the relation

$$E(R_{r+2}|R_r = x) = ax + b.$$

Nagaraja (1988) also characterized distributions by means of

$$E(R_r|R_{r+1} = x) = ax + b.$$

Other characterizations based on conditional expectations of non-adjacent record values are given in Raqab (2002), Wu (2004) and Wu and Lee (2001). Lopez-Blazquez and Moreno-Rebollo (1997) and Dembinska and Wesolowski (2000) characterized distributions by means of the relation

$$E(R_{r+j}|R_r = x) = ax + b,$$

as well as its dual under different set of conditions.

Gupta and Ahsanullah (2004) characterized distributions separately for order statistics and record values by means of $E\{\psi(X_{r+2,n})|X_{r,n} = x\} = g(x)$ and $E\{\psi(R_{r+2})|R_r = x\} = g(x)$, respectively.

Let $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ be n generalized order statistics. Then it is known that many ordered variables including order statistics, record values and k-record values are special cases of generalized order statistics. Thus, it will be interesting to characterize distributions based on generalized order statistics and to deduce the corresponding results of order statistics and record values as special cases. Keseling (1999) gave characterization of exponential distribution under the condition

$$E\{\psi(X(r+1,n,m,k)) - X(r,n,m,k)|X(r,n,m,k) = x\} = c,$$

where c is a constant. The general problem is to characterize distributions by means of

$$E\{\psi(X(r+j,n,m,k))|X(r,n,m,k) = x\} = g(x),$$

under some appropriate conditions on $\psi(.)$ and $g(.)$. Ahsanullah and Raqab (2004) proved that for $j = 2$, the above relation uniquely determines the distributions. The problem of characterization of distributions by means of

$$E\{\psi(X(r+j,n,m,k))|X(r,n,m,k) = x\} = g(x),$$

and its dual

$$E\{\psi(X(r,n,m,k))|X(r+j,n,m,k) = x\} = g(x),$$

for $1 \leq r \leq n-j,\ j \geq 1$, are unsolved.

In this paper we give general solution of these problems under some mild conditions on $\psi(.)$ and $g(.)$. It is shown that most of the known characterization results based on the conditional expectations of order statistics, record values and generalized order statistics are special cases of the results of this paper.

2. Main Results. Suppose $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ ($k \geq 1,\ m$ is a real number ≥ -1), are n generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df $F(x)$ and pdf $f(x)$. Their joint pdf $f_{1,2,\ldots,n}(x_1, x_2, \ldots, x_n)$ can be written as (see Kamps (1995), pp. 50-51)
(2.1)

$$f_{1,2,\ldots,n}(x_1,x_2,\ldots,x_n) = \begin{cases} k\prod_{j=1}^{n-1}\gamma_j \prod_{i=1}^{n-1}(\overline{F}(x_i))^m f(x_i)(\overline{F}(x_n))^{k-1} f(x_n), \\ F^{-1}(0) < x_1 < x_2 < \ldots < x_n < F^{-1}(1) \\ \\ 0, \text{otherwise} \end{cases}$$

where $\overline{F}(x) = 1 - F(x)$ and $\gamma_j = k + (n-j)(m+1),\ j = 1, 2, \cdots, n$.
The generalized order statistics are introduced by Kamps (1995) as a unified model for ordered random variables which includes among others order statistics, record values and k-record values as special cases. If $m = 0$ and $k = 1$, then $X(r, n, m, k)$ reduces to the r-th order statistic and (2.1) is the joint pdf of the n order statistics $X_{1,n} \leq X_{2,n} \leq \ldots \leq X_{n,n}$. If $k = 1$ and $m = -1$, then (2.1) is the joint pdf of the first n upper record values from a sequence of iid random variables with df $F(x)$ and pdf $f(x)$. For details of order statistics and upper record values, see David and Nagaraja (2003) and Ahsanullah (2004), respectively.

Integrating out $x_1, x_2, \ldots, x_{r-1}, x_{r+1}, \ldots,$ and x_n from (2.1), we get the pdf $f_{r,n,m,k}(x)$ of $X(r, n, m, k),\ 1 \leq r \leq n$ (see Kamps (1995), p. 64) as

$$f_{r,n,m,k}(x) = \frac{c_{r-1}}{(r-1)!}(\overline{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \tag{2.2}$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$,

$$g_m(x) = h_m(x) - h_m(0) = \begin{cases} \frac{1}{m+1}(1-(1-x)^{m+1}), & m \neq -1 \\ -ln(1-x), & m = -1,\ x \in [0,1) \end{cases}$$

and

$$h_m(x) = \begin{cases} -\frac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1,\ x \in [0,1). \end{cases}$$

Note, since $lim_{m \to -1}[\frac{1}{(m+1)}(1-(1-x)^{m+1})] = -ln(1-x)$, we will write $g_m(x) = [\frac{1}{(m+1)}(1-(1-x)^{m+1})]$, for all $x \in [0,1)$ and for all m with $g_{-1}(x) = lim_{m \to -1} g_m(x)$.

The joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k),\ 1 \leq r < s \leq n$, is given by (see Kamps (1995), p. 68)

$$\begin{aligned} f_{r,s,n,m,k}(x, y) &= \frac{c_{s-1}}{(r-1)!(s-r-1)!}(\overline{F}(x))^m f(x) g_m^{r-1}(F(x)) \\ &\quad \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}(\overline{F}(y))^{\gamma_s - 1} f(y),\ x < y. \end{aligned} \tag{2.3}$$

Noting, $m+1-\gamma_r = (m+1)-k-(n-r)(m+1) = -\gamma_{r+1}$, the conditional pdf of $X(s, n, m, k)$ given $X(r, n, m, k) = x,\ 1 \leq r < s \leq n$, have the following form

$$\begin{aligned} f_{s|r,n,m,k}(y|x) &= \frac{c_{s-1}}{c_{r-1}(s-r-1)!}[h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\quad \times \frac{(\overline{F}(y))^{\gamma_s - 1}}{(\overline{F}(x))^{\gamma_{r+1}}} f(y),\ x < y. \end{aligned} \tag{2.4}$$

We first prove a lemma which plays the pivotal role in our main result. We denote by $E\{\psi(Y)|X = x\}$, the conditional expectation of a function of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, i.e., $E\{\psi(X(s,n,m,k))|X(r,n,m,k) = x\}$, where $\psi(.)$ is an absolutely continuous and strictly monotonic function.

LEMMA 2.1. *If for two consecutive values r and $r+1$, $1 \le r < s-1 < n$,*

$$g_{s|r,n,m,k}(x) = E\{\psi(Y)|X = x\}$$

is finite and differentiable for all real x, then

$$\frac{g'_{s|r,n,m,k}(x)}{\gamma_{r+1}[g_{s|r,n,m,k}(x) - g_{s|r+1,n,m,k}(x)]} = \frac{f(x)}{\overline{F}(x)} \tag{2.5}$$

for all $x \in (\alpha,\beta)$, $\alpha = inf\{x : F(x) > 0\}$, $\beta = sup\{x : F(x) < 1\}$, where $\frac{f(x)}{\overline{F}(x)}$ is the failure rate.

PROOF. We have

$$E\{\psi(Y)|X = x\} = \int_x^\beta \psi(y) f_{s|r,n,m,k}(y|x)\, dy.$$

Using (2.4), we get

(2.6)

$$\begin{aligned}(\overline{F}(x))^{\gamma_{r+1}} g_{s|r,n,m,k}(x) &= \int_x^\beta \tfrac{c_{s-1}\psi(y)}{c_{r-1}(s-r-1)!}[h_m(F(y)) - h_m(F(x))]^{s-r-1} \\ &\quad \times (\overline{F}(y))^{\gamma_s - 1} f(y)\, dy.\end{aligned}$$

Differentiating both sides of (2.6), with respect to x, we get

$$\begin{aligned}&\gamma_{r+1}(\overline{F}(x))^{\gamma_{r+1}-1} g_{s|r,n,m,k}(x)(-f(x)) + (\overline{F}(x))^{\gamma_{r+1}} g'_{s|r,n,m,k}(x) \\ &= \int_x^\beta \tfrac{c_{s-1}\psi(y)}{c_{r-1}(s-r-2)!}[h_m(F(y)) - h_m(F(x))]^{s-r-2} \\ &\times \tfrac{d}{dx}(-h_m(F(x)))(\overline{F}(y))^{\gamma_s-1} f(y)\, dy.\end{aligned} \tag{2.7}$$

Observe that $-h_m(F(x)) = \frac{1}{m+1}(1 - F(x))^{m+1}$ and hence $\frac{d}{dx}[-h_m(F(x))] = -f(x)(\overline{F}(x))^m$. Using relation $c_r = c_{r-1}\gamma_{r+1}$, equation (2.7) becomes

$$\begin{aligned}&-\gamma_{r+1}(\overline{F}(x))^{\gamma_{r+1}-1} g_{s|r,n,m,k}(x) f(x) + (\overline{F}(x))^{\gamma_{r+1}} g'_{s|r,n,m,k}(x) \\ &= -\gamma_{r+1}(\overline{F}(x))^{\gamma_{r+2}+m} f(x) g_{s|r+1,n,m,k}(x).\end{aligned}$$

Further, using relation $\gamma_{r+2} + m = \gamma_{r+1} - 1$, the above equation gives

$$\gamma_{r+1}(\overline{F}(x))^{\gamma_{r+1}-1} f(x)(g_{s|r,n,m,k}(x) - g_{s|r+1,n,m,k}(x)) = (\overline{F}(x))^{\gamma_{r+1}} g'_{s|r,n,m,k}(x)$$

and hence,

$$\frac{g'_{s|r,n,m,k}(x)}{\gamma_{r+1}[g_{s|r,n,m,k}(x) - g_{s|r+1,n,m,k}(x)]} = \frac{f(x)}{\overline{F}(x)}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$. □

COROLLARY 2.1. *If* $g_{s|r,n,m,k}(x) = a_{s,r,n,m,k}\ \psi(x) + b_{s,r,n,m,k}$, *where* $a_{s,r,n,m,k} > 0$ *is a real number and* $b_{s,r,n,m,k}$ *is some constant, then*

$$\frac{a_{s,r,n,m,k}\psi'(x)}{\gamma_{r+1}[(a_{s,r,n,m,k} - a_{s,r+1,n,m,k})\psi(x) + (b_{s,r,n,m,k} - b_{s,r+1,n,m,k})]} = \frac{f(x)}{\overline{F}(x)}.$$

COROLLARY 2.2. *For order statistics (with* $k = 1$ *and* $m = 0$*), if* $g_{s|r,n,0,1}(x) = a_{s,r,n,0,1}\ \psi(x) + b_{s,r,n,0,1}$, *then*

$$\frac{a_{s,r,n,0,1}\ \psi'(x)}{(n-r)[(a_{s,r,n,0,1} - a_{s,r+1,n,0,1})\psi(x) + (b_{s,r,n,0,1} - b_{s,r+1,n,0,1})]} = \frac{f(x)}{\overline{F}(x)}$$

and $g_{s|r,n,0,1}(x) = E\{\psi(X_{s,n})|X_{r,n} = x\},\ s > r.$

COROLLARY 2.3. *For record values (with* $k = 1$ *and* $m = -1$*), if* $g_{s|r,n,-1,1}(x) = a_{s,r,n,-1,1}\ \psi(x) + b_{s,r,n,-1,1}$, *then*

$$\frac{a_{s,r,n,-1,1}\ \psi'(x)}{[(a_{s,r,n,-1,1} - a_{s,r+1,n,-1,1})\psi(x) + (b_{s,r,n,-1,1} - b_{s,r+1,n,-1,1})]} = \frac{f(x)}{\overline{F}(x)}$$

and $g_{s|r,n,-1,1}(x) = E\{\psi(R_s)|R_r = x\},\ s > r.$

If $\{X_i's, i \geq 1\}$ are iid from the exponential distribution $F(x) = 1 - exp\{-x\},\ x > 0$, denoted by $E(0,1)$, then $X(s,n,m,k) \stackrel{d}{=} \sum_{j=1}^{s} \frac{W_j}{\gamma_j}$, where $\stackrel{d}{=}$ denotes equal in distribution and $W_1, W_2, \ldots, W_s$ are iid with $W_i \in E(0,1)$. (see Ahsanullah (2000), p. 86-87). Also,

$$X(s,n,m,k)|X(r,n,m,k) \stackrel{d}{=} X(r,n,m,k) + \sum_{j=r+1}^{s} \frac{W_j}{\gamma_j}$$

and hence,

$$E\{X(s,n,m,k)|X(r,n,m,k) = x\} = x + \sum_{j=r+1}^{s} \frac{1}{\gamma_j}.$$

If we take $\psi(x) = x$, then $a_{s,r,n,m,k} = 1$ and $b_{s,r,n,m,k} = \sum_{j=r+1}^{s} \frac{1}{\gamma_j}$.

For order statistics (with $m = 0$ and $k = 1$), $\gamma_j = n-j+1,\ X(r,n,0,1) = X_{r,n},\ X(s,n,0,1) = X_{s,n}$ and

$$E(X_{s,n}|X_{r,n} = x) = x + \sum_{j=r+1}^{s} \frac{1}{n-j+1}.$$

If $s = r+1$, we get

$$E(X_{r+1,n}|X_{r,n} = x) = x + \frac{1}{n-r}.$$

For record values (with $m = -1$ and $k = 1$), $\gamma_j = 1,\ X(r,n,-1,1) = R_r,\ X(s,n,-1,1) = R_s$ and

$$E(R_s|R_r = x) = x + (s-r).$$

If $s = r+1$, we get

$$E(R_{r+1}|R_r = x) = x + 1.$$

Suppose $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ are n generalized order statistics from a Pareto distribution with df $F(x) = 1 - \frac{1}{(1+x)^\alpha},\ x > 0,\ \alpha > 0$. It can be shown that

$$1 + X(s,n,m,k) \overset{d}{=} (1+V_1)^{1/\gamma_1}(1+V_2)^{1/\gamma_2}\cdots(1+V_s)^{1/\gamma_s},$$

where V_i's are iid with $F(x) = 1 - \frac{1}{(1+x)^\alpha},\ x > 0,\ \alpha > 0$. Also,

$$X(s,n,m,k)|X(r,n,m,k) \overset{d}{=} (1+X(r,n,m,k))(1+V_{r+1})^{1/\gamma_{r+1}}$$
$$\times(1+V_{r+2})^{1/\gamma_{r+2}}\cdots(1+V_s)^{1/\gamma_s} - 1$$

and hence

$$E\{X(s,n,m,k)|X(r,n,m,k) = x\} = (1+x)\prod_{j=r+1}^{s}\frac{\alpha\gamma_j}{\alpha\gamma_j - 1} - 1$$
$$= x\prod_{j=r+1}^{s}\frac{\alpha\gamma_j}{\alpha\gamma_j - 1} + \prod_{j=r+1}^{s}\frac{\alpha\gamma_j}{\alpha\gamma_j} - 1$$

For order statistics (with $m = 0$ and $k = 1$), $\gamma_j = n-j+1$ and for $s > r$,

$$E(X_{s,n}|X_{r,n} = x) = (1+x)\prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)-1} - 1$$
$$= x\prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)-1} + \prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)} - 1.$$

For record values (with $m=-1$ and $k=1$), $\gamma_j = 1$ and for $s>r$,

$$\begin{aligned} E(R_s|R_r = x) &= x\prod_{j=r+1}^{s}\tfrac{\alpha}{\alpha-1} + \prod_{j=r+1}^{s}\tfrac{\alpha}{\alpha-1} - 1 \\ &= x(\tfrac{\alpha}{\alpha-1})^{s-r} + (\tfrac{\alpha}{\alpha-1})^{s-r} - 1,\ \alpha > 1. \end{aligned}$$

For power function distribution with cdf $F(x) = 1-(1-x)^{\alpha},\ 0<x<1,\ \alpha>0$,

$$1 - X(s,n,m,k) \stackrel{d}{=} (1-U_1)^{1/\gamma_1}(1-U_2)^{1/\gamma_2}\cdots(1-U_s)^{1/\gamma_s},$$

where U_i's are iid with df $F(x) = 1-(1-x)^{\alpha},\ 0<x<1,\ \alpha>0$. Also,

$$\begin{aligned} X(s,n,m,k)|X(r,n,m,k) &\stackrel{d}{=} 1-(1-X(r,n,m,k))(1-U_{r+1})^{1/\gamma_{r+1}} \\ &\quad \times(1-U_{r+2})^{1/\gamma_{r+2}}\cdots(1-U_s)^{1/\gamma_s} \end{aligned}$$

and hence

$$\begin{aligned} &E\{X(s,n,m,k)|X(r,n,m,k)=x\} = 1-(1-x)\prod_{j=r+1}^{s}\tfrac{\alpha\gamma_j}{\alpha\gamma_j+1} \\ &= x\prod_{j=r+1}^{s}\tfrac{\alpha\gamma_j}{\alpha\gamma_j+1} + 1 - \prod_{j=r+1}^{s}\tfrac{\alpha\gamma_j}{\alpha\gamma_j+1}. \end{aligned}$$

For order statistics (with $m=0$ and $k=1$), $\gamma_j = n-j+1$ and for $s>r$,

$$E(X_{s,n}|X_{r,n}=x) = x\prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)+1} + 1 - \prod_{j=r+1}^{s}\frac{\alpha(n-j+1)}{\alpha(n-j+1)+1}.$$

For record values (with $m=-1$ and $k=1$), $\gamma_j = 1$ and for $s>r$,

$$\begin{aligned} E(R_s|R_r = x) &= x\prod_{j=r+1}^{s}\tfrac{\alpha}{\alpha+1} + 1 - \prod_{j=r+1}^{s}\tfrac{\alpha}{\alpha+1} \\ &= x(\tfrac{\alpha}{\alpha+1})^{s-r} + 1 - (\tfrac{\alpha}{\alpha+1})^{s-r}. \end{aligned}$$

THEOREM 2.1. *Suppose $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ are n generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df $F(x)$ and pdf $f(x)$. If for an absolutely continuous and strictly monotonic function $\psi(.)$,*

$$E\{\psi(X(s,n,m,k))|X(r,n,m,k)=x\} = g_{s|r,n,m,k}(x)$$

for two consecutive values r *and* $r+1,\ 1 \le r < s-1 < n$ *and* $g_{s|r,n,m,k}(x)$ *is finite and differentiable for all real* x, *then*

$$\overline{F}(x) = exp\left\{-\int_{\alpha}^{x} M_{s,r,n,m,k}(u)\,du\right\}$$

for all $x \in (\alpha,\beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$, *where*

$$M_{s,r,n,m,k}(x) = \frac{g'_{s|r,n,m,k}(x)}{\gamma_{r+1}[g_{s|r,n,m,k}(x) - g_{s|r+1,n,m,k}(x)]}.$$

PROOF. By Lemma 2.1,

$$\frac{f(x)}{\overline{F}(x)} = \frac{g'_{s|r,n,m,k}(x)}{\gamma_{r+1}[g_{s|r,n,m,k}(x) - g_{s|r+1,n,m,k}(x)]} = M_{s,r,n,m,k}(x),\ \text{say.}$$

Thus

$$ln\overline{F}(x) = -\int_{\alpha}^{x} M_{s,r,n,m,k}(u)\,du$$

and hence

$$\overline{F}(x) = exp\left\{-\int_{\alpha}^{x} M_{s,r,n,m,k}(u)\,du\right\}$$

for all $x \in (\alpha,\beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$. □

In case of adjacent generalized order statistics we require only one conditional expectation in Theorem 2.1. For $s = r+1$,

$$\begin{aligned} g_{s|r+1,n,m,k}(x) = g_{r+1|r+1,n,m,k}(x) &= E\{\psi(X(r+1,n,m,k))|X(r+1,n,m,k) = x\} \\ &= \psi(x) \end{aligned}$$

and hence we have the following theorem.

THEOREM 2.2. *Suppose* $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ *are* n *generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df* $F(x)$ *and pdf* $f(x)$. *If for an absolutely continuous and strictly monotonic function* $\psi(.)$,

$$E\{\psi(X(r+1,n,m,k))|X(r,n,m,k) = x\} = g_{r+1|r,n,m,k}(x),$$

$1 \le r \le n$ *and* $g_{r+1|r,n,m,k}(x)$ *is finite and differentiable for all real* x, *then*

$$\overline{F}(x) = exp\left\{-\int_{\alpha}^{x} \frac{g'_{r+1|r,n,m,k}(x)}{\gamma_{r+1}[g_{r+1|r,n,m,k}(x) - \psi(x)]}\,du\right\}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$.

PROOF. We have from (2.6)

$$g_{r+1|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}} = \int_x^\beta \gamma_{r+1}\psi(y)(\overline{F}(y))^{\gamma_{r+1}-1} f(y)\, dy.$$

Differentiating the above equation with respect to x, we obtain

$$\begin{aligned} g'_{r+1|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}} &- \gamma_{r+1} g_{r+1|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}-1} f(x) \\ &= -\gamma_{r+1}\psi(x)(\overline{F}(x))^{\gamma_{r+1}-1} f(x) \end{aligned}$$

which gives

$$\frac{g'_{r+1|r,n,m,k}(x)}{\gamma_{r+1}[g_{r+1|r,n,m,k}(x) - \psi(x)]} = \frac{f(x)}{\overline{F}(x)}$$

and hence

$$\overline{F}(x) = exp\left\{ -\int_\alpha^x \frac{g'_{r+1|r,n,m,k}(x)}{\gamma_{r+1}[g_{r+1|r,n,m,k}(x) - \psi(x)]}\, du \right\}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$. $\square$

Furthermore, for a gap two also, we can get our characterization result in terms of only one conditional expectation but we have to assume $g_{s|r,n,m,k}(x)$ to be twice differentiable not once. We now have the following theorem.

THEOREM 2.3. *Suppose* $X(1, n, m, k), X(2, n, m, k), \ldots, X(n, n, m, k)$ *are* n *generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df* $F(x)$ *and pdf* $f(x)$*. If for an absolutely continuous and strictly monotonic function* $\psi(.)$,

$$E\{\psi(X(r + 2, n, m, k))|X(r, n, m, k) = x\} = g_{r+2|r,n,m,k}(x),$$

$1 \le r \le n$ *and* $g_{r+2|r,n,m,k}(x)$ *is finite and twice differentiable for all real* x*, then*

$$\overline{F}(x) = exp\left\{ -\int_\alpha^x h(u)\, du \right\}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$ *and* $r(x) = h(x)$ *is the solution of the equation*

$$\begin{aligned} g'_{r+2|r,n,m,k}(x)\frac{r'(x)}{r(x)} &- (\gamma_{r+1} + \gamma_{r+2}) g'_{r+2|r,n,m,k}(x) r(x) = g''_{r+2|r,n,m,k}(x) \\ &- \gamma_{r+1}\gamma_{r+2}(\psi(x) - g_{r+2|r,n,m,k}(x))(r(x))^2, \end{aligned}$$

where $r(x) = \frac{f(x)}{\overline{F}(x)}$.

PROOF. We have from (2.6)

$$\begin{aligned} g_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}} &= \int_x^\beta \gamma_{r+1}\gamma_{r+2}\psi(y)[h_m(F(y)) - h_m(F(x))] \\ &\quad \times (F(y))^{\gamma_{r+2}-1} f(y)\, dy. \end{aligned}$$

Differentiating the above equation with respect to x, we get

$$\begin{aligned} & g'_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}} - \gamma_{r+1} g_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+1}-1} f(x) \\ = & -\int_x^\beta \gamma_{r+1}\gamma_{r+2}\psi(y)(F(x))^m (F(y))^{\gamma_{r+2}-1} f(x) f(y)\, dy, \end{aligned}$$

which gives

$$\begin{aligned} & g'_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}}(\frac{1}{r(x)}) - \gamma_{r+1} g_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}} \\ = & -\int_x^\beta \gamma_{r+1}\gamma_{r+2}\psi(y)(F(y))^{\gamma_{r+2}-1} f(y)\, dy. \end{aligned}$$

Differentiating once again with respect to x, we obtain

$$\begin{aligned} & g''_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}}(\frac{1}{r(x)}) - \gamma_{r+2} g'_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}-1} f(x)(\frac{1}{r(x)}) \\ - & g'_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}} r'(x)(\frac{1}{r(x)})^2 - \gamma_{r+1} g'_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}} \\ + & \gamma_{r+1}\gamma_{r+2} g_{r+2|r,n,m,k}(x)(\overline{F}(x))^{\gamma_{r+2}-1} f(x) \\ = & \gamma_{r+1}\gamma_{r+2}\psi(x)(F(x))^{\gamma_{r+2}-1} f(x), \end{aligned}$$

which gives

$$\begin{aligned} & g'_{r+2|r,n,m,k}(x)\tfrac{r'(x)}{r(x)} + (\gamma_{r+1} + \gamma_{r+2}) g'_{r+2|r,n,m,k}(x) r(x) \\ = & g''_{r+2|r,n,m,k}(x) - \gamma_{r+1}\gamma_{r+2}(\psi(x) - g_{r+2|r,n,m,k}(x))(r(x))^2. \end{aligned} \tag{2.8}$$

With known functions $\psi(x)$ and $g_{r+2|r,n,m,k}(x)$, the above equation can be written as

$$r'(x) = H(x, r(x)). \tag{2.9}$$

We assume $r(x)$ and $r'(x)$ are continuous (equivalently, $f(x)$ and $f'(x)$ are continuous) in $x \in (\alpha, \beta)$. Then $H(x, r(x))$ as well as $\frac{d}{dr(x)} H(x, r(x))$ are continuous. Thus, the solution of $r(x) = h(x)$ is unique satisfying given boundary conditions (see Rabenstein (1966), Theorem 1, p.375). Let $r(x) = h(x)$ be the unique solution of (2.8), then $h(x) = \frac{f(x)}{\overline{F}(x)}$ and hence

$$\overline{F}(x) = exp\left\{- \int_{\alpha}^{x} h(u)\, du\right\}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$. We will use the boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$ to select the particular solution of $h(x)$. $\square$

Suppose $\psi(x) = x,\ g_{r+2|r,n,m,k}(x) = x + b\sigma,\ b > 0$, then we obtain from (2.8)

$$\frac{r'(x)}{r(x)} + (\gamma_{r+1} + \gamma_{r+2})r(x) = b\sigma\gamma_{r+1}\gamma_{r+2}(r(x))^2,$$

which gives

$$r'(x) = -(\gamma_{r+1} + \gamma_{r+2})(r(x))^2 + b\sigma\gamma_{r+1}\gamma_{r+2}(r(x))^3 \tag{2.10}$$

and hence $r(x) = \frac{(\gamma_{r+1}+\gamma_{r+2})}{b\sigma(\gamma_{r+1}\gamma_{r+2})}$ is the unique solution of (2.10).
The above relation gives a characterization of the exponential distribution. Note, for the exponential distribution $F(x) = 1 - exp\{-x/\sigma\},\ x > 0$ we have $r(x) = 1/\sigma$ and

$$E\{X(r+2, n, m, k)|X(r, n, m, k) = x\} = x + \frac{\sigma(\gamma_{r+1} + \gamma_{r+2})}{\gamma_{r+1}\gamma_{r+2}}.$$

For the general case (i.e., for a gap more than two), in terms of single conditional expectation, the problem becomes more complicated because of the resulting differential equation. Hence, we use Theorem 2.1.

2.1. *Applications.*

Most of the known characterization results based on conditional expectations of order statistics, record values and generalized order statistics can easily be deduced as special cases of Theorem 2.1, Theorem 2.2 and Theorem 2.3. We mention few.

1. If we take $\psi(x) = x,\ s = r + 1,\ g_{r+1|r,n,m,k}(x) = a_{r+1,r,n,m,k}\ x + b_{r+1,r,n,m,k}$ with $m = 0$ and $k = 1$, then using Theorem 2.2, we get the result of Ferguson (1967) for order statistics.

2. If we take $\psi(x) = x, \ s = r+1, \ g_{r+1|r,n,m,k}(x) = a_{r+1,r,n,m,k} \ x + b_{r+1,r,n,m,k}$ with $m = -1$ and $k = 1$, then using Theorem 2.2, we get the result of Nagaraja (1977) for record values.

3. If we take $\psi(x) = x, \ s = r+2, \ g_{r+2|r,n,m,k}(x) = a_{r+2,r,n,m,k} \ x + b_{r+2,r,n,m,k}$ with $m = 0$ and $k = 1$, then using Theorem 2.1 (or Theorem 2.3), we get the result of Wesolowski and Ahsanullah (1997) for order statistics.

4. If we take $\psi(x) = x, \ s = r+2, \ g_{r+2|r,n,m,k}(x) = a_{r+2,r,n,m,k} \ x + b_{r+2,r,n,m,k}$ with $m = -1$ and $k = 1$, then using Theorem 2.1 (or Theorem 2.3), we get the result of Ahsanullah and Wesolowski (1998) for record values.

5. If we take $\psi(x) = x, \ g_{s|r,n,m,k}(x) = a_{s,r,n,m,k} \ x + b_{s,r,n,m,k}$ with $m = 0$ and $k = 1$, then using Theorem 2.1, we get the result of Dembinska and Wesolowski (1998) for order statistics.

6. If we take $\psi(x) = x, \ g_{s|r,n,m,k}(x) = a_{s,r,n,m,k} \ x + b_{s,r,n,m,k}$ with $m = -1$ and $k = 1$, then using Theorem 2.1, we get the result of Dembinska and Wesolowski (2000) for record values.

7. Gupta and Ahsanullah (2004) gave expression for $E\{\psi(X_{k+s,n})|X_{k,n} = x\}$ and $E\{\psi(R_{k+s})|R_k = x\}$. They were successful in giving unique solutions for $s = 2$ by using relation $E\{\psi(X_{k+s,n})|X_{k,n} = x\} = g(x)$ for order statistics and by $E\{\psi(R_{k+s})|R_k = x\} = g(x)$ for record values. Their results are special cases of Theorem 2.1 (or Theorem 2.3) with $s = 2$ ($m = 0, \ k = 1$ for order statistics and $m = -1, \ k = 1$ for record values).

8. In Keseling (1999) the exponential distribution is proved to be the only continuous distribution with constant regression

$$E\{\psi(X(r+1, n, m, k)) - X(r, n, m, k)|X(r, n, m, k) = x\} = c$$

and $\psi(.)$ is a real monotonic function, using the result of Rao and Shanbhag (1994) for order statistics. This can be easily deduced from Theorem 2.2 with $s = r+1$ and $g_{r+1|r,n,m,k}(x) = x + b_{r+1,r,n,m,k}$. Keseling considered $m \neq -1$.

9. Ahsanullah and Raqab (2004) proved that

$$E\{\psi(X(r+2, n, m, k))|X(r, n, m, k) = x\} = g(x)$$

uniquely determines the distributions. Their result is a special case of Theorem 2.1 (or Theorem 2.3) with $s = r+2$.

3. Characterizations by reverse ordering. In this section we will characterize distributions by means of

(3.1) $E\{\psi(X(r,n,m,k))|X(s,n,m,k)=x\} = g_{r|s,n,m,k}(x),\ 1 \le r < s \le n.$

From (2.2) and (2.3), the conditional pdf of $X(r,n,m,k)$ given $X(s,n,m,k),\ 1 \le r < s \le n$ is given by

$$\begin{aligned} f_{r|s,n,m,k}(y|x) &= \tfrac{(s-1)!}{(r-1)!(s-r-1)!}(\overline{F}(y))^m \left[\tfrac{1-(\overline{F}(y))^{m+1}}{(m+1)}\right]^{r-1} \\ &\times \left[\tfrac{(\overline{F}(y))^{m+1}-(\overline{F}(x))^{m+1}}{(m+1)}\right]^{s-r-1} \left[\tfrac{(m+1)}{1-(\overline{F}(x))^{m+1}}\right]^{s-1} f(y),\ y<x. \end{aligned} \tag{3.2}$$

Note that

$$lim_{m\to-1}\left[\frac{1-(\overline{F}(y))^{m+1}}{(m+1)}\right] = -ln\overline{F}(y)$$

and

$$lim_{m\to-1}\left[\frac{(\overline{F}(y))^{m+1}-(\overline{F}(x))^{m+1}}{(m+1)}\right] = ln\overline{F}(y) - ln\overline{F}(x).$$

We will use (3.2) for all values of m and take the limiting values for $m=-1$. We now have the following lemma.

LEMMA 3.1. *If for two consecutive values $s-1$ and s, $1 \le r < s-1 < n$,*

$$g_{r|s,n,m,k}(x) = E\{\psi(Y)|X=x\}$$

is finite and differentiable for all real x, then

$$\frac{g'_{r|s,n,m,k}(x)}{(s-1)[g_{r|s-1,n,m,k}(x)-g_{r|s,n,m,k}(x)]} = \frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]} \tag{3.3}$$

for all $x \in (\alpha,\beta),\ \alpha = inf\{x : F(x)>0\},\ \beta = sup\{x : F(x)<1\}$.

PROOF. We have

$$E\{\psi(Y)|X=x\} = \int_\alpha^x \psi(y) f_{r|s,n,m,k}(y|x)\,dy.$$

Utilizing (3.2), we get

$$\begin{aligned} \left[\tfrac{1-(\overline{F}(x))^{m+1}}{(m+1)}\right]^{s-1} g_{r|s,n,m,k}(x) &= \textstyle\int_\alpha^x \psi(y) \tfrac{(s-1)!}{(r-1)!(s-r-1)!} \\ \times(\overline{F}(y))^m \left[\tfrac{1-(\overline{F}(y))^{m+1}}{(m+1)}\right]^{r-1} &\left[\tfrac{(\overline{F}(y))^{m+1}-(\overline{F}(x))^{m+1}}{(m+1)}\right]^{s-r-1} f(y)\,dy \end{aligned} \tag{3.4}$$

Let $\phi(x)=[\frac{1-(\overline{F}(x))^{m+1}}{(m+1)}]^{s-1}$, then differentiating both sides of (3.4) with respect to x, we get

$$
\begin{aligned}
&g'_{r|s,n,m,k}(x)\phi(x)+g_{r|s,n,m,k}(x)\phi'(x)=(s-1)g_{r|s-1,n,m,k}(x) \\
&\times\left[\tfrac{1-(\overline{F}(x))^{m+1}}{(m+1)}\right]^{s-2}(\overline{F}(x))^m f(x)
\end{aligned} \tag{3.5}
$$

On simplification , we get from (3.5)

$$
\begin{aligned}
&g'_{r|s,n,m,k}(x)+g_{r|s,n,m,k}(x)\left[\tfrac{\phi'(x)}{\phi(x)}\right]=\left[\tfrac{(s-1)}{\phi(x)}\right]g_{r|s-1,n,m,k}(x) \\
&\times\left[\tfrac{1-(\overline{F}(x))^{m+1}}{(m+1)}\right]^{s-2}(\overline{F}(x))^m f(x)
\end{aligned}
$$

and hence

$$
\frac{g'_{r|s,n,m,k}(x)}{(s-1)[g_{r|s-1,n,m,k}(x)-g_{r|s,n,m,k}(x)]}=\frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]}
$$

for all $x\in(\alpha,\beta),\ \alpha=inf\{x:F(x)>0\},\ \beta=sup\{x:F(x)<1\}.$ □

COROLLARY 3.1. *If* $g_{r|s,n,m,k}(x)=a_{r,s,n,m,k}\ \psi(x)+b_{r,s,n,m,k}$, *where* $a_{r,s,n,m,k}>0$ *is a real number and* $b_{r,s,n,m,k}$ *is some constant, then*

$$
\frac{a_{r,s,n,m,k}\ \psi'(x)}{(s-1)[(a_{r,s-1,n,m,k}-a_{r,s,n,m,k})\psi(x)+(b_{r,s-1,n,m,k}-b_{r,s,n,m,k})]}
$$
$$
=\frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]}.
$$

COROLLARY 3.2. *For order statistics (with* $k=1$ *and* $m=0$*), if* $g_{r|s,n,0,1}(x)=a_{r,s,n,0,1}\ \psi(x)+b_{r,s,n,0,1}$, *then*

$$
\frac{a_{r,s,n,0,1}\ \psi'(x)}{(s-1)[(a_{r,s-1,n,0,1}-a_{r,s,n,0,1})\psi(x)+(b_{r,s-1,n,0,1}-b_{r,s,n,0,1})]}=\frac{f(x)}{F(x)},
$$

which is the retro-hazard function and $g_{r|s,n,0,1}(x)=E\{\psi(X_{r,n})|X_{s,n}=x\},\ s>r.$

COROLLARY 3.3. *For record values (with* $k=1$ *and* $m=-1$*), if* $g_{r|s,n,-1,1}(x)=a_{r,s,n,-1,1}\ \psi(x)+b_{r,s,n,-1,1}$, *then*

$$
\frac{a_{r,s,n,-1,1}\ \psi'(x)}{(s-1)[(a_{r,s-1,n,-1,1}-a_{r,s,n,-1,1})\psi(x)+(b_{r,s-1,n,-1,1}-b_{r,s,n,-1,1})]}
$$

$$= \frac{f(x)}{\overline{F}(x)}[-\frac{1}{ln\overline{F}(x)}]$$

and $g_{r|s,n,-1,1}(x) = E\{\psi(R_r)|R_s = x\},\ s > r.$

THEOREM 3.1. *Suppose* $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ *are* n *generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df* $F(x)$ *and pdf* $f(x)$. *If for an absolutely continuous and strictly monotonic function* $\psi(.)$,

$$E\{\psi(X(r,n,m,k))|X(s,n,m,k) = x\} = g_{r|s,n,m,k}(x)$$

for two consecutive values $s-1$ *and* s, $1 \le r < s-1 < n$ *and* $g_{r|s,n,m,k}(x)$ *is finite and differentiable for all real* x, *then*

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta M_{r,s,n,m,k}(u)\,du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha,\beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$, *where*

$$M_{r,s,n,m,k}(x) = \frac{g'_{r|s,n,m,k}(x)}{(s-1)[g_{r|s-1,n,m,k}(x) - g_{r|s,n,m,k}(x)]}.$$

PROOF. By Lemma 3.1,

$$\frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]} = \frac{g'_{r|s,n,m,k}(x)}{(s-1)[g_{r|s-1,n,m,k}(x) - g_{r|s,n,m,k}(x)]} = M_{r,s,n,m,k}(x),\ \text{say.}$$

Thus

$$ln[1-(\overline{F}(x))^{m+1}] = \int_x^\beta M_{r,s,n,m,k}(u)\,du$$

and hence

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta M_{r,s,n,m,k}(u)\,du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha,\beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}.$ □

In reverse order case also, for adjacent generalized order statistics, we require only one conditional expectation in Theorem 3.1. For $s = r+1$,

$$\begin{aligned} g_{r|s-1,n,m,k}(x) = g_{r|r,n,m,k}(x) &= E\{\psi(X(r,n,m,k))|X(r,n,m,k) = x\} \\ &= \psi(x) \end{aligned}$$

and hence we have the following theorem.

THEOREM 3.2. *Suppose* $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ *are* n *generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df* $F(x)$ *and pdf* $f(x)$. *If for an absolutely continuous and strictly monotonic function* $\psi(.)$,

$$E\{\psi(X(r,n,m,k))|X(r+1,n,m,k)=x\} = g_{r|r+1,n,m,k}(x),$$

$1 \le r \le n$ *and* $g_{r|r+1,n,m,k}(x)$ *is finite and differentiable for all real* x, *then*

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta \frac{g'_{r|r+1,n,m,k}(x)}{r[\psi(x) - g_{r|r+1,n,m,k}(x)]}\, du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$.

PROOF. We have from (3.4)

$$g_{r|r+1,n,m,k}(x)\left[\frac{1-(\overline{F}(x))^{m+1}}{m+1}\right]^r = \int_\alpha^x r\psi(y)(\overline{F}(y))^m \left[\frac{1-(\overline{F}(y))^{m+1}}{m+1}\right]^{r-1} f(y)\, dy.$$

For $m = -1$, we will take the limiting values (see p.11). Differentiating the above equation with respect to x, we obtain

$$g'_{r|r+1,n,m,k}(x)\left[\frac{1-(\overline{F}(x))^{m+1}}{m+1}\right]^r + g_{r|r+1,n,m,k}(x) r \left[\frac{1-(\overline{F}(x))^{m+1}}{m+1}\right]^{r-1}$$
$$\times(\overline{F}(x))^m f(x) = r\psi(x)(\overline{F}(x))^m \left[\frac{1-(\overline{F}(x))^{m+1}}{m+1}\right]^{r-1} f(x),$$

which gives

$$\frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]} = \frac{g'_{r|r+1,n,m,k}(x)}{r[\psi(x) - g_{r|r+1,n,m,k}(x)]}$$

and hence

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta \frac{g'_{r|r+1,n,m,k}(u)}{r[\psi(u) - g_{r|r+1,n,m,k}(u)]}\, du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha, \beta),\ \alpha = inf\{x : F(x) > 0\},\ \beta = sup\{x : F(x) < 1\}$. □

Furthermore, for a gap two in reverse order case also, we can get our characterization result in terms of only one conditional expectation but we have to assume $g_{r|s,n,m,k}(x)$ to be twice differentiable not once. We now have the following theorem.

THEOREM 3.3. *Suppose $X(1,n,m,k), X(2,n,m,k), \ldots, X(n,n,m,k)$ are n generalized order statistics from an absolutely continuous (with respect to Lebesgue measure) df $F(x)$ and pdf $f(x)$. If for an absolutely continuous and strictly monotonic function $\psi(.)$,*

$$E\{\psi(X(r,n,m,k))|X(r+2,n,m,k)=x\} = g_{r|r+2,n,m,k}(x),$$

$1 \leq r \leq n$ and $g_{r|r+2,n,m,k}(x)$ is finite and twice differentiable for all real x, then

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta h(u)\, du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha, \beta)$, $\alpha = inf\{x : F(x) > 0\}$, $\beta = sup\{x : F(x) < 1\}$ and $v(x) = h(x)$ is the solution of the equation

$$\begin{aligned} g''_{r|r+2,n,m,k}(x)\frac{1}{v(x)} &+ 2(r+1)g'_{r|r+2,n,m,k}(x) - g'_{r|r+2,n,m,k}(x)\left[1 + \frac{v'(x)}{v^2(x)}\right] \\ &+ r(r+1)g_{r|r+2,n,m,k}(x)v(x) = r(r+1)\psi(x)v(x), \end{aligned}$$

where $v(x) = \frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]}$.

PROOF. We have from (3.4)

$$\begin{aligned} g_{r|r+2,n,m,k}(x)\left[\frac{1-(\overline{F}(x))^{m+1}}{m+1}\right]^{r+1} &= \int_\alpha^x r(r+1)\psi(y)(\overline{F}(y))^m \\ \times \left[\frac{1-(\overline{F}(y))^{m+1}}{m+1}\right]^{r-1} &\left[\frac{(\overline{F}(y))^{m+1} - (\overline{F}(x))^{m+1}}{m+1}\right] f(y)\, dy. \end{aligned}$$

Let $\varphi(x) = \frac{1-(\overline{F}(x))^{m+1}}{m+1}$, then the above equation becomes
(3.6)

$$g_{r|r+2,n,m,k}(x)(\varphi(x))^{r+1} = \int_\alpha^x r(r+1)\psi(y)(\overline{F}(y))^m(\varphi(y))^{r-1}(\varphi(x) - \varphi(y))f(y)\, dy.$$

Differentiating (3.6) with respect to x, we get

$$\begin{aligned} g'_{r|r+2,n,m,k}(x)\left[\frac{(\varphi(x))^{r+1}}{\varphi'(x)}\right] &+ (r+1)g_{r|r+2,n,m,k}(x)(\varphi(x))^r \\ &= \int_\alpha^x r(r+1)\psi(y)(\overline{F}(y))^m(\varphi(y))^{r-1}f(y)\, dy. \end{aligned}$$

Differentiating the above equation with respect to x, we obtain

$$\begin{aligned} & g''_{r|r+2,n,m,k}(x)\left[\frac{(\varphi(x))^{r+1}}{\varphi'(x)}\right] + (r+1)g'_{r|r+2,n,m,k}(x)(\varphi(x))^{r} \\ - & g'_{r|r+2,n,m,k}(x)\left[\frac{(\varphi(x))^{r+1}}{(\varphi'(x))^2}\right]\varphi''(x) + (r+1)g'_{r|r+2,n,m,k}(x)(\varphi(x))^{r} \\ + & r(r+1)g_{r|r+2,n,m,k}(x)(\varphi(x))^{r-1}(\varphi'(x)) \\ = & r(r+1)\psi(x)(\overline{F}(x))^m(\varphi(x))^{r-1}f(x), \end{aligned}$$

which gives

$$\begin{aligned} & g''_{r|r+2,n,m,k}(x)\left[\frac{(\varphi(x))}{\varphi'(x)}\right] + 2(r+1)g'_{r|r+2,n,m,k}(x) \\ - & g'_{r|r+2,n,m,k}(x)\left[\frac{(\varphi(x))}{(\varphi'(x))^2}\right]\varphi''(x) + r(r+1)g_{r|r+2,n,m,k}(x)\left[\frac{\varphi'(x)}{\varphi(x)}\right] \\ = & r(r+1)\psi(x)\left[\frac{\varphi'(x)}{\varphi(x)}\right]. \end{aligned}$$

Let $v(x) = \frac{\varphi'(x)}{\varphi(x)}$, then $v'(x) = \frac{\varphi''(x)}{\varphi(x)} - \left[\frac{\varphi'(x)}{\varphi(x)}\right]^2$ and we obtain from the above equation

(3.7)

$$\begin{aligned} & g''_{r|r+2,n,m,k}(x)\tfrac{1}{v(x)} + 2(r+1)g'_{r|r+2,n,m,k}(x) - g'_{r|r+2,n,m,k}(x)\left[1 + \tfrac{v'(x)}{v^2(x)}\right] \\ + & r(r+1)g_{r|r+2,n,m,k}(x)v(x) = r(r+1)\psi(x)v(x). \end{aligned}$$

With known functions $\psi(x)$ and $g_{r|r+2,n,m,k}(x)$, the above equation can be written as

$$v'(x) = G(x, v(x)),\ v(x) = \frac{\varphi'(x)}{\varphi(x)} = \frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]}. \tag{3.8}$$

We assume that $v(x)$ and $v'(x)$ are continuous (equivalently, $f(x)$ and $f'(x)$ are continuous) in $x \in (\alpha, \beta)$. Then $G(x, v(x))$ as well as $\frac{d}{dv(x)}G(x, v(x))$ are continuous. Thus, we have an unique solution for $v(x)$ satisfying given boundary conditions (see Rabenstein (1966), Theorem 1, p.375.). Suppose $v(x) = h(x)$ is the unique solution of (3.7), then

$$h(x) = \frac{(m+1)(\overline{F}(x))^m f(x)}{[1-(\overline{F}(x))^{m+1}]}$$

and hence

$$\overline{F}(x) = \left[1 - exp\left\{\int_x^\beta h(u)\, du\right\}\right]^{1/(m+1)}$$

for all $x \in (\alpha, \beta)$, $\alpha = inf\{x : F(x) > 0\}$, $\beta = sup\{x : F(x) < 1\}$. We will use the boundary conditions $F(\alpha) = 0$ and $F(\beta) = 1$ to select the particular solution of $h(x)$. $\square$

Suppose $\psi(x) = x$, $g_{r|r+2,n,m,k}(x) = x - b$, $b > 0$, then from (3.7) we obtain

$$2(r+1) - \left[1 + \frac{v'(x)}{v^2(x)}\right] + r(r+1)(x-b)v(x) = r(r+1)xv(x),$$

which gives

$$\frac{v'(x)}{v^2(x)} = 2r + 1 - r(r+1)bv(x). \tag{3.9}$$

Under the assumption that $v(x)$ and $v'(x)$ are continuous for all $x \in (\alpha, \beta)$, we have $v(x) = \frac{2r+1}{r(r+1)b}$, the unique solution of (3.9). Hence,

$$\frac{(m+1)(\overline{F}(x))^m f(x)}{[1 - (\overline{F}(x))^{m+1}]} = \frac{2r+1}{r(r+1)b}, \ b > 0. \tag{3.10}$$

In case of order statistics for a gap two with $m = 0$, we get

$$F(x) = c \ exp\left\{\frac{(2r+1)x}{r(r+1)b}\right\}.$$

Since $F(\alpha) = 0$ and $F(\beta) = 1$, we must have

$$F(x) = exp\left\{\frac{(2r+1)(x-\beta)}{r(r+1)b}\right\}, \ x \in (\alpha, \beta); \ \alpha = -\infty, \ \beta < \infty.$$

For the general case (i.e., for a gap more than two) in reverse order case also, in terms of single conditional expectation, the problem becomes more complicated because of the resulting differential equation. Hence, we use Theorem 3.1.

3.1. *Applications.*

In reverse order, most of the known results based on conditional expectations of order statistics, record values and generalized order statistics are special cases of Theorem 3.1, Theorem 3.2 and Theorem 3.3. We mention few.

1. If we take $\psi(x) = x,\ s = r+1,\ g_{r|r+1,n,m,k}(x) = a_{r,r+1,n,m,k}\ x + b_{r,r+1,n,m,k}$ with $m = 0$ and $k = 1$, then using Theorem 3.2, we get the result of Ferguson (1967) for order statistics.

2. If we take $\psi(x) = x,\ s = r+1,\ g_{r|r+1,n,m,k}(x) = a_{r,r+1,n,m,k}\ x + b_{r,r+1,n,m,k}$ with $m = -1$ and $k = 1$, then using Theorem 3.2, we get the result of Nagaraja (1988) for record values.

3. If we take $\psi(x) = x,\ g_{r|s,n,m,k}(x) = a_{r,s,n,m,k}\ x + b_{r,s,n,m,k}$ with $m = 0$ and $k = 1$ (with $m = -1$ and $k = 1$), then using Theorem 3.1, we get the result of Dembinska and Wesolowski (2000) for order statistics (record values).

4. If we take $\psi(x) = x,\ s = r+1,\ g_{r|r+1,n,m,k}(x) = a_{r,r+1,n,m,k}\ x + b_{r,r+1,n,m,k}$, then using Theorem 3.2, we get the result of Keseling (1999) for generalized order statistics. Keseling considered $m \neq -1$.

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