Technical Report No. 01/05, February 2005

EFFICIENT HEDGING METHODOLOGY APPLIED TO EQUITY-LINKED LIFE INSURANCE

Alexander Melnikov and Victoria Skornyakova

Efficient Hedging Methodology Applied to Equity-linked Life Insurance

Alexander Melnikov
Department of Mathematical and Statistical Sciences,
University of Alberta, Edmonton, Alberta, Canada
melnikov@ualberta.ca
Victoria Skornyakova
Mercer Investment Consulting,
Montreal, Quebec, Canada
victoria.skornyakova@mercer.com

Abstract

In this paper we study efficient hedging and its applications to the pricing of equity-linked life insurance contracts. We devote our attention to the pure endowment contracts with a flexible guarantee. In our setting, these insurance instruments are based on two risky assets of the market controlled by the Black-Scholes model during the contract period. The first asset is responsible for the maximal size of future profit while the second provides a flexible guarantee for the insured.
The insurance company is considered as a hedger of a maximum of two risky assets as a contingent claim in this market. The contract is exercised if the insured is still alive at the maturity time and cannot be perfectly hedged in view of a positive survival probability of a client. To provide an appropriate risk-management in connection of such a contract, the company should exploit some imperfect hedging forms. Here we propose the use of efficient hedging with a power loss function.
Specifying developments in this area, we create the pricing methodology for the insurance contracts under consideration in terms of a generalized Margrabe's formula. The results are illustrated by a numerical actuarial analysis with the indices Russell 2000 and Dow Jones Industrial Average.

Key words: equity-linked life insurance, efficient hedging, flexible guarantee, pure endowment, Margrabe's formula.

JEL Classification: G10, G12, D81.

1. Introduction

These innovative insurance contracts combine both financial and insurance risks and allow insurance companies to be competitive in the modern financial system. Therefore, the problem of the pricing of these contracts should be very important for various insurance institutions. The first papers on this topic (see Brennan and Schwartz (1976, 1979), Boyle and Schwartz (1977)) were devoted to pure endowment contracts with fixed guarantee. The authors of these papers recognized a close connection of equity-linked life insurance and option pricing discovered several years before by Black, Scholes, and Merton. They recognized that the payoff of such an insurance contract is identical to the payoff of a call option with the strike price equal to the guarantee plus some fixed amount. We should also mention the papers by Delbaen (1986), and Bacinello and Ortu (1993) where the contracts with guarantees modeled as deterministic functions were studied by an application of the Black-Scholes-Merton option pricing theory and the Monte-Carlo numerical technique. Aase and Person (1994) were probably the first who gave relatively strong mathematical theory of the pricing of equity-linked life insurance policies based on perfect hedging. Boyle and Hardy (1997) investigated advantages and disadvantages of methods available for such pricing including transaction costs. Moeller (1998, 2002) proposed another approach exploiting the mean-variance hedging theory by Follmer-Sondermann-Schweizer. He gave a full description of the optimal (in the mean-variance sense) initial prices and hedging strategies for the contracts with fixed guarantees.

To complete our brief historical description of the subject we would like to mention the books of Hardy (2003) and Melnikov (2004a). The first one can be regarded as rich source of general information about equity-linked contracts. The second one develops quantitative methods of risk-management in finance and insurance, and devoted to mathematical aspects of the pricing of equity-linked life insurance.

We study here the contracts pure endowment with flexible guarantees, and therefore, the contracts with fixed and deterministic guarantees can be regarded as particular cases. In our setting, these contracts are based on two risky assets of the Black-Scholes model on $[0,T]$. The first asset S_T^1 is more profitable while the second one S_T^2 is more reliable. So, S_t^1 can be considered as possible gains of insured and S_T^2 as a flexible guarantee. We consider the insurance company as a hedger of the claim $\max(S_T^1, S_T^2)$. It is exercised only if the insured is still alive at the time T. This contingent claim cannot be hedged perfectly due to survival probability, which is less than one. Therefore, the initial capital of any admissible strategy is strictly less than the initial capital of a perfect hedge. In this situation, we must use another type of hedging when the optimal strategy is obtained by minimization of the expected value of some loss function under the above restriction of the initial capital of the strategy. This is efficient hedging introduced and developed by Follmer and Leukert (2000). We apply their general results to the case of the power loss function and obtain the concrete formulae in case of our pricing problem in terms of Margrabe’s formula (1978). We formulate our results in a form that is most convenient for actuarial analysis where the survival probability is given as a quantitative characteristic of insurance component of risk, and is compared with a pure financial risk component. Finally, we give a numerical actuarial analysis based on our theoretical results and real financial data for assets S^1 (the Russell 2000) and S^2 (the Dow Jones Industrial Average).

2. Basic notions and auxiliary results

The paper presents new actuarial methodology of pricing equity-linked life insurance contracts with a flexible guarantee. Keeping in mind both, the theoretical and applied character of the paper, we consider the following Black-Scholes model for the underlying risky asset S^1 and flexible guarantee S^2:

$$dS_t^i = S_t^i(\mu_i dt + \sigma_i dW_t),\ i = 1,2,\ t \le T. \tag{2.1}$$

Here μ_i and σ_i are the rate of return and volatility of the asset S^i, $W = (W_t)_{t\le T}$ is a Wiener process defined on a standard stochastic basis $(\Omega, F, \mathbf{F} = (F_t)_{t\le T}, P)$.

We note that the model (2.1) can be rewritten with the help of Kolmogorov-Ito's formula as follows (see Shiryaev (1999)):

$$S_t^i = S_0^i \exp\left\{\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i W_t\right\} \tag{2.2}$$

For the non-risky asset B, we assume for simplicity that $B_t = 1$. Therefore, the risk-free interest rate is equal to zero.

We consider the model (2.1)-(2.2) under a technical condition $\mu_1\sigma_2 = \mu_2\sigma_1$ to make risky asset S^1 and guarantee S_t^2 by martingales over a measure P^* with the density with respect to P

$$Z_T = \exp\left\{-\frac{\mu_1}{\sigma_1}W_T - \frac{1}{2}\left(\frac{\mu_1}{\sigma_1}\right)^2 T\right\}. \tag{2.3}$$

Remark 2.1. According to the Girsanov theorem, the process $W_t^* = W_T + \frac{\mu_1}{\sigma_1}t$

is a Wiener process with respect to P^*.

Remark 2.2. The case of a constant guarantee can be included in our consideration with $\sigma_2 = \mu_2 = 0$.

Remark 2.3. We consider here the model where both the guarantee S_t^2 and risky asset S_t^1 are generated by the same Wiener process. The case where these assets are modeled by different correlated Wiener processes is studied in another paper by the first author. Finally, as a direct corollary of (2.2) we note the following useful representation of the guarantee S_t^2 through the underlying risky asset S_t^1:

$$
\begin{aligned}
S_t^2 &= S_0^2 \exp\left\{\sigma_2 W_t + \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t\right\} \\
&= S_0^2 \exp\left\{\frac{\sigma_2}{\sigma_1}\left(\sigma_1 W_t + \left(\mu_1 - \frac{\sigma_1^2}{2}\right)t\right) - \frac{\sigma_2}{\sigma_1}\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t\right\} \\
&= S_0^2 \left(S_0^1\right)^{-\sigma_2/\sigma_1} \left(S_t^1\right)^{\sigma_2/\sigma_1} \exp\left\{-\frac{\sigma_2}{\sigma_1}\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t\right\}.
\end{aligned}
$$

We are going to study the pure endowment contracts with flexible guarantee based on the model (2.1)-(2.2). Assume that a client at age x has the remaining lifetime $T(x)$. This random variable is defined on the probability space $\left(\tilde{\Omega}, \tilde{F}, \tilde{P}\right)$. The payoff of the contracts under consideration has the form

$$
H(T(x)) = \max\left\{S_T^1, S_T^2\right\} \cdot I_{\{T(x) > T\}}. \tag{2.4}
$$

Let us rewrite the pure financial component of (2.4) as follows:

$$
\max\left\{S_T^1, S_T^2\right\} = S_T^2 + \left(S_T^1 - S_T^2\right)^+, \tag{2.5}
$$

where $x^+ = \max(0, x),\ x \in R^1$.

Using (2.5) we can reduce the pricing of the claim (2.4) to the pricing of the option $\left(S_T^1 - S_T^2\right)^+$ provided $\{T(x) > T\}$. It is quite natural to consider the contract with the payoff (2.4) on the product probability space $\left(\Omega \times \tilde{\Omega}, F \times \tilde{F}, P \times \tilde{P}\right)$ emphasizing the independence of financial and insurance risks.

What price for the contract (2.4) would be appropriate? According to the option pricing theory, we can find it using (2.4)-(2.5) as follows:

$$
{}_T U_x = E^* \times \tilde{E} H(T(x)) = {}_T p_x E^*\left(S_T^2\right) + {}_T p_x E^*\left(S_T^1 - S_T^2\right)^+, \tag{2.6}
$$

where ${}_T p_x = \tilde{P}\{T(x) > T\}$ is a survival probability, $E^* \times \tilde{E}$ is expectation with respect to $P^* \times \tilde{P}$.

Rewriting (2.6) as

$$
{}_T U_x - {}_T p_x E^*\left(S_T^2\right) = {}_T p_x E^*\left(S_T^1 - S_T^2\right)^+, \tag{2.7}
$$

we arrive to the conclusion that the value ${}_T U_x - {}_T p_x E^*\left(S_T^2\right)$ is the maximal available initial price of the option $\left(S_T^1 - S_T^2\right)^+$. On the other hand, the equality (2.7) shows that this price is strictly less than the initial price $E^*\left(S_T^1 - S_T^2\right)^+$ of perfect hedge.

Recall (see Shiryaev (1999)) that for the market (2.1)-(2.2) with the basic risky asset $S_t = S_t^1$, any process $\pi_t = (\beta_t, \gamma_t)_{t \ge 0}$ adapted to the price evolution F_t is called a portfolio (strategy). Define its value (capital) as a sum $X_t^{\pi} = \beta_t + \gamma_t S_t$. We shall consider only self-financing strategies satisfying the following condition $dX_t^{\pi} = d\beta_t + \gamma_t dS_t$, where all stochastic differentials are well defined.

Every F_T-measurable nonnegative random variable H is called a contingent claim (c.c). A self-financing strategy π is a perfect hedge for H if

$$X_T^{\pi} \geq H \quad \text{(a.s.).} \tag{2.8}$$

The option pricing theory of Black-Scholes-Merton in the framework of the model (2.1)-(2.2) states that such a strategy does exist, is unique for a given contingent claim, and has the initial capital $E^* H$. The idea of hedging in the sense (2.8) should be reformed if

$$X_T^{\pi} \leq X_0 < E^* H . \tag{2.9}$$

Let's introduce a loss function $l : R_+ \to R_+ = [0, \infty)$. Using this function, we can consider $El\left(\left(H - X_T^{\pi}\right)^+\right)$ as another measure of closeness between X_T^{π} and H . In this setting, the optimal strategy π^* is defined from

$$El\left(\left(H - X_T^{\pi^*}\right)^+\right) = \inf_{\pi} El\left(\left(H - X_T^{\pi}\right)^+\right), \tag{2.10}$$

where $\inf$ is taken over all self-financing strategies with nonnegative values satisfying the budget restriction (2.9). The optimal strategy satisfying (2.10) is called the efficient hedge (see Foellmer and Leukert (2000)).

Below, we consider a power loss function $l(x) = x^p$, $p > 0, x \geq 0$. The main results in efficient hedging with this loss function are the following.

Efficient hedge π^* for the problem (2.10) exists and coincides with a perfect hedge for a modified c.c. H_p with the structure

$$H_p = H - a_p Z_T^{1/(p-1)} \wedge H \qquad \text{for } p > 1, \tag{2.11}$$

$$H_p = H \cdot I_{\left\{Z_T^{-1} > a_p H^{1-p}\right\}} \qquad \text{for } 0 < p < 1, \tag{2.12}$$

$$H_p = H \cdot I_{\left\{Z_T^{-1} > a_p\right\}} \qquad \text{for } p = 1, \tag{2.13}$$

where constants a_p are defined from the initial condition $E^* H_p = X_0$.

According to (2.5)-(2.7), pricing of (2.5) can be reduced to hedging of $\left(S_T^1 - S_T^2\right)^+$ under the initial restriction

$$X_0^{\pi} \leq X_0 = {}_T p_x E^* \left(S_T^1 - S_T^2\right)^+ < E^* \left(S_T^1 - S_T^2\right)^+ . \tag{2.14}$$

To provide the corresponding actuarial analysis of (2.5), we shall use efficient hedging methodology applied to $\left(S_T^1 - S_T^2\right)^+$.

3. Actuarial analysis based on efficient hedging

First of all, we would like to find a relationship between insurance and financial components of the contract (2.5). Using the definitions of perfect and efficient hedging, one can conclude from (2.6)-(2.7) that

$$X_0 = {}_T p_x E^* \left(S_T^1 - S_T^2\right)^+ = E^* \left(S_T^1 - S_T^2\right)_{p,\, p>0}^+ \tag{3.1}$$

where $\left(S_T^1 - S_T^2\right)_p^+$ is defined by (2.11)-(2.13).

Rewriting (3.1) to separate insurance and financial components of the contract, we obtain the following, convenient for our actuarial analysis, formula:

$${}_T p_x = \frac{E^* \left(S_T^1 - S_T^2\right)_p^+}{E^* \left(S_T^1 - S_T^2\right)^+}. \tag{3.2}$$

The left-hand side of (3.2) is equal to survival probability of the insured accumulating quantitatively insurance risk of the contract (2.5) while the right-hand side is related to pure financial risk. So, the equation (3.2) can be viewed as a key balance equation controlling the risks associated with the contract (2.5).
The main problem is how to calculate the ratio in (3.2). The denominator of the ratio can be represented as

$$\begin{aligned} E^* \left(S_T^1 - S_T^2\right)^+ &= E^* \left(S_T^1 - S_T^2\right) \cdot I_{\{Y_T > 1\}} \\ &= E^* S_T^1 - E^* S_T^2 - E^* \left(S_T^1 - S_T^2\right) \cdot I_{\{Y_T \le 1\}}. \end{aligned} \tag{3.3}$$

Because S^1, S^2 are martingales with respect to P^*, we have $E^* S_T^i = S_0^i$, i=1,2.
For the last term in (3.3), we get

$$E^* S_T^i \cdot I_{\{Y_T \le 1\}} = E^* \exp\{-\eta_i\} \cdot I_{\{\xi \le \ln 1\}} \tag{3.4}$$

where $\eta_i = -\ln S_T^i$, $\xi = \ln Y_T$ are Gaussian random variables. Applying Lemma 2.4 from Melnikov (2004a), we find from (3.4) that

$$E^* \exp\{-\eta_i\} \cdot I_{\{\xi \le \ln 1\}} = \exp\left\{ \frac{\sigma_{\eta_i}^2}{2} - \mu_{\eta_i} \right\} \Phi\left(\frac{\ln 1 - \left(\mu_\xi - \operatorname{cov}(\xi, \eta_i)\right)}{\sigma_\xi} \right) \tag{3.5}$$

where $\mu_{\eta_i} = E^* \eta_i, \sigma_{\eta_i}^2 = \operatorname{var}(\eta_i), \mu_\xi = E^* \xi, \sigma_\xi^2 = \operatorname{var} \xi,$

$\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} \exp(-y^2/2) dy$.

Using (3.4)-(3.5), we obtain a variant of Margrabe's formula for the case under consideration

$$E^* \left(S_T^1 - S_T^2\right)^+ = S_0^1 \Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2 \Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right) \tag{3.6}$$

where $b_{\pm}\left(S_0^1, S_0^2, T\right)=\dfrac{\ln\dfrac{S_0^1}{S_0^2} \pm (\sigma_1-\sigma_2)^2 \dfrac{T}{2}}{(\sigma_1-\sigma_2)\sqrt{T}}$.

To provide an appropriate analysis of the contract (2.5) using the formula (3.2), we need to estimate the numerator of the ratio in (3.2). The main technical idea is to represent $\left(S_T^1 - S_T^2\right)_p^+$ in terms of the ratio $Y_T = S_T^1 / S_T^2$. In order to do this, we rewrite W_T with the help a free parameter γ in the form

$$\begin{aligned} W_T &= (1+\gamma)W_T - \gamma W_T \\ &= \frac{1+\gamma}{\sigma_1}\left(\sigma_1 W_T + \left(\mu_1 - \frac{\sigma_1^2}{2}\right)T\right) - \frac{\gamma}{\sigma_2}\left(\sigma_2 W_T + \left(\mu_2 - \frac{\sigma_2^2}{2}\right)T\right) \\ &\quad - \frac{1+\gamma}{\sigma_1}\left(\mu_1 - \frac{\sigma_1^2}{2}\right)T + \frac{\gamma}{\sigma_2}\left(\mu_2 - \frac{\sigma_2^2}{2}\right)T. \end{aligned} \tag{3.7}$$

Using (2.3) and (3.7), we obtain the next representation of Z_T:

$$Z_T = G \cdot \left(S_T^1\right)^{-\frac{(1+\gamma)\mu_1}{\sigma_1^2}} \left(S_T^2\right)^{\frac{\gamma\mu_1}{\sigma_1\sigma_2}} \tag{3.8}$$

where

$$\begin{aligned} G = G(\gamma) &= \left(S_0^1\right)^{\frac{(1+\gamma)\mu_1}{\sigma_1^2}} \left(S_0^2\right)^{-\frac{\gamma\mu_1}{\sigma_1\sigma_2}} \\ &\quad \times \exp\left(\frac{(1+\gamma)\mu_1}{\sigma_1^2}\left(\mu_1 - \frac{\sigma_1^2}{2}\right)T - \frac{\gamma\mu_1}{\sigma_1\sigma_2}\left(\mu_2 - \frac{\sigma_2^2}{2}\right)T - \frac{1}{2}\left(\frac{\mu_1}{\sigma_1}\right)^2 T\right). \end{aligned}$$

Now we separate our analysis of the option $\left(S_T^1 - S_T^2\right)_p^+$ in three cases according to (2.11)-(2.13).

Case 1: $p > 1$.

According to (3.8), we have

$$\begin{aligned} Z_T^{1/(p-1)} \cdot \left(S_T^2\right)^{-1} &= G^{1/(p-1)} \cdot \left(S_T^1\right)^{-\frac{(1+\gamma)\mu_1}{\sigma_1^2(p-1)}} \left(S_T^2\right)^{-1+\frac{\gamma\mu_1}{\sigma_1\sigma_2(p-1)}} \\ &= G^{1/(p-1)} \cdot Y_T^{\alpha_p} \end{aligned} \tag{3.9}$$

with

$$\alpha_p = -\frac{(1+\gamma)\mu_1}{\sigma_1^2(p-1)} = 1 - \frac{\gamma\mu_1}{\sigma_1\sigma_2(p-1)} \ . \tag{3.10}$$

Equation (3.10) has the unique solution

$$\gamma = \gamma_p = \frac{\sigma_1^2\sigma_2(p-1)+\mu_1\sigma_2}{\mu_1(\sigma_1-\sigma_2)} \ . \tag{3.11}$$

The ratio in (3.11) is positive, therefore

$$\alpha_p = -\frac{(1+\gamma_p)\mu_1}{\sigma_1^2(p-1)} < 0$$

and the equation

$$a_p G^{1/(p-1)} y^{\alpha_p} = (y-1)^+, \ y \geq 1 \tag{3.12}$$

has the unique solution $C = C(p) \geq 1$.

Using (3.9)-(3.12), we can represent $\left(S_T^1 - S_T^2\right)_p^+$ as follows

$$\begin{aligned}\left(S_T^1 - S_T^2\right)_p^+ &= S_T^2(Y_T-1)^+ - \left(a_p G^{1/(p-1)} Y_T^{\alpha_p} S_T^2\right) \wedge S_T^2(Y_T-1)^+ \\ &= S_T^2\left((Y_T-1)^+ - \left(a_p G^{1/(p-1)} Y_T^{\alpha_p}\right) \wedge (Y_T-1)^+\right) \\ &= S_T^2\left((Y_T-1)^+ - (Y_T-1)^+ I_{\{Y_T \leq C(p)\}} - a_p G^{1/(p-1)} Y_T^{\alpha_p} I_{\{Y_T > C(p)\}}\right).\end{aligned}$$

Taking into account the equality $I_{\{Y_T > C(p)\}} = 1 - I_{\{Y_T \leq C(p)\}}$, we get

$$\begin{aligned}E^*\left(S_T^1 - S_T^2\right)_p^+ &= E^*\left(S_T^1 - S_T^2\right)^+ - E^*\left(S_T^1 - S_T^2\right)^+ I_{\{Y_T \leq C(p)\}} \\ &\quad - a_p G^{1/(p-1)}\left(E^* S_T^2 Y_T^{\alpha_p} - E^* S_T^2 Y_T^{\alpha_p} I_{\{Y_T \leq C(p)\}}\right).\end{aligned} \tag{3.13}$$

Because of $C(p) \geq 1$, we get

$$\begin{aligned}E^*\left(S_T^1 - S_T^2\right)^+ - E^*\left(S_T^1 - S_T^2\right)^+ I_{\{Y_T \leq C(p)\}} &= E^*\left(S_T^1 - S_T^2\right) I_{\{Y_T > C(p)\}} \\ &= E^*\left(S_T^1 - S_T^2\right) - E^*\left(S_T^1 - S_T^2\right) I_{\{Y_T \leq C(p)\}}.\end{aligned} \tag{3.14}$$

Using (3.14), we can calculate the difference between two terms in (3.13) reproducing exactly the same procedure as in (3.3)-(3.6) and replacing 1 by $C = C(p)$:

$$\begin{aligned}&E^*\left(S_T^1 - S_T^2\right)^+ - E^*\left(S_T^1 - S_T^2\right)^+ I_{\{Y_T \leq C(p)\}} \\ &\qquad = S_0^1\Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right).\end{aligned} \tag{3.15}$$

To calculate other two terms in (3.13), we represent the product $S_T^2 Y_T^{\alpha_p}$ as follows

$$S_T^2 Y_T^{\alpha_p} = \left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}$$
$$\times \exp\left\{\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)W_T^* - \frac{1}{2}\left(\sigma_1^2\alpha_p+\sigma_2^2\left(1-\alpha_p\right)\right)T + \frac{\mu_2\sigma_1-\mu_1\sigma_2}{\sigma_1}\left(1-\alpha_p\right)T\right\}$$
$$= \left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}$$
$$\times \exp\left\{\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)W_T^* - \frac{1}{2}\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)^2 T \right. \tag{3.16}$$
$$\left. + \frac{1}{2}\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)^2 T - \frac{1}{2}\left(\sigma_1^2\alpha_p+\sigma_2^2\left(1-\alpha_p\right)\right)T + \frac{\mu_2\sigma_1-\mu_1\sigma_2}{\sigma_1}\left(1-\alpha_p\right)T\right\}$$
$$= \left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}$$
$$\times \exp\left\{\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)W_T^* - \frac{1}{2}\left(\sigma_1\alpha_p+\sigma_2\left(1-\alpha_p\right)\right)^2 T \right.$$
$$\left. -\alpha_p\left(1-\alpha_p\right)\left(\sigma_1-\sigma_2\right)^2\frac{T}{2} + \frac{\mu_2\sigma_1-\mu_1\sigma_2}{\sigma_1}\left(1-\alpha_p\right)T\right\}.$$

Taking an expected value of (3.16) with respect to P^* and using our technical condition $\mu_1\sigma_2 = \mu_2\sigma_1$, we find that

$$E^* S_T^2 Y_T^{\alpha_p} = \left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}\exp\left\{-\alpha_p\left(1-\alpha_p\right)\left(\sigma_1-\sigma_2\right)^2\frac{T}{2}\right\}. \tag{3.17}$$

Exploiting (3.16)-(3.17), we can reproduce the same procedure as in (3.3)-(3.6) (as in (3.15), we should replace there 1 by $C = C(p)$) and obtain

$$-a_p G^{1/(p-1)}\left(E^* S_T^2 Y_T^{\alpha_p} - E^* S_T^2 Y_T^{\alpha_p} I_{\{Y_T \le C(p)\}}\right)$$
$$= -a_p G^{1/(p-1)}\left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}\exp\left\{-a_p\left(1-a_p\right)\left(\sigma_1-\sigma_2\right)^2\frac{T}{2}\right\} \tag{3.18}$$
$$\times \Phi\left(b_-\left(S_0^1, CS_0^2, T\right) + a_p\left(\sigma_1-\sigma_2\right)\sqrt{T}\right).$$

Combining (3.2), (3.6), (3.13), (3.15), and (3.18), we arrive to the final formula

$$_T p_x = \frac{E^*\left(S_T^1 - S_T^2\right)_p^+}{E^*\left(S_T^1 - S_T^2\right)^+} = \frac{S_0^1\Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right)}{S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)}$$
$$+ \frac{(C-1)^+}{C^{\alpha_p}}\left(S_0^1\right)^{\alpha_p}\left(S_0^2\right)^{1-\alpha_p}\exp\left\{\alpha_p\left(1-\alpha_p\right)\frac{\left(\sigma_1-\sigma_2\right)^2}{2}T\right\} \tag{3.19}$$
$$\times \frac{\Phi\left(b_-\left(S_0^1, CS_0^2, T\right) + \alpha_p\left(\sigma_1-\sigma_2\right)\sqrt{T}\right)}{S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)}$$

where we replaced a constant $a_p G^{1/(p-1)}$ by $\dfrac{(C-1)^+}{C^{\alpha_p}}$ according to the equation (3.12).

Remark 3.1. If ${}_T p_x$ is known, we can reconstruct the constant C (and a_p) from (3.19) and find the corresponding hedging strategy as a perfect hedge for a modified claim.

Case 2: $0 < p < 1$.

This case can be treated in the same way.
Taking into account the structure of $\left(S_T^1 - S_T^2\right)_p^+$ in (2.12), we express the product $Z_T \left(S_T^2\right)^{1-p}$ introducing a free parameter γ (see (3.7)-(3.12)) and get

$$Z_T \left(S_T^2\right)^{1-p} = G\left(S_T^1\right)^{-\frac{(1+\gamma)\mu_1}{\sigma_1^2}} \left(S_T^2\right)^{\frac{\gamma\mu_1}{\sigma_1\sigma_2}} \left(S_T^2\right)^{1-p} = GY_T^{\alpha_p}. \tag{3.20}$$

In the equality (3.20) $\alpha_p = -\dfrac{(1+\gamma)\mu_1}{\sigma_1^2} = -\dfrac{\gamma\mu_1}{\sigma_1\sigma_2} - (1-p)$ and, hence,

$$\begin{aligned} \gamma = \gamma_p &= \frac{\sigma_2\left(\mu_1 - (1-p)\sigma_1^2\right)}{\mu_1(\sigma_1 - \sigma_2)}, \\ -\alpha_p &= \frac{\mu_1}{\sigma_1^2}\left(1 + \frac{\sigma_2\left(\mu_1 - (1-p)\sigma_1^2\right)}{\mu_1(\sigma_1 - \sigma_2)}\right) = \frac{\mu_1}{\sigma_1^2} + \frac{\sigma_2}{(\sigma_1 - \sigma_2)}\left(\frac{\mu_1}{\sigma_1^2} - (1-p)\right). \end{aligned} \tag{3.21}$$

Consider the following analog of the characteristic equation (3.12):

$$y^{-\alpha_p} = \alpha_p G\left((y-1)^+\right)^{1-p}, \;\; y \geq 0. \tag{3.22}$$

Let us separate two possible situations.

If $-\alpha_p > 1-p$, then according to (3.21)

$$\frac{\mu_1}{\sigma_1^2} + \frac{\sigma_2}{(\sigma_1 - \sigma_2)}\left(\frac{\mu_1}{\sigma_1^2} - (1-p)\right) > 1-p \quad \text{or} \quad \frac{\mu_1}{\sigma_1^2} > 1-p. \tag{3.23}$$

If the condition (3.23) is fulfilled, then the equation (3.22) admits one, two or zero solutions. All these cases can be considered in a similar way as in Case 1.

If $-\alpha_p \leq 1-p$, then $\dfrac{\mu_1}{\sigma_1^2} \leq 1-p$ and, therefore, the equation (3.22) has only one solution $C = C(p)$. The corresponding modified contingent claim can be rewritten in the form

$$\left(S_T^1 - S_T^2\right)_p^+ = \left(S_T^1 - S_T^2\right)^+ I_{\{Y_T \leq C(p)\}} \tag{3.24}$$

and according to (3.2), we find (for $0 < p < 1$) that

$$ {}_T p_x = \frac{E^*\left(S_T^1 - S_T^2\right)_p^+}{E^*\left(S_T^1 - S_T^2\right)^+} = 1 - \frac{S_0^1\Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right)}{S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)}. \tag{3.25} $$

Remark 3.2. Due to (3.24) we can analyze (3.25) in another way. We can fix a probability $P(Y_T \le C(p)) = 1 - \varepsilon,\ \varepsilon > 0$, and, using log-normality of Y_T, find $C(p)$ through ε-quantile of a standard normal distribution.

Case 3: $p = 1$.

According to (2.13) to reconstruct this modified contingent claim, we represent (see (3.7)-(3.12)) Z_T as follows

$$ Z_T = G\left(S_T^1\right)^{-\frac{(1+\gamma)\mu_1}{\sigma_1^2}}\left(S_T^2\right)^{\frac{\gamma\mu_1}{\sigma_1\sigma_2}} = GY_T^{\alpha_p} \tag{3.26} $$

where $\alpha_p = -\frac{(1+\gamma)\mu_1}{\sigma_1^2} = -\frac{\gamma\mu_1}{\sigma_1\sigma_2}$ and, therefore,

$$ \gamma = \gamma_p = \frac{\sigma_1\sigma_2}{\sigma_1(\sigma_1 - \sigma_2)}, $$

$$ -\alpha_p = \frac{\mu_1}{\sigma_1\sigma_2}\gamma_p = \frac{\mu_1}{\sigma_1(\sigma_1 - \sigma_2)},\ \ \sigma_1 > \sigma_2. $$

Using $-\alpha_p$ we find from (2.13) and (3.26) that

$$ \left(S_T^1 - S_T^2\right)^+ I_{\left\{Y_T^{-\alpha_p} > Ga_p\right\}} = \left(S_T^1 - S_T^2\right)^+ I_{\left\{Y_T^{-\frac{\mu_1}{\sigma_1(\sigma_1-\sigma_2)}} > Ga_p\right\}} = \left(S_T^1 - S_T^2\right)^+ I_{\{Y_T > C\}} \tag{3.27} $$

where

$$ C = \left(G\alpha_p\right)^{\frac{\sigma_1(\sigma_1-\sigma_2)}{\mu_1}}. \tag{3.28} $$

Using (3.2)-(3.6), (3.14), (3.15), and (3.27), we find for the case $p = 1$ that

$$ {}_T p_x = \frac{E^*\left(S_T^1 - S_T^2\right)_p^+}{E^*\left(S_T^1 - S_T^2\right)^+} = 1 - \frac{S_0^1\Phi\left(b_+\left(S_0^1, CS_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, CS_0^2, T\right)\right)}{S_0^1\Phi\left(b_+\left(S_0^1, S_0^2, T\right)\right) - S_0^2\Phi\left(b_-\left(S_0^1, S_0^2, T\right)\right)} \tag{3.29} $$

where C is defined by (3.28).

Remark 3.3. To analyze the formula (3.29) one can reproduce the same arguments as in Remark 3.2. These results were shortly announced in Melnikov (2004b).

4. Concluding remarks and numerical example

According to actuarial tradition, insurance company deals with a group of l_x insureds of age x. Denote their future life-times as independent identically distributed random variables $T_i(x),\ i = 1,...,l_x$. Let ${}_t d_x = \sum_{i=1}^{l_x} I_{\{T_i(x) \le t\}}$ be a number of deaths in the group of insureds by the time $t \le T$.

In the beginning of the contract period (see (2.4)-(2.6)), the premium collected from the group is $l_x \cdot {}_t U_x$. At time $t \le T$, the price of the contract for the group should be equal to $(l_x - {}_t d_x) \cdot {}_{T-t} U_x$. Values $l_x - {}_t d_x = l_{x+t}$ can be found from actuarial statistics, and ${}_{T-t}U_x$ can be calculated by the same method as a premium ${}_T U_x$. Therefore, we will illustrate our analysis only for a single contract at initial time $t = 0$.

Consider the equation (2.1) as a theoretical model for financial indices the Russell 2000 (RUT-I) and the Dow Jones Industrial Average (DJIA) correspondingly for $i = 1, 2.$ The Russell 2000 is the index of small US companies' stocks, whereas the Dow Jones Industrial Average is based on the portfolio consisting of 30 blue-chip stocks in the USA. The first index, RUT-I, is supposed to be more risky than DJIA.

Using daily observations of prices from August 1, 1997, until July 31, 2003, we estimate empirically $(\mu_1,\ \sigma_1)$ and $(\mu_2,\ \sigma_2)$, the rate of return and volatility for RUT-I and DJIA. We get the following specification of the model:

$$\mu_1 = .0481, \quad \sigma_1 = .2232,$$
$$\mu_2 = .0417, \quad \sigma_2 = .2089.$$

Calculating $\mu_1 \sigma_2 = .0100$ and $\mu_2 \sigma_1 = .0093$ we can conclude that our technical condition $\mu_1 \sigma_2 = \mu_2 \sigma_1$ is approximately fulfilled.

The initial prices of these indices (August 1, 2003) are 468.08 and 9153.97. Therefore, we use $\frac{9153.97}{468.08} \cdot S_t^1$ as the value of the first asset to make the initial values of both assets the same.

The key formulae for our numerical analysis are (3.19), (3.25), (3.29) containing an unknown parameter $C(p)$. In our example, the ratio $\frac{\mu_1}{\sigma_1^2} = .9655$ and, therefore, for sufficiently small p (see (3.23)), the corresponding modified contingent claim has a quantile form (3.24). To calculate ${}_T p_x$, we will use (3.25) and Remark 3.2 for identification of $C(p)$. The results for risk levels $\varepsilon = 0.01,\ 0.025, 0.05,\ 0.1$ and $T = 1,\ 3,\ 5,\ 10$ are given in Table 1. Using Life Tables from Bowers et al (1997), we can find the ages of insureds for such contracts, which are displayed in Table 2.

When the level of risk ε increases, the company should restrict the group of insureds by attracting older clients. As a result, the company diminishes the insurance component of risk to compensate for the increasing financial risk.
Using contracts for a longer term T allows the insurance company to diminish insurance risk with fixed ε. Therefore, the company can afford to work with younger groups of clients.
We can consider the case $p = 1$ in the same way because of the similarity between (3.25) and (3.29). The difference will appear when we construct hedge exploiting different formulae for a_p : (3.22) for $0 < p < 1$ and (3.28) for $p = 1$.
For the case $p > 1$, the calculations with the help of formula (3.19) demonstrate results with very small values of ${}_T p_x$. In Table 3, the results for $p = 2$ are shown, however, the very similar data can be obtained for other values of $p > 1$. Therefore, we can conclude that the insurance company with the loss function x^p, $p > 1$, does not accept any financial risk transferring it to insurance risk.

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
$T = 1$	0.931898	0.877458	0.781251	0.621439
$T = 3$	0.93979	0.891087	0.804175	0.657577
$T = 5$	0.944328	0.898968	0.817541	0.678939
$T = 10$	0.951165	0.910903	0.837938	0.71195

Table 1. Survival Probabilities for $0 < p < 1$

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
$T = 1$	78	85	92	99
$T = 3$	64	70	77	85
$T = 5$	55	63	70	77
$T = 10$	43	51	58	65

Table 2. Age of Insureds

	$\varepsilon = 0.01$	$\varepsilon = 0.025$	$\varepsilon = 0.05$	$\varepsilon = 0.1$
$T = 1$	0.081596	0.145582	0.256894	0.437299
$T = 3$	0.061184	0.110569	0.198556	0.346621
$T = 5$	0.055751	0.101166	0.18268	0.321402
$T = 10$	0.048835	0.089097	0.162063	0.288051

Table 3. Survival Probabilities for $p > 1$

References

AASE, K. and PERSSON, S. 1994. "Pricing of unit-linked insurance policies." Scandinavian Actuarial Journal 1: 26-52.

BACINELLO, A.R. and ORTU, F. 1993. "Pricing of unit-linked life insurance with endogenous minimum guarantees." Insurance: Math. and Economics 12:245-257.

BOWERS, N.L., GERBER, H.U., HICKMAN, J.C., JONES D.A. and NESBITT, C.I. 1997. "Actuarial Mathematics." Society of Actuaries, Schaumburg, Illinois.

BOYLE, P.P. and SCHWARTZ, E.S. 1977. "Equilibrium prices of guarantees under equity-linked contracts." Journal of Risk and Insurance 44: 639-680.

BOYLE, P.P. and HARDY, M.R. 1997. "Reserving for maturity guarantees: Two approaches." Insurance: Math. and Economics 21: 113-127.

BRENNAN, M.J. and SCHWARTZ, E.S. 1976. "The pricing of equity-linked life insurance policies with an asset value guarantee." Journal of Financial Economics 3: 195-213.

BRENNAN, M.J. and SCHWARTZ, E.S. 1979. "Alternative investment strategies for the issuers of equity-linked life insurance with an asset value guarantee." Journal of Business 52: 63-93.

DELBAEN, F. 1986. "Equity-linked policies." Bulletin Association Royale Actuaires Belges 80: 33-52.

FOELLMER, H. and LEUKERT, P. 2000. "Efficient hedging: cost versus short-fall risk." Finance Stochast. 4: 117-146.

HARDY, M.R. 2003. "Investment guarantees: Modeling and risk-management for equity-linked life insurance." J.Wiley.

MARGRABE, W. 1978. "The value of an option to exchange one asset to another." J. of finance 33: 177-186.

MELNIKOV, A. 2004a. "Risk analysis in Finance and Insurance." Chapman&Hall/CRC.

MELNIKOV, A. 2004b. "On efficient hedging of equity-linked life insurance policies." Doklady Mathematics 69: 462-464.

MOELLER, T. 1998. "Risk-minimizing hedging strategies for unit-linked life-insurance contracts." Astin Bulletin 28: 17-47.

MOELLER, T. 2002. "Hedging equity-linked life insurance contracts." North American Actuarial Journal 5: 79-95.

SHIRYAEV, A.N. 1999. "Essentials of Stochastic Finance." World Scientific.

List of Recent Technical Reports

46. Yogendra P. Chaubey and Anthony Crisalli, *The Generalized Smoothing Estimator*, April 2002

47. Yogendra P. Chaubey and Pranab K. Sen, *Smooth Isotonic Estimation of Density, Hazard and MRL Functions*, April 2002

48. Pablo Olivares, *Maximum Likelihood Estimators for a Branching-Diffusion Process*, August 2002

49. Shuanming Li and José Garrido, *On Ruin for the Erlang*(n) *Risk Process*, June 2003

50. G. Jogesh Babu and Yogendra P. Chaubey, *Smooth Estimation of a Distribution and Density function on a Hypercube Using Bernstein Polynomials for Dependent Random Vectors* , August 2003

51. Shuanming Li and José Garrido, *On the Time Value of Ruin for a Sparre Anderson Risk Process Perturbed by Diffusion*, November 2003

52. Yogendra P. Chaubey, Cynthia M. DeSouza and Fassil Nebebe, *Bayesian Inference for Small Area Estimation under the Inverse Gaussian Model via Cibbs Sampling*, December 2003

53. Alexander Melnikov and Victoria Skornyakova, *Pricing of Equity–Linked Life Insurance Contracts with Flexible Guarantees*, May 2004

54. Yi Lu and José Garrido, *Regime–Switching Periodic Models for Claim Counts*, June 2004.

55. I. Urrutia-Romaní, R. Rodríguez-Ramos, J. Bravo-Castillero and R. Guinovart-Díaz, *Asymptotic Homogenization Method Applied to Linear Viscoelastic Composites. Examples*, August 2004.

56. Yi Lu and José Garrido, *Double Periodic Non-Homogeneous Poisson Models for Hurricanes Data*, September 2004.

57. M.I. Beg and M. Ahsanullah, *On Characterizing Distributions by Conditional Expectations of Functions of Generalized Order Statistics*, September, 2004.

58. M.I. Beg and M. Ahsanullah, *Concomitants of Generalized Order Statistics from Farlie-Gumbel-Morgenstern Distributions*, September, 2004.

59. Yogendra P. Chaubey and Debaraj Sen, *An investigation into properties of an estimator of mean of an inverse Gaussian population*, September, 2004.

60. Steven N. Evans and Xiaowen Zhou, *Balls-in-boxes duality for coalescing random walks and coalescing Brownian motions*, September, 2004.

61. Qihe Tang, *Asymptotic ruin probabilities of the renewal model with constant interest force and regular variation*, November, 2004.

62. Xiaowen Zhou, *On a classical risk model with a constant dividend barrier*, November, 2004.

63. K. Balasubramanian and M.I. Beg, *Three isomorphic vector spaces–II*, December, 2004.

64. Michael A. Kouritzin and Wei Sun, *Rates for branching particle approximations of continuous-discrete filters*, December, 2004.

65. Rob Kaas and Qihe Tang, *Introducing a dependence structure to the occurrences in studying precise large deviations for the total claim amount* , December, 2004.

66. Qihe Tang and Gurami Tsitsiashvili, *Finite and infinite time ruin probabilities in the presence of stochastic returns on investments*, December, 2004.

67. Alexander Melnikov and Victoria Skornyakova, *Efficient hedging methodology applied to equity-linked life insurance*, February, 2005.

Copies of technical reports can be requested from:

Prof. Xiaowen Zhou
Department of Mathematics and Statistics
Concordia University
7141, Sherbrooke Street West
Montréal (QC) H4B 1R6 CANADA