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ON THE RELATION BETWEEN THE LÉVY MEASURE AND THE JUMP FUNCTION OF A LÉVY PROCESS

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On the relation between the Lévy measure and the jump function of a Lévy process

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Abstract

The Lévy and jump measures are two key characteristics of Lévy processes. This paper fills what seems to be a simple gap in the literature, by giving an explicit relation between the jump measure, which is a Poisson random measure, and the Lévy measure. This relation paves the way to a simple proof of the classical result on path continuity of Lévy processes in Section 2.

The jump function in Paul Lévy's version of the Lévy–Khinchine formula and the Lévy measure in more recent characterizations essentially play the same role, but with different drift and Gaussian components. This point is shown in detail in Section 3, together with an explicit relation between the jump function and the Lévy measure.

1 Introduction

The characteristic function of a stochastically continuous process starting at zero and with stationary independent increments can be written as

$$\begin{aligned}\Phi_{X_t}(s) \quad &= \quad \mathbb{E}[e^{isX_t}] = \exp\left\{t\left[ias - \frac{s^2b^2}{2}\right.\right.\\ &\left.\left.\qquad + \int_{\mathbb{R}\setminus\{0\}}\left[e^{isx} - 1 - isx\mathbb{I}_{\{-1,1\}}(x)\right]\nu(dx)\right]\right\}, \qquad (1)\end{aligned}$$

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for $s \in \mathbb{R}$, $t \geq 0$ and with constants $a \in \mathbb{R}$, $b \in \mathbb{R}^+$, where ν is a measure defined on $\mathbb{R} \setminus \{0\}$ that satisfies:

$$\int_{\mathbb{R}\setminus\{0\}} \left(|x|^2 \wedge 1\right) \nu(dx) < \infty \,. \tag{2}$$

Equation (1) is the celebrated Lévy–Khinchine representation of a Lévy process. The theory of Lévy process is well established (see [1], [5], or [3]) and is becoming increasingly popular in applications.

Lévy processes being cadlag, the number of jumps exceeding a fixed threshold, i.e. $|\Delta X_s| \geq \epsilon$, before some time t, has to be finite for all $\epsilon > 0$. Hence if a Borel set $B \in \mathcal{B}(\mathbb{R})$, is bounded away from 0 (i.e. $0 \notin \bar{B}$, the closure of B), then for any $t \geq 0$ the cardinality

$$N_t^B = \sharp\{s \in [0,t]\,;\, \Delta X_s \in B\} = J_X([0,t] \times B)\,, \tag{3}$$

is well defined and a.s. finite. The process N_t^B is called the counting process of B. It inherits the Lévy properties from X and is a Poisson process with intensity $\nu_X(B) < \infty$. J_X is known as the jump measure associated with the process $X = \{X_t\,;\, t \geq 0\}$.

2 Jump measure and path continuity

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, $E \subset \mathbb{R}$ and μ a given positive Radon measure on $(E, \mathcal{E})$. Then the existence of a Poisson random measure on E is assured by the following proposition (for a proof see [2]).

Proposition 2.1 *For any Radon measure μ on $E \subset \mathbb{R}$, there exists a Poisson random measure M on E with intensity μ.*

B in (3) can be adapted to a threshold process. For a general Lévy process if the small jumps are truncated at some level say, ϵ, the resulting process is a compound Poisson process. The following proposition shows that the jump measure corresponding to a threshold process is a Poisson random measure (for a proof see [2]).

Proposition 2.2 *Let $X = \{X_t\,;\, t \geq 0\}$ be a compound Poisson process with intensity λ and jump size distribution F. Its jump measure J_X is a Poisson random measure on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mu(dx \times dt) = \nu(dx)\, dt = \lambda\, dF(x)\, dt$.*

Now setting $B = [\epsilon, \infty)$, from Proposition 2.2 it follows that for $k \in \mathbb{N}^+$

$$
\begin{aligned}
\mathbf{P}\big\{J_X(B \times [t_1, t_1^{'}]) = k\big\} &= \exp\Big[-\int_B \int_{t_1}^{t_1^{'}} \mu(dx \times dt)\Big] \frac{\Big[\int_B \int_{t_1}^{t_1^{'}} \mu(dx \times dt)\Big]^k}{k!} \\
&= \exp\Big[-\int_B \int_{t_1}^{t_1^{'}} \lambda \, dF(x) \, dt\Big] \frac{\Big[\int_B \int_{t_1}^{t_1^{'}} \lambda \, dF(x) \, dt\Big]^k}{k!} \\
&= \exp\Big[-|t_1^{'} - t_1| \int_B \nu(dx)\Big] \frac{\big[|t_1^{'} - t_1| \int_B \nu(dx)\big]^k}{k!} \\
&= e^{\big[-|t_1^{'} - t_1| \nu([\epsilon,\infty)\big]} \frac{\big[|t_1^{'} - t_1| \nu([\epsilon, \infty])\big]^k}{k!} . \qquad (4)
\end{aligned}
$$

Equation (4) characterizes the relation between the jump measure J_X and the Lévy measure ν of a Lévy process. This characterization leads to the following simple proof.

Proposition 2.3 *X has continuous sample paths if and only if $J_X = 0$ a.s., which implies that there are no jumps.*

Proof: $\mathbf{P}\{J_X(A \times [t_1, t_1^{'}]) = 0\} = 1$ implies, by (4), that

$$
\exp\Big[-|t_1^{'} - t_1| \, \nu([\epsilon, \infty])\Big] \frac{\big[|t_1^{'} - t_1| \, \nu([\epsilon, \infty])\big]^0}{0!} = 1 \,,
$$

for any choice of $t_1^{'}$, t_1 and ϵ. Hence it must be that $\nu(\epsilon, \infty) = 0$. But then by the Lévy–Khinchine representation of a Lévy process in (1) we see that X is Brownian motion with drift, that is it has continuous sample paths.

On the other hand, if X has continuous sample paths then its Lévy measure, which controls all jumps, should be identically zero. That is $\nu(\epsilon, \infty) = 0$ for any choice of ϵ. Then (4) implies for any positive integer k that

$$
\mathbf{P}\{J_X(A \times [t_1, t_1^{'}]) = k\} = 0
$$

and for $k = 0$

$$
\mathbf{P}\{J_X(A \times [t_1, t_1^{'}]) = 0\} = 1 \,,
$$

that is J_X is almost surely zero. ∎

3 Connection between different characterization of Lévy processes

Nowadays, the characterization of Lévy processes in (1) is most frequently used, but the original characterization, known as Kolmogorov's representation, is

$$\log \Phi_{X_t}(s) = iats + t\int_{-\infty}^{\infty} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2}\, dG(x)\,, \quad s \in \mathbb{R}\,, \tag{5}$$

where a is a real constant and G a bounded non–decreasing function with $G(-\infty) = 0$, known as the jump function.

Proposition 3.1 *The jump of G, in* (5)*, at $x = 0$ is the variance of the normal component, i.e. $\sigma^2 = G(0^+) - G(0^-)$.*

The objective of this section is to establish a correspondence between the characterizations in (1) and (5). From this correspondence an explicit relation between G and ν is obtained. This also yields the proof of Proposition 3.1.

We can split $\log \Phi_{X_t}$ in (5) to write it as:

$$\begin{aligned} \log \Phi_{X_t}(s) \quad = \quad & iats + t\int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2}\, dG(x) \\ & + t\int_{\{0\}} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2}\, dG(x)\,. \end{aligned} \tag{6}$$

Now, in the second term, the integrand has a limit (by l'Hospital's rule)

$$\lim_{x\to 0} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2} = \frac{-s^2}{2}\,,$$

and hence

$$\int_{\{0\}} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2}\, dG(x) = \frac{-s^2}{2}\left[G(0+) - G(0-)\right]. \tag{7}$$

Further, for the first integrand in (6) we observe that

$$\begin{aligned} & \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2} \\ & \quad = \left[e^{isx} - 1 - isx\,\mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2} - \frac{is}{x} + isx\,\mathbb{I}_{\left[|x|<1\right]} \frac{(1+x^2)}{x^2} \\ & \quad = \left[e^{isx} - 1 - isx\,\mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2} + \frac{is}{x}\left[\mathbb{I}_{\left[|x|<1\right]} - 1\right] + isx\,\mathbb{I}_{\left[|x|<1\right]}\,. \end{aligned}$$

Hence the first integral in (6) becomes:

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - \frac{isx}{(1+x^2)}\right] \frac{(1+x^2)}{x^2}\, dG(x) \\
&\quad = \int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - isx\, \mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2}\, dG(x) \\
&\qquad + is \int_{\mathbb{R}\setminus\{0\}} \frac{1}{x} \left[\mathbb{I}_{\left[|x|<1\right]} - 1\right] dG(x) + is \int_{\mathbb{R}\setminus\{0\}} x\, \mathbb{I}_{\left[|x|<1\right]}\, dG(x) \\
&\quad = \int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - isx\, \mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2}\, dG(x) \\
&\qquad - is \int_{\mathbb{R}\setminus(-1,1)} \frac{1}{x}\, dG(x) + is \int_{(-1,1)\setminus\{0\}} x\, dG(x)\,,
\end{aligned}
$$

and we can rewrite (5) as:

$$
\begin{aligned}
\Phi_{X_t}(s) \quad &= \exp\Big\{iats + t\int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - isx\, \mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2}\, dG(x) \\
&\qquad\qquad -ist \int_{\mathbb{R}\setminus(-1,1)} \frac{1}{x}\, dG(x)\Big\} \exp\Big\{ist \int_{(-1,1)\setminus\{0\}} x\, dG(x) \\
&\qquad\qquad -\frac{s^2}{2}\left[G(0+) - G(0-)\right] t\Big\} \\
&= \exp\Big\{is\Big[a - \int_{\mathbb{R}\setminus(-1,1)} \frac{1}{x}\, dG(x) + \int_{(-1,1)\setminus\{0\}} x\, dG(x)\Big] t \\
&\qquad\qquad -\frac{s^2}{2}\left[G(0+) - G(0-)\right] t\Big\} \qquad\qquad (8) \\
&\qquad\qquad \exp\Big\{t \int_{\mathbb{R}\setminus\{0\}} \left[e^{isx} - 1 - isx\, \mathbb{I}_{\left[|x|<1\right]}\right] \frac{(1+x^2)}{x^2}\, dG(x)\Big\}.
\end{aligned}
$$

Comparing (8) with (1) we see that for the Lévy–Khinchine characterization in terms of G, as in (5), the drift is given by $a - \int_{\mathbb{R}\setminus(-1,1)} \frac{1}{x}\, dG(x) + \int_{(-1,1)\setminus\{0\}} x\, dG(x)$, the Gaussian component is $\left[G(0+) - G(0-)\right]$ and the explicit relation between G and ν is $\nu(dx) = \frac{(1+x^2)}{x^2}\, dG(x)$.

Remark 3.1 *The set of points of increase of G in* (5)*, for $x \neq 0$, gives information as to the nature of the compound Poisson component, i.e. the relative density of the size of the discontinuities in the Lévy process sample paths.*

4 Conclusion

The jump measure, being a Poisson random measure, is explicitly related to the Lévy measure of a Lévy process. This relation yields a basic proof of the classical result on path continuity of the process. The relation between the jump function in Kolmogorov's representation of Lévy–Khinchine's formula and the Lévy measure in more recent Lévy–Khinchine formulas is derived in detail. This relation is useful in estimating the Lévy measure by estimating the jump function from observed sample paths. For further details on such an application see [4].

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