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CAPM AND APT LIKE MODELS WITH RISK MEASURES

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Abstract

The paper deals with optimal portfolio choice problems when risk levels are given by coherent risk measures, expectation bounded risk measures or general deviations. Both static and dynamic pricing models may be involved.

Unbounded problems are characterized by new notions such as compatibility and strong compatibility between pricing rules and risk measures. Surprisingly, it is pointed out that the lack of bounded optimal risk and/or return levels arises in practice for very important pricing models (for instance, the Black and Scholes model) and risk measures (VaR, $CVaR$, absolute deviation and downside semi-deviation, etc.).

Bounded problems will present a Market Price of Risk and generate a pair of benchmarks. From these benchmarks we will introduce APT and $CAPM$ like analyses, in the sense that the level of correlation between every available security and some economic factors will expalin the security expected return. On the contray, the risk level non correlated with these factors will have no influence on any return, despite we are dealing with very general risk functions that are beyond the standard deviation.

Key words. Risk Measure, Compatibility between Prices and Risks, Efficient Portfolio, APT and $CAPM$ like models.

J.E.L. Classification. G11, G13.

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1 Introduction

General risk functions are becoming very important in finance and insurance. Since Artzner *et al.* (1999) introduced the axioms and properties of the "Coherent Measures of Risk" many authors have extended the discussion. The recent development of new markets (insurance or weather linked derivatives, commodity derivatives, energy/electricity markets, etc.) and products (inflation-linked bonds, equity indexes annuities or unit-links, hedge funds, etc.), the necessity of managing new types of risk (credit risk, operational risk, etc.) and the (often legal) obligation of providing initial capital requirements have made it necessary to overcome the variance as the most used risk measure and to introduce more general risk functions. It has been proved that the variance is not compatible with the Second Order Stochastic Dominance if asymmetries and heavy tails are involved (Ogryczak and. Ruszczynski, 1999).

Hence, it is not surprising that the recent literature presents many interesting contributions focusing on new methods for measuring risk levels. Among others, Föllmer and Schied (2002) have defined the Convex Risk Measures, Goovaerts *et al.* (2004) have introduced the Consistent Risk Measures, Rockafellar *et al.* (2006a) have defined the General Deviations and the Expectation Bounded Risk Measures, and Brown and Sim (2009) have introduced the Satisfying Measures.

Many classical actuarial and financial problems have been revisited by using new risk functions. For instance, pricing and hedging issues in incomplete markets (Föllmer and Schied, 2002, Nakano, 2004, Staum, 2004, Balbás *et al.*, 2009a, etc.), as well as equity linked annuities hedging issues (Barbarin and Devolder, 2005), optimal reinsurance problems (Balbás *et al.*, 2009*b*), and other practical topics.

With regard to portfolio choice and asset allocation problems, among others, Alexander *et al.* (2006) compare the minimization of the Value at Risk (VaR) and the Conditional Value at Risk ($CVaR$) for a portfolio of derivatives (such a portfolio is obviously composed of asymmetric securities and therefore the standard deviation is not appropriate), Calafiore (2007) studies "robust" efficient portfolios in discrete probability spaces if the risk measure is the absolute deviation, Schied (2007) deals with optimal investment with convex risk measures, and Miller and Ruszczynski (2008) analyze efficient portfolios with coherent risk measures. Other authors have also dealt with generalizations of the Sharpe ratio, the introduction of benchmarks in the line of the Market Portfolio of the classical Capital Asset Pricing Model ($CAPM$), and the extension of formulas relating expected returns to some kind of generalized betas, also in the line of the $CAPM$. For instance, Stoyanov *et al.* (2007) have introduced new ratios related to many risk measures such as $CVaR$, and leading to various benchmark portfolios. Similarly, Rockafellar *et al.* (2006*b*) and (2006*c*) have analyzed portfolio choice problems when risk levels are given by deviation measures, have introduced benchmarks, and have defined new *betas* related to the deviation they are using that preserves the usual relationship between beta and expected return. Rockafellar *et al.* (2007) have also shown the possible existence of equilibrium if agents deal with general deviations.

The present paper considers a general measure of risk ρ. Both expectation bounded risk measures and deviations are included in the analysis, as well as coherent risk measures. Then we present a classical risk/return mathematical programming problem whose solutions will be the efficient portfolios. An important novelty is that this portfolio choice problem involves both ρ and the market pricing rule denoted by Π. From a theoretical point of view, considering Π seems to present some advantages with respect to the usual analysis focusing on the distributions of the available assets' returns. Indeed, Π will be characterized by the Stochastic Discount Factor (SDF) z_π of the economy (Chamberlain and Rothschild, 1983, or Duffie, 1988) which will permit us to study many properties by connecting the SDF z_π of Π and the sub-gradient Δ_ρ of ρ.

The paper's outline is as follows. Section 2 will present the notations and the general framework we are going to deal with. Section 3 will be devoted to study the properties of the efficient portfolios. The section is divided into three subsections. In the first one the general portfolio choice problem is discussed, and necessary and sufficient optimality conditions are provided (Theorem 3). This seems to be one of the first times that this kind of conditions are given for maybe infinite-dimensional portfolio choice problems. As said above, we use pricing rules rather than return distributions, which allows us to consider dynamic pricing models (the Black and Scholes model, for instance) leading to infinite-dimensional optimization problems.

Theorem 3 is used in the second subsection of Section 3 so as to present many cases leading to meaningless economic properties. So, though the notion of compatibility between pricing rules and risk measures has been defined in Balbás and Balbás (2009), this paper deals with its implications in portfolio choice. Theorem 4 points out that risk levels may tend to $-\infty$ while expected returns simultaneously tend to ∞ if the lack of compatibility applies. It is also pointed out that many important risk measures (VaR, $CVaR$, weighted $CVaR$ or $WCVaR$, Dual Power Transform or DPT, etc.) are not compatible with very important pricing models (Black and Scholes, Heston, etc.). All of these cases lead to unbounded risk and returns.

We will also introduce the new notion of strong compatibility between a pricing rule and a risk measure (Definition 2). Once again the lack of strong compatibility makes the expected return be unbounded, although the risk level remains bounded in this case. This pathological situation applies for very important compatible risk measures and deviations (the measure of Wang, the Compatible Conditional Value at Risk or $CCVaR$, the absolute deviation, the absolute down-side semi-deviation, etc.) along with important pricing models (Black and Scholes, Heston, etc.). Theorem 6 and its remarks clarify this finding, that may be very interesting to managers and traders. Indeed, many risk measures are used in practice so as to compute capital requirements, so an unbounded optimal risk/return problem may provide practitioners with practical tools to reach significant falls in the risk levels and the capital requirements, that are sometimes also understood as opportunity costs. Finally, there are two additional remarkable findings of this subsection. Firstly, the new deviation measure $\tilde{N}$ is introduced so as to overcome the incompatibility of the $CVaR$ and the $WCVaR$ with respect to the Black and Scholes model. Secondly, we will show that the standard deviation is strongly compatible with every pricing rule.

The third subsection of the third section is devoted to those situations presenting strong compatibility. In such a case we will introduce the benchmark and the Capital Market Line (CML) for a general couple (Π, ρ), as well as the Market Price of Risk.

The fourth section of the paper will deal with extensions of the classical Arbitrage Pricing Theory (APT) and $CAPM$. With respect to the important contributions of Rockafellar *et al.* (2006*b*) our approach seems to present four major novelties. First, it also applies for expectation bounded or coherent risk measures. Second, it also deals with the APT model. Third, it clarifies that this type of analysis only makes sense in presence of strong compatibility. An fourth, our betas are essentially different of those of Rockafellar *et al.* (2006*b*), and they are similar to those of the classical $CAPM$ and APT (that uses the standard deviation). We do not use the risk/deviation measure ρ so as to define the betas. On the contrary, they are given by the covariance between the returns of the analyzed security and the factors explaining the market (APT) or the benchmark generating the CML ($CAPM$). This fact seems to reflect an advantage since one can clearly see that the betas and the systematic risk are indicating correlation with the factors/market, while the specific risk and the specific noise have null correlation with the factors/market, and therefore cannot be explained by them. In this sense, the betas are reflecting the information about the analyzed security that is given by the factors/market, and the approach becomes quite parallel to that of the classical APT or $CAMP$. This could be another advantage provided by the use of pricing rules and the SDF, a major concept in Financial Economics.

Summarizing, there seems to be several contributions in this paper. So, we provide general optimality conditions in a portfolio choice problem that may involve static and dynamic pricing models. Second, we introduce the new notion of strong compatibility between prices and risks and study the effect of both the lack of compatibility and the lack of strong compatibility. It is pointed out that the lack of (strong) compatibility applies in very important models of Financial Economics. Third, models with a market price of risk are also characterized and analyzed, and they also may involve dynamic pricing models. Finally, APT and $CAPM$ like developments are presented for general risk measures, and they do not modify the classical definitions of the betas. On the contrary, systematic risks will depend on the correlations with the factors/market whereas idiosyncratic risks and noises will be non correlated with them.

Section 5 points out the most important conclusions of the paper.

2 Preliminaries and notations

Consider the probability space $(\Omega, \mathcal{F}, \mu)$ composed of the set of "states of the world" Ω, the $\sigma-$algebra $\mathcal{F}$ and the probability measure μ. If $p \in [1, \infty)$, L^p will denote the space of $\mathbb{R}-$valued random variables y on Ω such that $\mathbb{E}\left(|y|^p\right) < \infty$, $\mathbb{E}()$ representing the mathematical expectation. If $q \in (1, \infty]$ is its conjugate value (*i.e.*, $1/p + 1/q = 1$) then the Riesz Representation Theorem (Horvàth, 1966) guarantees that L^q is the dual space of L^p, where L^∞ is composed of the essentially bounded random variables. A special important case arises for $p = q = 2$.

Consider a time interval $[0, T]$, a subset $\mathcal{T} \subset [0, T]$ of trading dates containing 0 and T, and a filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ providing the arrival of information and such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We will denote by $Y \subset L^2$ a closed subspace composed of reachable pay-offs, *i.e.*, if $y \in Y$ there exists an adapted to the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ price process of a self-financing portfolio $(S_t)_{t \in \mathcal{T}}$, such that $S_T = y$, *a.s.* Then, if $\Pi(y) = S_0$, following usual conventions we will suppose that the pricing rule

$$\Pi : Y \longrightarrow \mathbb{R}$$

providing us with the price $\Pi(y)$ of every $y \in Y$ is linear and continuous.[1] As usual, the market will be said to be complete if $Y = L^2$.

Assume the existence of a riskless asset. Denote by $r_f \geq 0$ the risk-free rate. The equality

$$\Pi(k) = k e^{-r_f T} \tag{1}$$

must hold for every $k \in \mathbb{R}$.

Being Y a Hilbert space the Riesz Representation Theorem implies the existence of a unique $z_\pi \in Y$ such that

$$\Pi(y) = e^{-r_f T} \mathbb{E}\left(y z_\pi\right) \tag{2}$$

[1] The absence of arbitrage implies that S_0 must remain the same if there are more than one self-financing portfolio whose final value equals $y \in Y$.

for every $y \in Y$. z_π is usually called "Stochastic Discount Factor" (SDF), and it is closely related to the Market Portfolio of the $CAPM$ (Duffie, 1988).

Expression (1) implies that

$$ke^{-r_f T} = \Pi(k) = e^{-r_f T} k \mathbb{E}(z_\pi),$$

which leads to

$$\mathbb{E}(z_\pi) = 1. \tag{3}$$

Let $p \in [1, 2]$ and

$$\rho : L^p \longrightarrow \mathbb{R}$$

be the general risk function that a trader uses in order to control the risk level of his final wealth at T. Denote by

$$\Delta_\rho = \{z \in L^q; -\mathbb{E}(yz) \leq \rho(y),\ \forall y \in L^p\}.^2 \tag{4}$$

The set Δ_ρ is obviously convex. We will assume that Δ_ρ is also $\sigma(L^q, L^p)$ −compact,[3] and

$$\rho(y) = Max\ \{-\mathbb{E}(yz) : z \in \Delta_\rho\} \tag{5}$$

holds for every $y \in L^p$. Furthermore, we will also impose the existence of $\tilde{E} \in \mathbb{R}$, $\tilde{E} \geq 0$, such that

$$\Delta_\rho \subset \left\{z \in L^q; \mathbb{E}(z) = \tilde{E}\right\}. \tag{6}$$

Summarizing, we have:

Assumption 1. The set Δ_ρ given by (4) is convex and $\sigma(L^q, L^p)$ −compact, (5) holds for every $y \in L^p$ and (6) holds. □

The assumption above is closely related to the Representation Theorem of Risk Measures stated in Rockafellar *et al.* (2006a). Following their ideas, and bearing in mind the Representation Theorem 2.4.9 in Zalinescu (2002) for convex functions, it is easy to prove that the fulfillment of Assumption 1 holds if and only if ρ is continuous and satisfies:

a) Translation invariance

$$\rho(y + k) = \rho(y) - \tilde{E}k$$

[2] Notice that $q \in [2, \infty]$.

[3] See Horvàth (1966) for further details about $\sigma(L^q, L^p)$ −compact sets.

for every $y \in L^p$ and $k \in \mathbb{R}$.

b) Homogeneity

$$\rho(\alpha y) = \alpha \rho(y)$$

for every $y \in L^p$ and $\alpha > 0$.

c) Sub-additivity

$$\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$$

for every $y_1, y_2 \in L^p$.

d) Mean dominating

$$\rho(y) \geq -\tilde{E}\mathbb{E}(y) \tag{7}$$

for every $y \in L^p$.[4]

It is easy to see that if ρ is continuous and satisfies Properties a), b), c) and d) above with $\tilde{E} = 1$ then it is also coherent in the sense of Artzner *et al.* (1999) if and only if

$$\Delta_\rho \subset L^q_+ = \{z \in L^q; \mu(z \geq 0) = 1\}. \tag{8}$$

Particular interesting examples with $\tilde{E} = 1$ are the Expectation Bounded Risk Measures of Rockafellar *et al.* (2006a). For instance, the Conditional Value at Risk ($CVaR$) and the Weighted Conditional Value at Risk ($WCVaR$) (Rockafellar *et al.*, 2006a), the Compatible Conditional Value at Risk ($CCVaR$) of Balbás and Balbás (2009), the Dual Power Transform (DPT) of Wang (2000) and the Wang Measure (Wang, 2000), among many others. Particular interesting examples with $\tilde{E} = 0$ are the deviation measures of Rockafellar *et al.* (2006a). Among others, the classical $p-$deviation given by

$$\sigma_p(y) = \left[\mathbb{E}\left(|\mathbb{E}(y) - y|^p\right)\right]^{1/p}, \tag{9}$$

or the downside $p-$semi-deviation given by

$$\sigma_p^-(y) = \left[\mathbb{E}\left(|Max\ \{\mathbb{E}(y) - y, 0\}|^p\right)\right]^{1/p}. \tag{10}$$

[4] Actually, the properties above are almost similar to those used by Rockafellar *et al.* (2006a) in order to introduce their Expectation Bounded Risk Measures.

Finally, let us remark that L^2 being a Hilbert space there are orthogonal projections on every closed subspace. In particular, we will focus on φ_Y and φ_π, the orthogonal projections on Y and the linear manifold $\mathcal{L}(1, z_\pi) \subset Y$ respectively (see Maurin, 1967, for further details about the orthogonal projection in Hilbert spaces).

3 Portfolio choice

3.1 General approach

Let us consider the following portfolio choice problem,

$$\left\{ \begin{array}{c} Min\ \rho(y) \\ \mathbb{E}(yz_\pi) \leq e^{r_f T} \\ \mathbb{E}(y) \geq R \\ y \in Y \end{array} \right. . \tag{11}$$

where $R > e^{r_f T}$ represents the minimum required return. Bearing in mind (2), (11) minimizes the risk of a reachable pay-off whose global price is not higher than one and whose expected value is at least R. Thus it is a standard Risk/Return approach with ρ as the risk measure. Of course, higher quantities of money may be invested. Since ρ and $\mathbb{E}$ are homogeneous the solution of (11) will be multiplied by $C > 0$ if C denotes the value of the quantity to invest and the first and second constraints become $\mathbb{E}(yz_\pi) \leq Ce^{r_f T}$ and $\mathbb{E}(y) \geq RC$ respectively.

The minimization of risk measures is a complex problem that may be addressed with several methods. Among others, the approaches of Ruszczynski and Shapiro (2006) or Rockafellar *et al.* (2006*b*) appropriately overcome those problems generated by the lack of differentiability of ρ. Nevertheless, we will follow the method of Balbás *et al.* (2009*b*) and, accordingly, we will transform (11) in the new problem

$$\begin{cases} Min\ \theta \\ \theta + \mathbb{E}(yz) \geq 0, \quad \forall z \in \Delta_\rho \\ \mathbb{E}(yz_\pi) \leq e^{r_f T} \\ \mathbb{E}(y) \geq R \\ \theta \in \mathbb{R},\ y \in Y \end{cases} \tag{12}$$

$\theta \in \mathbb{R}$ and $y \in L^2$ being the decision variables. Following the paper above, (5) allows us to prove that y solves (11) if and only if there exists $\theta \in \mathbb{R}$ such that (θ, y) solves (12), in which case

$$\theta = \rho(y)$$

holds. Furthermore, with similar arguments as in Balbás *et al.* (2009*b*), one can show that Problem

$$\begin{cases} Max\ \ -e^{r_f T}\lambda_1 + R\lambda_2 \\ \mathbb{E}(y(\lambda_1 z_\pi - \lambda_2 - z)) = 0, \quad \forall y \in Y \\ z \in \Delta_\rho,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \end{cases}$$

is the dual of (12), $\lambda_1, \lambda_2 \in \mathbb{R}$ and $z \in \Delta_\rho$ being the decision variables. The first constraint means that $\lambda_1 z_\pi - \lambda_2 - z \in Y^T$, Y^T denoting the orthogonal subspace of Y. Then it is equivalent to $\varphi_Y(\lambda_1 z_\pi - \lambda_2 - z) = 0$, which, along with $1 \in Y$ and $z_\pi \in Y$, lead to the following dual problem

$$\begin{cases} Max\ \ -e^{r_f T}\lambda_1 + R\lambda_2 \\ \varphi_Y(z) = \lambda_1 z_\pi - \lambda_2, \\ z \in \Delta_\rho,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0 \end{cases} . \tag{13}$$

Proposition 1 $\mathbb{E}(\varphi_Y(z)) = \tilde{E}$ *for every* $z \in \Delta_\rho$.

Proof. Obviously $z - \varphi_Y(z) \in Y^T$, and $1 \in Y$, so $\mathbb{E}(\varphi_Y(z)) = \mathbb{E}(z)$. Therefore the conclusion follows from (6). □

Consequently we can simplify (13). Indeed, taking expectations in the first restriction of (13), and taking into account (3) we have

$$\lambda_1 = \lambda_2 + \tilde{E}.$$

Thus, changing the variable $\lambda_2 = \lambda$, $\lambda_1 = \lambda + \tilde{E}$ we have the following problem equivalent to (13)

$$\begin{cases} Max \ \left(R - e^{r_f T}\right)\lambda - \tilde{E}e^{r_f T} \\ \varphi_Y(z) = \left(\tilde{E} + \lambda\right) z_\pi - \lambda \\ z \in \Delta_\rho, \ \ \lambda \geq 0 \end{cases} \tag{14}$$

Problems (12) and (14) involve the infinite-dimensional Hilbert space L^2. Thus, the absence of the so called "duality gap" is not guaranteed, *i.e.*, the dual optimal value may be strictly lower than the primal one (Luenberger, 1969). To overcome this caveat we have to verify the fulfillment of the Slater qualification, which requires an additional assumption.

Assumption 2. There exists $y \in Y$ such that $\mathbb{E}(yz_\pi) \leq e^{r_f T}$ and $\mathbb{E}(y) > e^{r_f T}$.[5] □

Proposition 2 *Problem (12) is feasible and satisfies the Slater qualification, i.e., there exists $(\theta, y) \in \mathbb{R} \times Y$ satisfying the three constraints of (12) as strict inequalities.*

Proof. Consider the pay-off y satisfying the conditions of Assumption 2, a positive constant $C < 1$, and for $\alpha \in \mathbb{R}$, $\alpha > 0$, take

$$y_\alpha = \alpha y - \left(\alpha e^{r_f T} - C\right).$$

Then, (3) trivially shows that $\mathbb{E}(y_\alpha)$ tends to ∞ as so does α whereas $\mathbb{E}(y_\alpha z_\pi) \leq C$. Hence we can fix α large enough to guarantee the fulfillment of the second and third constraints as strict inequalities. Besides, the function $\Delta_\rho \ni z \to \mathbb{E}(y_\alpha z) \in \mathbb{R}$ is continuous and Δ_ρ is compact, so taking $\theta > Max\left\{-\mathbb{E}(yz); z \in \Delta_\rho\right\}$ the first constraint is satisfied as a strict inequality too. □

The Slater qualification ensures the absence of duality gap (Luenberger, 1969). Thus, one can give the Strong Duality Theorem below, whose proof is omitted because a similar one may be found in Balbás *et al.* (2009*b*).

[5] Since $\mathbb{E}(y_0 z_\pi) = 1$ and $\mathbb{E}(y_0) = 1$ if $y_0 = 1$ is a risless security, and $\mathbb{E}\left(ye^{-r_f T} z_\pi\right) \leq 1$ and $\mathbb{E}\left(ye^{-r_f T}\right) > 1$, actually Assumption 2 only imposes the existence of a risky security whose expected return is higher than the interest rate.

Theorem 3 *Suppose that* $y^* \in L^p$ *and* $(\lambda^*, z^*) \in \mathbb{R} \times L^q$*. Then, they solve (11) and (14) if and only if the following Karush-Kuhn-Tucker conditions*

$$\begin{cases} \left(\lambda^* + \tilde{E}\right)\left(\mathbb{E}\left(y^* z_\pi\right) - e^{r_f T}\right) = 0 \\ \lambda^*\left(\mathbb{E}\left(y^*\right) - R\right) = 0 \\ \mathbb{E}\left(y^* z_\pi\right) \leq e^{r_f T} \\ \mathbb{E}\left(y^*\right) \geq R \\ \varphi_Y\left(z^*\right) = \left(\tilde{E} + \lambda^*\right) z_\pi - \lambda^* \\ \mathbb{E}\left(y^* \varphi_Y\left(z^*\right)\right) \leq \mathbb{E}\left(y^* \varphi_Y\left(z\right)\right), \qquad \forall z \in \Delta_\rho \\ \lambda^* \geq 0, \ z^* \in \Delta_\rho \end{cases} \tag{15}$$

are fulfilled. Moreover, the dual solution is attainable if (11) is bounded, in which case the optimal value of both (11) and (14) becomes $\left(R - e^{r_f T}\right)\lambda^* - \tilde{E} e^{r_f T}$*.* □

3.2 Cases with unbounded optimal risk or return

This subsection is devoted to illustrate the existence of examples leading to meaningless situations from a economic point of view. Surprisingly, some of these examples will involve very important pricing models (for instance, Black and Scholes) and very important risk measures (for instance, $CVaR$). Non pathological cases will be analyzed in the next subsection.

We will consider two notions: Compatibility and strong compatibility between the pricing rule Π and the risk measure ρ.

Definition 1 *(Balbás and Balbás, 2009). The pricing rule* Π *and the risk measure* ρ *are said to be compatible if there are no sequences*

$$(y_n)_{n=1}^{\infty} \subset Y$$

such that $e^{r_f T}\Pi\left(y_n\right) = \mathbb{E}\left(y_n z_\pi\right) \leq 0$ *for every* $n \in \mathbb{N}$ *and* $Lim_n \rho\left(y_n\right) = -\infty$*.* □

Next let us show the existence of pathological situations

Theorem 4 Π *and* ρ *are not compatible if and only if Problem (11) is unbounded, i.e., if and only if for every* $R > e^{r_f T}$ *the risk level may tend to* $-\infty$ *whereas the expected return is at least* R*.*

Proof. Take y_0 (11)-feasible (the existence is guaranteed by Proposition 2) and the sequence $(y_n)_{n=1}^{\infty}$ of the definition above. Then, $\mathbb{E}\left((y_0 + y_n)z_\pi\right) \leq \mathbb{E}\left(y_0 z_\pi\right) \leq e^{r_f T}$ whereas $\rho(y_0 + y_n) \leq \rho\left(y_0\right) + \rho\left(y_n\right)$ obviously tends to $-\infty$. Hence, it is sufficient to show that $\mathbb{E}\left(y_0 + y_n\right) \geq R$. $\mathbb{E}\left(y_0 + y_n\right) = \mathbb{E}\left(y_0\right) + \mathbb{E}\left(y_n\right) \geq R + \mathbb{E}\left(y_n\right)$. (7) leads to $\mathbb{E}\left(y_n\right) \geq -\frac{1}{\tilde{E}}\rho\left(y_n\right) \geq 0$ because $\tilde{E} > 0$ and $\rho\left(y_n\right) \leq 0$ for $n \in \mathbb{N}$ large enough.[6]

Conversely, if Π and ρ are compatible then

$$\rho\left(y\right) \geq -\tilde{E}\mathbb{E}\left(y z_\pi\right)$$

for every $y \in Y$ (Balbás and Balbás, 2009), and this implies that $\rho\left(y\right) \geq -\tilde{E}e^{r_f T}$ for every $y \in Y$ such that $\mathbb{E}\left(y z_\pi\right) \leq e^{r_f T}$. □

Remark 1 *There are many examples that fit in the latter theorem. For instance, Balbás and Balbás (2009) have shown that the $CVaR$, the $WCVaR$ and the DPT are not compatible with the Black and Scholes model and the Heston models, among many other classical pricing models related to derivative securities. All of these cases lead to portfolio choice problems such that there are available strategies whose risk becomes $-\infty$ while their expected return becomes as large as desired. Moreover, since*

$$VaR_{\mu_0}\left(y\right) \leq CVaR_{\mu_0}\left(y\right)$$

holds for every level of confidence $\mu_0 \in (0,1)$ and every $y \in L^2$, if we fix $R > e^{r_f T}$ then for the Black and Scholes and for the Heston pricing model one can find a sequence of reachable pay-offs whose expected return remains higher than R while their VaR_{μ_0} tends to $-\infty$. □

Remark 2 *An obvious consequence of Theorems 3 and 4 is that the compatibility of Π and ρ is equivalent to the feasibility of (14), i.e., to the existence of $\lambda \geq 0$ and $z \in \Delta_\rho$ such that*

$$\varphi_Y\left(z\right) = \left(\tilde{E} + \lambda\right) z_\pi - \lambda \tag{16}$$

holds. □

The second important notion in this section is the "strong compatibility".

[6] Notice that $\tilde{E} = 0$ cannot hold because (7) would imply $\rho\left(y\right) \geq 0$ for every $y \in L^p$, and Definition 1 could not hold.

Definition 2 *The pricing rule* Π *and the risk measure* ρ *are said to be strongly compatible if there exist* $\lambda > 0$ *and* $z \in \Delta_\rho$ *such that (16) holds.* $\square$

The lack of strong compatibility will also lead to pathological situations.

Theorem 5 *Suppose that* Π *and* ρ *are compatible but they are not strongly compatible. Then:*

a) The dual solution (λ^*, z^*) *exists and satisfies* $\lambda^* = 0$.

b) The (11)-optimal value equals $-\tilde{E}e^{r_f T}$ *and does not depend on* R.

Proof. Since Π and ρ are compatible Theorem 4 shows that (11) is bounded, so Theorem 3 implies the existence of a dual solution (λ^*, z^*). The lack of strong compatibility implies that $\lambda^* = 0$, since there are no (14)-feasible solutions with strictly positive λ. Moreover, (14) makes it obvious that the optimal value equals $-\tilde{E}e^{r_f T}$ and does not depend on R. $\square$

Remark 3 *If the lack of strong compatibility occurs then once again we are facing a meaningless phenomenon from an economic point of view. Indeed, Theorem 5b points out that the minimum risk level will remain constant and equal to* $-\tilde{E}e^{r_f T}$ *while the expected return* R *may tend to* ∞. *As in the previous case of lack of compatibility, there is no market price of risk either, since the expected return may increase as desired without any increment of risk. The only difference between both scenarios is given by the behavior of the optimal risk level. If there is no compatibility it may go to* $-\infty$. *If there is compatibility but there is no strong compatibility then it remains the same* $(-\tilde{E}e^{r_f T})$. $\square$

Next let us see that the lack of strong compatibility frequently holds for complete markets.

Theorem 6 *Suppose that for every* $\delta > 0$

$$\mu\left(z_\pi < \delta\right) > 0. \tag{17}$$

If the market is complete ($Y = L^2$) *and* ρ *is coherent and expectation bounded then* Π *and* ρ *are not strongly compatible.* $\square$

Proof. Since the market is complete φ_Y becomes the identity map. Furthermore, $\tilde{E} = 1$ because ρ is expectation bounded. Therefore (16) becomes

$$z = (1 + \lambda)\, z_\pi - \lambda. \tag{18}$$

Suppose that Π and ρ are strongly compatible and take $\lambda > 0$ and $z \in \Delta_\rho$ satisfying (18). Given $\delta > 0$ one has that

$$\mu \left(z_\pi < \frac{\delta}{1 + \lambda} \right) > 0,$$

and (18) implies that $\mu\,(z < \delta - \lambda) > 0$. Taking $\delta < \lambda$ one has that $\mu\,(z < 0) > 0$. On the other hand, the coherence of ρ and (8) show that $z \geq 0$. Whence, we have a clear contradiction. □

Remark 4 *There are many examples of complete markets satisfying (17). For instance, the Black and Scholes model (Wang, 2000, or Hamada and Sherris, 2003). It may be also proved that the Heston model and other Stochastic Volatility models satisfy (17). All of these models are not strongly compatible with any coherent and expectation bounded risk measure. Very important examples of such a measures are the $CCVaR$ and the Wang measure, among others. For all of these cases there is no market price of risk, and the optimal value of (11) always equals $-e^{r_f T}$ and does not depend on R. In other words, one can construct a portfolio whose expected value is as large as desired and whose risk level remains bounded and constant.* □

Remark 5 *Balbás and Balbás (2009) have shown that the $CCVaR$ and the Black and Scholes model are compatible, but the latter remark has pointed out that they are not strongly compatible. For the Black and Scholes model, and for every level of confidence $\mu_0 \in (0, 1)$, one has that (Balbás and Balbás, 2009)*

$$CCVaR_{\mu_0}(y) = Max\ \left\{ CVaR_{\mu_0}(y)\,, -\Pi\,(y)\, e^{r_f T} \right\}.$$

Since the $CCVaR$ is coherent and expectation bounded, and following Rockafellar et al. (2006a) to construct deviations, one can define the new deviation measure

$$\tilde{N}_{\mu_0}(y) = Max\ \left\{ CVaR_{\mu_0}(y)\,, -\Pi\,(y)\, e^{r_f T} \right\} + \mathbb{E}\,(y)\,,$$

which satisfies the requirements of Assumption 1 for $\tilde{E} = 0$. It is easy to see that $\tilde{N}_{\mu_0}$ and the Black and Scholes model are strongly compatible. Indeed, otherwise we could find a sequence

$(y_n)_{n=1}^{\infty} \subset Y$ *with* $\mathbb{E}(y_n z_\pi) \leq e^{r_f T}$ *for every* $n \in \mathbb{N}$, $\left(\tilde{N}_{\mu_0}(y_n)\right)_{n=1}^{\infty}$ *bounded from above and* $(\mathbb{E}(y_n))_{n=1}^{\infty}$ *going to* ∞. *Therefore* $\left(CCVaR_{\mu_0}(y_n)\right)_{n=1}^{\infty} = \left(\tilde{N}_{\mu_0}(y_n)\right)_{n=1}^{\infty} - (\mathbb{E}(y_n))_{n=1}^{\infty}$ *would go to* $-\infty$. *Thus, (11) would be unbounded and Theorem* 4 *would imply that the* $CCVaR$ *would not be compatible with the Black and Scholes model.*

Finally, let us indicate that a similar remark applies if the role of the $CVaR$ *is played by the* $WCVaR$. □

Remark 6 *It is worth to remark that the absence of compatibility cannot hold if* ρ *is a deviation measure. Indeed, notice that* $\tilde{E} = 0$ *in such a case, so (7) points out that* ρ *does not achieve negative values and therefore it cannot tend to* $-\infty$, *i.e., Definition* 1 *cannot hold.*

However, the lack of strong compatibility may still hold. For instance, take the absolute deviation (9) for $p = 1$

$$\rho(y) = \sigma_1(y) = \mathbb{E}(|\mathbb{E}(y) - y|).$$

Then, according to Rockafellar et al. (2006a),

$$\Delta_\rho = \{z - \mathbb{E}(z);\ z \in L^\infty,\ \|z\|_\infty \leq 1\}.$$

Therefore Δ_ρ *is obviously composed of (essentially) bounded random variables. Besides, (16) and* $\tilde{E} = 0$ *lead to*

$$z = \lambda z_\pi - \lambda$$

for complete markets (φ_Y *is the identity map). Nevertheless, if* z_π *is unbounded (Black and Scholes, Heston etc.), the latter equality implies that* $\lambda = 0$, *i.e., there is no strong compatibility. This is also interesting to remark that the absolute deviation is the unique* $p-$*deviation (see (9)) compatible with the Second Order Stochastic Dominance and the standard utility functions (Ogryczak and Ruszczynski, 1999 and 2002). Finally, it is also easy to see that the absolute semi-deviation ((10),* $p = 1$*) is not strongly compatible with the Black and Scholes and the Heston models neither. It trivially follows from* $\sigma_1^-(y) = \sigma_1(y)/2$. □

Remark 7 *Finally, it is also remarkable that the standard deviation is strongly compatible with every pricing rule. Indeed, for* $\rho = \sigma_2$ *we have that (Balbás et al., 2009b)*

$$\Delta_\rho = \left\{z \in L^2;\ \mathbb{E}(z) = 0,\ \sigma_2^2(z) \leq 1\right\}. \tag{19}$$

Then, (3) shows that $\lambda z_\pi - \lambda \in \Delta_\rho$ if $\lambda > 0$ is small enough so as to satisfy $\lambda^2 \sigma_2^2 (z_\pi - 1) \leq 1$. Besides, the equality $\tilde{E} = 0$ and $\varphi_Y(\lambda z_\pi - \lambda) = \lambda z_\pi - \lambda$ point out that Equality (16) holds. □

3.3 Models with a market price of risk

This subsection will deal with models where the strong compatibility holds. Thus, henceforth we will assume the following

Assumption 3. There exists strong compatibility between Π and ρ. □

Theorem 7 *The dual solution (λ^*, z^*) exists, does not depend on $R > e^{r_f T}$ and satisfies $\lambda^* > 0$. The (11) and (14) optimal value equals $\left(R - e^{r_f T}\right)\lambda^* - \tilde{E} e^{r_f T}$.*

Proof. Assumption 3 and Theorem 4 show that (11) and (14) are bounded and Theorem 3 shows that (14) attains its optimal value. Moreover it is obvious that this optimal value coincides with the solution of

$$\left\{ \begin{array}{c} Max\ \lambda \\ \varphi_Y(z) = \left(\tilde{E} + \lambda\right) z_\pi - \lambda \\ z \in \Delta_\rho,\ \lambda \geq 0 \end{array} \right. . \tag{20}$$

The remaining statements are now trivial. □

Remark 8 *According to (15) the solutions of (11) and (14) are characterized by*

$$\left\{ \begin{array}{l} \mathbb{E}(y^* z_\pi) = e^{r_f T} \\ \mathbb{E}(y^*) = R \\ \varphi_Y(z^*) = \left(\tilde{E} + \lambda^*\right) z_\pi - \lambda^* \\ \mathbb{E}(y^* \varphi_Y(z^*)) \leq \mathbb{E}(y^* \varphi_Y(z)),\quad \forall z \in \Delta_\rho \\ \lambda^* > 0,\ z^* \in \Delta_\rho \end{array} \right. \tag{21}$$

since $\lambda^* + \tilde{E} \geq \lambda^* > 0$. The two first equalities show that the (11)-constraints are saturated, so R is the real expected return of the investment. □

Remark 9 *If y^* solves (11) the absence of duality gap (Theorem 3) and $\lambda^* > 0$ (Theorem 7) for the dual solution imply that*

$$\rho(y^*) = \left(R - e^{r_f T}\right)\lambda^* - \tilde{E}e^{r_f T},$$

and therefore

$$R = \frac{1}{\lambda^*}\left(\rho(y^*) + \tilde{E}e^{r_f T}\right) + e^{r_f T}. \tag{22}$$

One can interpret that $\frac{1}{\lambda^*}$ represents the Market Price of Risk, in the sense that there is an affine relationship between optimal risks and returns, and the expected return R increases $\frac{1}{\lambda^*}$ units per unit of the risk increment. Due to the analogy with the usual Capital Asset Pricing Model the affine function (22) will be called Capital Market Line (CML). □

Remark 10 *Henceforth fix*

$$R_0 > e^{r_f T} \tag{23}$$

and take y_0^*, solution of (11). Consider (λ^*, z^*) such that y_0^* and (λ^*, z^*) satisfy (21). Then bearing in mind (3) and Proposition 1 it is easy to see that

$$\alpha y^* - \left(\alpha e^{r_f T} - e^{r_f T}\right) \in Y$$

and (λ^*, z^*) also satisfy (21) if $\alpha\left(R_0 - e^{r_f T}\right) + e^{r_f T}$ replaces R_0. Since $\alpha\left(R_0 - e^{r_f T}\right) + e^{r_f T}$ takes all the values within $\left(e^{r_f T}, \infty\right)$ as so does α within the interval $(0, \infty)$, it is clear that the dual solution does not depend on R_0 and the primal one is a combination of y_0^* and the riskless asset leading to the required expected return.

The remaining efficient portfolios (solutions of (11)) that arises as R varies are combinations of the risk-free asset and the benchmark y_0^*. So, for $R > e^{r_f T}$ the discussion above shows that the proportion α to invest in the benchmark y_0^* must satisfy

$$R = \alpha\left(R_0 - e^{r_f T}\right) + e^{r_f T}$$

which leads to

$$\alpha = \frac{R - e^{r_f T}}{R_0 - e^{r_f T}},$$

and

$$1 - \alpha = \frac{R_0 - R}{R_0 - e^{r_f T}}$$

will be invested in the riskless security.

Every portfolio $y \in Y$ *with* $\Pi(y) = 1$ *and* $\mathbb{E}(y) = R > 1$ *must be replaced by*

$$\frac{R - e^{r_f T}}{R_0 - e^{r_f T}} y_0^* + \frac{R_0 - R}{R_0 - e^{r_f T}} e^{r_f T} = \frac{\mathbb{E}\left(y - e^{r_f T}\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} y_0^* + \frac{\mathbb{E}\left(y_0^* - y\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} e^{r_f T} \tag{24}$$

so as to reach an efficient portfolio with optimal risk level. The optimal risk level

$$\frac{\mathbb{E}\left(y - e^{r_f T}\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} \rho\left(y_0^*\right) - \tilde{E} \frac{\mathbb{E}\left(y_0^* - y\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} e^{r_f T} \tag{25}$$

will be called systematic risk of y*, and the remaining risk*

$$\rho(y) - \frac{\mathbb{E}\left(y - e^{r_f T}\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} \rho\left(y_0^*\right) + \tilde{E} \frac{\mathbb{E}\left(y_0^* - y\right)}{\mathbb{E}\left(y_0^* - e^{r_f T}\right)} e^{r_f T} \tag{26}$$

will be called idiosyncratic or specific. □

4 CAMP and APT like models

The object of this section is to prove that $CAPM$ and APT like formulas also hold in the general framework we are dealing with. To this purpose we will consider the portfolio

$$y_1^* = \varphi_\pi\left(y_0^*\right) \in \mathcal{L}\left(1, z_\pi\right),$$

orthogonal projection of the benchmark y_0^* on the linear manifold $\mathcal{L}(1, z_\pi)$ generated by the riskless asset and the SDF. Since $y_1^* - y_0^* \in \mathcal{L}(1, z_\pi)^T$, orthogonal subspace of $\mathcal{L}(1, z_\pi)$ (Maurin, 1967), one has that $\mathbb{E}\left(y_1^* - y_0^*\right) = \mathbb{E}\left(\left(y_1^* - y_0^*\right) z_\pi\right) = 0$, which gives

$$\mathbb{E}\left(y_1^*\right) = \mathbb{E}\left(y_0^*\right) = R_0 \tag{27}$$

and

$$\mathbb{E}\left(y_1^* z_\pi\right) = \mathbb{E}\left(y_0^* z_\pi\right) = e^{r_f T}. \tag{28}$$

In particular, y_1^* is (11)-feasible.

Hereafter the variance of a random variable $y \in L^2$ and the covariance between two random variables $y_1, y_2 \in L^2$ will be denoted by $\sigma_2^2(y)$ and $\mathbb{C}(y_1, y_2)$, respectively.

In the classical $CAPM$ and APT models one must assume that the market is not risk-neutral, which means that the Market Portfolio is not a riskless security (Duffie, 1988). Actually our Assumption 2 also imposes a non risk neutral market.

Proposition 8 *The market is not risk-neutral, i.e., the benchmarks y_0^* and y_1^* are not riskless securities (are not zero-variance). Therefore the SDF z_π is not a riskless security either and $\mathcal{L}(1, z_\pi) = \mathcal{L}(1, y_1^*)$.*

Proof. If y_0^* were riskless security then $y_0^* \in \mathcal{L}(1, z_\pi)$ and $y_1^* = \varphi_\pi(y_0^*) = y_0^*$ would be a riskless security too. Thus, let us show that y_1^* is not a riskless security. Indeed, suppose that y_1^* is constant. (23) and (27) show that $y_1^* > e^{r_f T}$, and therefore $\mathbb{E}(y_1^* z_\pi) = y_1^* \mathbb{E}(z_\pi) = y_1^*$ owing to (3), which contradicts (28).

Besides, $y_1^* \in \mathcal{L}(1, z_\pi)$ points out that z_π is not a riskless security either since otherwise the dimension of $\mathcal{L}(1, z_\pi)$ would equal one and y_1^* would have to be risk-free. Finally, the equality $\mathcal{L}(1, z_\pi) = \mathcal{L}(1, y_1^*)$ is already trivial. $\square$

Theorem 9 *(APT like formula). Suppose that $\{y_1, y_2, ..., y_k\} \subset Y$ is a linearly independent system such that $\mathbb{C}(y_i, y_j) = 0$ for $i \neq j$. Suppose also that the benchmark y_1^* satisfies*

$$y_1^* \in \mathcal{L}(1, y_1, y_2, ..., y_k).$$ [7]

Then, for every reachable pay-off $y \in Y$ we have that

$$y - \mathbb{E}(y z_\pi) = \sum_{j=1}^{k} \beta_j (y_j - \mathbb{E}(y_j z_\pi)) + \varepsilon_y \tag{29}$$

and

$$\mathbb{E}(y - y z_\pi) = \sum_{j=1}^{k} \beta_j (\mathbb{E}(y_j - y_j z_\pi)),$$

$\varepsilon_y \in Y$ satisfying

$$\mathbb{E}(\varepsilon_y) = 0, \quad \Pi(\varepsilon_y) = 0, \quad \text{and} \quad \mathbb{C}(\varepsilon_y, y_j) = 0, \; j = 1, 2, ..., k, \tag{30}$$

and β_j being the regression coefficient

$$\beta_j = \frac{\mathbb{C}(y, y_j)}{\sigma_2^2(y_j)}, \tag{31}$$

[7] Notice that this condition is equivalent to

$$z_\pi \in \mathcal{L}(1, y_1, y_2, ..., y_k)$$

due to $\mathcal{L}(1, z_\pi) = \mathcal{L}(1, y_1^*)$.

$j = 1, 2, ..., k$. In particular, if $\Pi(y) = \Pi(y_j) = 1$, $j = 1, 2, ..., k$, then

$$y - e^{r_f T} = \sum_{j=1}^{k} \beta_j \left(y_j - e^{r_f T} \right) + \varepsilon_y \tag{32}$$

and

$$\mathbb{E}\left(y - e^{r_f T} \right) = \sum_{j=1}^{k} \beta_j \left(\mathbb{E}\left(y_j - e^{r_f T} \right) \right). \tag{33}$$

Corollary 10 *(CAPM like formula). For every reachable pay-off $y \in Y$ we have that*

$$y - \mathbb{E}(y z_\pi) = \beta \left(y_1^* - e^{r_f T} \right) + \varepsilon_y$$

and

$$\mathbb{E}(y - y z_\pi) = \beta \left(\mathbb{E}\left(y_1^* - e^{r_f T} \right) \right),$$

$\varepsilon_y \in Y$ satisfying

$$\mathbb{E}(\varepsilon_y) = 0, \quad \Pi(\varepsilon_y) = 0, \quad \text{and} \quad \mathbb{C}(\varepsilon_y, y_1^*) = 0,$$

and β being the regression coefficient

$$\beta = \frac{\mathbb{C}(y, y_1^*)}{\sigma_2^2(y_1^*)}. \tag{34}$$

In particular, if $\Pi(y) = 1$, then

$$y - e^{r_f T} = \beta \left(y_1^* - e^{r_f T} \right) + \varepsilon_y \tag{35}$$

and

$$\mathbb{E}\left(y - e^{r_f T} \right) = \beta \left(\mathbb{E}\left(y_1^* - e^{r_f T} \right) \right). \tag{36}$$

Proof. Let us prove Theorem 9 since Corollary 10 is a trivial consequence if one bears in mind (2) and (28). Obviously, if $R_j = \mathbb{E}(y_j)$, $j = 1, 2, ..., k$,

$$\left\{ 1, \left(\frac{y_j - R_j}{\sigma_2(y_j)} \right)_{j=1}^{k} \right\} \subset Y$$

is a orthonormal system. Thus, the projection Lemma of Hilbert Spaces (Maurin, 1967) establishes the existence of $\varepsilon_y \in \mathcal{L}(1, y_1, y_2, ..., y_k)^T$ such that

$$y = \tilde{\beta}_0 + \sum_{j=1}^{k} \tilde{\beta}_j \left(\frac{y_j - R_j}{\sigma_2(y_j)} \right) + \varepsilon_y, \tag{37}$$

where

$$\tilde{\beta}_0 = \mathbb{E}(y)$$

and

$$\tilde{\beta}_j = \mathbb{E}\left(y\left(\frac{y_j - R_j}{\sigma_2(y_j)}\right)\right), \tag{38}$$

$j = 1, 2, ..., k$. $\varepsilon_y \in \mathcal{L}(1, y_1, y_2, ..., y_k)^T$ trivially leads to (30), and multiplying by z_π and taking expectations in (37) one has

$$\tilde{\beta}_0 = \Pi(y) e^{r_f T} - \sum_{j=1}^{k} \tilde{\beta}_j \left(\frac{\Pi(y_j) e^{r_f T} - R_j}{\sigma_2(y_j)}\right).$$

Whence (37) becomes

$$y - \Pi(y) e^{r_f T} = \sum_{j=1}^{k} \tilde{\beta}_j \left(\frac{y_j - \Pi(y_j) e^{r_f T}}{\sigma_2(y_j)}\right) + \varepsilon_y,$$

which, due to (2), leads to (29) if one takes $\tilde{\beta}_j = \beta_j / \sigma_2(y_j)$, $j = 1, 2, ..., k$. Moreover, (31) trivially follows from (38). The remaining expressions, (32) and (33), are now obvious. □

Remark 11 *Expressions (32) and (33) are clearly similar to those of the classical APT model. They indicate that the real $y - e^{r_f T}$ and the expected $\mathbb{E}\left(y - e^{r_f T}\right)$ risk premiums may be given by a family of non correlated factors that generate the benchmark y_1^* if one adds the riskless asset. One only needs to estimate the systematic risk levels β_j, given by (31), that yield the sensitivity of the security (pay-off) y with respect to the $j-th$ factor explaining the market. The committed error ε_y has neither correlation with the factors nor with the benchmark y_1^*, and therefore is something specific of the security y.*

Analogously, (35) and (36) indicate that the real $y - e^{r_f T}$ and the expected $\mathbb{E}\left(y - e^{r_f T}\right)$ risk premiums may also be given by the real $y_1^* - e^{r_f T}$ and the expected $\mathbb{E}\left(y_1^* - e^{r_f T}\right)$ risk premiums of the benchmark y_1^*. The relationship is given by the systematic risk level β given by (34). Once again the error ε_y has no correlation with the benchmark y_1^* and is specific of the asset/portfolio we are analyzing.

As said in Remark 10, given $y \in Y$ with $\Pi(y) = 1$ one can construct an efficient portfolio with the same expected return but lower risk. (24) and (36) show that the efficient portfolio will be

$$\beta y_0^* + (1 - \beta) e^{r_f T}$$

where β is given by (34). The systematic risk (25) of y becomes

$$\beta\rho\left(y_0^*\right)-\tilde{E}\left(1-\beta\right)e^{r_fT},$$

which is clearly given by β once $\rho\left(y_0^*\right)$ is known, i.e., β may be understood as a measure of the systematic risk.

ρ being an homogeneous, translation invariant and sub-additive risk measure implies that

$$\begin{aligned}\rho\left(y\right)&=\rho\left(e^{r_fT}+\beta\left(y_1^*-e^{r_fT}\right)+\varepsilon_y\right)\\&\leq\beta\rho\left(y_1^*\right)-\tilde{E}\left(1-\beta\right)e^{r_fT}+\rho\left(\varepsilon_y\right).\end{aligned}\tag{39}$$

Since y_0^* is efficient (28) and (27) point out that

$$\rho\left(y_0^*\right)\leq\rho\left(y_1^*\right),$$

with equality if and only if $y_1^*=y_0^*$. Thus, bearing in mind (39), if $\beta\geq0$ the specific (26) risk of y will be

$$\begin{aligned}&\rho\left(y\right)-\beta\rho\left(y_0^*\right)+\tilde{E}\left(1-\beta\right)e^{r_fT}\\&\leq\beta\rho\left(y_1^*\right)-\tilde{E}\left(1-\beta\right)e^{r_fT}+\rho\left(\varepsilon_y\right)-\beta\rho\left(y_0^*\right)+\tilde{E}\left(1-\beta\right)e^{r_fT}\\&=\beta\left(\rho\left(y_1^*\right)-\rho\left(y_0^*\right)\right)+\rho\left(\varepsilon_y\right),\end{aligned}$$

and therefore we have an upper bound for the idiosyncratic risk that depends on ε_y and the difference of risk between both benchmarks. The term $\beta\left(\rho\left(y_1^*\right)-\rho\left(y_0^*\right)\right)$ will vanish if and only if $y_1^*=y_0^*$. □

In the particular case of the Standard Deviation $\rho=\sigma_2$ (see (9)), if Y is generated by a static approach ($\mathcal{T}=\{0,T\}$, only one trading date), it is known that $y_0^*\in\mathcal{L}\left(1,z_\pi\right)$, which obviously implies that $y_1^*=y_0^*$, and both the benchmark y_0^* providing the efficient portfolios and the one y_1^* providing the $CAPM$-like formulas (35) and (36) coincide. Then, it may be interesting to characterize those properties leading to an identical situation if ρ is a more general risk measure or deviation and the pricing model may be dynamic.

Remark 12 *Consider the dual solution $\left(\lambda^*,z^*\right)$ that may obtained by solving the linear problem (20). $y_1^*=y_0^*$ holds if and only if there exist x_1, $x_2\in\mathbb{R}$ such that $y_0^*=x_1+x_2z_\pi$. Since $\left(y_0^*,\lambda^*,z^*\right)$ must satisfy (21) we have that*

$$\begin{cases}x_1+x_2\mathbb{E}\left(z_\pi^2\right)=e^{r_fT}\\x_1+x_2=R_0\end{cases}$$

must hold, which, taking into account Proposition 8 and $0 < \sigma_2^2(z_\pi) = \mathsf{E}(z_\pi^2) - \mathsf{E}(z_\pi)^2 = \mathsf{E}(z_\pi^2) - 1$*, shows that* $y_1^* = y_0^*$ *if and only if*

$$y_0^* = \frac{R_0\left(1+\sigma_2^2(z_\pi)\right) - e^{r_f T}}{\sigma_2^2(z_\pi)} - \frac{R_0 - e^{r_f T}}{\sigma_2^2(z_\pi)} z_\pi. \tag{40}$$

The fulfillment of (40) is easy to verify once y_0^* *has been computed, or by checking whether*

$$\left(\frac{R_0\left(1+\sigma_2^2(z_\pi)\right) - e^{r_f T}}{\sigma_2^2(z_\pi)} - \frac{R_0 - e^{r_f T}}{\sigma_2^2(z_\pi)} z_\pi, \lambda^*, z^*\right)$$

satisfies (21). □

Despite the latter remark characterizes the fulfillment of $y_1^* = y_0^*$, one can also give another conditions that only require to solve the linear problem (20).

Theorem 11 *Consider the dual solution* (λ^*, z^*)*.* $y_1^* = y_0^*$ *holds if and only if* $\rho(-z_\pi) = \mathbb{E}(z_\pi z^*)$*.*

Proof. The latter remark states that $y_1^* = y_0^*$ holds if and only if (40) holds. (21) implies that it is equivalent to the inequality

$$\mathbb{E}\left(\left(\frac{R_0\left(1+\sigma_2^2(z_\pi)\right) - e^{r_f T}}{\sigma_2^2(z_\pi)} - \frac{R_0 - e^{r_f T}}{\sigma_2^2(z_\pi)} z_\pi\right)\varphi_Y(z^*)\right) \leq$$

$$\mathbb{E}\left(\left(\frac{R_0\left(1+\sigma_2^2(z_\pi)\right) - e^{r_f T}}{\sigma_2^2(z_\pi)} - \frac{R_0 - e^{r_f T}}{\sigma_2^2(z_\pi)} z_\pi\right)\varphi_Y(z)\right), \qquad \forall z \in \Delta_\rho$$

Manipulating, and taking into account Proposition 1, the previous inequality is equivalent to

$$\mathbb{E}(z_\pi \varphi_Y(z^*)) \geq \mathbb{E}(z_\pi \varphi_Y(z)), \quad \forall z \in \Delta_\rho.$$

Since $z_\pi \in Y$ and $z - \varphi_Y(z) \in Y^T$ we have $\mathbb{E}(z_\pi \varphi_Y(z)) = \mathbb{E}(z_\pi z)$, $\forall z \in \Delta_\rho$, and the inequality is equivalent to

$$\mathbb{E}(z_\pi z^*) \geq \mathbb{E}(z_\pi z), \quad \forall z \in \Delta_\rho,$$

and result trivially follows from (5). □

Consequently, for the Standard Deviation, which is strongly compatible with every pricing model due to Remark 7, the equality $y_1^* = y_0^*$ also holds for dynamic approaches.

Corollary 12 *If* $\rho = \sigma_2$ *then* $y_1^* = y_0^*$.

Proof. Remark 7 shows that there are (14)-feasible solutions (λ, z) with $\lambda > 0$. Hence, if (λ^*, z^*) solves (14) $\lambda^* > 0$. Suppose that we prove that

$$(\lambda^*, z^*) = \left(\frac{1}{\sigma_2 (z_\pi)}, \frac{1}{\sigma_2 (z_\pi)} z_\pi - \frac{1}{\sigma_2 (z_\pi)} \right) \tag{41}$$

solves (14). Then

$$\mathbb{E} (z_\pi z^*) = \mathbb{E} \left(\frac{1}{\sigma_2 (z_\pi)} z_\pi^2 - \frac{1}{\sigma_2 (z_\pi)} z_\pi \right)$$

$$= \frac{\mathbb{E} (z_\pi^2) - 1}{\sigma_2 (z_\pi)} = \frac{\mathbb{E} (z_\pi^2) - \mathbb{E} (z_\pi)^2}{\sigma_2 (z_\pi)}$$

$$= \frac{\sigma_2^2 (z_\pi)^2}{\sigma_2 (z_\pi)} = \sigma_2 (z_\pi) = \sigma_2 (-z_\pi),$$

and the latter theorem applies. Let us now see (41). Since $z^* - \varphi_Y (z^*)$ and $\varphi_Y (z^*) = \lambda^* z_\pi - \lambda^*$ are orthogonal the Pythagorean Theorem of Hilbert Spaces (Maurin, 1967) and (6) lead to

$$\sigma_2^2 (z^*) = \|z^*\|_2^2 = \|z^* - \varphi_Y (z^*)\|_2^2 + \|\varphi_Y (z^*)\|_2^2 \geq \|\lambda^* z_\pi - \lambda^*\|_2^2 = \sigma_2^2 (\lambda^* z_\pi - \lambda^*).$$

Moreover, since $1 \geq \sigma_2^2 (z^*)$ due to (19), $\|z^* - \varphi_Y (z^*)\|_2^2 > 0$ would lead to $1 > (\lambda^*)^2 \sigma_2^2 (z_\pi - 1)$. Then for $\alpha > 1$ and small enough $\alpha \lambda^* (z_\pi - 1)$ would have zero expectation and a variance lower than one, *i.e.*, $\alpha \lambda^* (z_\pi - 1)$ would be (14)-feasible due to (19). Since $\alpha \lambda^* > \lambda^*$ because $\alpha > 1$ and $\lambda^* > 0$, we have a contradiction because (λ^*, z^*) cannot solve (14).

Consequently, $z^* - \varphi_Y (z^*) = 0$, and $z^* = \varphi_Y (z^*) = \lambda^* z_\pi - \lambda^*$. As above, $1 > (\lambda^*)^2 \sigma_2^2 (z_\pi - 1)$ cannot hold, so

$$(\lambda^*)^2 \sigma_2^2 (z_\pi - 1) = 1,$$

which ends the proof. □

5 Conclusions

This paper has dealt with the general portfolio choice problem and the classical *APT* and *CAPM* models when risk levels are given by risk measures beyond the variance. Expectation bounded risk measures, coherent risk measures and general deviation measures are included in the analysis. It seems to be an interesting topic since the variance has presented some drawbacks. For instance, it is not always compatible with the Second Order Stochastic Dominance. With respect to the reachable pay-offs, we have focused on the pricing rule and the SDF rather than the distribution of the returns of the available securities. Consequently, this analysis may apply for both static and dynamic pricing models.

First of all general optimality conditions have been given, despite the level of generality for both risks and prices. Secondly, new notions such as strong compatibility between prices and risks have been introduced. Surprisingly, the lack of (strong) compatibility leads to unbounded portfolio choice problems, despite it is complex to reach economic interpretations of that. Nevertheless, the lack of (strong) compatibility holds for very important risk measures (VaR, $CVaR$, $WCVaR$, $CCVaR$, DPT, Wang, absolute deviation, absolute downside semi-deviation, etc.) and pricing models (Black Scholes, Heston, other complete derivative-linked pricing models, etc.). Thirdly, models with a market price of risk have also been characterized and analyzed, and they also may involve dynamic pricing models. A CML have been generated and two major benchmarks have been introduced. Finally, APT and $CAPM$ like developments have been presented, and they do not modify the classical definitions of the betas. On the contrary, systematic risks will depend on the correlations with the factors/market, whereas idiosyncratic risks and noises will have null correlation with the factors/market.

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