Structural Controllability of Multi-Agent Systems Subject to Partial Failure

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A Thesis

in

The Department

of

Electrical and Computer Engineering

Presented in Partial Fulfillment of the Requirements

for the Degree of Master of Applied Science at

Concordia University

Montréal, Québec, Canada

July 2010

CONCORDIA UNIVERSITY

School of Graduate Studies

This is to certify that the thesis proposal prepared

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Entitled: **Structural Controllability of Multi-Agent Systems Subject to Partial Failure**

and submitted in partial fulfilment of the requirements for the degree of

Master of Applied Science

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

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Abstract

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Formation control of multi-agent systems has emerged as a topic of major interest during the last decade, and has been studied from various perspectives using different approaches. This work considers the structural controllability of multi-agent systems with leader-follower architecture. To this end, graphical conditions are first obtained based on the information flow graph of the system. Then, the notions of p-link, q-agent, and joint-(p, q) controllability are introduced as quantitative measures for the controllability of the system subject to failure in communication links or/and agents. Necessary and sufficient conditions for the system to remain structurally controllable in the case of the failure of some of the communication links or/and loss of some agents are derived in terms of the topology of the information flow graph. Moreover, a polynomial-time algorithm for determining the maximum number of failed communication links under which the system remains structurally controllable is presented. The proposed algorithm is analogously extended to the case of loss of agents, using the node-duplication technique.

The above results are subsequently extended to the multiple-leader case, i.e., when more than one agent can act as the leader. Then, leader localization problem is investigated, where it is desired to achieve p-link or q-agent controllability in a multi-agent system. This problem is concerned with finding a minimal set of agents whose selection as leaders results in a p-link or q-agent controllable system. Polynomial-time algorithms to find such minimal sets for both undirected and directed information flow graphs are presented.

To my parents,

for their tireless support throughout my life

Acknowledgements

I would like to express my gratitude to Dr. Amir Aghdam, my supervisor, for his encouragement, guidance, and support during my Master study.
I would also like to thank Amir Ajorlou, PhD candidate, who helped me all the time. His novel ideas and suggestions were essential for the results of this thesis; and it was a great honor for me to work with him.

Finally, I dedicate this thesis to my family for their support that has enabled me to complete my Master degree.

This research was supported in part by the Natural Sciences and Engineering Council of Canada (NSERC) under Grant STPGP-364892-08, and in part by Motion Metrics International Corp. They are really appreciated.

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Chapter 1

Introduction

1.1 Motivation and Related Work

Collective behavior in large groups of entities is common in the real world and can be viewed as a special behavior of large number of agents interacting together. Such behavior has certain advantages; for example, collective motions of ant colonies, bird flocks, and fish schools can increase chance of finding food and avoiding predators and other risks [1]. Such natural behavior has inspired study of multi-agent systems [2–6]. A multi-agent system is a collection of dynamic units that interact over an information exchange network for its operation. The applications of such systems have been expanding to the areas such as rescue missions, firespotting, ocean exploration, space science missions, terrain mapping, surveillance, and even military applications with various types of agents [7–13]. Due to vast applications of multi-agent systems, their control and coordination has emerged as a topic of major interest during the last decades [14–18]. Examples include cooperative control of unmanned aerial/ground/underwater vehicles, scheduling of automated highway systems, formation control of satellite clusters, and congestion control in communication networks.

The use of multiple agents has more potential advantages than a single-agent application including robustness, flexibility, and adaptability to unknown dynamic environments [19]. Spacecraft formation flying is an example in which a mission is performed by a virtual spacecraft consisting of a group of simple, low-cost, highly coordinated small satellites. A wide range of scientific, military, and commercial space applications can potentially benefit from such a technology to perform distributed observations for surveillance, magnetosphere sensing, interferometry, and a variety of other missions. This approach provides significant advantages such as (*i*) autonomous operations with minimal ground involvement; (*ii*) increased coverage due to the separation between the agents, and (*iii*) creating a more flexible structure by regarding the group of agents as one large complex agent, with no single point of failure (as in the case of a single agent) [20–22].

Along with these advantages, there exist a number of challenges in designing control algorithms because the individual agents have limited computational, communications, sensing, and mobility resources. In particular, due to the importance of information sharing in the coordination of a multi-agent system, the information flow structure between the agents needs to be taken into account in control design [23]. In [24–27], a number of approaches are proposed to address the above problem. The role of the interconnection topology in the information flow and formation stability is studied in a number of papers (e.g. see [28, 29]). In [30], leader-follower local control laws with vision-based feedback are employed to stabilize a formation of multi-agent systems. On the other hand, behavioral schemes are investigated in [31, 32] to stabilize formations of vehicles.

Motivated by recent technological advances in the areas of wireless communications, embedded computation, and micro fabrication, several results are reported in the literature on the formation control of multiple autonomous mobile agents [33–38]. Graph-theoretic approaches have recently been recognized as effective tools for the analysis of multi-agent systems in the control literature. Such mathematical tools are employed for the analysis of a number of relevant problems such as consensus, rendezvous, flocking, leader-follower formation control, and containment [39–45]. In the consensus problem, all agent should converge to the same point in the state space. In the rendezvous problem which is an instantiation of the consensus problem, a collection of agents are supposed to meet at an unspecified common location. Flocking problem, on the other hand, is concerned with the convergence of the velocity vectors and orientations of the agents to a common value at steady state. In the leader-follower formation control, a subset of agents act as leaders and influence the behavior of the other agents (referred to as the followers) to achieve prescribed objectives such as formation reconfiguration. In the containment problem, the leaders move in such a way that the followers remain in the convex leader-polytope at all times.

Another issue concerning the multi-agent systems coordination is their controllability. The controllability problem in the leader-follower multi-agent systems was first introduced in [46], where the classical notion of controllability for a leader-based multi-agent system was studied. In this problem, one or more agents acting as the leaders are influenced by an external control input while the other agents are governed by a decentralized averaging rule and produce their control input based on the information they receive from their neighbors. It is aimed to steer the interconnected system to specific positions by regulating the motion of the leader. Since the dynamics of the system relies on its interconnection topology, it is clear that the controllability should depend on the network topology. In [46], necessary and sufficient conditions are provided for the controllability of a system in terms of the eigenvalues and eigenvectors of a sub-matrix of the Laplacian of the information flow graph. It is also substantiated in [46] that increasing the size of the information flow graph would not necessarily improve the controllability of the system. Subsequently, in [47, 48], the controllability of the leader-follower multi-agent systems is characterized by graph theory. It is shown in [47] that a leader-symmetric interconnection network is uncontrollable. The above paper also investigates the dependency of the rate of convergence of the system on the position of the leader. *Network equitable partitions* are introduced in [49] to present a new necessary condition for the controllability of a multi-agent system. Using this notion, the controllability characterizations are extended to a multiple-leader setting in [50]. More recently, the notion of *relaxed equitable partitions* is introduced in [51] to provide a graph-theoretic interpretation for the controllability subspace when the network is not completely controllable. The controllability of a single-leader multi-agent system under fixed and switching topologies for both continuous-time and discrete-time cases is studied in [52, 53], where it is shown that the controllability of the overall system does not require that the network be controllable for a fixed topology; in other words, even if the network interconnection switches between a number of uncontrollable networks with fixed topologies, the overall system can be controllable.

While the aforementioned results provide efficient techniques to check the controllability of multi-agent systems, they are mainly concerned with undirected interconnection graphs. Furthermore, they assume that the followers obey a consensus-like control strategy, and then investigate the controllability of the system for the given control law. For example, such a control law is considered in [54]; however, the corresponding information flow graph is assumed to be weighted. It is shown that these weights can be adjusted properly to obtain a controllable system if and only if the underlying graph is connected.

1.2 Thesis Contributions

In this thesis, a novel approach to the controllability verification of multi-agent systems is introduced. The proposed method is concerned with the structural controllability, as opposed to the controllability of a fixed topology. To this end, a quantitative measure is provided for the controllability of any given directed information flow graph. It is primarily aimed to obtain graphical conditions for the structural controllability of a leader-follower multi-agent system with a single leader. The structural controllability is then quantified for both cases of communication links failure and loss of agents. Necessary and sufficient conditions are derived for controllability preservation under such failures. These results are extended to the case of multiple leaders, and then the problem of finding a minimal set of agents whose selection as leaders results in a structurally controllable system is investigated.

1.3 Thesis Outline and Publications

The rest of the thesis is organized as follows. Chapter 2 provides the basic theoretical background, where some preliminaries from graph theory are presented, and the concept of structural controllability is introduced. In Chapter 3, graphical conditions for the controllability of a single leader multi-agent system based on the topology of its information flow graph is derived. Moreover, conditions for controllability preservation in the case of failure of some communication links or loss of some agents are provided. The results are then extended to a multiple-leader case. Chapter 4 considers the leader localization problem on both undirected and directed information flow graphs. Finally, conclusions and future research directions are presented in Chapter 5.

The results of this research are are published in/submitted to the following conferences and journals:

[**1**] **S. Jafari**, A. Ajorlou, A. G. Aghdam, and S. Tafazoli, "On the Structural Controllability of Multi-Agent Systems Subject to Failure: A Graph-Theoretic Approach," in *Proceedings of the 49*th *IEEE Conference on Decision and Control*, Atlanta, USA, 2010 (to appear).

[**2**] **S. Jafari**, A. Ajorlou, and A. G. Aghdam, "Leader Localization in Multi-Agent Systems Subject to Failure: A Graph-Theoretic Approach," submitted to a journal.

[**3**] **S. Jafari**, A. Ajorlou, A. G. Aghdam, and S. Tafazoli, "Structural Controllability of Multi-Agent Systems Subject to Partial Failure," submitted to a journal.

[**4**] **S. Jafari**, A. Ajorlou, and A. G. Aghdam, "Leader Selection in Multi-Agent Systems Subject to Partial Failure," to be submitted to a conference.

Some other results obtained during the author's master's program are published in:

[**5**] **S. Jafari**, A. Ajorlou, A. G. Aghdam, and S. Tafazoli, "Distributed Control of Formation Flying Spacecraft using Deterministic Communication Schedulers," in *Proceedings of the 49*th *IEEE Conference on Decision and Control*, Atlanta, USA, 2010 (to appear).

Chapter 2

Background

This chapter provides a background to the problem under study in this dissertation. First, basic concepts of graph theory are introduced, and then the notions of structured systems and structural controllability are studied. Throughout this thesis, the set of integers $\{1, 2, \ldots, k\}$ is denoted by $\mathbb{N}_k$. The difference of the set X and the Y which is the set containing all elements of X that do not belong to Y is denoted by $X\backslash Y$. The size of a set X is the number of its elements, and is represented by $|X|$. The ith member of an ordered set X is denoted by $X(i)$. Two sets X and Y are *intersecting* if the sets $X\backslash Y$, $Y\backslash X$, and $X \cap Y$ are all nonempty.

It is to be noted that most of the material presented in Section 2.1 are drawn from [55–57].

2.1 A Brief Introduction to Graph Theory

Graphs are so named because they can be represented graphically, and this graphical representation helps us understand many of their properties. An *undirected graph* $\mathcal{G}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a non-empty finite set of elements called *vertices*, and $\mathcal{E}$ is a set of unordered pairs of elements of $\mathcal{V}$ called *edges*. If $e_{ij} = \{i, j\} \in \mathcal{E}$ is an edge with end vertices i and j, then e_{ij} is said to join i and j. The size of the vertex set and the edge set of a graph are called the *order* and *size* of the graph, respectively. A graph can be represented pictorially as in Fig. 2.1(a) and (b). The black circles or nodes represent the vertices, and each edge $\{i, j\}$ is represented by a line joining the nodes corresponding to the vertices i and j. For example, Fig. 2.1(a) represents a graph of order four and size six with the vertex set $\mathcal{V} = \{1, 2, 3, 4\}$ and edge set $\mathcal{E} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.

The ends of an edge are said to be *incident* with the edge, and vice versa. Two

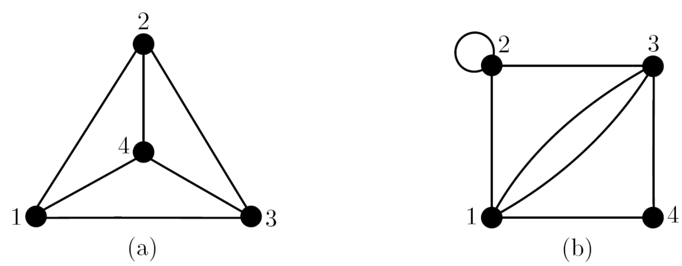


Figure 2.1: Examples of: (a) an undirected graph, and (b) an undirected graph with a self-loop and a pair of parallel edges.

vertices (edges) are *adjacent* if they are incident with a common edge (vertex). Two distinct adjacent vertices are called *neighbors*; clearly, in an undirected graph if vertex i is a neighbor of vertex j, then j is a neighbor of i as well. N_i denotes the set of all neighbors of a vertex i, i.e. $N_i := \{j \mid \{i, j\} \in \mathcal{E}\}$. Edges incident with only one vertex (i.e., those with identical ends) are called *self-loops*, while distinct edges incident with the same vertices are called *parallel edges*. For example, the graph in Fig. 2.1(b) has a self-loop on vertex 2, and a pair of parallel edges between vertices 1 and 3. A graph is *simple* if it has no self-loops and no parallel edges. A *path* from vertex i_1 to vertex i_k, denoted by $P_{i_1 i_k} = (i_1, i_2, \ldots, i_k)$, is a sequence of distinct vertices starting with i_1 and ending with i_k such that any pair of vertices in $\{i_1, i_2, \ldots, i_k\}$ are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Similarly, a *cycle* is a sequence of vertices $C = (i_1, i_2, \ldots, i_k, i_1)$ starting and ending with the same vertex such that any pair of vertices in $\{i_1, i_2, \ldots, i_k\}$ are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Two paths (cycles) are called *disjoint* if they consist of disjoint sets of vertices. Each vertex on the path $P_{i_1 i_k}$ (including the vertex i_k itself) is called an *ancestor* of i_k, and each vertex of which i_k is an ancestor is a *descendant* of i_k. An ancestor or descendant of a vertex is *proper* if it is not the vertex itself. The immediate proper ancestor of a vertex i_j on $P_{i_1 i_k}$ ($j \neq 1$) is its *predecessor* or *parent*, denoted by $\zeta(i_j)$, and the vertices whose predecessor is i_j are its *successors* or *children*.

The *length* of a path or a cycle is the number of its edges. A self-loop is a cycle of length one, and a pair of parallel edges constructs a cycle of length two. A *chord* of a path (cycle) is an edge between two vertices of the path (cycle) that is not an edge of the path (cycle). A path (cycle) is *chordless* if it contains no chords. A graph $\mathcal{G}$ is *chordal* if each cycle in $\mathcal{G}$ of length greater than three has at least one chord; otherwise it is called *unchordal*. Chordal graphs are also called *triangulated* and *perfect elimination* graphs. Fig. 2.2 depicts a chordal and an unchordal graph. A *clique* of a graph $\mathcal{G}$ is a set of mutually adjacent vertices. In other words, $\mathcal{V}' \subseteq \mathcal{V}$

Figure 2.2: (a) A chordal graph, and (b) an unchordal graph.

is a clique in $\mathcal{G}$ if for all distinct vertices $i, j \in \mathcal{V}'$, $\{i, j\} \in \mathcal{E}$. A *maximal clique* is a clique that is not a subset of any other clique. A *clique cover* is a family of cliques that includes every vertex of the graph. Finding the minimum number of cliques which cover all vertices of a graph is known as *minimum clique cover problem.* A set of vertices or edges is called *independent* (or *stable*) if there is no pair of adjacent in it. A *complete graph* is a simple graph in which any two vertices are neighbors; a complete graph of order n is denoted by $\mathcal{K}_n$. Two complete graphs are shown in Fig. 2.3.

A graph is *connected* if there exists a path between each pair of vertices; otherwise

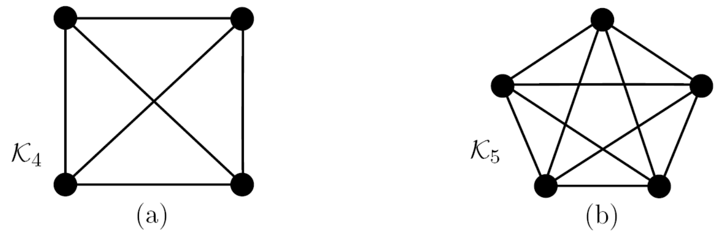


Figure 2.3: (a) A complete graph of order four, and (b) a complete graph of order five.

the graph is *disconnected*. A *disconnecting set* in a connected graph $\mathcal{G}$ is a set of edges whose removal disconnects $\mathcal{G}$. A *cutset* of $\mathcal{G}$ is a disconnecting set, such that none of its proper subsets is a disconnecting set. The *edge-connectivity* of $\mathcal{G}$ is the size of the smallest cutset in $\mathcal{G}$. A *separating set* in $\mathcal{G}$ is a set of vertices whose deletion disconnects $\mathcal{G}$. The *vertex-connectivity* of $\mathcal{G}$ is the size of the smallest separating set in $\mathcal{G}$.

Let X be a set of vertices, the *edge cut* of $\mathcal{G}$ associated with X is the set of edges of G with one end in X and the other end in $\mathcal{V}\backslash X$, and is denoted by $\partial_{\mathcal{G}}(X)$. Fig. 2.4 shows a simple graph in which the edge cut associated with vertex set $X = \{1, 2, 6\}$ is $\partial_{\mathcal{G}}(X) = \{\{2, 3\}, \{5, 6\}\}$.

The *degree* (or *valency*) of a vertex x of $\mathcal{G}$ is the number of edges incident with x,

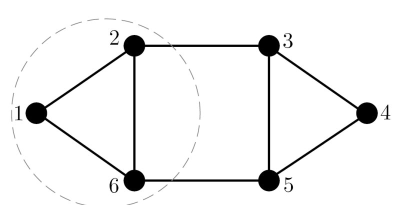


Figure 2.4: The edge cut associated with a vertex set in a simple graph.

and is denoted by $d_{\mathcal{G}}(\{x\})$. In general, the degree of a set $X \subset \mathcal{V}$ is the number of edges of $\mathcal{G}$ with one end in X and the other end in $\mathcal{V}\backslash X$, that is $d_{\mathcal{G}}(X) = |\partial_{\mathcal{G}}(X)|$.

Although it is convenient to represent any graph by a diagram consisting of a set of points (vertices) joined by lines (edges), such a representation may be unsuitable when it comes to storing information of a large graph in a computer. A useful representation of graphs involves matrices. Let $\mathcal{G}$ be a graph of order n and of size m, with the vertex set $\mathcal{V} = \{1, 2, \ldots, n\}$. The *adjacency matrix* $\mathcal{A}(\mathcal{G})$ of this graph is an $n \times n$ matrix whose (i, j) entry is the number of edges joining vertices i and j. An adjacency matrix of a simple graph has entries 0 or 1, where the (i, j) entry $\mathcal{A}(\mathcal{G})$ is 1 if $\{i, j\} \in \mathcal{E}$, and is 0 otherwise; diagonal elements are also zero. The adjacency matrix of an undirected graph is symmetric, and also diagonalizable over $\mathbb{R}$; that is, there is a basis of $\mathbb{R}^n$ consisting of the eigenvectors of $\mathcal{A}(\mathcal{G})$. The *incidence matrix* of $\mathcal{G}$, denoted by $\mathcal{I}(\mathcal{G})$, is an $n \times m$ matrix whose (i, j) entry is 1 if vertex i is incident to edge j, and is 0 otherwise. The *degree matrix* of $\mathcal{G}$, denoted by $\Delta(\mathcal{G})$, is an $n \times n$ diagonal matrix whose (i, i) entry is the degree of vertex i, and the other entries are all zero. The *Laplacian matrix* $\mathcal{L}(\mathcal{G})$ is defined as $\mathcal{L}(\mathcal{G}) = \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G})$. In an undirected graph, the Laplacian matrix is always symmetric and positive semi-definite. It can be shown that $\mathcal{L}(\mathcal{G}) = \mathcal{I}(\mathcal{G})\mathcal{I}(\mathcal{G})^T$. The algebraic multiplicity of the zero eigenvalue of $\mathcal{L}(\mathcal{G})$ is equal to the number of connected components in the graph. In a connected graph of order n, the n-dimensional eigenvector associated with the single zero eigenvalue of $\mathcal{L}(\mathcal{G})$ is the vector of ones. Its second smallest eigenvalue is positive and is referred to as the *algebraic connectivity* of the graph, because it is directly related to how vertices are interconnected.

A *directed graph* or *digraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is characterized by a set of vertices $\mathcal{V}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $\mathcal{G}$ is denoted by $e_{ij} := (i, j) \in \mathcal{E}$, which is a directed arc from vertex i to vertex j. In such an ordered pair, the first vertex i is called *tail* and the second one j is called *head*. In a digraph, a *self-loop* $e_{ii} = (i, i)$ is an edge connecting vertex i to itself. Two edges are *anti-parallel* if one's head/tail is the other's tail/head. Fig. 2.5 shows a digraph containing a pair of anti-parallel edges between vertices 2 and vertex 6, and a pair of parallel edges between vertices

3 and 5.

The set of all neighbors of vertex i is defined as $N_i := \{j \mid e_{ji} \in \mathcal{E}\}$. A sequence

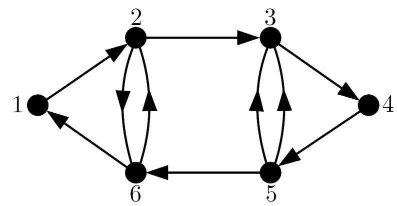


Figure 2.5: Parallel and anti-parallel edges in a digraph.

of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$ is referred to as a *directed* $i_1 i_k$*-path* ($i_j \in \mathcal{V}$, $j \in \mathbb{N}_k$). The *parent function* of this path is defined as $\zeta(i_j) = i_{j-1}$, for $j \in \mathbb{N}_k \backslash \mathbb{N}_1$. The vertex i_1 is called the *origin* or *root* of the path, and the vertex i_k is called the *end* of the path. A vertex i is called *reachable* from vertex j if there exists a path whose origin and end are j and i, respectively. An R*-rooted path* is a path whose origin is in the set $R \subset \mathcal{V}$; the set R associated with such a path is called the *root set*. A group of mutually disjoint R-rooted paths is called an R*-rooted path family*. Fig. 2.6 shows three pairwise disjoint R-rooted paths which construct an R*-rooted path family*. A closed path consisting of distinct vertices is called a *cycle* and a set

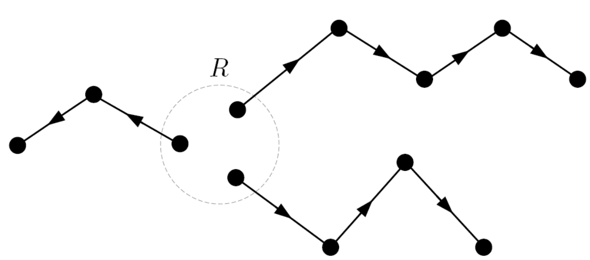


Figure 2.6: Three mutually disjoint R-rooted paths.

of disjoint cycles is called a *cycle family*. The set of all edges of $\mathcal{G}$ entering $X \subseteq \mathcal{V}$ is denoted by $\partial^-_{\mathcal{G}}(X)$ and is called the *incut* of X, and the set of all edges of G leaving X is denoted by $\partial^+_{\mathcal{G}}(X)$ and is referred to as the *outcut* of X. In other words, incut (outcut) of X is the set of edges of $\mathcal{G}$ whose heads (tails) lie in X and whose tails (heads) lie in $\mathcal{V} \backslash X$.

In the digraph shown in Fig. 2.5, the incut and outcut of vertex set $X = \{3, 4, 5, 6\}$ are $\partial^-_{\mathcal{G}}(X) = \{e_{23}, e_{26}\}$ and $\partial^+_{\mathcal{G}}(X) = \{e_{62}, e_{61}\}$, respectively. The size of the incut and outcut associated with X are called the *indegree* and *outdegree* of X, respectively, and are denoted by $d^-_{\mathcal{G}}(X)$ and $d^+_{\mathcal{G}}(X)$. For two disjoint sets $X, Y \subset \mathcal{V}$, let $\partial^-_{\mathcal{G}_Y}(X) \subseteq \partial^-_{\mathcal{G}}(X)$ be the set of all edges of $\mathcal{G}$ whose tails lie in Y and whose heads lie in X; also, let the size of this set be denoted by $d^-_{\mathcal{G}_Y}(X)$.

A digraph is *weakly connected* if there is an undirected path between any pair of vertices, that is, the corresponding undirected graph is connected. A digraph is *strongly connected* if there is a directed path between every pair of vertices.

2.2 Controllability of Structured Systems

Controllability is an important concept in the analysis and design of control system. Consider an LTI system described by the following standard state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and input of the system, respectively, and A and B are real valued matrices of appropriate dimensions. The LTI system (2.1) is said to be *controllable* if there exists an input $u(t)$ which will drive the system from any initial state $x(0) = x_0$ to any final state $x(t_f) = x_f$ in a finite time interval $[0 \;\; t_f]$ [58]. It is to be noted that in this definition no constraint is imposed on the input.

In a controllable system of the form (2.1), the matrix $W_c(t) = \int_0^t e^{A\tau} BB^T e^{A^T\tau} d\tau$ is nonsingular for any $t > 0$. It can be shown that the input

$$u(t) = B^T e^{A^T(t_f - t)} W_c^{-1}(t_f)(x_f - e^{At_f} x_0) \tag{2.2}$$

will transfer any initial state $x(0) = x_0$ to any final state $x(t_f) = x_f$ [58] (note that such an input is not unique). It can be shown that the input $u(t)$ given by (2.2) minimizes the cost function $J = \int_0^{t_f} ||u(\tau)||^2 d\tau$, where $||.||$ denotes the Euclidean norm.

In many real-world problem, in modeling problems the matrices A and B have a number of fixed zero entries determined by the physical structure of the system, and the remaining entries are not known exactly. In other words, the parameters of a system are usually not known precisely, with the exception of zeros that are fixed (e.g., due to the absence of physical connections between certain parts of a system) [59].

Typical analysis and design techniques for linear systems are often based on the premise that full knowledge of the system parameters is available. Although a number of methods have been developed in the past three decades to deal with uncertainty in the analysis and design of control systems, such techniques are known to have important drawbacks in practice. For example, they are complex in general, and can lead to very conservative results (e.g. sufficient conditions for robust stability) [60, 61]. Moreover, robustness analysis techniques rely on the numerical values of the physical model, and are usually effective for a certain range of parameter variation only. As an alternative to conventional methods discussed above, one can develop an analysis methodology based on the structure of the system, as opposed to its exact numerical values.

The structured system approach was first introduced in [62], and was further developed later in [63–65]. Several control problems have been studied since then in the framework of structured systems. Input-output decoupling of structured systems is studied in [66,67]. In [68,69], disturbance rejection for structured systems is investigated using state feedback. The problem of fault diagnosis for structured systems is considered in [70]. Decentralized control of structured systems, on the other hand, is investigated in [71].

A matrix is called *structured* if its entries are either fixed zeros or independent free parameters. Two matrices are said to be *structurally equivalent* if there is a one-to-one correspondence between their zeros. A system of the form (2.1) is called *structured* if both A and B are structured matrices, and also the union of the nonzero entries of A and B is algebraically independent. In the remainder of this chapter, let an LTI system of the form (2.1) be represented by the pair (A, B).

Definition 2.1. [59] *A system (A, B) is structurally controllable if there exists a system structurally equivalent to (A, B) which is controllable in the usual sense.*

Proposition 2.1. [62] *A system (A, B) is structurally controllable if and only if for any $\varepsilon > 0$, there exists a controllable system (A_0, B_0) of the same structure as (A, B) such that $||A - A_0|| < \varepsilon$ and $||B - B_0|| < \varepsilon$ (where $||.||$ denotes any arbitrary norm).*

Definition 2.2. [65] *The structured matrix $[A \mid B] \in \mathbb{R}^{n\times(n+m)}$ is said to be irreducible if there exists no permutation matrix P such that*

$$PAP^{-1} = \begin{bmatrix} A_{11} & \mathbf{0} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } PB = \begin{bmatrix} \mathbf{0} \\ B_{21} \end{bmatrix}$$

where $A_{11} \in \mathbb{R}^{k\times k}$, $A_{21} \in \mathbb{R}^{(n-k)\times k}$, $A_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$, and $B_{21} \in \mathbb{R}^{(n-k)\times m}$ are structured matrices, with $k \in \mathbb{N}_n$, and $\mathbf{0}$ is the zero matrix of appropriate dimension.

Definition 2.3. [63] *The generic rank of a structured matrix is the maximum rank of its structurally equivalent matrices. Therefore, a structured matrix has full generic rank if and only if there exists a matrix of full rank obtained by fixing the free parameters at some particular values.*

Remark 2.1. *Neither stability nor instability is a structural property, so these properties cannot be handled with structured systems. As a simple example consider the first order system $\dot{x}(t) = \alpha x(t)$, in which for any value of α, it can be stable or unstable.*

Structured systems can be represented by digraphs. This enables one to study such systems from a graph theoretic prospective. An m-input n-dimensional linear structured system defined by the pair (A, B) can be represented by a digraph of order $n+m$ denoted by $\mathcal{G}(A, B) = (\mathcal{V}, \mathcal{E})$. The vertex set of $\mathcal{G}(A, B)$ is given by $\mathcal{V} = X \cup U$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $U = \{u_1, u_2, \ldots, u_m\}$ are the sets of state and input vertices, respectively. The (i, j) entry of the matrix $[A \mid B]$ corresponding to a nonzero parameter is associated with a directed edge from vertex j to vertex i. Denote these edges by $\mathcal{E}$, i.e. $\mathcal{E} = \{(j, i) \mid \text{The } (i, j) \text{ entry of } [A \mid B] \text{ is a free parameter}\}$. As an example, consider the following structured system

$$\dot{x}(t) = \begin{bmatrix} \times & 0 & 0 & 0 \\ \times & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \times & \times \end{bmatrix} x(t) + \begin{bmatrix} \times & 0 \\ 0 & \times \\ \times & 0 \\ 0 & \times \end{bmatrix} u(t) \tag{2.3}$$

where $\times$ denotes parameters of the system that can be chosen freely. The digraph $\mathcal{G}(A, B)$ of the system is of order six with the vertex set $\mathcal{V} = \{x_1, x_2, x_3, x_4, u_1, u_2\}$. The resulting digraph is depicted in Fig. 2.7.

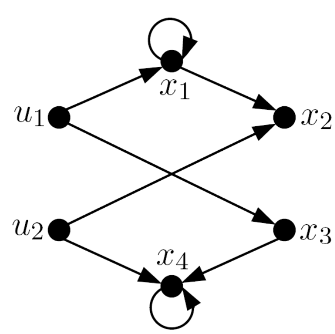


Figure 2.7: A digraph associated with system (2.3).

In $\mathcal{G}(A, B)$, a path whose origin is in U is called a *stem*. The terminal vertex of a stem is called *top* of the stem. Consider a cycle C and an edge $e = (i, j)$ such that vertex j is contained in C but vertex i is not; then, $C \cup \{e\}$ is called a *bud*, and e is called the *distinguished edge* of the bud. Consider a stem $\mathcal{S}$ and a sequence of buds $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_\ell$; the digraph $\mathcal{C}_\mathcal{G} = \mathcal{S} \cup \left(\cup_{j=1}^{\ell} \mathcal{B}_j\right)$ is called a *cactus* if for any $i \in \mathbb{N}_\ell$, the tail of the distinguished edge of $\mathcal{B}_i$ is not the top of $\mathcal{S}$, and is the only vertex belonging to $\mathcal{B}_i$ and $\mathcal{S} \cup \left(\cup_{j=1}^{i-1} \mathcal{B}_j\right)$. Fig. 2.8 gives an illustration for the case $\ell = 3$.

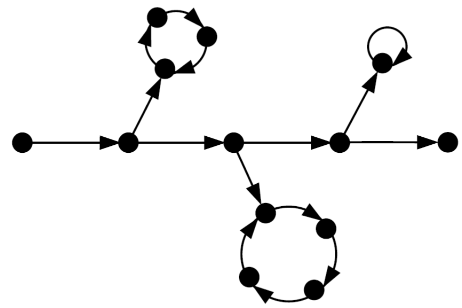


Figure 2.8: A cactus with three buds.

Theorem 2.1. [65, 72] *The following statements for a structured system (A, B) are equivalent.*

i. (A, B) *is structurally controllable.*

ii. $[A \mid B]$ *is irreducible and its generic rank is* n.

iii. In $\mathcal{G}(A, B)$, *there exists a disjoint union of cacti that covers all state vertices.*

iv. In $\mathcal{G}(A, B)$, *every state vertex is the end vertex of a U-rooted path, and there exists a disjoint union of a U-rooted path family and a cycle family that covers all state vertices.*

Using this theorem, one can determine if a linear time-invariant (LTI) system with a state-space representation of the standard form (2.1) is structurally controllable. It is to be noted that analogous results hold for structural observability [72].

Chapter 3

Structural Controllability of Multi-Agent Systems

In this chapter, a novel approach to the controllability verification of multi-agent systems is introduced. The proposed method is concerned with the structural controllability, as opposed to the controllability of the fixed system. To this end, a quantitative measure is provided for the controllability of any given directed information flow graph. It is primarily aimed to obtain graphical conditions for the structural controllability of a leader-follower multi-agent system with a single leader. The structural controllability is then quantified for both cases of communication links failure and loss of agents. Necessary and sufficient conditions are derived for preserving structural controllability under such failures. Polynomial-time algorithms are provided subsequently to find the maximum number of such failures for which the system remains structurally controllable. The case of a single-leader is considered first, and the results are then extended to the multiple-leader setting. We consider a directed information flow graph; clearly, the results can also be applied to undirected information flow graphs by replacing each undirected edge with a pair of anti-parallel edges. Throughout this thesis it is assumed, unless otherwise stated, that the information flow graph is a digraph with no parallel edges.

3.1 Problem Statement

Consider a team of single integrator agents given by

$$\dot{x}_i(t) = u_i(t), \qquad i \in \mathbb{N}_n \tag{3.1}$$

where $x_i(t)$ and $u_i(t)$ are the state and control input of agent i, respectively. Let the interaction between the agents be specified by a given information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. This graph, which is assumed to be static and directed, has n vertices (one for each agent) and an edge from vertex j to vertex i ($i, j \in \mathbb{N}_n, \ \ i \neq j$) if agent j is capable of transmitting its state to agent i. In a static information flow graph, the movements of the agents would not result in any new edge to be generated, or any existing edge to be eliminated. Fig. 3.1 shows an information flow graph for a group of five agents in which, for instance, agents 3 and agent 4 have access to state of agent 1, while agent 1 does not have any information about them.

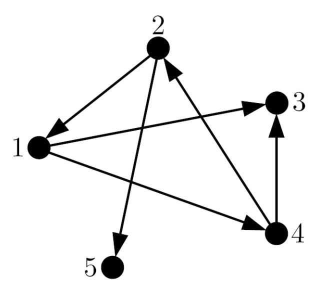


Figure 3.1: An information flow graph for a group of five agents.

Assume that one of the agents, say agent n, acts as the leader, and is influenced by an external control input, denoted by $u_n(t) = u_{\text{ext}}(t)$, enabling it to move without any constraint. Let the remaining four agents, called followers, obey a control law of the following form

$$u_i(t) = \sum_{j \in N_i \cup \{i\}} \alpha_{ij} x_j(t), \qquad i \in \mathbb{N}_{n-1} \tag{3.2}$$

where the coefficients $\alpha_{ij} \in \mathbb{R}$ are fixed. Therefore, the state of each agent is updated based on its interaction with the neighbors under the above rule. For example, in the information flow graph shown in Fig. 3.1 the control input of agent 3 is expressed as

$$u_3(t) = \alpha_{31}x_1(t) + \alpha_{33}x_3(t) + \alpha_{34}x_4(t).$$

The state of each agent is defined to be its absolute position with respect to an inertial reference frame, which can often be measured with sufficient accuracy by using GPS-based systems [73]. Throughout the paper, it is assumed that the agent dynamics is decoupled along each dimension, allowing for one-dimensional state representation for each agent.

Definition 3.1. *The information flow graph $\mathcal{G}$ is called controllable if there exist α_{ij}'s, such that by moving the leader properly, the followers can take any desired configuration.*

In the information flow graph $\mathcal{G}$ of a leader-follower multi-agent system with a vertex representing each agent, the vertex corresponding to the leading agent is called the *root*, and is denoted by r. It is assumed that $d^-_G(\{r\}) = 0$.

Under the control law (3.2), the dynamics of the followers can be described as

$$\dot{x}(t) = Ax(t) + bu(t) \tag{3.3}$$

where $x(t) = [x_1(t) \ ... \ x_{n-1}(t)]^T \in \mathbb{R}^{n-1}$, $u(t) = x_n(t) \in \mathbb{R}$, and $A = [a_{ij}]$, $b = [b_i]$ are structured matrices of proper dimensions. For any $i, j \in \mathbb{N}_{n-1}$, $a_{ij} = \alpha_{ij}$ if $j \in N_i \cup \{i\}$; otherwise, $a_{ij} = 0$. Also, for any $i \in \mathbb{N}_{n-1}$, $b_i = \alpha_{in}$ if $n \in N_i$; otherwise $b_i = 0$. Therefore, (3.3) describes a structured system whose controllability under the control input $u(t)$ is equivalent to the controllability of the underlying information flow graph.

It is desired to graphically interpret the necessary and sufficient conditions for controllability of a digraph and to find conditions under which it remains controllable after the elimination of a number of edges or vertices. It is also aimed to quantitatively measure the controllability of an information flow graph subject to failure in the agents or communication links.

3.2 Controllability of an Information Flow Graph

Consider a structured system described by the following state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.4}$$

where $x(t) \in \mathbb{R}^n$ denotes the state, $u(t) \in \mathbb{R}^m$ is the input, and A, B are structured matrices of appropriate dimensions. Let $\mathcal{G}(A, B)$ be the digraph representing the system (3.4). The following theorem gives the necessary and sufficient conditions on the digraph $\mathcal{G}(A, B)$ for the structural controllability of the system (3.4).

Theorem 3.1. [72] *System (3.4) with the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is structurally controllable if and only if both of the following conditions hold: i) Every vertex in $\mathcal{G}$ is the end vertex of an R-rooted path; ii) There exists a disjoint union of an R-rooted path family and a cycle family that covers all vertices, where $R \subset \mathcal{V}$ is the set of vertices corresponding to the columns of the matrix B.*

One can use the above theorem to find graphical conditions for the controllability of the information flow graph (see Definition 3.1). Since each of the followers has access to its own local state, one can assume that there exists a self-loop on each vertex associated with a follower. This means that α_{ii} in the control law (3.2) is nonzero for any $i \in \mathbb{N}_{n-1}$ (note that this would not require additional communication links). This assumption along with the fact that the root set R is a singleton, i.e. $R = \{r\}$, gives rise to the following theorem.

Theorem 3.2. *The information flow graph $\mathcal{G}$ is controllable if and only if any vertex in $\mathcal{G}$ is reachable from the root.*

Proof: The reachability of all vertices from the root implies that each vertex is the end vertex of a rooted path. The vertex corresponding to the root can be considered as a rooted path of length zero, and a self-loop on each vertex constructs a cycle family whose union with the rooted path spans the vertex set $\mathcal{V}$. Thus, the proof follows from Theorem 3.1. $\square$

Fig. 3.2 depicts two information flow graphs for a six-agent group in which the vertex labeled r represents the leading agent. In Fig. 3.2(a), vertex 5 is not the end of an r-rooted path, i.e., it is unreachable from the root. Hence, from Theorem 3.2 the digraph is uncontrollable. Fig. 3.2(b) represents a controllable information flow graph, since each vertex is reachable from the root.

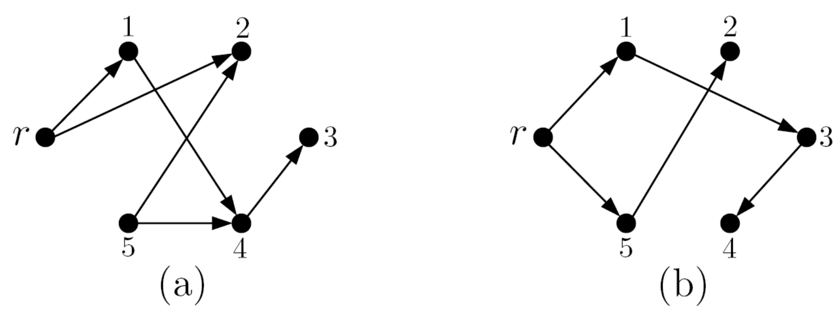


Figure 3.2: (a) An uncontrollable information flow graph; (b) a controllable information flow graph.

In the next sections, controllability preservation of an information flow graph in case of failure of some communication links or loss of agents is investigated.

3.3 Controllability under Communication Links Failure

The goal of this subsection is to present graph-based necessary and sufficient conditions under which an information flow graph remains controllable in the presence of a certain number of communication link failures. In other words, it is desired to verify whether the whole system remains controllable by using any subset of a given size of the communication links. To this end, the notion of *p-link controllability* is introduced in the sequel.

Definition 3.2. *The information flow graph G is said to be p-link controllable if p is the largest number such that the controllability of the digraph is preserved after removing any group of at most $p-1$ edges. In this case, p is called the link-controllability degree.*

The above definition implies that in a p-link controllable digraph, the minimum number of edges whose removal makes the digraph uncontrollable is p. This number will hereafter be denoted by $lc(\mathcal{G})$. The minimal set of links whose failure makes the digraph uncontrollable is called *critical links set* (this set is not unique, in general). The value of $lc(\mathcal{G})$ provides a quantitative measure of reliability with respect to communication failure.

Theorem 3.3. *The information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is p-link controllable if and only if*

$$\min_{\substack{X \subset \mathcal{V} \\ r \in X}} d^+_{\mathcal{G}}(X) = p.$$

Proof: It is clear from the definition of outcut that removing the set $\partial^+_{\mathcal{G}}(X)$ from the edge set $\mathcal{E}$ for every $X \subset \mathcal{V}$ with $r \in X$ makes the set $\mathcal{V} \backslash X$ unreachable from the root. On the other hand, suppose that F is the minimal set of edges whose removal makes at least one of the vertices unreachable from the root. Let X_F be the set of reachable vertices from r after the removal of those edges which belong to F. The proof follows now on noting that F includes all members of the outcut of X_F. $\square$

Given a digraph $\mathcal{G}$, one may use Theorem 3.3 to find the value of $lc(\mathcal{G})$. However, calculating the outdegree of all possible subsets of the vertex set $\mathcal{V}$ takes exponential time. In other words, Theorem 3.3 provides an algorithm whose complexity is exponential in the size of the input, and hence is intractable for high-order digraphs. The complexity of an algorithm is defined to be the number of basic computational steps (e.g., arithmetical operations and comparisons) required for its execution. This number clearly depends on the size and nature of the input. In polynomial-time algorithms, complexity is bounded above by a polynomial in the input size. The significance of such algorithms is that normally they are computationally feasible, even for large input graphs. On the contrary, exponential-time algorithms have high running times which render them unusable even on inputs of moderate size. Therefore, it is desired to develop a polynomial-time algorithm to determine the value of $lc(\mathcal{G})$ for any digraph.

Let $lc(\mathcal{G}, x)$ be the minimum number of edges whose deletion makes the vertex x unreachable from the root. Note that by definition $lc(\mathcal{G}) = \min_{x \in \mathcal{V} \setminus \{r\}} lc(\mathcal{G}, x)$.

Lemma 3.1. *For a specified vertex y in a digraph $\mathcal{G}$,*

$$lc(\mathcal{G}, y) = \min_{\substack{X \subset \mathcal{V} \\ r \in X, y \notin X}} d^+_{\mathcal{G}}(X).$$

Proof: The proof is similar to that of Theorem 3.3. $\square$

Lemma 3.2. *Given a digraph $\mathcal{G}$, consider a specified vertex y and an ry-path denoted by P_{ry}. Let $\mathcal{G}_{\text{new}}$ be the digraph obtained from $\mathcal{G}$ by reversing the direction of the edges of P_{ry}. Then the following relation holds*

$$lc(\mathcal{G}_{\text{new}}, y) = lc(\mathcal{G}, y) - 1.$$

Proof: It is straightforward to show that for any set of vertices X which includes the root and excludes vertex y, $|P_{ry} \cap \partial_{\mathcal{G}}^{+}(X)| - |P_{ry} \cap \partial_{\mathcal{G}}^{-}(X)| = 1$, that is, the number of edges of P_{ry} leaving X is one more than those entering X (see Fig. 3.3). In other words, reversing an ry-path decreases $d_{\mathcal{G}}^{+}(X)$ by one, for every $X \subset \mathcal{V}$ with $r \in X$ and $y \notin X$. The proof follows directly from this result and Lemma 3.1. $\square$

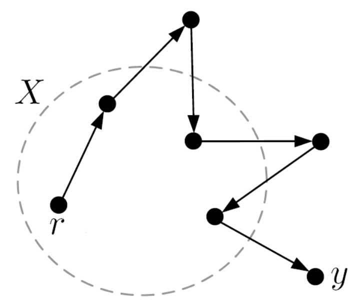


Figure 3.3: An illustrative example given for the proof of Lemma 3.2.

Theorem 3.4. *Consider a digraph $\mathcal{G}$ with a specified vertex y. Construct a new digraph $\mathcal{G}_{\text{new}}$ by reversing the direction of the edges of an ry-path in $\mathcal{G}$, if any. Repeat the same procedure for $\mathcal{G}_{\text{new}}$ and continue until a digraph $\mathcal{G}_{\text{final}}$ is obtained in which y is unreachable from the root. Denote with $Y \subset \mathcal{V}$ the reachable vertices from r in $\mathcal{G}_{\text{final}}$. Then, the outcut of Y in $\mathcal{G}$ is a minimal set whose deletion makes y unreachable from the root; in particular, $d_{\mathcal{G}}^{+}(Y) = lc(\mathcal{G}, y)$.*

Proof: From Lemma 3.2, each time the direction of the edges of an ry-path is revered, the outdegree of Y decreases by one. The proof follows now from the fact that for the final digraph $\mathcal{G}_{\text{final}}$, $\partial_{\mathcal{G}_{\text{final}}}^{+}(Y) = \emptyset$. $\square$

One of the important outcomes of Theorem 3.4 is that one can use it to develop a polynomial-time procedure for finding $lc(\mathcal{G})$. An algorithm of complexity $\mathcal{O}(|\mathcal{V}|^2|\mathcal{E}|)$ is developed below whose input is an information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order n, and outputs are $lc(\mathcal{G})$ and a set of critical links $\mathcal{C}$.

Algorithm 3.1.

$\mathcal{C} = \emptyset$.

$lc(\mathcal{G}) = n$.

for $i = 1$ to $n-1$,

$\mathcal{H} = \mathcal{G}$.

Main: $Y = \{r\}$ and $\zeta(j) = \emptyset$ $(\forall j \in \mathcal{V})$.

while $\exists\, e_{xy} \in \partial^+_{\mathcal{H}}(Y)$,

$Y = Y \cup \{y\}$.

$\zeta(y) = x$.

end while

if $i \in Y$,

In H, reverse the direction of all the edges in the ri-path obtained by using the parent function ζ, and then jump to *Main*.

end if

if $d^+_{\mathcal{G}}(Y) < lc(\mathcal{G})$,

$lc(\mathcal{G}) = d^+_{\mathcal{G}}(Y)$.

$\mathcal{C} = \partial^+_{\mathcal{G}}(Y)$.

end if

end for

return $lc(\mathcal{G})$ and $\mathcal{C}$.

As an example, consider the digraph shown in Fig. 3.4(a). Let Algorithm 3.1 be applied to this digraph to find the value of $lc(\mathcal{G}, i)$ for a specific vertex, say $i = 3$, and its corresponding critical links set, $\mathcal{C}_3$ (i.e., a set of edges whose deletion makes vertex 3 unreachable from the root). The evolution of set Y at each step is as follows.

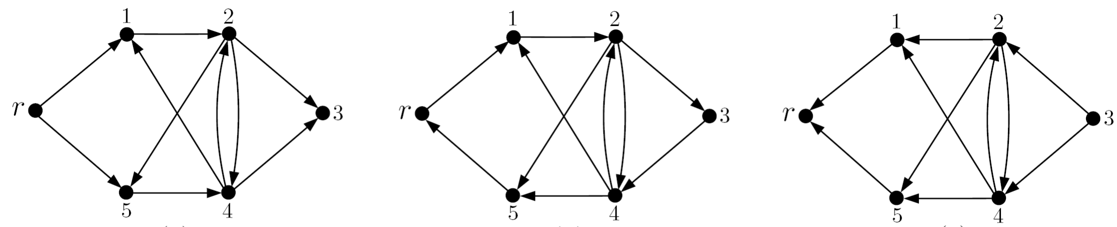


Figure 3.4: (a) A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; (b) the digraph $\mathcal{H}$ obtained by reversing the first $r3$-path, (c) the digraph $\mathcal{H}$ obtained by reversing the second $r3$-path.

Step 1: $Y = \{r\}$; $Y = \{r, 1\}$, $\zeta(1) = r$; $Y = \{r, 1, 5\}$, $\zeta(5) = r$; $Y = \{r, 1, 5, 2\}$, $\zeta(2) = 1$; $Y = \{r, 1, 5, 2, 4\}$, $\zeta(4) = 5$; $Y = \{r, 1, 5, 2, 4, 3\}$, $\zeta(3) = 4$. In this digraph, $3 \in Y$ and the $r3$-path is $((r, 5), (5, 4), (4, 3))$. Fig. 3.4(b) shows the digraph $\mathcal{H}$ after reversing the $r3$-path obtained in this step.

Step 2: $Y = \{r\}$; $Y = \{r, 1\}$, $\zeta(1) = r$; $Y = \{r, 1, 2\}$, $\zeta(2) = 1$; Y $\{r, 1, 2, 4\}$, $\zeta(4) = 2$; $Y = \{r, 1, 2, 4, 3\}$, $\zeta(3) = 2$; $Y = \{r, 1, 2, 4, 3, 5\}$, $\zeta(5) = 2$. In the resultant digraph, $3 \in Y$ and the $r3$-path is $((r, 1), (1, 2), (2, 3))$. Fig. 3.4(c) shows the digraph $\mathcal{H}$ after reversing the $r3$-path obtained in this step.

Step 3: $Y = \{r\}$. Here, $3 \notin Y$; therefore, $lc(\mathcal{G}, 3) = d^+_{\mathcal{G}}(Y) = 2$ and $\mathcal{C}_3 = \partial^+_{\mathcal{G}}(Y) = \{(r, 1), (r, 5)\}$.

By applying this procedure to all vertices, one obtains $lc(\mathcal{G}) = 2$, that is the digraph shown in Fig. 3.4(a) is 2-link controllable.

As noted before, the set of critical links is not unique, in general, and hence Algorithm 3.1 gives one of possibly multiple critical sets. For instance, Fig. 3.5 depicts nine possible sets of critical links of the digraph illustrated in Fig. 3.4(a) by dashed arrows. For this digraph, Algorithm 3.1 gives the set shown in Fig. 3.5(a) as a critical links set.

The next theorem provides the minimum number of edges required for p-link controllability of a digraph.

Theorem 3.5. *Any p-link controllable digraph of order n has at least $(n-1)p$ edges. Also, there exists a p-link controllable digraph whose size attains this lower bound.*

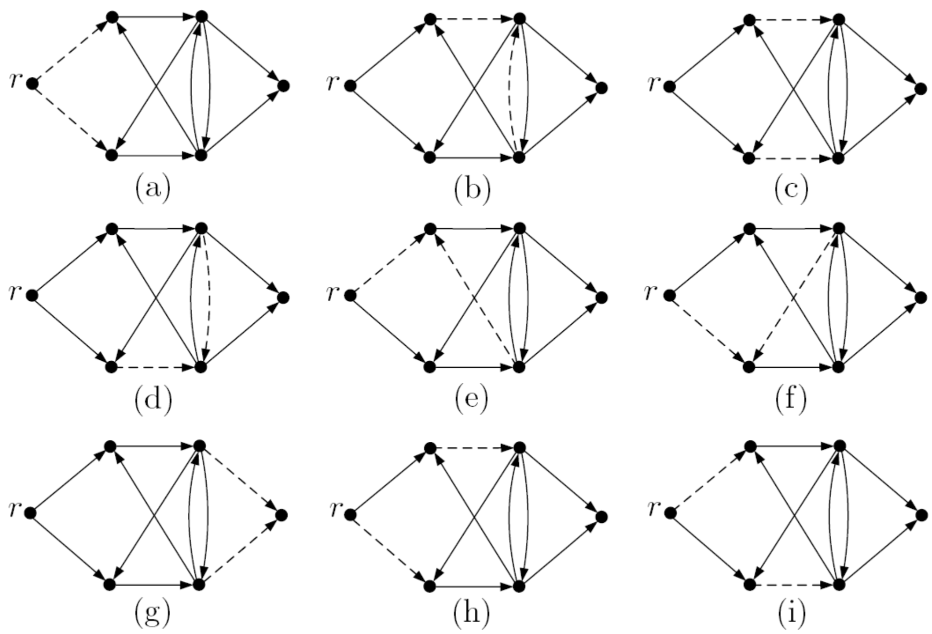


Figure 3.5: Nine sets of critical links for a 2-link controllable digraph. Edges associated with critical links are depicted by dashed arrows.

Proof: In a p-link controllable digraph, the indegree of any vertex $x \in \mathcal{V}\backslash\{r\}$ is at least p. Therefore,

$$|\mathcal{E}| = \sum_{x \in \mathcal{V}\backslash\{r\}} d_{\mathcal{G}}^{-}(\{x\}) \geq (n-1)p$$

Let $X \subset \mathcal{V}\backslash\{r\}$ be a set of p vertices and $Y = \mathcal{V}\backslash\{X \cup \{r\}\}$. Construct a complete digraph of order p whose set of vertices is X. Then, create the p edges (r, x), for all $x \in X$. Create also edges from any vertex in X to any vertex in Y. The resulting digraph has $p + p(p-1) + p(n-p-1) = (n-1)p$ edges (Fig. 3.6 gives an example with $p = 3$ and $n = 6$). To prove that this digraph is p-link controllable, it suffices to show that there exists p edge-disjoint paths from r to any vertex $x \in \mathcal{V}\backslash\{r\}$. If $x \in X$, then the p edge-disjoint paths are given by the $p-1$ paths $(r, y), (y, x)$, for all $y \in X\backslash\{x\}$, together with the path (r, x). If $x \in Y$, the paths are given by $(r, y), (y, x)$, for all $y \in X$. $\square$

The rest of this section is dedicated to finding the value of p for some standard digraphs.

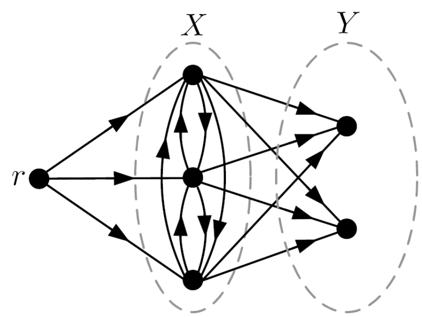


Figure 3.6: An illustrative example given for the proof of Theorem 3.5.

Proposition 3.1. *Choose any vertex of a complete digraph $\mathcal{K}_n = (\mathcal{V}, \mathcal{E})$ of order n as the root, and remove all of its incoming edges. Then, the resultant digraph is $(n-1)$-link controllable.*

Proof: Since the outdegree of each vertex (including the root) is equal to $n-1$, thus $p \leq n-1$. Moreover, $P_i = (r,i),(i,j)$ (for $i \in \mathcal{V}\backslash\{j\}$) and $P_j = (r,j)$ provide $n-1$ edge-disjoint paths from the root to vertex j. This yields that any critical links set should have at least $n-1$ edges, and this completes the proof. □

Remark 3.1. *Since a complete digraph has the maximum number of edges, Proposition 3.1 implies that the maximum possible value for $lc(\mathcal{G})$ in an information flow graph $\mathcal{G}$ with n vertices is $n-1$.*

Three complete digraphs of order three, four, and five are shown in Fig. 3.7, where in each digraph a vertex is chosen as the root and its incoming edges are removed. From Proposition 3.1, the digraph shown in Fig. 3.7(a) is 2-link controllable, that in Fig. 3.7(b) is 3-link controllable, and that in Fig. 3.7(c) is 4-link controllable.

Kautz digraph is known to be an ideal candidate to represent the interconnection network for parallel system architectures [74]. The Kautz digraph $\mathcal{K}(d,\kappa)$ for $d \geq 2$ and $\kappa \geq 1$ consists of the vertex set $\mathcal{V} = \{0, 1, \dots, n-1\}$, and the edge set $\mathcal{E} = \{(i,j)|\ j \equiv -(id+\tau) \pmod{n},\ \tau = 1, \dots, d\}$, where $n = d^{\kappa} + d^{\kappa-1}$. This

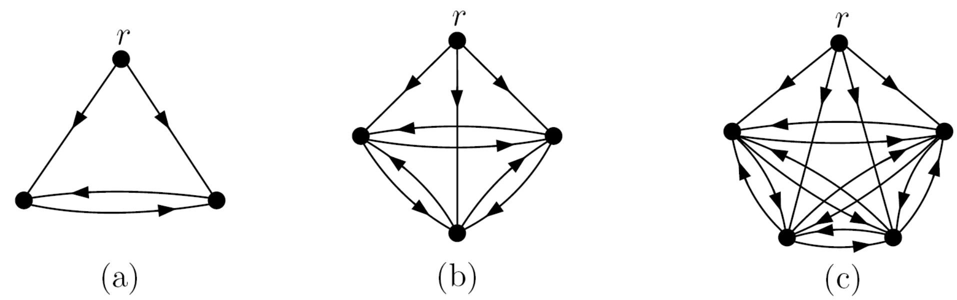


Figure 3.7: Complete digraphs of order (a) three, (b) four, and (c) five.

digraph has $|\mathcal{V}| = n$ vertices and $|\mathcal{E}| = nd$ edges. The link-controllability degree for a Kautz digraph is obtained in the next proposition.

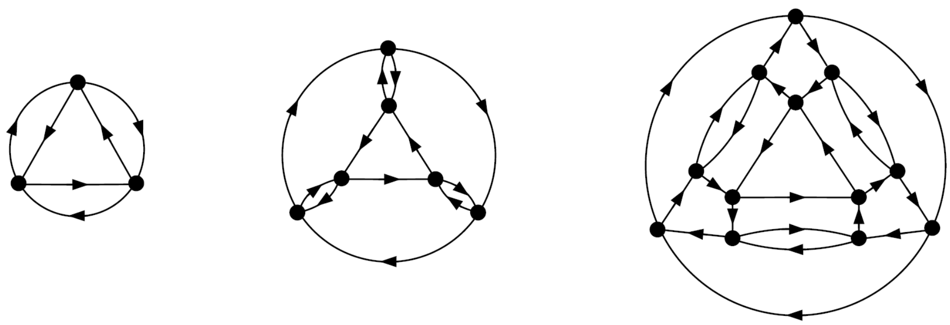
Proposition 3.2. *Choose any vertex of a Kautz digraph $\mathcal{K}(d, \kappa) = (\mathcal{V}, \mathcal{E})$ as the root, and remove all of its incoming edges. Then, the resultant digraph is d-link controllable.*

Proof: In a Kautz digraph $\mathcal{K}(d, \kappa)$, the indegree and outdegree of each vertex are both equal to d. It is shown in [75] that for any two distinct vertices x and y in $\mathcal{K}(d, \kappa)$, there exist d paths from x to y whose common vertices are exactly x and y. From this result, one can conclude (in a way similar to the proof of Proposition 3.1) that $p = d$. $\square$

Three Kautz digraphs with $d = 2$ and different values for κ ($\kappa = 1$, 2, and 3) are depicted in Fig. 3.8. In these digraphs, by choosing any of the vertices as the root and removing its incoming edges, the resultant digraph will be 2-link controllable.

3.4 Controllability under Agents Failure

This section presents a condition under which an information flow graph remains controllable after a number of agents fail, i.e. they stop operating. It is to be noted that the failure of an agent implies that all communication links from/to the corresponding agent also fail. However, in certain type of agent failures the



(a) (b) (c)

Figure 3.8: Three Kautz digraphs: (a) $\mathcal{K}(2,1)$, (b) $\mathcal{K}(2,2)$, and (c) $\mathcal{K}(2,3)$.

communication links of the failed agent may remain operational [76]. Note also that it is assumed in this section that the leader is not subject to failure.

Definition 3.3. *Analogously to the definition of p-link controllability, the information flow graph $\mathcal{G}$ is said to be q-agent controllable if q is the largest number such that the controllability of the digraph is preserved after removing any group of at most $q-1$ vertices. In this case, q is called the agent-controllability degree.*

In other words, in a q-agent controllable digraph, q is the minimum number of vertices whose removal makes the digraph uncontrollable. This number will hereafter be denoted by $ac(\mathcal{G})$. Moreover, the minimal set of agents whose failure makes the digraph uncontrollable is called *critical agents set* (this set is not unique, in general). The larger the value of $ac(\mathcal{G})$ is, the more reliable the corresponding networked system is to agents failure. It is to be noted that a digraph resulted by removing all of the followers is known to be uncontrollable. This implies that $q \leq n-1$.

The problem of finding agent-controllability degree of an information flow graph $\mathcal{G}$ can be converted to that of finding link-controllability degree by creating a digraph $\tilde{\mathcal{G}}$ from $\mathcal{G}$ using node-duplication technique [77] as follows. Replace any vertex i of G, except the root with two vertices $\tilde{i}_1$ and $\tilde{i}_2$, and draw the edge $(\tilde{i}_1, \tilde{i}_2)$ subsequently. Then, replace each edge (i,j) of $\mathcal{G}$ with $(\tilde{i}_2, \tilde{j}_1)$. Note that the edge (r,i) in $\mathcal{G}$ is replaced by $(r, \tilde{i}_1)$. As an example, a digraph with its node-duplicated counterpart are shown in Fig. 3.9(a) and (b), respectively.

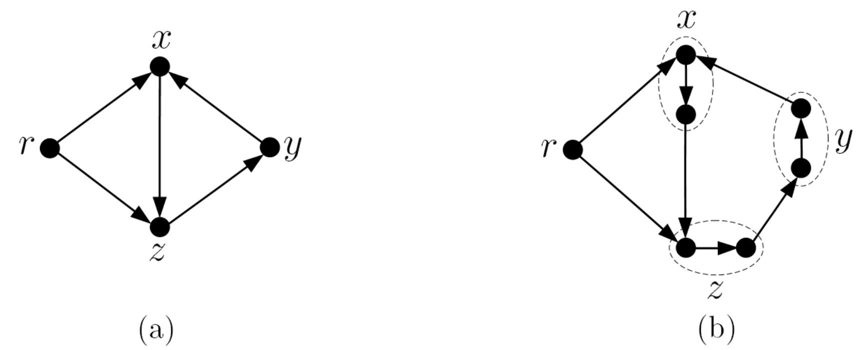


Figure 3.9: An example of a digraph (a) and its node-duplicated counterpart (b).

Lemma 3.3. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$; consider a specified vertex y, where $(r, y) \notin \mathcal{E}$. Let $ac(\mathcal{G}, y)$ be the minimum number of vertices of $\mathcal{G}$ whose removal makes the vertex $y \in \mathcal{V} \backslash \{r\}$ unreachable from the root. Then*

$$ac(\mathcal{G}, y) = lc(\tilde{\mathcal{G}}, \tilde{y}_1).$$

Proof: Let X be a minimal set of vertices of $\mathcal{G}$ such that any path from r to y in $\mathcal{G}$ passes through at least one of these vertices. Let also F be a minimal set of edges of $\tilde{\mathcal{G}}$ such that any path from r to $\tilde{y}_1$ in $\tilde{\mathcal{G}}$ includes at least one of the edges of F. It follows from the definitions of q-agent and p-link controllability that $|X| = ac(\mathcal{G}, y)$ and $|F| = lc(\tilde{\mathcal{G}}, \tilde{y}_1)$. Consequently, to prove this lemma it suffices to show that $|X| = |F|$. For any $x \in X$, consider the edge $(\tilde{x}_1, \tilde{x}_2) \in \tilde{\mathcal{G}}$ and let F_X be the union of all such edges. It is straightforward to show that $|X| = |F_X|$, and that every path from r to $\tilde{y}_1$ in $\tilde{\mathcal{G}}$ includes at least one edge in F_X. This yields that $|X| \geq |F|$. Now, it is aimed to map F to a set of vertices X_F (of size $|F|$) in G in such a way that every path from r to y passes through at least one vertex in X_F. Every edge in F has one of the forms $(r, \tilde{i}_1)$ or $(\tilde{i}_1, \tilde{i}_2)$ or $(\tilde{j}_2, \tilde{i}_1)$ or $(\tilde{i}_2, \tilde{y}_1)$, and is mapped to vertex i in $\mathcal{G}$ (see Fig. 3.10). This map is injective because if two edges in F are mapped to the same vertex i, then both edges can be replaced by $(\tilde{i}_1, \tilde{i}_2)$ which contradicts the initial assumption of the minimality of F. Under this mapping, one

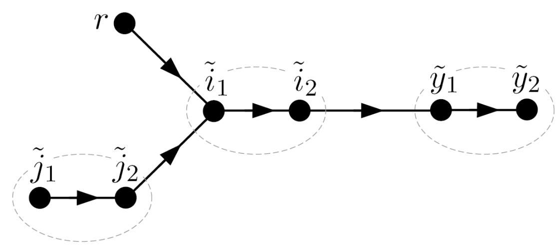


Figure 3.10: An illustrative example given for the proof of Lemma 3.3.

reaches a set of vertices X_F in $\mathcal{G}$ of the same size as F, with the property that any path from r to y in $\mathcal{G}$ passes through at least one of these vertices. This implies that $|F| \geq |X|$, which completes the proof. $\square$

It is to be noted that for a vertex y directly connected to the root, removal of any set of vertices does not affect its reachability from the root. For any such vertex, define $ac(\mathcal{G}, y) = n - 1$. It is straightforward to show that $ac(\mathcal{G}) = \min_{y \in \mathcal{V} \setminus \{r\}} ac(\mathcal{G}, y)$. Therefore, from Lemma 3.3 one can find the value of q using the results presented for the p-link controllability. In order to find a set of critical agents, one can apply the same mapping used in the proof of Lemma 3.3 to the set of critical edges $\mathcal{C}$ of $\tilde{\mathcal{G}}$ given by Algorithm 3.1. Fig. 3.11(a) shows a 1-agent controllable digraph with a unique critical agents set ($\{1\}$). The digraph shown in Fig. 3.11(b) is 2-agent controllable and has multiple critical agents sets (e.g., $\{1, 5\}$ and $\{2, 4\}$).

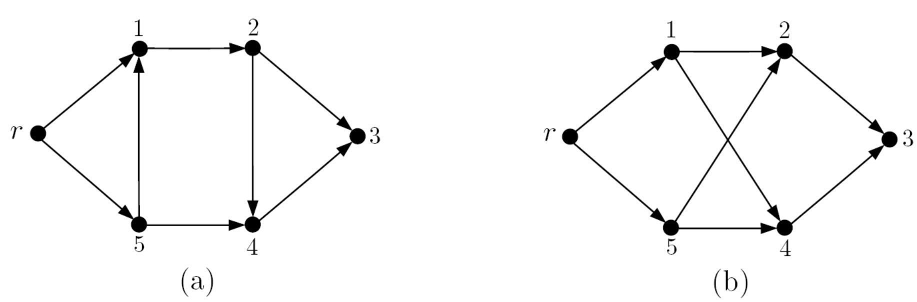


Figure 3.11: (a) A 1-agent controllable digraph with a unique critical agent set, (b) a 2-agent controllable digraph with multiple critical agents sets.

It is clear that in an $(n-1)$-agent controllable digraph of order n, all vertices are directly connected to the root. The next theorem presents a lower bound on the number of edges of a q-agent controllable digraph, for $1 \leq q \leq n-2$.

Theorem 3.6. *Any q-agent controllable digraph of order n with $1 \leq q \leq n-2$ has at least $n+q-2$ edges. Also, there exists a q-agent controllable digraph whose size attains this lower bound.*

Proof: In a q-agent controllable digraph of order n with $1 \leq q \leq n-2$, there exist $k \geq 1$ vertices which are not directly connected to the root. Let $X_1 \subset \mathcal{V}\backslash\{r\}$ be the set of such k vertices, and $X_2 = \mathcal{V}\backslash\{X_1 \cup \{r\}\}$ be the set of all vertices that are directly connected to the root. Note that the indegree of any vertex in X_1 is at least q, while that of any vertex in X_2 is at least one. Therefore,

$$\begin{aligned}
|\mathcal{E}| =& \sum_{x \in \mathcal{V}\backslash\{r\}} d^-_{\mathcal{G}}(\{x\}) = \sum_{x \in X_1} d^-_{\mathcal{G}}(\{x\}) + \sum_{x \in X_2} d^-_{\mathcal{G}}(\{x\}) \\
\geq & \;\; kq + (n-k-1) = n-1+k(q-1) \\
\geq & \;\; n+q-2
\end{aligned}$$

To construct a q-agent controllable digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order n with $n+q-2$ edges, consider a vertex $z \in \mathcal{V}\backslash\{r\}$ and let X be the set of all vertices except r and z (clearly, $|X| = n-2$). Draw an edges (r, x) for every vertex $x \in X$. Let Y be a set of q arbitrary vertices of X. From each vertex $y \in Y$ create an edge (y, z). The resulting digraph has $(n-2)+q$ edges (see Fig. 3.12 for $q = 2$ and $n = 6$). To show that this digraph is q-agent controllable, it suffices to show that $ac(\mathcal{G}, z) = q$ because $ac(\mathcal{G}, x) = n-1$ for any $x \in X$. Since only the removal of the vertex set Y can make the vertex z unreachable from the root, this implies that $ac(\mathcal{G}, z) = q$, which completes the proof. □

Proposition 3.3. *Choose arbitrarily a vertex of a complete digraph $\mathcal{K}_n = (\mathcal{V}, \mathcal{E})$ of order n as the root, and remove all of its incoming edges. Then, the resultant digraph $\mathcal{G}$ is $(n-1)$-agent controllable.*

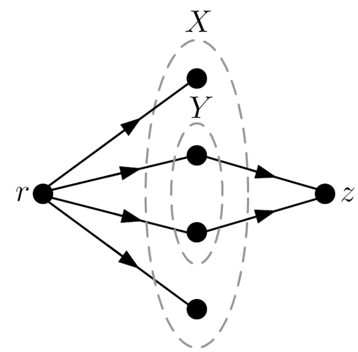


Figure 3.12: An illustrative figure for the proof of Theorem 3.6.

Proof: The proof is straightforward on noting that in a complete digraph, any vertex (including the root) is directly connected to all other vertices. □

Proposition 3.4. *Choose arbitrarily a vertex of a Kautz digraph $\mathcal{K}(d, \kappa) = (\mathcal{V}, \mathcal{E})$ with $\kappa > 1$ as the root, and remove all of its incoming edges. Then, the resultant digraph $\mathcal{G}$ is d-agent controllable.*

Proof: In a Kautz digraph, the outdegree of any vertex is d. For $\kappa > 1$, the number of vertices is greater than $d + 1$, and hence there exists at least one vertex $y \in \mathcal{V} \backslash \{r\}$ such that $(r, y) \notin \mathcal{E}$. It results from the fact $d^+(\{r\}) = d$ that $ac(\mathcal{G}, y) \leq d$, for any y not directly connected to r. This result along with the fact that there exist d paths from r to y whose common vertices are exactly r and y implies that $ac(\mathcal{G}, y) = d$. □

3.5 Joint Controllability

This section investigates the controllability of the information flow graph under simultaneous failure of some of the communication links and agents.

Definition 3.4. *An information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be joint (p, q)-controllable if in case of simultaneous failure of any set of agents of size $y \leq q$ and any set of links of size $x \leq p$, such that $x + y < p + q$, it remains controllable.*

The above definition states that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is joint (p, q)-controllable if for all points $(a, b) \in \mathcal{D}$, where $\mathcal{D} := \{(x, y)|\ x, y \in \mathbb{Z}^+, x \leq p,\ y \leq q,\ \text{and } x + y < p + q\}$ (see Fig. 3.13), for any vertex set $X \subset \mathcal{V}$ of size b and any edge set $F \subset \mathcal{E}$ of size a, the digraph $\mathcal{H} = (\mathcal{V} \backslash X, \mathcal{E} \backslash F)$ is controllable.

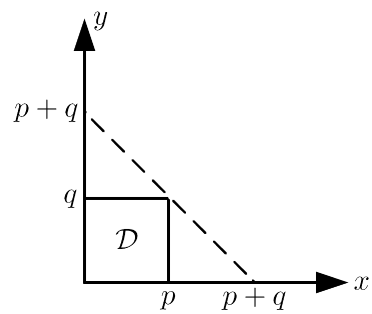


Figure 3.13: An illustrative figure for the definition of joint (p, q)-controllability.

Let Ω be the set of all pairs (p, q) such that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is joint (p, q)-controllable. Also, let Ω_i be the set of all pairs (p, q) such that in case of the removal of any set of vertices of size $y \leq q$ (excluding r and i) and any set of edges of size $x \leq p$ at the same time, such that $x + y < p + q$, vertex i remains reachable from the root. It is clear that $\Omega \subseteq \cup_{i=1}^{n-1} \Omega_i$; thus, if one finds the sets Ω_i's for any vertex $i \in \mathcal{V} \backslash \{r\}$, the set Ω can then be obtained.

Proposition 3.5. *A complete digraph of order n is joint (p, q)-controllable for any pair $(p, q) \in \Omega$, where $\Omega = \{(x, y)|\ x, y \in \mathbb{Z}^+,\ y = -x + n - 1\}$.*

Proof: In a complete digraph of order n, there exist $n - 1$ ri-paths (P_j's) for any $i \in \mathcal{V} \backslash \{r\}$, where

$$P_j = \begin{cases} (r, j)(j, i) & i \neq j \\ (r, i) & i = j \end{cases}, \qquad j = 1, 2, \ldots, n - 1$$

Any pair of distinct paths intersect at exactly r and i (see Fig. 3.15). Assume that an arbitrary set of vertices of size y (excluding r and i) and an arbitrary set of edges of size x are subject to removal at the same time. It can be easily seen that while

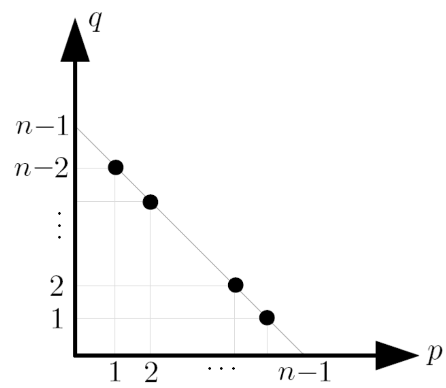


Figure 3.14: Joint (p, q)-controllability of a complete digraph of order n.

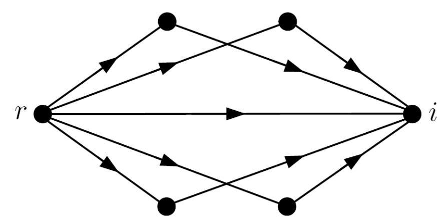


Figure 3.15: An illustrative figure for the proof of proposition 3.5, ri-paths in a complete digraph of order six.

the number of removed elements (i.e. $x + y$) is less than $n - 1$, vertex i remains reachable from the root. Moreover, in the case of removal of one element (vertex or edge) from each ri-path (i.e. $x + y = n - 1$), vertex i becomes unreachable from the root. The proof follows from the definition of joint (p, q)-controllability. □

Fig. 3.16 shows three complete digraphs of order three, four and five, along with their corresponding (p, q) pairs.

Proposition 3.6. *Choose an arbitrary vertex of a Kautz digraph $\mathcal{K}(d, \kappa) = (\mathcal{V}, \mathcal{E})$ with $\kappa > 1$ as the root, and remove all of its incoming edges. Then, the resultant digraph is joint (p, q)-controllable for any pair $(p, q) \in \Omega$, where $\Omega = \{(x, y)|\ x, y \in \mathbb{Z}^+,\ y = -x + d\}$.*

Proof: In a Kautz digraph, there exist d ri-paths for any $i \in \mathcal{V}\backslash\{r\}$ whose common vertices are exactly r and i [75]. The proof follows now, using an approach similar to the one in Proposition 3.5. □

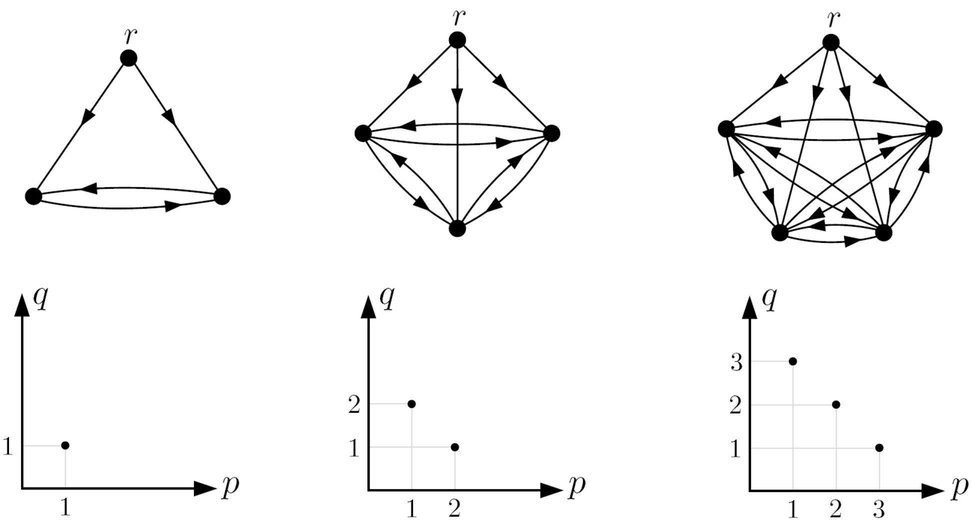


Figure 3.16: The set of pairs (p, q) for three complete digraphs.

3.6 Examples

This section considers a number of digraphs (which can potentially be the information flow graphs of multi-agent systems) and gives the values of p and q and the pair (p, q) for each one.

A *simple loop* digraph of order n is a directed cycle consisting of the vertex set $\mathcal{V} = \{0, 1, \ldots, n-1\}$, and the edge set $\mathcal{E} = \{(i, j)|\ j - i \equiv 1 \pmod{n}\}$ [74]. Fig. 3.17(a) shows a simple loop digraph of order five. In a simple loop digraph with $n > 2$, by choosing an arbitrary vertex as the root, the digraph will be 1-link and 1-agent controllable. However, there is no pair (p, q) with $p, q > 0$ to achieve joint (p, q)-controllability.

A *distributed double loop* digraph of order n consists of the vertex set $\mathcal{V} = \{0, 1, \ldots, n-1\}$, and the edge set $\mathcal{E} = \{(i, j)|\ j - i \equiv 1, n-1 \pmod{n}\}$ [78]. Fig. 3.17(b) shows a distributed double loop digraph of order five. In a distributed double loop digraph with $n > 3$, by choosing any vertex as the root, the digraph will be 2-link and 2-agent controllable. Also, it is joint $(1, 1)$-controllable.

A *daisy chain loop* digraph of order n consists of the vertex set $\mathcal{V} = \{0, 1, \ldots, n-$

1}, and the edge set $\mathcal{E} = \{(i,j) |\; j - i \equiv 1, n-2 \pmod{n}\}$ [79]. Fig. 3.17(c) shows a daisy chain loop digraph of order five. In a daisy chain loop digraph with $n > 3$, by choosing any vertex as the root, the digraph will be 2-link and 2-agent controllable. Also, it is a joint $(1,1)$-controllable digraph.

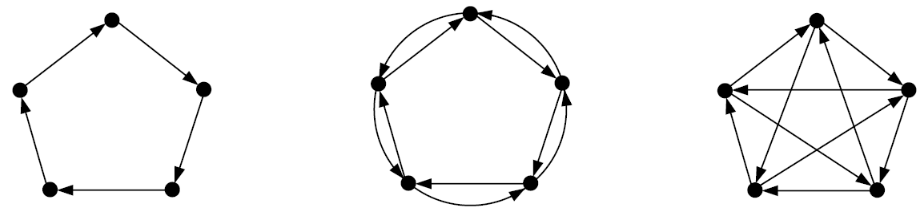


Figure 3.17: (a) Simple loop digraph of order five; (b) distributed double loop digraph of order five, and (c) daisy chain loop digraph of order five.

In all the examples provided in the chapter, the value of p is equal to the value of q; however this is not always the case. As an example, consider the digraph shown in Fig. 3.18 for which $p = 2$ and $q = 1$. In this digraph, the critical agents set is unique (agent 4) while the critical links set is not, that is, there exist several pairs of edges whose removal makes at least one vertex unreachable from the root.

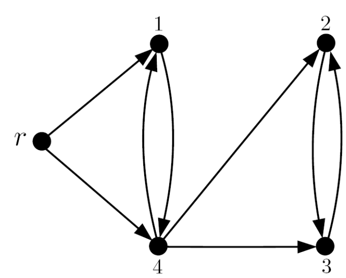


Figure 3.18: A 2-link and 1-agent controllable information flow graph.

Circulant digraph of order n consists of the vertex set $\mathcal{V} = \{0, 1, \ldots, n-1\}$, and the edge set $\mathcal{E} = \{(i,j) |\; j - i \equiv \beta \pmod{n},\; \beta \in B\}$ where $B \subseteq \mathbb{N}_{n-1}$ [80]. In a weakly connected circulant digraph of order $n > 3$, by choosing a vertex as the root, one has $p = |B|$ and $q \leq |B|$. Fig. 3.19 shows a circulant digraph of order six with $B = \{2, 3, 5\}$. In this digraph, by choosing any vertex as the root and removing its incoming edges, the resultant digraph will be 3-link and 2-agent controllable.

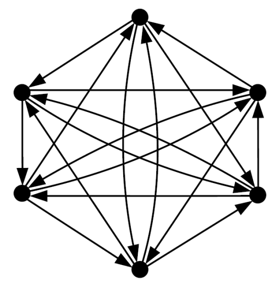


Figure 3.19: A circulant digraph of order six with $p = 3$ and $q = 2$.

3.7 Multiple-Leader Case

In the previous sections, the controllability of the information flow graph of a multi-agent system with single leader is studied. This section aims to extend the results obtained so far to a multiple leader setting, i.e., when more than one agent can act as the leader.

Consider a team of single integrator agents given by

$$\dot{x}_i(t) = u_i(t), \qquad i \in \mathbb{N}_n \tag{3.5}$$

and assume some of the agents, say agents $n-m+1$, $m-m+2$, …, n act as the leaders and are influenced by unconstrained external control inputs $u_i(t) = u^i_{\text{ext}}(t)$, $i \in \mathbb{N}_n \backslash \mathbb{N}_{n-m}$. The remaining agents, i.e. followers, obey the following control law

$$u_i(t) = \sum_{j \in N_i \cup \{i\}} \alpha_{ij} x_j(t), \qquad i \in \mathbb{N}_{n-m} \tag{3.6}$$

Therefore, the dynamics of the followers is described as

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.7}$$

where $x(t) = [x_1(t) \; ... \; x_{n-m}(t)]^T \in \mathbb{R}^{n-m}$, $u(t) = [x_{n-m+1}(t) \; ... \; x_n(t)]^T \in \mathbb{R}^m$, and $A = [a_{ij}]$ and $B = [b_{ij}]$ are structured matrices of proper dimensions. Let $\mathcal{G}$ be the digraph representing the structured system (3.7), and R be the set of vertices corresponding to the leading agents in this digraph.

Theorem 3.7. *The information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is controllable if and only if each vertex in $\mathcal{V}\backslash R$ is reachable from the root set R.*

Proof: The reachability of each vertex of the set $\mathcal{V}\backslash R$ from the root set R is equivalent to each member of the above set being the end vertex of an R-rooted path. The vertices in R can be considered as R-rooted paths of length zero, and a self-loop on each vertex $x \in \mathcal{V}\backslash R$ constructs a cycle family whose union with these zero-length R-rooted paths span the vertex set $\mathcal{V}$. The proof follows now from Theorem 3.1. □

Given a leader-follower multi-agent system with multiple leaders, let $lc(\mathcal{G}; R)$ be the minimum number of edges whose removal results in a digraph in which at lease one vertex is unreachable from the root set R. Let also $lc(\mathcal{G}, x; R)$ be the minimum number of edges whose removal makes the vertex $x \in \mathcal{V}\backslash R$ unreachable from R. Clearly, $lc(\mathcal{G}; R) = \min_{x\in\mathcal{V}\backslash R} lc(\mathcal{G}, x; R)$; therefore, in a p-link controllable information flow graph $\mathcal{G}$, the relation $lc(\mathcal{G}, x; R) \geq p$ holds, for any $x \in \mathcal{V}\backslash R$. The following theorem gives a necessary and sufficient condition for the p-link controllability in multi-agent systems with multiple leaders.

Theorem 3.8. *The information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with the root set R is p-link controllable if and only if*

$$\min_{R\subseteq X\subset\mathcal{V}} d^+_{\mathcal{G}}(X) = p.$$

Proof: The proof is similar to that of Theorem 3.3, and is omitted here. □

In order to find the value of $lc(\mathcal{G}, x; R)$, let a new digraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ be constructed from $\mathcal{G}$ by extending the sets $\mathcal{V}$ and $\mathcal{E}$ as follows: Consider a new vertex

r, and define

$$\mathcal{V}' = \mathcal{V} \cup \{r\} \quad \text{and} \quad \mathcal{E}' = \mathcal{E} \cup \{(r, i), \ \ \forall i \in R\}.$$

The digraph $\mathcal{G}'$ will be referred to as the expanded digraph of $\mathcal{G}$ with respect to R. As an illustrative example, Fig. 3.20(a) shows a digraph with a root set R of size three, and Fig. 3.20(b) demonstrates how the digraph $\mathcal{G}'$ is constructed from $\mathcal{G}$.

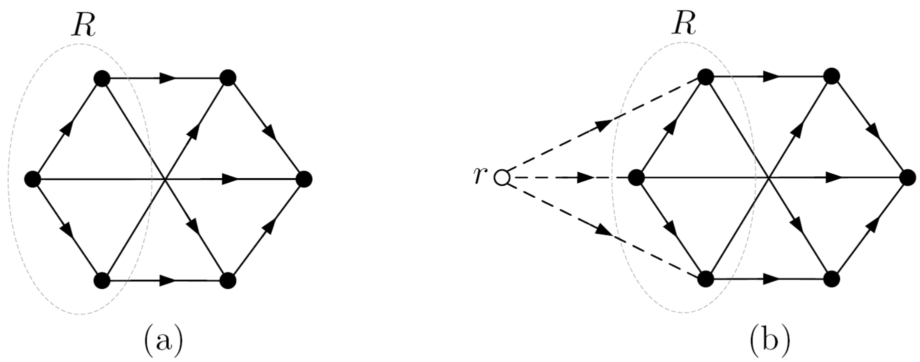


Figure 3.20: (a) An information flow graph $\mathcal{G}$ and, (b) the corresponding expanded digraph $\mathcal{G}'$ with respect to R.

Consider the expanded digraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ corresponding to a given digraph $\mathcal{G}$, and let $x \in \mathcal{V}' \backslash \{R \cup \{r\}\}$ be a specified vertex of $\mathcal{G}'$. Construct a new digraph $\mathcal{G}'_{\text{new}}$ by reversing the direction of all edges of any rx-path, except for those edges which belong to $\{r\} \times R$, if any. Repeat the same procedure for $\mathcal{G}'_{\text{new}}$ and continue until a digraph $\mathcal{G}'_{\text{final}}$ is obtained in which x is unreachable from the root r. Denote with $X_{r,\mathcal{G}'}$ the set of all reachable vertices from r in $\mathcal{G}'_{\text{final}}$ (note that $X_{r,\mathcal{G}'} \subset \mathcal{V}'$).

Theorem 3.9. *The outcut of $X_{r,\mathcal{G}'}$ in $\mathcal{G}'$ is a minimal set whose deletion makes vertex $x \in \mathcal{V}' \backslash \{R \cup \{r\}\}$ unreachable from R. In particular, $d^+_{\mathcal{G}'}(X_{r,\mathcal{G}'}) = lc(\mathcal{G}, x; R)$.*

Proof: The proof is similar to that of Theorem 3.4. □

One can use the result of Theorem 3.9 to develop a polynomial-time procedure for finding the value of $lc(\mathcal{G}, x; R)$. The following algorithm is presented for this purpose.

Algorithm 3.2.

$\mathcal{H} = \mathcal{G}'$.

Main: $Y = \{r\}$ and $\zeta(j) = \emptyset$ $(\forall j \in \mathcal{V}')$.

while $\exists\, e_{uv} \in \partial^+_{\mathcal{H}}(Y)$,

$Y = Y \cup \{v\}$.

$\zeta(v) = u$.

end while

if $x \in Y$,

In $\mathcal{H}$, reverse the direction of all the edges in the rx-path obtained by using the parent function ζ, except that the paths of the form (r, i), $i \in R$, and then go to *Main*.

end if

$lc(\mathcal{G}, x; R) = d^+_{\mathcal{G}'}(Y)$

return $lc(\mathcal{G}, x; R)$.

Using the above algorithm, the notion of q-agent and joint (p, q)-controllability can be extended to multiple leader settings.

Proposition 3.7. *In a leader-follower multi-agent system with a controllable information flow graph, all agents (including the leaders) can take any desired position.*

Proof: As noted before, the controllability of the information flow graph $\mathcal{G}$ is equivalent to the controllability of the corresponding structured system (3.7) under the control input $u(t) = [x_{n-m+1}(t) \ \dots \ x_n(t)]^T$, where $\dot{x}_i(t) = u^i_{\text{ext}}(t)$, $i \in \mathbb{N}_n \backslash \mathbb{N}_{n-m}$. Let $z(t) = [x^T(t) \ \ u^T(t)]^T = [x_1(t) \ \dots \ x_n(t)]^T$ and $\dot{u}(t) = [u^{n-m+1}_{\text{ext}}(t) \ \ \dots \ \ u^n_{\text{ext}}(t)]^T = u_{\text{ext}}(t)$; then

$$\dot{z}(t) = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u_{\text{ext}}(t) \tag{3.8}$$

where 0 and I are the zero and identity matrices of proper dimensions. If the pair (A, B) is controllable, then $\mathrm{rank}([A - \lambda I \mid B]) = n - m$, for all complex numbers λ. Therefore, the matrix

$$\left[\begin{array}{cc|c} A - \lambda I & B & 0 \\ 0 & \lambda I & I \end{array}\right] \tag{3.9}$$

has rank n, for all complex numbers λ. This means that one can find the control input $u_{\mathrm{ext}}(t)$ such that the state of system (3.8) (i.e., the positions of all agents including the leaders) can be transferred from any initial state to any final state in a finite time interval. □

As an example consider five robots moving in vw-plane, where the state of agent i, $x_i(t)$, is defined to be its absolute position with respect to the origin. The information flow graph of the system is shown in Fig. 3.21. It can be easily seen that the action of agents 4 and 5 as leaders makes the system controllable. Thus,

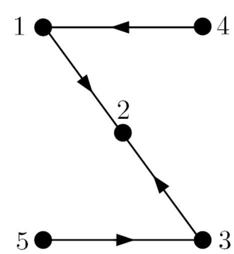


Figure 3.21: The information flow graph of a five-agent system.

the equations of the system can be written as follows: $\dot{x}_1(t) = \alpha_{14} x_4(t)$, $\dot{x}_2(t) = \alpha_{21} x_1(t) + \alpha_{23} x_3(t)$, $\dot{x}_3(t) = \alpha_{35} x_5(t)$, $\dot{x}_4(t) = u_{\mathrm{ext}}^4(t)$, and $\dot{x}_5(t) = u_{\mathrm{ext}}^5(t)$, where $x_i(t) \in \mathbb{R}^2$, for $i = 1, 2, \ldots, 5$. These equations can be represented in a matrix form as follows, in which I and 0 are 2×2 identity and zero matrices, respectively, and

$x(t) = [x_1^T(t), x_2^T(t), \ \dots\ , x_5^T(t)]^T$.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & \alpha_{14}I & 0 \\ \alpha_{21}I & 0 & \alpha_{23}I & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_{35}I \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} u_{\text{ext}}^4(t) \\ u_{\text{ext}}^5(t) \end{bmatrix} \tag{3.10}$$

It is aimed to bring the agents from configuration shown in Fig. 3.22(a) to that of Fig. 3.22(b). Assume $\alpha_{14} = 0.2$, $\alpha_{21} = 0.1$, $\alpha_{23} = 0.3$, and $\alpha_{35} = 0.5$, and

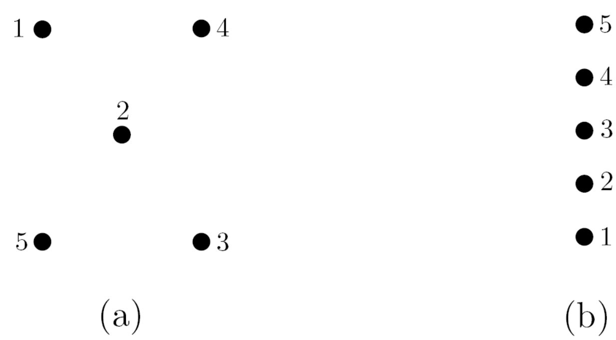


Figure 3.22: (a) Initial configuration, (b) final configuration.

let the initial and final states be $x_0 = [1\ 13\ 7\ 7\ 13\ 1\ 13\ 13\ 1\ 1]^T$ and $x_f = [30\ 1\ 30\ 4\ 30\ 7\ 30\ 10\ 30\ 13]^T$, respectively. The external control inputs (in two-dimensional space) shown in Fig. 3.23 can drive the state of the system from x_0 to x_f in 10 seconds. Fig. 3.24 shows the initial and the final positions of the agents along with their respective trajectories in vw-plane, in which the blank circle represent the leading agents.

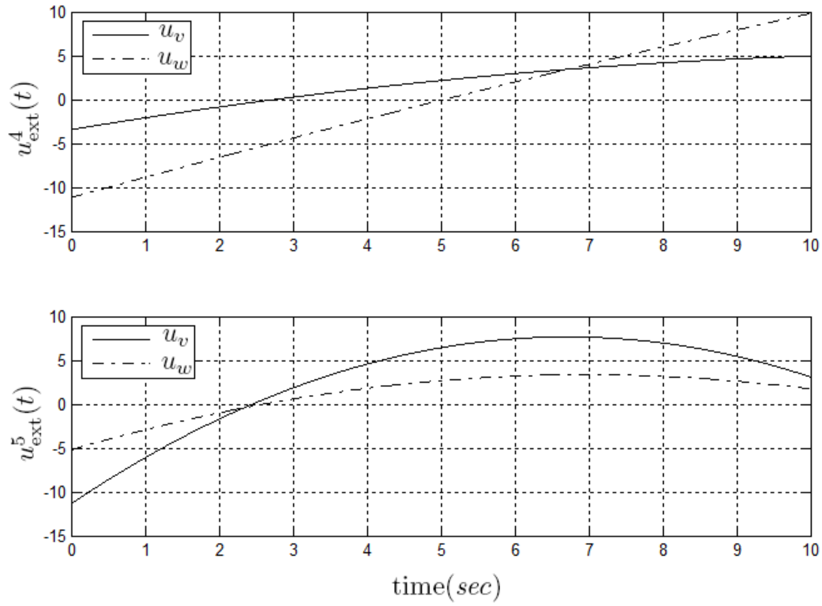


Figure 3.23: The external control inputs in two-dimensional space.

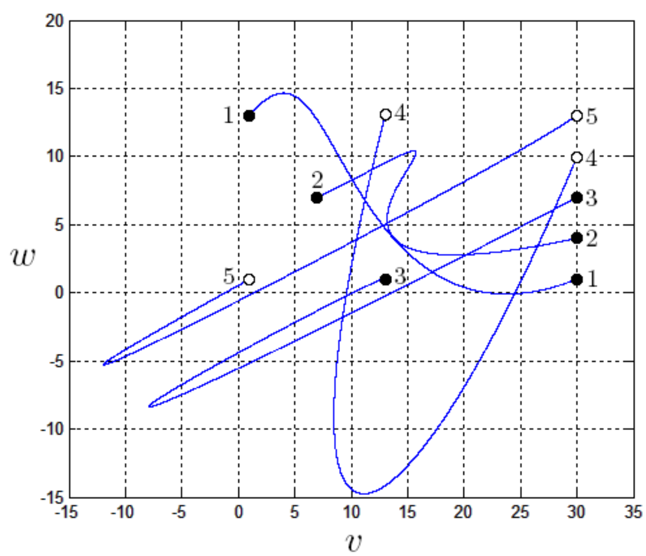


Figure 3.24: The state trajectories of the system.

Chapter 4

Leader Localization

The location problem in the computer science literature deals with finding the best location of facilities such as shops, telecommunication centers, factories, warehouses, and computer servers in a network to achieve certain specifications, and is often formulated as an optimization problem [81]. This problem has been extensively studied in the past few decades [82–84]. Although in a typical location problem the distance plays an important role in defining constraints and objective functions, in a class of location problems called source location, flow or connectivity requirements are also taken into consideration. Various types of source location problem with different constraints and objectives are investigated in the literature [85–89]. This chapter investigates a special case of this problem for multi-agent system, which will be referred to as the *leader localization* problem.

Leader localization problem in a multi-agent system, on the other hand, is concerned with selecting a minimum number of leaders among the agents, in order for the information flow graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to be controllable. Also, it is aimed to find a vertex set $R \subseteq \mathcal{V}$ of the smallest size, such that the information flow graph is p-link or q-agent controllable. It is also desired to present polynomial-time algorithms to find such minimal sets in both undirected and directed information flow graphs.

Definition 4.1. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a set $X \subset \mathcal{V}$ is called p-deficient if $d^-_{\mathcal{G}}(X) < p$. A p-deficient set is minimal if none of its proper subsets is p-deficient.*

Fig. 4.1 shows a digraph with two minimal 2-deficient sets, $X_1 = \{1, 2, 3\}$ and $X_2 = \{4\}$.

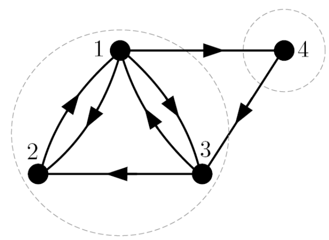


Figure 4.1: A digraph with two minimal 2-deficient sets.

Definition 4.2. *The set $R \subseteq \mathcal{V}$ is called a p-link root set if $lc(\mathcal{G}; R) \geq p$.*

Theorem 4.1. *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a set $R \subseteq \mathcal{V}$ is a p-link root set if and only if any p-deficient set X in $\mathcal{G}$ intersects R.*

Proof: Assume X is a p-deficient set disjoint from R. Since $d^-_{\mathcal{G}}(X) < p$, hence for every vertex $x \in X$, $lc(\mathcal{G}, x; R) < p$. This contradicts the initial assumption that R is a p-link root set. Consider now a set R ($R \subseteq \mathcal{V}$) for which $lc(\mathcal{G}; R) < p$, and assume R intersects any p-deficient set of $\mathcal{G}$. According to Theorem 3.8, there exists a set $X \subset \mathcal{V}$ with $R \subseteq X$, such that $d^+_{\mathcal{G}}(X) < p$, or equivalently $d^-_{\mathcal{G}}(\mathcal{V}\backslash X) < p$. This means that $\mathcal{V}\backslash X$ is a p-deficient set disjoint from R, which contradicts the assumption that R intersects any p-deficient set of $\mathcal{G}$. This contradiction completes the proof. $\square$

It is deduced from Theorem 4.1 that a set R is a p-link root set if and only if any minimal p-deficient set intersects R.

Although undirected graphs can be viewed as a special case of digraphs, because of certain properties they have, one can develop a simpler procedure to solve the leader localization problem for undirected graphs. Therefore, these two cases are considered separately in the sequel.

4.1 Leader Localization in Undirected Information Flow Graphs

This section investigates the leader localization problem in multi-agent systems with bidirectional communication links, represented by the undirected information flow graph $\mathcal{G}_0 = (\mathcal{V}, \mathcal{E}_0)$.

Let $\vec{\mathcal{G}} = (\mathcal{V}, \vec{\mathcal{E}})$ be a digraph obtained by replacing each edge of $\mathcal{G}_0$ with two anti-parallel directed edges (as an illustrative example, see Fig. 4.2). It is straightforward to show that $lc(\mathcal{G}_0, x; R) = lc(\vec{\mathcal{G}}, x; R)$, for any $R \subseteq \mathcal{V}$ and $x \in \mathcal{V} \backslash R$. The symbol $\vec{\mathcal{G}}$ will hereafter be used to represent the directed counterpart of an undirected graph $\mathcal{G}_0$. It is to be noted that the main idea of this section is borrowed from [86].

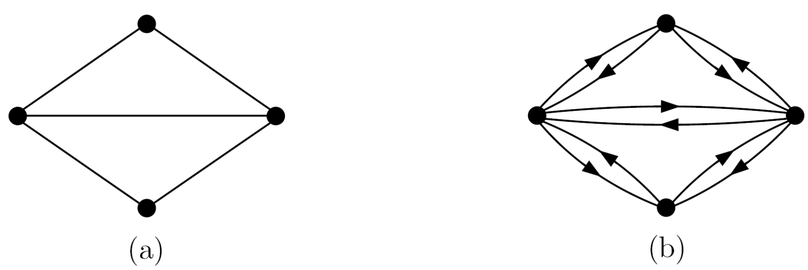


Figure 4.2: (a) An undirected graph $\mathcal{G}_0$, and (b) its directed counterpart $\vec{\mathcal{G}}$.

Lemma 4.1. *Let X and Y be two disjoint subsets of the vertex set of $\vec{\mathcal{G}}$; then*

$$d^-_{\vec{\mathcal{G}}_Y}(X) = d^-_{\vec{\mathcal{G}}_X}(Y).$$

Proof: The proof follows immediately on noting that for any edge from Y to X in $\vec{\mathcal{G}}$, there exists an edge in the opposite direction (from X to Y). $\square$

Lemma 4.2. *Let X and Y be two intersecting subsets of $\mathcal{V}$ and define $Z = \mathcal{V}\backslash\{X \cup Y\}$; then*

$$d^-_{\vec{\mathcal{G}}}(X) + d^-_{\vec{\mathcal{G}}}(Y) = d^-_{\vec{\mathcal{G}}}(X\backslash Y) + d^-_{\vec{\mathcal{G}}}(Y\backslash X) + 2d^-_{\vec{\mathcal{G}}_Z}(X \cap Y).$$

Proof: It is straightforward to show that

$$\begin{aligned}
d^-_{\vec{\mathcal{G}}}(X) &= d^-_{\vec{\mathcal{G}}_Z}(X\backslash Y) + d^-_{\vec{\mathcal{G}}_{Y\backslash X}}(X\backslash Y) + d^-_{\vec{\mathcal{G}}_Z}(X \cap Y) + d^-_{\vec{\mathcal{G}}_{Y\backslash X}}(X \cap Y) \\
d^-_{\vec{\mathcal{G}}}(Y) &= d^-_{\vec{\mathcal{G}}_Z}(Y\backslash X) + d^-_{\vec{\mathcal{G}}_{X\backslash Y}}(Y\backslash X) + d^-_{\vec{\mathcal{G}}_Z}(X \cap Y) + d^-_{\vec{\mathcal{G}}_{X\backslash Y}}(X \cap Y) \\
d^-_{\vec{\mathcal{G}}}(X\backslash Y) &= d^-_{\vec{\mathcal{G}}_Z}(X\backslash Y) + d^-_{\vec{\mathcal{G}}_{Y\backslash X}}(X\backslash Y) + d^-_{\vec{\mathcal{G}}_{X\cap Y}}(X\backslash Y) \\
d^-_{\vec{\mathcal{G}}}(Y\backslash X) &= d^-_{\vec{\mathcal{G}}_Z}(Y\backslash X) + d^-_{\vec{\mathcal{G}}_{X\backslash Y}}(Y\backslash X) + d^-_{\vec{\mathcal{G}}_{X\cap Y}}(Y\backslash X)
\end{aligned} \tag{4.1}$$

From Lemma 4.1, we have

$$d^-_{\vec{\mathcal{G}}_{Y\backslash X}}(X \cap Y) = d^-_{\vec{\mathcal{G}}_{X\cap Y}}(Y\backslash X) \quad \text{and} \quad d^-_{\vec{\mathcal{G}}_{X\backslash Y}}(X \cap Y) = d^-_{\vec{\mathcal{G}}_{X\cap Y}}(X\backslash Y) \tag{4.2}$$

Therefore, the proof follows from (4.1) and (4.2). □

Theorem 4.2. *Let $\vec{\mathcal{G}} = (\mathcal{V}, \vec{\mathcal{E}})$ be the directed counterpart of a given undirected graph $\mathcal{G}_0 = (\mathcal{V}, \mathcal{E}_0)$. Then, all minimal p-deficient sets of $\vec{\mathcal{G}}$ are pairwise disjoint.*

Proof: Let $X_1, X_2 \subset \mathcal{V}$ be two distinct minimal p-deficient sets, and assume $X_1 \cap X_2 \neq \emptyset$. It follows from the definition of a minimal p-deficient set that X_1 and X_2 are intersecting. On the other hand, since $X_1\backslash X_2 \subset X_1$ and $X_2\backslash X_1 \subset X_2$, hence the sets $X_1\backslash X_2$ and $X_2\backslash X_1$ are not p-deficient. From Lemma 4.2

$$d^-_{\vec{\mathcal{G}}}(X_1) + d^-_{\vec{\mathcal{G}}}(X_2) = d^-_{\vec{\mathcal{G}}}(X_1\backslash X_2) + d^-_{\vec{\mathcal{G}}}(X_2\backslash X_1) + 2d^-_{\vec{\mathcal{G}}_Z}(X_1 \cap X_2)$$

where $Z = \mathcal{V}\backslash\{X_1 \cup X_2\}$. The facts that $d^-_{\vec{\mathcal{G}}}(X_1) < p$, $d^-_{\vec{\mathcal{G}}}(X_2) < p$, $d^-_{\vec{\mathcal{G}}}(X_1\backslash X_2) \geq p$, and $d^-_{\vec{\mathcal{G}}}(X_2\backslash X_1) \geq p$ imply $d^-_{\vec{\mathcal{G}}_Z}(X_1 \cap X_2) < 0$. This result is not true because the indegree of a set cannot be negative. This contradiction completes the proof. □

One can use Theorem 4.2 to find the minimal p-link root set without explicitly identifying the minimal p-deficient sets. This is spelled out in the sequel.

Theorem 4.3. *Let R be a p-link root set of $\vec{\mathcal{G}}$. For a vertex $x \in R$, if $lc(\vec{\mathcal{G}}, x; R\backslash\{x\}) \geq p$, then $R\backslash\{x\}$ is a p-link root set as well. Moreover, if $lc(\vec{\mathcal{G}}; R\backslash\{x\}) < p$, then $|R \cap X| = 1$, where X is the minimal p-deficient set and $x \in X$.*

Proof: Assume that $lc(\vec{\mathcal{G}}, x; R\backslash\{x\}) \geq p$. Since R is a p-link root set, it intersects any p-deficient set. It can be shown that either $R\backslash\{x\}$ intersects any p-deficient set too, or there exists a p-deficient set X with $x \in X$ disjoint from $R\backslash\{x\}$. This implies that $lc(\vec{\mathcal{G}}, x; R\backslash\{x\}) \leq d^-_{\vec{\mathcal{G}}}(X) < p$ which is a contradiction. The proof in this case follows from Theorem 4.1. Assume now that $lc(\vec{\mathcal{G}}; R\backslash\{x\}) < p$. This implies that $R\backslash\{x\}$ is not a p-link root set. Since R is a p-link root set, x should belong to a minimal p-deficient set; otherwise, $R\backslash\{x\}$ will intersect any minimal p-deficient set as well. Clearly, $R \cap X \neq \emptyset$ and $R\backslash\{x\} \cap X = \emptyset$, which completes the proof. $\square$

Theorems 4.1 and 4.2 imply that in an undirected information flow graph $\mathcal{G}_0$, a minimal p-link root set contains one vertex from each minimal p-deficient set of $\vec{\mathcal{G}}$. Theorem 4.3 is used next to develop a polynomial-time procedure for finding a minimal p-link root set. It is to be noted that the minimal p-link root set is not necessarily unique.

Algorithm 4.1.

$R = \mathcal{V}$.

for $i = 1$ to n,

  if $lc(\vec{\mathcal{G}}, i; R\backslash\{i\}) \geq p$,

    $R = R\backslash\{i\}$.

  end if

end for

return R.

Using the node duplication technique discussed in Section 3.4 along with the above algorithm, one can find a minimal root set to achieve q-agent controllability in undirected information flow graphs.

4.2 Leader Localization in Directed Information Flow Graphs

The result of Theorem 4.2 is not valid for a general digraph $\mathcal{G}$, that is, the minimal p-deficient sets of $\mathcal{G}$ are not mutually disjoint in general. Therefore, Algorithm 4.1 cannot be employed to find a minimal p-link root set for an arbitrary digraph. As as example, consider the digraph shown in Fig. 4.3. This digraph has two intersecting minimal 2-deficient sets $X_1 = \{1, 2, 3\}$ and $X_2 = \{1, 4, 5\}$. From Theorem 4.1, it can be easily concluded that in this example the minimal 2-link root set is $R = \{1\} \subseteq X_1 \cap X_2$, while Algorithm 4.1 may give a 2-link root set of size more than one, e.g., $R = \{2, 5\}$.

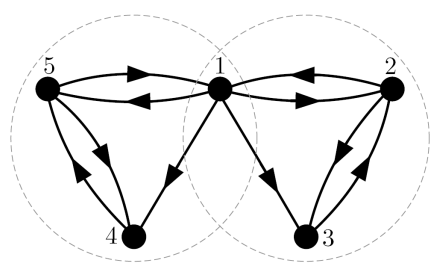


Figure 4.3: A digraph with two intersecting minimal 2-deficient sets.

According to Theorem 4.1, a minimal p-link root set is a minimal set intersecting all the minimal p-deficient sets. Therefore, one approach to find a minimal p-link root set in a digraph is constructing all minimal p-deficient sets, and then finding a minimal set R that intersects all of them.

As an efficient approach for finding p-deficient sets of a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, one can consider all sets $F \subset \mathcal{E}$ of size less than p, and find all sets $X \subset \mathcal{V}$ for which $F = \partial^-_{\mathcal{G}}(X)$. Using this approach, the following algorithm of complexity $\mathcal{O}(|\mathcal{E}|^p)$ gives the family of all p-deficient sets S, where $S(i) = X_i$ denotes the ith p-deficient set in S. Although the complexity of the algorithm grows exponentially with the value of p, the set S can be constructed in polynomial-time in terms of the size of the graph, because p is a fixed and typically small number.

Algorithm 4.2.

$S = \emptyset$ and $j = 1$.

for $k = 1$ to $p - 1$,

  Let Z be the family of all subsets of $\mathcal{E}$ of size k.

  for $i = 1$ to $|Z|$,

    Let $H = \{y|(x, y) \in Z(i)\}$ and $T = \{x|(x, y) \in Z(i)\}$.

    while there exists an edge $(u, v) \in \partial^-_{\mathcal{G}}(H)$ and $(u, v) \notin Z(i)$,

      $H = H \cup \{u\}$.

    end while

    if $T \cap H = \emptyset$,

      $S(j) = H$.

      $j = j + 1$.

    end if

  end for

end for

return S.

The members of S are not necessarily minimal, i.e., it is possible that $X_i \subset X_j$, for some i and j. To construct the family of all minimal p-deficient sets of $\mathcal{G}$, the set S must be modified by removing all sets X_j, if there exists a set X_i such that $X_i \subset X_j$. Clearly, this can be performed in polynomial-time. The modified version of S consisting of only minimal p-deficient sets is hereafter denoted by $\hat{S}$. By applying Algorithm 4.2 to the digraph shown in Fig. 4.3, one arrives at $S = \{\{1,2,3\},\{1,2,3,5\},\{1,4,5\},\{1,2,4,5\}\}$. Since $X_1 \subset X_2$ and $X_3 \subset X_4$, the family of all minimal p-deficient sets will be $\hat{S} = \{X_1, X_3\} = \{\{1,2,3\},\{1,4,5\}\}$.

The following two theorems present some useful properties of minimal p-deficient sets of a digraph $\mathcal{G}$, and will be used later to find a minimal p-link root set of $\mathcal{G}$. It is to be noted that the main idea of these theorems is borrowed from [87].

Theorem 4.4. *Let $\{X_1, X_2, \ldots, X_k\} \subseteq \hat{S}$ be any group of pairwise intersecting members of $\hat{S}$; then, $\bigcap_{i=1}^{k} X_i \neq \emptyset$.*

Proof: To prove the theorem by contradiction, assume $W = \{X_1, \ldots, X_\ell\} \subseteq \hat{S}$ is a set with the smallest size ℓ for which the statement of the theorem does not hold. This implies that $\bigcap_{X_i \in W} X_i = \emptyset$, and that for any $W' \subset W$, $\bigcap_{X_i \in W'} X_i \neq \emptyset$. Let $Y_j = \bigcap_{X_i \in W (i \neq j)} X_i$, for $j = 1, \ldots, \ell$. It is clear that Y_i's are nonempty pairwise disjoint sets. Also, $\partial_{\mathcal{G}}^{-}(Y_i) \cap \partial_{\mathcal{G}}^{-}(Y_j) = \emptyset$ for $i \neq j$. This, along with the facts that $\partial_{\mathcal{G}}^{-}(Y_i) \subseteq \bigcup_{j \neq i} \partial_{\mathcal{G}}^{-}(X_j)$ and X_i's are p-deficient, implies that

$$\sum_{i=1}^{\ell} d_{\mathcal{G}}^{-}(Y_i) \leq \sum_{i=1}^{\ell} d_{\mathcal{G}}^{-}(X_i) < p\ell.$$

This means that $d_{\mathcal{G}}^{-}(Y_i) < p$ for some $i \in \mathbb{N}_\ell$; i.e., Y_i is also p-deficient. This contradicts the minimality assumption for X_j $(i \neq j)$, on noting that $Y_i \subseteq X_j$ for $i \neq j$. This contradiction completes the proof. $\square$

Let $\mathcal{G}^* = (\mathcal{V}^*, \mathcal{E}^*)$ be an undirected graph of order $|\hat{S}|$, with a one-to-one correspondence between its vertices and the members of $\hat{S}$, that is, $X_i \in \hat{S}$ corresponds to vertex i in $\mathcal{G}^*$. In this graph, two distinct vertices $i, j \in \mathcal{V}^*$ are adjacent if the corresponding members of $\hat{S}$ (i.e., X_i and X_j) have a nonempty intersection. Then, finding the minimal p-link root set is equivalent to finding the minimum covering by cliques for $\mathcal{G}^*$. Such a minimal root set can be obtained by considering a common element of the sets corresponding to each clique in this minimum covering (the existence of such a common element for each clique is guaranteed by Theorem 4.4). For instance, consider the information flow graph $\mathcal{G}$ shown in Fig. 4.4, and assume it is desired to find a minimal 2-link root set in it. From Algorithm 4.2, the family of all minimal 2-deficient sets is given by $\hat{S} = \{X_1, X_2, X_3\}$, where $X_1 = \{1, 6, 7\}$, $X_2 = \{3, 4, 8\}$, and $X_3 = \{8, 9, 13, 14\}$. Since the sets X_2 and X_3 are intersecting,

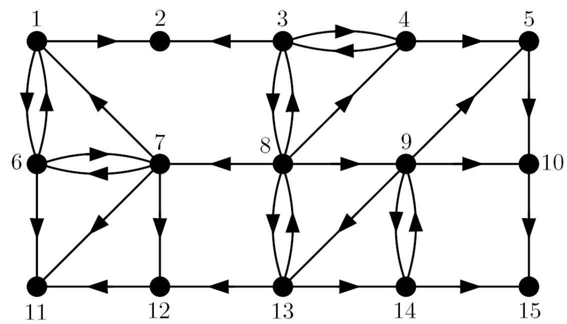


Figure 4.4: An information flow graph of a group of 15 agents.

taking two vertices, one from X_1 and another one from $X_2 \cap X_3$, say $\{6, 8\}$, gives a minimal 2-link root set for $\mathcal{G}$. A special property of $\mathcal{G}^*$ presented in the next theorem enables one to develop a polynomial-time procedure for finding a minimum covering by cliques for $\mathcal{G}^*$.

Theorem 4.5. *The graph $\mathcal{G}^*$ constructed from the family of all minimal p-deficient sets of a digraph $\mathcal{G}$ is chordal.*

Proof: Assume $\mathcal{G}^*$ is not chordal, and has a cycle of length greater than three without any chord. Let the sequence of vertices $1, 2, \dots, \ell$ represent this chordless cycle, and define $Y_i = X_i \cap X_{i+1}$ for any $i \in \mathbb{N}_\ell$. Let also $X_{\ell+1} = X_1$, where X_i is the minimal p-deficient set of $\mathcal{G}$ corresponding to vertex $i \in \mathcal{V}^*$. It is evident that Y_i's are nonempty pairwise disjoint sets. This implies that $\partial^-_{\mathcal{G}}(Y_i) \cap \partial^-_{\mathcal{G}}(Y_j) = \emptyset$ for $i \neq j$. Also, $\partial^-_{\mathcal{G}}(Y_i) \subseteq \partial^-_{\mathcal{G}}(X_i) \cup \partial^-_{\mathcal{G}}(X_{i+1})$. The above derivation together with the fact that X_i's are p-deficient, yields

$$\sum_{i=1}^{\ell} d_{\mathcal{G}}^{-}(Y_i) \leq \sum_{i=1}^{\ell} d_{\mathcal{G}}^{-}(X_i) < p\ell.$$

This implies that $d_{\mathcal{G}}^{-}(Y_i) < p$, for some $i \in \mathbb{N}_\ell$, i.e., Y_i is p-deficient. This contradicts the initial assumption of the minimality of X_i, as $Y_i \subseteq X_i$. This contradiction completes the proof. □

In [90], a polynomial-time algorithm is proposed to find the minimum covering by cliques for chordal graphs. In order to use this algorithm, it is first required to rename the vertices and orient the edges of $\mathcal{G}^*$ as follows.

Algorithm 4.3.

$\upsilon = |\mathcal{V}^*|$.

Mark all vertices of $\mathcal{G}^*$.

while $\mathcal{G}^*$ has more than one marked vertex,

    if there exists a marked vertex i such that all of its marked neighbors form a clique, then,

        Rename i to υ and let $\phi(\upsilon) = i$.

        Unmark i and let $\upsilon = \upsilon - 1$.

    end if

end while

Rename the remaining vertex, j, to 1 and let $\phi(1) = j$.

for any edge $e_{ij} \in \mathcal{E}^*$,

    Orient e_{ij} from $\min(i, j)$ to $\max(i, j)$.

end for

Let $\vec{\mathcal{G}}^* = \mathcal{G}^*$.

return $\vec{\mathcal{G}}^*$.

Let N_i^* be the set of all neighbors of vertex i in $\vec{\mathcal{G}}^*$, and define a sequence of vertices $n_1, n_2, \ldots, n_\nu$ as follows:

$$n_1 = |\mathcal{V}^*| \text{ and } n_k = \max\{i \in \mathcal{V}^* |\; i < n_{k-1},\ i \notin \cup_{j=1}^{k-1} N_{n_j}^*\}.$$

For any $i \in \mathbb{N}_\nu$, let $P_i = N_{n_i}^* \cup \{n_i\}$; then, $C_i = \bigcup_{j\in P_i} \phi(j)$ is a clique and the family of $\{C_1, C_2, \ldots, C_\nu\}$ is a minimum covering by cliques for the graph $\mathcal{G}^*$.

By retrieving the original labels of the vertices of $\mathcal{G}^*$ using the function ϕ, one can construct a minimal p-link root set of $\mathcal{G}$ as follows: Let $Y_i = \cap_{j\in C_i} X_j$, for any $i \in \mathbb{N}_\nu$. From Theorem 4.4, $Y_i \neq \emptyset$, for any $i \in \mathbb{N}_\nu$; therefore, by taking one element from each Y_i, a minimal p-link root set of $\mathcal{G}$ is obtained.

As an example, consider a digraph $\mathcal{G}$ with eight minimal p-deficient sets $X_1, \ldots, X_8$, where the graph $\mathcal{G}^*$ corresponding to its minimal p-deficient sets is shown in Fig. 4.5. Algorithm 4.3 gives a digraph $\vec{\mathcal{G}}^*$ obtained from $\mathcal{G}^*$ by renaming

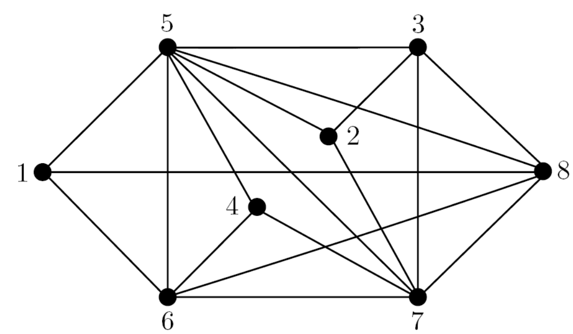


Figure 4.5: The graph $\mathcal{G}^*$ obtained from the family of all minimal p-deficient sets of the digraph $\mathcal{G}$.

its vertices and orienting the edges. Choose the vertices of $\mathcal{G}^*$ in the order given by $(1, 2, 3, 4, 5, 6, 7, 8)$. Then, $\phi(8) = 1$, $\phi(7) = 2$, $\phi(6) = 3$, $\phi(5) = 4$, $\phi(4) = 5$, $\phi(3) = 6$, $\phi(2) = 7$, and $\phi(1) = 8$. The digraph $\vec{\mathcal{G}}^*$ shown in Fig. 4.6 is the outcome of the above procedure. Note in $\vec{\mathcal{G}}^*$ that $n_1 = 8$, $N_8^* = \{1, 3, 4\}$, $n_2 = 7$, $N_7^* = \{2, 4, 6\}$, and $n_3 = 5$, $N_5^* = \{2, 3, 4\}$. Then, $P_1 = \{1, 3, 4, 8\}$, $P_2 = \{2, 4, 6, 7\}$, and $P_3 = \{2, 3, 4, 5\}$. Therefore,

$$C_1 = \{\phi(1), \phi(3), \phi(4), \phi(8)\} = \{8, 6, 5, 1\}$$

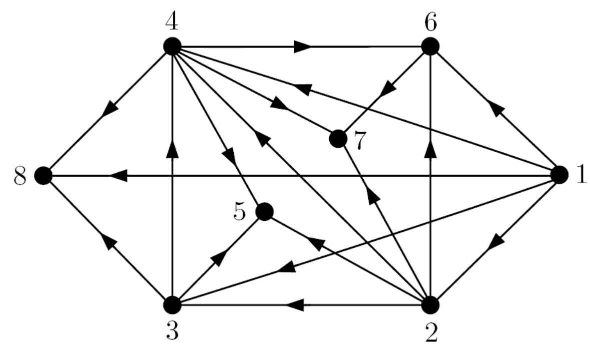


Figure 4.6: The digraph $\vec{\mathcal{G}}^*$ obtained from $\mathcal{G}^*$.

$$C_2 = \{\phi(2), \phi(4), \phi(6), \phi(7)\} = \{7, 5, 3, 2\}$$
$$C_3 = \{\phi(2), \phi(3), \phi(4), \phi(5)\} = \{7, 6, 5, 4\}$$

Now, let $Y_1 = X_8 \cap X_6 \cap X_5 \cap X_1$, $Y_2 = X_7 \cap X_5 \cap X_3 \cap X_2$, $Y_3 = X_7 \cap X_6 \cap X_5 \cap X_4$, and assume $r_1 \in Y_1$, $r_2 \in Y_2$ and $r_3 \in Y_3$, Therefore, $R = \{r_1, r_2, r_3\} \subset \mathcal{V}$ is a minimal p-link root set for $\mathcal{G}$.

Using the technique discussed in Section 3.4, one can find a minimal root set to achieve q-agent controllability in an arbitrary digraph.

Chapter 5

Conclusions and Future Work

5.1 Summary

The structural controllability of a leader-follower team of single integrator agents is investigated in this work. The information flow graph of the system is assumed to be directed, as a general representation of this type of interconnected network. Graphical interpretations of structural controllability of a single leader system based on the information flow graph are provided.

Three notions are introduced in Chapter 3 as quantitative measures for the controllability of the system subject to communication links and/or agents failure. The first notion is p-link controllability, which deals with controllability preservation in the case of communication links failure. The second one is q-agent controllability, which is concerned with agents' loss. Finally, controllability of the network in the case of simultaneous failure in some communication links and agents is studied by introducing the notion of joint-(p, q) controllability. A polynomial-time algorithm is subsequently presented to find the values of p and q, for any given p-link and q-agent controllable information flow graph. Necessary and sufficient conditions on the structure of the information flow graph are also derived to ensure the structural controllability in the case of communication links or agents failure.

The results obtained are then extended to the multiple-leader case. In particular, in Chapter 4, it is shown that a proper selection of agents as leaders can improve the reliability of the network by increasing the degree of controllability of the network. The problem of leader localization is introduced, which is concerned with finding a minimal set of agents whose selection as leaders results in a p-link or q-agent controllable system, for given p and q. Polynomial-time algorithms are also provided to find the underlying minimal sets. The problem is investigated for directed information flow graphs first, and is then extended to the case of undirected graphs.

5.2 Suggestions for Future Work

The research presented in this thesis provides a foundation for future research in the field of cooperative control systems, where it is desired to design a reliable and fault tolerant network of multi-agent systems. Finding minimum number of communication links whose addition to a given information flow graph results in a p-link, q-agent, or joint-(p, q) controllable system, considering agents with double-integrator or nonholonomic dynamics, weight assignment for the communication links to achieve the fastest rate of convergence to desired configuration, and extending the results to information flow graphs with (randomly) switching topologies are some of the problems which can be addressed in the future.

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