Lower Bounds on Broadcast Function for $n = 23,\ 24$ and 25.

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ABSTRACT

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The exponential growth of interconnection networks transformed the communication process into an important area of research. One of the fundamental types of communications is one-to-all communication, i.e. broadcasting. Broadcasting is an information dissemination process, in which one node, called the originator, disseminates a message to all other nodes by placing a series of calls along the communication lines of the network. One of the main problems in broadcasting is the minimum broadcast graph problem. Many studies attempted to investigate ways to construct sparse networks, in which broadcasting can be completed in the minimum possible time from any originator. Minimum broadcast graphs are the sparsest possible networks of this type, which have the minimum number of communication lines denoted by $B(n)$ where n is the number of nodes in networks. Until now, $B(n)$ is known only for a few general broadcast graph families and for some particular, mostly small values of n. For all $n \leq 32$ $B(n)$ is known, except for $n = 23,\ 24$ and 25. These cases were subject of our study. In this thesis we introduce the case-by-case analysis of each of these graphs in order to determine the lower bound on $B(n)$. Our results improved previously known lower bounds and can be used in future studies to construct minimum broadcast networks on $23,\ 24$ and 25 nodes.

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Contents

List of Figures

Chapter 1

Introduction

It is almost impossible to imagine today's modern society without computer networks. Communication between people over the Internet has become an integral part of our daily lives. Calling from wireless cellular phones, sending emails or simply surfing the net; all require passing information over computer networks. Moreover, computer networks are very useful for parallel computing and the design of distributed systems. Since the efficiency of a single CPU has virtually reached its limit [38], computer architecture has moved more and more toward parallelism. In parallelism a complex problem is divided into multiple subproblems; each subproblem is handled by a single processor. This approach enables speedy resolution of very difficult tasks. However, the different processors working in parallel eventually need to exchange the data among themselves, which is achieved by sending a message using an interconnection network.

An interconnection network organization has a significant impact on multiprocessor system performance and often determines its overall efficiency. Two methods exist to facilitate the information dissemination process. One is to reduce the amount of data being transferred, which can be achieved by data compression or by eliminating redundant information. The second method is to minimize the delay in information spreading by either designing efficient algorithms of message transfer or by designing efficient network topologies.

One of the fundamental types of information dissemination is broadcasting. Broadcasting is the process in which a message is distributed from one node, called the originator, to all other nodes, that are members of the network. Broadcasting is accomplished by placing series of calls over the communication channels of the network. Once informed, the nodes distribute the message to other uninformed neighbors. Normally, this process takes place in discrete time units, sometimes called rounds. The goal of efficient network architecture and efficient broadcasting algorithm is to ensure that this type of communication is completed as fast as possible, subject to the following constraints:

- Each call involves only two nodes;
- Each call requires one time unit;
- Each node can participate in only one call per unit of time;
- Each node can distribute a message only to an adjacent node.

Generally, a network can be modeled as a connected graph $G = (V, E)$, where set V represents the set of all nodes and set E represents the set of all communication lines in a network. Two vertices $u, v \in V$ are called *adjacent* nodes if there is an edge $e \in E$, such that $e = (u, v)$. Vertices u and v are called *neighbors* in a given network. The *degree* of a vertex v is defined as the number of incident edges of that vertex and is denoted by $deg(v)$. The *degree* of a graph G is the maximum degree among all vertices in this graph. A *path* in a network is a sequence of communication lines and nodes, in which a message can travel from the originator node along the sequence of lines and nodes to the target node. All of the nodes and communication lines in a path are connected to one another. The number of communication lines in a path is called the *length* of the path. The shortest path between a vertex u and a vertex v is called the *distance* between u and v, which is denoted by $dist(u, v)$. The *diameter* of a graph G, denoted by D is the maximum distance between any pair of vertices in the graph G. If there is at least a path between any two nodes in G, G is said to be a *connected network*. Two networks (graphs) G and H are *isomorphic* if H can be obtained from G by relabeling the nodes - that is a one-to-one correspondence between the nodes of G and those of H, such that the number of communication lines joining any pair of nodes in G is equal to the number of edges joining the corresponding pair of nodes in H. A *Hamilton cycle* is a cycle that passes through every vertex in a graph. A graph with such a cycle is called *Hamiltonian graph*. Typically one thinks of a Hamiltonian graph as a cycle with a number of other edges (called cords of the cycle). A *star graph* (or star network) consists of one vertex v which is incident with every edge and all other vertices incident on exactly one edge whose other vertex is v. v is said to be the center of the star graph. In a graph G, a set $S \subset V(G)$ is a *dominating set* if every vertex not in S has a neighbor in S. And if a node v is a member of S or v has at least a neighbor in S, v is said to be covered by S.

Let vertex u be a message originator. The minimum number of time units required to distribute a message from u to all other vertices is called the *broadcast time* of vertex u and is denoted as $b(u)$. The broadcast time of the graph G is defined as the maximum broadcast time from any vertex in the graph: $b(G) = max\{b(u)|u \in V\}$. The broadcast time of any graph G is $b(G) \geq \lceil \log_2 n \rceil$, since during each time unit the number of informed nodes can be at most double. A graph in which broadcasting may be accomplished in $\lceil \log_2 n \rceil$ time units is called a *broadcast graph*. The classic example of a broadcast graph, where broadcasting is completed in the shortest time possible, is the complete graph K_n, $b(K_n) = \lceil \log_2 n \rceil$, yet K_n is not minimal with respect to the number of edges. Hence, we can remove edges from K_n and still have a graph G with n vertices such that $b(G) = \lceil \log_2 n \rceil$.

A *Broadcast function* $B(n)$ is defined as a minimum number of edges in any broadcast graph on n vertices. The graph that contains only the minimum number of edges is called minimum broadcast graph - *mbg*. A minimum broadcast graph has very important practical implications; it represents the cheapest possible architecture to build a network, in which broadcasting is accomplished as quickly as possible. A set of all calls during a broadcast is called a *broadcast scheme* or *broadcast protocol*. A broadcast protocol shows how a message is disseminated in a network from an originator. Let node u in graph G be the originator of the message. The broadcast protocol of u is a rooted spanning tree in which u is the root and all the communication lines are labeled with the transition time. In a broadcast protocol, each communication line is used exactly once and the message is always transmitted from a parent to a child. Research for efficient broadcasting has focused on two main problems. The first is finding a broadcast time in a given graph and the second is designing minimum broadcast graphs where broadcast time is known to be $b(G) = \lceil \log_2 n \rceil$. Determining $b(u)$ for an arbitrary originator u in an arbitrary graph G is NP-complete problem, which was proved in [37]. The second problem is equally difficult. There is no known feasible method for determining $B(n)$ for an arbitrary graph G with n nodes and there is no known method of direct construction of mbg. In fact, this is a very difficult process, since in the worst case, one must check that every vertex has broadcast protocol with broadcast time $\lceil \log_2 n \rceil$. The most difficult part is to prove that there is no other broadcast graphs on n vertices with less number of edges. As such, most researchers in this area have concentrated on constructions which combine several known mbgs to create new ones. For this reason, it is very important to design mbg and determine values of $B(n)$ for small n, so that it can be used to construct larger mbgs. Until now, values of $B(n)$ have been determined for all $n \leq 32$ except the three values: when $n = 23$, $n = 24$ and $n = 25$. In [31] the authors analyzed minimum degree graph property and indicated that a lower bound on $B(23)$ is 33, while for $n = 24$ and $n = 25$, good lower bounds on $B(n)$ are still unknown. The best results for lower bound on $B(n)$ known so far are received using the formula from [36], described in section 2.4. The upper bound on $B(n)$ is obtained based on the graphs constructed in [3].

n	Previous best lower bounds	Upper bounds	Ref.
23	33	34	[3]
24	27	36	[3]
25	28	40	[3]

Table 1: Known lower and upper bounds on $B(n)$ for $n = 23,\ 24$ and 25.

The research conducted for this thesis significantly improves the existing lower bounds. In fact, our newly obtained lower bounds are very close to the upper bound on $B(23),\ B(24)$ and $B(25)$. Dispensing with a number of possibilities of constructing such graphs which existed earlier, significantly increases the chances of generating mbgs.

n	Our new lower bounds	Previous best lower bounds	Ref.
23	33	33	[31]
24	35	27	[36]
25	38	28	[36]

Table 2: New and previous lower bounds on $B(n)$ for $n = 23,\ 24$ and 25.

The thesis is organized as follows: In chapter 2 the related literature with previously obtained results is reviewed. In chapter 3 our research work is introduced, which is the improvement of theoretically calculated lower bound on $B(23)$, $B(24)$ and $B(25)$. In chapter 4 the pseudocode of the implemented program is presented, which was used to get the results. Lastly, in chapter 5 the conclusion and future work are discussed.

Chapter 2

Previous Results

It is obvious that for a complete graph K_n with $n \geq 2$ vertices, $b(K_n) = \lceil \log_2 n \rceil$, since all vertices are connected and the number of informed nodes at every round of calls can be doubled. However, a complete graph is not an optimal graph with respect to the number of edges. Many studies have been initiated to design graphs and find $B(n)$, in an effort to ensure minimum broadcasting time $b(G) = \lceil \log_2 n \rceil$.

2.1 Broadcast Graphs for known values of $B(n)$

Previous work has shown that there is no general method to find $B(n)$ for an arbitrary value of n, and there is not a known method to construct mbgs. For small values of n mbg can be found by exhaustive case analysis, but when n becomes large, the number of possible graphs grows exponentially and this technique is no longer useful

[9].

The structures of mbg and $B(n)$ are known only for several values of n. It can be proved that for $n = 2^k$, $B(n) = k2^{k-1}$. In any mbg with 2^k vertices each vertex must have a degree at least k in order to complete the broadcasting in k rounds. If a vertex has degree less than k then in k time units that vertex may inform at most $2^k - 1$ vertices. To prove this we let $G = (V, E)$ be a mbg on 2^k vertices and u be a broadcast originator with $deg(u) = k - 1$. Let i be the time unit when each of those neighbours is informed from u. The maximum number of vertices that can be informed from such neighbour is 2^{k-i}, therefore the total number of informed vertices from u in k rounds would be: $\sum_{i=1}^{k-1} 2^{k-i} + 1 = 2^k - 1$, less than the number of nodes in a graph - 2^k. Thus, $deg(u) \geq k$ for any vertex $u \in V$.

Therefore, since each vertex must have a degree at least k, number of edges is: $B(2^k) \geq k(2^k)/2 = k2^{k-1}$. This lower bound is attained by three non-isomorphic families of minimum broadcast graphs:

- the hypercube of dimension k [9]
- the recursive circulant $G(2^k, 4)$ [34]
- Knödel graph $W_{k,2^k}$ [26]

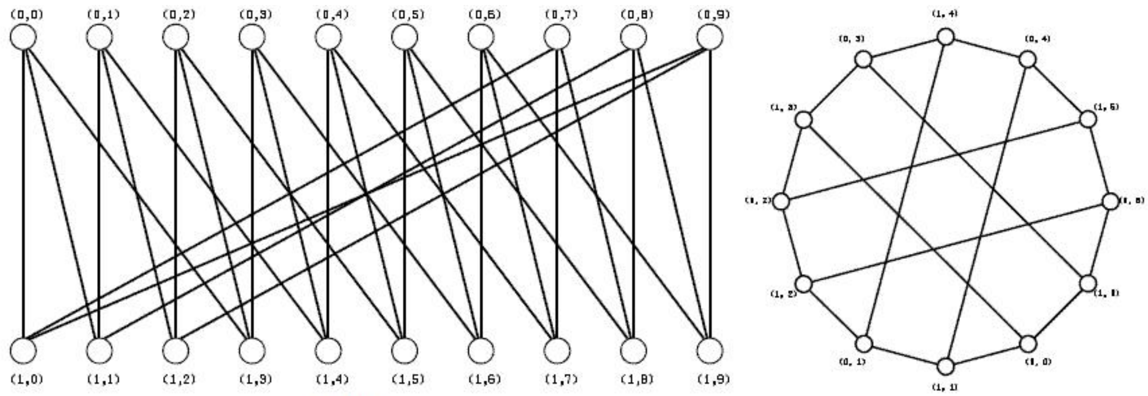
For $k = 0$ the mbg is a single vertex. For all $k > 0$ we take two copies of minimum broadcast graphs on 2^{k-1} vertices and connect every vertex from one graph to the corresponding vertex in another graph. The result of such construction is hypercube H_k of dimension k. The broadcast scheme for such a graph was described in [9],

[26], [29]. Let H_k be the k dimensional hypercube on 2^k vertices, where each vertex corresponds to a k-bit binary string. The two vertices are linked with an edge if and only if their binary strings differ by precisely one bit. In k-dimensional hypercube graph, the broadcast can be performed in $k = \lceil \log_2 n \rceil$ rounds by using the following scheme: in round i, each informed vertex sends a message to its neighbour that differs in the i-th bit. These vertices are called i-th dimensional neighbours.

The second family of mbgs includes modified Knödel graphs on $2^k - 2$ vertices.

Knödel graphs have been originally introduced in 1975 [26]. However, the family of Knödel graphs has been formally defined by Fraigniaud and Peters [12]. Since then, they have been widely studied as useful topologies of communication networks, mainly because of their good properties in terms of broadcasting and gossiping [2], [10], [11], [12], [13], [15], [16], [17], [18], [21], [22], [25], [36].

Definition 1. *The Knödel graph on $n \geq 2$ vertices (n is even) and of maximum degree d, $1 \leq d \leq \lfloor log_2 n \rfloor$, is denoted $W_{d,n}$. The vertices of $W_{d,n}$ are the couples (i, j), with $i = 0, 1$ and $0 \leq j \leq \frac{n}{2} - 1$. For every $0 \leq j \leq \frac{n}{2} - 1$ there is an edge between vertex $(0, j)$ and every vertex $(1, j + 2^k - 1 \mod \frac{n}{2})$, for $k = 0,\ 1,\ \ldots,\ d - 1$.*



(a) $W_{3,20}$ (b) $W_{3,12}$

Figure 1: Examples of Knödel graphs

A modified Knödel graph on n nodes is isomorphic to a Knödel graph with degree of $\lfloor \log_2 n \rfloor$ for any even n which is not a power of 2. Modified Knödel graph can be formally defined in the following way [2]:

Definition 2. *Let W_n denote a modified Knödel graph on n vertices (n is even and not a power of 2), $W_n = (V(W_n), E(W_n))$. $V(W_n) = \{(i,j) | i \in \{0,1\}$ and $j \in \{0,\ 1,\ 2,\ \ldots,\ \frac{n}{2}-1\}\}$, $E(W_n) = \{((0,j),\ (1,\ j+2^k-1 \mod \frac{n}{2})) | k = 0,\ 1,\ \ldots,\ \lfloor \log_2 n \rfloor - 1\}$.*

So, based on Definition 1, a modified Knödel graph W_n is a Knödel graph $W_{\lfloor \log_2 n \rfloor, n}$ for any even n which is not a power of 2. Modified Knödel graphs are broadcast networks [27]. Modified Knödel graphs on $2^k - 2$ nodes are minimum broadcast networks [25], [6].

Every vertex in any mbg on $2^k - 2$ vertices must have degree at least $k-1$, therefore the lower bound on $B(n)$ for $n = 2^k - 2$ for all $k \geq 2$: $B(2^k - 2) \geq (k-1)(2^{k-1} - 1)$ [25], [6]. To prove that we let $G = (V, E)$ be a mbg on $2^k - 2$ vertices, let u be the broadcast originator with degree less than $k-1$ and let v_i, $(i = 1, \ \dots, \ k-1)$, be the neighbour of u, which is informed at time i. The number of informed vertices from v_1 is at most 2^{k-1}, the number of informed vertices from v_2 is at most 2^{k-2} and etc., the number of informed vertices of v_{k-2} is at most 2^2. Therefore the total number of vertices that u can inform in k rounds is at most:

$2^{k-1} + 2^{k-2} + \ldots + 2^2 + 1 = \sum_{i=1}^{k-2} 2^{k-i} + 1 = 2^k - 3 < 2^k - 2$

Thus, $deg(u) \geq k-1$ for any vertex $u \in V$

Since every vertex has degree at least $k-1$:

$B(2^k - 2) \geq \frac{(k-1)(2^k-2)}{2}$.

The broadcast protocol for a modified Knödel graph on $2^k - 2$ vertices was described in [25]. Let $H_k = (V, E)$ be a modified Knödel graph on $2^k - 2$ vertices, where V is the set of all vertices: $V = \{v_0, v_1, \ldots, v_{2^k-3}\}$ and E the set of all edges: $E = \{(v_i, v_j); v_i \in V, v_j \in V, (i+j) = (2^r - 1) \mod (2^k - 2), 1 \leq r \leq k-1\}$. v_j is called a neighbour of v_i in dimension j. In order to inform all vertices in k rounds, at round r an informed vertex v_i sends the message to vertex v_j, where $(i+j) = (2^r - 1) \mod (2^k - 2)$ for $1 \leq r \leq k-1$ and $i + j = 1 \mod (2^k - 2)$ for the last round $r = k$. In other words all informed vertices call neighbours in dimension j at time j for $1 \leq j \leq k-1$ and neighbours at dimension 1 at time k. This is called a dimensional broadcast scheme $(1, 2, \ldots, k-1, 1)$.

Another general graph family for which lower bound on $B(n)$ has been determined is mbg on $2^k - 1$ vertices. It has been presented by Labahn [28].

$B(2^k - 1) = \frac{k^2(2^k-1)}{2(k+1)}$.

There are two types of vertices in these broadcast graphs on $n = 2^k - 1$: vertices with degree $k-1$ and vertices with degree at least k. Moreover, for each vertex u with degree $k-1$, there must exist a neighbour v of u such that $d(v) \geq k$, where $d(v)$ stands for degree of vertex v [3, 28]. The number of vertices of degree at least k is at least $\frac{2^k-1}{k+1}$ and the number of vertices with degree $k-1$ is at most $\frac{k(2^k-1)}{k+1}$. All vertices with degree $k-1$ form a Hamiltonian cycle of length $\frac{k(2^k-1)}{k+1}$ and are called ring vertices. Each vertex with degree k forms a star with alternative connection to the ring vertices. In other words this mbg is composed of one ring and $\frac{2^k-1}{k+1}$ copies of star graphs. Vertex of degree k, which is the center of the star, has k neighbours of degree $k-1$; while a vertex with degree $k-1$ has exactly one neighbour of degree k [36]. For cases where $k > 4$, in addition to one edge between ring and star vertices, each ring vertex has also $k-2$ chords, edges that connect one ring vertex with another ring vertex. Figures 2, 3, 4 show mbgs with 7, 15, 31, 63 and 127 vertices ($k = 3, 4, 5, 6, 7$). Consider for example a graph on 31 nodes; we can see the cycle of 25 vertices of degree 4 and 6 star vertices with degree 5. Every star vertex is connected with 5 cycle vertices. Every cycle vertex is connected with two neighbors, one star vertex and one additional vertex on the cycle by a chord $(i, i+4)$ or $(i, i+12), i \in \{0, 1, 2, \ldots, 24\}$

Graph on 63 vertices ($k = 6$) has a similar structure [28] (see Figure 3). It contains 9 vertices with degree 6 $Y = \{54, 55, \ldots, 62\}$, and 53 vertices with degree 5 $X = \{0, 1, 2, \ldots, 53\}$. Those 53 vertices form a Hamiltonian cycle, where every vertex is connected with two cycle neighbors, one star vertex and two more cycle nodes

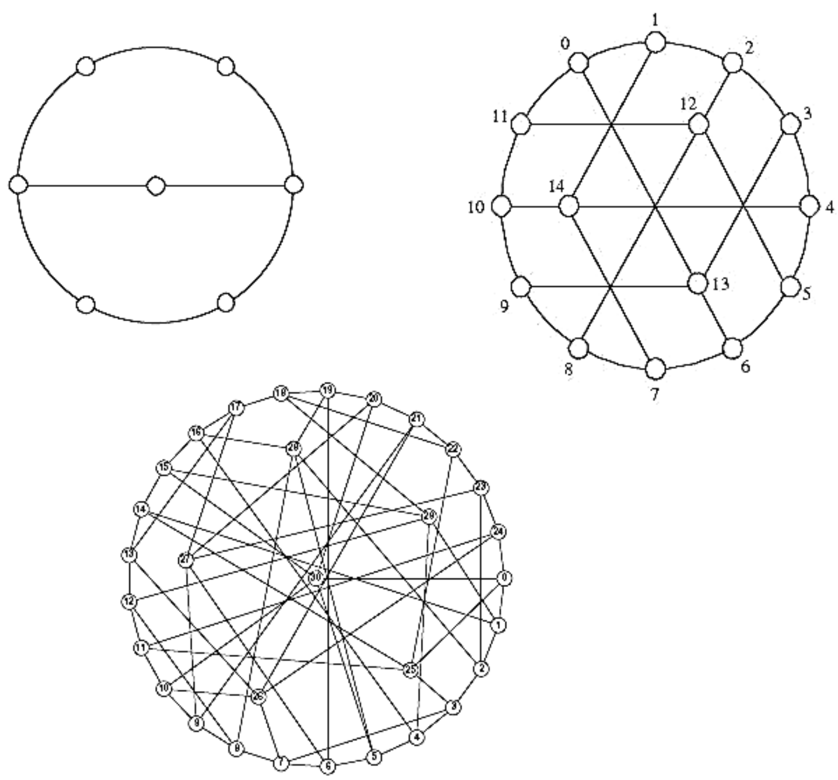


Figure 2: mbgs with 7,15,31 vertices

with interval of 4: $E(X) = \{\{x, x+1\}, \{x+4\}; x \in X\}$. Every star vertex is connected with 6 cycle nodes: $E'(Y) = \{\{x, y\}; x \in X, y \in Y, x = y \bmod 9\}$.

Based on the aforementioned properties, we can calculate the lower bound on $B(n)$ when $n = 2^k - 1$.

Number of vertices with degree at least k:

$n_k \geq \left\lceil \frac{2^k - 1}{k+1} \right\rceil$

Number of vertices with degree $k-1$:

$n_{k-1} \leq (2^k - 1) - n_k = (2^k - 1) - \left\lceil \frac{2^k - 1}{k+1} \right\rceil$

The number of incident edges of vertices with degree at least k is $e_k \geq kn_k$.

The number of incident edges of vertices with degree $k-1$ is $e_{k-1} = (k-1)n_{k-1}$.

Thus, the total number of edges:

$(e_k + e_{k-1})/2 =$

$\frac{1}{2}(\left\lceil \frac{2^k-1}{k+1} \right\rceil k + ((2^k-1) - \left\lceil \frac{2^k-1}{k+1} \right\rceil)(k-1))$

So, for $n = 2^k - 1$, $B(n) \geq \frac{1}{2}(\left\lceil \frac{2^k-1}{k+1} \right\rceil k + ((2^k-1) - \left\lceil \frac{2^k-1}{k+1} \right\rceil)(k-1))$. If $k+1$ is a prime number then $\frac{2^k-1}{k+1}$ is an integer, and we get $B(n) \geq \frac{k^2(2^k-1)}{2(k+1)}$. The upper bound on $B(2^k-1) \leq 2^{k-1}(k-\frac{1}{2})$ was presented in [18]. This upper bound was further improved to $B(2^k-1) \leq 2^{k-1}(k-\frac{1}{2})-(k-1)$ and to $B(2^k-1) \leq 2^{k-1}(k-1)+2\left\lceil \frac{2^{k-1}-1}{5} \right\rceil - \left\lfloor \frac{p+1}{4} \right\rfloor$ in [17] and [41] respectively. Using a similar construction, the minimum broadcast

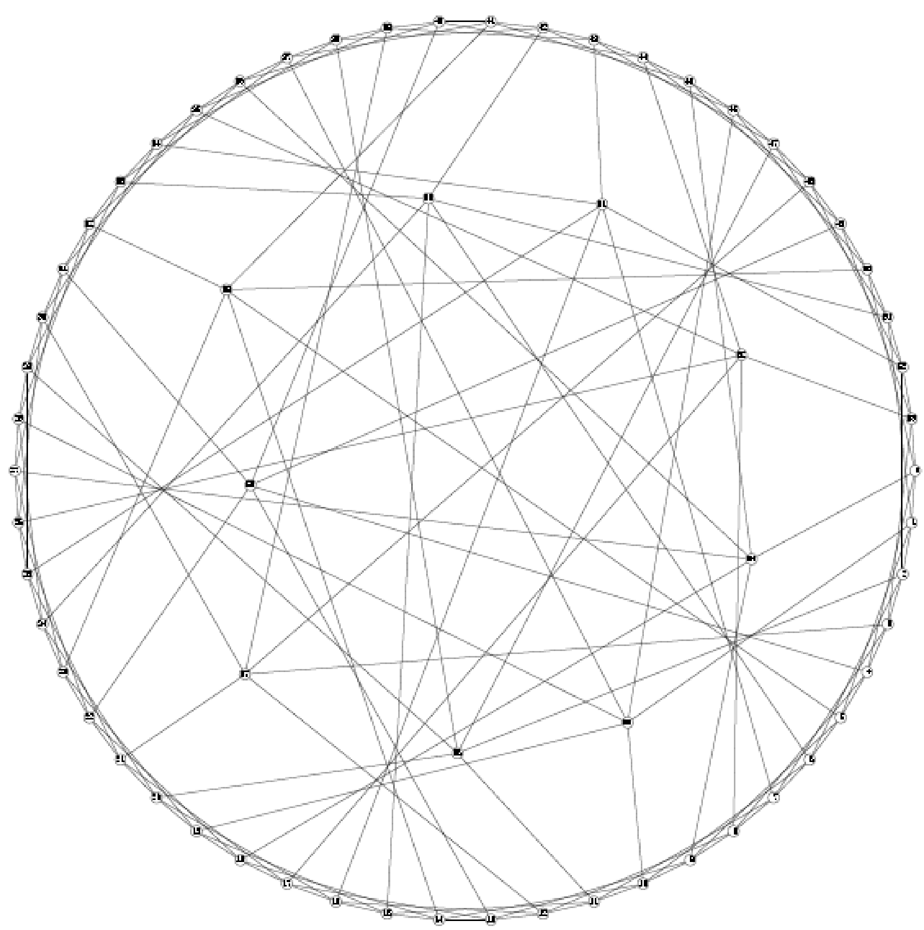


Figure 3: mbg with 63 vertices

graph on 127 vertices was obtained [17], as shown in Fig. 4.

Another very interesting aspect of studying mbgs on $2^k - 1$ vertices is to construct mbg on $2^k - 1$ for all values of k. The main question is how the edges between the ring vertices are and which particular vertices of the ring have edges to a star vertex. Solving this question may help to determine a broadcasting scheme. Having observed graphs where $k + 1$ is a prime number, one can easily see their symmetric structure. This fact suggests the existence of a symmetric way of broadcasting from any originator. This is the subject of the ongoing research and up until now the broadcast scheme for all ring-star mbgs has not been found.

There are two constructed mbgs with a bigger number of vertices, for $k = 10,\ n = 1023$ and for $k = 12,\ n = 4095$ [36]. These graphs have the same structural property as the graphs on 63 and 127 vertices. Consider a graph on 1023 nodes. Let $G = (V, E)$ be a graph $G(1023)$. There are two types of vertices. $\frac{1023}{11} = 93$ star vertices with degree 10 and $1023 - 93 = 930$ cycle vertices with degree 9. Each cycle node has 8 chords that connect a node with its adjacent nodes on the cycle. Let E' be a set of chords. $E' = \{(v, (v \pm 2^0) \bmod 930), (v, (v \pm 2^2) \bmod 930), (v, (v \pm 2^4) \bmod 930), (v, (v \pm 2^6) \bmod 930)\}$. In addition, every cycle vertex is connected to a star vertex. Let E'' be a set of edges between star and cycle vertices. $E'' = \{(v, v \bmod 930 + 93i)\}$, where $0 \leq i \leq 9$, $E' \bigcup E'' = E$.

The broadcasting scheme for these graphs were generated by a computer using the TBA (Tree Bases Algorithm) [36], [24]. However, based on the full symmetry in graph structure, there is a solid foundation to believe that a systematic, well-defined and predictable scheme can be discovered. Finding such protocol of message dissemination for the graphs on 1023 and 4096 graphs may allow us to generalize and define the broadcasting scheme for any other mbg on $2^k - 1$ vertices where $(k+1)$ is a prime number.

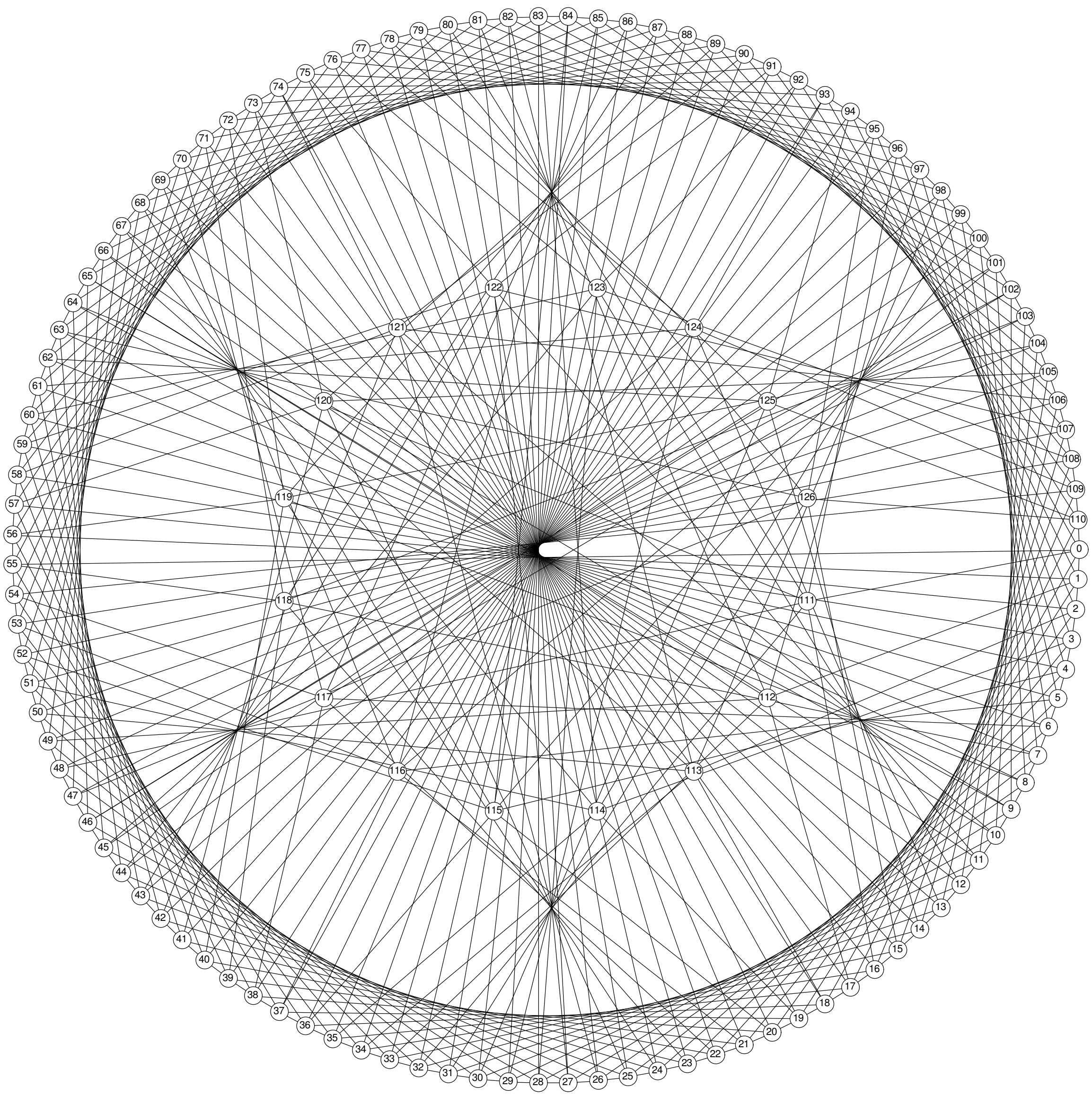


Figure 4: mbg with 127 vertices

Apart from these findings, the number of edges in a minimum broadcast graph is known for only some particular values of n. Several authors studied sparse broadcast graphs, i.e. graphs with a small number of edges. Each of these studies presented non trivial case analysis based on the number of vertices of various degree for each n. In [9], Farley, Hedetniemi, Mitchell and Proskurowski determined the values of $B(n)$ for $n \leq 15$. Figure 5 demonstrates some minimum broadcast networks for small n [12].

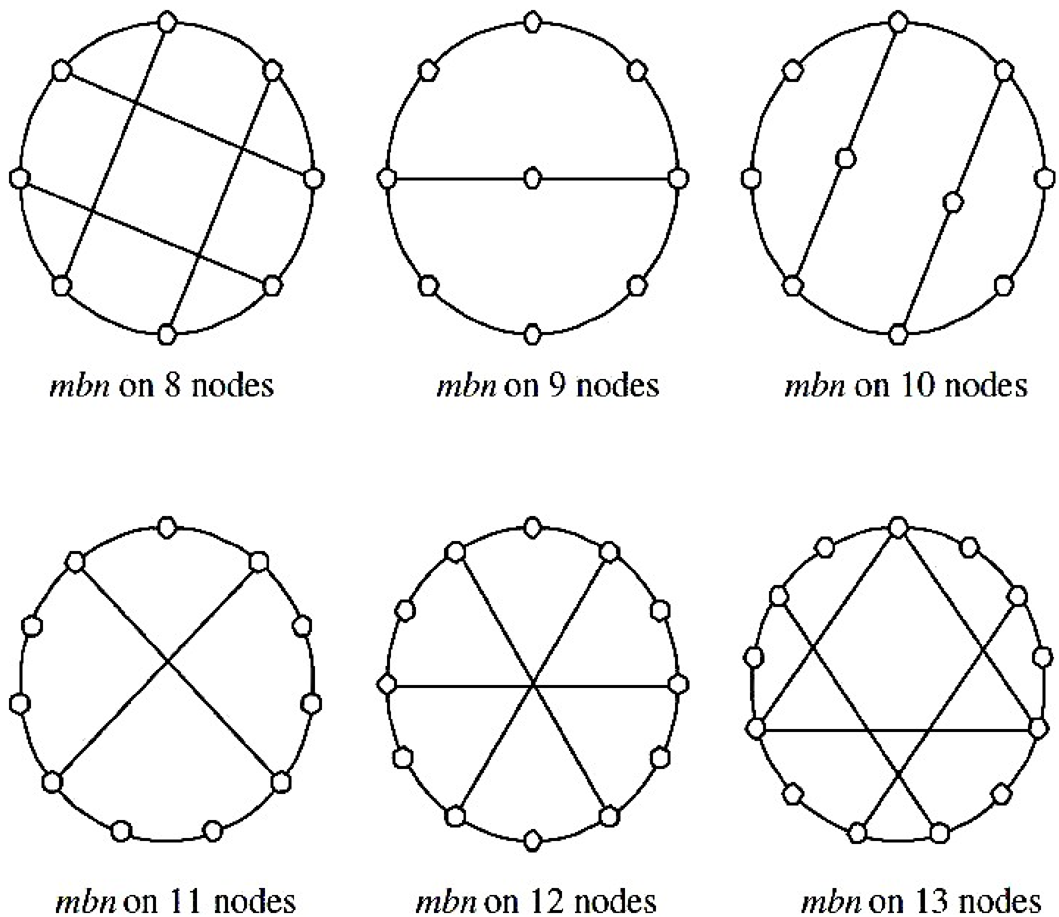


Figure 5: Some minimum broadcast graphs for small n

Mitchell and Hedetniemi [33] determined the value for $B(17)$. Bermond at el. [3] established the values of $B(n)$ for $n = 18,\ 19,\ 30$ and 31, as shown in Figure 6.

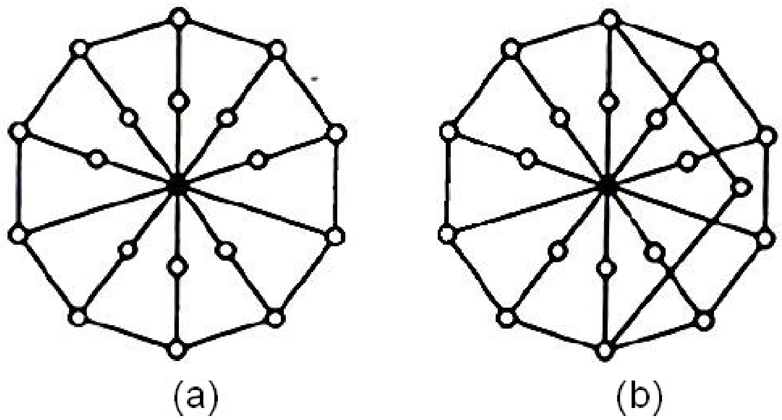


Figure 6: (a) mbg on 18 vertices, (b) mbg on 19 vertices

M. Mahéo and J.-F. Saclé in [32] determined $B(n)$ for $n = 20,\ 21$ and 22. For the next 3 values, $n = 23,\ 24$ and 25, the exact value of $B(n)$ was not determined yet. Bermond at el. [3] constructed broadcast graphs on $n = 23$ with 34 edges, on $n = 24$ with 36 edges and on $n = 25$ with 40 edges. From this we conclude that the upper bounds on $B(23) \leq 34$, $B(24) \leq 36$ and $B(25) \leq 40$. Later in this thesis we will present lower bound on $B(n)$ for $n = 23,\ 24$ and 25, which we have found in our study.

J. G. Zhou and K. M. Zhang constructed mbg on 26 vertices and proved that $B(26) = 42$ [42] as shown in Fig.7.

$B(n)$ for $n = 27,\ 28,\ 29$ was found, in general, by studying other families of minimum broadcast graphs by J.-F. Saclé in [35]. It was proved that for graphs with $n = 2^k - 4$ minimum number of edges is bounded by:

$\left\lceil (k - 2 + \frac{4}{2k+1})\frac{n}{2} \right\rceil \leq B(2^k - 4) \leq (k - 2 + \frac{1}{2})\frac{n}{2}$

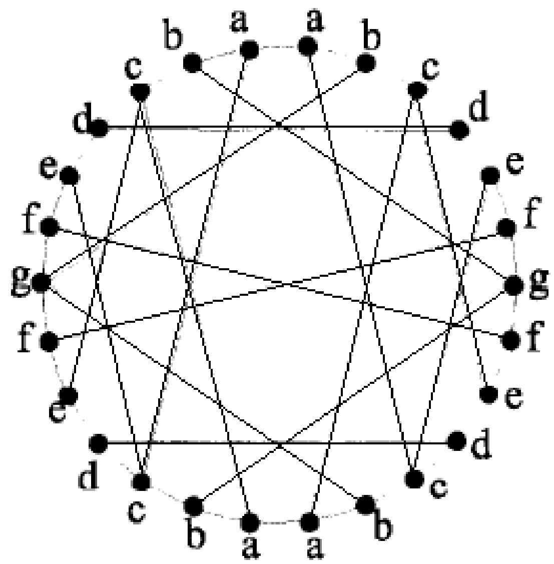


Figure 7: Minimum Broadcast Graph with 26 vertices

For example $B(28) = 48, k = 5$, $B(60) = 130, k = 6$. For graphs with $n = 2^k - 6$ minimum number of edges is bounded by.

$\lceil (k - 2 + \frac{1}{k})\frac{n}{2} \rceil \leq B(2^k - 6)$

For example $B(26) = 42, k = 5$, $B(58) = 121, k = 6$. For graphs with $n = 2^k - 3$ minimum number of edges is bounded by.

$\lceil (k - 2 + \frac{3k-5}{k^2-k-1})\frac{n}{2} \rceil \leq B(2^k - 3)$

For example $B(29) = 52$, $B(61) = 136$. For graphs with $n = 2^k - 5$ minimum number of edges is bounded by.

$\lceil (k - 2 + \frac{2}{2k-1})\frac{n}{2} \rceil \leq B(2^k - 5)$

For example $B(27) = 44$, $B(59) = 124$.

Table 3 summarizes all known values of $B(n)$ along with their references for $n \leq 32$. As we can see, $B(n)$ was known for all $n \leq 32$, except to $n = 23,\ 24$ and 25.

n	B(n)	Ref.
1	0	[9]
2	1	[9]
3	2	[9]
4	4	[9]
5	5	[9]
6	6	[9]
7	8	[9]
9	10	[9]
10	12	[9]
11	13	[9]
12	15	[9]
13	18	[9]
14	21	[9]
15	24	[9]
16	32	[9]
17	22	[33]
18	23	[3]
19	25	[3]
20	26	[32]
21	28	[32]
22	31	[32]
23	33,34	this thesis
24	35,36	this thesis
25	38,39,40	this thesis
26	42	[42]
27	44	[35]
28	48	[35]
29	52	[35]
30	60	[3]
31	65	[3]
32	80	[9]

Table 3: The values of $B(n)$ for $n \leq 32$

Table 4 shows some other known values of $B(n)$ for $n > 32$.

58	121	[35]
59	124	[35]
60	130	[35]
61	136	[35]
62	155	[8]
63	162	[28]
127	389	[17]
1023	4650	[36]
4095	22680	[36]

Table 4: The values of $B(n)$ for $n \leq 32$

2.2 Construction Methods

Since the exact values of $B(n)$ are known only for some particular values, many previous papers have been devoted to finding sparse broadcast graphs and their construction methods. Most authors have focused on constructions which combine several broadcast graphs of smaller size to create new ones of a bigger size. Farley in 1979 [8] introduced a recursive algorithm to construct broadcast graphs on an arbitrary number of vertices n, and proved that the number of edges in the broadcast graph produced by that algorithm is bounded by $(n/2)\lceil \log_2 n \rceil$. Chau and Liestman in [4] proposed a method of constructing broadcast graphs by combining 5, 6 and 7 smaller broadcast graphs. Gargano and Vaccaro developed a method of interconnecting small hypercubes to build up larger broadcast graphs [13]. Bermond et al. [3] discovered four methods to construct broadcast graphs and used them for constructing broadcast graphs for $18 \leq n \leq 63$. Ventura and Weng [39] developed a method based on the concept of aggregated nodes and aggregated edges (which are used to replace ordinary nodes and edges, respectively, of known mbgs, for $9 \leq n \leq 15$) to construct sparse broadcast graphs. Several papers have shown methods to construct broadcast graphs by forming the compound of two or more known broadcast graphs [1], [7], [18], [25]. These methods are proven to be effective for the graphs on n_1 and n_2 vertices. The result of the combined construction is the graph on $n_1 n_2$ vertices. However, these methods cannot be used to form broadcast graphs on n vertices for every n, for example when n is a prime number.

Weng and Ventura [40] proposed a general method that allows systematic vertex deletion. The main idea of this method, called the *doubling procedure*, is a center set node, defined via the so-called official broadcasting. This construction method was also investigated by Dinneen et al. [7], who treated official broadcasting and center node sets in greater detail, using iterative algorithms for these constructions.

Harutyunyan and Liestman in [18] an effective method of constructing broadcast graphs by combining known minimum broadcast graphs on $2^k - 2$ vertices with other known broadcast graphs. The main idea behind this method is to make multiple copies of the same graph on $2^k - 2$, which is called a modified Knödel graph, and combine it with another broadcast graph. The resulted compound is on $c(2^k - 2)$ vertices and m edges, where c is the number of copies of mbg on $2^k - 2$ vertices. The $r + 1$ vertices are then merged into one vertex to form a graph with $c(2^k - 2) - r$ vertices but still m edges. For example, let H_p be a minimum broadcast graph on $(2^k - 2)$ vertices. We take 4 copies of $H_3, p = 3, c = 4$. We also take a cycle graph on 4 vertices $G = C_4$ to connect vertices of each copy of H_3^i in such a way that all odd vertices are connected $(1^i, 3^i, 5^i)$. Such connection produces 3 cycles. Each cycle has 4 edges, providing the total of 12 additional edges. As such, the total number of edges in the combined graph is $4 \times 6 + 12 = 36$. The final step of the construction is to merge the vertices labeled 0 into one vertex. Let $r = 2$. We merge 3 vertices $0^1, 0^2, 0^3$ to form a graph on 22 vertices. If $r = 3$, we merge all $0^1, 0^2, 0^3, 0^4$ to form a graph on 21 vertices as shown in Fig. 8, if $r = 1$. The result is the graph on 23 vertices. All these graphs have 36 edges.

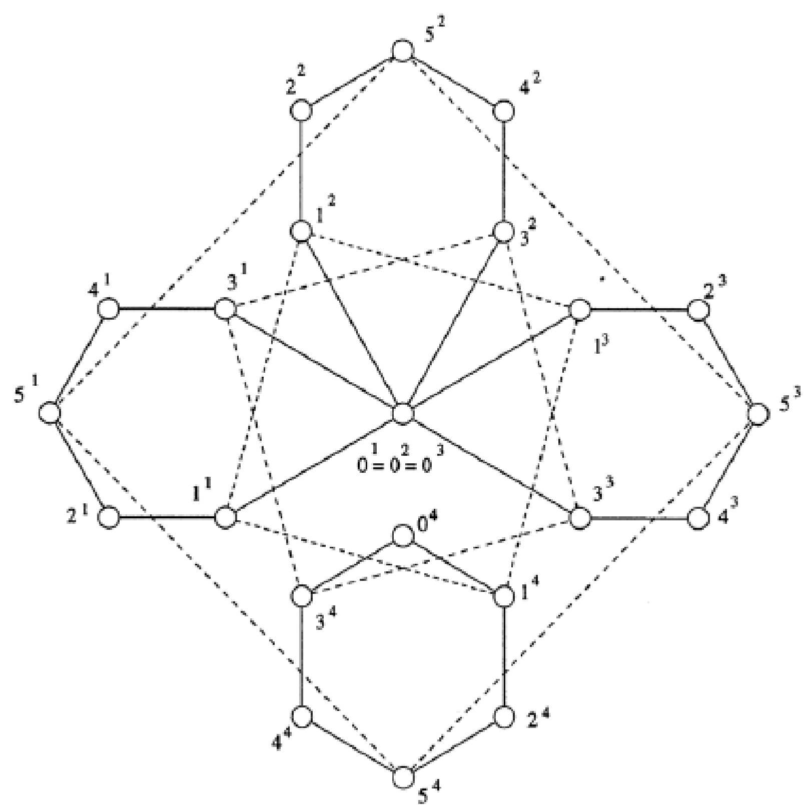


Figure 8: Compound method of construction broadcast graphs

Another method to construct broadcast graphs is described in [17]. This method is based on a vertex addition approach. The idea is to take a modified Knödel graph on $2^k - 2$ vertices. Let $G = (V, E)$ be such graph. We first find a dominating set of the graph $S \subseteq V$, then add a new vertex and connect it with all vertices in dominating set S. The result is the graph with an odd number of vertices $2^k - 1$.

2.3 k-broadcasting Model

The broadcast process we reviewed so far has one constraint: every informed vertex can call only one neighbor at every time unit. This model is called 1-broadcasting. This is a simple model of a more general k-broadcasting process. k refers to number of calls generated by any informed node in a graph at every time unit. Many previous studies were dedicated to this general model [19], [27], [20], [23], [36].

Let $G = (V, E)$ be an undirected, connected graph and u be a message originator $u \in V$. The broadcast time of vertex u, $b_k(u)$ is the minimum time units required to complete broadcasting from vertex u. At every time unit any informed vertex informs up to k its adjacent nodes. The broadcast time of a graph G, $b_k(G)$ is defined as $b_k(G) = max\{b_k(u) | u \in V)\}$. The minimum broadcast time of a graph G can be achieved if during every time unit the number of informed vertices are multiplied by $k + 1$. As such, $b_k(G) \geq \lceil \log_{k+1} n \rceil$.

$B_k(n)$ for some particular values of k and n were presented in [29], [36] and [19]. Table 2.3 presents these results. From the application point of view, the graph represents a network architecture where every node consists of a processor and memory. Edge represents the channels that connect the node to one of its neighbors. Such channels

n	$B_2(n)$	$B_3(n)$	$B_4(n)$	$B_7(n)$	Ref.
1	0	0	0		[29]
2	1	1	1		[29]
3	3	3	3		[29]
4	3	6	6		[29]
5	5	4	10		[29]
6	7	7	5		[29]
7	10	9	9		[29]
8	12	11	11		[29]
9	18				[29]
10	12	15			[29] [19]
11	13	18			[29] [19]
12	15				[29]
24	48				[19]
50				175	[19]

Table 5: The values of $B_k(n)$ for some particular values of k and n

are called DMA (Direct Memory Access) channels, since they are used to transmit the information between the memory of one node to the memory of an adjacent node [29]. The processor communicates with a neighbor by writing the information to memory. The network where processing time dominates over transmission time is called a processor-bound system. Systems with massive computations tasks are one example of such processor-bound systems. In such systems, every processor can communicate with only one neighbor at every time unit. This is a 1-broadcasting model. On the opposite side, the network, where communication time dominates over processing time is called a DMA-bound system, because the efficiency of such network communication depends on DMA-channels. In such systems, the processor can communicate with more than one of its neighbors. This is a k-broadcasting model, where $k > 1$. k is the number of data streams that can be handled simultaneously by a single node.

2.4 Previous Lower Bounds on $B_k(n)$

As it was mentioned before, 1-mbg in particular and k-mbg in general are very difficult to find. In the result, many previous studies presented sparse broadcast graphs with a specific number of edges, which provide upper bound on $B_k(n)$ [3], [4], [8], [14], [18], [19], [30], [5]. Some of these papers also provided lower bounds on $B_k(n)$ [14], [19], [36], [31]. When lower and upper bounds match, a new mbg is determined. This section presents some general findings of lower bound on $B_k(n)$

Grigni and Peleg [14] showed that:

$B_k((k+1)^p) = \frac{1}{2}kp(k+1)^p(k \geq 1)$

A similar result was presented in [29].

Consider the $(k+1)$-ary representation of an integer $n-1$:

$n-1 = (\gamma_{m-1}\gamma_{m-2}\ldots\gamma_0)_{k+1}$, where $0 \leq \gamma_i \leq k$ for $i = 0, 1, \ldots, m-1$ and $\gamma_{m-1} \neq 0$.

Let p be the index of the leftmost digit which is not equal to k. Then, $n-1 = k(k+1)^{m-1} + \ldots + k(k+1)^{p+1} + \gamma_p(k+1)^p + \gamma_{p-1}(k+1)^{p-1} + \ldots + \gamma_0$. Given n and k, $\beta = 0$ if $p = 0$ or if $\gamma_0 = \gamma_1 = \ldots = \gamma_{p-1} = 0$. Otherwise $\beta = \gamma_p + 1$.

Thus, $\beta 00\ldots0$ (p digits) $\geq \gamma_p\gamma_{p-1}\ldots\gamma_0$. $B_k(n) \geq \frac{nk}{2}(m-p-1)$.

Harutyunyan and Liestman [19] improved this lower bound and showed that:

$B_k(n) \geq \frac{nk}{2}(m-p-1) + \frac{n}{2}\beta$

In order to inform at least n vertices in $\lceil \log_{k+1} n \rceil$ rounds, a vertex must send the message to at least $k(m-p-1)+\beta$ neighbours during the k-broadcasting [19]. It follows that the degree of each vertex in a k-mbg is at least $k(m-p-1)+\beta$.

In [36] this result was further improved. Let $D(n)$ stands for $k(m-p-1)+\beta$. Bin Shao showed that the minimum possible number of edges for k-broadcast graph is: $B_k(n) \geq \frac{n}{2}\frac{D^2+2D+1}{D+2} = \frac{n}{2}(D+\frac{1}{D+2}) = \frac{n}{2}D + \frac{n}{2(D+2)}$.

Chapter 3

Improved Lower Bounds on $B(n)$ for $n = 23,\ 24$ and 25 vertices

This chapter presents lower bounds on broadcast function $B(n)$ for $n = 23,\ 24$ and 25.

3.1 Common Definitions

Definition 3. *Let G be a minimum broadcast graph on n vertices. $G = (V, E)$, where V is a set of all vertices and E is a set of all edges in the graph. Let V_i represent the set of vertices with degree $i, i \geq 2$.*

$|V| = |V_2| + |V_3| + |V_4| + |V_5| + |V_{>5}|$

$V_{>5}$ - represents the set of vertices with degree greater than 5.

Let E_{ij} represent the set of edges between V_i and V_j as illustrated in Figure 9.

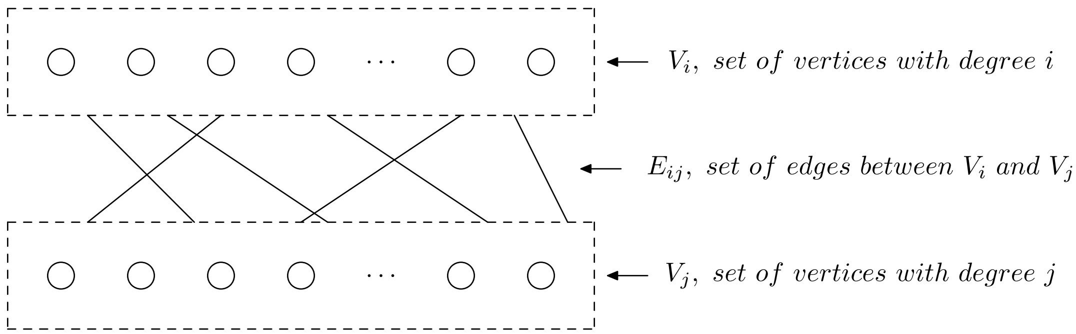


Figure 9: The set of vertices V_i and V_j with set of edges E_{ij}.

Definition 4. *The minimum degree of a graph G, denoted by $\delta(G)$, is a minimum degree of its vertices. The maximum degree of a graph G, denoted by $\Delta(G)$, is a maximum degree of its vertices.*

3.2 Lower bound on $B(24)$.

The broadcast in any mbg on 24 vertices must be completed in 5 time units.

For $n = 24$, $b(G) = \lceil \log_2(2^k - 8) \rceil$. Since $k = 5$, $b(G) = 5$

A broadcast graph on 24 vertices was constructed by Bermond et al. [3] using 36 edges, as shown in Figure 10, that gives an upper bound on $B(24) \leq 36$.

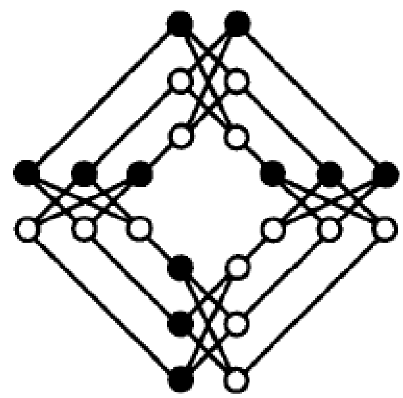


Figure 10: Broadcast graph on 24 vertices.

Our approach to find a lower bound for the number of edges in a broadcast graph will be based on the minimum degree requirement of vertices in any broadcast graph. As mentioned in section 2.4, the best theoretical lower bound on $B_k(n)$ known so far is $B_k(n) \geq \frac{n(D(n)+1)^2}{2(D(n)+2)}$, where $D(n)$ stands for $k(m-p-1)+\beta$. For the particular case of graph on 24 vertices $k=1, m=5, p=3, \beta=1$, we get $D(24)=2$, thus $B(24) \geq 27$.

This thesis shows that $B(24) \geq 35$, which is much closer to the upper bound on $B(24) \leq 36$, meaning that mbg on 24 vertices exists either on 35 or 36 edges.

Proposition 1. *The minimum degree of a broadcast graph on 24 vertices is equal to 2,* $\delta(G) = 2$.

Proof. Let G be a broadcast graph on 24 vertices. If some vertex u has degree 1, it informs its neighbour v in the first round, then in the remaining 4 rounds v can inform at most 16 vertices. Therefore, in 5 rounds u can inform at most 17 vertices, as shown in Figure 11 (the maximum number of informed nodes in a particular tree T is denoted $n(T)$). Therefore, in order to inform 24 vertices in minimum possible number of rounds, the minimum degree in G must be at least 2. □

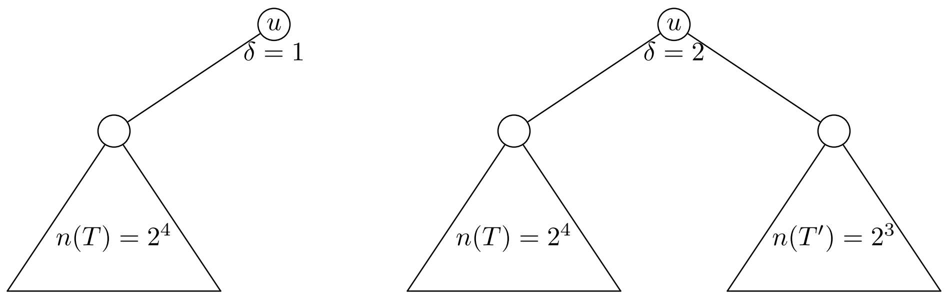


Figure 11: Broadcast tree in the graphs where minimum degree 1 and 2.

Proposition 2. *The maximum degree of a broadcast graph on 24 vertices is greater or equal to 4, $\Delta(G) \geq 4$. If graph G has at least one 2-degree vertex, denoted u, there are two possible options to produce a broadcast tree rooted at vertex u - $T(u)$: Figure 12 and Figure 13.*

In the Figure 12 maximum degree in graph G is 4. The number inside the node represents the time unit at which the node is being informed.

In the figure 13 the maximum degree in the graph G is 5 or greater.

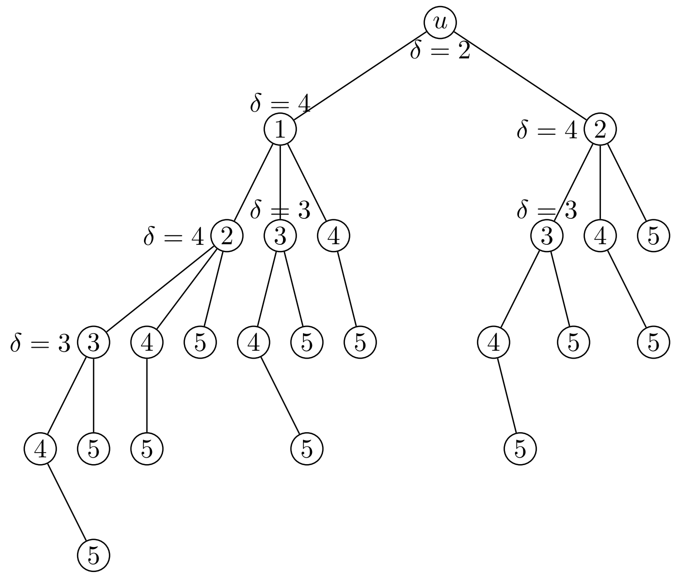


Figure 12: Broadcast tree $T(u)$ in the graph G, $\Delta(G) = 4$.

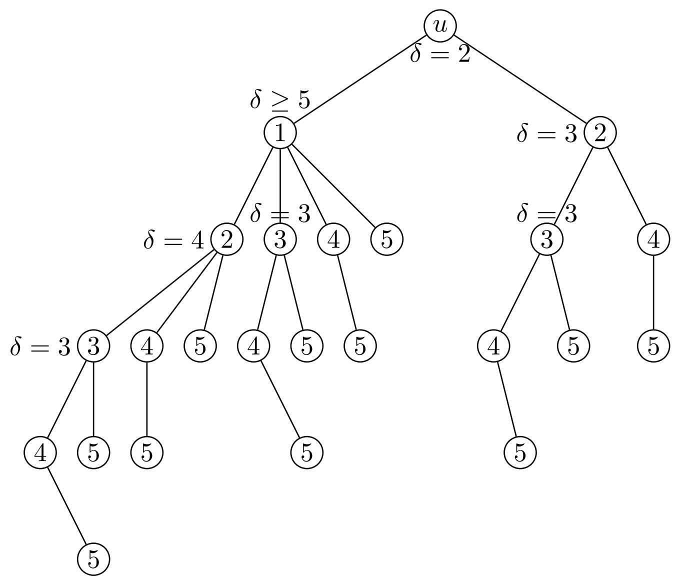


Figure 13: Broadcast tree $T(u)$ in the graph G, $\Delta(G) \geq 5$.

Proposition 3. *Each vertex of degree 2 must have only neighbours of degree 3 or higher. Moreover, none of the vertices from V_3, V_4, V_5 or $V_{>5}$ can have all neighbours of degree 2.* $|E_{23}| \leq 2|V_3|$, $|E_{24}| \leq 3|V_4|$, $|E_{25}| \leq 4|V_5|$.

Proof. Let u be the originator of broadcasting with all neighbours of degree 2. The number of informed vertices from any vertex with degree 2 is at most $2^{i-1}+1$ for $i = 1, \ldots, 4$, i is a number of remaining rounds to complete a broadcast. For $i = 0$, this number is equal to 1. For instance, if 2-degree vertex v is informed at round 1, the number of informed vertices from v is at most 9, if v is informed at round 2, the number of informed vertices from v is at most 5 and etc.. If 2-degree originator has both neighbours with degree 2, the total number of informed vertices in such broadcast tree is at most 15. The broadcast trees for other 3 cases are shown in Figures 14, 15, 16. The total number of informed vertices from originator u is 18 if $deg(u) = 3$, 20 if $deg(u) = 4$ and 21 if $deg(u) \geq 5$. It follows that in order to inform 24 vertices in 5 rounds each vertex in V_3 may have at most 2 neighbours in V_2, each vertex in V_4 may have at most 3 neighbours in V_2 and each vertex in V_5 may have at most 4 neighbours in V_2. □

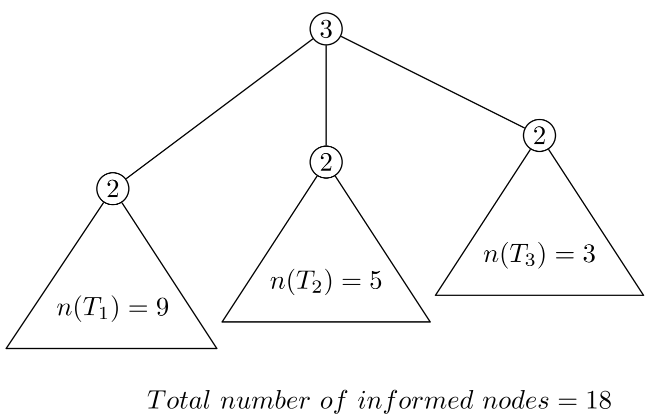


Figure 14: Broadcast tree from originator of degree 3 with all neighbours of degree 2.

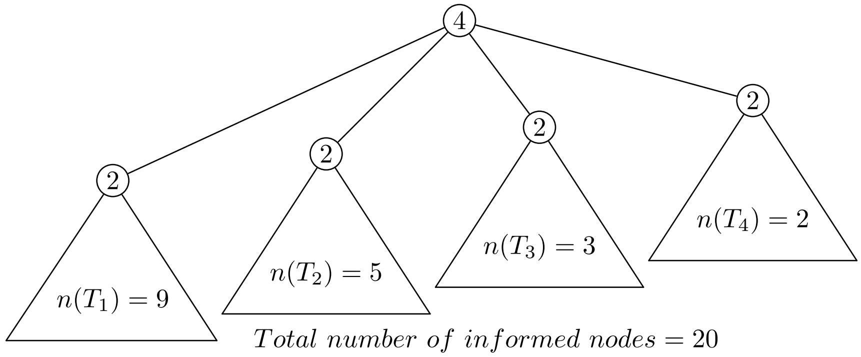


Figure 15: Broadcast tree from originator of degree 4 with all neighbours of degree 2.

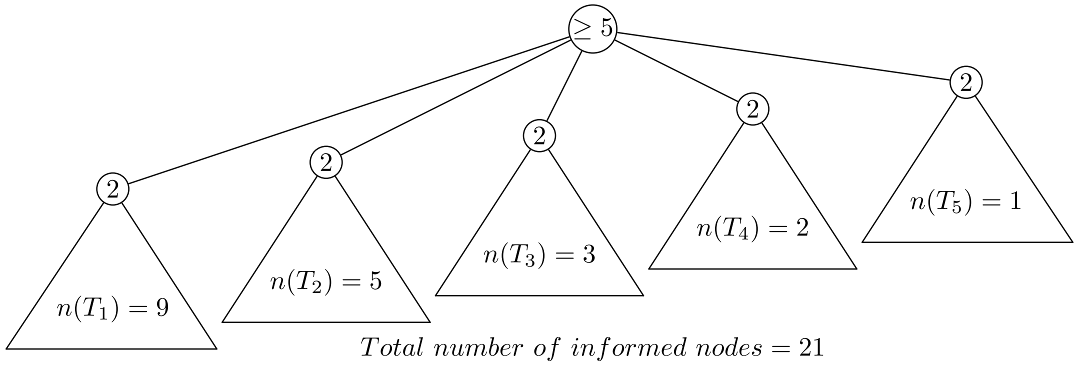


Figure 16: Broadcast tree from originator of degree 5 with all neighbours of degree 2.

Proposition 4. *The number of edges between vertices with degree 2 and 3 is bounded:* $|E_{23}| \leq 2 * (|E_{34}| + |E_{35}| + |E_{3>5}|)$.

Proof. Let us consider the scenario when broadcast originator u is a vertex with degree 3.

As illustrated in the Figure 17, for any vertex u in the graph $G = (V, E), u \in V, deg(u) = 3$ there must be one neighbour of degree at least 4 if other two neighbours are vertices of degree 2. In other words, vertex of degree 3 cannot have two neighbours of degree 2 and one neighbour of degree 3. Let E_{34} be a set of edges between V_3 and V_4, E_{35} be a set of edges between V_3 and V_5 and $E_{3>5}$ be a set of edges between V_3 and $V_{>5}$. Let x be the number of vertices in the graph $G = (V, E), S \subset V_3, u \in S$, such that all neighbours of u are vertices of either degree 2 or 3, $x = |S|$: $x \geq |V_3| - |E_{34}| - |E_{35}| - |E_{3>5}|$.

From the Proposition 3 follows that the maximum number of edges between set V_2 and set V_3 is $2 * |V_3|$. Since x number of vertices from V_3 are not allowed to have 2 neighbours of degree 2, we have to subtract this number from $2 * |V_3|$. Therefore,

$|E_{23}| \leq 2 * |V_3| - 2 * x \leq 2 * |V_3| - 2 * (|V_3| - |E_{34}| - |E_{35}| - |E_{3>5}|)$

$|E_{23}| \leq 2 * (|E_{34}| + |E_{35}| + |E_{3>5}|)$ □

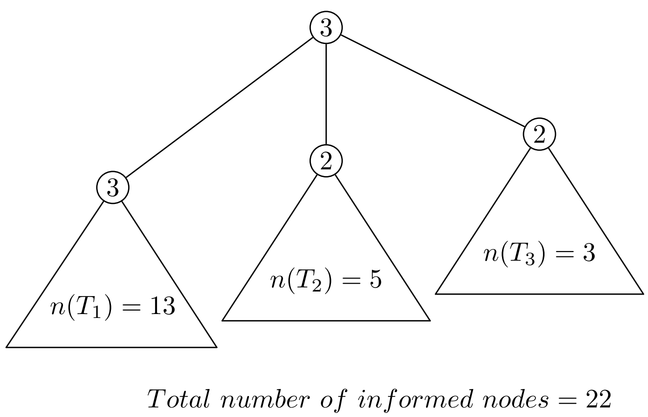


Figure 17: Broadcast tree from originator of degree 3 with two neighbours of degree 2 and one neighbour of degree 3.

Proposition 5. *The number of edges between vertices with degree 2 and 5 is bounded:* $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{23}|}{3} \rceil$.

Proof. Let E_{23} be a set of edges between V_2 and V_3, E_{25} be a set of edges between V_2 and V_5. For each edge $x \in E_{23}$ there must be an edge $y \in E_{25}$, since vertex with degree 2 cannot have both neighbours of degree 3. Moreover, it cannot have one neighbour of degree 3 and another neighbour of degree 4. In order to complete broadcast from originator of degree 2 with one neighbour of degree 3, second neighbour must be a vertex of degree 5: $|E_{23}| \leq |E_{25}|$.

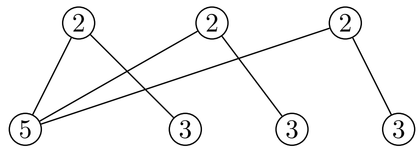


Figure 18: Tree with originator of degree 2 and neighbour of degree 3 (first level).

In addition, this broadcast tree implies that when vertex $u \in V_5$ is connected to $v \in V_2$, while another edge from v is going to V_3, u must also be connected to V_4 and V_3. Thus, vertex u can have at most 3 neighbours of degree 2.

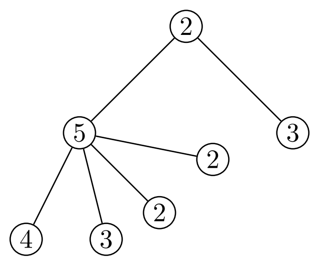


Figure 19: Tree with originator of degree 2 and neighbour of degree 5 (two levels).

Let V_5' be the set of such vertices u, $V_5' \subset V_5$. Since each vertex u, when $u \in V_5'$, may serve up to 3 vertices from V_2, $|V_5'| = \lceil \frac{|E_{23}|}{3} \rceil$. Therefore the number of edges between V_2 and V_5 cannot be larger than the previously calculated maximum $4 * |V_5|$ minus $|V_5'|$. As such, $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{23}|}{3} \rceil$.

$\square$

Proposition 6. *The number of edges between vertices with degree 5 and vertices with degree 4 or greater is bounded:*

$|E_{45}| + |E_{55}| + |E_{5>5}| \geq \lceil \frac{|E_{23}|}{3} \rceil$.

Proof. Let E_{23} be a set of edges between V_2 and V_3; E_{45} be a set of edges between V_4 and V_5; E_{55} be a set of edges inside V_5 and $E_{5>5}$ be a set of edges between V_5 and $V_{>5}$. As discussed in the Proposition 5, we know that for each vertex of degree 2 with one neighbour of degree 3, the only possible broadcast tree is illustrated in Figure 19. This broadcast tree suggests that for each edge $x \in E_{23}$ there must be at least one edge $y \in E_{45} \cup E_{55} \cup E_{5>5}$. In other words, a subtree rooted at the vertex of degree 5 when that vertex is connected to the broadcast originator of degree 2, must be a complete subtree as shown in Figure 20. Therefore, $(|E_{45}| + |E_{55}| + |E_{5>5}|) \geq \lceil \frac{|E_{23}|}{3} \rceil$ □

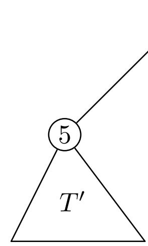


Figure 20: Left side neighbour of originator of degree 2 must be a root of complete subtree.

Proposition 7. *The number of vertices with degree 4 with 2-degree neighbours is bounded:* $|V_4'| \geq \frac{5}{2}|E_{24}'|$

Proof. Let V_2' be a set of vertices with degree 2, where each vertex has both its neighbours with degree 4, as shown in Figure 12. $V_2' \in V_2$. Number of incident edges connected to V_2' is denoted E_{24}'. Let x be the number of vertices in a set V_2, where each vertex has at least one edge connected to either V_3 or V_5. $x = |V_2 \backslash V_2'|$. Therefore $|V_2'| = |V_2| - x$ and $|E_{24}'| = 2(|V_2| - x)$.

Let $|V_4'|$ be the number of vertices in a set V_4, $V_4' \in V_4$, where 2-degree adjacent vertex has both its neighbours with degree 4. The difference between $|E_{25}|$ and $|E_{23}|$ represents the number of 2-degree vertices that has one edge between V_2 and V_5 and the second edge may connect V_2 and V_4. In such case, each 4-degree vertex may serve up to 3 2-degree vertices. Therefore $|V_4| - \lfloor \frac{|E_{25}|-|E_{23}|}{3} \rfloor$ results in upper bound of $|V_4'|$.

Let u denote an originator of the tree and v', v'' denote neighbours of u with degree 4. Analyzing the broadcast tree rooted at u, Figure 21, we can see that v' may have at most 2 neighbours with degree 2 and v'' may have at most 3 neighbours with degree 2. The numbers inside the nodes indicate the minimum degree of a node.

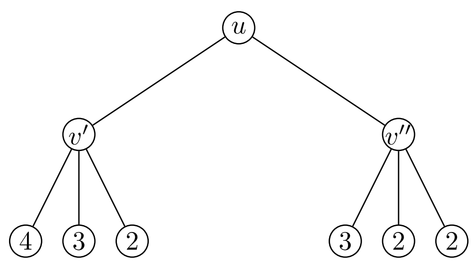


Figure 21: Broadcast tree where originator with degree 2 is connected to two 4-degree neighbours.

This observation suggests that $|E_{24}'|$ can be at most an average, i.e. 5/2, thus

$|E_{24}'| \leq 5/2|V_4'|$

$|E_{24}| - (|E_{25}| - |E_{23}|) \leq 5/2 * (|V_4| - \lfloor \frac{|E_{25}|-|E_{23}|}{3} \rfloor)$

$\square$

The following list summarizes all aforementioned inequalities.

1. $|E_{23}| \leq |E_{25}|$

2. $|E_{23}| \leq 2|V_3|$

3. $|E_{24}| \leq 3|V_4|$

4. $|E_{25}| \leq 4|V_5|$

5. $|E_{23}| \leq 2 * (|E_{34}| + |E_{35}| + |E_{3>5}|)$

6. $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{23}|}{3} \rceil$

7. $|E_{45}| + |E_{55}| + |E_{5>5}| \geq \lceil \frac{|E_{23}|}{3} \rceil$

8. $|E_{24}| - (|E_{25}| - |E_{23}|) \leq 5/2 * (|V_4| - \lfloor \frac{|E_{25}| - |E_{23}|}{3} \rfloor)$

The program introduced in Chapter 4 was designed to generate all possible graphs on 24 vertices. The initial parameters used were: the total number of vertices, the upper bound on $B(n)$ and 5 possible types of vertices. The number of graphs generated was more than $54,000$. In our study, we analyzed broadcast trees from different originators, and came up with the number of propositions, introduced in this chapter. Applying these propositions to the program, we succeeded to eliminate absolutely all theoretically possible graphs with $B(24) < 35$. Therefore, $B(24) \geq 35$.

3.3 Lower bound on $B(25)$.

The broadcast in any mbg on 25 vertices must be completed in 5 time units.

For $n = 25$, $b(G) = \lceil \log_2(2^k - 7) \rceil$. Since $k = 5$, $b(G) = 5$.

A broadcast graph on 25 vertices was constructed by Bermond et al. [3] using 40 edges, as shown in Figure 22, that gives an upper bound on $B(25) \leq 40$.

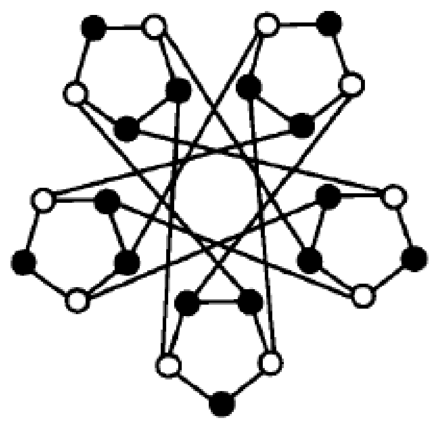


Figure 22: Broadcast graph on 25 vertices.

We can calculate the theoretical lower bound using the formula from [36]: $B_k(n) \geq \frac{n(D(n)+1)^2}{2(D(n)+2)}$, where $D(n)$ stands for $k(m-p-1)+\beta$. For the particular case of the graph on 25 vertices $k = 1, m = 5, p = 3, \beta = 1$, we get $D(25) = 2$. Thus, $B(25) \geq 28$. This thesis shows that $B(25) \geq 38$, which is much closer to upper bound $B(25) \leq 40$. It means that mbg on 25 vertices exists on 38, 39 or 40 edges.

Proposition 8. *The minimum degree of a broadcast graph on 25 vertices is equal to*

2, $\delta(G) = 2$*.*

Proof. Let G be a broadcast graph on 25 vertices. If some vertex u has degree 1, it informs its neighbour v in the first round, then in the remaining 4 rounds v can inform at most 16 vertices. Therefore, in 5 rounds u can inform at most 17 vertices, as shown in Figure 11. In order to inform 25 vertices in minimum possible number of rounds, the minimum degree in G must be at least 2. □

Proposition 9. *The maximum degree of a broadcast graph on 25 vertices is greater or equal to 5,* $\Delta(G) \geq 5$*. If graph G has at least one 2-degree vertex, denoted u, then there is only one possible option to produce a broadcast tree $T(u)$ rooted at vertex u, Figure 23.*

The number inside the node represents the time unit at which the node is being informed.

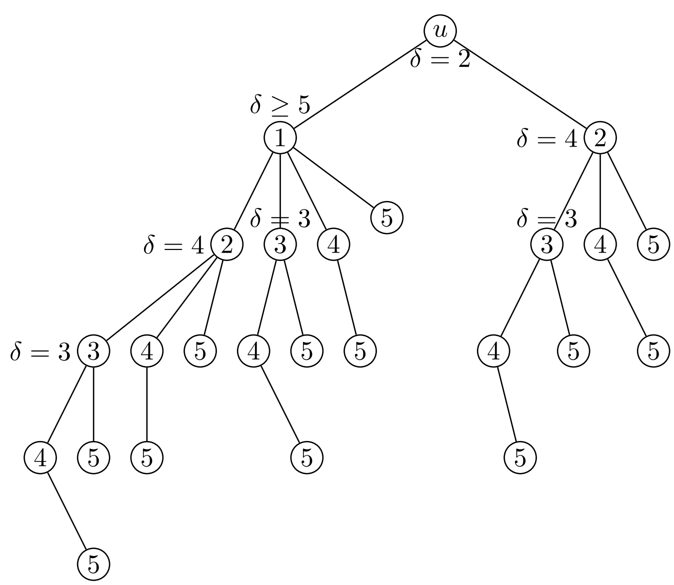


Figure 23: Broadcast tree in the graphs on 25 vertices.

Proposition 10. *In any mbg on 25 vertices there is no edges between vertices with degree 2 and 3.* $|E_{23}| = 0$

Proof. As shown in Figure 23, this is the only possible broadcast tree for broadcast graph $G = (V, E)$ from originator $u, u \in V$ when $deg(u) = 2$. Let us assume that u has one neighbour v, such that $deg(v) < 4$.

If u sends a message to v at the first round, then in remaining 4 rounds v may inform at most $2^4 - 3$ vertices. The second neighbour of u may inform at most 2^3 vertices. Therefore, in such case the maximum number of informed vertices from u is 22.

If u sends a message to v at the second round, then v may inform at most $2^3 - 1$, while another neighbour of u may inform at most 2^4, which leads in total to a maximum of 24 informed vertices from u. The conclusion is that in broadcast graph on 25 vertices 2-degree vertex must have neighbours only with degree 4 and higher. □

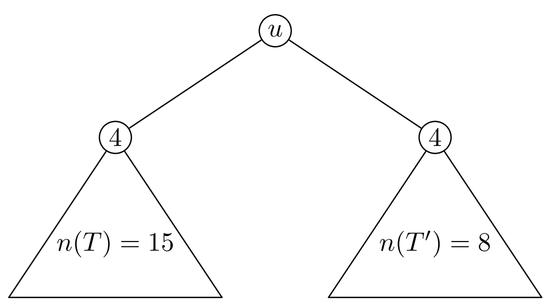
Proposition 11. *The number of edges between vertices with degree 2 and any of their neighbours is bounded:* $|E_{24}| \leq 3|V_4|$, $|E_{25}| \leq 4|V_5|$. *The maximum number of 2-degree vertices which are adjacent to* $|V_{>5}|$ *is not determined precisely, but the lower bound is* $5|V_{>5}|$.

Proof. From the previous proposition follows that each vertex of degree 2 must have a neighbour of either degree 4 or higher. None of the vertices from V_4, V_5 or $V_{>5}$ can have all neighbours of degree 2. According to the diagrams displayed in Figures 15, 16, the total number of informed vertices from originator u is 20 if $deg(u) = 4$, and 21 if $deg(u) \geq 5$. Each vertex in V_4, then, may have at most 3 neighbours in V_2 and each vertex in V_5 may have at most 4 neighbours in V_2. $\square$

Proposition 12. *The number of edges between vertices with degree 2 and 4 cannot be greater than the number of edges between vertices with degree 2 and 5,* $|E_{24}| \leq |E_{25}|$

Proof. Let E_{24} be a set of edges between V_2 and V_4 and E_{25} be a set of edges between V_2 and V_5. For each edge $x \in E_{24}$ there must be an edge $y \in E_{25}$. Figure 24 shows the broadcast tree from u, $deg(u) = 2$ with both neighbours of degree 4. The maximum number of informed nodes in such tree, rooted at u is 24. Therefore, in order to complete a broadcast in $G(n)$ for $n = 25$ in 5 time units from originator u with one neighbour of degree 4, the second neighbour must be a vertex of degree 5. That means :

$|E_{24}| \leq |E_{25}|$ $\square$



$Total\ number\ of\ informed\ nodes = 24$

Figure 24: Broadcast tree with originator of degree 2 and both its neighbours of degree 4.

Proposition 13. *The number of edges between vertices with degree 3 is bounded:* $|E_{33}| \leq \frac{8*|V_3|}{3}$

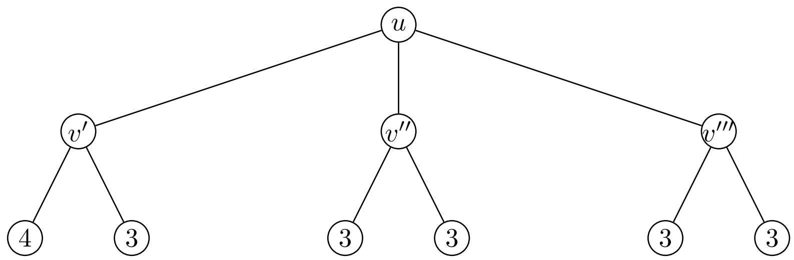


Figure 25: Tree rooted at originator of degree 3 and all neighbours of degree 3.

Proof. Now we consider the case when broadcasting originator u in graph $G = (V, E), u \in V$ has degree 3, $deg(u) = 3$. Let us denote $T(u)$ to be a broadcast spanning tree rooted at u. $T(u)$ cannot have all vertices with degree 3, otherwise the total number of informed nodes in such a tree would be at most 24. At least one edge between V_3 and either V_4 or V_5 must be present at the first or second level of $T(u)$. Let us denote v', v'', v''' neighbours of u. We want to calculate the maximum number of edges in E_{33}. Broadcasting in $T(u)$ can be successful only if v' has at least one edge in E_{34} or in E_{35}. Analyzing the graph shown in Figure 25, we see that if v'' and v''' both are adjacent to neighbours of only degree 3, then v' can have only 2 neighbours of degree 3.

$max(|E_{33}|) = \frac{3*|V_3|+3*|V_3|+2*|V_3|}{3}$

$|E_{33}| \leq \frac{8*|V_3|}{3}$ □

Proposition 14. *Each vertex with degree 3 must be adjacent to a vertex with degree at least 4. If* $|V_3| > 0$ *then* $|E_{34}| + |E_{35}| + |E_{3>5}| > 0$

Proof. As already mentioned, each vertex u, $deg(u) = 3$ cannot have neighbours of degree 2 and, moreover, all incident edges coming from V_3 cannot come to V_3 only.

Formally, it means $3 * |V_3| > |E_{33}|$ and therefore the sum of V_3 incident edges going to V_4, V_5 or $V_{>5}$ must be positive: $|E_{34}| + |E_{35}| + |E_{3>5}| > 0$. □

Proposition 15. *The number of edges between vertices with degree 2 and 5 is bounded:* $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{24}|}{3} \rceil$

Proof. Let E_{24} be a set of edges between V_2 and V_4, E_{25} be a set of edges between V_2 and V_5. For each edge $x \in E_{24}$ there must be an edge $y \in E_{25}$, since vertex with degree 2 cannot have both neighbours of degree 4. In order to complete broadcast from originator of degree 2 with one neighbour of degree 4, the second neighbour must be a vertex of degree 5.

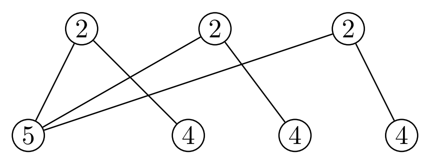


Figure 26: Tree with originator of degree 2 and neighbour of degree 4 (first level).

In addition, this broadcast tree implies that if vertex $u \in V_5$ is connected to $v \in V_2$, such that another edge from v is going to V_4, u must be adjacent to one vertex with degree at least 4 and to another vertex with degree at least 3. Thus, vertex u can have at most 3 neighbours of degree 2.

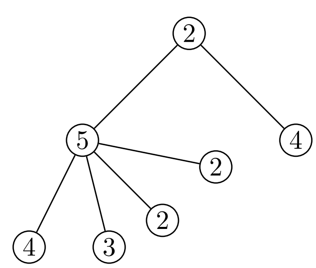


Figure 27: Tree with originator of degree 2 and neighbours of degree 5 and 4 (two levels).

Let V_5' be the set of such vertices u, $V_5' \subset V_5$. Since each vertex u may serve up to 3 vertices from V_2, $|V_5'| = \lceil \frac{|E_{24}|}{3} \rceil; u \in V_5'; V_5' \subset V_5$, therefore the number of edges between V_2 and V_5 cannot be bigger than the previously calculated maximum $4 * |V_5|$ minus $|V_5'|$, thus $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{24}|}{3} \rceil$.

□

Proposition 16. *The number of edges between vertices with degree 5 and vertices with degree 4 or greater is bounded:*

$|E_{45}| + |E_{55}| + |E_{5>5}| \geq \lceil \frac{|E_{24}|}{3} \rceil$

Proof. Let E_{24} be a set of edges between V_2 and V_4; E_{45} be a set of edges between V_4 and V_5; E_{55} be a set of edges inside V_5; $E_{5>5}$ be a set of edges between V_5 and $V_{>5}$. As discussed in the previous proposition, we know that for each vertex of degree 2 with one neighbour with degree 4, another neighbour must be with degree 5. We also know that the neighbour of degree 5 should be adjacent to at least one neighbour of degree at least 4, so for each edge $x \in E_{24}$ there must be at least one edge $y \in E_{45} \cup E_{55} \cup E_{5>5}$. In other words, a subtree rooted at the left side neighbour of 2-degree broadcast originator must be a complete subtree, as shown in Figure 23. Therefore, $|E_{45}| + |E_{55}| + |E_{5>5}| \geq \lceil \frac{|E_{24}|}{3} \rceil$ □

The following list summarizes all aforementioned inequalities.

1. $|E_{23}| = 0$

2. $|E_{24}| \leq 3|V_4|$

3. $|E_{25}| \leq 4|V_5|$

4. $|E_{24}| \leq |E_{25}|$

5. $|E_{33}| \leq \frac{8*|V_3|}{3}$

6. $|E_{34}| + |E_{35}| + |E_{3>5}| > 0$

7. $|E_{25}| \leq 4 * |V_5| - \lceil \frac{|E_{24}|}{3} \rceil$

8. $|E_{45}| + |E_{55}| + |E_{5>5}| \geq \lceil \frac{|E_{24}|}{3} \rceil$

Using the program described in Chapter 4, we found that in the case of graph on 25 vertices, the number of possible graphs was around $300,000$.

Based on the analysis and observation of broadcast trees of theoretically possible cases, we came up with the propositions presented in this section. Applying these propositions to the program, we succeeded to eliminate absolutely all theoretically possible graphs with number of edges < 38. Therefore, $B(25) \geq 38$.

3.4 Lower bound on $B(23)$.

The broadcast in any mbg on 23 vertices must be completed in 5 time units.

For $n = 23$, $b(G) = \lceil \log_2(2^k - 9) \rceil$. Since $k = 5$, $b(G) = 5$.

A broadcast graph on 23 vertices was constructed by Bermond et al. [3] using 34 edges, as shown in Figure 28, that gives an upper bound on $B(23) \leq 34$.

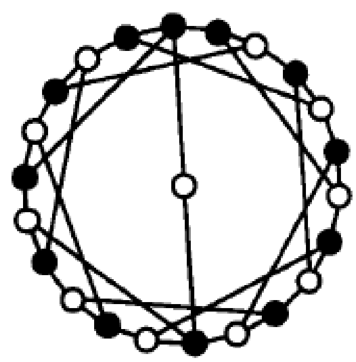


Figure 28: Broadcast graph on 23 vertices.

We can calculate the theoretical lower bound using the formula from [36]: $B_k(n) \geq \frac{n(D(n)+1)^2}{2(D(n)+2)}$, where $D(n)$ stands for $k(m - p - 1) + \beta$. For the particular case of the graph on 23 vertices $k = 1, m = 5, p = 3, \beta = 1$, we get $D(23) = 2$. Thus, $B(23) \geq 26$. In this thesis I will show that $B(26) \geq 33$, which is much more closer to upper bound $B(23) \leq 34$. It means that mbg on 23 vertices exists on 33 or 34 edges.

The same result was presented also in [31], while the authors used slightly different approach. Their work was also based on minimum graph degree property, but they did not use software program to get all possible graph variations.

Proposition 17. *The minimum degree of a broadcast graph on 23 vertices is equal to 2,* $\delta(G) = 2$.

Proof. Let G be a broadcast graph on 23 vertices. If some vertex u has degree 1, it informs its neighbour v in the first round, then in the remaining 4 rounds v can inform at most 16 vertices. Therefore, in 5 rounds u can inform at most 17 vertices, as shown in Figure 11. In order to inform 23 vertices in minimum possible number of rounds, the minimum degree in G must be at least 2. □

Proposition 18. *The maximum degree of a broadcast graph on 23 vertices is greater or equal to 4,* $\Delta(G) \geq 4$. *If graph G has at least one 2-degree vertex, denoted u, then there is only one possible option to produce the broadcast tree $T(u)$ rooted at vertex u, see Figure 29.*

The number inside the node represents the time unit at which the node is being informed.

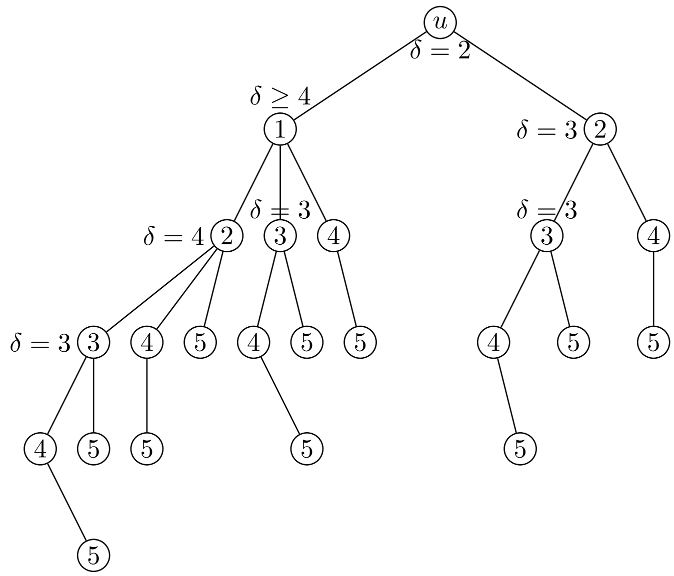


Figure 29: Broadcast tree in the graph on 23 vertices from 2-degree originator.

Proposition 19. *The number of edges between vertices with degree 2 and 3 cannot be greater than the number of edges between vertices with degree 2 and 4,* $|E_{23}| \leq |E_{24}|$.

Proof. Let u be a broadcast originator with $deg(u) = 2$. If u is adjacent to one vertex with degree 3, denoted v', then another neighbour of u, denoted v'', must be a vertex with degree at least 4, as shown in the Figure 29. Otherwise, let us assume that $deg(v'') < 4$. In such case v' may inform at most $2^3 - 1$ vertices and v'' may inform at most $2^4 - 3$ vertices. The total number of vertices informed by u is 21. Therefore, in order to inform 23 vertices, $deg(v'') \geq 4$. In other words, the number of edges going from V_2 to V_3 cannot be larger than the number of edges going from V_2 to V_4, thus $|E_{23}| \leq |E_{24}|$). □

Proposition 20. *A vertex with degree 3 can have at most 2 neighbours with degree 2,* $|E_{23}| \leq 2 * |V_3|$.

Proof. Let u be a broadcast originator with degree 2 and let v be its neighbour with degree 3. In such case, in order to complete broadcasting in 5 time units, v must inform at least 7 vertices, as illustrated in the Figure 30. 7 vertices can be informed from v if and only if v has at most two neighbours of degree 2 and one neighbour of degree at least 3. In other words v can serve at most 2 vertices of degree 2, thus: $|E_{23}| \leq 2 * |V_3|$. $\square$

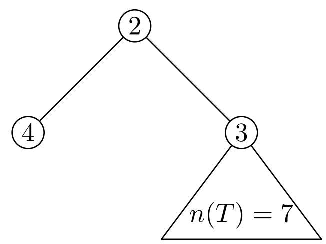


Figure 30: Broadcast tree from originator of degree 2 with neighbour of degree 3.

Proposition 21. *The number of edges between vertices with degree 2 and 3 is bounded:* $|E_{23}| \leq 2 * (|E_{34}| + |E_{3>4}|)$

Proof. Let us consider the scenario when broadcast originator u is a vertex of degree 3, $deg(u) = 3$. As illustrated in the Figure 17, we see that for any vertex u in the graph $G = (V, E), u \in V, deg(u) = 3$ there must be one neighbour of degree at least 4 if other two neighbours are vertices of degree 2. In other words, vertex of degree 3 cannot have two neighbours of degree 2 and one neighbour of degree 3. Let x be the number of vertices in the graph $G = (V, E), S \subset V_3, u \in S$, $x = |S|$, such that all neighbours of u are vertices of either degree 2 or 3, and let $E_{3>4}$ be a set of edges between V_3 and $V_{>4}$, which is set of vertices with degree greater than 4: $x = |V_3| - |E_{34}| - |E_{3>4}|$.

From the previous proposition it follows that the maximum number of edges between set V_2 and set V_3 is $2*|V_3|$. Since x number of vertices from V_3 are not allowed to have 2 neighbours of degree 2, we have to substruct this number from $2*|V_3|$. Therefore, $|E_{23}| \leq 2*|V_3| - 2*x = 2*|V_3| - 2*(|V_3| - |E_{34}| - |E_{3>4}|) = 2*(|E_{34}| + |E_{3>4}|)$

□

Proposition 22. *The number of edges between vertices with degree 2 and 4 is bounded:* $|E_{24}| \leq 2*|V_4|$.

Proof. Let u be a broadcast originator with degree 2 and let v be a neighbour of u, $deg(v) = 4$. In order to complete broadcasting in 5 time units, v must have one neighbour of at least degree 4 and another neighbour of degree at least 3, see Figure 31. It means that v may have at most 2 2-degree neighbours. Therefore, $|E_{24}| \leq 2*|V_4|$. □

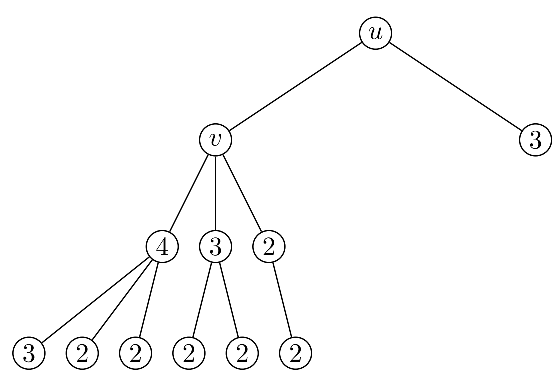


Figure 31: Vertex of degree 4 can have at most 2 neighbours of degree 2.

Proposition 23. *The number of edges between vertices with degree 4 and vertices with degree 4 or greater is bounded:*

$|E_{44}| + |E_{4>4}| \geq \lceil \frac{|E_{23}|}{2} \rceil$

Proof. Let E_{44} be a set of edges going from V_4 to V_4. As discussed previously, we know that for each vertex of degree 2 with one neighbour with degree 3 the only possible broadcast tree is illustrated in Figure 29. This broadcast tree suggests that for each edge $x \in E_{23}$ there must be at least one edge $y \in E_{44} \cup E_{4>4}$. Since each vertex with degree 4 can serve at most 2 vertices with degree 2, based on the Proposition 21, the number of edges in $E_{44} \cup E_{4>4}$ must be greater or equal to $\lceil \frac{|E_{23}|}{2} \rceil$. Therefore, $|E_{44}| + |E_{4>4}| \geq \lceil \frac{|E_{23}|}{2} \rceil$. □

The following list summarizes all aforementioned inequalities.

1. $|E_{23}| \leq |E_{24}|$
2. $|E_{23}| \leq 2 * |V_3|$
3. $|E_{23}| \leq 2 * (|E_{34}| + |E_{3>4}|)$
4. $|E_{24}| \leq 2 * |V_4|$
5. $|E_{44}| + |E_{4>4}| \geq \lceil \frac{|E_{23}|}{2} \rceil$

Using the program described in Chapter 4, we found that in the case of graph on 23 vertices, the number of possible graphs was around $10,000$.

Based on the analysis and observation of broadcast trees of theoretically possible cases, we came up with the propositions presented in this section. Applying these propositions to the program, we succeeded to eliminate absolutely all theoretically possible graphs with number of edges < 33. Therefore, $B(23) \geq 33$.

Chapter 4

Programming Implementation

This chapter presents pseudocode of the program which was used to find all possible graph variation.

Let N be a total number of vertices in the graph.

Let M be a sum of all degrees in the graph: $2 * |E|$.

Let v_i be a number of i-degree vertices, $2 \leq i \leq 5$.

Let v_6 be a number of all vertices with degree greater than 5.

Let e_{ij} be a number of egdes between sets V_i and V_j, $2 \leq i, j \leq 6$.

4.1 Broadcast Graph on 24 vertices

The following procedure $Main()$ produces all possible combinations of vertices and edges for a given $N = 24$ and $M = 70$:

Procedure: MAIN()

$N \leftarrow 24$
$M \leftarrow 70\ (2 * 35)$
for $v_2 \leftarrow 0$ **to** N
do $\left\{\begin{array}{l}\textbf{for } v_3 \leftarrow 0 \textbf{ to } N - v_2 \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } v_4 \leftarrow 0 \textbf{ to } N - v_2 - v_3 \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } v_5 \leftarrow 0 \textbf{ to } N - v_2 - v_3 - v_4 \\ \textbf{do} \left\{\begin{array}{l} v_6 \leftarrow N - (v_2 + v_3 + v_4 + v_5) \\ m \geq 2v_2 + 3v_3 + 4v_4 + 5v_5 + 6v_6 \\ \textbf{if } m \leq M \\ \quad \textbf{then } \left\{\text{EDGES}(m, v_2, v_3, v_4, v_5, v_6)\right. \end{array}\right.\end{array}\right.\end{array}\right.\end{array}\right.$

Procedure: EDGES$(m, v_2, v_3, v_4, v_5, v_6)$

for $e_{23} \leftarrow 0$ **to** v_2
do $\left\{\begin{array}{l}\textbf{for } e_{24} \leftarrow 0 \textbf{ to } 2v_2 - e_{23} \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } e_{25} \leftarrow 0 \textbf{ to } 2v_2 - e_{23} - e_{24} \\ \textbf{do} \left\{\begin{array}{l} e_{26} \leftarrow 2v_2 - (e_{23} + e_{24} + e_{25}) \\ \textbf{for } e_{33} \leftarrow 0 \textbf{ to } 3v_3 - e_{23} \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } e_{34} \leftarrow 0 \textbf{ to } 3v_3 - e_{23} - e_{33} \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } e_{35} \leftarrow 0 \textbf{ to } 3v_3 - e_{23} - e_{33} - e_{34} \\ \textbf{do} \left\{\begin{array}{l} e_{36} \leftarrow 3v_3 - (e_{23} + e_{33} + e_{34} + e_{35}) \\ \textbf{for } e_{44} \leftarrow 0 \textbf{ to } 4v_4 - e_{24} - e_{34} \\ \textbf{do} \left\{\begin{array}{l}\textbf{for } e_{45} \leftarrow 0 \textbf{ to } 4v_4 - e_{24} - e_{34} - e_{44} \\ \quad \textbf{do } \left\{\text{TEMP}()\right.\end{array}\right.\end{array}\right.\end{array}\right.\end{array}\right.\end{array}\right.\end{array}\right.\end{array}\right.$

Procedure: TEMP()

$e_{46} \leftarrow 4v_4 - (e_{24} + e_{34} + e_{44} + e_{45})$
for $e_{55} \leftarrow 0$ **to** $5v_5 - e_{25} - e_{35} - e_{45}$
do $\left\{\begin{array}{l} e_{56} \leftarrow 5v_5 - (e_{25} + e_{35} + e_{45} + e_{55}) \\ e_{66} \geq 6v_6 - (e_{26} + e_{36} + e_{46} + e_{56}) \\ \textbf{if } m = 2(e_{23} + e_{24} + e_{25} + e_{34} + e_{35} + e_{45} + e_{26} + e_{36} + e_{46} + e_{56}) + e_{33} + e_{44} + e_{55} + e_{66} \\ \;\; \textbf{and } e_{33} \textit{ mod } 2 = 0 \textbf{ and } e_{44} \textit{ mod } 2 = 0 \textbf{ and } e_{55} \textit{ mod } 2 = 0 \textbf{ and } e_{66} \textit{ mod } 2 = 0 \\ \quad \textbf{then} \left\{\begin{array}{l} \textbf{comment: } \text{Constraints} \\ \textbf{output } (m, v_2, v_3, v_4, v_5, v_6) \\ \textbf{output } (e_{23}, e_{24}, e_{25}, e_{26}, e_{33}, e_{34}, e_{35}, e_{36}, e_{44}, e_{45}, e_{46}, e_{55}, e_{56}, e_{66}) \end{array}\right.\end{array}\right.$

Propositions presented in the chapter 3 were translated into inequalities and these inequalities were applied in *Procedure Edges*. The modified procedure is listed below (only the constraints). The final output of the modified procedure consists of 971 cases of the graph where $m = 70$ ($|E| = 35$), meaning that all cases where $m < 70$ ($|E| < 35$) were eliminated, therefore $B(24) \geq 35$.

Procedure: $\textsc{Edges}(m, v_2, v_3, v_4, v_5, v_6)$

...
if $e_{23} \leq e_{25}$ $\quad$ $(/ * \, Prop \; 4 * /)$
and $e_{23} \leq 2v_3$ **and** $e_{24} \leq 3v_4$ **and** $e_{25} \leq 4 * v_5$ $\quad$ $(/ * \, Prop \; 2 * /)$
and $e_{23} \leq 2(e_{34} + e_{35} + e_{36})$ $\quad$ $(/ * \, Prop \; 3 * /)$
and $e_{25} \leq 4v_5 - \lceil \frac{e_{23}}{3} \rceil$ $\quad$ $(/ * \, Prop \; 4 * /)$
and $e_{45} + e_{55}/2 + e_{56} \geq \lceil \frac{e_{23}}{3} \rceil$ $\quad$ $(/ * \, Prop \; 5 * /)$
and $e_{24} - (e_{25} - e_{23}) \leq 2.5(v_4 - \lfloor \frac{e_{25} - e_{23}}{3} \rfloor)$ $\quad$ $(/ * \, Prop \; 6 * /)$

$$\textbf{then} \begin{cases} \textbf{output } (m, v_2, v_3, v_4, v_5, v_6) \\ \textbf{output } (e_{23}, e_{24}, e_{25}, e_{26}, e_{33}, e_{34}, e_{35}, e_{36}, e_{44}, e_{45}, e_{46}, e_{55}, e_{56}, e_{66}) \end{cases}$$

4.2 Broadcast Graph on 25 vertices

The following procedure *Main*() produces all possible combinations of vertices and edges for a given $N = 25$ and $M = 78$:

Procedure: MAIN()

$$
\begin{array}{l}
N \leftarrow 25 \\
M \leftarrow 78\ (2*39) \\
\textbf{for } v_2 \leftarrow 0 \textbf{ to } N \\
\quad \textbf{do } \left\{ \begin{array}{l}
\textbf{for } v_3 \leftarrow 0 \textbf{ to } N - v_2 \\
\quad \textbf{do } \left\{ \begin{array}{l}
\textbf{for } v_4 \leftarrow 0 \textbf{ to } N - v_2 - v_3 \\
\quad \textbf{do } \left\{ \begin{array}{l}
\textbf{for } v_5 \leftarrow 0 \textbf{ to } N - v_2 - v_3 - v_4 \\
\quad \textbf{do } \left\{ \begin{array}{l}
v_6 \leftarrow N - (v_2 + v_3 + v_4 + v_5) \\
m \geq 2v_2 + 3v_3 + 4v_4 + 5v_5 + 6v_6 \\
\textbf{if } m \leq M \\
\quad \textbf{then } \{ \text{EDGES}(m, v_2, v_3, v_4, v_5, v_6)
\end{array} \right.
\end{array} \right.
\end{array} \right.
\end{array} \right.
\end{array}
$$

Procedure *Edges* for a broadcast graph on 25 is the same procedure as we used for ahi broadcast graph on 24 vertices.

Propositions presented in the chapter 3 were translated into inequalities and these inequalities were applied in *Procedure Edges*. The modified procedure is listed below (the constraints only). The final output of the modified procedure consists of 820 cases of the graph where $m = 76$ ($|E| = 38$) and 7515 cases of the graph where $m = 78$ ($|E| = 39$) , meaning that all cases where $m < 76$ ($|E| < 38$) were eliminated. Therefore $B(25) \geq 38$.

Procedure: EDGES$(m, v_2, v_3, v_4, v_5, v_6)$

$$
\begin{array}{ll}
\ldots & \\
\textbf{if } e_{23} = 0 & (/*\ \mathit{Prop}\ 8\ */) \\
\quad \textbf{and } e_{24} \leq e_{25} & (/*\ \mathit{Prop}\ 10\ */) \\
\quad \textbf{and } e_{24} \leq 3v_4 \textbf{ and } e_{25} \leq 4v_5 & (/*\ \mathit{Prop}\ 9\ */) \\
\quad \textbf{and } e_{33} \leq \frac{8v_3}{3} & (/*\ \mathit{Prop}\ 11\ */) \\
\quad \textbf{and } v_3 = 0 \textbf{ or } (v_3 > 0 \textbf{ and } (e_{34} + e_{35} + e_{36}) > 0)) & (/*\ \mathit{Prop}\ 12\ */) \\
\quad \textbf{and } e_{25} \leq 4v_5 - \lceil \frac{e_{24}}{3} \rceil & (/*\ \mathit{Prop}\ 13\ */) \\
\quad \textbf{and } e_{45} + e_{55}/2 + e_{56} \geq \lceil \frac{e_{24}}{3} \rceil & (/*\ \mathit{Prop}\ 14\ */) \\
\quad\quad \textbf{then } \left\{ \begin{array}{l}
\textbf{output } (m, v_2, v_3, v_4, v_5, v_6) \\
\textbf{output } (e_{23}, e_{24}, e_{25}, e_{26}, e_{33}, e_{34}, e_{35}, e_{36}, e_{44}, e_{45}, e_{46}, e_{55}, e_{56}, e_{66})
\end{array} \right. &
\end{array}
$$

4.3 Broadcast Graph on 23 vertices

The following procedure *Main*() produces all possible combinations of vertices and edges for a given $N = 23$ and $M = 66$:

Procedure: MAIN()

$N \leftarrow 23$
$M \leftarrow 66\ (2 * 33)$
for $v_2 \leftarrow 0$ **to** N
  do **for** $v_3 \leftarrow 0$ **to** $N - v_2$
    do **for** $v_4 \leftarrow 0$ **to** $N - v_2 - v_3$
      do $v_5 \leftarrow 0$ **to** $N - v_2 - v_3 - v_4$
        do $m \geq 2v_2 + 3v_3 + 4v_4 + 5v_5$
      if $m \leq M$
        then EDGES(m, v_2, v_3, v_4, v_5)

Procedure: EDGES(m, v_2, v_3, v_4, v_5)

for $e_{23} \leftarrow 0$ **to** v_2
  do **for** $e_{24} \leftarrow 0$ **to** $2v_2 - e_{23}$
    do **for** $e_{25} \leftarrow 0$ **to** $2v_2 - e_{23} - e_{24}$
      do $e_{26} \leftarrow 2v_2 - (e_{23} + e_{24} + e_{25})$
      for $e_{33} \leftarrow 0$ **to** $3v_3 - e_{23}$
        do **for** $e_{34} \leftarrow 0$ **to** $3v_3 - e_{23} - e_{33}$
          do **for** $e_{35} \leftarrow 0$ **to** $3v_3 - e_{23} - e_{33} - e_{34}$
            do $e_{36} \leftarrow 3v_3 - (e_{23} + e_{33} + e_{34} + e_{35})$
            for $e_{44} \leftarrow 0$ **to** $4v_4 - e_{24} - e_{34}$
              do TEMP()

Procedure: TEMP()

$e_{45} \leftarrow 0$ **to** $4v_4 - e_{24} - e_{34} - e_{44}$
$e_{55} \leftarrow 0$ **to** $5v_5 - e_{25} - e_{35} - e_{45}$
if $m = 2 * (e_{23} + e_{24} + e_{25} + e_{34} + e_{35} + e_{45}) + e_{33} + e_{44} + e_{55}$
 and $e_{33}\ mod\ 2 = 0$ **and** $e_{44}\ mod\ 2 = 0$ **and** $e_{55}\ mod\ 2 = 0$
  then **comment:** Constraints
    output (m, v_2, v_3, v_4, v_5)
    output $(e_{23}, e_{24}, e_{25}, e_{33}, e_{34}, e_{35}, e_{44}, e_{45}, e_{55})$

Propositions presented in the chapter 3 were translated into inequalities and these inequalities were applied in *Procedure Edges*. The modified procedure is listed below (only the constraints). The final output of the modified procedure consists of 24 cases of the graph where $m = 66$ ($|E| = 33$), meaning that all cases where $m < 66$ ($|E| < 33$) were eliminated. Therefore $B(23) \geq 33$.

Procedure: $\textsc{Edges}(m, v_2, v_3, v_4, v_5)$

...
if $e_{23} \leq e_{24}$ $\quad$ $(/ * \mathit{Prop}\ 16 * /)$
and $e_{23} \leq 2 * v_3$ $\quad$ $(/ * \mathit{Prop}\ 17 * /)$
and $e_{23} \leq 2 * (e_{34} + e_{35})$ $\quad$ $(/ * \mathit{Prop}\ 18 * /)$
and $e_{24} \leq 2 * v_4$ $\quad$ $(/ * \mathit{Prop}\ 19 * /)$
and $e_{44}/2 + e_{45} \geq \lceil \frac{e_{23}}{2} \rceil$ $\quad$ $(/ * \mathit{Prop}\ 20 * /)$

$$\textbf{then} \begin{cases} \textbf{output}\ (m, v_2, v_3, v_4, v_5) \\ \textbf{output}\ (e_{23}, e_{24}, e_{25}, e_{33}, e_{34}, e_{35}, e_{44}, e_{45}, e_{55}) \end{cases}$$

Chapter 5

Conclusions and Future Work

Up until now, no general method has been found that allows us to determine values of minimum broadcast function $B(n)$ for an arbitrary graph G with n vertices. In fact, this is a very difficult process , which is not even NP-hard problem and therefore studies have focused on finding $B(n)$ for particular values. Knowing the minimum possible number of edges in broadcast graphs has a very practical implication, as it allows to construct minimum broadcast networks and, in many cases allows us to predict broadcast protocol in a given network. All values of $B(n)$ for $n \leq 32$ are known, with the exception of $n = 23, 24, 25$.

Given this context, this thesis determines the lower bound of broadcast functions: $B(23)$, $B(24)$, $B(25)$. $B(23)$ was determined in [31] - 33 or 34, but for $B(24)$ and $B(25)$ the best results obtained so far using the general mathematical method of calculating the minimum broadcast function presented in [36] show that:

$B(24) \geq 27$; $B(25) \geq 28$

This thesis improves the above results significantly through studies of the minimum and maximum degree of the graphs and through deep analysis of broadcast trees from originators of different degrees. The results obtained in our study are very close to known upper bound values of $B(n)$ for $n = 23, 24, 25$. Table 6 demonstrates previously known lower bound values, new values and upper bound values.

	$n = 23$	$n = 24$	$n = 25$
Previously Known Lower Bound on B(n)	33	27	28
New Lower Bound on B(n)	**33**	**35**	**38**
Upper Bound on B(n)	34	36	40

Table 6: The table of the results on B(n).

Although the best desired result would be to find a match between the lower bound and known upper bound, meaning that the new minimum broadcast graph has been determined for a given n, or alternatively to design a broadcast graph with the number of edges equal to the new lower bound on $B(n)$, I was not able to reach this goal. This is definitely area of future research. Despite the fact that new mbgs were not discovered during my study, finding lower bound of $B(n)$ very close to upper bounds, compared to previously known values, significantly reduces the number of possibilities of constructing minimum broadcast networks with 23, 24 or 25 vertices. Table 7 compares the number of possibilities.

	$n = 23$	$n = 24$	$n = 25$
# of possibilities based on previous lower bound on $B(n)$	10,000	54,000	300,000
# of possibilities based on new lower bound on $B(n)$	**23 (33 edges)**	**971 (35 edges)**	**820 (38 edges)** **7515(39 edges)**

Table 7: The number of possibilities to construct mbg.

The proposed method of analyzing broadcast trees from a different originator based on the study of a minimum possible degree in the graph can be used in further studies of broadcast graphs with unknown $B(n)$. It will also be worthwhile to investigate if this method can be generalized to give tight lower bound on $B(2^k - 7)$, $B(2^k - 8)$ and $B(2^k - 9)$.

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