Generalized Feynman-kac Transformation And Fukushima's Decomposition For Nearly Symmetric Markov Processes

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Abstract

Generalized Feynman-Kac transformation and Fukushima's decomposition for nearly symmetric Markov processes

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In this thesis, we study some problems about nearly symmetric Markov processes, which are associated with non-symmetric Dirichlet forms or semi-Dirichlet forms.

For a Markov process (X_t, P_x) associated with a non-symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$, we study the strong continuity of the generalized Feynman-Kac semigroup $(P_t^u)_{t\geq 0}$, which is defined by

$$P_t^u f(x) := E_x[e^{N_t^u} f(X_t)], \quad f \geq 0 \text{ and } t \geq 0.$$

Here $u \in D(\mathcal{E})$, N_t^u is the continuous additive functional of zero energy in the Fukushima's decomposition. We give two sufficient conditions for $(P_t^u)_{t\geq 0}$ to be strongly continuous.

The first sufficient condition is that there exists a constant $\alpha_0 \geq 0$ such that for any $f \in D(\mathcal{E})_b$, $Q^u(f,f) \geq -\alpha_0 (f,f)_m$, where $(Q^u, D(\mathcal{E})_b)$ is defined by

$$Q^u(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg), \quad f,g \in D(\mathcal{E})_b := D(\mathcal{E}) \cap L^\infty(E;m).$$

The second sufficient condition is that there exists a constant $\alpha_0 \geq 0$ such that

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0.$$

For a Markov process associated with a semi-Dirichlet form, we establish Fukushima's decomposition and give a transformation formula for local martingale additive functionals.

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Contents

Chapter 1

Introduction

The one-to-one correspondence between Dirichlet forms and Markov processes provides a bridge between the classical potential theory and stochastic analysis, by which we can transfer between some analytic problems and stochastic problems. The Dirichlet form theory has been developed very quickly and has been used widely. It is an effective machinery for studying various stochastic models, especially those with non-smooth coefficients, on fractal-like spaces or spaces of infinite dimensions.

The notion of Dirichlet form was introduced by A. Beurling and J. Deny in 1958-1959, who essentially established the analytic part of the Dirichlet space theory. The more recent probabilistic part was initiated by M. Fukushima and M.L. Silverstein, who connected the regular symmetric Dirichlet forms with Hunt processes on locally compact separable metric spaces. Later, S. Carillo-Menende and Y. LeJan extended Dirichlet forms to the non-symmetric case. Then, S. Albeverio and Z.M. Ma showed that a Dirichlet form on a Lusin space is associated with a pair of right processes if and only if the Dirichlet form is quasi-regular. One advantage of the correspondence between Markov processes and Dirichlet forms is that some sample path properties of the Markov processes can be described by the associated Dirichlet forms. For example, the continuity of the sample paths of Markov processes is equivalent to the local property of Dirichlet forms.

Although many researchers have worked on Dirichlet form theory and have gotten lots of beautiful results, there are still some unsolved problems in the field. In this thesis, we will focus on two interesting problems.

Let (X_t, P_x) be a Markov process associated with the (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ and $(N_t^u)_{t\geq 0}$ be the continuous additive functionals of zero energy in the Fukushima's decomposition. Defined $P_t^u f(x) := E_x[e^{N_t^u} f(X_t)]$, $f \geq 0$ and $t \geq 0$.The first problem is the strong continuity of the generalized Feynman-Kac semigroups $(P_t^u)_{t\geq 0}$.

The strong continuity of generalized Feynman-Kac semigroups for symmetric Markov processes has been studied extensively by many people. Note that $(N_t^u)_{t\geq 0}$ is not of bounded variation (cf. (FOT1994, Example 5.5.2)). Hence the classical results of S. Albeverio and Z.M. Ma given in (AM1991) do not apply directly. Under the assumption that X is the standard d-dimensional Brownian motion, u is a bounded continuous function on $\mathbf{R}^d$ and $|\nabla u|^2$ belongs to the Kato class, J. Glover et al. proved in (GRSS1994) that $(P_t^u)_{t\geq 0}$ is a strongly continuous semigroup on $L^2(\mathbf{R}^d; dx)$. Moreover, they gave an explicit representation for the closed quadratic form corresponding to $(P_t^u)_{t\geq 0}$. (T2001) generalized the results of (GRSS1994) to symmetric Lévy processes on $\mathbf{R}^d$ and removed the assumption that u is bounded continuous. Furthermore, Z.Q. Chen and T.S. Zhang established in (CZ2002) the corresponding results for general symmetric Markov processes via the Girsanov transformation. They proved that if $\mu_{\langle u\rangle}$, the energy measure of u, is a measure in the Kato class, then $(P_t^u)_{t\geq 0}$ is a strongly continuous semigroup on $L^2(E; m)$. Also, they characterized the closed quadratic form corresponding to $(P_t^u)_{t\geq 0}$. P.J. Fitzsimmons and K. Kuwae (FK2004) established the strong continuity of $(P_t^u)_{t\geq 0}$ under the assumption that X is a symmetric diffusion process and $\mu_{\langle u\rangle}$ is a measure in the Hardy class. Furthermore, Z.Q. Chen et al. (CFKZ2008b) established the strong continuity of $(P_t^u)_{t\geq 0}$ for general symmetric Markov processes under the assumption that $\mu_{\langle u\rangle}$ is a measure in the Hardy class.

All the results mentioned above only give sufficient conditions for $(P_t^u)_{t\geq 0}$ to be strongly continuous, where $\mu_{\langle u\rangle}$ is assumed to be in the Hardy class. In (CS2006), under the assumption that X is a symmetric diffusion process, C.Z. Chen and W. Sun showed that the semigroup $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E; m)$ if and only if the bilinear form $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded, where

$$Q^u(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg), \quad f,g \in D(\mathcal{E})_b := D(\mathcal{E}) \cap L^\infty(E;m).$$

Furthermore, C.Z. Chen et al. (CMS2007) generalized this result to general symmetric Markov processes. Z.Q. Chen et al. (CFKZ2009) studied general perturbations of symmetric Markov processes and gave another proof for the equivalence of the strong continuity of $(P_t^u)_{t\geq 0}$ and the lower semi-boundedness of $(Q^u, D(\mathcal{E})_b)$.

In the first part of this thesis, by a localization method and the Beurling-Deny formula of non-symmetric Dirichlet form, which was developed very recently, we give two sufficient conditions for $(P_t^u)_{t\geq 0}$ to be strongly continuous. Our results generalize all the previous results on the strong continuity of the generalized Feynman-Kac semigroup.

The second problem is Fukushima's decomposition in the framework of semi-Dirichlet forms. Suppose that X is a right process which is associated with a non-symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$. Fukushima's decomposition tells us that for $u \in D(\mathcal{E})$, $u(X_t) - u(X_0) = M_t^u + N_t^u$, where M_t^u is a martingale additive functional of finite energy and N_t^u is a continuous additive functional of zero energy. Fukushima's decomposition is a generalization of Itô's formula for semi-martingales and Doob-Meyer decomposition for super-martingales.

Fukushima's decomposition is very useful. For example, by defining the stochastic integrals with respect to continuous additive functionals of zero energy, we can define the stochastic integrals with respect to Dirichlet processes and thus generalize Itô's formula. Also, for symmetric Dirichlet forms, by using the time reversal operator, we have the Lyons-Zheng decomposition, which is a summation of backward and forward martingales. Then by using martingale inequalities, we can get many good estimates on additive functionals.

There are many references on Fukushima's decomposition in the Dirichlet forms setting. (FOT1994, Theorem 5.2.2) gives Fukushima's decomposition for $u \in D(\mathcal{E})_e$, the extended Dirichlet space, in the case of regular Dirichlet forms. Then (FOT1994, Theorem 5.5.1) gives Fukushima's decomposition for u which is locally in $D(\mathcal{E})$ in the broad sense (see (FOT1994, page 226)) in the framework of regular local Dirichlet forms (in other words, the associated Markov processes have no jumping parts). Later, (MR1992, Chapter VI Theorem 2.5) generalizes Fukushima's decomposition to the quasi-regular case by the transfer method. Recently (K2010, Theorem 4.2) gives Fukushima's decomposition for $u \in D(\mathcal{E})_{loc}$ in the case of general symmetric Dirichlet forms by generalizing stochastic calculus.

Up to now, there is no paper concerning Fukushima's decomposition in the semi-Dirichlet forms case. There are big differences between Dirichlet forms and semi-Dirichlet forms. For example, for Dirichlet forms, the set of bounded functions in the domain of the Dirichlet forms is an algebra, while this is not true for semi-Dirichlet forms. Also, there is a pair of Markov processes associated with a Dirichlet form, but there is only one Markov process associated with a semi-Dirichlet form.

The notations and terminologies of this thesis follow (FOT1994), (MR1992) and (MS2010b). For the convenience of the reader, we will give a brief introduction to semi-Dirichlet forms in the first section of this chapter. In the second section, we will present the main results of this thesis. In the last section, we will describe the organization of this thesis.

1.1 Introduction to semi-Dirichlet forms

In this section, we recall some basic facts on semigroups, resolvents, generators, semi-Dirichlet forms and the associated Markov processes. We refer the reader to (MR1992, Chapter 1), (FOT1994) and (MS2010b) for the proofs and more details. Throughout this section, we fix a real Hilbert space H with inner product $(\ ,\ )$ and norm $\|\ \| := (\ ,\ )^{1/2}$.

Definition 1.1. *(strongly continuous contraction semigroups) A family $(T_t)_{t>0}$ of linear operators on H whose domain is $D(T_t) = H$ for all $t > 0$ is called a strongly continuous contraction semigroup on H (abbreviated by semigroup) if $(T_t)_{t>0}$ satisfies the following three conditions,*

(i) $\lim_{t\downarrow 0} \| T_t f - f \| = 0,\ \forall f \in H$ *(strong continuity).*

(ii) $||T_t f|| \leq ||f||,\ \forall f \in H$ *(contraction).*

(iii) $T_t(T_s f) = T_{t+s} f,\ \forall t, s > 0, f \in H$ *(semigroup property).*

Definition 1.2. *(strongly continuous contraction resolvents) A family* $(G_\alpha)_{\alpha>0}$ *of linear operators on* H *with domain* $D(G_\alpha) = H$ *for any* $\alpha > 0$ *is called a strongly continuous contraction resolvent on* H *if*

(i) $\lim_{\alpha\uparrow\infty} ||\alpha G_\alpha f - f|| = 0,\ \forall f \in H$ *(strong continuity).*

(ii) $||\alpha G_\alpha f|| \leq ||f||,\ \forall f \in H$ *(contraction).*

(iii) $G_\alpha f - G_\beta f = (\beta - \alpha) G_\alpha(G_\beta f),\ \forall \alpha, \beta > 0, f \in H$ *(resolvent equation).*

Proposition 1.1. *(the relationship between strongly continuous contraction semigroups and strongly continuous contraction resolvents)*
(i) Given a strongly continuous contraction semigroup $(T_t)_{t>0}$*, define for* $\alpha > 0$

$$G_\alpha f := \int_0^\infty e^{-\alpha t} T_t f\, dt, \quad \forall f \in H. \tag{1.1}$$

Then $(G_\alpha)_{\alpha>0}$ *is a strongly continuous contraction resolvent.*

(ii) Let $(G_\alpha)_{\alpha>0}$ *be a strongly continuous contraction resolvent on* H*. Define for* $t > 0$

$$T_t f := \lim_{\alpha\to\infty} e^{t\alpha(\alpha G_\alpha - 1)} f := \lim_{\alpha\to\infty} e^{-t\alpha} \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{n!} (\alpha G_\alpha)^n f, \quad \forall f \in H.$$

Then $(T_t)_{t>0}$ *is a strongly continuous contraction semigroup and* G_α *is expressed by (1.1) for* $\alpha > 0$.

Definition 1.3. *(generators of semigroups) Let* $(T_t)_{t>0}$ *be a strongly continuous contraction semigroup on* H*. Define*

$$\begin{aligned} D(L) &:= \{f \in H \mid \lim_{t\to 0} \frac{1}{t}(T_t f - f) \text{ exists in } H\}, \\ Lf &:= \lim_{t\to 0} \frac{1}{t}(T_t f - f),\ f \in D(L). \end{aligned} \tag{1.2}$$

and we call $(L, D(L))$ *the generator of* $(T_t)_{t>0}$.

Proposition 1.2. *(the relationship between resolvents and generators) Given a strongly continuous contraction semigroups* $(T_t)_{t>0}$*, define* $(L, D(L))$ *by (1.2) and* $(G_\alpha)_{\alpha>0}$ *by*

(1.1). Then, for $\alpha > 0$,

$$\begin{aligned} G_\alpha &= (\alpha - L)^{-1}, \\ L &= \alpha - G_\alpha^{-1}. \end{aligned}$$

Definition 1.4. *(resolvent set) Let* L *be a linear operator on* H*. If a real number* α *satisfies the following condition*

(i) $(\alpha - L) : D(L) \to H$ *is one-to-one,*

(ii) the range of $(\alpha - L)$ *is* H*,*

(iii) the inverse $(\alpha - L)^{-1}$ *is continuous on* H*,*

then we say α *is in the resolvent set of* L*, denoted by* $\alpha \in \rho(L)$*.*

Theorem 1.1. *(Hille-Yosida) Let* $(L, D(L))$ *be a dense (that is,* $D(L)$ *is dense in* H*) linear operator on* H*. Then a necessary and sufficient condition for* $(L, D(L))$ *to be the generator of a strongly continuous contraction semigroup* $(T_t)_{t>0}$ *on* H *is that* $(L, D(L))$ *satisfies the following properties:*

(L1) $(0, \infty) \subset \rho(L)$*.*

(L2) $||\alpha(\alpha - L)^{-1} f|| \leq ||f||, \ \forall \alpha > 0, f \in H$*.*

In this case $(T_t)_{t>0}$ *is uniquely determined by* L*.*

Proposition 1.3. *Let* $(T_t)_{t>0}$ *be a strongly continuous contraction semigroup,* $(G_\alpha)_{\alpha>0}$ *and* $(L, D(L))$ *be its resolvent and generator, respectively. Then the following three assertions are equivalent to each other:*

(i) $(T_t)_{t>0}$ *is analytic, that is, the complexification of* $(e^{-t}T_t)_{t>0}$ *is the restriction of a holomorphic contraction semigroup on some sector region* $S(K)$ $(K > 0)$ *of the complex plane* $\mathbb{C}$*. Here* $S(K)$ *is defined by* $S(K) := \{z \in \mathbb{C} \,|\, |Imz| \leq K Rez\}$*.*

(ii) G_α *satisfies the sector condition for one (hence for all)* $\alpha > 0$*. (We say that a positive definite linear operator* $(A, D(A))$ *satisfies the (strong) sector condition if there exists* $K > 0$ *such that*

$$|(Au, v)| \leq K(Au, u)^{1/2}(Av, v)^{1/2}, \quad \forall u, v \in D(A)).$$

(L3) $I - L$ *(*$I :=$ *the identity map) satisfies the sector condition.*

Definition 1.5. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a bilinear form on H (that is, $\mathcal{E}$ is a bilinear map from $D(\mathcal{E}) \times D(\mathcal{E})$). We say that $(\mathcal{E}, D(\mathcal{E}))$ is a coercive closed form, if it satisfies the following conditions:*

(i) for every $f \in D(\mathcal{E}), \mathcal{E}(f,f) \geq 0$ (nonnegative definite).

(ii) $D(\mathcal{E})$ is dense in H.

(iii) $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a symmetric closed form, that is, $D(\mathcal{E})$ is complete under the norm $\tilde{\mathcal{E}}_1^{1/2}$, here $\tilde{\mathcal{E}}(u,v) = 1/2(\mathcal{E}(u,v) + \mathcal{E}(v,u))$.

(iv) there is a constant $K > 0$ (called it continuity constant), such that

$$|\mathcal{E}_1(u,v)| \leq K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2},$$

where $\mathcal{E}_\alpha(u,v) = \mathcal{E}(u,v) + \alpha(u,v), \quad \forall u,v \in D(\mathcal{E})$ (section condition).

Remark 1.1. *$(\mathcal{E}, D(\mathcal{E}))$ is said to satisfy the (strong) sector condition if there exists $K > 0$ such that*

$$|\mathcal{E}(u,v)| \leq K\mathcal{E}(u,u)^{1/2}\mathcal{E}(v,v)^{1/2}, \quad \forall u,v \in D(\mathcal{E}).$$

Lemma 1.1. *(MR1992, Lemma 2.12) Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive closed form on $L^2(E;m)$ and $f_n \in D(\mathcal{E}), n \geq 1$ such that*

$$\sup_{n\geq 1} \mathcal{E}(f_n, f_n) < \infty.$$

If $f \in H$ such that $f_n \to f$ in H as $n \to \infty$, then $f \in D(\mathcal{E})$ and f_n converges weakly to f in the Hilbert space $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ and there exists a subsequence f_{n_k} of $\{f_n\}$ such that its Cesaro mean $w_n = \frac{1}{n}\sum_{k=1}^n f_{n_k} \to f$ in $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ as $n \to \infty$. Moreover,

$$\mathcal{E}(f,f) \leq \liminf_{n\to\infty} \mathcal{E}(f_n, f_n).$$

Theorem 1.2. *(i) There is a one-to-one correspondence between all the strongly continuous contraction resolvents $(G_\alpha)_{\alpha>0}$ satisfying sector condition and all the coercive closed forms $(\mathcal{E}, D(\mathcal{E}))$. The correspondence is given by*

$$G_\alpha(H) \subset D(\mathcal{E}) \quad \text{and} \quad \mathcal{E}_\alpha(G_\alpha u, v) = (u,v) \quad \text{for all } u \in H, v \in D(\mathcal{E}), \alpha > 0. \tag{1.3}$$

(ii) Given a coercive closed form $(\mathcal{E}, D(\mathcal{E}))$, the corresponding resolvent $(G_\alpha)_{\alpha>0}$ is uniquely determined by (1.3).

(iii) Given a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ satisfying sector condition, the corresponding coercive closed form is uniquely determined by

$$\begin{aligned} D(\mathcal{E}) &= \{u \in H \,|\, \sup_{\beta>0} \beta(u - \beta G_\beta u, u) < \infty\}, \\ \mathcal{E}(u,v) &= \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, v),\ \forall u, v \in D(\mathcal{E}). \end{aligned}$$

Theorem 1.3. *(i) There is a one-to-one correspondence between all the dense linear operators $(L, D(L))$ satisfying (L.1)–(L.3) and all the coercive closed forms $(\mathcal{E}, D(\mathcal{E}))$. The correspondence is given by*

$$D(L) \subset D(\mathcal{E}) \ \ \textit{and} \ \ \mathcal{E}(u,v) = (-Lu, v) \ \ \textit{for all}\ u \in D(L), v \in D(\mathcal{E}).$$

In this case $(L, D(L))$ is called the generator of $(\mathcal{E}, D(\mathcal{E}))$.

(ii) Given a coercive closed form $(\mathcal{E}, D(\mathcal{E}))$, the corresponding generator $(L, D(L))$ is uniquely determined by

$$\begin{aligned} D(L) &= \{u \in D(\mathcal{E}) \,|\, \exists\ w \in H \ \textit{such that}\ \mathcal{E}(u,v) = (-w, v),\ \forall v \in D(\mathcal{E})\}, \\ Lu &= w,\ \textit{if}\ u \in D(L)\ \textit{and}\ w\ \textit{is as above}. \end{aligned}$$

(iii) Given a dense linear operator $(L, D(L))$ satisfying (L.1)–(L.3), the corresponding coercive closed form is uniquely determined by

$$\begin{aligned} \mathcal{E}(u,v) &= (-Lu, v), \quad \forall u, v \in D(L), \\ D(\mathcal{E}) &= \overline{D(L)}^{\tilde{\mathcal{E}}_1}, \end{aligned}$$

where $\overline{D(L)}^{\tilde{\mathcal{E}}_1}$ is the completion of $D(L)$ w.r.t. the norm induced by $\tilde{\mathcal{E}}_1$.

Therefore there is a one-to-one correspondence among the strongly continuous contraction analytic semigroups $(T_t)_{t>0}$, the strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ satisfying sector condition, the dense linear operator $(L, D(L))$ with $(L.1)$-$(L.3)$ and coercive closed form $(\mathcal{E}, D(\mathcal{E}))$. In addition, given a the strongly continuous contraction analytic semigroup $(T_t)_{t>0}$, the corresponding coercive closed form is uniquely determined by (see (AFRS1995))

$$
\begin{aligned}
D(\mathcal{E}) &= \{u \in H \mid \sup_{t>0} \frac{1}{t}(u - T_t u, u) < \infty\}, \\
\mathcal{E}(u,v) &= \lim_{t \to 0} \frac{1}{t}(u - T_t u, v), \ \forall u, v \in D(\mathcal{E}).
\end{aligned}
$$

We now replace H by the concrete Hilbert space $L^2(E;m) := L^2(E;\mathcal{B};m)$ with usual inner product $(\ ,\ )$, where $(E;\mathcal{B};m)$ is a measure space. As usual we set for $u, v : E \to \mathbb{R}$

$$
u \vee v := \sup(u,v),\ u \wedge v := \inf(u,v),\ u^+ := u \vee 0.
$$

Definition 1.6. *(i) A strongly continuous contraction semigroup $(T_t)_{t>0}$ on $L^2(E;m)$ is sub-Markovian if*

$$
f \in L^2(E;m), 0 \le f \le 1 \ m\text{-a.e.} \Rightarrow 0 \le T_t f \le 1 \ m\text{-a.e.},\ \forall t > 0. \tag{1.4}
$$

(ii) A strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on $L^2(E;m)$ is sub-Markovian if

$$
f \in L^2(E;m), 0 \le f \le 1 \ m\text{-a.e.} \Rightarrow 0 \le \alpha G_\alpha f \le 1 \ m\text{-a.e.},\ \forall \alpha > 0. \tag{1.5}
$$

(iii) A densely defined linear operator $(L, D(L))$ on $L^2(E;m)$ is Dirichlet if

$$
(Lu, (u-1)^+) \le 0, \quad \forall u \in D(L). \tag{1.6}
$$

(iv) A coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is semi-Dirichlet if

$$
u \in D(\mathcal{E}) \Rightarrow u^+ \wedge 1 \in D(\mathcal{E}) \text{ and } \mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \ge 0. \tag{1.7}
$$

If

$$
\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0
$$

also holds, then we say $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form.

Theorem 1.4. *Let $(T_t)_{t>0}$, $(G_\alpha)_{\alpha>0}$ and $(L, D(L))$ be the semigroups, resolvents and generators of a coercive closed form $(\mathcal{E}, D(\mathcal{E}))$ respectively. Then (1.4)$\Leftrightarrow$(1.5) $\Leftrightarrow$ (1.6) $\Leftrightarrow$(1.7).*

Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E;m)$, F be a closed subset E, define

$$D(\mathcal{E})_F = \{f \in D(\mathcal{E}) \mid f(x) = 0, \quad m.a.e. \text{ for } \ x \in E - F\}.$$

In the following, we will give the definitions of nest, quasi-continuous and exceptional sets in the framework of semi-Dirichlet forms, which are used frequently in this thesis.

Definition 1.7. *(i) Let $(\mathcal{E}, D(\mathcal{E}))$ be a semi-Dirichlet form on $L^2(E;m)$, $\{F_k\}_{k\geq 1}$ be an increasing sequence of closed sets, if $\cup_{k\geq 1} D(\mathcal{E})_{F_k}$ is $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$-dense in $D(\mathcal{E})$, then we say that $\{F_k\}_{k\geq 1}$ is an $\mathcal{E}$-nest.*
(ii) We say $u \in D(\mathcal{E})$ is quasi-continuous if there exists an $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ such that for any $k \geq 1$, f is continuous on F_k.
(iii) We say $N \subset E$ is an $\mathcal{E}-$exceptional set if there is an $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ such that $N \subset \bigcap_{k\geq 1} F_k^c$. We say that a property of points in E holds $\mathcal{E}-$quasi-everywhere (abbreviated $\mathcal{E} - q.e.$), if the property holds outside some $\mathcal{E}-$exceptional set.

Definition 1.8. *(MR1992, IV, Definition 1.8) Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a Markov process with state space E, life time ζ, cemetery Δ, and shift operators θ_t, $t \geq 0$. $\mathbf{M}$ is called a right process if it satisfies the following three conditions:*

(i) $\mathbf{M}$ is normal, i.e., $P_x(X_0 = x) = 1$ for all $x \in E_\Delta$.

(ii) $\mathbf{M}$ is right continuous, i.e., for each $\omega \in \Omega$, $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$.

(iii) $P_x(R_\alpha f(X_t)$ is right continuous on $[0,\infty)$with respect to $t) = 1$ for all $x \in E, \alpha > 0$, and nonnegative $f \in C_b(E)$. (Hereafter $C_b(E)$ denotes the set of all bounded continuous functions on E, $R_\alpha f := E.[\int_0^\infty e^{-\alpha t} f(X_t)dt]$.)

Let $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a right process, denote the transition semigroup of M by

$$P_t f(x) := E_x[f(X_t)], \quad t \geq 0, \quad f \in \mathcal{B}^+(E).$$

Definition 1.9. *(MOR1995, Definition 3.3) A right process $\mathbf{M}$ with state space E is said to be (properly) associated with a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$*

if and only if $P_t f$ *is an* ($\mathcal{E}$*-quasi-continuous*) m*-version of* $T_t f$ *for all* $f \in \mathcal{B}_b(E) \cap L^2(E;m)$ *and all* $t > 0$.

Definition 1.10. *(MOR1995, Definition 3.5) A semi-Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(E;m)$ *is called quasi-regular if:*

(i) There exists an $\mathcal{E}$*-nest* $\{E_k\}_{k\in\mathbb{N}}$ *consisting of metrizable compact sets.*

(ii) There exists an $\tilde{\mathcal{E}}_1^{1/2}$*-dense subset of* $D(\mathcal{E})$ *whose elements have* $\mathcal{E}$*-quasi-continuous* m*-versions.*

(iii) There exist $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, *having* $\mathcal{E}$*-quasi-continuous* m*-versions* $\tilde{u}_n$, $n \in \mathbb{N}$, *and an* $\mathcal{E}$*-exceptional set* $N \subset E$ *such that* $\{\tilde{u}_n \,|\, n \in \mathbb{N}\}$ *separates the points of* $E \backslash N$.

Theorem 1.5. *(MS2010b, Theorem 1.40) Let* $(\mathcal{E}, D(\mathcal{E}))$ *be a semi-Dirichlet form on* $L^2(E;m)$*, where* E *is a Lusin metrizable space. Then there exists a right process* $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ *associated with* $(\mathcal{E}, D(\mathcal{E}))$ *if and only if* $(\mathcal{E}, D(\mathcal{E}))$ *is quasi-regular. Moreover,* $\mathbf{M}$ *is always properly associated with* $(\mathcal{E}, D(\mathcal{E}))$.

Definition 1.11. *(HMS2006, Definition 3.7) A semi-Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(E,m)$ *is said to be quasi-homeomorphic to a semi-Dirichlet form* $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ *on* $L^2(E^\sharp; m^\sharp)$*, if there exists a map* $j : \bigcup_{k\geq 1} F_k \to \bigcup_{k\geq 1} F_k^\sharp$*, where* $\{F_k\}_{k\in\mathbb{N}}$ *is an* $\mathcal{E}$*-nest in* E *and* $\{F_k^\sharp\}_{k\in\mathbb{N}}$ *an* $\mathcal{E}^\sharp$*-nest in* $E^\sharp$*, such that*

(i) j *is a topological homeomorphism from* F_k *onto* $F_k^\sharp$ *for each* $k \in \mathbb{N}$.

(ii) $m^\sharp = m \circ j^{-1}$.

(iii) $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp)) = (\mathcal{E}^j, D(\mathcal{E}^j))$*, where* $(\mathcal{E}^j, D(\mathcal{E}^j))$ *is the image of* $(\mathcal{E}, D(\mathcal{E}))$ *under* j.

The map j *is called a quasi-homeomorphism from* $(\mathcal{E}, D(\mathcal{E}))$ *to* $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$.

Let E be a locally compact separable metric space and m be a positive Radon measure on E with $\mathrm{supp}[m] = E$. We say that a semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is regular if $C_0(E) \cap D(\mathcal{E})$ is dense in $D(\mathcal{E})$ with respect to the $\mathcal{E}_1$-norm and $C_0(E) \cap D(\mathcal{E})$ is dense in $C_0(E)$ with respect to the uniform norm $\| \ \|_\infty$. Hereafter $C_0(E)$ denotes the set of all continuous functions on E with compact supports.

Theorem 1.6. *(HMS2006, Theorem 3.8) A semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;m)$ is quasi-regular if and only if it is quasi-homeomorphic to a regular semi-Dirichlet form $(\mathcal{E}^\sharp, D(\mathcal{E}^\sharp))$ on $L^2(E^\sharp; m^\sharp)$.*

Therefore many results established for regular semi-Dirichlet forms are applicable to quasi-regular semi-Dirichlet forms.

Next, we introduce Beurling-Deny formula for non-symmetric Dirichlet form.

Definition 1.12. *(MS2010b, Definition 1.95) Let Q be a σ-finite positive Borel measure on $E \times E\backslash d$. A measurable function f on $E \times E\backslash d$ is said to be integrable w.r.t. Q in the sense of symmetric principle value (abbreviated by SPV integrable) if there exists an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of $E \times E\backslash d$ satisfying $Q((E \times E\backslash d) \setminus (\bigcup_{n\geq 1} A_n)) = 0$, $I_{A_n}(x,y) = I_{A_n}(y,x)$ for all $x, y \in E$, f is integrable on each A_n, $n \geq 1$, and for any sequence $\{A_n\}_{n\in\mathbb{N}}$ with these properties, the limit*

$$SPV \int_{E\times E\backslash d} f(x,y)Q(dx,dy) := \lim_{n\to\infty} \int_{A_n} f(x,y)Q(dx,dy)$$

exists and is independent of the specific choice of the sequence $\{A_n\}_{n\in\mathbb{N}}$.

Theorem 1.7. *(HMS2010, Theorem 1.3 (i) (ii)) Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular non-symmetric Dirichlet form on $L^2(E;m)$.*
(i) There exist a unique σ-finite positive Borel measure J on $E \times E\backslash d$ and a unique positive Radon measure K on E such that for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in I(v)$,

$$\mathcal{E}(u,v) = \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dx,dy) + \int_E u(x)v(x)K(dx).$$

where $I_q(v) := \{u \in C_0(E) \cap D(\mathcal{E}) \,|\, u$ is constant on a neighbourhood of $supp[v]\}$.
(ii) Define

$$\mathcal{A}(v) := \{u \in C_0(E) \cap D(\mathcal{E}) \,|\, (u(y) - u(x))v(y) \text{ is SPV integrable w.r.t. } J(dx,dy)\}.$$

Then for $v \in C_0(E) \cap D(\mathcal{E})$ and $u \in \mathcal{A}(v)$, we have the following unique decomposition:

$$\begin{aligned}\mathcal{E}(u,v) &= \mathcal{E}^c(u,v) + SPV \int_{E\times E\backslash d} 2(u(y) - u(x))v(y)J(dx,dy) \\ &\quad + \int_E u(x)v(x)K(dx),\end{aligned}$$

where $\mathcal{E}^c$ *satisfies the left strong local property in the sense that* $I_q[v] \subset \mathcal{A}(v)$ *and* $\mathcal{E}^c(u, v) = 0$ *whenever* $v \in C_0(E) \cap D(\mathcal{E})$ *and* $u \in I(v)$. $\mathcal{E}^c$, J *and* K *are called the diffusion part, jumping measure and killing measure of* $(\mathcal{E}, D(\mathcal{E}))$, *respectively.*

1.2 Main results

For the generalized Feynman-Kac semigroups associated with nearly symmetric Markov processes, we have the following results:

Theorem 1.8. *Suppose that* X *is a right process which is associated with a (non-symmetric) Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(E; m)$. *Let* $u \in D(\mathcal{E})$. *Assume that* $J_1(E \times E \backslash d) < \infty$, *where* J_1 *is the anti-symmetric part of the jumping measure in the Beurling-Deny decomposition of* $(\mathcal{E}, D(\mathcal{E}))$ *and* d *means the elements on the diagonal. Then the following two conditions are equivalent to each other:*

(i) There exists a constant $\alpha_0 \geq 0$ *such that*

$$Q^u(f, f) \geq -\alpha_0 (f, f)_m, \quad \forall f \in D(\mathcal{E})_b,$$

where $D(\mathcal{E})_b = D(\mathcal{E}) \cap L^\infty(E, m)$.

(ii) There exists a constant $\alpha_0 \geq 0$ *such that*

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0,$$

where $\|P_t^u\|_2$ *means the operator norm of* P_t^u *from* $L^2(E, m)$ *to* $L^2(E, m)$.

Furthermore, if one of these conditions holds, then the semigroup $(P_t^u)_{t \geq 0}$ *is strongly continuous on* $L^2(E; m)$.

Theorem 1.9. *Let* U *be an open set of* $\mathbf{R}^d$ *and* m *be a positive Radon measure on* U *with* $\mathrm{supp}[m] = U$. *Suppose that* X *is a right process which is associated with a (non-symmetric) Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(U; m)$ *such that* $C_0^\infty(U)$ *is dense in* $D(\mathcal{E})$. *Then the conclusions of Theorem 1.1 remain valid without assuming that* $J_1(E \times E \backslash d) < \infty$.

To get Fukushima's decomposition in the semi-Dirichlet forms setting, we need to put one assumption on the quasi-regular semi-Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$.

Fix a function $\phi \in L^2(E; m)$ with $0 < \phi \leq 1$ m-$a.e.$ and set $\hat{h} = \hat{G}_1\phi$. Let V be a quasi-open subset of E. Define $\tau_V = \inf\{t \geq 0 \mid X_t \notin V\}$. Define the part process $X^V = (X_t^V)_{t\geq 0}$ of X on V as follows

$$X_t^V = X_t \text{ for } t < \tau_V, \quad X_t^V = \Delta \text{ for } t \geq \tau_V.$$

Denote $(\mathcal{E}^V, D(\mathcal{E})_V)$ the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V; m)$. Denote by $(G_\alpha^V)_{\alpha\geq 0}$ and $(\hat{G}_\alpha^V)_{\alpha\geq 0}$ the resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$, respectively. Define $\bar{h}^V := \hat{h}|_V \wedge \hat{G}_1^V\phi$.

For an additive functional (abbreviate as AF) $A = (A_t)_{t\geq 0}$ of X^V, we define

$$e^V(A) := \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}^V\cdot m}(A_t^2)$$

whenever the limit exists in $[0, \infty]$. Define

$$\begin{aligned} \dot{\mathcal{M}}^V &:= \{M \mid M \text{ is an AF of } X^V,\ E_x(M_t^2) < \infty, E_x(M_t) = 0 \\ &\qquad \text{for all } t \geq 0 \text{ and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(M) < \infty\}, \end{aligned}$$

$$\begin{aligned} \mathcal{N}_c^V &:= \{N \mid N \text{ is a CAF of } X^V, E_x(|N_t|) < \infty \text{ for all } t \geq 0 \\ &\qquad \text{and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(N) = 0\}, \end{aligned}$$

$$\begin{aligned} \Theta &:= \{\{V_n\} \mid V_n \text{ is } \mathcal{E}\text{-quasi-open},\ V_n \subset V_{n+1}\ \mathcal{E}\text{-}q.e., \\ &\qquad \forall\ n \in \mathbb{N},\ \text{and } E = \cup_{n=1}^\infty V_n\ \mathcal{E}\text{-}q.e.\}, \end{aligned}$$

and

$$\begin{aligned} D(\mathcal{E})_{loc} &:= \{u \mid \exists\ \{V_n\} \in \Theta \text{ and } \{u_n\} \subset D(\mathcal{E}) \\ &\qquad \text{such that } u = u_n\ m\text{-}a.e. \text{ on } V_n,\ \forall\ n \in \mathbb{N}\}. \end{aligned}$$

Define

$$\begin{aligned} \dot{\mathcal{M}}_{loc} &:= \{M \mid M \text{ is a local AF of } \mathbf{M},\ \exists\ \{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \mid M^n \in \dot{\mathcal{M}}^{V_n}\} \\ &\qquad \text{such that } E_n \subset V_n,\ M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\} \end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{c,loc} \quad := \quad & \{N \,|\, N \text{ is a local AF of } \mathbf{M},\ \exists\, \{V_n\}, \{E_n\} \in \Theta \text{ and } \{N^n \,|\, N^n \in \mathcal{N}_c^{V_n}\} \\ & \qquad\qquad \text{such that } E_n \subset V_n,\ N_{t\wedge\tau_{E_n}} = N^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}.\end{aligned}$$

We use $\mathcal{M}_{loc}^{[\![0,\zeta[\![}$ to denote the family of all local martingales on $[\![0,\zeta[\![$ (cf. (HWY1992, §8.3)).

We put the following assumption:

Assumption 1.1. *There exists $\{V_n\} \in \Theta$ such that, for each $n \in \mathbb{N}$, there exists a Dirichlet form $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n; m)$ and a constant $C_n > 1$ such that $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and for any $u \in D(\mathcal{E})_{V_n}$,*

$$\frac{1}{C_n}\eta_1^{(n)}(u,u) \leq \mathcal{E}_1(u,u) \leq C_n \eta_1^{(n)}(u,u).$$

Theorem 1.10. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 1.1. Then, for any $u \in D(\mathcal{E})_{loc}$, there exist $M^{[u]} \in \dot{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$ such that*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \ \text{ for } \mathcal{E}\text{-}q.e.\ x \in E. \tag{1.8}$$

Moreover, $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$. Decomposition (1.8) is unique up to the equivalence of local AFs.

For local martingale additive functionals, we have the following result.

Theorem 1.11. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 1.1. Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{loc}$, $1 \leq i \leq m$. Then $\Phi(u) \in D(\mathcal{E})_{loc}$ and*

$$M^{[\Phi(u)],c} = \sum_{i=1}^m \Phi_{x_i}(u) \cdot M^{[u_i],c} \text{ on } [0,\zeta), \quad P_x\text{-}a.s. \ \text{ for } \mathcal{E}\text{-}q.e.\ x \in E.$$

1.3 Organization of the thesis

This thesis is organized as follows. In Chapter 2, we will give the results on the strong continuity of the generalized Feynman-Kac semigroups for Markov processes which are associated with (non-symmetric) Dirichlet forms. In Chapter 3, we will present the results on Fukushima's decomposition and a transform formula for local martingale additive functionals in the semi-Dirichlet forms case. In Chapter 4, we will state the future work.

Chapter 2

Generalized Feynman-Kac transformation

Let E be a metrizable Lusin space and $X = ((X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a right (continuous strong Markov) process on E. Suppose that X is associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$, where m is a σ-finite measure on the Borel σ-algebra $\mathcal{B}(E)$ of E. Then, by (MR1992, IV, Theorem 6.7) (cf. also (F2001, Theorem 3.22)), $(\mathcal{E}, D(\mathcal{E}))$ is quasi-regular. Moreover, $(\mathcal{E}, D(\mathcal{E}))$ is quasi-homeomorphic to a regular Dirichlet form (see (CMR1994)).

Let $u \in D(\mathcal{E})$. Then, we have Fukushima's decomposition (cf. (MR1992, VI, Theorem 2.5))

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^u + N_t^u,$$

where $\tilde{u}$ is a quasi-continuous m-version of u, M_t^u is a square integrable martingale additive functional (MAF) and N_t^u is a continuous additive functional (CAF) of zero energy. For $x \in E$, denote by E_x the expectation with respect to P_x. Define the generalized Feynman-Kac transformation

$$P_t^u f(x) = E_x[e^{N_t^u} f(X_t)], \quad f \geq 0 \text{ and } t \geq 0.$$

In this chapter, we will investigate the strong continuity of the semigroup $(P_t^u)_{t\geq 0}$ on $L^2(E; m)$. This part of the thesis is based on the paper (MS2010a), which will appear in the Journal of Theoretical Probability.

Note that many useful tools of symmetric Dirichlet forms, e.g. time reversal and Lyons-Zheng decomposition, do not apply well to the non-symmetric Dirichlet forms setting. That makes the problem more difficult. Also, we would like to point out that the Girsanov transformed process of X induced by M_t^u and the Girsanov transformed process of $\hat{X}$ induced by $\hat{M}_t^u$ are not in duality in general (cf. (CS2009)), where $\hat{X}$ is the dual process of X and $\hat{M}_t^u$ is the martingale part of $\tilde{u}(\hat{X}_t) - \tilde{u}(\hat{X}_0)$. The method of this part is inspired by (CMS2007) and (CFKZ2009). We will combine the h-transform method of (CMS2007) and the localization method used in(CFKZ2009). It is worth to point out that the Beurling-Deny formula given in (HMS2006) and LeJan's transformation rule developed in (HMS2010) play a crucial role here.

Denote by J and K the jumping and killing measures of $(\mathcal{E}, D(\mathcal{E}))$, respectively. Write $\hat{J}(dx, dy) = J(dy, dx)$. Denote by $J_1 := (J - \hat{J})^+$ the positive part of the Jordan decomposition of $J - \hat{J}$. J_1 is called the dissymmetric part of J. Note that $J_0 := J - J_1$ is the largest symmetric σ-finite positive measure dominated by J. Denote by d the diagonal of the product space $E \times E$; and denote by $\|\cdot\|_2$ and $(\cdot, \cdot)_m$ the norm and inner product of $L^2(E; m)$, respectively.

Now we can state the main results of this chapter.

Theorem 2.1. *Suppose that X is a right process which is associated with a (non-symmetric) Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$. Let $u \in D(\mathcal{E})$. Assume that $J_1(E \times E \backslash d) < \infty$. Then the following two conditions are equivalent to each other:*

(i) There exists a constant $\alpha_0 \geq 0$ such that

$$Q^u(f, f) \geq -\alpha_0 (f, f)_m, \quad \forall f \in D(\mathcal{E})_b.$$

(ii) There exists a constant $\alpha_0 \geq 0$ such that

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0.$$

Furthermore, if one of these conditions holds, then the semigroup $(P_t^u)_{t \geq 0}$ is strongly continuous on $L^2(E; m)$.

Theorem 2.2. *Let U be an open set of $\mathbf{R}^d$ and m be a positive Radon measure on U with $\mathrm{supp}[m] = U$. Suppose that X is a right process which is associated with* *a (non-symmetric) Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $L^2(U;m)$ *such that* $C_0^\infty(U)$ *is dense in* $D(\mathcal{E})$*. Then the conclusions of Theorem 2.1 remain valid without assuming that* $J_1(E \times E\backslash d) < \infty$*.*

The rest of this chapter is organized as follows. In the first section, we will make necessary preparations. In the second section, we will prove the main results and give some remarks. In the last section, we will apply the results to some examples.

2.1 Preliminaries

By quasi-homeomorphism, we assume without loss of generality that X is a Hunt process and $(\mathcal{E}, D(\mathcal{E}))$ is a regular (non-symmetric) Dirichlet form on $L^2(E;m)$, where E is a locally compact separable metric space and m is a positive Radon measure on E with $\mathrm{supp}[m] = E$. We denote by Δ and ζ the cemetery and lifetime of X, respectively. It is known that every $f \in D(\mathcal{E})$ has a quasi-continuous m-version. To simplify notation, we still denote this version by f.

Let $u \in D(\mathcal{E})$. By (MR1992, III, Proposition 1.5), there exists $|u|_E \in D(\mathcal{E})$ such that $|u|_E \geq |u|$ m-a.e. on E and $\mathcal{E}_1(|u|_E, w) \geq 0$ for all $w \in D(\mathcal{E})$ with $w \geq 0$ m-a.e. on E. Similar to (FOT1994, Theorems 2.2.1 and 2.2.2), we can show that there exists a positive Radon measure η_u on E such that η_u charges no $\mathcal{E}$-exceptional set and

$$\mathcal{E}_1(|u|_E, w) = \int_E w d\eta_u, \quad w \in D(\mathcal{E}). \tag{2.1}$$

Define

$$u^* := u + |u|_E. \tag{2.2}$$

Then, u^* has a quasi-continuous m-version which is nonnegative q.e. on E. Moreover, there exists an $\mathcal{E}$-nest $\{F_n\}_{n\in\mathbf{N}}$ consisting of compact sets of E such that u^* is continuous and hence bounded on F_n for each $n \in \mathbf{N}$. Define $\tau_{F_n} = \inf\{t > 0 \,|\, X_t \notin F_n\}$. By (MR1992, IV, Proposition 5.30), $\lim_{n\to\infty} \tau_{F_n} = \zeta$ P_x-a.s. for q.e. $x \in E$.

Let (N, H) be a Lévy system of X, that is, $N(x, dy)$ is a kernel on $(E_\Delta, \mathcal{B}(E_\Delta))$ and H_t is a positive continuous additive functional (abbreviated as PCAF) with bounded

1-potential such that for any nonnegative Borel function f on $E_\Delta \times E_\Delta$ vanishing on the diagonal and any $x \in E_\Delta$,

$$E_x[\sum_{s\leq t} f(X_{s-}, X_s)] = E_x[\int_0^t \int_{E_\Delta} f(X_s, y)N(X_s, dy)dH_s].$$

Let ν be the Revuz measure of H. Define

$$B_t = \sum_{s\leq t} \left[e^{(u^*(X_{s-})-u^*(X_s))} - 1 - (u^*(X_{s-}) - u^*(X_s))\right]. \tag{2.3}$$

Note that for any $M > 0$ there exists $C_M > 0$ such that $(e^x - 1 - x) \leq C_M x^2$ for all x satisfying $x \leq M$. Since $(u^*(X_{t-}))_{t\geq 0}$ is locally bounded, $(u^*(X_t))_{t\geq 0}$ is nonnegative and M^{-u^*} is a P_x-square integrable martingale for q.e. $x \in E$, hence $(B_t)_{t\geq 0}$ is locally P_x-integrable on $[0, \zeta)$ for q.e. $x \in E$. Here and henceforth the phrase "on $[0, \zeta)$" is understood as "on the optional set $[\![0, \zeta[\![$ of interval type" in the sense of (HWY1992, Chap. VIII, 3). By (FOT1994, (A.3.23)), one finds that the dual predictable projection of $(B_t)_{t\geq 0}$ is given by

$$B_t^p = \int_0^t \int_{E_\Delta} [e^{(u^*(X_s)-u^*(y))} - 1 - (u^*(X_s) - u^*(y))]N(X_s, dy)dH_s.$$

We set

$$M_t^d = B_t - B_t^p \tag{2.4}$$

and denote

$$M_t = M_t^{-u^*} + M_t^d. \tag{2.5}$$

Note that for any $M > 0$ there exists $D_M > 0$ such that $(e^x - 1 - x)^2 \leq D_M x^2$ for all x satisfying $x \leq M$. Since $(u^*(X_{t-}))_{t\geq 0}$ is locally bounded, $(u^*(X_t))_{t\geq 0}$ is nonnegative and M^{-u^*} is a P_x-square integrable martingale for q.e. $x \in E$, hence $(M_t^d)_{t\geq 0}$ is a locally square integrable martingale additive functional (abbreviated as MAF) on $[0, \zeta)$ by (HWY1992, Theorem 7.40). Therefore $(M_t)_{t\geq 0}$ is a locally square integrable MAF on $[0, \zeta)$. We denote the Revuz measure of $(< M >_t)_{t\geq 0}$ by $\mu_{<M>}$ (cf. (CFKZ2008a, Remark 2.2)).

Let $M_t^{-u^*,c}$ be the continuous part of $M_t^{-u^*}$. Define

$$A_t^{-u^*} = B_t^p + \frac{1}{2} < M^{-u^*,c} >_t. \tag{2.6}$$

Then $(A_t^{-u^*})_{t\geq 0}$ is a PCAF. Denote by μ_{-u^*} the Revuz measure of $(A_t^{-u^*})_{t\geq 0}$. Then

$$\begin{aligned} \mu_{-u^*}(dx) &= \int_{E_\triangle}[e^{(u^*(x)-u^*(y))}-1-(u^*(x)-u^*(y))]N(x,dy)\nu(dx) \\ &\quad +\frac{1}{2}\mu_{<M^{-u^*,c}>}(dx). \end{aligned} \tag{2.7}$$

Define

$$\mu_{-u} := \mu_{-u^*} + \eta_u - |u|_E m \tag{2.8}$$

and

$$\mu'_{-u} := \mu_{-u^*} + \eta_u + |u|_E m.$$

Lemma 2.1. *There exists an $\mathcal{E}$-nest $\{F'_n\}_{n\in\mathbf{N}}$ consisting of compact sets of E which satisfies the following condition: $\forall\varepsilon>0$, there exists a constant $A_\varepsilon^n>0$ such that $\forall f\in D(\mathcal{E})$,*

$$\int_E f^2 I_{F'_n} d(\mu_{<M>}+\mu'_{-u}) \leq \varepsilon\mathcal{E}(f,f)+A_\varepsilon^n(f,f).$$

Proof. Let $(\tilde{\mathcal{E}},D(\mathcal{E}))$ be the symmetric part of $(\mathcal{E},D(\mathcal{E}))$. Denote by $\{\tilde{G}_\alpha\}_{\alpha\geq 0}$ the resolvent of $(\tilde{\mathcal{E}},D(\mathcal{E}))$. Let $\phi\in L^2(E,m)$ and $0<\phi\leq 1$ $m.a.e$, set $\tilde{h}=\tilde{G}_1\phi$. Define $(\tilde{\mathcal{E}}^{\tilde{h}},D(\tilde{\mathcal{E}}^{\tilde{h}}))$, the $\tilde{h}-$transform of $(\tilde{\mathcal{E}},D(\mathcal{E}))$, by

$$\begin{aligned} D(\tilde{\mathcal{E}}^{\tilde{h}}) &:= \{u\in L^2(E;\tilde{h}^2m)|\ u\tilde{h}\in D(\mathcal{E})\} \\ \tilde{\mathcal{E}}^{\tilde{h}}(u,v) &:= \tilde{\mathcal{E}}(\tilde{h}u,\tilde{h}v). \end{aligned}$$

To simplify notation, we denote $\mu:=\mu_{<M>}+\mu'_{-u}$. By (MR1995, Proposition 4.2), we know that an $\tilde{\mathcal{E}}_1^{\tilde{h}}$-nest is also an $\mathcal{E}_1$-nest, so $\mu\in S(\tilde{\mathcal{E}}_1^{\tilde{h}})$. Note that $(\tilde{\mathcal{E}}_1^{\tilde{h}},D(\tilde{\mathcal{E}}^{\tilde{h}}))$ is a symmetric Dirichlet form on $L^2(E;\tilde{h}^2m)$. By (AM1992, Theorem 2.4), there is an $\tilde{\mathcal{E}}_1^{\tilde{h}}-$nest $\{F_k^1\}_{k\geq 1}$ consisting of compact sets such that $I_{F_k^1}\mu\in S_K(\tilde{\mathcal{E}}_1^{\tilde{h}})$ (the Kato class of smooth measure). Set $F'_k=F_k^1\bigcap F_k$, where $\{F_k\}_{k\geq 1}$ is an $\tilde{\mathcal{E}}_1^{\tilde{h}}-$nest such that $\tilde{h}$ is continuous on each F_k. Then $\{F'_k\}_{k\geq 1}$ is an $\tilde{\mathcal{E}}_1^{\tilde{h}}-$nest and hence an $\mathcal{E}_1-$nest. By (AM1991, Proposition 3.1(i)), for any $\varepsilon>0$, there is a constant $\sigma>0$ such that for $g\in D(\tilde{\mathcal{E}}^{\tilde{h}})$,

$$\int g^2 I_{F'_k} d\mu \leq \frac{\varepsilon}{\|\ \tilde{h}\ |_{F'_k}\|_\infty^2}\tilde{\mathcal{E}}^{\tilde{h}}(g,g)+\sigma(g,g)_{\tilde{h}^2m}.$$

Let $f \in D(\mathcal{E})$. Then $\frac{f}{\tilde{h}} \in D(\tilde{\mathcal{E}}^{\tilde{h}})$. Note that any smooth measure dose not charge set of zero capacity. Then

$$\begin{aligned}\int f^2 I_{F'_k} d\mu &= \int (\frac{f}{\tilde{h}})^2 (\tilde{h})^2 I_{F'_k} d\mu \leq \| \tilde{h} \mid_{F'_k} \|_\infty^2 \int (\frac{f}{\tilde{h}})^2 I_{F'_k} d\mu \\ &\leq \varepsilon \tilde{\mathcal{E}}^{\tilde{h}} (\frac{f}{\tilde{h}}, \frac{f}{\tilde{h}}) + \sigma(\frac{f}{\tilde{h}}, \frac{f}{\tilde{h}})_{\tilde{h}^2 m} = \varepsilon \mathcal{E}(f, f) + \sigma(f, f).\end{aligned}$$

□

Remark 2.1. *Here we use $\tilde{h}$-transform to prove the following result: Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular Dirichlet form and $\mu \in S$. Then there is an $\mathcal{E}$-nest $\{F_k\}_{k\geq 1}$ satisfying the following conditions: for any $\varepsilon > 0$, there exist a constant A_ε^n such that for any $f \in D(\mathcal{E})$,*

$$\int_E I_{F_n} f^2 d\mu \leq \varepsilon \mathcal{E}(f, f) + A_\varepsilon^n \|f\|_2^2. \tag{2.9}$$

In fact, (2.9) hold for any quasi-regular semi-Dirichlet form. We will prove it in the Appendix by another method.

To simplify notation, we still use F_n to denote $F_n \cap F'_n$ for $n \in \mathbf{N}$. Let E_n be the fine interior of F_n with respect to X. Define $D(\mathcal{E})_n := \{f \in D(\mathcal{E}) \mid f = 0 \text{ q.e. on } E_n^c\}$, $\tau_{E_n} = \inf\{t > 0 \mid X_t \notin E_n\}$ and

$$\bar{P}_t^{u,n} f(x) := E_x[e^{M_t^{-u^*} - N_t^{|u|_E}} f(X_t); t < \tau_{E_n}].$$

2.2 Proofs of the main results

2.2.1 The bilinear form associated with $(\bar{\mathbf{P}}_{\mathbf{t}}^{\mathbf{u,n}})_{\mathbf{t}\geq \mathbf{0}}$ on $\mathbf{L^2(E_n; m)}$

For $n \in \mathbf{N}$, we define the bilinear form $(\bar{Q}^{u,n}, D(\mathcal{E})_n)$ by

$$\bar{Q}^{u,n}(f, g) = \mathcal{E}(f, g) - \int_E g d\mu_{<M^f, M>} - \int_E fg d\mu_{-u}, \quad f, g \in D(\mathcal{E})_n. \tag{2.10}$$

By Lemma 2.1 and the choice of $\{F_n\}_{n\geq 1}$, we know that for every $\varepsilon > 0$, there exists a constant $A_\varepsilon^n > 0$ such that

$$\int_E w^2 d(\mu_{<M>} + \mu'_{-u}) \leq \varepsilon \mathcal{E}(w, w) + A_\varepsilon^n \|w\|_2^2, \quad w \in D(\mathcal{E})_n.$$

Suppose that $|\mathcal{E}(f,g)| \leq k_1 \mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}}$ for all $f,g \in D(\mathcal{E})$ and some constant $k_1 > 0$. Then

$$\begin{aligned}
|\bar{Q}^{u,n}(f,g)| &\leq k_1 \mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}} + \left(\int_E d\mu_{<M^f>}\right)^{\frac{1}{2}} \left(\int_E g^2 d\mu_{<M>}\right)^{\frac{1}{2}} \\
&\quad + \left(\int_E f^2 d\mu'_{-u}\right)^{\frac{1}{2}} \left(\int_E g^2 d\mu'_{-u}\right)^{\frac{1}{2}} \\
&\leq k_1 \mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}} + (\max(\varepsilon, A^n_\varepsilon))^{\frac{1}{2}} [2\mathcal{E}(f,f)]^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}} \\
&\quad + \max(\varepsilon, A^n_\varepsilon) \cdot \mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}} \\
&\leq \theta_n \mathcal{E}_1(f,f)^{\frac{1}{2}} \mathcal{E}_1(g,g)^{\frac{1}{2}}, \qquad (2.11)
\end{aligned}$$

where $\theta_n := (k_1 + \sqrt{2\max(\varepsilon, A^n_\varepsilon)} + \max(\varepsilon, A^n_\varepsilon))$.

Fix an $\varepsilon < (\sqrt{2}-1)/(\sqrt{2}+1)$ and set $\alpha_n := 2A^n_\varepsilon$. Then

$$\begin{aligned}
\bar{Q}^{u,n}_{\alpha_n}(f,f) &:= \bar{Q}^{u,n}(f,f) + \alpha_n(f,f) \\
&\geq \mathcal{E}(f,f) - \left(\int_E d\mu_{<M^f>}\right)^{\frac{1}{2}} \left(\int_E f^2 d\mu_{<M>}\right)^{\frac{1}{2}} \\
&\quad - \int_E f^2 d\mu'_{-u} + \alpha_n(f,f) \\
&\geq \mathcal{E}(f,f) - (\varepsilon\ \mathcal{E}(f,f) + A^n_\varepsilon \|f\|_2^2)^{\frac{1}{2}} [2\mathcal{E}(f,f)]^{\frac{1}{2}} \\
&\qquad - (\varepsilon\ \mathcal{E}(f,f) + A^n_\varepsilon \|f\|_2^2) + \alpha_n(f,f) \\
&\geq \mathcal{E}(f,f) - \frac{1}{\sqrt{2}}((1+\varepsilon)\mathcal{E}(f,f) + A^n_\varepsilon \|f\|_2^2) \\
&\qquad - (\varepsilon\ \mathcal{E}(f,f) + A^n_\varepsilon \|f\|_2^2) + \alpha_n(f,f) \\
&\geq \frac{\sqrt{2} - 1 - (\sqrt{2}+1)\varepsilon}{\sqrt{2}} \mathcal{E}(f,f) + \frac{(\sqrt{2}-1)A^n_\varepsilon}{\sqrt{2}} \|f\|_2^2. \qquad (2.12)
\end{aligned}$$

By (2.11), (2.12) and (MR1992, I, Proposition 3.5), we know that $(\bar{Q}^{u,n}_{\alpha_n}, D(\mathcal{E}))$ is a coercive closed form on $L^2(E_n; m)$.

Theorem 2.3. *For each $n \in \mathbf{N}$, $(\bar{P}^{u,n}_t)_{t\geq 0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n; m)$ with $\|\bar{P}^{u,n}_t\|_2 \leq e^{\beta_n t}$ for every $t > 0$ and some constant $\beta_n > 0$. Moreover, the coercive closed form associated with $(e^{-\beta_n t}\bar{P}^{u,n}_t)_{t\geq 0}$ is given by $(\bar{Q}^{u,n}_{\beta_n}, D(\mathcal{E})_n)$.*

Proof. The proof is similar to that of (FK2004, Theorem 1.1), which is based on a key lemma (see (FK2004, Lemma 3.2)) and a remarkable localization method. In fact, the proof of our Theorem 2.3 is simpler since $I_{F_n}(\mu_{<M>}+\mu'_{-u})$ is in the Kato class instead of the Hardy class and there is no time reversal part in the semigroup $(\bar{P}_t^{u,n})_{t\geq 0}$. We omit the details of the proof here and only give the following key lemma, which is the counterpart of (FK2004, Lemma 3.2). $\square$

Lemma 2.2. *Let $(L^{\bar{Q}^{u,n}}, D(L^{\bar{Q}^{u.n}}))$ be the generator of $(\bar{Q}^{u,n}, D(\mathcal{E})_n)$. Then, for any $f \in D(L^{\bar{Q}^{u,n}})$, we have*

$$\begin{aligned} f(X_t)e^{M_t^{-u^*}-N_t^{|u|_E}} &= f(X_0)+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}dM_s^f \\ &+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}f(X_{s-})dM_s \\ &+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}}L^{\bar{Q}^{u,n}}f(X_s)ds \end{aligned}$$

P_m-a.s. on $\{t<\tau_{E_n}\}$.

Proof. Let $f \in D(L^{\bar{Q}^{u,n}})$ and $g \in D(\mathcal{E})_n$. Then, by (2.10), we get

$$\begin{aligned} \mathcal{E}(f,g) &= \bar{Q}^{u,n}(f,g)+\int_E g d\mu_{<M^f,M>}+\int_E fgd\mu_{-u} \\ &= -(L^{\bar{Q}^{u,n}}f,g)+\int_E g d\mu_{<M^f,M>}+\int_E fgd\mu_{-u}. \end{aligned} \tag{2.13}$$

By (2.1), (2.13) and (O1988, Theorem 5.2.7), we find that $(N_t^{|u|_E})_{t\geq 0}$ is a CAF of bounded variation and

$$N_t^f = \int_0^t L^{\bar{Q}^{u,n}}f(X_s)ds-<M^f,M>_t-\int_0^t f(X_s)d(A_s^{-u^*}-N_s^{|u|_E})$$

for $t<\tau_{E_n}$. Therefore, for $t<\tau_{E_n}$, we have

$$\begin{aligned} f(X_t)-f(X_0) &= M_t^f+N_t^f \\ &= M_t^f+\int_0^t L^{\bar{Q}^{u,n}}f(X_s)ds-<M^f,M>_t \\ &\quad -\int_0^t f(X_s)d(A_s^{-u^*}-N_s^{|u|_E}). \end{aligned} \tag{2.14}$$

By Itô's formula (cf. (P2005, II, Theorem 33)), (2.14) and (2.4)−(2.6), we obtain that for $t < \tau_{E_n}$

$$
\begin{aligned}
& f(X_t)e^{M_t^{-u^*}-N_t^{|u|_E}} \\
= \; & f(X_0) + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} df(X_s) + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) d(M_s^{-u^*} - N_s^{|u|_E}) \\
& + \frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) d < M^{-u^*,c} >_s + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^{f,c}, M^{-u^*,c} >_s \\
& + \sum_{s \le t} [f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}} - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \\
& \qquad - e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle f(X_s) - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle M_s^{-u^*}] \\
= \; & f(X_0) + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} dM_s^f + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} L^{\bar{Q}^{u,n}} f(X_s) ds \\
& - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^f, M >_s - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_s) d(A_s^{-u^*} - N_s^{|u|_E}) \\
& + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) d(M_s^{-u^*} - N_s^{|u|_E}) \\
& + \frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) d < M^{-u^*,c} >_s + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^{f,c}, M^{-u^*,c} >_s \\
& + \sum_{s \le t} [f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}} - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \\
& \qquad - e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle f(X_s) - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle M_s^{-u^*}] \\
= \; & \Bigg\{ f(X_0) + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} dM_s^f + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} L^{\bar{Q}^{u,n}} f(X_s) ds \\
& \qquad + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) dM_s^{-u^*} \Bigg\} \\
& + \Bigg\{ - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_s) dA_s^{-u^*} + \frac{1}{2}\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) d < M^{-u^*,c} >_s \\
& - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^f, M >_s + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^{f,c}, M^{-u^*,c} >_s \\
& + \sum_{s \le t} [f(X_s)e^{M_s^{-u^*}-N_s^{|u|_E}} - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \\
& \qquad - e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle f(X_s) - f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} \triangle M_s^{-u^*}] \Bigg\} \\
= \; & \Bigg\{ f(X_0) + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} dM_s^f + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} L^{\bar{Q}^{u,n}} f(X_s) ds \\
& \qquad + \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) dM_s^{-u^*} \Bigg\} \\
& + \Bigg\{ - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} f(X_{s-}) dB_s^p - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^{f,d}, M^d >_s \\
& - \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|_E}} d < M^{f,d}, M^{-u^*,d} >_s
\end{aligned}
$$

$$
\begin{aligned}
& +\sum_{s\le t}[f(X_s)e^{M_s^{-u^*}-N_s^{|u|E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}} \\
& \qquad -e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle M_s^{-u^*}]\Big\} \\
:= \; & I+II. \qquad (2.15)
\end{aligned}
$$

Note that

$$
\begin{aligned}
II \;=\; & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dB_s^p+\sum_{s\le t}[-e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)\triangle B_s \\
& -e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)\triangle M_s^{-u^*}]+\sum_{s\le t}[f(X_s)e^{M_s^{-u}-N_s^{|u|E}} \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}-e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle M_s^{-u^*}] \\
= \; & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dB_s^p+\sum_{s\le t}[-e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)(e^{\triangle M_s^{-u^*}}-1) \\
& +f(X_s)e^{M_s^{-u^*}-N_s^{|u|E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}-e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle f(X_s)) \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle M_s^{-u^*}] \\
= \; & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dB_s^p+\sum_{s\le t}[e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})e^{\triangle M_s^{-u^*}} \\
& -f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}-f(X_{s-})e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}\triangle M_s^{-u^*}] \\
= \; & -\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dB_s^p+\int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dB_s \\
= \; & \int_0^t e^{M_{s-}^{-u^*}-N_{s-}^{|u|E}}f(X_{s-})dM_s^d. \qquad (2.16)
\end{aligned}
$$

Therefore (2.13) follows from (2.15) and (2.16). □

2.2.2 The bilinear form associated with $(\mathbf{\bar{P}_t^{u,n}})_{\mathbf{t\ge 0}}$ on $\mathbf{L^2(E_n; e^{-2u^*}m)}$

For $n\in\mathbf{N}$, since $u^*\cdot I_{E_n}$ is bounded, $(\bar{P}_t^{u,n})_{t\ge 0}$ is also a strongly continuous semigroup on $L^2(E_n;e^{-2u^*}m)$ by Theorem 2.3. In the following, we will study the bilinear form associated with $(\bar{P}_t^{u,n})_{t\ge 0}$ on $L^2(E_n;e^{-2u^*}m)$.

Define $D(\mathcal{E})_{n,b}:=D(\mathcal{E})_n\cap L^\infty(E;m)$. Let $f,g\in D(\mathcal{E})_{n,b}$. Note that $e^{-2u^*}g=(e^{-2u^*}-1)g+g\in D(\mathcal{E})_{n,b}$. Define

$$
\mathcal{E}^{u,n}(f,g):=\bar{Q}^{u,n}(f,e^{-2u^*}g),\quad f,g\in D(\mathcal{E})_{n,b}. \qquad (2.17)
$$

Then, by Theorem 2.3, we get

$$\mathcal{E}^{u,n}(f,g) = \lim_{t\to 0}\frac{1}{t}(f-\bar{P}_t^{u,n}f, e^{-2u^*}g)_m = \lim_{t\to 0}\frac{1}{t}(f-\bar{P}_t^{u.n}f,g)_{e^{-2u^*}m}. \tag{2.18}$$

$(\mathcal{E}^{u,n}, D(\mathcal{E})_{n,b})$ is called the bilinear from associated with $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n; e^{-2u^*}m)$.

Note that

$$\begin{aligned} &< M^f, M^d >_t \\ &\quad = [M^f, M^d]_t^p \\ &\quad = \left\{\sum_{s\leq t}[f(X_s)-f(X_{s-})][e^{(u^*(X_{s-})-u^*(X_s))}-1-(u^*(X_{s-})-u^*(X_s))]\right\}^p \\ &\quad = \int_0^t\int_{E_\triangle}[f(y)-f(X_s)][e^{(u^*(X_s)-u^*(y))}-1-(u^*(X_s)-u^*(y))]N(X_s,dy)dH_s. \end{aligned}$$

Then

$$\begin{aligned} &\int_E g d\mu_{<M^f,M^d>} \\ &\quad = \int_E\int_{E_\triangle} g(x)[f(y)-f(x)][e^{(u^*(x)-u^*(y))}-1-(u^*(x)-u^*(y))]N(x,dy)\nu(dx). \end{aligned} \tag{2.19}$$

By (2.7) and (2.8), we get

$$\begin{aligned} \int_E fg d\mu_{-u} &= \int_E\int_{E_\triangle} f(x)g(x)[e^{(u^*(x)-u^*(y))}-1-(u^*(x)-u^*(y))]N(x,dy)\nu(dx) \\ &\quad +\frac{1}{2}\int_E fg d\mu_{<M^{-u^*,c}>}+\int_E fg d\eta_u-\int_E fg|u|_E dm. \end{aligned} \tag{2.20}$$

Similar to (FOT1994, Theorem 5.3.1) (cf. also (O1988, Chapter 5)), we can show that $J(dx,dy)=\frac{1}{2}N(y,dx)\nu(dy)$ and $K(dx)=N(x,\triangle)\nu(dx)$. Therefore, we obtain by (2.17), (2.10), (2.19) and (2.20) that

$$\begin{aligned} \mathcal{E}^{u,n}(f,g) &= \bar{Q}^{u.n}(f,e^{-2u^*}g) \\ &= \mathcal{E}(f,e^{-2u^*}g)-\int_E e^{-2u^*}g d\mu_{<M^f,M>}-\int_E e^{-2u^*}fg d\mu_{-u} \\ &= \mathcal{E}(f,e^{-2u^*}g)-\int_E e^{-2u^*}g d\mu_{<M^f,M^{-u^*}>}-\int_E e^{-2u^*}g d\mu_{<M^f,M^d>} \\ &\quad -\int_E e^{-2u^*}fg d\mu_{-u} \end{aligned}$$

$$\begin{aligned}
&= \mathcal{E}(f, e^{-2u^*} g) - \int_E e^{-2u^*} g d\mu_{<M^f, M^{-u^*}>} \\
&\quad -2 \int_{E\times E\backslash d} e^{-2u^*(y)} g(y) f(x) [e^{(u^*(y)-u^*(x))} - 1 - (u^*(y) - u^*(x))] J(dx, dy) \\
&\quad -\frac{1}{2} \int_E e^{-2u^*} f g d\mu_{<M^{-u^*}, c>} - \mathcal{E}(|u|_E, e^{-2u^*} f g).
\end{aligned} \tag{2.21}$$

By using Beurling-Deny formula given in (HMS2006) and LeJan's transformation rule developed in (HMS2010), we will prove the following result.

Theorem 2.4. *For each $n \in \mathbf{N}$, under the assumption of Theorem 2.1 or Theorem 2.2, we have*

$$\mathcal{E}^{u,n}(f, g) = Q^u(f e^{-u^*}, g e^{-u^*}), \quad f, g \in D(\mathcal{E})_{n,b}. \tag{2.22}$$

Proof. We fix an $n \in \mathbf{N}$. Define

$$\begin{aligned}
\Psi^{u^*,n}(f, g) &:= \mathcal{E}(f, e^{-2u^*} g) - \int_E e^{-2u^*} g d\mu_{<M^f, M^{-u^*}>} \\
&\quad -2 \int_{E\times E\backslash d} e^{-2u^*(y)} g(y) f(x) [e^{(u^*(y)-u^*(x))} - 1 - (u^*(y) - u^*(x))] J(dx, dy) \\
&\quad -\frac{1}{2} \int_E e^{-2u^*} f g d\mu_{<M^{-u^*}, c>}, \quad f, g \in D(\mathcal{E})_{n,b}.
\end{aligned} \tag{2.23}$$

Then, by (2.21) and (2.18), we find that (2.22) is equivalent to

$$\Psi^{u^*,n}(f, g) = \mathcal{E}(f e^{-u^*}, g e^{-u^*}) + \mathcal{E}(u^*, e^{-2u^*} f g), \quad f, g \in D(\mathcal{E})_{n,b}. \tag{2.24}$$

Since $u^* \cdot I_{E_n}$ is bounded, there exists $l_0 \in \mathbf{N}$ such that $|u^*(x)| \leq l_0$ for all $x \in E_n$. For $l \in \mathbf{N}$, define $u_l^* := ((-l) \vee u^*) \wedge l$. Then $u_l^* \in D(\mathcal{E})_b$ and $u^* = u_l^*$ on E_n for $l \geq l_0$. Similar to (FOT1994, Lemma 5.3.1), we can show that $\mu_{<M^{-u^*}, c>}|_{E_n} = \mu_{<M^{-u_l^*}, c>}|_{E_n}$ for $l \geq l_0$. For $\phi \in D(\mathcal{E})_b$, we define

$$\begin{aligned}
\Psi^{\phi,n}(f, g) &:= \mathcal{E}(f, e^{-2\phi} g) - \int_E e^{-2\phi} g d\mu_{<M^f, M^{-\phi}>} \\
&\quad -2 \int_{E\times E\backslash d} e^{-2\phi(y)} g(y) f(x) [e^{(\phi(y)-\phi(x))} - 1 - (\phi(y) - \phi(x))] J(dx, dy) \\
&\quad -\frac{1}{2} \int_E e^{-2\phi} f g d\mu_{<M^{-\phi}, c>}, \quad f, g \in D(\mathcal{E})_{n,b}.
\end{aligned} \tag{2.25}$$

Then, by (2.23) and (2.25), we find that for $l \geq l_0$

$$\Psi^{u^*,n}(f,g) = \Psi^{u_l^*,n}(f,g) + \int_E e^{-2u^*} g d\mu_{<M^f, M^{u^*-u_l^*}>}, \quad f,g \in D(\mathcal{E})_{n,b}.$$

Note that by (O1988, (5.1.3))

$$\begin{aligned} \left| \int_E e^{-2u^*} g d\mu_{<M^f, M^{u^*-u_l^*}>} \right| &\leq 2e^{2l_0} \|g\|_\infty \mathcal{E}(f,f)^{\frac{1}{2}} \mathcal{E}(u^* - u_l^*, u^* - u_l^*)^{\frac{1}{2}} \\ &\rightarrow 0 \text{ as } l \rightarrow \infty, \end{aligned}$$

and

$$\mathcal{E}(u_l^*, e^{-2u^*} fg) \rightarrow \mathcal{E}(u^*, e^{-2u^*} fg) \text{ as } l \rightarrow \infty.$$

Hence, to establish (2.24), it is sufficient to show that for any $\phi \in D(\mathcal{E})_b$ and $f, g \in D(\mathcal{E})_{n,b}$

$$\Psi^{\phi,n}(f,g) = \mathcal{E}(fe^{-\phi}, ge^{-\phi}) + \mathcal{E}(\phi, e^{-2\phi} fg). \tag{2.26}$$

Let $\phi \in D(\mathcal{E})_b$. By (O1988, (5.3.2)), we have

$$\int g d\mu_{<M^f, M^{-\phi}>} = -\mathcal{E}(f, g\phi) - \mathcal{E}(\phi, gf) + \mathcal{E}(f\phi, g). \tag{2.27}$$

By (2.25) and (2.27), we find that (2.26) is equivalent to

$$\begin{aligned} &\mathcal{E}(f, e^{-2\phi} g) + \mathcal{E}(f, e^{-2\phi} g\phi) - \mathcal{E}(f\phi, e^{-2\phi} g) \\ &\quad -2 \int_{E\times E \backslash d} e^{-2\phi(y)} g(y) f(x) [e^{(\phi(y)-\phi(x))} - 1 - (\phi(y) - \phi(x))] J(dx, dy) \\ &\quad -\frac{1}{2} \int_E e^{-2\phi} fg d\mu_{<M^{-\phi,c}>} \\ &= \mathcal{E}(fe^{-\phi}, ge^{-\phi}). \end{aligned} \tag{2.28}$$

Denote by $M_t^{-\phi,j}$ and $M_t^{-\phi,k}$ the jumping and killing parts of $M_t^{-\phi}$, respectively. Then, similar to (FOT1994, (5.3.9) and (5.3.10)), we get

$$\mu_{<M^{-\phi,j}>}(dx) = 2 \int_E (\phi(x) - \phi(y))^2 J(dy, dx) \text{ and } \mu_{<M^{-\phi,k}>}(dx) = \phi^2(x) K(dx).$$

Thus, for any $w \in D(\mathcal{E})_b$, we have

$$\begin{aligned} \int_E w d\mu_{<M^{-\phi,c}>} &= \int_E w d(\mu_{<M^{-\phi}>} - \mu_{<M^{-\phi,j}>} - \mu_{<M^{-\phi,k}>}) \\ &= 2\mathcal{E}(\phi, \phi w) - \mathcal{E}(\phi^2, w) \\ &\quad -2 \int_{E\times E\backslash d} (\phi(y) - \phi(x))^2 w(y) J(dx, dy) - \int_E w\phi^2 dK. \end{aligned} \tag{2.29}$$

By (2.29), we find that (2.28) is equivalent to

$$\begin{aligned}
&\mathcal{E}(f, e^{-2\phi}g) + \mathcal{E}(f, e^{-2\phi}g\phi) - \mathcal{E}(f\phi, e^{-2\phi}g) - \mathcal{E}(\phi, e^{-2\phi}\phi fg) + \frac{1}{2}\mathcal{E}(\phi^2, e^{-2\phi}fg) \\
&\quad -2\int_{E\times E\backslash d} e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]J(dx,dy) \\
&\quad + \int_{E\times E\backslash d}(\phi(y)-\phi(x))^2 e^{-2\phi(y)}f(y)g(y)J(dx,dy) + \frac{1}{2}\int_E e^{-2\phi}fg\phi^2 dK \\
&= \mathcal{E}(fe^{-\phi}, ge^{-\phi}). \qquad (2.30)
\end{aligned}$$

Proof of (2.30) under the assumption of Theorem 2.1.

Denote by $\tilde{\mathcal{E}}$ the symmetric part of $\mathcal{E}$. Then $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a regular symmetric Dirichlet form. Denote by $\tilde{J}$ and $\tilde{K}$ the jumping and killing measures of $(\tilde{\mathcal{E}}, D(\mathcal{E}))$, respectively. Then

$$\begin{aligned}
&\int_{E\times E\backslash d}(\phi(y)-\phi(x))^2 J(dx,dy) + \int_E \phi^2 dK \\
&\quad \leq 2\left\{\int_{E\times E\backslash d}(\phi(y)-\phi(x))^2\tilde{J}(dx,dy) + \int_E \phi^2 d\tilde{K}\right\} \\
&\quad \leq 2\mathcal{E}(\phi,\phi) \qquad (2.31)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{E\times E\backslash d}[e^{(\phi(y)-\phi(x))} - 1 - (\phi(y)-\phi(x))]J(dx,dy) \\
&\quad \leq C_{\|\phi\|_\infty}\int_{E\times E\backslash d}(\phi(y)-\phi(x))^2 J(dx,dy) \\
&\quad \leq C_{\|\phi\|_\infty}\mathcal{E}(\phi,\phi) \qquad (2.32)
\end{aligned}$$

for some constant $C_{\|\phi\|_\infty} > 0$. Hence, to establish (2.30) for $\phi \in D(\mathcal{E})_b$ and $f, g \in D(\mathcal{E})_{n,b}$, it is sufficient to establish (2.30) for $\phi, f, g \in D := C_0(E) \cap D(\mathcal{E})$ by virtue of the density of D in $D(\mathcal{E})$ and approximation.

By (HMS2006, Theorem 4.8 and Proposition 5.1), we have the following Beurling-Deny decomposition

$$\begin{aligned}
\mathcal{E}(f,g) &= \mathcal{E}^c(f,g) + SPV\int_{E\times E\backslash d} 2(f(y)-f(x))g(y)J(dx,dy) \\
&\quad + \int_E fg dK, \quad f, g \in D(\mathcal{E})_b, \qquad (2.33)
\end{aligned}$$

where $SPV\int$ denotes the symmetric principle value integral (see (HMS2006, Definition 2.5)) and $\mathcal{E}^c(f,g)$ satisfies the left strong local property in the sense that $\mathcal{E}^c(f,g)=0$ if f is constant $\mathcal{E}$-q.e. on a quasi-open set containing the quasi-support of g (see (HMS2006, Theorem 4.1)). By (2.33), we obtain that for any $w\in D(\mathcal{E})_b$,

$$\begin{aligned}
&2\mathcal{E}(\phi,\phi w)-\mathcal{E}(\phi^2,w)\\
&\qquad -2\int_{E\times E\backslash d}(\phi(y)-\phi(x))^2 w(y)J(dx,dy)-\int_E w\phi^2 dK\\
&= \quad 2\mathcal{E}^c(\phi,\phi w)-\mathcal{E}^c(\phi^2,w).
\end{aligned}$$

Hence (2.30) is equivalent to

$$\begin{aligned}
&\mathcal{E}(f,e^{-2\phi}g)+\mathcal{E}(f,e^{-2\phi}g\phi)-\mathcal{E}(f\phi,e^{-2\phi}g)-\mathcal{E}^c(\phi,e^{-2\phi}\phi fg)+\frac{1}{2}\mathcal{E}^c(\phi^2,e^{-2\phi}fg)\\
&\qquad -2\int_{E\times E\backslash d}e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]J(dx,dy)\\
&= \quad \mathcal{E}(fe^{-\phi},ge^{-\phi}). \qquad (2.34)
\end{aligned}$$

In the following, we will establish (2.34) by showing that its left hand side and its right hand side have the same diffusion, jumping and killing parts. We assume without loss of generality that $\phi,f,g\in D$.

First, let us consider the diffusion parts of both sides of (2.34). Following (HMS2010, (3.4)), we introduce a linear functional $< L(w,v),\cdot >$ for $w,v\in D$ by

$$< L(w,v),f >:=\check{\mathcal{E}}^c(w,vf):=\frac{1}{2}(\mathcal{E}^c(w,vf)-\hat{\mathcal{E}}^c(w,vf)),\quad f\in D, \qquad (2.35)$$

where $\hat{\mathcal{E}}^c$ is the left strong local part of the dual Dirichlet form $(\hat{\mathcal{E}},D(\mathcal{E}))$. Define

$$\begin{aligned}
D_{\rm loc}:=\{w\,|\ &\text{for any relatively compact open set } G \text{ of } E,\text{there}\\
&\text{exists a function } v\in D \text{ such that } w=v \text{ on } G\}.
\end{aligned}$$

Then, the linear functional $< L(w,v),\cdot >$ can be extended and defined for any $w,v\in D_{\rm loc}$ (cf. (HMS2010, Definition 3.6)). Note that J_1 is assumed to be finite. Similar to (HMS2010, Theorem 3.8), we can prove the following lemma.

Lemma 2.3. *Let $w_1,\dots,w_l,v\in D_{\rm loc}$ and $f\in D$. Denote $w:=(w_1,\dots,w_l)$. If $\psi\in C^2(\mathbf{R}^l)$, then $\psi(w)\in D_{\rm loc}$, $\psi_{x_i}(w)\in D_{\rm loc}$, $1\le i\le l$, and*

$$< L(\psi(w),v),f >=\sum_{i=1}^{l}< L(w_i,v),\psi_{x_i}(w)f > . \qquad (2.36)$$

By (2.35) and (2.36), we get

$$\begin{aligned}
&\check{\mathcal{E}}^c(f,e^{-2\phi}g)+\check{\mathcal{E}}^c(f,e^{-2\phi}g\phi)-\check{\mathcal{E}}^c(f\phi,e^{-2\phi}g)\\
&\quad-\check{\mathcal{E}}^c(\phi,e^{-2\phi}\phi fg)+\frac{1}{2}\check{\mathcal{E}}^c(\phi^2,e^{-2\phi}fg)\\
&=\check{\mathcal{E}}^c(f,e^{-2\phi}g)+\check{\mathcal{E}}^c(f,e^{-2\phi}g\phi)-\check{\mathcal{E}}^c(f\phi,e^{-2\phi}g)\\
&=\check{\mathcal{E}}^c(f,e^{-2\phi}g)-\check{\mathcal{E}}^c(\phi,e^{-2\phi}fg)\\
&=\check{\mathcal{E}}^c(f,e^{-2\phi}g)+\check{\mathcal{E}}^c(e^{-\phi},e^{-\phi}fg)\\
&=\check{\mathcal{E}}^c(fe^{-\phi},ge^{-\phi}).
\end{aligned}\tag{2.37}$$

By LeJan's formula (cf. (FOT1994, Theorem 3.2.2 and Page 117), we can check that

$$\begin{aligned}
&\tilde{\mathcal{E}}^c(f,e^{-2\phi}g)+\tilde{\mathcal{E}}^c(f,e^{-2\phi}g\phi)-\tilde{\mathcal{E}}^c(f\phi,e^{-2\phi}g)\\
&\quad-\tilde{\mathcal{E}}^c(\phi,e^{-2\phi}\phi fg)+\frac{1}{2}\tilde{\mathcal{E}}^c(\phi^2,e^{-2\phi}fg)\\
&=\frac{1}{2}\int_E d\tilde{\mu}^c_{<f,e^{-2\phi}g>}+\frac{1}{2}\int_E d\tilde{\mu}^c_{<f,e^{-2\phi}g\phi>}-\frac{1}{2}\int_E d\tilde{\mu}^c_{<f\phi,e^{-2\phi}g>}\\
&\quad-\frac{1}{2}\int_E d\tilde{\mu}^c_{<\phi,e^{-2\phi}\phi fg>}+\frac{1}{4}\int_E d\tilde{\mu}^c_{<\phi^2,e^{-2\phi}fg>}\\
&=\frac{1}{2}\int_E d\tilde{\mu}^c_{<fe^{-\phi},ge^{-\phi}>}\\
&=\tilde{\mathcal{E}}^c(fe^{-\phi},ge^{-\phi}),
\end{aligned}\tag{2.38}$$

where $\tilde{\mathcal{E}}^c$ denotes the strong local part of $(\tilde{\mathcal{E}},D(\mathcal{E}))$ and $\tilde{\mu}^c$ denotes the local part of energy measure w.r.t. $(\tilde{\mathcal{E}},D(\mathcal{E}))$. Then the diffusion parts of both sides of (2.34) are equal by (2.37) and (2.38).

For the jumping parts of (2.34), we have

$$\begin{aligned}
&\mathcal{E}^j(f,e^{-2\phi}g)+\mathcal{E}^j(f,e^{-2\phi}g\phi)-\mathcal{E}^j(f\phi,e^{-2\phi}g)-\mathcal{E}^j(fe^{-\phi},ge^{-\phi})\\
&\quad-2\int_{E\times E\backslash d}e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]J(dx,dy)\\
&=2SPV\int_{E\times E\backslash d}\{(f(y)-f(x))e^{-2\phi(y)}g(y)+(f(y)-f(x))\phi(y)e^{-2\phi(y)}g(y)\\
&\quad-(f(y)\phi(y)-f(x)\phi(x))e^{-2\phi(y)}g(y)-(f(y)e^{-\phi(y)}-f(x)e^{-\phi(x)})e^{-\phi(y)}g(y)\\
&\quad-e^{-2\phi(y)}g(y)f(x)[e^{(\phi(y)-\phi(x))}-1-(\phi(y)-\phi(x))]\}J(dx,dy)\\
&=0.
\end{aligned}$$

For the killing parts of (2.34), we have

$$
\begin{aligned}
&\mathcal{E}^k(f,e^{-2\phi}g)+\mathcal{E}^k(f,e^{-2\phi}g\phi)-\mathcal{E}^k(f\phi,e^{-2\phi}g)-\mathcal{E}^k(fe^{-\phi},ge^{-\phi})\\
&= \int_E (fe^{-2\phi}g+fe^{-2\phi}g\phi-f\phi e^{-2\phi}g-fe^{-2\phi}g)dK\\
&= 0.
\end{aligned}
$$

The proof is complete.

Proof of (2.30) under the assumption of Theorem 2.2.

Let G be a relatively compact open subset of U such that the distance between the boundary of G and that of U is greater than some constant $\delta > 0$. Then, similar to (HMS2010, Theorem 4.8), we can show that $(\mathcal{E}, C_0^\infty(G))$ has the following representation:

$$
\begin{aligned}
\mathcal{E}(w,v) &= \sum_{i,j=1}^d \int_U \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_j}d\nu_{ij}^G+\sum_{i=1}^d < F_i^G, \frac{\partial w}{\partial x_i}v >\\
&\quad + SPV\int_{U\times U\backslash d} 2\left(\sum_{i=1}^d (y_i-x_i)\frac{\partial w}{\partial y_i}(y)I_{\{|x-y|\le\frac{\delta}{2}\}}(x,y)\right)v(y)\tilde{J}(dx,dy)\\
&\quad +\int_{U\times U\backslash d} 2\left(w(y)-w(x)-\sum_{i=1}^d (y_i-x_i)\frac{\partial w}{\partial y_i}(y)I_{\{|x-y|\le\frac{\delta}{2}\}}(x,y)\right)v(y)J(dx,dy)\\
&\quad +\int_U wvdK, \qquad w,v\in C_0^\infty(G),
\end{aligned}
\tag{2.39}
$$

where $\{\nu_{ij}^G\}_{1\le i,j\le d}$ are signed Radon measures on U such that for every $K\subset U$, K is compact, $\nu_{ij}^G(K)=\nu_{ji}^G(K)$ and $\sum_{i,j=1}^d \xi_i\xi_j\nu_{ij}^G(K)\ge 0$ for all $\xi=(\xi_1,\dots,\xi_d)\in\mathbf{R}^d$, $\{F_i^G\}_{1\le i\le d}$ are generalized functions on U.

By (2.39), we can check that (2.30) holds for all $\phi, f, g\in C_0^\infty(U)$. Therefore (2.30) holds for $\phi\in D(\mathcal{E})_b$ and $f,g\in D(\mathcal{E})_{n,b}$ by (2.31), (2.32) and approximation. The proof is complete. □

2.2.3 Proofs of the main results and some remarks

Proof of the main Results

Proof. By Theorem 2.3, for each $n\in\mathbf{N}$, $(\bar{P}_t^{u,n})_{t\ge 0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n;m)$ with $\|\bar{P}_t^{u,n}\|_2\le e^{\beta_n t}$ for every $t>0$ and some constant $\beta_n > 0$. Moreover, the coercive closed form associated with $(e^{-\beta_n t}\bar{P}_t^{u,n})_{t\geq 0}$ is given by $(\bar{Q}_{\beta_n}^{u,n}, D(\mathcal{E})_n)$. Note that $(\bar{P}_t^{u,n})_{t\geq 0}$ is also a strongly continuous semigroup of bounded operators on $L^2(E_n; e^{-2u^*}m)$ and the bilinear from associated with $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n; e^{-2u^*}m)$ is given by $(\mathcal{E}^{u,n}, D(\mathcal{E})_{n,b})$ (see (**??**)). Define

$$P_t^{u,n}f(x) := E_x[e^{N_t^u}f(X_t); t < \tau_{E_n}].$$

Then

$$\begin{aligned} P_t^{u,n}f(x) &= E_x[e^{N_t^{u^*}-N_t^{|u|_E}}f(X_t); t < \tau_{E_n}] \\ &= E_x[e^{u^*(X_t)-u^*(X_0)+M_t^{-u^*}-N_t^{|u|_E}}f(X_t); t < \tau_{E_n}] \\ &= e^{-u^*(x)}\bar{P}_t^{u,n}(e^{u^*}f)(x). \end{aligned} \tag{2.40}$$

Hence $(P_t^{u,n})_{t\geq 0}$ is a strongly continuous semigroup of bounded operators on $L^2(E_n; m)$. Let $(Q^{u,n}, D(\mathcal{E})_{n,b})$ be the restriction of Q^u to $D(\mathcal{E})_{n,b}$. Then, by (2.40), (2.18) and Theorem 2.4, we know that the bilinear from associated with $(P_t^{u,n})_{t\geq 0}$ on $L^2(E_n; m)$ is given by $(Q^{u,n}, D(\mathcal{E})_{n,b})$. That is,

$$Q^{u,n}(f,g) = \lim_{t\to 0}\frac{1}{t}(f - P_t^{u,n}f, g)_m, \quad f, g \in D(\mathcal{E})_{n,b}. \tag{2.41}$$

(i) Suppose that there exists a constant $\alpha_0 \geq 0$ such that

$$Q^u(f,f) \geq -\alpha_0(f,f)_m, \quad \forall f \in D(\mathcal{E})_b.$$

For $n \in \mathbf{N}$, let $(L^n, D(L^n))$ be the generator of $(P_t^{u,n})_{t\geq 0}$ on $L^2(E_n; m)$. Then $D(L^n - \alpha_0)$ is dense in $L^2(E_n; m)$.

Define

$$\bar{L}^n f(x) = e^{u^*(x)}L^n(e^{-u^*}f)(x), \quad f \in D(\bar{L}^n) := \{e^{u^*}g \,|\, g \in D(L^n)\}. \tag{2.42}$$

Then, by (2.40), $(\bar{L}^n, D(\bar{L}^n))$ is the generator of $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n; e^{-2u^*}m)$. $(\bar{L}^n, D(\bar{L}^n))$ is also the generator of $(\bar{P}_t^{u,n})_{t\geq 0}$ on $L^2(E_n; m)$ due to the boundedness of u^* on E_n. Since $(e^{-\beta_n t}\bar{P}_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n; m)$, Range$(\lambda - \bar{L}^n) = L^2(E_n; m)$ for all $\lambda > \beta_n$. Hence Range$(\lambda - (L^n - \alpha_0)) = L^2(E_n; m)$ for all $\lambda > \beta_n - \alpha_0$ by (2.42).

Let $f \in L^2(E_n;m)$. Then, for any $\alpha > 0$, we obtain by (2.41) that

$$\begin{aligned}
\|[\alpha-(L^n-\alpha_0)]f\|_2\cdot\|f\|_2 &= \|[(\alpha+\alpha_0)-L^n]f\|_2\cdot\|f\|_2\\
&\geq ([(\alpha+\alpha_0)-L^n]f,f)_m\\
&= Q^{u,n}(f,f)+(\alpha+\alpha_0)(f,f)_m\\
&\geq \alpha(f,f)_m.
\end{aligned}$$

Hence $L^n-\alpha_0$ is dissipative on $L^2(E_n;m)$. Therefore $(e^{-\alpha_0 t}P_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n;m)$ by the Hille-Yosida theorem.

Let $g \in L^2(E;m)$ and $t > 0$. Then

$$\begin{aligned}
\|P_t^u g\|_2 &\leq \|P_t^u|g|\,\|_2\\
&= \lim_{l\to\infty}\|P_t^u|g\cdot I_{E_l}|\,\|_2\\
&\leq \liminf_{l\to\infty}\liminf_{n\to\infty}\|P_t^{u,n}|g\cdot I_{E_l}|\,\|_2\\
&\leq e^{\alpha_0 t}\|g\|_2.
\end{aligned}$$

Since $g \in L^2(E;m)$ is arbitrary, we get

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0.$$

(ii) Suppose that there exists a constant $\alpha_0 \geq 0$ such that

$$\|P_t^u\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0. \tag{2.43}$$

Let $n \in \mathbf{N}$ and $f \in L^2(E_n;m)$. Then

$$\|P_t^{u,n}f\|_2 \leq \|P_t^{u,n}|f|\,\|_2 \leq \|P_t^u|f|\,\|_2 \leq e^{\alpha_0 t}\|f\|_2.$$

Hence $(e^{-\alpha_0 t}P_t^{u,n})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E_n;m)$. By (2.41), we get

$$Q^{u,n}(f,f)+\alpha_0(f,f)_m = \lim_{t\to 0}\frac{1}{t}(f-e^{-\alpha_0 t}P_t^{u,n}f,f)_m \geq 0, \quad \forall f \in D(\mathcal{E})_{n,b}. \tag{2.44}$$

By (2.44) and approximation, we find that

$$Q^u(f,f) \geq -\alpha_0(f,f)_m, \quad \forall f \in D(\mathcal{E})_b.$$

Now we show that $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$. Let $n\in\mathbf{N}$ and $f\in L^2(E_n;m)$ satisfying $f\geq 0$. Then, we obtain by (2.43) and the strong continuity of $(P_t^{u,n})_{t\geq 0}$ that

$$\begin{aligned}
&\limsup_{t\to 0}\|f-e^{-\alpha_0 t}P_t^u f\|_2^2\\
&\quad= \limsup_{t\to 0}\{2(f-e^{-\alpha_0 t}P_t^u f,f)_m-[(f,f)_m-\|e^{-\alpha_0 t}P_t^u f\|_2^2]\}\\
&\quad\leq 2\limsup_{t\to 0}(f-e^{-\alpha_0 t}P_t^u f,f)_m\\
&\quad\leq 2\limsup_{t\to 0}(f-e^{-\alpha_0 t}P_t^{u,n} f,f)_m\\
&\quad= 0.
\end{aligned}$$

Since f and n are arbitrary, $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$ by (2.43). The proof is complete. □

Remark 2.2. *Let $u\in D(\mathcal{E})$. Define*

$$B_t^u=\sum_{s\leq t}\left[e^{(u(X_{s-})-u(X_s))}-1-(u(X_{s-})-u(X_s))\right]. \tag{2.45}$$

Note that $(B_t^u)_{t\geq 0}$ may not be locally integrable (cf. (CMS2007, Theorem 3.3)). To overcome this difficulty, we introduced the nonnegative function u^* and the locally integrable increasing process $(B_t)_{t\geq 0}$ (see (2.2) and (2.3)). This technique has been used in (CMS2007) to show that if X is symmetric and $u\in D(\mathcal{E})_e$, then $(P_t^u)_{t\geq 0}$ is strongly continuous if and only if $(Q^u,D(\mathcal{E})_b)$ is lower semi-bounded. Here and henceforth $D(\mathcal{E})_e$ denotes the extended Dirichlet space of $(\mathcal{E},D(\mathcal{E}))$.

In fact, if we assume that $(\mathcal{E},D(\mathcal{E}))$ satisfies the strong sector condition instead of the weak sector condition (cf. (MR1992, Pages 15 and 16) for the definitions), then similar to (CMS2007, Page 158) we can introduce a function $|u|_E^g$ for each $u\in D(\mathcal{E})_e$. Define $u^*:=u+|u|_E^g$. Using this defined u^*, similar to the above proof of this section, we can show that Theorems 2.1 and 2.2 hold for all $u\in D(\mathcal{E})_e$.

On the other hand, suppose we still assume that $(\mathcal{E},D(\mathcal{E}))$ satisfies the weak sector condition and $u\in D(\mathcal{E})_e$. Define

$$F_t^u=\sum_{s\leq t}\left[e^{(u(X_{s-})-u(X_s))}-1-(u(X_{s-})-u(X_s))\right]^2.$$

If $(F_t^u)_{t\geq 0}$ is locally P_x-integrable on $[0,\zeta)$ for q.e. $x \in E$, then we can show that Theorems 2.1 and 2.2 still hold. The proof is similar to the above proof of this section but we directly apply the $(B_t^u)_{t\geq 0}$ defined in (2.45) instead of the $(B_t)_{t\geq 0}$ defined in (2.3). Note that if u is lower semi-bounded or $e^u \in D(\mathcal{E})_e$ (cf. (CMS2007, Example 3.4 (iii)), then $(F_t^u)_{t\geq 0}$ is locally P_x-integrable on $[0,\zeta)$ for q.e. $x \in E$.

Remark 2.3. *If $(\mathcal{E}, D(\mathcal{E}))$ is a symmetric Dirichlet form, then the assumption of Theorem 2.1 is automatically satisfied. Note that $(P_t^u)_{t\geq 0}$ is symmetric on $L^2(E;m)$. If $(P_t^u)_{t\geq 0}$ is strongly continuous, then (2.43) holds (cf. (CFKZ2009, Remark 1.6(ii))). Therefore, the following three assertions are equivalent to each other:*

(i) $(Q^u, D(\mathcal{E})_b)$ is lower semi-bounded.

(ii) There exists a constant $\alpha_0 \geq 0$ such that $\|P_t^u\|_2 \leq e^{\alpha_0 t}$ for $t > 0$.

(iii) $(P_t^u)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$.

Remark 2.4. *Denote by S the set of all smooth measures on $(E, \mathcal{B}(E))$. Let $\mu = \mu_1 - \mu_2 \in S - S$, $(A_t^1)_{t\geq 0}$ and $(A_t^2)_{t\geq 0}$ be PCAFs with Revuz measures μ_1 and μ_2, respectively. Define*

$$\bar{P}_t^A f(x) = E_x[e^{A_t^2 - A_t^1} f(X_t)], \quad f \geq 0 \text{ and } t \geq 0,$$

and

$$\begin{cases} \mathcal{E}^\mu(f,g) := \mathcal{E}(f,g) + \int_E fg d\mu, \\ f, g \in D(\mathcal{E}^\mu) := \{w \in D(\mathcal{E}) \,|\, w \text{ is } (\mu_1 + \mu_2)\text{-square integrable}\}. \end{cases}$$

Then, by a localization argument similar to that used in the proof of Theorems 2.1 and 2.2 (cf. also (Z2005), we can show that the following two conditions are equivalent to each other:

(i) There exists a constant $\alpha_0 \geq 0$ such that

$$\mathcal{E}^\mu(f,f) \geq -\alpha_0 (f,f)_m, \quad \forall f \in D(\mathcal{E}^\mu).$$

(ii) There exists a constant $\alpha_0 \geq 0$ such that

$$\|\bar{P}_t^A\|_2 \leq e^{\alpha_0 t}, \quad \forall t > 0.$$

Furthermore, if one of these conditions holds, then the semigroup $(\bar{P}_t^A)_{t\geq 0}$ is strongly continuous on $L^2(E;m)$.

This result generalizes the corresponding results of (AM1991) and (C2007). Note that, similar to Theorems 2.1 and 2.2, it is not necessary to assume that the bilinear form $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ satisfies the sector condition.

2.3 Some applications

Example 2.3.1

Let $d \geq 3$, U be an open set of $\mathbf{R}^d$ and $m = dx$, the Lebesgue measure on U. Let $a_{ij} \in L^1_{\mathrm{loc}}(U;dx)$, $1 \leq i,j \leq d$, $b_i, d_i \in L^d_{\mathrm{loc}}(U;dx)$, $d_i - b_i \in L^d(U;dx) \cup L^\infty(U;dx)$, $1 \leq i \leq d$, $c \in L^{d/2}_{\mathrm{loc}}(U;dx)$. Define for $f, g \in C_0^\infty(U)$

$$
\begin{aligned}
\mathcal{E}(f,g) \;\; &= \;\; \sum_{i,j=1}^d \int_U \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} a_{ij}dx + \sum_{i=1}^d \int_U f \frac{\partial g}{\partial x_i} d_i dx \\
&\quad + \sum_{i=1}^d \int_U \frac{\partial f}{\partial x_i} g b_i dx + \int_U fgcdx.
\end{aligned}
$$

Denote $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$ and $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $1 \leq i,j \leq d$. Suppose that the following conditions hold:

(C1) There exists $\gamma \in (0,\infty)$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \gamma \sum_{i=1}^d |\xi_i|^2$, $\forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbf{R}^d$.

(C2) $|\check{a}_{ij}| \leq M \in (0,\infty)$ for $1 \leq i,j \leq d$.

(C3) $cdx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i} \geq 0$ and $cdx - \sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \geq 0$ (in the sense of Schwartz distributions, i.e., $\int_U (cf + \sum_{i=1}^d d_i \frac{\partial f}{\partial x_i})dx, \int_U (cf + \sum_{i=1}^d b_i \frac{\partial f}{\partial x_i})dx \geq 0$ for all $f \in C_0^\infty(U)$ with $f \geq 0$).

Then $(\mathcal{E}, C_0^\infty(U))$ is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular Dirichlet form on $L^2(U;dx)$ (see MR(1992) II, Proposition 2.11).

Let $u \in C_0^\infty(U)$. Then, for $f \in C_0^\infty(U)$, we have

$$\begin{aligned} Q^u(f,f) &= \mathcal{E}(f,f) + \mathcal{E}(u,f^2) \\ &= \sum_{i,j=1}^d \int_U \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} a_{ij}dx + \int_U f^2\left(c(1+u) + \sum_{i=1}^d \frac{\partial u}{\partial x_i} b_i\right)dx \\ &\quad + \int_U \sum_{i=1}^d \frac{\partial f^2}{\partial x_i}\left(\frac{d_i+b_i}{2} + ud_i + \sum_{j=1}^d \frac{\partial u}{\partial x_j} a_{ji}\right)dx. \end{aligned}$$

Suppose that the following condition holds:

(C4) There exists a constant $\alpha_0 \geq 0$ such that

$$\left(\alpha_0 + c(1+u) + \sum_{i=1}^d \frac{\partial u}{\partial x_i} b_i\right)dx - \sum_{i=1}^d \frac{\partial\left(\frac{d_i+b_i}{2} + ud_i + \sum_{j=1}^d \frac{\partial u}{\partial x_j} a_{ji}\right)}{\partial x_i} \geq 0$$

in the sense of Schwartz distribution.

Then $Q^u(f,f) \geq -\alpha_0(f,f)_m$ for any $f \in C_0^\infty(U)$ and thus for any $f \in D(\mathcal{E})_b$ by approximation.

Let X be a Hunt process associated with $(\mathcal{E}, D(\mathcal{E}))$ and $(P_t^u)_{t\geq 0}$ be the generalized Feynman-Kac semigroup induced by u. Then, by Theorem 2.1 or Theorem 2.2, $(e^{-\alpha_0 t}P_t^u)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(U;dx)$.

Example 2.3.2 In this example, we study the generalized Feynman-Kac semigroup for the non-symmetric Dirichlet form given in (MR1992, II, 3 e)).

Let E be a locally convex topological real vector space which is a (topological) Souslin space. Let $m := \mu$ be a finite positive measure on $\mathcal{B}(E)$ such that $\operatorname{supp}\mu = E$. Let E' denote the dual of E and ${}_{E'}\langle , \rangle_E : E' \times E \to \mathbf{R}$ the corresponding dualization. Define

$$\mathcal{F}C_b^\infty := \{f(l_1,\dots,l_m)\,|\, m \in \mathbf{N}, f \in C_b^\infty(\mathbf{R}^m), l_1,\dots,l_m \in E'\}.$$

Assume that there exists a separable real Hilbert space $(H, \langle , \rangle_H)$ densely and continuously embedded into E. Identifying H with its dual H' we have that

$$E' \subset H \subset E \text{ densely and continuously,}$$

and ${}_{E'}\langle , \rangle_E$ restricted to $E' \times H$ coincides with $\langle , \rangle_H$. For $f \in \mathcal{F}C_b^\infty$ and $z \in E$, define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), h\rangle_H = \frac{\partial u}{\partial h}(z), \quad h \in H.$$

Let $(\mathcal{E}_\mu, \mathcal{F}C_b^\infty)$, defined by

$$\mathcal{E}_\mu(f,g) = \int_E \langle \nabla f, \nabla g\rangle_H d\mu, \quad f,g \in \mathcal{F}C_b^\infty,$$

be closable on $L^2(E;\mu)$ (cf. (MR1992, II, Proposition 3.8 and Corollary 3.13)). Let $\mathcal{L}_\infty(H)$ denote the set of all bounded linear operators on H with operator norm $\|\ \|$. Suppose $z \mapsto A(z), z \in E$, is a map from E to $\mathcal{L}_\infty(H)$ such that $z \mapsto \langle A(z)h_1, h_2\rangle_H$ is $\mathcal{B}(E)$-measurable for all $h_1, h_2 \in H$. Furthermore, suppose that the following conditions hold:

(C1) There exists $\gamma \in (0,\infty)$ such that $\langle A(z)h, h\rangle_H \geq \gamma \|h\|_H^2$ for all $h \in H$.

(C2) $\|\tilde{A}\|_\infty \in L^1(E;\mu)$ and $\|\check{A}\|_\infty \in L^\infty(E;\mu)$, where $\tilde{A} := \frac{1}{2}(A+\hat{A}), \check{A} := \frac{1}{2}(A-\hat{A})$ and $\hat{A}(z)$ denotes the adjoint of $A(z), z \in E$.

(C3) Let $c \in L^\infty(E,\mu)$ and $b,d \in L^\infty(E \to H;\mu)$ such that for $u \in \mathcal{F}C_b^\infty$ with $u \geq 0$

$$\int_E (\langle d, \nabla u\rangle_H + cu) d\mu \geq 0, \int_E (\langle b, \nabla u\rangle_H + cu) d\mu \geq 0.$$

Define for $f,g \in \mathcal{F}C_b^\infty$

$$\begin{aligned} \mathcal{E}(f,g) &= \int_E \langle A\nabla f, \nabla g\rangle_H d\mu + \int_E f\langle d, \nabla g\rangle_H d\mu \\ &\quad + \int_E \langle b, \nabla f\rangle_H g d\mu + \int_E fgc d\mu. \end{aligned}$$

Then $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;\mu)$ (see by (MR1992, II, 3 e)).

Let $u \in \mathcal{F}C_b^\infty$. Then, for $f \in \mathcal{F}C_b^\infty$, we have

$$\begin{aligned} Q^u(f,f) &= \mathcal{E}(f,f) + \mathcal{E}(u, f^2) \\ &= \int_E \langle A\nabla f, \nabla f\rangle_H d\mu + \int_E (c(1+u) + \langle b, \nabla u\rangle_H) f^2 dx \\ &\quad + \int_E \left\langle \frac{d+b}{2} + ud + A\nabla u, \nabla f^2 \right\rangle_H d\mu. \end{aligned}$$

Suppose that the following condition holds:

(C4) There exists a constant $\alpha_0 \geq 0$ such that

$$\int_E \left\{ (\alpha_0 + c(1+u) + \langle b, \nabla u\rangle_H) f + \left\langle \frac{d+b}{2} + ud + A\nabla u, \nabla f \right\rangle_H \right\} d\mu \geq 0$$

for all $f \in \mathcal{F}C_b^\infty$ with $f \geq 0$.

Then $Q^u(f,f) \geq -\alpha_0(f,f)_m$ for any $f \in \mathcal{F}C_b^\infty$ and thus for any $f \in D(\mathcal{E})_b$ by approximation.

Let X be a μ-tight special standard diffusion process associated with $(\mathcal{E}, D(\mathcal{E}))$ and $(P_t^u)_{t\geq 0}$ be the generalized Feynman-Kac semigroup induced by u. Then, by Theorem 2.1, $(e^{-\alpha_0 t}P_t^u)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E;\mu)$.

Chapter 3

Fukushima's decomposition in the semi-Dirichlet forms setting

The classical decomposition of Fukushima was originally established for regular symmetric Dirichlet forms (cf. (F1979) and (FOT1994, Theorem 5.2.2)). Later it was extended to the non-symmetric and quasi-regular cases, respectively (cf. (O1988, Theorem 5.1.3) and (MR1992, Theorem VI.2.5)). Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular Dirichlet form on $L^2(E;m)$ with associated Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$. If $u \in D(\mathcal{E})$, then there exist unique martingale additive functional (abbreviated by MAF) $M^{[u]}$ of finite energy and continuous additive functional (abbreviated by CAF) $N^{[u]}$ of zero energy such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \tag{3.1}$$

where $\tilde{u}$ is an $\mathcal{E}$-quasi-continuous m-version of u and the energy of an AF $A := (A_t)_{t\geq 0}$ is defined to be

$$e(A) := \lim_{t\to 0} \frac{1}{2t} E_m[A_t^2] \tag{3.2}$$

whenever the limit exists in $[0, \infty]$.

The aim of this chapter is to establish Fukushima's decomposition for some Markov processes associated with semi-Dirichlet forms. Note that the assumption of the existence of dual Markov process (a Markov process $\hat{X}$ is said to be a dual process of the Markov process X if any $f, g \in \mathcal{B}_b^+(E)$, $(P_t f, g)_m = (f, \hat{P}_t g)_m$, where $\{P_t\}_{t\geq 0}$

and $\{\hat{P}_t\}_{t\geq 0}$ are the semigroup of X and $\hat{X}$ respectively) plays a crucial role in all the Fukushima-type decompositions known up to now. In fact, without that assumption, the usual definition (3.2) of energy of AFs is questionable. First, let us consider a concrete semi-Dirichlet form as follows.

Let $d \geq 3$, U be an open subset of $\mathbb{R}^d$ and $m = dx$, the Lebesgue measure, on U. Let $a_{ij} \in L^1_{loc}(U; dx)$, $1 \leq i, j \leq d$, $b_i, d_i \in L^d_{loc}(U; dx)$, $b_i - d_i \in L^\infty(U; dx) \cup L^d(U; dx)$, $1 \leq i \leq d$, $c \in L^{d/2}_{loc}(U; dx)$. Define for $u, v \in C_0^\infty(U)$ (:= the set of all infinitely differentiable functions with compact supports in U)

$$\begin{aligned}\mathcal{E}(u,v) &= \sum_{i,j=1}^d \int_U \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}dx + \sum_{i=1}^d \int_U \frac{\partial u}{\partial x_i} v b_i dx \\ &\quad + \sum_{i=1}^d \int_U u \frac{\partial v}{\partial x_i} d_i dx + \int_U uvcdx. \qquad (3.3)\end{aligned}$$

Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$ and $\check{a}_{ij} := \frac{1}{2}(a_{ij} - a_{ji})$, $1 \leq i, j \leq d$. Suppose that the following conditions hold:

(C.1) There exists $\eta > 0$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \geq \eta|\xi|^2,\ \forall \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$.

(C.2) $\check{a}_{ij} \in L^\infty(U; dx)$ for $1 \leq i, j \leq d$.

(C.3) $cdx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i} \geq 0$ and $cdx - \sum_{i=1}^d \frac{\partial \gamma_i}{\partial x_i} \geq 0$ (in the sense of Schwartz distributions, i.e., $\int_U (cu + \sum_{i=1}^d d_i \frac{\partial u}{\partial x_i})dx, \int_U (cu + \sum_{i=1}^d \gamma_i \frac{\partial u}{\partial x_i})dx \geq 0$ for all $u \in C_0^\infty(U)$ with $u \geq 0$), where $b_i = \beta_i + \gamma_i$ with $\beta_i \in L^\infty(U; dx) \cup L^p(U; dx)$ for some $p \geq d$, $\gamma_i \in L^1_{loc}(U; dx)$, $1 \leq i \leq d$.

Then, $(\mathcal{E}, C_0^\infty(U))$ is closable on $L^2(U; dx)$ and its closure $(\mathcal{E}, D(\mathcal{E}))$ is a regular local semi-Dirichlet form on $L^2(U; dx)$. If $\beta \neq 0$, $(\mathcal{E}, D(\mathcal{E}))$ is in general not a Dirichlet form. For $u \in D(\mathcal{E})$, it is natural to ask whether a decomposition similar to (3.1) holds. Based on the results that developed in this chapter, we will see that the answer is affirmative. Note that the Doob-Meyer decomposition for supermartingales and Itô's formula for semimartingales do not apply to this particular case.

The rest of this chapter is organized as follows. In Section 2, we present results on the potential theory for semi-Dirichlet forms, which are necessary to deriving Fukushima's decomposition in the semi-Dirichlet forms setting. In Section 3, we use a localization method to obtain Fukushima's decomposition for diffusions associated with semi-Dirichlet forms (see Theorem 3.4 below). Also, we give some concrete examples. In Section 4, we prove a transformation formula for local MAFs (see Theorem 3.8 below). Since so far there is no analog of LeJan's transformation rule available for semi-Dirichlet forms, a lot of extra efforts are made (cf. Theorem 3.5 and Remark 3.2 below).

This part of the thesis is based on the paper (MMS2011), which has been submitted for publication.

3.1 Revuz correspondence in the semi-Dirichlet forms setting

Let E be a metrizable Lusin space (i.e., E is topologically isomorphic to a Borel subset of a complete separable metric space) and m be a σ-finite positive measure on its Borel σ-algebra $\mathcal{B}(E)$. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular semi-Dirichlet form on $L^2(E;m)$. Let $K > 0$ be a continuity constant of $(\mathcal{E}, D(\mathcal{E}))$, i.e.,

$$|\mathcal{E}_1(u,v)| \leq K\mathcal{E}_1(u,u)^{1/2}\mathcal{E}_1(v,v)^{1/2}, \quad \forall u, v \in D(\mathcal{E}).$$

Denote by $(T_t)_{t\geq 0}$ and $(G_\alpha)_{\alpha\geq 0}$ (resp. $(\hat{T}_t)_{t\geq 0}$ and $(\hat{G}_\alpha)_{\alpha\geq 0}$) the semigroup and resolvent (resp. co-semigroup and co-resolvent) associated with $(\mathcal{E}, D(\mathcal{E}))$. Then there exists an m-tight special standard process $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ in the sense that $P_t f$ is an $\mathcal{E}$-quasi-continuous m-version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap L^2(E;m)$ and all $t > 0$, where $(P_t)_{t\geq 0}$ denotes the semigroups associated with $\mathbf{M}$ (cf. (MOR1995, Theorem 3.8)). It is known that any quasi-regular semi-Dirichlet form is quasi-homeomorphic to a regular semi-Dirichlet form (cf. (HMS2006, Theorem 3.8)).

Let $A \subset E$ and $f \in D(\mathcal{E})$. Denote by f_A (resp. $\hat{f}_A$) the 1-balayaged (resp. 1-cobalayaged) function of f on A. *Throughout this chapter, we fix $\phi \in L^2(E;m)$ with $0 < \phi \leq 1$ m-a.e. and set $h = G_1\phi$, $\hat{h} = \hat{G}_1\phi$.* Define for $U \subset E$, U open,

$$\mathrm{cap}_\phi(U) := (h_U, \phi)$$

and for any $A \subset E$,

$$\mathrm{cap}_\phi(A) := \inf\{\mathrm{cap}_\phi(U) \mid A \subset U, U \text{ open}\}.$$

Hereafter, $(\cdot, \cdot)$ denotes the usual inner product of $L^2(E; m)$. By (MOR1995, Theorem 2.20), we get

$$\mathrm{cap}_\phi(A) = (h_A, \phi) = \mathcal{E}_1(h_A, \hat{G}_1\phi).$$

Definition 3.1. *A positive measure μ on $(E, \mathcal{B}(E))$ is said to be of finite energy integral if $\mu(N) = 0$ whenever $N \in \mathcal{B}(E)$ is $\mathcal{E}$-exceptional and*

$$\int_E |\tilde{v}(x)|\mu(dx) \le C\mathcal{E}_1(v, v)^{1/2}, \quad \forall v \in D(\mathcal{E}),$$

for some positive constant C.

We denote by S_0 the set of all measures of finite energy integral.

Remark 3.1. *(i) Assume that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form. Let μ be a positive Radon measure on E satisfying*

$$\int_E |v(x)|\mu(dx) \le C\mathcal{E}_1(v, v)^{1/2}, \quad \forall v \in C_0(E) \cap D(\mathcal{E})$$

for some positive constant C, where $C_0(E)$ denotes the set of all continuous functions on E with compact supports. Then one can show that μ charges no $\mathcal{E}$-exceptional set (cf. (HS2010, Lemma 3.5)) and thus $\mu \in S_0$.

(ii) Let $\mu \in S_0$ and $\alpha > 0$. Then there exist unique $U_\alpha\mu \in D(\mathcal{E})$ and $\hat{U}_\alpha\mu \in D(\mathcal{E})$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E \tilde{v}(x)\mu(dx) = \mathcal{E}_\alpha(v, \hat{U}_\alpha\mu). \tag{3.4}$$

We call $U_\alpha\mu$ and $\hat{U}_\alpha\mu$ α-potential and α-co-potential, respectively.

Let $u \in D(\mathcal{E})$. By quasi-homeomorphism and similar to (FOT1994, Theorem 2.2.1) (cf. (HS2010, Lemma 1.2)), one can show that the following conditions are equivalent to each other:

(i) u is α-excessive (resp. α-co-excessive).

(ii) u is an α-potential (resp. α-co-potential).

(iii) $\mathcal{E}_\alpha(u, v) \ge 0$ (resp. $\mathcal{E}_\alpha(v, u) \ge 0$), $\forall v \in D(\mathcal{E}),\ v \ge 0$.

Theorem 3.1. *Define*

$$\hat{S}_{00}^{*} := \{\mu \in S_0 \,|\, \hat{U}_1\mu \leq c\hat{G}_1\phi \text{ for some constant } \ c > 0\}.$$

Let $A \in \mathcal{B}(E)$. If $\mu(A) = 0$ for all $\mu \in \hat{S}_{00}^{*}$, then $cap_{\phi}(A) = 0$.

Proof. By quasi-homeomorphism, without loss of generality, we suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form. Assume that $A \in \mathcal{B}(E)$ satisfying $\mu(A) = 0$ for all $\mu \in \hat{S}_{00}^{*}$. We will prove that $\text{cap}_{\phi}(A) = 0$.

Step 1. We first show that $\mu(A) = 0$ for all $\mu \in S_0$. Suppose that $\mu \in S_0$. By (MR1995, Proposition 4.13), there exists an $\mathcal{E}$-nest $\{F_k\}$ such that $\widetilde{\hat{G}_1\phi}$, $\widetilde{\hat{U}_1\mu} \in C(\{F_k\})$ and $\widetilde{\hat{G}_1\phi} > 0$ on F_k for each $k \in \mathbb{N}$. Then, there exists a sequences of positive constants $\{a_k\}$ such that

$$\widetilde{\hat{U}_1\mu} \leq a_k\widetilde{\hat{G}_1\phi} \ \text{ on } F_k \text{ for each } k \in \mathbb{N}.$$

Define $u_k = \hat{U}_1(1_{F_k}\cdot\mu)$ and set $v_k = u_k \wedge a_k\hat{G}_1\phi$ for $k \in \mathbb{N}$. Then $\widetilde{u_k} \leq \widetilde{\hat{U}_1\mu} \leq a_k\widetilde{\hat{G}_1\phi}$ $\mathcal{E}$-*q.e.* on F_k. By (3.4), we get

$$\mathcal{E}_1(v_k, u_k) = \int_{F_k} \widetilde{v_k}(x)\mu(dx) = \int_{F_k} \widetilde{u_k}(x)\mu(dx) = \mathcal{E}_1(u_k, u_k).$$

Since v_k is a 1-co-potential and $v_k \leq u_k$ m-*a.e.*, $\mathcal{E}_1(v_k - u_k, v_k - u_k) = \mathcal{E}_1(v_k - u_k, v_k) - \mathcal{E}_1(v_k - u_k, u_k) \leq 0$, proving that $u_k = v_k \leq a_k\hat{G}_1\phi$ m-*a.e.* Hence $1_{F_k} \cdot \mu \in \hat{S}_{00}^{*}$. Therefore $\mu(A) = 0$ by the assumption that A is not charged by each measure of $\hat{S}_{00}^{*}$.

Step 2. Suppose that $\text{cap}_{\phi}(A) > 0$. By (MOR1995, Corollary 2.22), there exists a compact set $K \subset B$ such that $\text{cap}_{\phi}(K) > 0$. Note that $\widehat{(\hat{G}_1\phi)_K} \in D(\mathcal{E})$ is 1-co-excessive. By Remark 3.1(ii), there exists $\mu_{\widehat{(\hat{G}_1\phi)_K}} \in S_0$ such that

$$\begin{aligned}
\text{cap}_{\phi}(K) &= \mathcal{E}_1((G_1\phi)_K, \hat{G}_1\phi) \\
&= \mathcal{E}_1(G_1\phi, \widehat{(\hat{G}_1\phi)_K}) \\
&= \int_E \widetilde{G_1\phi}\, d\mu_{\widehat{(\hat{G}_1\phi)_K}} \\
&\leq \mu_{\widehat{(\hat{G}_1\phi)_K}}(E).
\end{aligned} \tag{3.5}$$

For any $v \in C_0(K^c) \bigcap D(\mathcal{E})$, we have $\int \tilde{v}\, d\mu_{\widehat{(\hat{G}_1\phi)_K}} = \mathcal{E}_1(v, \widehat{(\hat{G}_1\phi)_K}) = 0$. Since $C_0(K^c) \bigcap D(\mathcal{E})$ is dense in $C_0(K^c)$, the support of $\mu_{\widehat{(\hat{G}_1\phi)_K}}$ is contained in K. Thus, by (3.5), we get $\mu_{\widehat{(\hat{G}_1\phi)_K}}(K) > 0$. Therefore $\text{cap}_{\phi}(A) = 0$ by Step 1. □

Definition 3.2. *A positive measure* μ *on* $(E, \mathcal{B}(E))$ *is called smooth (w.r.t.* $(\mathcal{E}, D(\mathcal{E}))$*) if* $\mu(N) = 0$ *whenever* $N \in \mathcal{B}(E)$ *is* $\mathcal{E}$*-exceptional and there exists an* $\mathcal{E}$*-nest* $\{F_k\}$ *of compact subsets of* E *such that*

$$\mu(F_k) < \infty \text{ for all } k \in \mathbb{N}.$$

We denote by S the set of all smooth measures on E.

Theorem 3.2. *The following conditions are equivalent for a positive measure* μ *on* $(E, \mathcal{B}(E))$.

(i) $\mu \in S$.

(ii) There exists an $\mathcal{E}$*-nest* $\{F_k\}$ *satisfying* $1_{F_k} \cdot \mu \in S_0$ *for each* $k \in \mathbb{N}$.

Proof. (ii) $\Rightarrow$ (i) is clear. We only prove (i) $\Rightarrow$ (ii). Let $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ be the symmetric part of $(\mathcal{E}, D(\mathcal{E}))$. Then $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ is a symmetric positivity preserving form. Denote by $(\tilde{G}_\alpha)_{\alpha \geq 0}$ the resolvent associated with $(\tilde{\mathcal{E}}, D(\mathcal{E}))$ and set $\bar{h} := \tilde{G}_1 \varphi$. Then $(\tilde{\mathcal{E}}_1^{\bar{h}}, D(\mathcal{E}^{\bar{h}}))$ is a quasi-regular symmetric Dirichlet form on $L^2(E; \bar{h}^2 m)$ (the $\bar{h}$-transform of $(\tilde{\mathcal{E}}_1, D(\mathcal{E})))$.

By (K2008, page 838-839), for an increasing sequence $\{F_k\}$ of closed sets, $\{F_k\}$ is an $\mathcal{E}$-nest if and only if it is an $\tilde{\mathcal{E}}_1^{\bar{h}}$-nest. We select a compact $\tilde{\mathcal{E}}_1^{\bar{h}}$-nest $\{F_k\}$ such that $\tilde{\bar{h}}$ is bounded on each F_k. Let $\mu \in S(\mathcal{E})$, the family of smooth measures w.r.t. $(E, \mathcal{B}(E))$. Then $\mu \in S(\tilde{\mathcal{E}}_1^{\bar{h}})$, the family of smooth measures w.r.t. $(\tilde{\mathcal{E}}_1^{\bar{h}}, D(\mathcal{E}^{\bar{h}}))$. By (FOT1994, Theorem 2.2.4) and quasi-homeomorphism, we know that there exists a compact $\tilde{\mathcal{E}}_1^{\bar{h}}$-nest (hence $\mathcal{E}$-nest) $\{J_k\}$ such that $I_{J_k} \cdot \mu \in S_0(\tilde{\mathcal{E}}_1^{\bar{h}})$. Then, there exists a sequence of positive constants $\{C_k\}$ such that

$$\int_E |\tilde{g}| I_{J_k} d\mu \leq C_k \tilde{\mathcal{E}}_1^{\bar{h}}(g, g)^{1/2}, \quad \forall g \in D(\mathcal{E}^{\bar{h}}).$$

We now show that each $1_{F_k \cap J_k} \cdot \mu \in S_0(\mathcal{E})$ and the proof is done. In fact, let

$f \in D(\mathcal{E})$. We have $\frac{f}{h} \in D(\mathcal{E}^{\bar{h}})$. Then

$$\begin{aligned}
\int_E |\tilde{f}| 1_{F_k \cap J_k} d\mu &\leq \| \bar{h}|_{F_k} \|_\infty \int_E |\frac{\tilde{f}}{\bar{h}}| 1_{F_k \cap J_k} d\mu \\
&\leq \| \bar{h}|_{F_k} \|_\infty \int_E |\frac{\tilde{f}}{\bar{h}}| I_{J_k} d\mu \\
&\leq \| h|_{F_n} \|_\infty C_k \tilde{\mathcal{E}}_1^{\bar{h}}(f/\bar{h}, f/\bar{h})^{1/2} \\
&= \| \bar{h}|_{F_k} \|_\infty C_k \mathcal{E}_1(f,f)^{1/2}.
\end{aligned}$$

Since $f \in D(\mathcal{E})$ is arbitrary, this implies that $I_{F_k \cap J_k} \cdot \mu \in S_0(\mathcal{E})$. □

Lemma 3.1. *For any $u \in D(\mathcal{E})$, $\nu \in S_0$, $0 < T < \infty$ and $\varepsilon > 0$,*

$$P_\nu(\sup_{0 \leq t \leq T} |\tilde{u}(X_t)| > \varepsilon) \leq \frac{2K^3 e^T}{\varepsilon} \mathcal{E}_1(u,u)^{1/2} \mathcal{E}_1(\hat{U}_1\nu, \hat{U}_1\nu)^{1/2}.$$

Proof. We take an $\mathcal{E}$-quasi-continuous Borel version $\tilde{u}$ of u. Let $A = \{x \in E \,|\, |\tilde{u}(x)| > \varepsilon\}$ and $\sigma_A := \inf\{t > 0 \,|\, X_t \in A\}$. By (K2008, Theorem 4.4), $H_A^1|u| := E_\cdot[e^{-\sigma_A}|u|(X_{\sigma_A})]$ is an $\mathcal{E}$-quasi-continuous version of $|u|_A$. Then, by (MOR1995, Proposition 2.8(i) and (2.1)), we get

$$\begin{aligned}
P_\nu(\sup_{0 \leq t \leq T} |\tilde{u}(X_t)| > \varepsilon) &\leq \frac{e^T E_\nu[e^{-\sigma_A}|u|(X_{\sigma_A})]}{\varepsilon} \\
&= \frac{e^T}{\varepsilon} \int_E |u|_A d\nu \\
&= \frac{e^T}{\varepsilon} \mathcal{E}_1(|u|_A, \hat{U}_1\nu) \\
&\leq \frac{Ke^T}{\varepsilon} \mathcal{E}_1(|u|_A, |u|_A)^{1/2} \mathcal{E}_1(\hat{U}_1\nu, \hat{U}_1\nu)^{1/2} \\
&\leq \frac{K^2 e^T}{\varepsilon} \mathcal{E}_1(|u|, |u|)^{1/2} \mathcal{E}_1(\hat{U}_1\nu, \hat{U}_1\nu)^{1/2} \\
&\leq \frac{2K^3 e^T}{\varepsilon} \mathcal{E}_1(u,u)^{1/2} \mathcal{E}_1(\hat{U}_1\nu, \hat{U}_1\nu)^{1/2}.
\end{aligned}$$

□

By Lemma 3.1 and Theorem 3.1, similar to (FOT1994, Lemma 5.1.2), we can prove the following lemma.

Lemma 3.2. *Let $\{u_n\}$ be a sequence of $\mathcal{E}$-quasi continuous functions in $D(\mathcal{E})$. If $\{u_n\}$ is an $\mathcal{E}_1$-Cauchy sequence, then there exists a subsequence $\{u_{n_k}\}$ satisfying the condition that for $\mathcal{E}$-q.e. $x \in E$*

$$P_x(u_{n_k}(X_t) \ \text{ converges uniformly in } t \text{ on each compact interval of } \ [0,\infty)) = 1.$$

Definition 3.3. *A family $(A_t)_{t\geq 0}$ of functions on Ω is said to be an additive functional (AF) of $\mathbf{M}$ if:*

(i) A_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

(ii) There exists a defining set $\Lambda \in \mathcal{F}$ and an exceptional set $N \subset E$ which is $\mathcal{E}$-exceptional such that $P_x[\Lambda] = 1$ for all $x \in E \backslash N$, $\theta_t(\Lambda) \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Lambda$, $t \to A_t(\omega)$ is right continuous on $(0,\infty)$ and has left limits on $(0,\zeta(\omega))$, $A_0(\omega) = 0$, $|A_t(\omega)| < \infty$ for $t < \zeta(\omega)$, $A_t(\omega) = A_\zeta(\omega)$ for $t \geq \zeta(\omega)$, and $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $s,t \geq 0$.

Two AFs $A = (A_t)_{t\geq 0}$ and $B = (B_t)_{t\geq 0}$ are called equivalent and we write $A = B$ if they have a common defining set Λ and a common exceptional set N such that $A_t(\omega) = B_t(\omega)$ for all $\omega \in \Lambda$ and $t \geq 0$. An AF is called a continuous AF (CAF) if $t \to A_t(\omega)$ is continuous on $(0,\infty)$ and a positive continuous AF (PCAF) if $A_t(\omega) \geq 0$ for all $t \geq 0$, $\omega \in \Lambda$.

In (F2001), Fitzsimmons has extended the smooth measure characterization of PCAFs from the Dirichlet forms setting to the semi-Dirichlet forms setting (see (F2001, Theorem 4.22)). In particular, the following proposition holds.

Proposition 3.1. *(cf. (F2001, Proposition 4.12)) For any $\mu \in S_0$, there is a unique finite PCAF A such that $E_x(\int_0^\infty e^{-t}dA_t)$ is an $\mathcal{E}$-quasi-continuous version of $U_1\mu$.*

By Proposition 3.1 and Theorem 3.2, following the arguments of (FOT1994, Theorems 5.1.3 and 5.1.4) (with necessary, slight modifications by virtue of (MOR1995; MR1995; K2008)), we can obtain the following theorem, which will play an important role in developing Fukushima's decomposition of semi-Dirichlet forms.

Theorem 3.3. *Let $\mu \in S$ and A be a PCAF. Then the following conditions are equivalent to each other:*

(i) For any γ-co-excessive function g ($\gamma \geq 0$) in $D(\mathcal{E})$ and $f \in \mathcal{B}^+(E)$,

$$\lim_{t\downarrow 0}\frac{1}{t}E_{g\cdot m}((fA)_t) =< f\cdot\mu, \tilde{g} > . \tag{3.6}$$

(ii)For any γ-co-excessive function g ($\gamma \geq 0$) in $D(\mathcal{E})$ and $f \in \mathcal{B}^+(E)$,

$$\alpha(g, U_A^{\alpha+\gamma}f) \ \uparrow < f\cdot\mu, \tilde{g} >, \quad \alpha\uparrow\infty,$$

where $U_A^{\alpha}f(x) := E_x(\int_0^\infty e^{-\alpha t}f(X_t)dA_t)$.

(iii) For any $t > 0$, $g \in \mathcal{B}^+(E)\bigcap L^2(E;m)$ and $f \in \mathcal{B}^+(E)$,

$$E_{g\cdot m}((fA)_t) = \int_0^t < f\cdot\mu, \widetilde{\hat{T}_s g} > ds.$$

(iv) For any $\alpha > 0$, $g \in \mathcal{B}^+(E)\bigcap L^2(E;m)$ and $f \in \mathcal{B}^+(E)$,

$$(g, U_A^{\alpha}f) =< f\cdot\mu, \widetilde{\hat{G}_\alpha g} > .$$

When $\mu \in S_0$, each of the above four conditions is also equivalent to each of the following three conditions:

(v) $U_A^1 1$ is an $\mathcal{E}$-quasi-continuous version of $U_1\mu$.

(vi) For any $g \in \mathcal{B}^+(E)\bigcap D(\mathcal{E})$ and $f \in \mathcal{B}_b^+(E)$,

$$\lim_{t\downarrow 0}\frac{1}{t}E_{g\cdot m}((fA)_t) =< f\cdot\mu, \tilde{g} > .$$

The family of all equivalent classes of PCAFs and the family S are in one to one correspondence under the Revuz correspondence (3.6).

Given a PCAF A, we denote by μ_A the Revuz measure of A.

Lemma 3.3. *Let A be a PCAF and $\nu \in \hat{S}_{00}^*$. Then there exists a positive constant C_ν such that for any $t > 0$*

$$E_\nu(A_t) \leq C_\nu(1+t)\int_E \tilde{\hat{h}} d\mu_A.$$

Proof. By Theorem 3.2, we may assume without loss of generality that $\mu \in S_0$. Set $c_t(x) = E_x(A_t)$. Similar to (O1988, page 137), we can show that for any $v \in D(\mathcal{E})$

$$\mathcal{E}(c_t, v) =< \mu_A, v - \hat{T}_t v > .$$

Let $\nu \in \hat{S}^*_{00}$. Then

$$\begin{aligned} E_\nu(A_t) &= <\nu, c_t> \\ &= \mathcal{E}_1(c_t, \hat{U}_1\nu) \\ &\leq <\mu_A, \hat{U}_1\nu> + <c_t, \hat{U}_1\nu> \\ &\leq c_\nu[<\mu_A, \hat{h}> + E_{\hat{h}\cdot m}(A_t)]. \end{aligned}$$

Therefore the proof is completed by (3.6). □

A subset F of E is said to be an $\hat{\mathcal{E}}$-quasi-open set if there is an $\hat{\mathcal{E}}$-nest $\{F_n\}_{n\geq 1}$ such that $F \cap F_n$ is open with respect to the relative topology on F_n for each $n \geq 1$. For a nearly Borel set B, denote the $\hat{\mathcal{E}}$-quasi interior of B by B^o, which is the union of all $\hat{\mathcal{E}}$-quasi-open subsets contained in B. One finds that $\hat{\mathcal{E}}$-quasi interior is same as $\mathcal{E}$-quasi interior.

Lemma 3.4. *Let $\mu \in S_0$ and A be a PCAF with Revuz measure μ. Then for any nearly Borel set $B \subset E$,*

$$\alpha E_{h\cdot m}[\int_0^{\sigma_B} e^{-(\alpha+\gamma)t} f(X_t)dA_t] \uparrow \int_{(E-B)^o} \tilde{h}(x)f(x)\mu(dx), \quad \alpha \uparrow \infty,$$

where h is any γ-co-excessive function ($\gamma \geq 0$) in $D(\mathcal{E})$, $f \in \mathcal{B}^+(E)$.

Proof. It is enough to consider the case that $\gamma = 0$, h is a bounded $L^2(E;m)$-function in $D(\mathcal{E})$ and $f \in \mathcal{B}^+_b(E)$. Let $(\hat{G}^{E-B}_\alpha)_{\alpha\geq 0}$ be the co-resolvent of the part form $(\mathcal{E}^{E-B}, D(\mathcal{E})_{E-B})$. Denote by $h\,|_{(E-B)^o}$ the restriction of h on $(E-B)^o$. Define $H^\alpha_B u(x) := E_x[u(X_{\sigma_B})e^{-\alpha\sigma_B}; \sigma_B < \infty]$. Then by (K2008, Proposition 3.3), we get

$$\begin{aligned} \alpha E_{h\cdot m}[\int_0^{\sigma_B} e^{-\alpha t} f(X_t)dA_t] &= \alpha(U^\alpha_A f - H^\alpha_B U^\alpha_A f, h) \\ &= \alpha\mathcal{E}_\alpha(U^\alpha_A f - H^\alpha_B U^\alpha_A f, \hat{G}_\alpha h) \\ &= \alpha\mathcal{E}_\alpha(U^\alpha_A f, \hat{G}_\alpha h - \widehat{(\hat{G}_\alpha h)}^\alpha_B) \\ &= \alpha\mathcal{E}_\alpha(U_\alpha(f\mu), \hat{G}^{E-B}_\alpha h) \\ &= \alpha\int_E \widetilde{\hat{G}^{E-B}_\alpha h}(x)f(x)\mu(dx) \\ &= \alpha\int_E \hat{G}^{E-B}_\alpha \widetilde{(h\,|_{(E-B)^o})}(x)f(x)\mu(dx), \end{aligned}$$

Similar to the proof of Lemma 3.7 below, we know that $h\mid_{(E-B)^o}$ is a 0-order co-excessive function with respect to the part form $(\mathcal{E}^{E-B}, D(\mathcal{E})_{E-B})$. Hence, by the monotone convergence theorem, we get

$$\alpha E_{h\cdot m}[\int_0^{\sigma_B} e^{-\alpha t} f(X_t)dA_t] \uparrow \alpha \int_{(E-B)^o} \tilde{h}(x)f(x)\mu(dx)$$

as $\alpha \uparrow \infty$. □

Let G be a nearly Borel finely open set. Denote by $(\widehat{T_s^G})_{s\geq 0}$ be the co-semigroup of the part form $(\mathcal{E}^G, D(\mathcal{E})_G)$. For a PCAF A and a non-negative Borel measurable function f, define $U_A^{G,\alpha}f(x) := E_x(\int_0^{\sigma_{E-G}} e^{-\alpha t}f(X_t)dA_t)$.

Lemma 3.5. *Let A be a PCAF and G be a nearly Borel finely open set.*
(i) If h is $\gamma-$co-excessive ($\gamma \geq 0$) on G with respect to $(\widehat{T_s^G})_{s\geq 0}$, $h \in D(\mathcal{E})$ and $f \in \mathcal{B}^+(E)$, then

$$\alpha(h, U_A^{G,\alpha+\gamma}f)_m \uparrow (fI_G\cdot\mu_A, \tilde{h}), \quad \alpha\uparrow\infty. \tag{3.7}$$

(ii) For any $t > 0$, $h \in \mathcal{B}^+(E)\cap L^2(E;m)$ and $f \in \mathcal{B}^+(E)$,

$$E_{h\cdot m}(\int_0^t f(X_s)dA_{s\wedge\sigma_{E-G}}) = \int_0^t (fI_G\cdot\mu_A, \widetilde{\widehat{T_s^G}h})ds.$$

(iii) Suppose that for m-a.e. $x \in E$,

$$P_x(A_t = 0,\ \forall t < \tau_G) = 1. \tag{3.8}$$

Then $\mu_A(G) = 0$ and (3.8) holds for $\mathcal{E}$-q.e. $x \in E$.

Proof. (i) For $\mu_A \in S_0$, this has been proved in Lemma 3.4. For general $\mu \in S$, by Theorem 3.2(ii), we can find an $\mathcal{E}-$nest $\{F_n\}_{n\geq 1}$ such that $I_{F_n}\mu_A \in S_0$. Substituting μ_A with $I_{F_n}\mu_A$ and A with $I_{F_n}A$ in (3.7), then by letting n tend to the infinity, we get (i).

(ii) By the uniqueness of Laplace transform, we find that (i) and (ii) are equivalent.

(iii) Note that $\tau_G = \sigma_{E-G}\wedge\zeta$. By the continuity of A, from (3.8) we know that for m-a.e. $x \in E$,

$$P_x(A_t = 0,\ \forall t < \sigma_{E-G}) = 1.$$

Then, by (ii), we know that $\mu_A(G) = 0$. Note that $\tau_G \le \sigma_{E-G}$, hence by Lemma 3.3, we get that for any $\nu \in \hat{S}^*_{00}$

$$0 \le E_\nu(A_t; t < \tau_G) \le E_\nu(A_t; t < \sigma_{E-G}) \le E_\nu(A_{t\wedge\sigma_{E-G}}) \le C_\nu(1+t)\int_G \widetilde{\hat{h}} d\mu_A = 0.$$

Therefore, by Theorem 3.1, we know that (3.8) holds for $\mathcal{E}$-q.e. $x \in E$. □

Similar to (FOT1994, Lemma 5.5.2), we can prove the following lemma by noting that for a semi-Dirichlet form any semi-polar set is exceptional (cf. (F2001, Theorem 4.3)).

Lemma 3.6. *For an AF A and a nearly Borel finely open set G,*

$$A_{(t+s)\wedge\tau_G} = A_{s\wedge\tau_G} + A_{t\wedge\tau_G} \circ \theta_{s\wedge\tau_G}, \quad P_x - a.s., \ \forall x \in E - N,$$

where N is any properly exceptional set containing $(E-G)-(E-G)^r$ and an exceptional set for A.

3.2 Fukushima's decomposition in the semi-Dirichlet forms setting

Throughout this section, we suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$. Here "local" means that $\mathcal{E}(u,v) = 0$ for all $u, v \in D(\mathcal{E})$ with $\mathrm{supp}[u] \cap \mathrm{supp}[v] = \emptyset$. Then, there exists a diffusion $\mathbf{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, (X_t)_{t\ge0}, (P_x)_{x\in E_\Delta})$ which is properly associated with $(\mathcal{E}, D(\mathcal{E}))$ (cf. (K2008, Theorem 4.5)). Here "diffusion" means that $\mathbf{M}$ is a right process satisfying

$$P_x[t \to X_t \text{ is continuous on } [0,\zeta)] = 1 \text{ for all } x \in E.$$

We fix $\phi \in L^2(E;m)$ with $0 < \phi \le 1$ m-a.e. and set $h = G_1\phi$, $\hat{h} = \hat{G}_1\phi$. Denote $\tau_B := \inf\{t > 0 \,|\, X_t \notin B\}$ for $B \subset E$.

Let V be a quasi-open subset of E. We denote by $X^V = (X^V_t)_{t\ge0}$ the part process of X on V and denote by $(\mathcal{E}^V, D(\mathcal{E})_V)$ the part form of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(V;m)$. It is known that X^V is a diffusion process and $(\mathcal{E}^V, D(\mathcal{E})_V)$ is a quasi-regular local semi-Dirichlet form (cf. (K2008)). Denote by $(T_t^V)_{t\geq 0}$, $(\hat{T}_t^V)_{t\geq 0}$, $(G_\alpha^V)_{\alpha\geq 0}$ and $(\hat{G}_\alpha^V)_{\alpha\geq 0}$ the semigroup, co-semigroup, resolvent and co-resolvent associated with $(\mathcal{E}^V, D(\mathcal{E})_V)$, respectively.

Lemma 3.7. *$\hat{h}|_V$ is 1-co-excessive w.r.t. $(\mathcal{E}^V, D(\mathcal{E})_V)$.*

Proof. It is easy to see that $\hat{h}|_V \geq 0$ m-*a.e.* on V. Let g be a positive measurable function on V. Then

$$\begin{aligned}
\int_V g e^{-t}\hat{T}_t^V(\hat{h}|_V)dm &= \int_V e^{-t}(T_t^V g)\hat{h}dm \\
&= \int_V e^{-t}E_x[g(X_t); t < \tau_V]\hat{h}(x)m(dx) \\
&\leq \int_E e^{-t}T_t g\hat{h}dm \\
&= \int_E g e^{-t}\hat{T}_t\hat{h}dm \\
&\leq \int_V g\hat{h}|_V dm.
\end{aligned}$$

Since g is arbitrary, $e^{-t}\hat{T}_t^V(\hat{h}|_V) \leq \hat{h}|_V$ m-*a.e.* on V. Therefore $\hat{h}|_V$ is 1-co-excessive w.r.t. $(\mathcal{E}^V, D(\mathcal{E})_V)$. □

Define $\bar{h}^V := \hat{h}|_V \wedge \hat{G}_1^V\phi$. Then $\bar{h}^V \in D(\mathcal{E})_V$ and $\bar{h}^V$ is 1-co-excessive. For an AF $A = (A_t)_{t\geq 0}$ of X^V, we define

$$e^V(A) := \lim_{t\downarrow 0}\frac{1}{2t}E_{\bar{h}^V\cdot m}(A_t^2) \tag{3.9}$$

whenever the limit exists in $[0,\infty]$. Define

$$\begin{aligned}
\dot{\mathcal{M}}^V \;&:=\; \{M \,|\, M \text{ is an AF of } X^V,\ E_x(M_t^2) < \infty, E_x(M_t) = 0 \\
&\qquad \text{for all } t \geq 0 \text{ and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(M) < \infty\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{N}_c^V \;&:=\; \{N \,|\, N \text{ is a CAF of } X^V, E_x(|N_t|) < \infty \text{ for all } t \geq 0 \\
&\qquad \text{and } \mathcal{E}\text{-}q.e.\ x \in V, e^V(N) = 0\},
\end{aligned}$$

$$\begin{aligned}\Theta &:= \{\{V_n\} \,|\, V_n \text{ is } \mathcal{E}\text{-quasi-open},\ V_n \subset V_{n+1}\ \mathcal{E}\text{-}q.e.,\\ &\qquad\qquad \text{for all } n \in \mathbb{N},\ \text{and } E = \cup_{n=1}^{\infty} V_n\ \mathcal{E}\text{-}q.e.\},\end{aligned}$$

and

$$\begin{aligned}D(\mathcal{E})_{loc} &:= \{u \,|\, \exists\, \{V_n\} \in \Theta \text{ and } \{u_n\} \subset D(\mathcal{E})\\ &\qquad\qquad \text{such that } \ u = u_n\ m\text{-}a.e. \text{ on } V_n\}.\end{aligned}$$

We call $A = (A_t)_{t\geq 0}$ a local AF of $\mathbf{M}$ if A satisfies all requirements for an AF stated in Definition 3.3 except that the additivity $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for $\omega \in \Lambda$ is required only for $s, t \geq 0$ with $t + s < \zeta(\omega)$. Two local AFs $A^{(1)}$, $A^{(2)}$ are said to be equivalent if for each $t \geq 0$ and $\mathcal{E}$-$q.e.$ $x \in E$,

$$P_x(A_t^{(1)} = A_t^{(2)}; t < \zeta) = P_x(t < \zeta).$$

Define

$$\begin{aligned}\dot{\mathcal{M}}_{loc} &:= \{M \,|\, M \text{ is a local AF of } \mathbf{M},\ \exists\, \{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \,|\, M^n \in \dot{\mathcal{M}}^{V_n}\}\\ &\qquad \text{such that } E_n \subset V_n,\ M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}_{c,loc} &:= \{N \,|\, N \text{ is a local AF of } \mathbf{M},\ \exists\, \{V_n\}, \{E_n\} \in \Theta \text{ and } \{N^n \,|\, N^n \in \mathcal{N}_c^{V_n}\}\\ &\qquad \text{such that } E_n \subset V_n,\ N_{t\wedge\tau_{E_n}} = N^n_{t\wedge\tau_{E_n}},\ t \geq 0,\ n \in \mathbb{N}\}.\end{aligned}$$

We use $\mathcal{M}_{loc}^{[\![0,\zeta[\![}$ to denote the family of local martingales on $[\![0,\zeta[\![$ (cf. (HWY1992, §8.3)).

We put the following assumption:

Assumption 3.1. *There exists $\{V_n\} \in \Theta$ such that, for each $n \in \mathbb{N}$, there exists a Dirichlet form $(\eta^{(n)}, D(\eta^{(n)}))$ on $L^2(V_n; m)$ and a constant $C_n > 1$ such that $D(\eta^{(n)}) = D(\mathcal{E})_{V_n}$ and for any $u \in D(\mathcal{E})_{V_n}$,*

$$\frac{1}{C_n}\eta_1^{(n)}(u,u) \leq \mathcal{E}_1(u,u) \leq C_n\eta_1^{(n)}(u,u).$$

Now we can state the main result of this section.

Theorem 3.4. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 3.1. Then, for any $u \in D(\mathcal{E})_{loc}$, there exist $M^{[u]} \in \dot{\mathcal{M}}_{loc}$ and $N^{[u]} \in \mathcal{N}_{c,loc}$ such that*

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^{[u]} + N_t^{[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E. \tag{3.10}$$

Moreover, $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$. Decomposition (3.10) is unique up to the equivalence of local AFs.

Before proving Theorem 3.4, we present some lemmas.

We fix a $\{V_n\} \in \Theta$ satisfying Assumption 3.1. Without loss of generality, we assume that $\widetilde{\bar{h}}$ is bounded on each V_n. Denote $D(\mathcal{E})_{V_n,b} := \mathcal{B}_b(E) \cap D(\mathcal{E})_{V_n}$. To simplify notation, we define $\bar{h}_n := \bar{h}^{V_n}$.

By Lemma 3.3, (3.9), Theorem 3.3 and Theorem 3.1, similar to (FOT1994, Theorem 5.2.1), we can prove the following lemma.

Lemma 3.8. *$\dot{\mathcal{M}}^{V_n}$ is a real Hilbert space with inner product e^{V_n}. Moreover, if $\{M_l\} \subset \dot{\mathcal{M}}^{V_n}$ is e^{V_n}-Cauchy, then there exist a unique $M \in \dot{\mathcal{M}}^{V_n}$ and a subsequence $\{l_k\}$ such that $\lim_{k\to\infty} e^{V_n}(M_{l_k} - M) = 0$ and for $\mathcal{E}$-q.e. $x \in V_n$,*

$$P_x(\lim_{k\to\infty} M_{l_k}(t) = M(t) \text{ uniformly on each compact interval of } [0,\infty)) = 1.$$

Next we give Fukushima's decomposition for the part process X^{V_n}.

Lemma 3.9. *Let $u \in D(\mathcal{E})_{V_n,b}$. Then there exist unique $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$ and $N^{n,[u]} \in \mathcal{N}_c^{V_n}$ such that* for *$\mathcal{E}$-q.e. $x \in V_n$,*

$$\tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) = M_t^{n,[u]} + N_t^{n,[u]}, \quad t \geq 0, \quad P_x\text{-}a.s. \tag{3.11}$$

Proof. Note that if an AF $A \in \dot{\mathcal{M}}^{V_n}$ with $e^{V_n}(A) = 0$ then $\mu_{<A>}^{(n)}(\widetilde{\bar{h}_n}) = 2e^{V_n}(A) = 0$ by Theorem 3.3 and (3.9). Here $\mu_{<A>}^{(n)}$ denotes the Revuz measure of A w.r.t. X^{V_n}. Hence $< A >= 0$ since $\widetilde{\bar{h}_n} > 0$ $\mathcal{E}$-q.e. on V_n. Therefore $\dot{\mathcal{M}}^{V_n} \cap \mathcal{N}_c^{V_n} = \{0\}$ and the proof of the uniqueness of decomposition (3.11) is complete.

To obtain the existence of decomposition (3.11), we start with the special case that $u = R_1^{V_n} f$ for some bounded Borel function $f \in L^2(V_n; m)$, where $(R_t^{V_n})_{t\geq 0}$ is the resolvent of X^{V_n}. Set

$$\begin{cases} N_t^{n,[u]} = \int_0^t (u(X_s^{V_n}) - f(X_s^{V_n}))ds, \\ M_t^{n,[u]} = u(X_t^{V_n}) - u(X_0^{V_n}) - N_t^{n,[u]}, \qquad t \geq 0. \end{cases} \tag{3.12}$$

Then $N^{n,[u]} \in \mathcal{N}_c^{V_n}$ and $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$. In fact,

$$\begin{aligned} e^{V_n}(N^{n,[u]}) &= \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}_n\cdot m}[(\int_0^t (u-f)(X_s^{V_n})ds)^2] \\ &\leq \lim_{t\downarrow 0} \frac{1}{2} E_{\bar{h}_n\cdot m}[\int_0^t (u-f)^2(X_s^{V_n})ds] \\ &= \lim_{t\downarrow 0} \frac{1}{2}[\int_0^t \int_{V_n} \bar{h}_n T_s^{V_n}(u-f)^2 dmds] \\ &= \lim_{t\downarrow 0} \frac{1}{2}[\int_0^t \int_{V_n} (u-f)^2 \hat{T}_s^{V_n}\bar{h}_n dmds] \\ &\leq \|u-f\|_\infty \lim_{t\downarrow 0} \frac{1}{2}[\int_0^t \int_{V_n} |u-f|\hat{T}_s^{V_n}\bar{h}_n dmds] \\ &\leq \|u-f\|_\infty \lim_{t\downarrow 0} \frac{1}{2}[\int_0^t (\int_{V_n} (u-f)^2 dm)^{1/2} (\int_{V_n} (\hat{T}_s^{V_n}\bar{h}_n)^2 dm)^{1/2} ds] \\ &\leq \|u-f\|_\infty (\int_{V_n} (u-f)^2 dm)^{1/2} (\int_{V_n} \bar{h}_n^2 dm)^{1/2} \lim_{t\downarrow 0} \frac{t}{2} \\ &= 0. \end{aligned} \tag{3.13}$$

By Assumption 3.1, $u^2 \in D(\mathcal{E})_{V_n,b}$ and $u\bar{h}_n \in D(\mathcal{E})_{V_n,b}$. Then, by (3.12), (3.13), (AFRS1995, Theorem 3.4) and Assumption 3.1, we get

$$\begin{aligned} e^{V_n}(M^{n,[u]}) &= \lim_{t\downarrow 0} \frac{1}{2t} E_{\bar{h}_n\cdot m}[(u(X_t^{V_n}) - u(X_0^{V_n}))^2] \\ &= \lim_{t\downarrow 0} \{\frac{1}{t}(u\bar{h}_n, u - T_t^{V_n}u) - \frac{1}{2t}(\bar{h}_n, u^2 - T_t^{V_n}u^2)\} \\ &= \mathcal{E}^{V_n}(u, u\bar{h}_n) - \frac{1}{2}\mathcal{E}^{V_n}(u^2, \bar{h}_n) \\ &\leq \mathcal{E}_1^{V_n}(u, u\bar{h}_n) \\ &\leq K\mathcal{E}_1^{V_n}(u,u)^{1/2}\mathcal{E}_1^{V_n}(u\bar{h}_n, u\bar{h}_n)^{1/2} \\ &\leq KC_n^{1/2}\mathcal{E}_1^{V_n}(u,u)^{1/2}\eta_1^{(n)}(u\bar{h}_n, u\bar{h}_n)^{1/2} \\ &\leq KC_n^{1/2}\mathcal{E}_1^{V_n}(u,u)^{1/2}(\|u\|_\infty \eta_1^{(n)}(\bar{h}_n, \bar{h}_n)^{1/2} + \|\bar{h}_n\|_\infty \eta_1^{(n)}(u,u)^{1/2}) \\ &\leq KC_n\mathcal{E}_1^{V_n}(u,u)^{1/2}(\|u\|_\infty \mathcal{E}_1^{V_n}(\bar{h}_n, \bar{h}_n)^{1/2} + \|\bar{h}_n\|_\infty \mathcal{E}_1^{V_n}(u,u)^{1/2}). \end{aligned} \tag{3.14}$$

Next, take any bounded Borel function $u \in D(\mathcal{E})_{V_n}$. Define

$$u_l = lR_{l+1}^{V_n}u = R_1^{V_n}g_l, \qquad g_l = l(u - lR_{l+1}^{V_n}u).$$

By the uniqueness of decomposition (3.11) for u_l's, we have $M^{n,[u_l]} - M^{n,[u_k]} = M^{n,[u_l-u_k]}$. Then, by (3.14), we get

$$
\begin{aligned}
&e^{V_n}(M^{n,[u_l]} - M^{n,[u_k]}) \\
&\quad = e^{V_n}(M^{n,[u_l-u_k]}) \\
&\quad \leq KC_n\mathcal{E}_1^{V_n}(u_l-u_k,u_l-u_k)^{1/2}(\|u_l-u_k\|_\infty\mathcal{E}_1^{V_n}(\bar{h}_n,\bar{h}_n)^{1/2} \\
&\qquad\qquad +\|\bar{h}_n\|_\infty\mathcal{E}_1^{V_n}(u_l-u_k,u_l-u_k)^{1/2}).
\end{aligned}
$$

Since $u_l \in D(\mathcal{E})_{V_n}$, bounded by $\|u\|_\infty$, and $\mathcal{E}_1^{V_n}$-convergent to u, we conclude that $\{M^{n,[u_l]}\}$ is an e^{V_n}-Cauchy sequence in the space $\dot{\mathcal{M}}^{V_n}$. Define

$$
M^{n,[u]} = \lim_{l\to\infty} M^{n,[u_l]} \text{ in } (\dot{\mathcal{M}}^{V_n}, e^{V_n}), \qquad N^{n,[u]} = \tilde{u}(X_t^{V_n}) - \tilde{u}(X_0^{V_n}) - M^{n,[u]}.
$$

Then $M^{n,[u]} \in \dot{\mathcal{M}}^{V_n}$ by Lemma 3.8.

It only remains to show that $N^{n,[u]} \in \mathcal{N}_c^{V_n}$. By Lemmas 3.2 and 3.8, there exists a subsequence $\{l_k\}$ such that for $\mathcal{E}$-*q.e.* $x \in V_n$,

$$
P_x(N^{n,[u_{l_k}]} \text{ converges to } N^{n,[u]} \text{ uniformly on each compact interval of } [0,\infty)) = 1.
$$

From this and (3.12), we know that $N^{n,[u]}$ is a CAF. On the other hand, by

$$
N_t^{n,[u]} = A_t^{n,[u-u_l]} - (M_t^{n,[u]} - M_t^{n,[u_l]}) + N_t^{n,[u_l]},
$$

we get

$$
e^{V_n}(N^{n,[u]}) \leq 3e^{V_n}(A^{n,[u-u_l]}) + 3e^{V_n}(M^{n,[u]} - M^{n,[u_l]}),
$$

which can be made arbitrarily small with large l by (3.14). Therefore $e^{V_n}(N^{n,[u]}) = 0$ and $N^{n,[u]} \in \mathcal{N}_c^{V_n}$. □

We now fix a $u \in D(\mathcal{E})_{loc}$. Then there exist $\{V_n^1\} \in \Theta$ and $\{u_n\} \subset D(\mathcal{E})$ such that $u = u_n$ m-*a.e.* on V_n^1. By (MOR1995, Proposition 3.6), we may assume without loss of generality that each u_n is $\mathcal{E}$-quasi-continuous. By (MOR1995, Proposition 2.16), there exists an $\mathcal{E}$-nest $\{F_n^2\}$ consisting of compact subsets of E such that $\{u_n\} \subset C\{F_n^2\}$. Denote by V_n^2 the finely interior of F_n^2 for $n \in \mathbb{N}$. Then $\{V_n^2\} \in \Theta$. Define $V_n' = V_n^1 \cap V_n^2$. Then $\{V_n'\} \in \Theta$ and each u_n is bounded on V_n'. To simplify notation, we still use V_n to denote $V_n \cap V_n'$ for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we define $E_n = \{x \in E \,|\, \widetilde{h_n}(x) > \frac{1}{n}\}$, where $h_n := G_1^{V_n}\phi$. Then $\{E_n\} \in \Theta$ satisfying $\overline{E}_n^{\mathcal{E}} \subset E_{n+1}$ $\mathcal{E}$-$q.e.$ and $E_n \subset V_n$ $\mathcal{E}$-$q.e.$ for each $n \in \mathbb{N}$ (cf. (K2008, Lemma 3.8)). Here $\overline{E}_n^{\mathcal{E}}$ denotes the $\mathcal{E}$-quasi-closure of E_n. Define $f_n = n\widetilde{h_n} \wedge 1$. Then $f_n = 1$ on E_n and $f_n = 0$ on V_n^c. Since f_n is a 1-excessive function of $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$ and $f_n \leq n\widetilde{h_n} \in D(\mathcal{E})_{V_n}$, hence $f_n \in D(\mathcal{E})_{V_n}$ by (MR1995, Remark 3.4(ii)). Denote by Q_n the bound of $|u_n|$ on V_n. Then $u_n f_n = ((-Q_n) \vee u_n \wedge Q_n) f_n \in D(\eta)_{V_n,b} = D(\mathcal{E})_{V_n,b}$.

For $n \in \mathbb{N}$, we denote by $\{\mathcal{F}_t^n\}$ the minimum completed admissible filtration of X^{V_n}. For $n < l$, $\mathcal{F}_t^n \subset \mathcal{F}_t^l \subset \mathcal{F}_t$. Since $E_n \subset V_n$, τ_{E_n} is an $\{\mathcal{F}_t^n\}$-stopping time.

Lemma 3.10. *For $n < l$, we have $M_{t\wedge\tau_{E_n}}^{n,[u_n f_n]} = M_{t\wedge\tau_{E_n}}^{l,[u_l f_l]}$ and $N_{t\wedge\tau_{E_n}}^{n,[u_n f_n]} = N_{t\wedge\tau_{E_n}}^{l,[u_l f_l]}$, $t \geq 0$, P_x-a.s.* for *$\mathcal{E}$-q.e. $x \in V_n$.*

Proof. Let $n < l$. Since $M^{n,[u_n f_n]} \in \dot{\mathcal{M}}^{V_n}$, $M^{n,[u_n f_n]}$ is an $\{\mathcal{F}_t^n\}$-martingale by the Markov property. Since τ_{E_n} is an $\{\mathcal{F}_t^n\}$-stopping time, $\{M_{t\wedge\tau_{E_n}}^{n,[u_n f_n]}\}$ is an $\{\mathcal{F}_{t\wedge\tau_{E_n}}^n\}$-martingale. Denote $\Upsilon_t^n = \sigma\{X_{s\wedge\tau_{E_n}}^{V_n} \,|\, 0 \leq s \leq t\}$. Then $\{M_{t\wedge\tau_{E_n}}^{n,[u_n f_n]}\}$ is a $\{\Upsilon_t^n\}$-martingale. Denote $\Upsilon_t^{n,l} = \sigma\{X_{s\wedge\tau_{E_n}}^{V_l} \,|\, 0 \leq s \leq t\}$. Similarly, we can show that $\{M_{t\wedge\tau_{E_n}}^{l,[u_n f_n]}\}$ is a $\{\Upsilon_t^{n,l}\}$-martingale. By the assumption that $\mathbf{M}$ is a diffusion, the fact that f_n is quasi-continuous and $f_n = 1$ on E_n, we get $f_n(X_{s\wedge\tau_{E_n}}) = 1$ if $0 < s\wedge\tau_{E_n} < \zeta$. Hence $X_{s\wedge\tau_{E_n}} \in V_n$, if $0 < s \wedge \tau_{E_n} < \zeta$, since $f_n = 0$ on V_n^c. Therefore

$$X_{s\wedge\tau_{E_n}}^{V_l} = X_{s\wedge\tau_{E_n}} = X_{s\wedge\tau_{E_n}}^{V_n}, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in V_n, \tag{3.15}$$

which implies that $\{M_{t\wedge\tau_{E_n}}^{l,[u_n f_n]}\}$ is a $\{\Upsilon_t^n\}$-martingale.

Let $N \in \mathcal{N}_c^{V_j}$ for some $j \in \mathbb{N}$. Then, for any $T > 0$,

$$\begin{aligned}
\sum_{k=1}^{[rT]} E_{\bar{h}_j\cdot m}[(N_{\frac{k+1}{r}} - N_{\frac{k}{r}})^2] &\leq \sum_{k=1}^{[rT]} e^T (E_\cdot(N_{\frac{1}{r}}^2), e^{-\frac{k}{r}} \hat{T}_{\frac{k}{r}}^{V_j} \bar{h}_j) \\
&\leq \sum_{k=1}^{[rT]} e^T (E_\cdot(N_{\frac{1}{r}}^2), \bar{h}_j) \\
&\leq rTe^T E_{\bar{h}_j\cdot m}(N_{\frac{1}{r}}^2) \to 0 \quad \text{as} \quad r \to \infty.
\end{aligned}$$

Hence

$$\sum_{k=1}^{[rT]} (N_{\frac{k+1}{r}} - N_{\frac{k}{r}})^2 \to 0, \quad r \to \infty, \quad \text{in} \quad P_m,$$

which implies that the quadratic variation process of N w.r.t. P_m is 0.

By (K2008, Proposition 3.3), $\widehat{(\hat{G}_1\phi)}^1_{V_n^c} = \hat{G}_1\phi - \hat{G}_1^{V_n}\phi$. Since $V_n^c \supset V_l^c$, $\widehat{(\hat{G}_1\phi)}^1_{V_n^c} \geq \widehat{(\hat{G}_1\phi)}^1_{V_l^c}$. Then $\hat{G}_1^{V_n}\phi \leq \hat{G}_1^{V_l}\phi$ and thus

$$\bar{h}_n \leq \bar{h}_l. \tag{3.16}$$

Therefore

$$e^{V_n}(A) \leq e^{V_l}(A) \tag{3.17}$$

for any AF $A = (A_t)_{t\geq 0}$ of X^{V_n}.

Note that $N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}} = \widetilde{(u_n f_n)}(X^{V_l}_{t\wedge\tau_{E_n}}) - \widetilde{(u_n f_n)}(X^{V_l}_0) - M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}} \in \Upsilon^{n,l}_t = \Upsilon^n_t \subset \mathcal{F}^n_{t\wedge\tau_{E_n}}$. By Lemma 3.6 $\{N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ is a CAF of X^{V_n}. By (3.17), $e^{V_n}(N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}) \leq e^{V_l}(N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}) = 0$. Hence $(N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}})_{t\geq 0} \in \mathcal{N}^{V_n}_c$, which implies that the quadratic variation process of $\{N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ w.r.t. P_m is 0. Since for $\mathcal{E}$-$q.e.$ $x \in V_n$, by (3.15),

$$\begin{aligned} M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} + N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} &= \widetilde{u_n f_n}(X^{V_n}_{t\wedge\tau_{E_n}}) - \widetilde{u_n f_n}(X^{V_n}_0) \\ &= \widetilde{u_n f_n}(X^{V_l}_{t\wedge\tau_{E_n}}) - \widetilde{u_n f_n}(X^{V_l}_0) \\ &= M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}} + N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}, \quad P_x - a.s., \end{aligned}$$

and both $\{M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ and $\{M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}\}$ are $\{\Upsilon^n_t\}$-martingale, hence $M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}$ and $N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}$, P_x-$a.s.$ for m-$a.e.$ $x \in V_n$. This implies that $E_m(< M^{n,[u_n f_n]}_{\cdot\wedge\tau_{E_n}} - M^{l,[u_n f_n]}_{\cdot\wedge\tau_{E_n}} >_t) = 0$, $\forall t \geq 0$. Then, by Theorem 3.3(i), $M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}$, $\forall t \geq 0$, P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in V_n$. Hence $N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = N^{l,[u_n f_n]}_{t\wedge\tau_{E_n}}$, $\forall t \geq 0$, P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in V_n$.

Since $u_n f_n = u_l f_l = u$ on E_n, similar to (K2010, Lemma 2.4), we can show that $M^{l,[u_n f_n]}_t = M^{l,[u_l f_l]}_t$ when $t < \tau_{E_n}$, P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in V_l$. If $\tau_{E_n} = \zeta$, then by the fact $u_n f_n(X^{V_l}_\zeta) = u_l f_l(X^{V_l}_\zeta) = 0$ and the continuity of $N^{l,[u_n f_n]}_t$ and $N^{l,[u_l f_l]}_t$, one finds that $M^{l,[u_n f_n]}_{t\wedge\tau_{E_n}} = M^{l,[u_l f_l]}_{t\wedge\tau_{E_n}}$. By the quasi-continuity of $u_n f_n$, $u_l f_l$ and the assumption that $\mathbf{M}$ is a diffusion, one finds that $M^{l,[u_n f_n]}$ and $M^{l,[u_l f_l]}$ are continuous on $[0,\zeta)$, P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in V_l$. Hence, if $\tau_{E_n} < \zeta$ we have $M^{l,[u_n f_n]}_{\tau_{E_n}} = M^{l,[u_l f_l]}_{\tau_{E_n}}$. Therefore $M^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = M^{l,[u_l f_l]}_{t\wedge\tau_{E_n}}$ and $N^{n,[u_n f_n]}_{t\wedge\tau_{E_n}} = N^{l,[u_l f_l]}_{t\wedge\tau_{E_n}}$, $t \geq 0$, P_x-$a.s.$ for $\mathcal{E}$-$q.e.$ $x \in V_n$. □

Proof of Theorem 3.4 We define $M^{[u]}_{t\wedge\tau_{E_n}} := \lim_{l\to\infty} M^{l,[u_l f_l]}_{t\wedge\tau_{E_n}}$ and $M^{[u]}_t := 0$ for $t > \zeta$ if there exists some n such that $\tau_{E_n} = \zeta$ and $\zeta < \infty$; or $M^{[u]}_t := 0$ for $t \geq \zeta$, otherwise. By Lemma 3.10, $M^{[u]}$ is well defined. Define $M_t^n := M_{t\wedge\tau_{E_n}}^{n+1,[u_{n+1}f_{n+1}]}$ for $t \geq 0$ and $n \in \mathbb{N}$. Then $M_{t\wedge\tau_{E_n}}^{[u]} = M_{t\wedge\tau_{E_n}}^n$ P_x-*a.s.* for $\mathcal{E}$-*q.e.* $x \in V_{n+1}$ by Lemma 3.10. Since $\overline{E}_n^{\mathcal{E}} \subset E_{n+1} \subset V_{n+1}$ $\mathcal{E}$-*q.e.* implies that $P_x(\tau_{E_n} = 0) = 1$ for $x \notin V_{n+1}$, $M_{t\wedge\tau_{E_n}}^{[u]} = M_{t\wedge\tau_{E_n}}^n$ P_x-*a.s.* for $\mathcal{E}$-*q.e.* $x \in E$. Similar to (3.16) and (3.17), we can show that $e^{V_n}(M^n) \leq e^{V_{n+1}}(M^n)$ for each $n \in \mathbb{N}$. Then $M^n \in \dot{\mathcal{M}}^{V_n}$ and hence $M^{[u]} \in \dot{\mathcal{M}}_{loc}$.

Next we show that M^n is also an $\{\mathcal{F}_t\}$-martingale, which implies that $M^{[u]} \in \mathcal{M}_{loc}^{[\![0,\zeta[\![}$. In fact, by the fact that τ_{E_n} is an $\{\mathcal{F}_t^{n+1}\}$-stopping time, we find that $I_{\tau_{E_n}\leq s}$ is $\mathcal{F}_{s\wedge\tau_{E_n}}^{n+1}$-measurable for any $s \geq 0$. Let $0 \leq s_1 < \cdots < s_k \leq s < t$ and $g \in \mathcal{B}_b(\mathbb{R}^k)$. Then, we obtain by (3.15) and the fact $M^{n+1,[u_{n+1}f_{n+1}]} \in \dot{\mathcal{M}}^{V_{n+1}}$ that for $\mathcal{E}$-*q.e.* $x \in V_{n+1}$,

$$\begin{aligned}
&\int_\Omega M_t^n g(X_{s_1},\ldots,X_{s_k})dP_x \\
&= \int_{\tau_{E_n}\leq s} M_t^n g(X_{s_1},\ldots,X_{s_k})dP_x + \int_{\tau_{E_n}>s} M_t^n g(X_{s_1},\ldots,X_{s_k})dP_x \\
&= \int_{\tau_{E_n}\leq s} M_s^n g(X_{s_1},\ldots,X_{s_k})dP_x \\
&\quad + \int_\Omega M_{t\wedge\tau_{E_n}}^{n+1,[u_{n+1}f_{n+1}]} g(X_{s_1\wedge\tau_{E_n}}^{V_{n+1}},\ldots,X_{s_k\wedge\tau_{E_n}}^{V_{n+1}})I_{\tau_{E_n}>s}dP_x \\
&= \int_{\tau_{E_n}\leq s} M_s^n g(X_{s_1},\ldots,X_{s_k})dP_x \\
&\quad + \int_\Omega M_{s\wedge\tau_{E_n}}^{n+1,[u_{n+1}f_{n+1}]} g(X_{s_1\wedge\tau_{E_n}}^{V_{n+1}},\ldots,X_{s_k\wedge\tau_{E_n}}^{V_{n+1}})I_{\tau_{E_n}>s}dP_x \\
&= \int_{\tau_{E_n}\leq s} M_s^n g(X_{s_1},\ldots,X_{s_k})dP_x + \int_{\tau_{E_n}>s} M_s^n g(X_{s_1},\ldots,X_{s_k})dP_x \\
&= \int_\Omega M_s^n g(X_{s_1},\ldots,X_{s_k})dP_x.
\end{aligned}$$

Obviously, the equality holds for $x \notin V_{n+1}$. Therefore, M^n is an $\{\mathcal{F}_t\}$-martingale.

Define $N_t^{[u]} = \tilde{u}(X_t) - \tilde{u}(X_0) - M_t^{[u]}$. Then, we have $N_{t\wedge\tau_{E_n}}^{[u]} = \lim_{l\to\infty} N_{t\wedge\tau_{E_n}}^{l,[u_l f_l]}$. Moreover $N^{[u]} \in \mathcal{N}_{c,loc}$.

Finally, we prove the uniqueness of decomposition (3.10). Suppose that $M^1 \in \dot{\mathcal{M}}_{loc}$ and $N^1 \in \mathcal{N}_{c,loc}$ such that

$$\tilde{u}(X_t) - \tilde{u}(X_0) = M_t^1 + N_t^1, \quad t \geq 0, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E.$$

Then, there exists $\{E_n\} \in \Theta$ such that, for each $n \in \mathbb{N}$, $\{(M^{[u]} - M^1)I_{[\![0,\tau_{E_n}]\!]}\}$ is a square integrable martingale and a zero quadratic variation process w.r.t. P_m. This implies that $P_m(<(M^{[u]}-M^1)I_{[\![0,\tau_{E_n}]\!]}>_t=0,\forall t\in[0,\infty))=0$. Consequently by Lemma 3.5(iii), $P_x(<(M^{[u]}-M^1)I_{[\![0,\tau_{E_n}]\!]}>_t=0,\forall t\in[0,\infty))=0$ for $\mathcal{E}$-*q.e.* $x\in E$. Therefore $M_t^{[u]}=M_t^1$, $0\le t\le\tau_{E_n}$, P_x-*a.s.* for $\mathcal{E}$-*q.e.* $x\in E$. Since n is arbitrary, we obtain the uniqueness of decomposition (3.10) up to the equivalence of local AFs. □

In the rest of this section, we investigate some concrete examples.

Example 3.1. *Consider the following bilinear form*

$$\mathcal{E}(u,v)=\int_0^1 u'v'dx+\int_0^1 bu'vdx,\quad u,v\in D(\mathcal{E}):=H_0^{1,2}(0,1).$$

(i) Suppose that $b(x)=x^2$. Then one can show that $(\mathcal{E},D(\mathcal{E}))$ is a regular local semi-Dirichlet form (but not a Dirichlet form) on $L^2((0,1);dx)$ (cf. (MOR1995, Remark 2.2(ii))). Note that any $u\in D(\mathcal{E})$ is bounded and $\frac{1}{2}-$Hölder continuous by the Sobolev embedding theorem. Then we obtain Fukushima's decomposition, $u(X_t)-u(X_0)=M_t^{[u]}+N_t^{[u]}$, by Lemma 3.9, where X is the diffusion process associated with $(\mathcal{E},D(\mathcal{E}))$, $M^{[u]}$ is an MAF of finite energy and $N^{[u]}$ is a CAF of zero energy.
(ii) Suppose that $b(x)=\sqrt{x}$. By (MOR1995, Remark 2.2(ii)), $(\mathcal{E},D(\mathcal{E}))$ is a regular local semi-Dirichlet form but not a Dirichlet form. Let $u\in D(\mathcal{E})_{loc}$. Then we obtain Fukushima's decomposition (3.10) by Theorem 3.4.

If $u\in D(\mathcal{E})$ satisfying $\mathrm{supp}[u]\subset(0,1)$*, then we may choose an open subset V of $(0,1)$ such that* $\mathrm{supp}[u]\subset V\subset(0,1)$*. Let X^V be the part process of X w.r.t. V. Then we obtain Fukushima's decomposition, $u(X_t^V)-u(X_0^V)=M_t^{V,[u]}+N_t^{V,[u]}$, by Lemma 3.9, where $M^{V,[u]}$ is an MAF of finite energy and $N^{V,[u]}$ is a CAF of zero energy w.r.t. X^V.*

Example 3.2. *Let $d\ge 3$, U be an open subset of $\mathbb{R}^d$, $\sigma,\rho\in L^1_{loc}(U;dx)$, $\sigma,\rho>0$ dx-a.e. For $u,v\in C_0^\infty(U)$, we define*

$$\mathcal{E}_\rho(u,v)=\sum_{i,j=1}^d\int_U\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\rho dx.$$

Assume that

$$(\mathcal{E}_\rho,C_0^\infty(U))\ \textit{is closable on}\ L^2(U;\sigma dx).$$

Let $a_{ij}, b_i, d_i \in L^1_{loc}(U;dx)$, $1 \le i,j \le d$. For $u, v \in C_0^\infty(U)$, we define

$$\begin{aligned}\mathcal{E}(u,v) &= \sum_{i,j=1}^d \int_U \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} a_{ij}dx + \sum_{i=1}^d \int_U \frac{\partial u}{\partial x_i} v b_i dx \\ &\quad + \sum_{i=1}^d \int_U u\frac{\partial v}{\partial x_i} d_i dx + \int_U uvcdx.\end{aligned}$$

Set $\tilde{a}_{ij} := \frac{1}{2}(a_{ij}+a_{ji})$, $\check{a}_{ij} := \frac{1}{2}(a_{ij}-a_{ji})$, $\underline{b} := (b_1,\ldots,b_d)$, and $\underline{d} := (d_1,\ldots,d_d)$. Define F to be the set of all functions $g \in L^1_{loc}(U;dx)$ such that the distributional derivatives $\frac{\partial g}{\partial x_i}$, $1 \le i \le d$, are in $L^1_{loc}(U;dx)$ such that $\|\nabla g\|(g\sigma)^{-\frac{1}{2}} \in L^\infty(U;dx)$ or $\|\nabla g\|^p(g^{p+1}\sigma^{p/q})^{-\frac{1}{2}} \in L^d(U;dx)$ for some $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, $p<\infty$, where $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^d$. We say that a $\mathcal{B}(U)-$measurable function f has property $(A_{\rho,\sigma})$ if one of the following conditions holds:

(i) $f(\rho\sigma)^{-\frac{1}{2}} \in L^\infty(U;dx)$.

(ii) $f^p(\rho^{p+1}\sigma^{p/q})^{-\frac{1}{2}} \in L^d(U,dx)$ for some $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, $p<\infty$, and $\rho \in F$.

Suppose that

(C.I) There exists $\eta > 0$ such that $\sum_{i,j=1}^d \tilde{a}_{ij}\xi_i\xi_j \ge \eta|\underline{\xi}|^2$, $\forall \underline{\xi} = (\xi_1,\ldots,\xi_d) \in \mathbb{R}^d$.

(C.II) $\check{a}_{ij}\rho^{-1} \in L^\infty(U;dx)$ for $1 \le i,j \le d$.

(C.III) For all $K \subset U$, K compact, $1_K\|\underline{b}+\underline{d}\|$ and $1_K c^{1/2}$ have property $(A_{\rho,\sigma})$, and $(c+\alpha_0\sigma)dx - \sum_{i=1}^d \frac{\partial d_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ for some $\alpha_0 \in (0,\infty)$.

(C.IV) $||\underline{b}-\underline{d}||$ has property $(A_{\rho,\sigma})$.

(C.V) $\underline{b} = \underline{\beta}+\underline{\gamma}$ such that $\|\underline{\beta}\|, \|\underline{\gamma}\| \in L^1_{loc}(U,dx)$, $(\alpha_0\sigma+c)dx - \sum_1^d \frac{\partial\gamma_i}{\partial x_i}$ is a positive measure on $\mathcal{B}(U)$ and $\|\underline{\beta}\|$ has property $(A_{\rho,\sigma})$.

Then, by (RS1995, Theorem 1.2), there exists $\alpha > 0$ such that $(\mathcal{E}_\alpha, C_0^\infty(U))$ is closable on $L^2(U;dx)$ and its closure $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$ is a regular local semi-Dirichlet form on $L^2(U;dx)$. Define $\eta_\alpha(u,u) := \mathcal{E}_\alpha(u,u) - \int\langle\nabla u, \underline{\beta}\rangle u dx$ for $u \in D(\mathcal{E}_\alpha)$. By (RS1995, Theorem 1.2 (ii) and (1.28)), we know $(\eta_\alpha, D(\mathcal{E})_\alpha)$ is a Dirichlet form and there exists $C > 1$ such that for any $u \in D(\mathcal{E}_\alpha)$,

$$\frac{1}{C}\eta_\alpha(u,u) \le \mathcal{E}_\alpha(u,u) \le C\eta_\alpha(u,u).$$

Let X be the diffusion process associated with $(\mathcal{E}_\alpha, D(\mathcal{E}_\alpha))$. Then, by Theorem 3.4, *Fukushima's decomposition holds for any* $u \in D(\mathcal{E})_{loc}$. *In particular, if* $\rho = \sigma = 1$ *then* $(\mathcal{E}, D(\mathcal{E}))$ *is the same as that given by (3.3).*

Example 3.3. *Let* S *be a Polish space. Denote by* $\mathcal{B}(S)$ *the Borel* σ*-algebra of* S. *Let* $E := \mathcal{M}_1(S)$ *be the space of probability measures on* $(S, \mathcal{B}(S))$. *For bounded* $\mathcal{B}(S)$*-measurable functions* f, g *on* S *and* $\mu \in E$, *we define*

$$\mu(f) := \int_S f d\mu, \quad \langle f, g\rangle_\mu := \mu(fg) - \mu(f)\cdot\mu(g), \quad \|f\|_\mu := \langle f, f\rangle_\mu^{1/2}.$$

Denote by $\mathcal{F}C_b^\infty$ *the family of all functions on* E *with the following expression:*

$$u(\mu) = \varphi(\mu(f_1), \dots, \mu(f_k)), \quad f_i \in C_b(S), 1 \le i \le k, \varphi \in C_0^\infty(\mathbb{R}^k), k \in \mathbb{N}.$$

Let m *be a finite positive measure on* $(E, \mathcal{B}(E))$, *where* $\mathcal{B}(E)$ *denotes the Borel* σ*-algebra of* E. *We suppose that* $\mathrm{supp}[m] = E$. *Let* $b : S \times E \to \mathbb{R}$ *be a measurable function such that*

$$\sup_{\mu \in E} \|b(\mu)\|_\mu < \infty,$$

where $b(\mu)(x) := b(x, \mu)$.

For $u, v \in \mathcal{F}C_b^\infty$, *we define*

$$\mathcal{E}^b(u, v) := \int_E (\langle \nabla u(\mu), \nabla v(\mu)\rangle_\mu + \langle b(\mu), \nabla u(\mu)\rangle_\mu v(\mu)) m(d\mu),$$

where

$$\nabla u(\mu) := (\nabla_x u(\mu))_{x\in S} := \left(\frac{d}{ds} u(\mu + s\varepsilon_x)\Big|_{s=0}\right)_{x \in S}.$$

We suppose that $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ *is closable on* $L^2(E; m)$. *Then, by (ORS1995, Theorem 3.5), there exists* $\alpha > 0$ *such that* $(\mathcal{E}_\alpha^b, \mathcal{F}C_b^\infty)$ *is closable on* $L^2(E; m)$ *and its closure* $(\mathcal{E}_\alpha^b, D(\mathcal{E}_\alpha^b))$ *is a quasi-regular local semi-Dirichlet form on* $L^2(E; m)$. *Moreover, by (ORS1995, Lemma 2.5), there exists* $C > 1$ *such that for any* $u \in D(\mathcal{E}_\alpha^b)$,

$$\frac{1}{C}\mathcal{E}_\alpha^0(u, u) \le \mathcal{E}_\alpha^b(u, u) \le C\mathcal{E}_\alpha^0(u, u).$$

Let X *be the diffusion process associated with* $(\mathcal{E}_\alpha^b, D(\mathcal{E}_\alpha^b))$, *which is a Fleming-Viot type process with interactive selection. Then, by Theorem 3.4, Fukushima's decomposition holds for any* $u \in D(\mathcal{E}^b)_{loc}$.

3.3 Transformation formula for local MAFs

In this section, we adopt the setting of § 3.2. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E; m)$ satisfying Assumption 3.1. We fix a $\{V_n\} \in \Theta$ satisfying Assumption 3.1 and $\widetilde{\hat{h}}$ is bounded on each V_n. Let X^{V_n}, $(\mathcal{E}^{V_n}, D(\mathcal{E})_{V_n})$, $\bar{h}_n$, etc. be the same as in § 3.2. For $u \in D(\mathcal{E})_{V_n,b}$, we denote by $\mu^{(n)}_{<u>}$ the Revuz measure of $< M^{n,[u]} >$ (cf. Lemma 3.9 and Theorem 3.3). For $u, v \in D(\mathcal{E})_{V_n,b}$, we define

$$\mu^{(n)}_{<u,v>} := \frac{1}{2}(\mu^{(n)}_{<u+v>} - \mu^{(n)}_{<u>} - \mu^{(n)}_{<v>}). \tag{3.18}$$

Lemma 3.11. *Let $u, v, f \in D(\mathcal{E})_{V_n,b}$. Then*

$$\int_{V_n} \tilde{f} d\mu^{(n)}_{<u,v>} = \mathcal{E}(u, vf) + \mathcal{E}(v, uf) - \mathcal{E}(uv, f). \tag{3.19}$$

Proof. By the polarization identity, (3.19) holds for $u, v, f \in D(\mathcal{E})_{V_n,b}$ is equivalent to

$$\int_{V_n} \tilde{f} d\mu^{(n)}_{<u>} = 2\mathcal{E}(u, uf) - \mathcal{E}(u^2, f), \quad \forall u, f \in D(\mathcal{E})_{V_n,b}. \tag{3.20}$$

Below, we will prove (3.20). Without loss of generality, we assume that $f \geq 0$.

For $k, l \in \mathbb{N}$, we define $f_k := f \wedge (k\bar{h}_n)$ and $f_{k,l} := l\hat{G}^{V_n}_{l+1} f_k$. By (MR1995, (3.9)), $f_k \in D(\mathcal{E})_{V_n,b}$ and

$$\mathcal{E}_1(f_k, f_k) \leq \mathcal{E}_1(f, f_k). \tag{3.21}$$

By (MR1992, Proposition III.1.2), $f_{k,l}$ is $(l+1)$-co-excessive. Since $\bar{h}_n$ is 1-co-excessive,

$$0 \leq f_{k,l} \leq k\bar{h}_n. \tag{3.22}$$

Hence $f_{k,l} \in D(\mathcal{E})_{V_n,b}$ by noting that $\bar{h}_n$ is bounded.

Note that by (3.22)

$$\lim_{t\downarrow 0} \frac{1}{t} E_{f_{k,l}\cdot m}[(N^{n,[u]}_t)^2] \leq k \lim_{t\downarrow 0} \frac{1}{t} E_{\bar{h}_n\cdot m}[(N^{n,[u]}_t)^2] = 2ke^{V_n}(N^{n,[u]}) = 0. \tag{3.23}$$

Then, by Theorem 3.3(i) and (3.23), we get

$$\begin{aligned}
\int_{V_n} \widetilde{f_{k,l}} d\mu^{(n)}_{<u>} &= \lim_{t\downarrow 0} \frac{1}{t} E_{f_{k,l}\cdot m}[< M^{n,[u]} >_t] \\
&= \lim_{t\downarrow 0} \frac{1}{t} E_{f_{k,l}\cdot m}[(\tilde{u}(X^{V_n}_t) - \tilde{u}(X^{V_n}_0))^2] \\
&= \lim_{t\downarrow 0} \frac{2}{t}(uf_{k,l}, u - P^{V_n}_t u) - \lim_{t\downarrow 0} \frac{1}{t}(f_{k,l}, u^2 - P^{V_n}_t u^2) \\
&= 2\mathcal{E}(u, uf_{k,l}) - \mathcal{E}(u^2, f_{k,l}).
\end{aligned} \tag{3.24}$$

By (MR1992, Theorem I.2.13), for each $k \in \mathbb{N}$, $f_{k,l} \to f_k$ in $D(\mathcal{E})_{V_n}$ as $l \to \infty$. Furthermore, by Assumption 3.1, (MR1992, Corollary I.4.15) and (3.22), we can show that $\sup_{l\geq 1} \mathcal{E}(uf_{k,l}, uf_{k,l}) < \infty$. Thus, we obtain by (MR1992, Lemma I.2.12) that $uf_{k,l} \to uf_k$ weakly in $D(\mathcal{E})_{V_n}$ as $l \to \infty$. Note that $\int_{V_n} \widetilde{\bar{h}_n} d\mu^{(n)}_{<u>} = 2e^{V_n}(M^{n,[u]}) < \infty$ for any $u \in D(\mathcal{E})_{V_n,b}$. Therefore, we obtain by (3.24), (3.22) and the dominated convergence theorem that

$$\int_{V_n} \widetilde{f}_k d\mu^{(n)}_{<u>} = 2\mathcal{E}(u, uf_k) - \mathcal{E}(u^2, f_k), \quad \forall u \in D(\mathcal{E})_{V_n,b}. \tag{3.25}$$

By (3.21) and the weak sector condition, we get $\sup_{k\geq 1} \mathcal{E}_1(f_k, f_k) < \infty$. Furthermore, by Assumption 3.1 and (MR1992, Corollary I.4.15), we can show that $\sup_{k\geq 1} \mathcal{E}(uf_k, uf_k) < \infty$. Thus, we obtain by (MR1992, Lemma I.2.12) that $f_k \to f$ and $uf_k \to uf$ weakly in $D(\mathcal{E})_{V_n}$ as $k \to \infty$. Therefore (3.20) holds by (3.25) and the monotone convergence theorem. □

For $u \in D(\mathcal{E})_{V_n,b}$, we denote by $M^{n,[u],c}$ and $M^{n,[u],k}$ the continuous and killing parts of $M^{n,[u]}$, respectively; denote by $\mu^{n,c}_{<u>}$ and $\mu^{n,k}_{<u>}$ the Revuz measures of $< M^{n,[u],c} >$ and $< M^{n,[u],k} >$, respectively. Then $M^{n,[u]} = M^{n,[u],c} + M^{n,[u],k}$ with

$$M^{n,[u],k} = -\tilde{u}(X^{V_n}_{\zeta^{(n)}-})I_{\{\zeta^{(n)}\leq t\}} - (-\tilde{u}(X^{V_n}_{\zeta^{(n)}-})I_{\{\zeta^{(n)}\leq t\}})^p,$$

where $\zeta^{(n)}$ denotes the life time of X^{V_n} and p denotes the dual predictable projection, and

$$\mu^{(n)}_{<u>} = \mu^{n,c}_{<u>} + \mu^{n,k}_{<u>}. \tag{3.26}$$

Let $(N^{(n)}(x, dy), H^{(n)})$ be a Lévy system of X^{V_n} and $\nu^{(n)}$ be the Revuz measure of $H^{(n)}$. Define $K^{(n)}(dx) := N^{(n)}(x, \Delta)\nu^{(n)}(dx)$. Similar to (FOT1994, (5.3.8) and (5.3.10)), we can show that

$$\begin{aligned} < M^{n,[u],k} >_t &= (\tilde{u}^2(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p \\ &= \int_0^t \tilde{u}^2(X^{V_n}_s)N^{(n)}(X^{V_n}_s, \Delta)dH^{(n)}_s \end{aligned} \tag{3.27}$$

and

$$\mu^{n,k}_{<u>}(dx) = \tilde{u}^2(x)K^{(n)}(dx). \tag{3.28}$$

For $u, v \in D(\mathcal{E})_{V_n,b}$, we define

$$\mu^{n,c}_{<u,v>} := \frac{1}{2}(\mu^{n,c}_{<u+v>} - \mu^{n,c}_{<u>} - \mu^{n,c}_{<v>}), \quad \mu^{n,k}_{<u,v>} := \frac{1}{2}(\mu^{n,k}_{<u+v>} - \mu^{n,k}_{<u>} - \mu^{n,k}_{<v>}). \quad (3.29)$$

Theorem 3.5. *Let $u, v, w \in D(\mathcal{E})_{V_n,b}$. Then*

$$d\mu^{n,c}_{<uv,w>} = \tilde{u} d\mu^{n,c}_{<v,w>} + \tilde{v} d\mu^{n,c}_{<u,w>}. \quad (3.30)$$

Proof. By quasi-homeomorphism and the polarization identity, (3.30) holds for $u, v, w \in D(\mathcal{E})_{V_n,b}$ is equivalent to

$$\int_{V_n} \tilde{f} d\mu^{n,c}_{<u^2,w>} = 2\int_{V_n} \tilde{f}\tilde{u} d\mu^{n,c}_{<u,w>}, \quad \forall f, u, w \in D(\mathcal{E})_{V_n,b}. \quad (3.31)$$

By (3.18) and (3.26)-(3.29), we find that (3.31) is equivalent to

$$\int_{V_n} \tilde{f} d\mu^{(n)}_{<u^2,w>} + \int_{V_n} \tilde{f}\tilde{u}^2\tilde{w} dK^{(n)} = 2\int_{V_n} \tilde{f}\tilde{u} d\mu^{(n)}_{<u,w>}, \quad \forall f, u, w \in D(\mathcal{E})_{V_n,b}. \quad (3.32)$$

For $k \in \mathbb{N}$, we define $v_k := kR^{V_n}_{k+1}u$. Then $v_k \to u$ in $D(\mathcal{E})_{V_n}$ as $k \to \infty$. By Assumption 3.1 and (MR1992, Corollary I.4.15), we can show that $\sup_{k\geq 1}\mathcal{E}(v_k w, v_k w) < \infty$. Then, by (MR1992, Lemma I.2.12), there exists a subsequence $\{(v_{k_l})\}_{l\in\mathbb{N}}$ of $\{v_k\}_{k\in\mathbb{N}}$ such that $u_k w \to uw$ in $D(\mathcal{E})_{V_n}$ as $k \to \infty$, where $u_k := \frac{1}{k}\sum_{l=1}^{k} v_{k_l}$. Note that $u_k \to u$ in $D(\mathcal{E})_{V_n}$ as $k \to \infty$ and $\|u_k\|_\infty \leq \|u\|_\infty$ for $k \in \mathbb{N}$. Moreover, $\|L^{V_n}u_k\|_\infty < \infty$ for $k \in \mathbb{N}$, where L^{V_n} is the generator of X^{V_n}.

By Assumption 3.1 and (MR1992, Corollary I.4.15), we can show that

$$\sup_{k\geq 1}[\mathcal{E}(u_k f w, u_k f w) + \mathcal{E}(u_k^2 f, u_k^2 f) + \mathcal{E}(u_k f, u_k f)] < \infty.$$

Then, we obtain by (MR1992, Lemma I.2.12) that $u_k f w \to u f w$, $u_k^2 f \to u^2 f$ and $u_k f \to u f$ weakly in $D(\mathcal{E})_{V_n}$ as $k \to \infty$. Hence by (3.19) and the fact

$$\sup_{k\geq 1}[\mathcal{E}(u_k f w, u_k f w) + \mathcal{E}(u_k f, u_k f)] < \infty$$

we get

$$\begin{aligned}
\int_{V_n} \tilde{f}\tilde{u} d\mu^{(n)}_{<u,w>} &= \mathcal{E}(u, ufw) + \mathcal{E}(w, u^2 f) - \mathcal{E}(uw, uf) \\
&= \lim_{k\to\infty}[\mathcal{E}(u, u_k f w) + \mathcal{E}(w, u_k^2 f) - \mathcal{E}(uw, u_k f)] \\
&= \lim_{k\to\infty}[\mathcal{E}(u_k, u_k f w) + \mathcal{E}(w, u_k^2 f) - \mathcal{E}(u_k w, u_k f)] \\
&= \lim_{k\to\infty}\int_{V_n} \tilde{f}\widetilde{u_k} d\mu^{(n)}_{<u_k,w>}. \quad (3.33)
\end{aligned}$$

By Assumption 3.1 and (MR1992, Corollary I.4.15), we can show that $\sup_{k\geq 1}[\mathcal{E}(u_k^2,u_k^2)+\mathcal{E}(u_k^2f,u_k^2f)+\mathcal{E}(u_k^2w,u_k^2w)]<\infty$. Then, we obtain by (MR1992, Lemma I.2.12) that $u_k^2\to u^2$, $u_k^2f\to u^2f$ and $u_k^2w\to u^2w$ weakly in $D(\mathcal{E})_{V_n}$ as $k\to\infty$. Hence by (3.19) we get

$$\begin{aligned}\int_{V_n}\tilde{f}d\mu^{(n)}_{<u^2,w>} &= \mathcal{E}(u^2,fw)+\mathcal{E}(w,u^2f)-\mathcal{E}(u^2w,f)\\ &= \lim_{k\to\infty}[\mathcal{E}(u_k^2,fw)+\mathcal{E}(w,u_k^2f)-\mathcal{E}(u_k^2w,f)]\\ &= \lim_{k\to\infty}\int_{V_n}\tilde{f}d\mu^{(n)}_{<u_k^2,w>}. \end{aligned}\tag{3.34}$$

By (3.33), (3.34) and the dominated convergence theorem, to prove (3.32), we may assume without loss of generality that u is equal to some u_k. Moreover, we assume without loss of generality that $f\geq 0$.

For $k,l\in\mathbb{N}$, we define $f_k:=f\wedge(k\bar{h}_n)$ and $f_{k,l}:=l\hat{G}^{V_n}_{l+1}f_k$. By (MR1995, (3.9)), $f_k\in D(\mathcal{E})_{V_n,b}$; by (MR1992, Proposition III.1.2), $f_{k,l}$ is $(l+1)$-co-excessive. Since $\bar{h}_n$ is 1-co-excessive,

$$0\leq f_{k,l}\leq k\bar{h}_n.$$

Hence $f_{k,l}\in D(\mathcal{E})_{V_n,b}$ by noting that $\bar{h}_n$ is bounded. By the dominated convergence theorem, to prove that (3.32) holds for any $f\in D(\mathcal{E})_{V_n,b}$, it suffices to prove that (3.32) holds for any $f_{k,l}$.

Below, we will prove (3.32) for $u=u_k$ and $f=f_{k,l}$.

Note that for any $g\in D(\mathcal{E})_{V_n,b}$,

$$\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[(N_t^{n,[g]})^2]\leq k\lim_{t\downarrow 0}\frac{1}{t}E_{\bar{h}_n\cdot m}[(N_t^{n,[g]})^2]=2ke^{V_n}(N^{n,[g]})=0.\tag{3.35}$$

By Theorem 3.3(i) and (3.35), we get

$$\begin{aligned}\int_{V_n}\widetilde{f_{k,l}}d\mu^{(n)}_{<u_k^2,w>} &= \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[<M^{n,[u_k^2]},M^{n,[w]}>_t]\\ &= \lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[(\tilde{u_k}^2(X_t^{V_n})-\tilde{u_k}^2(X_0^{V_n}))(\tilde{w}(X_t^{V_n})-\tilde{w}(X_0^{V_n}))]\\ &= \lim_{t\downarrow 0}\frac{2}{t}E_{(f_{k,l}u_k)\cdot m}[(\tilde{u_k}(X_t^{V_n})-\tilde{u_k}(X_0^{V_n}))(\tilde{w}(X_t^{V_n})-\tilde{w}(X_0^{V_n}))]\\ &\quad+\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[(\tilde{u_k}(X_t^{V_n})-\tilde{u_k}(X_0^{V_n}))^2(\tilde{w}(X_t^{V_n})-\tilde{w}(X_0^{V_n}))]\\ &:= \lim_{t\downarrow 0}[I(t)+II(t)]. \end{aligned}\tag{3.36}$$

By (3.35), Theorem 3.3(iii) and (3.19), we get

$$\begin{aligned}
\lim_{t\downarrow 0} I(t) &= \lim_{t\downarrow 0}\frac{2}{t}E_{(f_{k,l}u_k)\cdot m}(< M^{n,[u_k]}, M^{n,[w]} >_t) \\
&= \lim_{t\downarrow 0}\frac{2}{t}\int_0^t < \mu^{(n)}_{<u_k,w>}, \widetilde{\hat{T}^{V_n}_s(f_{k,l}u_k)} > ds \\
&= \lim_{t\downarrow 0}\frac{2}{t}\int_0^t [\mathcal{E}(u_k, w\hat{T}^{V_n}_s(f_{k,l}u_k)) + \mathcal{E}(w, u_k\hat{T}^{V_n}_s(f_{k,l}u_k)) \\
&\qquad -\mathcal{E}(u_k w, \hat{T}^{V_n}_s(f_{k,l}u_k))]ds. \qquad (3.37)
\end{aligned}$$

By (AFRS1995, Theorem 3.4), $\hat{T}^{V_n}_s(f_{k,l}u_k) \to f_{k,l}u_k$ in $D(\mathcal{E})_{V_n}$ as $s \to 0$. Furthermore, by Assumption 3.1, (MR1992, Corollary I.4.15) and the fact that $|e^{-s}\hat{T}^{V_n}_s(f_{k,l}u_k)| \leq k\|u_k\|_\infty \bar{h}_n$, $s > 0$, we can show that $\sup_{s>0}\mathcal{E}(w\hat{T}^{V_n}_s(f_{k,l}u_k), w\hat{T}^{V_n}_s(f_{k,l}u_k)) < \infty$. Thus, we obtain by (MR1992, Lemma I.2.12) that $w\hat{T}^{V_n}_s(f_{k,l}u_k) \to wf_{k,l}u_k$ weakly in $D(\mathcal{E})_{V_n}$ as $s \to 0$. Similarly, we get $u_k\hat{T}^{V_n}_s(f_{k,l}u_k) \to u_k f_{k,l}u$ weakly in $D(\mathcal{E})_{V_n}$ as $s \to 0$. Therefore, by (3.37) and (3.19), we get

$$\lim_{t\downarrow 0} I(t) = 2\int_{V_n} \widetilde{f_{k,l}}\widetilde{u_k} d\mu^{(n)}_{<u_k,w>}. \qquad (3.38)$$

Note that

$$\begin{aligned}
II(t) &= \frac{1}{t}E_{f_{k,l}\cdot m}[(M_t^{n,[u_k],c})^2 M_t^{n,[w],c}] + \frac{1}{t}E_{f_{k,l}\cdot m}[(M_t^{n,[u_k],k})^2 M_t^{n,[w],k}] \\
&:= III(t) + IV(t). \qquad (3.39)
\end{aligned}$$

By Burkholder-Davis-Gunday inequality, we get

$$\begin{aligned}
\lim_{t\downarrow 0} III(t) &\leq (\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[(M_t^{n,[u_k],c})^4])^{1/2}(\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[< M^{n,[v],c} >_t])^{1/2} \\
&\leq C(2ke^{V_n}(M^{n,[v]}))^{1/2}(\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[< M^{n,[u_k],c} >_t^2])^{1/2} \qquad (3.40)
\end{aligned}$$

for some constant $C > 0$, which is independent of t.

By Theorem 3.3(i), for any $\delta > 0$, we get

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[< M^{n,[u_k],c} >_t^2] \\
&= \lim_{t\downarrow 0}\frac{2}{t}E_{f_{k,l}\cdot m}[\int_0^t < M^{n,[u_k],c} >_{(t-s)} \circ\theta_s d < M^{n,[u_k],c} >_s] \\
&= \lim_{t\downarrow 0}\frac{2}{t}E_{f_{k,l}\cdot m}[\int_0^t E_{X_s^{V_n}}[< M^{n,[u_k],c} >_{(t-s)}]d < M^{n,[u_k],c} >_s] \\
&\leq 2 < E_\cdot[< M^{n,[u_k]} >_\delta] \cdot \mu^{(n)}_{<u_k>}, \widetilde{f_{k,l}} > . \qquad (3.41)
\end{aligned}$$

Note that by our choice of u_k, there exists a constant $C_k > 0$ such that $E_x(< M^{n,[u_k]} >_\delta) = E_x[(M_\delta^{n,[u_k]})^2] = E_x[(\widetilde{u_k}(X_\delta^{V_n}) - \widetilde{u_k}(X_0^{V_n}) - \int_0^\delta L^{V_n} u_k(X_s^{V_n})ds)^2] \leq C_k$ for any $\delta \leq 1$ and $\mathcal{E}$-$q.e.$ $x \in V_n$. Letting $\delta \to 0$, by (3.41), the dominated convergence theorem and (3.40), we get

$$\lim_{t\downarrow 0} III(t) = 0. \tag{3.42}$$

By (P2005, Theorem II.33, integration by parts (page 68) and Theorem II.28), we get

$$\begin{aligned}
IV(t) &= \frac{1}{t} E_{f_{k,l}\cdot m}[I_{\zeta^{(n)}\leq t}\{-(\widetilde{u_k}^2\tilde{w})(X^{V_n}_{\zeta^{(n)}-}) \\
&\qquad +2(\widetilde{u_k}\tilde{w})(X^{V_n}_{\zeta^{(n)}-})(\widetilde{u_k}(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p \\
&\qquad +(\widetilde{u_k}^2)(X^{V_n}_{\zeta^{(n)}-})(\tilde{w}(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p\}] \\
&= \frac{1}{t} E_{f_{k,l}\cdot m}[-((\widetilde{u_k}^2\tilde{w})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p \\
&\qquad +2((\widetilde{u_k}\tilde{w})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p M_t^{n,[u_k],k} \\
&\qquad +(\widetilde{u_k}^2(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p M_t^{n,[w],k}]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{t} E_{f_{k,l}\cdot m}[-((\widetilde{u_k}^2\tilde{w})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p] \\
&\quad +\frac{2}{t} E^{1/2}_{f_{k,l}\cdot m}[\{((\widetilde{u_k}\tilde{w})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p\}^2] E^{1/2}_{f_{k,l}\cdot m}[< M^{n,[u_k],k} >_t] \\
&\quad +E^{1/2}_{f_{k,l}\cdot m}[\{(\widetilde{u_k}^2(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p\}^2] E^{1/2}_{f_{k,l}\cdot m}[< M^{n,[w],k} >_t].
\end{aligned} \tag{3.43}$$

By Theorem 3.3(i), (3.27)-(3.29), we obtain that for $\psi_1, \psi_2 \in D(\mathcal{E})_{V_n,b}$,

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{1}{t} E_{f_{k,l}\cdot m}[((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\leq t})^p] &= \int_{V_n} \widetilde{f_{k,l}} d\mu^{n,k}_{<\psi_1,\psi_2>} \\
&= \int_{V_n} \widetilde{f_{k,l}}\widetilde{\psi_1}\widetilde{\psi_2} dK^{(n)}
\end{aligned} \tag{3.44}$$

and

$$\lim_{t\downarrow 0} \frac{1}{t} E_{f_{k,l}\cdot m}[< M^{n,[\psi_1],k} >_t] = \int_{V_n} \widetilde{f_{k,l}} d\mu^{n,k}_{<\psi_1>}. \tag{3.45}$$

Furthermore, for any $\delta > 0$,

$$
\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[\{((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le t})^p\}^2]\\
&=\lim_{t\downarrow 0}\frac{2}{t}E_{f_{k,l}\cdot m}[\int_0^t((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le (t-s)})^p\circ\theta_s d((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le s})^p]\\
&=\lim_{t\downarrow 0}\frac{2}{t}E_{f_{k,l}\cdot m}[\int_0^t E_{X^{V_n}_s}[((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le (t-s)})^p]d((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le s})^p]\\
&\le < E_\cdot[(\widetilde{|\psi_1\psi_2|}(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le\delta})^p]\cdot\mu^{n,k}_{<|\psi_1|,|\psi_2|>},\widetilde{f_{k,l}}>\\
&=< E_\cdot[\widetilde{|\psi_1\psi_2|}(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le\delta}]\cdot\mu^{n,k}_{<|\psi_1|,|\psi_2|>},\widetilde{f_{k,l}}>.
\end{aligned}\tag{3.46}
$$

Letting $\delta\to 0$, by (3.46) and the dominated convergence theorem, we get

$$\lim_{t\downarrow 0}\frac{1}{t}E_{f_{k,l}\cdot m}[\{((\widetilde{\psi_1}\widetilde{\psi_2})(X^{V_n}_{\zeta^{(n)}-})I_{\zeta^{(n)}\le t})^p\}^2]=0.\tag{3.47}$$

By (3.43)-(3.45) and (3.47), we get

$$\lim_{t\downarrow 0}IV(t)=-\int_{V_n}\widetilde{f_{k,l}}\widetilde{u_k}^2\tilde{w}dK^{(n)}.\tag{3.48}$$

Therefore, the proof is completed by (3.36), (3.38), (3.39), (3.42) and (3.48). □

Remark 3.2. *When deriving formula (3.30) for non-symmetric Markov processes, we cannot apply Theorem 3.3(vi) or (vii) to smooth measures which are not of finite energy integral. To overcome that difficulty and obtain (3.30) in the semi-Dirichlet forms setting, we have to make some extra efforts as shown in the above proof. The proof uses some ideas of (K1987, Theorem 5.4) and (O1988, Theorem 5.3.2).*

Theorem 3.6. *Let $m\in\mathbb{N}$, $\Phi\in C^1(\mathbb{R}^m)$ with $\Phi(0)=0$, and $u=(u_1,u_2,\dots,u_m)$ with $u_i\in D(\mathcal{E})_{V_n,b}$, $1\le i\le m$. Then $\Phi(u)\in D(\mathcal{E})_{V_n,b}$ and for any $v\in D(\mathcal{E})_{V_n,b}$,*

$$d\mu^{n,c}_{<\Phi(u),v>}=\sum_{i=1}^m\Phi_{x_i}(\tilde{u})d\mu^{n,c}_{<u_i,v>}.\tag{3.49}$$

Proof. $\Phi(u)\in D(\mathcal{E})_{V_n,b}$ is a direct consequence of Assumption 3.1 and the corresponding property of Dirichlet form. Below we only prove (3.49). Let $v\in D(\mathcal{E})_{V_n,b}$. Then (3.49) is equivalent to

$$\int_{V_n}\tilde{f}\bar{h}_n d\mu^{n,c}_{<\Phi(u),v>}=\sum_{i=1}^m\int_{V_n}\tilde{f}\bar{h}_n\Phi_{x_i}(\tilde{u})d\mu^{n,c}_{<u_i,v>},\quad\forall f\in D(\mathcal{E})_{V_n,b}.\tag{3.50}$$

Let $\mathcal{A}$ be the family of all $\Phi \in C^1(\mathbb{R}^m)$ satisfying (3.49). If $\Phi, \Psi \in \mathcal{A}$, then $\Phi\Psi \in \mathcal{A}$ by Theorem 3.5. Hence $\mathcal{A}$ contains all polynomials vanishing at the origin. Let O be a finite cube containing the range of $u(x) = (u_1(x), \ldots, u_m(x))$. We take a sequence $\{\Phi^k\}$ of polynomials vanishing at the origin such that $\Phi^k \to \Phi$, $\Phi^k_{x_i} \to \Phi_{x_i}$, $1 \leq i \leq m$, uniformly on O. By Assumption 3.1 and (FOT1994, (3.2.27)), $\Phi^k(u)$ converges to $\Phi(u)$ w.r.t. $\mathcal{E}_1^{V_n}$ as $k \to \infty$. Then, by (3.14), we get

$$
\begin{aligned}
&|\int_{V_n} \tilde{f}\bar{h}_n d\mu^{n,c}_{<\Phi(u),v>} - \int_{V_n} \tilde{f}\bar{h}_n d\mu^{n,c}_{<\Phi^k(u),v>}|\\
&\quad \leq \|f\|_\infty |\int_{V_n} \bar{h}_n d\mu^{n,c}_{<\Phi(u)-\Phi^k(u)>}|^{1/2} |\int_{V_n} \bar{h}_n d\mu^{n,c}_{<v>}|^{1/2}\\
&\quad \leq \|f\|_\infty |\int_{V_n} \bar{h}_n d\mu^{(n)}_{<\Phi(u)-\Phi^k(u)>}|^{1/2} |\int_{V_n} \bar{h}_n d\mu^{(n)}_{<v>}|^{1/2}\\
&\quad = 2\|f\|_\infty e^{V_n}(M^{n,[\Phi(u)-\Phi^k(u)]})^{1/2} e^{V_n}(M^{n,[v]})^{1/2}\\
&\quad \leq 2\|f\|_\infty e^{V_n}(M^{n,[v]})^{1/2} [KC_n \mathcal{E}_1^{V_n}(\Phi(u)-\Phi^k(u), \Phi(u)-\Phi^k(u))^{1/2}\\
&\qquad \cdot(\|\Phi(u)-\Phi^k(u)\|_\infty \mathcal{E}_1^{V_n}(\bar{h}_n, \bar{h}_n)^{1/2}\\
&\qquad\quad + \|\bar{h}_n\|_\infty \mathcal{E}_1^{V_n}(\Phi(u)-\Phi^k(u), \Phi(u)-\Phi^k(u))^{1/2})]^{1/2}.
\end{aligned}
$$

Hence

$$\int_{V_n} \tilde{f}\bar{h}_n d\mu^{n,c}_{<\Phi(u),v>} = \lim_{k\to\infty} \int_{V_n} \tilde{f}\bar{h}_n d\mu^{n,c}_{<\Phi^k(u),v>}.$$

It is easy to see that

$$\int_{V_n} \tilde{f}\bar{h}_n \Phi_{x_i}(\tilde{u}) d\mu^{n,c}_{<u_i,v>} = \lim_{k\to\infty} \int_{V_n} \tilde{f}\bar{h}_n \Phi^k_{x_i}(\tilde{u}) d\mu^{n,c}_{<u_i,v>}, \quad 1 \leq i \leq m.$$

Therefore (3.50) holds. $\square$

For $M, L \in \dot{\mathcal{M}}^{V_n}$, there exists a unique CAF $< M, L >$ of bounded variation such that

$$E_x(M_t L_t) = E_x(< M, L >_t), \quad t \geq 0, \ \mathcal{E}\text{-}q.e.\ x \in V_n.$$

Denote by $\mu^{(n)}_{<M,L>}$ the Revuz measure of $< M, L >$. Then, similar to (FOT1994, Lemma 5.6.1), we can prove the following lemma.

Lemma 3.12. *If $f \in L^2(V_n; \mu^{(n)}_{<M>})$ and $g \in L^2(V_n; \mu^{(n)}_{<L>})$, then fg is integrable w.r.t. $|\mu^{(n)}_{<M,L>}|$ and*

$$(\int_{V_n} | fg | \, d \mid \mu^{(n)}_{<M,L>} \mid)^2 \le \int_{V_n} f^2 d\mu^{(n)}_{<M>} \int_{V_n} g^2 d\mu^{(n)}_{<L>}.$$

Lemma 3.13. *Let $M \in \dot{\mathcal{M}}^{V_n}$ and $f \in L^2(V_n; \mu^{(n)}_{<M>})$. Then there exists a unique element $f \cdot M \in \dot{\mathcal{M}}^{V_n}$ such that*

$$e^{V_n}(f \cdot M, L) = \frac{1}{2} \int_{V_n} f \bar{h}_n d\mu^{(n)}_{<M,L>}, \ \forall L \in \dot{\mathcal{M}}^{V_n}. \tag{3.51}$$

The mapping $f \to f \cdot M$ is continuous and linear from $L^2(V_n; \mu^{(n)}_{<M>})$ into the Hilbert space $(\dot{\mathcal{M}}^{V_n}; e^{V_n})$.

Proof. Let $L \in \dot{\mathcal{M}}^{V_n}$. Then, by Lemma 3.12, we get

$$\begin{aligned} \mid \frac{1}{2} \int_{V_n} f \bar{h}_n d\mu^{(n)}_{<M,L>} \mid &\le \frac{1}{\sqrt{2}} (\int_{V_n} f^2 \bar{h}_n d\mu^{(n)}_{<M>})^{1/2} (1/2 \int_{V_n} \bar{h}_n d\mu^{(n)}_{<L>})^{1/2} \\ &\le \frac{\|\bar{h}_n\|_\infty}{\sqrt{2}} \parallel f \parallel_{L^2(V_n; \mu^{(n)}_{<M>})} \sqrt{e^{V_n}(L)}. \end{aligned}$$

Therefore, the proof is completed by Lemma 3.8. $\square$

Similar to (FOT1994, Lemma 5.6.2, Corollary 5.6.1 and Lemma 5.6.3), we can prove the following two lemmas.

Lemma 3.14. *Let $M, L \in \dot{\mathcal{M}}^{V_n}$. Then*

(i) $d\mu^{(n)}_{<f \cdot M,L>} = f d\mu^{(n)}_{<M,L>}$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$.

(ii) $g \cdot (f \cdot M) = (gf) \cdot M$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$ and $g \in L^2(V_n; f^2 d\mu^{(n)}_{<M>})$.

(iii) $e^{V_n}(f \cdot M, g \cdot L) = \frac{1}{2} \int f g \bar{h}_n d\mu^{(n)}_{<M,L>}$ for $f \in L^2(V_n; \mu^{(n)}_{<M>})$ and $g \in L^2(V_n; \mu^{(n)}_{<L>})$.

Lemma 3.15. *The family $\{\tilde{f} \cdot M^u \,|\, f \in D(\mathcal{E})_{V_n,b}\}$ is dense in $(\dot{\mathcal{M}}^{V_n}, e^{V_n})$.*

Theorem 3.7. *Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$ with $\Phi(0) = 0$, and $u = (u_1, u_2, \dots, u_m)$ with $u_i \in D(\mathcal{E})_{V_n,b}$, $1 \le i \le m$. Then*

$$M^{[\Phi(u)],c} = \sum_{i=1}^{m} \Phi_{x_i}(u) \cdot M^{[u_i],c}, \quad P_x\text{-a.s. for } \mathcal{E}\text{-q.e. } x \in V_n. \tag{3.52}$$

Proof. Let $v \in D(\mathcal{E})_{V_n,b}$ and $f, g \in D(\mathcal{E})_{V_n,b}$. Then, by Lemma 3.14(iii) and Theorem 3.6, we get

$$\begin{aligned}
e^{V_n}(\tilde{f} \cdot M^{n,[\Phi(u)],c}, \tilde{g} \cdot M^{n,[v]}) &= \frac{1}{2}\int_{V_n} \tilde{f}\tilde{g}\bar{h}_n d\mu^{(n)}_{<M^{n,[\Phi(u)],c},M^{n,[v]}>} \\
&= \frac{1}{2}\int_{V_n} \tilde{f}\tilde{g}\bar{h}_n d\mu^{n,c}_{<\Phi(u),v>} \\
&= \frac{1}{2}\sum_{i=1}^{m}\int_{V_n} \tilde{f}\tilde{g}\bar{h}_n \Phi_{x_i}(u) d\mu^{n,c}_{<u_i,v>} \\
&= \frac{1}{2}\sum_{i=1}^{m}\int_{V_n} \tilde{f}\tilde{g}\bar{h}_n \Phi_{x_i}(u) d\mu^{(n)}_{<M^{n,[u_i],c},M^{n,[v]}>} \\
&= e^{V_n}(\sum_{i=1}^{m}(\tilde{f}\Phi_{x_i}(u)) \cdot M^{n,[u_i],c}, \tilde{g} \cdot M^{n,[v]}).
\end{aligned}$$

By Lemma 3.15, we get

$$\tilde{f} \cdot M^{n,[\Phi(u)],c} = \sum_{i=1}^{m}(\tilde{f}\Phi_{x_i}(u)) \cdot M^{n,[u_i],c}, \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in V_n.$$

Therefore, (3.52) is satisfied by Lemma 3.14(ii), since $f \in D(\mathcal{E})_{V_n,b}$ is arbitrary. $\square$

Let $M \in \dot{\mathcal{M}}_{loc}$. Then, there exist $\{V_n\}, \{E_n\} \in \Theta$ and $\{M^n \mid M^n \in \dot{\mathcal{M}}^{V_n}\}$ such that $E_n \subset V_n$, $M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}}$, $t \geq 0$, $n \in \mathbb{N}$. We define

$$< M >_{t\wedge\tau_{E_n}} :=< M^n >_{t\wedge\tau_{E_n}}; \quad < M >_t := \lim_{s\uparrow\zeta} < M >_s \quad \text{for } t \geq \zeta.$$

Then, we can see that $< M >$ is well-defined and $< M >$ is a PCAF. Denote by $\mu_{<M>}$ the Revuz measure of $< M >$. We define

$$\begin{aligned}
L^2_{loc}(E; \mu_{<M>}) := \{f \mid \exists\ &\{V_n\}, \{E_n\} \in \Theta \text{ and } \{M^n \mid M^n \in \dot{\mathcal{M}}^{V_n}\} \text{ such that} \\
&E_n \subset V_n, M_{t\wedge\tau_{E_n}} = M^n_{t\wedge\tau_{E_n}},\ f \cdot I_{E_n} \in L^2(E_n; \mu^{(n)}_{<M^n>}),\ t \geq 0,\ n \in \mathbb{N}\}
\end{aligned}$$

For $f \in L^2_{loc}(E; \mu_{<M>})$, we define $f \cdot M$ on $[\![0, \zeta[\![$ by

$$(f \cdot M)_{t\wedge\tau_{E_n}} := ((f \cdot I_{E_n}) \cdot M^n)_{t\wedge\tau_{E_n}}, \quad t \geq 0,\ n \in \mathbb{N}.$$

Then, we can see that $f \cdot M$ is well-defined and $f \cdot M \in \mathcal{M}^{[\![0,\zeta[\![}_{loc}$. Denote by M^c the continuous part of M.

Finally, we obtain the main result of this section.

Theorem 3.8. *Suppose that $(\mathcal{E}, D(\mathcal{E}))$ is a quasi-regular local semi-Dirichlet form on $L^2(E;m)$ satisfying Assumption 3.1. Let $m \in \mathbb{N}$, $\Phi \in C^1(\mathbb{R}^m)$, and $u = (u_1, u_2, \ldots, u_m)$ with $u_i \in D(\mathcal{E})_{loc}$, $1 \leq i \leq m$. Then $\Phi(u) \in D(\mathcal{E})_{loc}$ and*

$$M^{[\Phi(u)],c} = \sum_{i=1}^{m} \Phi_{x_i}(u) \cdot M^{[u_i],c} \text{ on } [0, \zeta), \quad P_x\text{-}a.s. \text{ for } \mathcal{E}\text{-}q.e.\ x \in E. \tag{3.53}$$

Proof. Since $1 \in D(\mathcal{E})_{loc}$, $\Phi(u) \in D(\mathcal{E})_{loc}$ by Theorem 3.6. (3.53) is a direct consequence of (3.52). □

Chapter 4

Future research

In the future, we will focus on the following three topics that are closely related to this thesis.

1. Let (X_t, P_x) be a right process associated with a quasi-regular semi-Dirichlet form. We are interested in the strong continuity of the generalized Feynman-Kac semigroup $P_t^u f(x) = E_x[e^{N_t^u} f(X_t)]$. Different from the non-symmetric Dirichlet forms case, we need to overcome two problems.

For a quasi-regular semi-Dirichlet forms $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E; m)$, a smooth measure μ is said to be in the Kato class if

$$\lim_{t \to 0} \inf_{\mathrm{Cap}_\phi(N)=0} \sup_{x \in E \setminus N} E_x[A_t^\mu] = 0,$$

where $(A_t^\mu)_{t \geq 0}$ is the PCAF associated with μ. Denote by S_K the Kato class of smooth measures.

The first problem is whether the following property holds: Let $\mu \in S_K$, then for any $\varepsilon > 0$, there exists a constant A_ε such that for any $f \in D(\mathcal{E})$,

$$\int_E \tilde{f}^2 d\mu \leq \varepsilon \mathcal{E}(f, f) + A_\varepsilon \|f\|_2^2. \tag{4.1}$$

We have solved this problem and include it in the Appendix.

Define $(\bar{Q}^{u,n}, D(\mathcal{E})_n)$ as in section 2.1. The second problem is whether this form is a coercive closed form or not. In semi-Dirichlet forms case, it is not true that $\frac{1}{2} \int_E d\mu_{<M^f>} \leq \mathcal{E}(f, f)$, so we can't follow the same method (see (2.11) and (2.12)) to get that $(\bar{Q}^{u,n}, D(\mathcal{E})_n)$ is a coercive closed form. New method should be considered. If this problem can be solved, then there is hope to generalize the results of Chapter 2 to the semi-Dirichlet forms case.

2. Fukushima's decomposition for semi-Dirichlet forms with jumping parts. In Chapter 3, we use the localization method to get Fukushima's decomposition for local semi-Dirichlet forms. If the semi-Dirichlet form has a jumping part, then the method does not work since the continuity of sample paths is essentially used in the proof. Recently, in (FU2010), jump-type Hunt processes generated by lower bounded semi-Dirichlet forms are considered. In that paper, C_0^{lip}, the space of uniformly Lipschitz continuous functions with compact supports, is contained in the domain of the semi-Dirichlet form, but C_0^{lip} is not a subset of the domain of the generator. We hope to further our method so as to get Fukushima's decomposition for general semi-Dirichlet forms which include diffusion, jumping and killing parts.

3. Large deviations problems. In Chapter 2, three transformations: Feynman-Kac transformation, h-transformation and Girsanov transformation, have been considered. This method can be used to study large deviations of additive functionals. We hope to make use of the method of Chapter 2 and paper (CHM2010) to study asymptotic behavior of additive functionals associated with nearly symmetric Markov processes.

Appendix

Let $(\mathcal{E}, D(\mathcal{E}))$ be a quasi-regular semi-Dirichlet form on $L^2(E;m)$. In this appendix, we will derive a useful inequality for measures in the Kato class.

Definition 4.1. *A smooth measure μ is said to be in the Kato class if*

$$\lim_{t\to 0} \inf_{\mathrm{Cap}_\phi(N)=0} \sup_{x\in E\setminus N} E_x[A_t^\mu] = 0,$$

where $(A_t^\mu)_{t\geq 0}$ is the PCAF associated with μ. Denote by S_K the Kato class of smooth measures.

Theorem 4.1. *Let $\mu \in S_K$. Then for any $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that for any $f \in D(\mathcal{E})$,*

$$\int_E \tilde{f}^2 d\mu \leq \varepsilon \mathcal{E}(f,f) + A_\varepsilon \|f\|_2^2. \tag{4.2}$$

Proof. By quasi-regular homoemorphism, without loss of generality, we may assume that $(\mathcal{E}, D(\mathcal{E}))$ is a regular semi-Dirichlet form on $L^2(E;m)$.

(i) First we assume that $\mu \in S_0 \cap S_K$, we will show that for $\alpha \geq 0$, $f \in D(\mathcal{E})$, the following inequality holds:

$$\int_E \tilde{f}^2 d\mu \leq 16(K+1)^2 ||U_\alpha \mu||_\infty \mathcal{E}_\alpha(f,f). \tag{4.3}$$

For $t > 0$, let $K_t = \{x \in E \mid \ |\tilde{f}(x)| \geq t\}$ and

$$\mathcal{L}_{K_t} := \{v \in D(\mathcal{E}) \mid \tilde{v} \geq 1 \ \mathcal{E} - q.e. \text{ on } K_t\}.$$

By (MOR1995, Remark 2.2 (iii)), we get $|f| \in D(\mathcal{E})$ and $\mathcal{L}_{K_t} \neq \emptyset$. Let $\hat{e}_{K_t}$ be the $\alpha-$order equilibrium co-potential, e_{K_t} be the $\alpha-$order equilibrium potential and $\bar{e}_{K_t}$

be the symmetric $\alpha-$order equilibrium potential. Then

$$\begin{aligned}
\mathcal{E}_\alpha(\bar{e}_{K_t},\hat{e}_{K_t}) &\leq (K+1)\mathcal{E}_\alpha(\bar{e}_{K_t},\bar{e}_{K_t})^{\frac{1}{2}}\mathcal{E}_\alpha(\hat{e}_{K_t},\hat{e}_{K_t})^{\frac{1}{2}} \\
&= (K+1)\tilde{\mathcal{E}}_\alpha(\bar{e}_{K_t},\bar{e}_{K_t})^{\frac{1}{2}}\mathcal{E}_\alpha(\hat{e}_{K_t},\hat{e}_{K_t})^{\frac{1}{2}} \\
&\leq (K+1)\tilde{\mathcal{E}}_\alpha(\bar{e}_{K_t},\bar{e}_{K_t})^{\frac{1}{2}}\mathcal{E}_\alpha(\bar{e}_{K_t},\hat{e}_{K_t})^{\frac{1}{2}}
\end{aligned}$$

where K is the continuity constant. Thus $\mathcal{E}_\alpha(\bar{e}_{K_t},\hat{e}_{K_t}) \leq (K+1)^2\tilde{\mathcal{E}}_\alpha(\bar{e}_{K_t},\bar{e}_{K_t})$. By (MR1995, page 832), similar to (FOT1994, Theorem 2.2.1) , we can show that for $u \in D(\mathcal{E})$, u is an α-potential if and only if u is an $\alpha-$excessive function in the positive preserving forms setting. Hence similar to (V1991, Proposition 1), we can show that

$$\int_0^\infty t\tilde{\mathcal{E}}_\alpha(\bar{e}_{K_t},\bar{e}_{K_t})dt \leq 2\tilde{\mathcal{E}}_\alpha(|f|,|f|)$$

Then, by (MOR1995, equation (2.1)), for $\alpha \geq 1$, we get

$$\int_0^\infty t\mathcal{E}_\alpha(\bar{e}_{K_t},\hat{e}_{K_t})dt \leq 2(K+1)^2\tilde{\mathcal{E}}_\alpha(|f|,|f|) \leq 8(K+1)^2\mathcal{E}_\alpha(f,f).$$

Define

$$\hat{\mathcal{E}}(u,v) := \mathcal{E}(v,u).$$

Then $(\hat{\mathcal{E}}, D(\mathcal{E}))$ is a regular positivity preserving form and $\hat{e}_{K_t}$ is an $\alpha-$potential with respect to $(\hat{\mathcal{E}}, D(\mathcal{E}))$. Hence there is a smooth measure ν such that $\hat{e}_{K_t} = \hat{U}_\alpha\nu$. Since $supp[\nu] \subset K_t$,

$$\begin{aligned}
\int_E \widetilde{f}(x)^2\mu(dx) &= 2\int_0^\infty t\int_E I_{K_t}\mu(dx)dt \\
&\leq 2\int_0^\infty t\int_E \widetilde{\hat{e}_{K_t}}\mu(dx)dt \\
&= 2\int_0^\infty t\mathcal{E}_\alpha(U_\alpha\mu,\hat{e}_{K_t})dt \\
&= 2\int_0^\infty t\int_E \widetilde{(U_\alpha\mu)}(x)\nu(dx)dt \\
&\leq 2||\widetilde{U_\alpha\mu}||_\infty\int_0^\infty t\int_E \widetilde{\bar{e}_{K_t}}\nu(dx)dt \\
&= 2||\widetilde{U_\alpha\mu}||_\infty\int_0^\infty t\mathcal{E}_\alpha(\bar{e}_{K_t},\hat{e}_{K_t})dt \\
&\leq 16(K+1)^2||\widetilde{U_\alpha\mu}||_\infty\mathcal{E}_\alpha(f,f).
\end{aligned}$$

(ii) Now we consider general $\mu \in S_K$. By Theorem 3.2, there exists an $\mathcal{E}$-nest $\{F_n\}_{n\geq 1}$ such that $\mu_n := I_{F_n}\mu \in S_0$. Denote by A the PCAF whose Revuz measure is μ. Then $A_t^n := \int_0^t I_{F_n}(X_s)dA_s^n$ is the PCAF corresponding to μ_n. By Theorem 3.3, we know that $U^\alpha\mu_n$ is a quasi-continuous version of $U^\alpha_{A_n}1$. Hence for any n, $||\widetilde{U^\alpha\mu_n}||_\infty = ||U^\alpha_{A^n}1||_\infty \leq ||U^\alpha_A 1||_\infty$. Then for any $f \in D(\mathcal{E})$, we have

$$\begin{aligned} \int \tilde{f}^2 d\mu &= \lim_{n\to\infty}\int \tilde{f}^2 d\mu_n \leq \lim_{n\to\infty} 16(K+1)^2||\widetilde{U^\alpha\mu_n}||_\infty\mathcal{E}_\alpha(f,f) \\ &\leq 16(K+1)^2||U^\alpha_A 1||_\infty\mathcal{E}_\alpha(f,f). \end{aligned} \tag{4.4}$$

Similar to (AM1992, Theorem 4.1), we can show that for $\mu \in S_K$, $\lim_{\alpha\to\infty}||U^\alpha_A 1||_\infty = 0$. Therefore (4.2) holds by (4.4). □

Remark 4.1. *The above proof is based on (V1991, Proposition 2). However, we have made some modifications since there are many differences between symmetric Dirichlet forms and (non-symmetric) positive preserving forms. (F2001, Proposition 4.2) also gives an inequality, which is similar to (4.3), by using a different method under the condition that μ satisfies $\mu U \leq C_0 m$, where $C_0 > 0$ is a constant. For a non-symmetric Dirichlet form, (CS2003, Proposition 4.3) also gives (4.2) by using Green functions of the dual process, however, the method does not work for semi-Dirichlet forms since there is no dual process in the semi-Dirichlet forms setting.*

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