Optimal Control and Estimation Strategies for Nonlinear and Switched Systems

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ABSTRACT

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This dissertation includes two main parts. In the first part, the main contribution is to use an inverse optimality approach to analytically solve the Hamilton-Jacobi-Bellman equation of a third order nonlinear optimal control problem for which the dynamics are affine and the cost is quadratic in the input. One special advantage of this work is that the solution is directly obtained for the control input without finding a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function. Furthermore, the developed controller is combined with a Continuous-Discrete Extended Kalman Filter (CDEKF) as an approach to deal with noisy measurements and provide an estimate of the states for feedback. The proposed technique is illustrated by its application to a path following problem of a Wheeled Mobile Robot (WMR).

The main contribution of the second part of this thesis is the development of two recursive state estimation algorithms for discrete-time piecewise affine (PWA) singular systems with simulation evidence that the idea works for both uncorrelated and correlated process and measurement noise. The proposed algorithms are derived based on successive QR decompositions and Maximum Likelihood (ML) estimation theory. Numerical examples are presented for the case of a PWA system with an unknown input, transformed to a PWA singular system.

In the Name of *Allah,*

the Beneficent, the Merciful

Dedicated to

my mother for her love and measureless support throughout my life

and to the memory of my father, Farhad.

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Chapter 1

Introduction

This chapter includes the motivation and related work on the two topics of the thesis: optimal control of a system based on an inverse optimality method and state estimation algorithms for piecewise affine singular systems. Thesis contributions, outline and publications are also stated in this chapter.

1.1 Motivation and Related Work

This section consists of two subsections in which the motivation and related work for the two parts of the thesis are presented.

1.1.1 Inverse Optimal Control Problems

Optimal control of nonlinear systems is one of the most challenging and difficult subjects in control theory. The control approaches can be divided into two main categories: direct optimal and inverse optimal control. In the direct nonlinear optimal control problem, a controller is developed to minimize an a priori given cost function, which ultimately results in finding a solution to a Hamilton-Jacobi-Bellman (HJB) equation. This equation is unfortunately hard to solve for a general nonlinear system. This obstacle motivated the development of inverse optimal nonlinear control design methods [1]- [16]. In particular, reference [2] finds the dynamics that verify the Hamilton-Jacobi-Bellman equation given the running cost and a value function. In [3] an analytical expression for a stabilizing controller is obtained for feedback linearizable dynamics given the coordinate transformation that feedback linearizes the system, a control Lyapunov function obtained as the solution of the Riccati equation for the linearized dynamics and a bound on the decay rate of the Lyapunov function. It is shown that the controller is optimal relative to a cost involving a control penalty bias. Reference [6] uses Youngs inequality, which was used before in [5] for the design of input-to-state stabilizing controllers, to find an analytical expression for the solution to a class of nonlinear optimal control problems. An expression for the cost that makes the controller optimal was also found. However, there is no indication as to what conditions must be verified such that the obtained cost is a sensible cost, namely, such that it is non-negative. This is shown on a case-by-case basis in the examples. In [10], an inverse optimal approach is applied to the autopilot design for the pitch control of a longitudinal missile model. The stability, convergence and transient performance are evaluated using numerical simulations with several levels of uncertainty in the aerodynamics and the control effectiveness of the missile model. The robustness of the control law to sensor noise and (unmodeled) actuator dynamics including rate, magnitude and bandwidth limits is also examined. In [11] an optimal feedback controller for bilinear systems is obtained that minimizes a quadratic cost function. This inverse optimal control design is applied to the problem of the stabilization of an inverted pendulum on a cart with horizontal and vertical movement. In [12], Nakamura's inverse optimal controller is applied to the magnetic levitation system. A hardware implementation of real-time chaos stabilization by means of inverse optimal control is presented in [13], [14]

Furthermore, an interesting well known result is that using a control Lyapunov function (CLF), many control laws can be calculated which globally asymptotically stabilize the system and can be inverse optimal relative to a meaningful cost functional not specified beforehand by the control designer. However, the main drawback of the CLF concept, as a design tool, is that there is no systematic way to find a control Lyapunov function for general nonlinear systems [17]. Another well known result is that when the cost is quadratic and the dynamics are affine in the input there is an explicit solution for the input as a function of the derivatives of the value function. This fact will be used in this thesis together with the structure of affine dynamics in the input to develop a method to solve the Hamilton-Jacobi-Bellman equation for a class of third order systems.

Based on the concept of inverse optimality, a new solution method that can determine at the same time a controller and a sensible nonnegative cost rendering the controller optimal was developed recently in [16]. In this method, the analytical solution for the control input is obtained directly, without needing to first assume or compute any coordinate transformation, value function, or Lyapunov function. The value function and a Lyapunov function can however be computed once the optimal control input has been found. It was then applied in [19] to solve a specific class of nonlinear third order optimal control problems. This work presents an extension of the aforementioned work to optimal control of a different third order nonlinear system, which does not in general fall in the class of problems considered in [19].

The motivation for this work is that when designers are faced with a control engineering problem and want to formulate it in the optimal control framework, the choice of the most appropriate cost is a difficult task. However, quite often the following three properties are required for the design:

1. The closed loop system should be asymptotically stable to a desired equilibrium point
2. The system should have enough damping so that the trajectories do not take too long to settle around the desired equilibrium point
3. The control energy should be penalized in the cost to avoid high control inputs that can saturate actuators

The particular functions involved in the cost are not usually pre-defined, except possibly the requirement on the control energy that is usually represented by a quadratic cost on the input. This work attempts to find a controller and a cost that together meet the requirements 1–3 and render the controller optimal relative to that cost. To that aim, the cost will be fixed to be quadratic in the input and the state plus an unknown term that shall be determined. The solution is based on the concept of inverse optimality. One special feature of this method, as compared to other methods in the literature, is that the solution is obtained directly for the control input without needing to assume or compute a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function.

1.1.2 State Estimation Algorithms for Piecewise Affine Singular Systems

Piecewise affine (PWA) systems represent a class of hybrid systems, which provide a powerful modelling framework for complex dynamical systems involving nonlinear phenomena [26], [27]. One of the motivations for studying PWA systems is that a broad range of nonlinear systems that appear frequently in engineering applications are either already PWA or can be accurately approximated by PWA systems. These include, but are not limited to, dead-zones, saturations, relays and hysteresis. In recent years, considerable attention has been devoted to the state estimation problem of both continuous-time and discrete-time PWA and hybrid systems, and a great number of results have been obtained [28]- [40]. In particular, state observer design for general PWA systems was first considered in [28] and later addressed for bimodal PWA systems in [29]- [32] and for fault detection in [33], through deterministic approaches. Luenberger observers for switching discrete-time linear systems were addressed in [34]. Another approach to state estimation for discrete time hybrid systems, based on moving horizon estimation, was considered in [36]. This approach is applicable to the general class of piecewise affine systems, but it is computationally demanding (mixed integer quadratic programming problems have to be solved online), which may be an obstacle for implementing it in applications with limited computational resources. In [38], a switching Kalman filter was proposed for the state estimation problem of switched affine systems. In [39], an unscented Kalman filter (UKF) was used for state estimation of an electronic throttle body (ETB) modeled as a discrete time PWA system. The problem of set-membership estimation for discrete-time PWA systems subject to unknown-but-bounded disturbances was addressed in [40]. A maximum-likelihood Kalman filtering approach for switching discrete-time linear systems was introduced in [41]- [42].

Singular systems, which are also referred to as descriptor and implicit systems, have been attracting the attention of many researchers over recent decades due to their capacity to preserve the structure of physical systems and to describe non-dynamic constraints and impulsive behaviors [45], [46]. They find applications in economical systems, network analysis, large scale systems [47], circuits [48], boundary control system [49], power networks, chemical processes, image modeling and robotics [46], [54]. In these models, the state variables may be related algebraically, resulting in a more general class of systems. Thus, a singular system model is described by a set of coupled differential equations and algebraic equations, which incorporate information on the static as well as dynamic constraints of a real plant [55]. A number of control issues have been successfully extended to descriptor systems and the related results have been reported, for instance, in [45], [46], [50]- [53] and the references therein. The state estimation problem for singular systems has been widely considered in the literature [56]- [63]. Most of these works present the generalized Kalman filter as a solution for recursive state estimation problem. In particular, filtering and LQG problems for discrete-time stochastic singular systems have been investigated in [56]. Gaussian descriptor systems have been treated in [57], where an optimal filter, according to the Maximum Likelihood (ML) criterion, has been presented. The Kalman filtering problem for discrete-time descriptor systems is studied through a deterministic approach in [58].

Recently, discrete-time PWA singular systems have been addressed in [64], where the stability and stabilization problems are discussed. However, to the best of our knowledge, there has not been any research in the literature for state estimation of PWA singular systems. This problem is important in both theory and practice and will be addressed in this thesis with simulation evidence that the idea works. First, a recursive estimation algorithm for discrete-time PWA stochastic singular systems will be developed, where our derivation of the algorithm assumes that the process noise and measurement noise are uncorrelated. Then we also note that in practical applications, the process noise and measurement noise may be correlated. As an example given in [72], suppose that our system is an airplane and winds are buffeting the plane. We are using an anemometer to measure wind speed as an input to our estimation algorithm. Thus, the random gusts of wind affect both the process (i.e.,the airplane dynamics) and the measurement (i.e.,the sensed wind speed). In this case, there is a correlation between the process noise and the measurement noise. Therefore, a second recursive estimation algorithm will be developed for discrete-time PWA stochastic singular systems with correlated process and measurement noise. The proposed algorithms are derived using successive QR decompositions and Maximum Likelihood (ML) estimation theory. One special feature of these algorithms is that the covariance matrix is determined from the computation of its square root and is always guaranteed to be nonnegative.

1.2 Contributions of the Thesis

The main contributions of this thesis are the following:

- To propose an inverse optimality approach to analytically solve the Hamilton-Jacobi-Bellman equation of a third order nonlinear optimal control problem for which the dynamics are affine and the cost is quadratic in the input. One special advantage of the proposed method is that the solution is directly obtained for the control input without finding a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function. Furthermore, the developed controller is combined with a Continuous-Discrete Extended Kalman Filter (CDEKF) as an approach to deal with noisy measurements and provide an estimate of the states for feedback. In particular, the controller and observer are designed separately and then the estimated states are included in the controller.

- To develop two recursive state estimation algorithms for discrete-time piecewise affine (PWA) singular systems with simulation evidence that the idea works for both uncorrelated and correlated process and measurement noise. The proposed algorithms are derived based on successive QR decompositions and Maximum Likelihood (ML) estimation theory. One special feature of these algorithms is that the covariance matrix is determined from the computation of its square root and is always guaranteed to be nonnegative.

1.3 Thesis Outline and Publications

The rest of the thesis is organized as follows. Chapter 2 provides some preliminaries, where the maximum likelihood estimation problem for a linear observation is studied, and the algorithmic details of the Continuous-Discrete Extended Kalman Filter (CDEKF) are described. Next, in chapter 3 a nonlinear optimal control problem for a third order system is defined and solved using an inverse optimality method. The developed controller is then combined with a Continuous-Discrete Extended Kalman Filter (CDEKF). Moreover, simulation results are illustrated for the combined observer-controller synthesis by its application to a path following problem of a Wheeled Mobile Robot (WMR) in chapter 3. Subsequently, chapter 4 presents a recursive estimation algorithm for discrete-time PWA stochastic singular systems with uncorrelated process noise and measurement noise. A second algorithm is also developed for discrete-time PWA stochastic singular systems with correlated process noise and measurement noise. A numerical example is also presented for each algorithm, where a system with an unknown input is transformed to a singular system and the performance of the proposed algorithms are validated through simulations. Finally, conclusions and future research directions are presented in chapter 5.

This thesis is mainly based on the following paper:

- Mehdi Abedinpour Fallah and Luis Rodrigues, ”Optimal Control of a Third Order Nonlinear System Based on an Inverse Optimality Method”, *American Control Conference (ACC2011)*, San Francisco, California, USA, June 29–July 01, 2011, pp. 900–904.

Chapter 2

Preliminaries

This chapter of the dissertation includes two main sections. First, the maximum likelihood estimation problem for a linear observation is studied , and then the algorithmic details of the Continuous-Discrete Extended Kalman Filter (CDEKF) are described.

2.1 Maximum Likelihood Estimation Problem

In this section, we consider the maximum likelihood estimation problem of an unknown vector x based on the measurement vector y for a linear observation of the form [67]

$$y = Hx + \nu \tag{2.1}$$

where $H \in \mathbb{R}^{m\times n}$, the measurement y and parameter vector x are known (observed) and unknown, respectively. The noise $\nu \sim \mathcal{N}(0, R)$ represents a zero-mean Gaussian random vector with covariance matrix R. The unknown vector x can be estimated from the measurements if and only if matrix H is full column rank [66]. Given a conditional probability density function $f_{y|x}(y \mid x)$ and a full column rank matrix H in (2.1), the Maximum Likelihood (ML) estimated value for x, is chosen as [67], [68]

$$\hat{x}_{ML}(y) = \arg\max_{x} f_{y|x}(y \mid x) \tag{2.2}$$

In practice it is often more convenient to work with the logarithm of the conditional density function, that is,

$$\hat{x}_{ML}(y) = \arg\max_{x} \ln f_{y|x}(y \mid x) \tag{2.3}$$

It can be shown that for the Gaussian distributions x and y and a positive definite covariance matrix R [67]

$$f_{y|x}(y \mid x) = \frac{1}{(2\pi)^{n/2}\,|R|^{1/2}} \exp\left(-\frac{1}{2}\left[(y-Hx)^T R^{-1}(y-Hx)\right]\right) \tag{2.4}$$

If H is full column rank, using (2.3) the ML estimate can be determined as

$$\hat{x}_{ML}(y) = \arg\min_{x}(y-Hx)^T R^{-1}(y-Hx) \tag{2.5}$$

which yields

$$\hat{x}_{ML}(y) = (H^T R^{-1} H)^{-1} H^T R^{-1} y \tag{2.6}$$

Notice that

$$b = E\left[\hat{x}_{ML} - x\right] = 0 \tag{2.7}$$

such estimators are known as unbiased estimators. Moreover, the covariance of the estimation error is given by [67]

$$P_e(x) = E\left[(\hat{x}_{ML}-x)(\hat{x}_{ML}-x)^T\right] = (H^T R^{-1} H)^{-1} \tag{2.8}$$

2.2 Continuous-Discrete Extended Kalman Filter

The Kalman filter addresses the general problem of trying to estimate the state $x \in \mathbb{R}^n$ of a process that is governed by a linear stochastic equation. There are several different versions of the Kalman filter. This section briefly describes what is called the Continuous-Discrete Extended Kalman Filter, where the label Continuous refers to the fact that the system dynamics are continuous, the label Discrete refers to the fact that the measurements are taken at discrete times and the label Extended refers to the fact that linear Kalman filter algorithm has been extended to a nonlinear model. The complete algorithm of CDEKF is

Table 2.1: Continuous-Discrete Extended Kalman Filter Algorithm

System model:

$\dot{x} = f(x,u) + G\,w$

$z_k = h(x_k) + v_k$

Initial Condition $x(0)$

Assumptions:

Knowledge of f, z_k, u,

$w \sim \mathcal{N}(0,Q)$,

$v_k \sim \mathcal{N}(0,R)$,

$x(0) \sim \mathcal{N}(\bar{x}_0, P_0)$,

$x(0)$, w and z_k are uncorrelated with each other.

Initialization:

$\hat{x}(0) = \bar{x}_0$,

$P(0) = P_0$.

Prediction:

Pick an output sample rate T_{out}, and the number of iterations N.

At each sample time T_{out}:

for $i = 1$ to N **do** {Propagate the equations.}

$\hat{x} = \hat{x} + (\frac{T_{out}}{N}) f(\hat{x},u)$

$A = \frac{\partial f}{\partial x}|_{x=\hat{x}}$

$P = P + (\frac{T_{out}}{N})(AP + PA^T + GQG^T)$

end for

Correction:

if a measurement has been received from sensor i **then**

{Measurement Update}

$H_i = \frac{\partial h_i}{\partial x}|_{x=\hat{x}}$

$K_i = PH_i^T(R_i + H_iPH_i^T)^{-1}$

$P = (I - K_iH_i)P$

$\hat{x} = \hat{x} + K_i(z_i - H_i\hat{x})$

end if

given in Table I using the notation from [73] and [74], where $x_k = x(t_k)$ is the k^{th} sample of x, $z_k = z(t_k)$ is the k^{th} measurement, u is the control vector, w is a zero-mean Gaussian process with covariance Q, and v_k which is a zero-mean Gaussian random variable with covariance R represents the measurement noise at time t_k. The covariance R can usually be estimated from sensor calibration, but the covariance Q is generally unknown and therefore becomes a system gain that can be tuned to improve the performance of the observer. Moreover, A and H denote the Jacobians of f and h respectively. It is noted that in Table I the prediction equations

$$\dot{\hat{x}} = f(\hat{x}, u) \tag{2.9}$$

$$\dot{P} = AP + PA^T + GQG^T \tag{2.10}$$

are approximated by Euler discrete propagation method between measurements. The objective of the Kalman filter is to reconstruct x for all time, given the measurement z_k.

Chapter 3

Optimal Control of a Third Order System Using an Inverse Optimality Method

3.1 Introduction

In this chapter, a nonlinear optimal control problem for a third order system is defined and solved. The optimal control law is found using an inverse optimality approach to analytically solve the Hamilton-Jacobi-Bellman equation of a third order nonlinear optimal control problem for which the dynamics are affine and the cost is quadratic in the input. The main idea was first proposed in [16] to solve a class of second order problems. It was then applied in [19] to solve a class of nonlinear third order optimal control problems. This chapter presents an extension of the aforementioned work to optimal control of a different class of third order nonlinear systems, which does not in general fall in the class of problems considered in [19]; for instance, see Example 3.3.1. One special advantage of this work is that the solution is obtained directly for the control input without needing to assume or compute a value function first. However, the value function can be obtained after one solves for the control input and it is shown to be at least a local Lyapunov function. Moreover, the developed controller will be combined with a Continuous-Discrete Extended Kalman Filter (CDEKF) as an approach to deal with noisy measurements and provide an estimate of the states for feedback. In particular, the controller and observer are designed separately and then the estimated states are included in the controller.

The remainder of this chapter is organized as follows. First the optimal control problem will be defined and solved. Then the technique will be applied to a path following problem of a Wheeled Mobile Robot (WMR) using simulations performed in MATLAB/Simulink. Next, adding a Kalman filter to the optimal controller is explained in a simulation example. Finally, a summary section will close the chapter.

3.2 Problem Definition and Solution

In this section we define an optimal control problem for a class of third order systems. The path following example of a Wheeled Mobile Robot (WMR) is the main motivation for choosing this class of systems, see Example 3.3.2. So consider the following optimal control problem

Problem 3.2.1.

$$\begin{aligned} V(x_0) = \inf \quad & \int_0^\infty \left\{ q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + r u^2 \right\} dt \\ s.t. \quad & \dot{x}_1(t) = f(x_2) \\ & \dot{x}_2(t) = d f(x_2) + x_3 \\ & \dot{x}_3(t) = bu \\ & x(0) = x_0 \ , u \in \mathscr{U} \end{aligned} \tag{3.1}$$

where it is assumed that $q_1 \geq 0, q_2 \geq 0, q_3 > 0, b \neq 0$*,* $r > 0$*,* $x(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T \in \mathbb{R}^3$ *and* $u \in \mathbb{R}$*. The set* $\mathscr{U}$ *represents the allowable inputs, which are considered to be Lebesgue integrable functions. The function* f *with bounded derivative* $f'(x_2)$ *is assumed to be* *measurable, bounded on any compact set with $f(0) = 0$ and with a finite or at most a countable set of zeros. The term*

$$L(x_1, x_2, x_3, u) = q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + r u^2 \tag{3.2}$$

is called the running cost.The problem is to find if possible a control $u^\star$, and a cost L of the form (3.2) such that $u^\star$ will be the optimal solution of (3.1) with finite cost and (3.2) is nonnegative and has a minimum at $x_1 = x_2 = x_3 = u = 0$.

If the function f is linear then from the Linear Quadratic Regulator theory [18] we know that a solution of the form $u = -k_1 x_1 - k_2 x_2 - k_3 x_3$ exists for the case $Q(x) = 0$. Motivated by this result, we will search for solutions of the form $u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3 - k g(x_2)$ where, for reasons that will become apparent in the proof of the main result, $g(x_2) = f(x_2)$. We start by presenting necessary conditions that the value function V must verify for such a solution to exist.

Theorem 3.2.1. *Assume that a control solution of the form*

$$u(x) = -k_1 x_1 - k_2 x_2 - k_3 x_3 - k f(x_2) \tag{3.3}$$

exists for problem 3.2.1 and that a class C^1 function V exists that verifies the corresponding HJB equation

$$\inf_u H(x_1, x_2, x_3, u, V_{x_1}, V_{x_2}, V_{x_3}) = 0 \tag{3.4}$$

where

$$\begin{aligned} H =& q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + Q(x) + r u^2 \\ &+ V_{x_1} f(x_2) + V_{x_2}(d f(x_2) + x_3) + V_{x_3}(b u) \end{aligned} \tag{3.5}$$

and

$$V_{x_i} = \frac{\partial V}{\partial x_i}, \textit{for } i = 1, 2, 3. \tag{3.6}$$

and with boundary condition $V(0) = 0$. Then V must be of the form

$$V(x) = 2rb^{-1}\left(k_1x_1x_3 + k_2x_2x_3 + k_3\frac{x_3^2}{2} + kx_3f(x_2)\right) + h(x_1,x_2) \tag{3.7}$$

where $h(x_1,x_2)$ is an arbitrary integration function of class C^1 with

$$h(0,0) = 0 \tag{3.8}$$

Proof. Consider the HJB equation (3.4) associated with (3.1). The necessary condition on u to be a minimizer is

$$\frac{\partial V}{\partial x_3} = -2rb^{-1}u(x) \tag{3.9}$$

and therefore

$$V(x) = -2rb^{-1}\int u(x)dx_3 + h(x_1,x_2) \tag{3.10}$$

where $h(x_1,x_2)$ is an arbitrary integration function of x_1 and x_2. Searching for a solution of the form $u = u_1(x_1,x_2) + u_2(x_3)$, expression (3.10) becomes

$$V(x) = -2rb^{-1}x_3u_1(x_1,x_2) - 2rb^{-1}\int u_2(x_3)dx_3 + h(x_1,x_2) \tag{3.11}$$

Replacing

$$u_1(x_1,x_2) = -k_1x_1 - k_2x_2 - kf(x_2) \tag{3.12}$$

$$u_2(x_3) = -k_3x_3 \tag{3.13}$$

yields (3.7) after integration. From the boundary condition $V(0) = 0$ one obtains the constraint (3.8). □

Remark 3.2.1. *It is important that the value function has cross terms or otherwise, from (3.9), the controller will only depend on x_3, which will considerably limit the class of systems for which a solution can be found.*

The main result is now stated in the next theorem.

Theorem 3.2.2. *If $q_1 \geq 0,\ q_2 \geq 0,\ q_3 > 0,\ b \neq 0,\ r > 0,$*

$$\begin{aligned} Q(x) = r(k_{32}^2 - 2b^{-1}kf'(x_2))x_3^2 + 2rk_1k_2x_1x_2 + 2rk_1k_{31}x_1x_3 \\ + 2rk_2k_{31}x_2x_3 + rk(k - 2k_{32}d + 2b^{-1}d^2f'(x_2))f^2 \end{aligned} \tag{3.14}$$

and if the gains

$$k_1 = \pm\sqrt{\frac{q_1}{r}} \tag{3.15}$$

$$k_2 = \pm\sqrt{\frac{q_2}{r}} \tag{3.16}$$

$$k_3 = \pm\sqrt{\frac{q_3}{r}} + b^{-1}\sqrt{\frac{q_2}{q_3}} \tag{3.17}$$

$$k = b^{-1}\left(\sqrt{\frac{q_1}{q_3}} + d\sqrt{\frac{q_2}{q_3}}\right) \tag{3.18}$$

verify

$$(k_{32}^2 - 2b^{-1}kf'(x_2))x_3^2 + k(k - 2k_{32}d + 2b^{-1}d^2f'(x_2))f^2 \geq 0 \tag{3.19}$$

with

$$k_{31} = \pm\sqrt{\frac{q_3}{r}}, \quad k_{32} = b^{-1}\sqrt{\frac{q_2}{q_3}} \tag{3.20}$$

then the control input (3.3) is a solution of the HJB equation (3.4) associated with (3.1) with value function

$$\begin{aligned} V(x) = r\left(\sqrt{k_1(k - dk_{32})}x_1 + \sqrt{k_2k_{32}}x_2 + \sqrt{b^{-1}k_{31}}x_3\right)^2 + \\ + rb^{-1}k_{32}x_3^2 + 2rb^{-1}kx_3f(x_2) + \\ - rb^{-1}kdf^2(x_2) + (2rkk_{32})\int f(x_2)dx_2 + \alpha \end{aligned} \tag{3.21}$$

where α is an integration constant verifying

$$\alpha = -(2rkk_{32})\left[\int f(x_2)dx_2\right]_{x_2=0} \tag{3.22}$$

The function V is also a local Lyapunov function provided it is locally positive definite in a region around the origin. Moreover, if V is globally positive definite and radially unbounded then it is a Lyapunov function. Finally, the trajectories will converge *to one of the minimizers of* $L(x_1,x_2,x_3,u(x_1,x_2,x_3))$*, i.e, to a point* (x_1,x_2,x_3) *such that* $L(x_1,x_2,x_3,u(x_1,x_2,x_3)) = 0$*. If* $L(x_1,x_2,x_3,u(x_1,x_2,x_3))$ *is convex, then the trajectories will converge to the origin for all initial conditions.*

Proof. Using the results of theorem 1 and replacing

$$k_3 = k_{31} + k_{32} \tag{3.23}$$

the HJB equation (3.4) yields after rearranging

$$\begin{aligned}
0 = {} & (q_1 - rk_1^2)x_1^2 + (q_2 - rk_2^2)x_2^2 + (q_3 - rk_{31}^2 - rk_{32}^2 + 2rb^{-1}k_2 \\
& - 2rk_{31}k_{32} + 2rb^{-1}kf')x_3^2 + Q(x) - 2rk_1k_2x_1x_2 \\
& - 2rk_1k_{31}x_1x_3 - 2rk_1k_{32}x_1x_3 - 2rk_2k_{31}x_2x_3 - 2rk_2k_{32}x_2x_3 \\
& - rk^2f^2 - 2rkk_{31}x_3f - 2rkk_{32}x_3f + 2rb^{-1}k_1x_3f \\
& - 2rkk_1x_1f - 2rkk_2x_2f + \frac{\partial h}{\partial x_1}f + \frac{\partial h}{\partial x_2}x_3 + 2rb^{-1}dk_2x_3f \\
& + 2rb^{-1}dkf'fx_3 + \frac{\partial h}{\partial x_2}df
\end{aligned} \tag{3.24}$$

where the arguments of the functions were omitted for simplicity. Making

$$\frac{\partial h}{\partial x_2} = 2rk_1k_{32}x_1 + 2rk_2k_{32}x_2 + 2rkk_{32}f(x_2) - 2rb^{-1}dkf'f \tag{3.25}$$

and using (3.25) together with (3.15)-(3.18) and (3.20) in (3.24) and noting that (3.18) and (3.20) can also be expressed as

$$\begin{aligned}
kk_{31} - b^{-1}k_1 - b^{-1}dk_2 = 0 \\
b^{-1}k_2 - k_{31}k_{32} = 0
\end{aligned} \tag{3.26}$$

then the expression (3.24) can be written as

$$\begin{aligned}
0 = {} & (-rk_{32}^2 + 2rb^{-1}kf')x_3^2 + Q(x) - 2rk_1k_2x_1x_2 - 2rk_1k_{31}x_1x_3 \\
& - 2rk_2k_{31}x_2x_3 - rk^2f^2 - 2rkk_1x_1f - 2rkk_2x_2f + \frac{\partial h}{\partial x_1}f \\
& + 2rk_2k_{32}dx_2f + 2rk_1k_{32}dx_1f + 2rkk_{32}df^2 \\
& - 2rb^{-1}kd^2f'f^2
\end{aligned} \tag{3.27}$$

Finally, adding and subtracting $2rk_1k_{32}x_2f(x_2)$ to the right hand side of the above expression, making

$$\frac{\partial h}{\partial x_1} = 2rk_1(k - k_{32}d)x_1 + 2rk_1k_{32}x_2 \tag{3.28}$$

and choosing $Q(x)$ as (3.14), one finds that all terms in (3.27) vanish and therefore the HJB equation is verified. This is a sufficient condition for the control input (3.3) to be a solution that minimizes the cost of problem 3.2.1 because the second derivative of the Hamiltonian (3.5) with respect to u is equal to $2r > 0$. The running cost is a sensible cost because from (3.2), (3.15)-(3.18) and (3.20) it is given by

$$\begin{aligned} L = & r(k_1x_1 + k_2x_2 + k_{31}x_3)^2 + r(k_{32}^2 - 2b^{-1}kf'(x_2))x_3^2 \\ & + rk(k - 2k_{32}d + 2b^{-1}d^2f'(x_2))f^2 + ru^2 \end{aligned} \tag{3.29}$$

and it is non-negative with a minimun at $x_1 = x_2 = x_3 = u = 0$ under the assumption (3.19). Replacing (3.12)-(3.13), (3.23) and the integral of resulting expressions from (3.25) and (3.28) in (3.11) yields the value function (3.21) after integration, considering (3.26). Furthermore, the boundary condition $V(x(\infty)) = 0$ yields (3.22). Observe that

$$\begin{aligned} V(x(0)) &= \int_0^\infty L(x_1, x_2, x_3, u^\star)\, dt = -[V(x(\infty)) - V(x(0))] \\ &= -\int_0^\infty \dot{V}(x)\, dt \end{aligned} \tag{3.30}$$

so when the optimal control law is used, L and $-\dot{V}$ coincide. Hence,

$$\dot{V} = -L(x_1, x_2, x_3, u^\star) \leq 0 \tag{3.31}$$

which makes V a local Lyapunov function for the system if it is positive definite in a region around the origin. If V is globally positive definite and radially unbounded then it is a Lyapunov function. Finally, since the optimal cost (3.21) is finite for all initial conditions, then the trajectories will converge to one of the minimizers of $L(x_1, x_2, x_3, u(x_1, x_2, x_3))$ because $L \geq 0$ and $\lim_{t\to\infty} L = 0$ for integrability. If L is convex, then the trajectories must converge to the origin because the origin is the only minimizer of L. This finishes the proof. □

Remark 3.2.2. *It is interesting that the square of the nonlinearity comes naturally as a term in the cost, although this would be difficult to predict based on a general tendency to always construct costs that have only quadratic terms on the state.*

Remark 3.2.3. *It is noted that by setting $d = 0$, this approach yields the results obtained in [19]; for an instance, see Example 3.3.2.*

3.3 Examples and Simulation Results

In this section, the effectiveness of the proposed method will be shown in several examples.

Example 3.3.1. Linear System

Consider a third order system with

$$b = d = 1 \quad , \quad f(x_2) = x_2 \,. \tag{3.32}$$

and with equilibrium points at

$$x_1 = constant \quad , \quad x_2 = x_3 = 0 \,. \tag{3.33}$$

Assuming $q_1 = q_3 = r = 1$ and $q_2 = 9$, yields the control input

$$u = -x_1 - 7x_2 - 4x_3 \tag{3.34}$$

and the running cost is given by

$$L(x_1, x_2, x_3, u) = (x_1 + 3x_2 + x_3)^2 + x_3^2 + u^2 \,. \tag{3.35}$$

It is noted that the running cost function (3.35) is strictly convex because its Hessian matrix is positive definite as follows

$$\nabla^2 L(x) = \begin{bmatrix} 4 & 20 & 10 \\ 20 & 116 & 62 \\ 10 & 62 & 36 \end{bmatrix} > 0 \,. \tag{3.36}$$

The resulted value function is

$$V(x) = (x_1 + 3x_2 + x_3)^2 + 12x_2^2 + 2x_3^2 + (2x_2 + x_3)^2 \tag{3.37}$$

which is positive definite and radially unbounded. Moreover, the derivative of the value function is

$$\begin{aligned}\dot{V}(x) =& -(x_1 + 3x_2 + x_3)^2 - x_3^2 - (x_1 + 7x_2 + 4x_3)^2 \\ =& x^T N x = x^T \begin{bmatrix} -2 & -10 & -5 \\ -10 & -58 & -31 \\ -5 & -31 & -18 \end{bmatrix} x\,. \end{aligned} \tag{3.38}$$

which is negative definite since $N < 0$. Therefore, the value function is a global Lyapunov function, and the system is globally asymptotically stable.

Example 3.3.2. Path Following of a WMR

In this example, the developed controller is applied to a path following problem of a WMR.

a) Dynamic Model: Fig.3.1 shows a schematic of the WMR, which is assumed to be rigid and to be driven by a torque T to control the heading angle ψ which is measured from the positive x-axis in the inertial frame. The forward velocity $V_0 = 1m/s$ is assumed to be already made constant by the proper design of a cruise controller. The dynamic model of the WMR is composed of two parts: kinematics and dynamics. The kinematics equations are

$$\begin{aligned}\dot{y} &= V_0 \sin\psi \\ \dot{\psi} &= \omega\end{aligned} \tag{3.39}$$

and the dynamics equation is

$$\dot{\omega} = \frac{1}{I} T \tag{3.40}$$

where T is the input torque generated by the DC motors. The moment of inertia of the WMR with respect to the center of mass is $I = 1kg.m^2$. It is desired that the WMR follows the path $y = 0$.

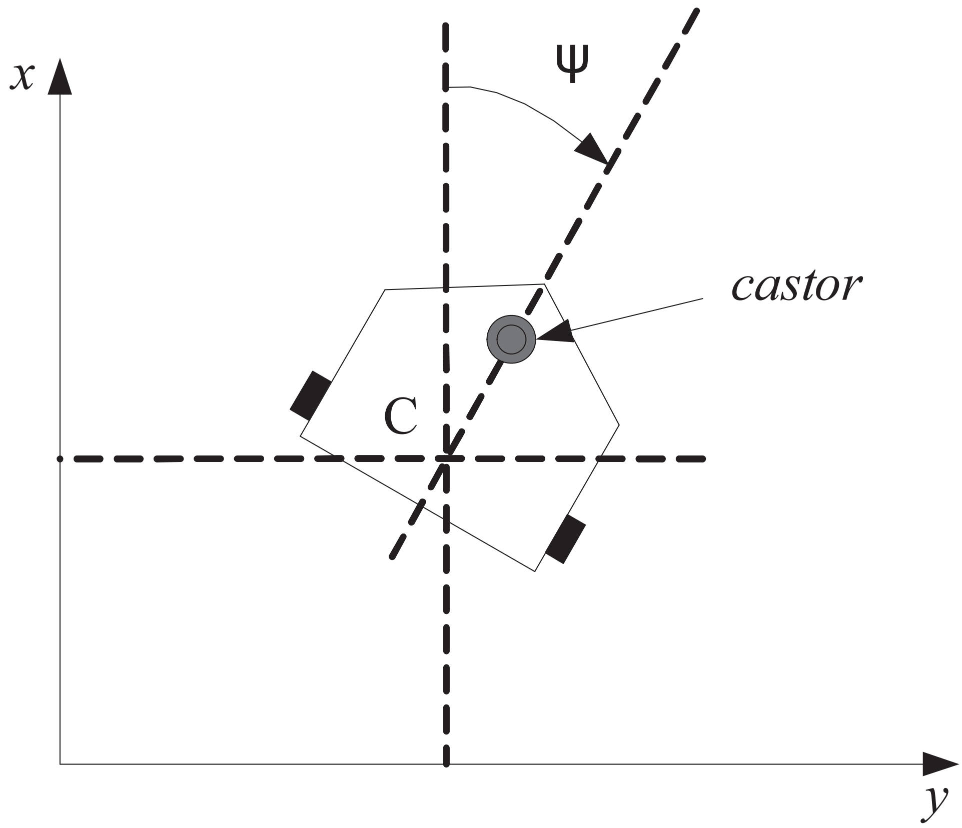

Figure 3.1: Schematic of the Wheeled Mobile Robot (WMR)

b) Controller Design and Simulation Results: The differential equations (3.39) and (3.40) are cast in the form of *Problem 3.2.1* with $f(x_2) = \sin(x_2)$, $b = 1$, $d = 0$, where the states are defined by $x_1 = y$, $x_2 = \psi$ and $x_3 = \omega$. If $q_1 = q_3 = r = 1$ and $q_2 = 4$, then the optimal controller is

$$u = -x_1 - 2x_2 - 3x_3 - sin(x_2) \tag{3.41}$$

the running cost is

$$L = (x_1 + 2x_2 + x_3)^2 + (4 - 2cos(x_2))x_3^2 + \sin^2(x_2) + u^2 \tag{3.42}$$

and the value function is

$$V(x) = (x_1 + 2x_2 + x_3)^2 + 2x_3^2 + 2x_3 \sin(x_2) - 4\cos(x_2) + 4 \tag{3.43}$$

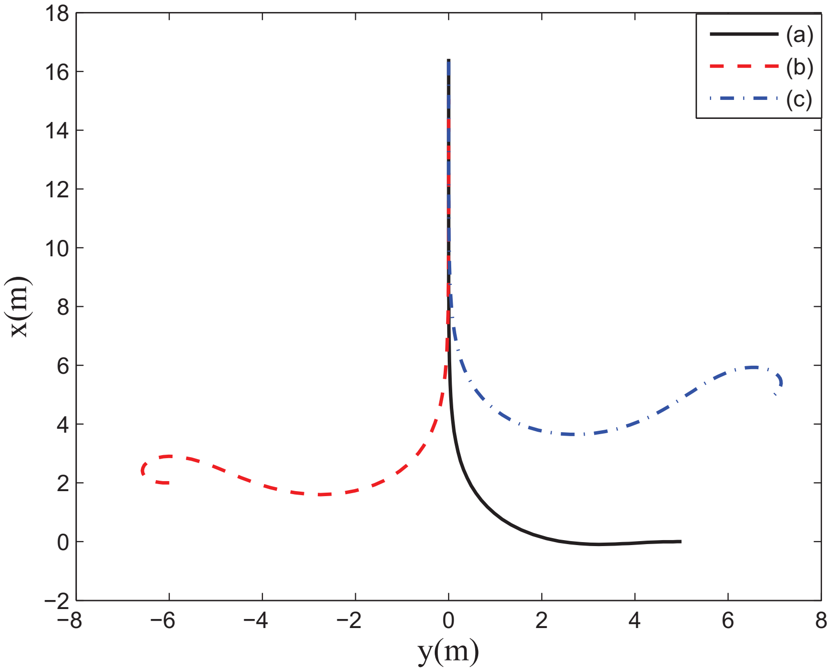

Figure 3.2: Trajectories of WMR following path $y = 0$

Moreover, the derivative of the value function

$$\begin{aligned}\dot{V} = &-(4-2cos(x_2))x_3^2 - (x_1+2x_2+x_3)^2 - \sin^2(x_2)\\ &-(x_1+2x_2+3x_3+sin(x_2))^2\end{aligned} \tag{3.44}$$

is negative definite for $x_2 \in (-\pi, \pi)$. Therefore, the value function is a local Lyapunov function in the largest invariant set contained in $\{(x_1,x_2,x_3) \in \mathbb{R}^3 \,|\, |x_2| < \pi\} \cap \{(x_1,x_2,x_3) \in \mathbb{R}^3 | V > 0\}$ where > 0 stands for positive definite. Note that one cannot guarantee convergence to the origin from any initial condition because L is not convex. Simulations were performed using the optimal controller (3.41) for the following different initial conditions:

case (a): $x_0 = 0, y_0 = 5$, $\psi_0 = -\frac{\pi}{2}$ and $\omega_0 = 0$

case (b): $x_0 = 2, y_0 = -6$, $\psi_0 = -\frac{\pi}{2}$ and $\omega_0 = 0$

case (c): $x_0 = 5, y_0 = 7$, $\psi_0 = \frac{\pi}{6}$ and $\omega_0 = 0$

Convergence to the desired path is clearly seen in $Fig.$3.2. Moreover, $Figures$(3.3)-(3.5) show the time variations of the position, heading angle and angular velocity. The input signal is depicted in $Fig.$3.6.

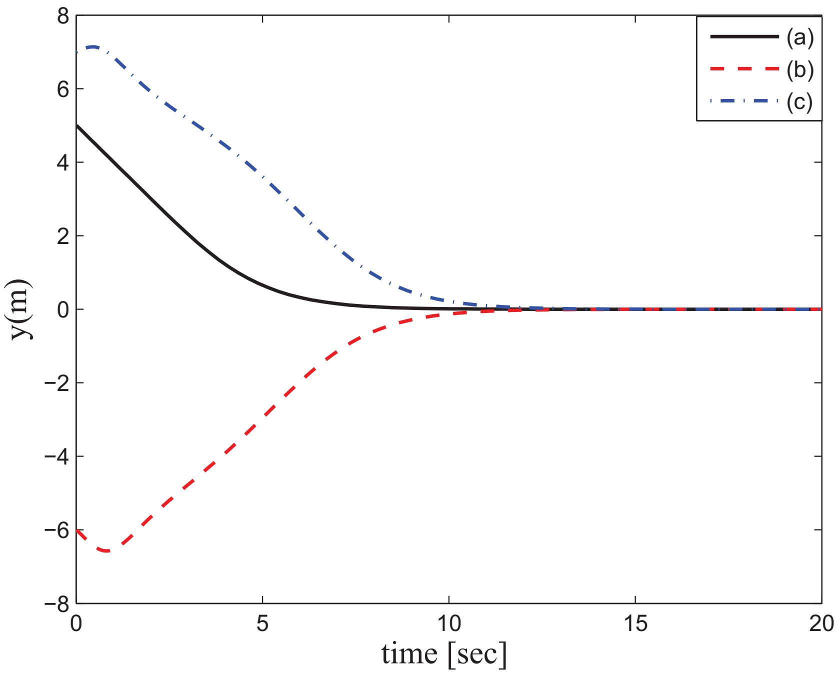

Figure 3.3: Position y

c) Adding Kalman Filter to the Optimal Control:

In practical applications with noisy sensor measurements, the necessary sensors for full-state feedback may not always be available and using numerical differentiation to obtain velocities can be problematic if position measurements are noisy. Therefore, here we propose to use a Continuous-Discrete Extended Kalman Filter (CDEKF) as described in section 2.2, to deal with the noisy measurements and to provide an estimate of the states for feedback. In the combined observer-controller synthesis, the controller and observer are designed separately. For example, suppose that we do not have any sensors available to measure the third state ω in the path following problem. From the dynamic model of the WMR the torque T is assumed to be the input while the commanded torque T^c comes from the control law. Let $e_T = T - T^c$ denote the error between torque and commanded

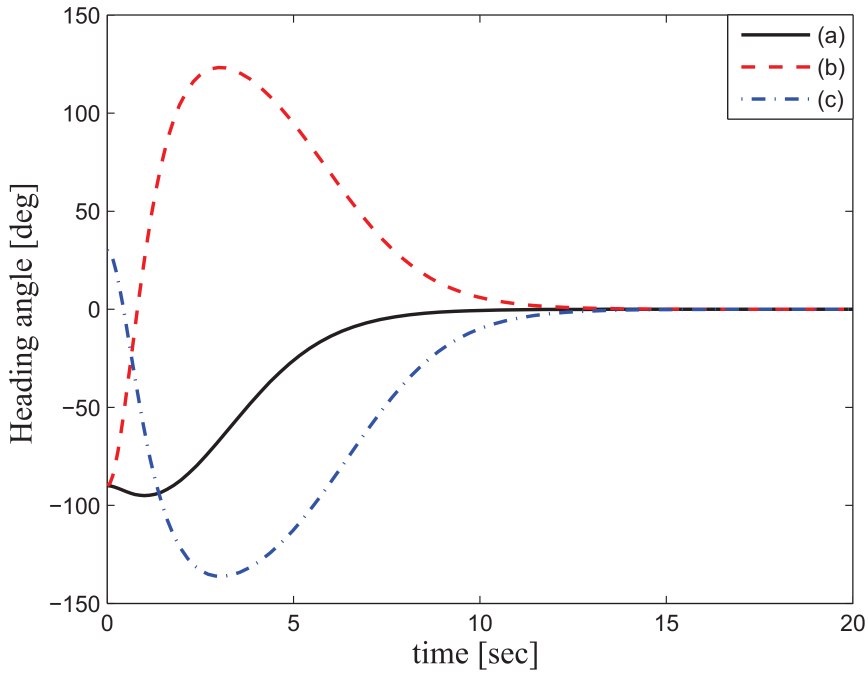

Figure 3.4: Heading angle ψ

torque. Then equations (3.39) and (3.40) with $I = 1Kgm^2$ can be written as

$$\begin{bmatrix} \dot{y} \\ \dot{\psi} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \sin\psi \\ \omega \\ T^c \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e_T \tag{3.45}$$

Suppose that only the first two states of the robot are measured directly. Hence,

$$z_k = \begin{bmatrix} y_k \\ \psi_k \end{bmatrix} + v_k \tag{3.46}$$

and the Jacobian matrices are computed as

$$A = \begin{bmatrix} 0 & \cos\hat{\psi} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.47}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.48}$$

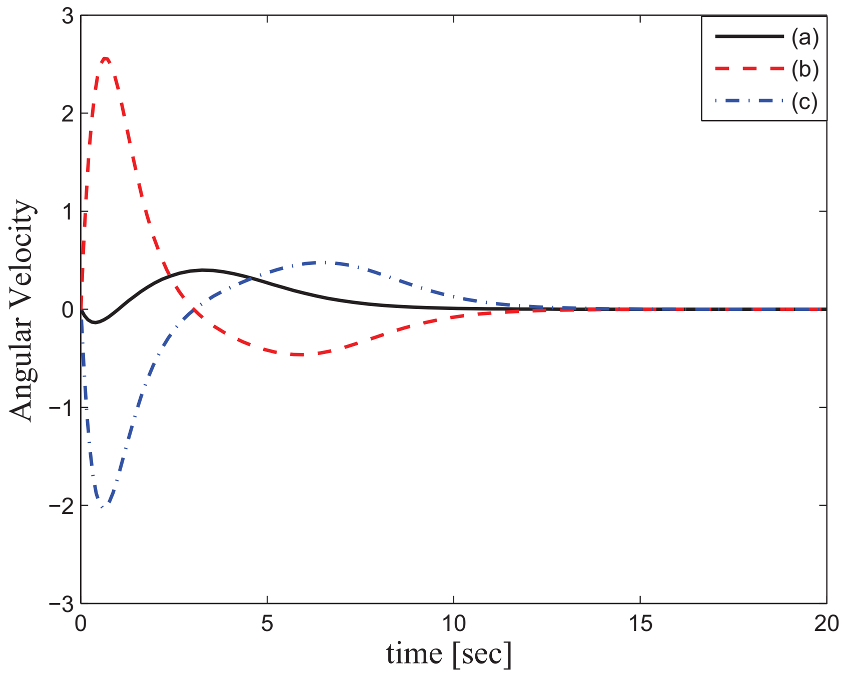

Figure 3.5: Angular Velocity

The initial condition of CDEKF is selected as $x_0 = \begin{bmatrix} 5 & \frac{\pi}{2} & 0 \end{bmatrix}^T$ for the state vector and $P_0 = I_3$ adds uncertainty to this choice, where I_3 is a 3×3 identity matrix. Finally, Q and R are chosen as

$$Q = 0.01 \tag{3.49}$$

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & \frac{\pi}{180} \end{bmatrix} \tag{3.50}$$

Including the estimated states in the control law (3.41) yields

$$\hat{u} = -\hat{x}_1 - 2\hat{x}_2 - 3\hat{x}_3 - sin(\hat{x}_2) \tag{3.51}$$

Simulations are performed using the controller (3.51) for the following different initial conditions:

case (a): $x_0 = 0, y_0 = 5,\ \psi_0 = -\frac{\pi}{2}$ and $\omega_0 = 0$

case (b): $x_0 = 2, y_0 = -6,\ \psi_0 = -\frac{\pi}{2}$ and $\omega_0 = 0$

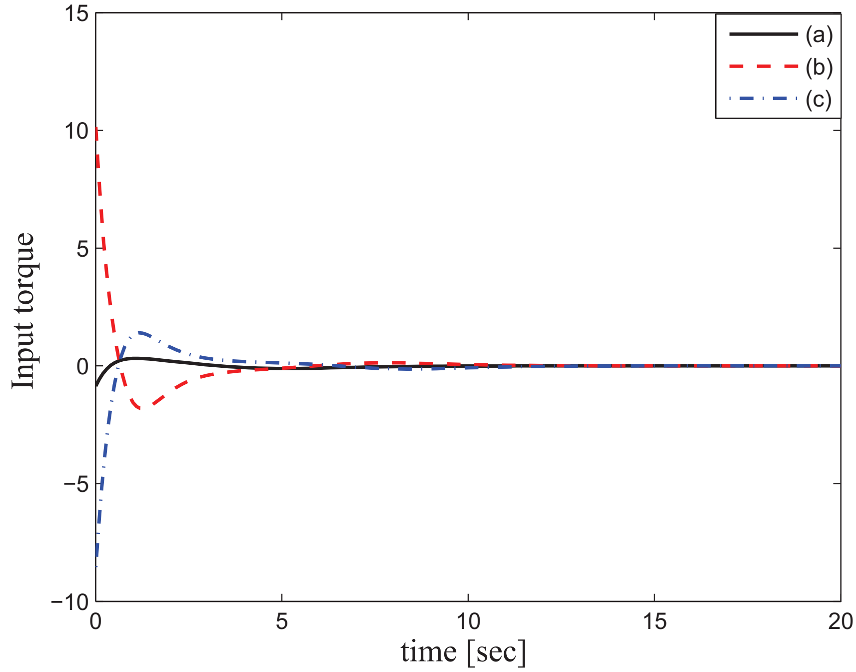

Figure 3.6: Input Signal

case (c): $x_0 = 5, y_0 = 7$, $\psi_0 = \frac{\pi}{6}$ and $\omega_0 = 0$

The resulting path for the WMR using the optimal controller (3.51) is illustrated in *Fig*.3.7. Moreover, *Figures*(3.8)-(3.10) show the time variations of the estimated position $\hat{y}$, heading angle $\hat{\psi}$ and angular velocity $\hat{\omega}$. And finally, *Figures*(3.11)-(3.13) show the true values and the CDEKF-estimates of the state variables. The simulation results verify the good performance of the combined observer-controller synthesis.

3.4 Summary

The solution to a third order nonlinear optimal control problem has been presented in this chapter extending an inverse optimality method originally developed for a class of second order systems. The important feature of this approach is that the analytical solution for the control input is obtained directly without needing to assume or compute a coordinate transformation, value function or Lyapunov function. The value function and a

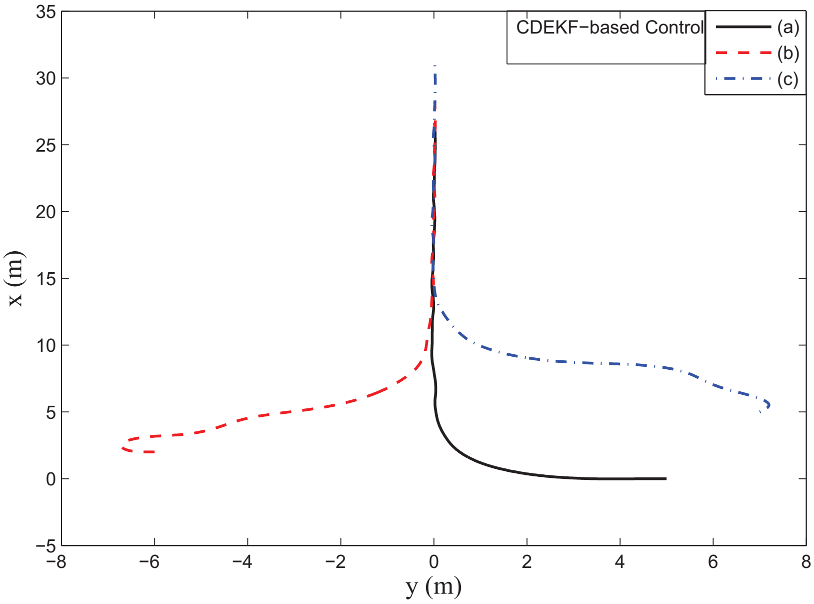

Figure 3.7: Trajectories of WMR following path $y = 0$

Lyapunov function can however be computed after the control input has been found. The controller was applied to a path following problem of a Wheeled Mobile Robot (WMR). A Continuous-Discrete Extended Kalman Filter (CDEKF) was also designed to provide the estimated states from noisy measurements for constructing the control law. The simulation results verified the effectiveness of the method.

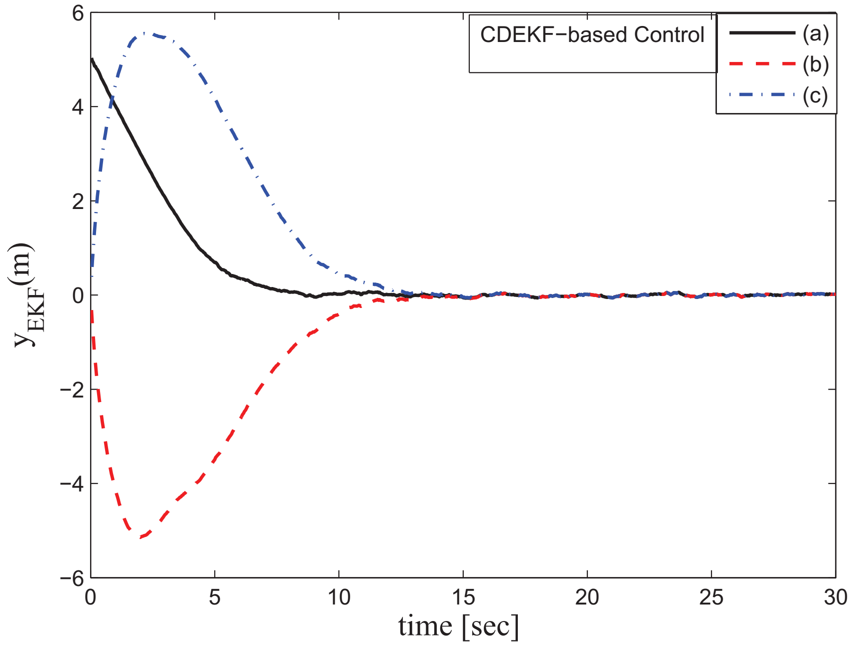

Figure 3.8: Estimated Position $\hat{y}$

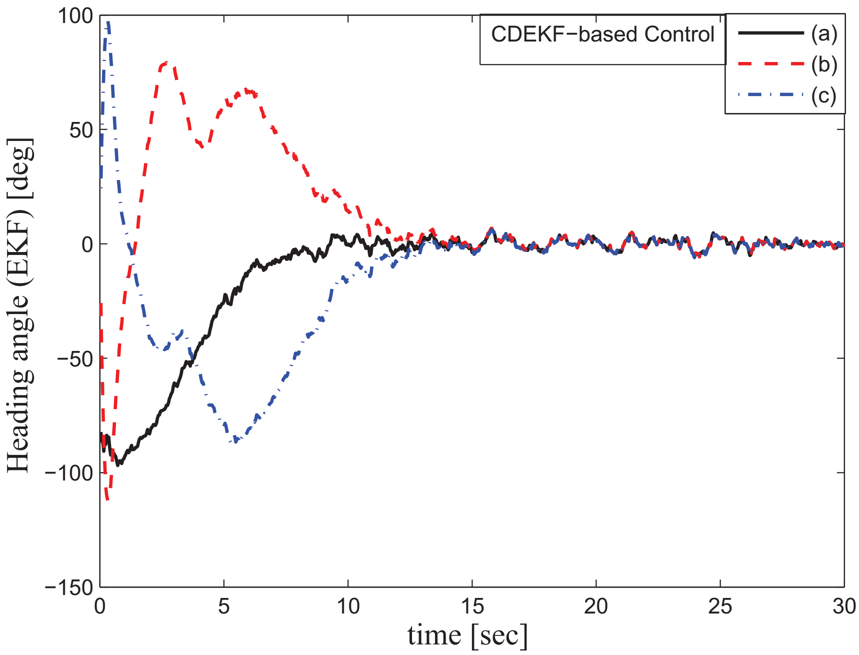

Figure 3.9: Estimated Heading angle $\hat{\psi}$

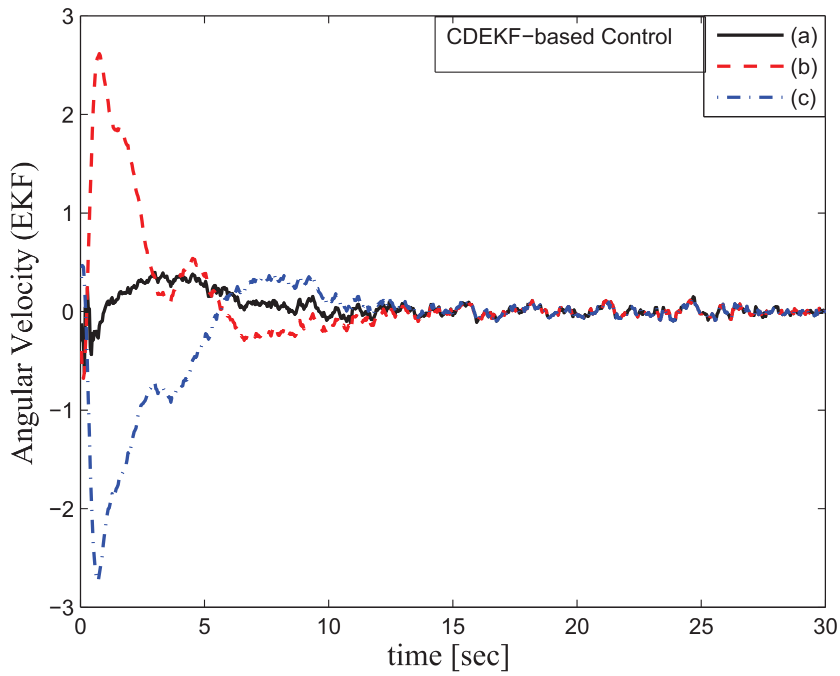

Figure 3.10: Estimated Angular Velocity $\hat{\omega}$

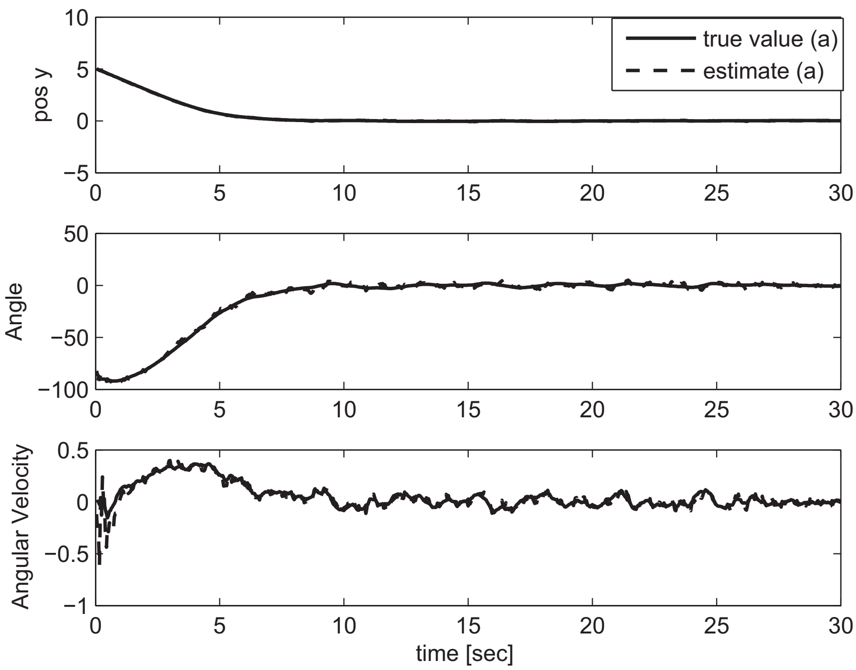

Figure 3.11: The true values and the estimates, case (a)

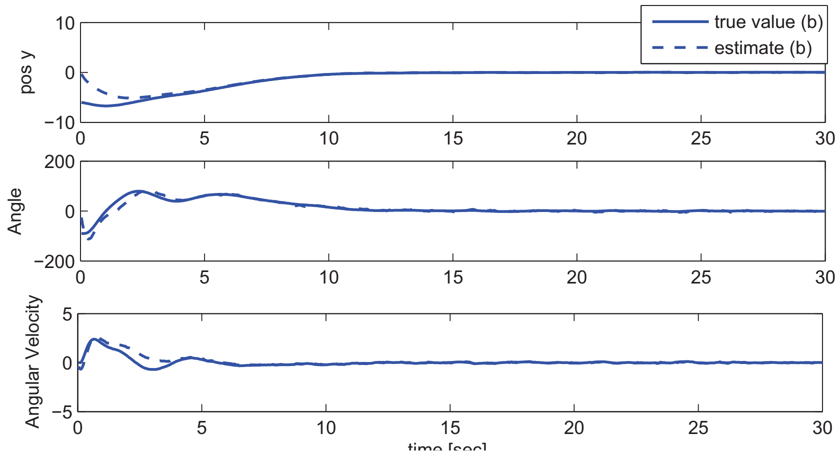

Figure 3.12: The true values and the estimates, case (b)

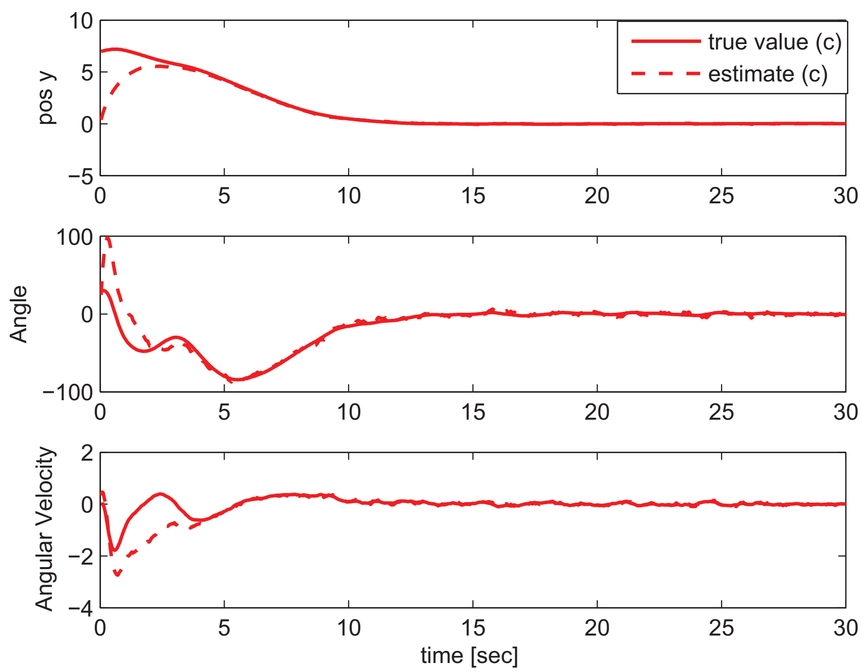

Figure 3.13: The true values and the estimates, case (c)

Chapter 4

Estimation Algorithms for Piecewise Affine Singular Systems

4.1 Introduction

The main contribution of this chapter is the development of two recursive state estimation algorithms for discrete-time piecewise affine stochastic singular systems with simulation evidence that the idea works. The proposed algorithms are derived based on successive QR decompositions and Maximum Likelihood (ML) estimation theory. One special feature of these methods is that the covariance matrix is determined from the computation of its square root and is always guaranteed to be nonnegative. Two numerical examples are also considered where a system with an unknown input is transformed to a singular system. The simulation results illustrate the effectiveness of the proposed algorithms.

The remainder of this chapter is organized as follows. First, a recursive estimation algorithm for discrete-time PWA stochastic singular systems with uncorrelated process noise and measurement noise is developed. Then, a second algorithm is developed for discrete-time PWA stochastic singular systems with correlated process noise and measurement noise. A numerical example is also presented for each algorithm, where a system with an unknown input is transformed to a singular system and the performance of the proposed algorithms are validated through simulations. Finally, a summary is given.

4.2 An Estimation Algorithm for PWA Singular Systems with Uncorrelated Noise

This section presents the first recursive estimation algorithm for discrete-time PWA stochastic singular systems, where the process noise and measurement noise are assumed to be uncorrelated.

4.2.1 Discrete-time PWA Singular Systems with Uncorrelated Noise

The discrete-time PWA stochastic singular system considered in this section is a state-based switched system described by the following dynamics

$$\begin{cases} E_{\sigma_k}x[k+1] = A_{\sigma_k}x[k] + a_{\sigma_k} + B_{\sigma_k}u[k] + W_{\sigma_k}w[k] \\ \quad y[k+1] = C_{\sigma_k}x[k+1] + b_{\sigma_k} + V_{\sigma_k}v[k] \\ \quad \forall x \in \mathscr{R}_{\sigma_k}, \sigma_k \in I = \{1, \ldots, M\} \end{cases} \tag{4.1}$$

where $E_{\sigma_k} \in \mathbb{R}^{q\times n}$, $A_{\sigma_k} \in \mathbb{R}^{q\times n}$, $B_{\sigma_k} \in \mathbb{R}^{q\times m}$, $W_{\sigma_k} \in \mathbb{R}^{q\times q}$, $C_{\sigma_k} \in \mathbb{R}^{p\times n}$ and $V_{\sigma_k} \in \mathbb{R}^{p\times p}$ are real matrices, $x[k] \in \mathbb{R}^n$ is the state vector, $u[k] \in \mathbb{R}^m$ is the control input and $y[k] \in \mathbb{R}^p$ is the measured output.

The switching signal σ_k defines the system mode at the time instant k. The constant vectors a_{σ_k} and b_{σ_k} are the affine terms for each affine model. Moreover, $\mathscr{R}_{\sigma_k}$, $\sigma_k \in \{1, \ldots, M\}$ is a polytopic partition of the state space defined as [26], [27]

$$\mathscr{R}_{\sigma_k} = \left\{x \mid \overline{L}_{\sigma_k}\overline{x} > 0\right\} \tag{4.2}$$

where $\overline{L}_{\sigma_k} = \begin{bmatrix} L_{\sigma_k} & l_{\sigma_k} \end{bmatrix}, \overline{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$.

Assumptions

1. We assume that for every value of σ, the following matrices are full column rank

$$\begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} \tag{4.3}$$

$$\begin{bmatrix} sE_{\sigma_k} - A_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} \tag{4.4}$$

where s is a complex variable.

2. It is assumed that $q+p \geq n$, V_{σ_k} is invertible and the random vectors $w[k] \sim \mathcal{N}(0, I_q)$ and $v[k] \sim \mathcal{N}(0, I_p)$ represent independent Gaussian white noise vectors.

Remark 4.2.1. *Condition 1) guarantees the existence of a unique solution to the state estimation problem for each mode of the class of singular systems (4.1) with non-square matrices E_{σ_k} and A_{σ_k}. It also corresponds to the assumption of regularity for systems in that class (see [71] for more details).*

4.2.2 Algorithm Derivation

In this section we develop a recursive estimation algorithm for discrete-time PWA singular systems of the form (4.1). The estimation problem is to reconstruct the state $x[k+1]$ at time $k+1$ from the given input and output sequences and the estimated operation mode. It is assumed that the state does not jump during the change of the operation mode.

Suppose that given knowledge of the process up to and including step k, there is an initial estimate for $x[k]$ denoted by $\hat{x}[k]$. We assume that the estimation error can be written as $e[k] = S(k)w_e[k]$ where $w_e[k] \sim \mathcal{N}(0, I_n)$ is a Gaussian white noise vector independent of $w[k]$ and $v[k]$. We can then write

$$\hat{x}[k] = x[k] + S(k)w_e[k] \tag{4.5}$$

Hence,

$$E\left[(\hat{x}-x)(\hat{x}-x)^T\right] = P_e = SS^T \tag{4.6}$$

holds for the covariance of the estimation error if (4.5) is verified. Then substituting for $x[k]$ from relation (4.5) into the system equations (4.1) yields

$$\begin{cases} E_{\sigma_k}x[k+1] = A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} - A_{\sigma_k}S[k]w_e[k] + B_{\sigma_k}u[k] + W_{\sigma_k}w[k] \\ \quad y[k+1] = C_{\sigma_k}x[k+1] + b_{\sigma_k} + V_{\sigma_k}v[k] \\ \quad \forall \hat{x} \in \mathscr{R}_{\sigma_k}, \sigma_k \in I = \{1,\ldots,M\} \end{cases} \tag{4.7}$$

where using (4.2) the operation mode σ_k is estimated by finding the polytopic region to which $\hat{x}$ belongs. Note that because the estimated and true values of $x[k]$ might not be the same, equations (4.7) and (4.1) are not equivalent. Stacking the noise vectors as

$$w_o[k] = \begin{bmatrix} w_e^T[k] & w^T[k] & v^T[k] \end{bmatrix}^T \sim \mathscr{N}(0, I_{n+q+p}) \tag{4.8}$$

equation (4.7) can be rewritten as

$$\begin{aligned} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} x[k+1] + \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] \\ + \begin{bmatrix} A_{\sigma_k}S[k] & -W_{\sigma_k} & 0 \\ 0 & 0 & V_{\sigma_k} \end{bmatrix} w_o[k] \end{aligned} \tag{4.9}$$

Note that according to equation (4.9), the unknown vector $x[k+1]$ of the system can be estimated if and only if the matrix (4.3) has full column rank. Performing the following QR decomposition

$$\begin{bmatrix} A_{\sigma_k}S[k] & -W_{\sigma_k} \end{bmatrix} = \begin{bmatrix} R_{b\sigma_k} & 0 \end{bmatrix} Q_{b\sigma_k} \tag{4.10}$$

transforms the matrix $\begin{bmatrix} A_{\sigma_k}S[k] & -W_{\sigma_k} \end{bmatrix}$ into the product of two matrices, where $R_{b\sigma_k} \in \mathbb{R}^{q\times q}$ is an invertible triangular matrix and $Q_{b\sigma_k} \in \mathbb{R}^{(n+q)\times(n+q)}$ is an orthogonal matrix. Then writing

$$Q_{b\sigma_k} \begin{bmatrix} w_e[k] \\ w[k] \end{bmatrix} = \begin{bmatrix} w_{a\sigma_k}[k] \\ w_{b\sigma_k}[k] \end{bmatrix} \tag{4.11}$$

where

$$
\begin{aligned}
w_{a\sigma_k}[k] &\sim \mathcal{N}(0, I_q) \\
w_{b\sigma_k}[k] &\sim \mathcal{N}(0, I_n)
\end{aligned} \tag{4.12}
$$

we have

$$
\begin{bmatrix} A_{\sigma_k} S[k] & -W_{\sigma_k} \end{bmatrix} \begin{bmatrix} w_e[k] \\ w[k] \end{bmatrix} = R_{b\sigma_k} w_{a\sigma_k}[k] \tag{4.13}
$$

Note that $w_{a\sigma_k}[k]$ and $w_{b\sigma_k}[k]$ are independent of each other because $Q_{b\sigma_k}$ is an orthogonal matrix and also $w_e[k]$ and $w[k]$ are independent of each other. So the equation (4.9) can be written as

$$
\begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} x[k+1] + \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] + \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix} w_{c\sigma_k}[k] \tag{4.14}
$$

where

$$
w_{c\sigma_k}[k] = \begin{bmatrix} w_{a\sigma_k}[k] \\ v[k] \end{bmatrix} \sim \mathcal{N}(0, I_{q+p}) \tag{4.15}
$$

Hence we get

$$
\begin{aligned}
\begin{bmatrix} R_{b\sigma_k}^{-1} & 0 \\ 0 & V_{\sigma_k}^{-1} \end{bmatrix} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} &= \begin{bmatrix} R_{b\sigma_k}^{-1} E_{\sigma_k} \\ V_{\sigma_k}^{-1} C_{\sigma_k} \end{bmatrix} x[k+1] + \begin{bmatrix} -R_{b\sigma_k}^{-1} a_{\sigma_k} \\ V_{\sigma_k}^{-1} b_{\sigma_k} \end{bmatrix} \\
&\quad + \begin{bmatrix} -R_{b\sigma_k}^{-1} B_{\sigma_k} \\ 0 \end{bmatrix} u[k] + w_{c\sigma_k}[k]
\end{aligned} \tag{4.16}
$$

Applying the following QR decomposition

$$
\begin{bmatrix} R_{b\sigma_k}^{-1} E_{\sigma_k} \\ V_{\sigma_k}^{-1} C_{\sigma_k} \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} \tag{4.17}
$$

gives an invertible triangular matrix $R_{a\sigma_k} \in \mathbb{R}^{n \times n}$ and an orthogonal matrix $Q_{a\sigma_k} \in \mathbb{R}^{(q+p)\times(q+p)}$.

So replacing equation (4.17) in (4.16) yields

$$\begin{bmatrix} R_{b\sigma_k}^{-1} & 0 \\ 0 & V_{\sigma_k}^{-1} \end{bmatrix} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} x[k+1] + \begin{bmatrix} -R_{b\sigma_k}^{-1} a_{\sigma_k} \\ V_{\sigma_k}^{-1} b_{\sigma_k} \end{bmatrix} + \begin{bmatrix} -R_{b\sigma_k}^{-1} B_{\sigma_k} \\ 0 \end{bmatrix} u[k] + w_{c\sigma_k}[k] \tag{4.18}$$

which can be rewritten as

$$\begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k} u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix} = \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix} w_{c\sigma_k}[k] + \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix} Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} x[k+1] \tag{4.19}$$

If matrix (4.3) is full column rank, then the unknown vector $x[k+1]$ of the obsevation (4.19) can be estimated. Note that the rank of $\begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix}$ is equal to the rank of $R_{a\sigma_k}$ because rank of the product of several matrices is equal to the minimum rank. So treating (4.19) as an observation of the form (2.1) with

$$H = \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix} Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} \tag{4.20}$$

and noise covariance matrix

$$R = \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix} \begin{bmatrix} R_{b\sigma_k} & 0 \\ 0 & V_{\sigma_k} \end{bmatrix}^T \tag{4.21}$$

and applying relations (2.6) and (2.8), $\hat{x}[k+1]$ and $P_e[k+1]$ are found as

$$\hat{x}[k+1] = \Xi_k \begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k} u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix} \tag{4.22}$$

$$P_e[k+1] = S[k+1]S^T[k+1] = (R_{a\sigma_k}^T R_{a\sigma_k})^{-1} \tag{4.23}$$

where

$$\Xi_k = \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}^{\dagger} Q_{a\sigma_k}^T \begin{bmatrix} R_{b\sigma_k}^{-1} & 0 \\ 0 & V_{\sigma_k}^{-1} \end{bmatrix} \tag{4.24}$$

and † denotes the Moore-Penrose pseudoinverse of a matrix. Hence $\hat{x}[k]$ and $S[k]$ can be computed recursively. Assuming the matrix given by (4.3) is full column rank, the complete operation of the proposed estimation method is summarized in Algorithm 1.

Problem 4.2.1. *Given $\hat{x}[0]$, $S[0]$ and $y[k]$, estimate the state of a PWA singular system of the form (4.1).*

Solution Algorithm 1

- Initialization

  $\hat{x}[0] = \hat{x}_0$

  $S[0] = S_0$

- Iterative Step

  – Estimate active operation mode $\sigma_k \in \{1, ..., M\}$ based on the region to which $\hat{x}[k]$ belongs. The regions are defined as

  $$\mathscr{R}_{\sigma_k} = \{\hat{x} \mid \overline{L}_{\sigma_k}\overline{\hat{x}} > 0\}$$

  where $\overline{L}_{\sigma_k} = \begin{bmatrix} L_{\sigma_k} & l_{\sigma_k} \end{bmatrix}, \overline{\hat{x}} = \begin{bmatrix} \hat{x} \\ 1 \end{bmatrix}$.

  – Given $S[k]$, find $R_{b\sigma_k}$ and $Q_{b\sigma_k}$ using

  $$\begin{bmatrix} A_{\sigma_k}S[k] & -W_{\sigma_k} \end{bmatrix} = \begin{bmatrix} R_{b\sigma_k} & 0 \end{bmatrix} Q_{b\sigma_k}$$

  – Given $R_{b\sigma_k}$, find $Q_{a\sigma_k}$ and $R_{a\sigma_k}$ using

  $$\begin{bmatrix} R_{b\sigma_k}^{-1}E_{\sigma_k} \\ V_{\sigma_k}^{-1}C_{\sigma_k} \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}$$

– Given $\hat{x}[k]$, compute $\hat{x}[k+1]$ and $S[k+1]$

$$\Xi_k = \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}^{\dagger} Q_{a\sigma_k}^{T} \begin{bmatrix} R_{b\sigma_k}^{-1} & 0 \\ 0 & V_{\sigma_k}^{-1} \end{bmatrix}$$

$$\hat{x}[k+1] = \Xi_k \begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k}u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix}$$

$$S[k+1] = R_{a\sigma_k}^{-1}$$

4.2.3 Numerical Example

This section presents some simulation results obtained by applying the proposed algorithm 1 on an unknown-input system modeled as a singular system. Consider the following bimodal discrete-time PWA system taken from [40] and modified by adding an unknown input signal $d[k]$,

$$\begin{cases} x[k+1] = A_{\sigma_k}x[k] + a_{\sigma_k} + D_1 d[k] + W w[k] \\ y[k+1] = C_{\sigma_k}x[k+1] + b_{\sigma_k} + D_2 d[k] + V v[k] \\ \quad \sigma_k \in I = \{1,2\} \end{cases} \tag{4.25}$$

with

$$A_1 = \begin{bmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4969 & -0.2247 \\ 0.0798 & 0.9767 \end{bmatrix} \tag{4.26}$$

$$a_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix} \tag{4.27}$$

$$C_1 = C_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.28}$$

$$b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, D_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{4.29}$$

$$W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, V = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix} \tag{4.30}$$

$$w[k] \sim \mathcal{N}(0, I_2), \quad v[k] \sim \mathcal{N}(0, I_2) \tag{4.31}$$

$$\sigma_k = \begin{cases} 1 & if\, x_1 \leq 1.4 \\ 2 & if\, x_1 > 1.4 \end{cases} \tag{4.32}$$

We want to use the algorithm 1 to estimate the value of the unknown input signal $d[k]$. Therefore, defining the augmented state $\bar{x}[k] = \begin{bmatrix} x^T[k] & d^T[k] \end{bmatrix}^T$ then the previous system may be transformed into the following singular form

$$\begin{cases} E\bar{x}[k+1] = \breve{A}_{\sigma_k}\bar{x}[k] + a_{\sigma_k} + Ww[k] \\ y[k+1] = \breve{C}_{\sigma_k}\bar{x}[k+1] + b_{\sigma_k} + Vv[k] \\ \quad \sigma_k \in I = \{1, 2\} \end{cases} \tag{4.33}$$

where $E = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}$, $\breve{A}_{\sigma_k} = \begin{bmatrix} A_{\sigma_k} & D_1 \end{bmatrix}$ and $\breve{C}_{\sigma_k} = \begin{bmatrix} C_{\sigma_k} & D_2 \end{bmatrix}$. Note that since the matrices $\begin{bmatrix} E \\ \breve{C}_{\sigma_k} \end{bmatrix}$ have full column rank, then the state of the singular system (4.33) can be estimated. If we choose $\bar{x}[0] = \begin{bmatrix} 4.8 & -9 & 2.5 \end{bmatrix}^T$ as the initial condition for the real system together with $\hat{\bar{x}}[0] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and $S[0] = diag[0.1, 0.1, 10]$ as the initial conditions for the estimator, then the simulation results based on the proposed algorithm are shown in $Figures$4.1–4.4 which also verify the good performance of the proposed algorithm. In particular, the unknown input signal $d[k]$ with the following expression

$$d[k] = \begin{cases} 2.5 & k \leq 70 \\ -2sin(0.3k) & 70 < k \leq 150 \\ 3 & 150 < k \end{cases} \tag{4.34}$$

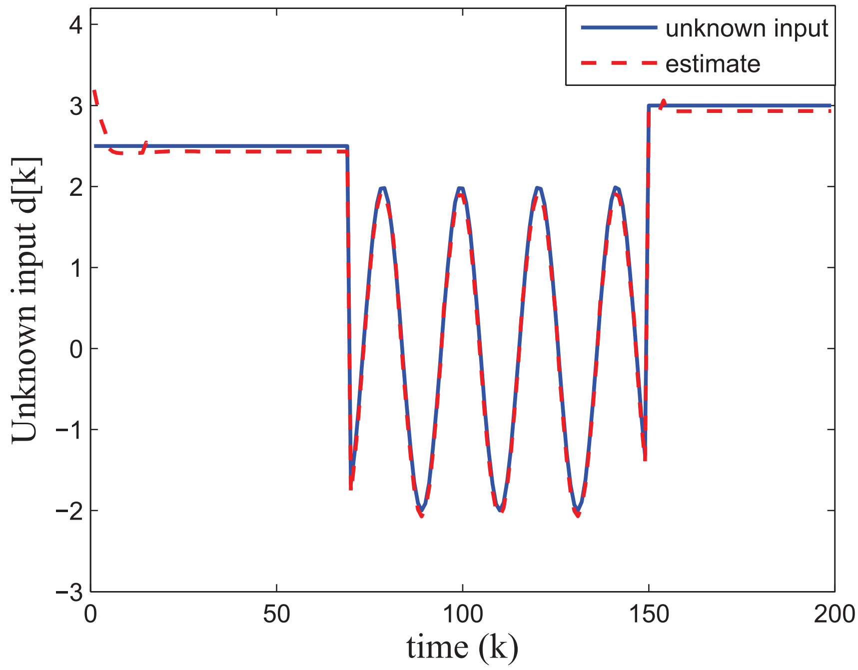

Figure 4.1: Unknown input and its estimate

is well estimated as depicted in *Figure* 4.1.

4.3 An Estimation Algorithm for PWA Singular Systems with Correlated Noise

This section presents the second recursive estimation algorithm for discrete-time PWA stochastic singular systems, where the process noise and measurement noise are assumed to be correlated.

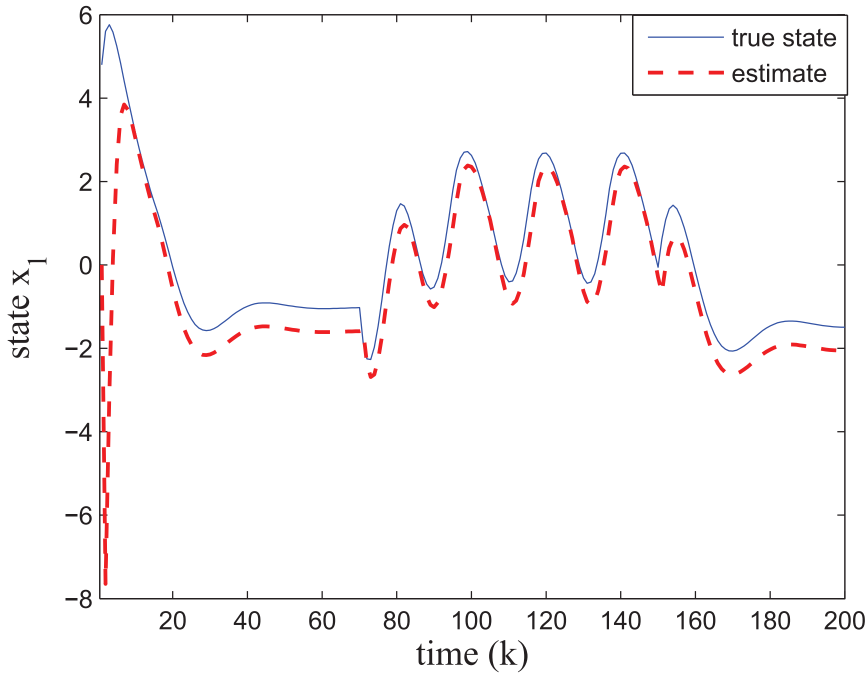

Figure 4.2: The true value of state $x_1[k]$ and its estimate

4.3.1 Discrete-time PWA Singular Systems with Correlated Noise

The discrete-time PWA stochastic singular system considered in this section is a state-based switched system described by the following dynamics

$$\begin{cases} E_{\sigma_k}x[k+1] = A_{\sigma_k}x[k] + a_{\sigma_k} + B_{\sigma_k}u[k] + \eta^p_{\sigma_k}[k] \\ \quad y[k+1] = C_{\sigma_k}x[k+1] + b_{\sigma_k} + \eta^m_{\sigma_k}[k] \\ \quad \forall x \in \mathscr{R}_{\sigma_k}, \sigma_k \in I = \{1, \ldots, M\} \end{cases} \tag{4.35}$$

where $E_{\sigma_k} \in \mathbb{R}^{q\times n}$, $A_{\sigma_k} \in \mathbb{R}^{q\times n}$, $B_{\sigma_k} \in \mathbb{R}^{q\times m}$, $C_{\sigma_k} \in \mathbb{R}^{p\times n}$ are real matrices, $x[k] \in \mathbb{R}^n$ is the state vector, $u[k] \in \mathbb{R}^m$ is the control input and $y[k] \in \mathbb{R}^p$ is the measured output. The random vectors $\eta^p_{\sigma_k}[k]$ and $\eta^m_{\sigma_k}[k]$ represent the correlated process and measurement noise,

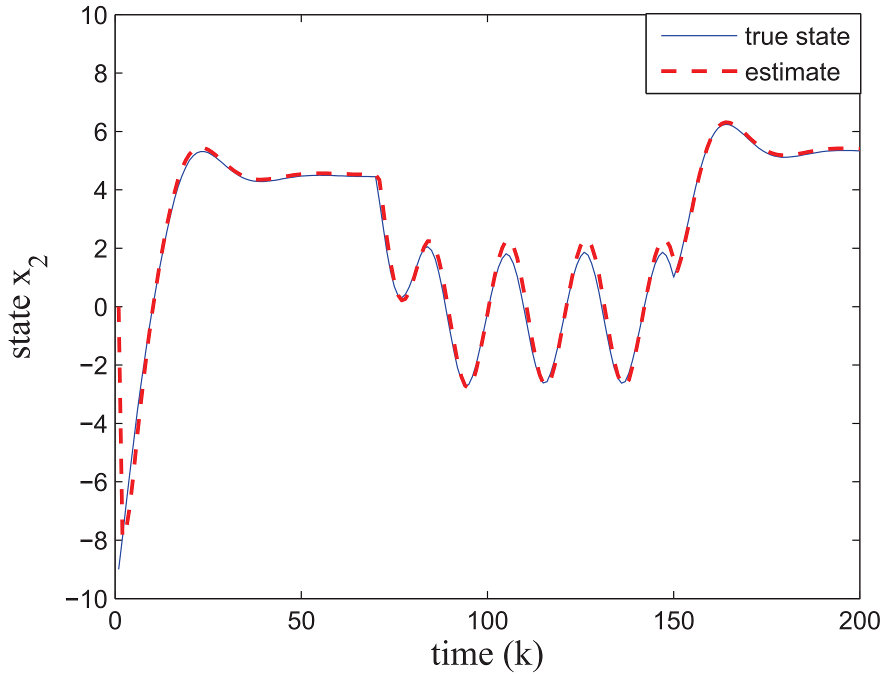

Figure 4.3: The true value of state $x_2[k]$ and its estimate

respectively, which can be modeled by

$$\begin{aligned} \eta^{p}_{\sigma_k}[k] &= \begin{bmatrix} W_{\sigma_k} & N_{\sigma_k} \end{bmatrix} \begin{bmatrix} w[k] \\ v[k] \end{bmatrix} \\ \eta^{m}_{\sigma_k}[k] &= \begin{bmatrix} N^{T}_{\sigma_k} & V_{\sigma_k} \end{bmatrix} \begin{bmatrix} w[k] \\ v[k] \end{bmatrix} \end{aligned} \tag{4.36}$$

where $W_{\sigma_k} \in \mathbb{R}^{q\times q}$, $V_{\sigma_k} \in \mathbb{R}^{p\times p}$ and $N_{\sigma_k} \in \mathbb{R}^{q\times p}$ are real matrices, $w[k] \sim \mathscr{N}(0, I_q)$ and $v[k] \sim \mathscr{N}(0, I_p)$ are Gaussian white noise vectors.

The switching signal σ_k defines the system mode at the time instant k. The constant vectors a_{σ_k} and b_{σ_k} are the affine terms for each affine model. Moreover, $\mathscr{R}_{\sigma_k}$, $\sigma_k \in \{1, ..., M\}$ is a polytopic partition of the state space defined as (4.2).

Assumptions:

1. Assumptions (4.3) and (4.4) hold.

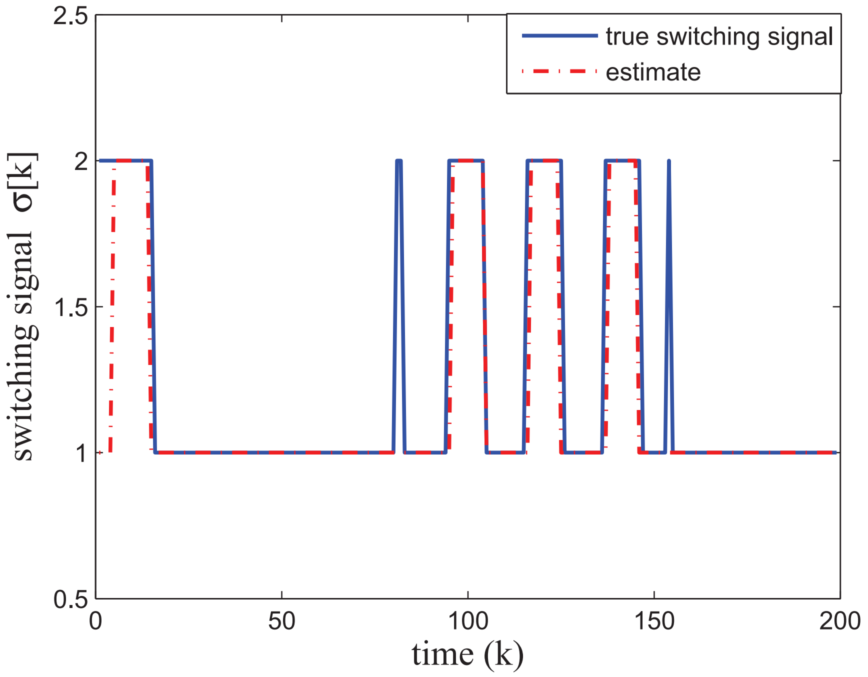

Figure 4.4: The true switching signal σ_k and its estimate

2. It is assumed that $q+p \geq n$, and also

$$\begin{bmatrix} W_{\sigma_k} & N_{\sigma_k} \\ N_{\sigma_k}^T & V_{\sigma_k} \end{bmatrix} \tag{4.37}$$

has full rank. Moreover, the random vectors $w[k] \sim \mathcal{N}(0, I_q)$ and $v[k] \sim \mathcal{N}(0, I_p)$ represent Gaussian white noise vectors.

4.3.2 Algorithm Derivation

In this section we develop a recursive estimation algorithm for discrete-time PWA singular systems of the form (4.35). The estimation problem is to reconstruct the state $x[k+1]$ at time $k+1$ from the given input and output sequences and the estimated operation mode. It is assumed that the state does not jump during the change of the operation mode.

Suppose that given knowledge of the process up to and including step k, there is an initial estimate for $x[k]$ denoted by $\hat{x}[k]$. As before, we assume that the estimation error can be written as $e[k] = S(k)w_e[k]$ where $w_e[k] \sim \mathcal{N}(0, I_n)$ is a Gaussian white noise vector independent of $w[k]$ and $v[k]$. Substituting (4.36) and $x[k]$ from relation (4.5) into (4.35) and using $\hat{x}$ to determine the region of operation yields

$$\begin{cases} E_{\sigma_k} x[k+1] = A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} - A_{\sigma_k} S[k] w_e[k] \\ \qquad\qquad + B_{\sigma_k} u[k] + W_{\sigma_k} w[k] + N_{\sigma_k} v[k] \\ y[k+1] = C_{\sigma_k} x[k+1] + b_{\sigma_k} + N_{\sigma_k}^T w[k] + V_{\sigma_k} v[k] \\ \forall \hat{x} \in \mathscr{R}_{\sigma_k}, \sigma_k \in I = \{1, ..., M\} \end{cases} \tag{4.38}$$

Then stacking the noise vectors as

$$w_o[k] = \begin{bmatrix} w_e^T[k] & w^T[k] & v^T[k] \end{bmatrix}^T \sim \mathcal{N}(0, I_{n+q+p}) \tag{4.39}$$

equation (4.38) can be rewritten as

$$\begin{aligned} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} &= \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} x[k+1] + \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] \\ &+ \begin{bmatrix} A_{\sigma_k} S[k] & -W_{\sigma_k} & -N_{\sigma_k} \\ 0 & N_{\sigma_k}^T & V_{\sigma_k} \end{bmatrix} w_o[k] \end{aligned} \tag{4.40}$$

According to equation (4.40), the unknown vector of the system can be estimated if and only if the following matrix

$$\begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} \tag{4.41}$$

is full column rank. Performing the following QR decomposition

$$\begin{bmatrix} A_{\sigma_k} S[k] & -W_{\sigma_k} & -N_{\sigma_k} \\ 0 & N_{\sigma_k}^T & V_{\sigma_k} \end{bmatrix} = \begin{bmatrix} R_{b\sigma_k} & 0 \end{bmatrix} Q_{b\sigma_k} \tag{4.42}$$

yields the triangular matrix $R_{b\sigma_k} \in \mathbb{R}^{(q+p)\times(q+p)}$ and the orthogonal matrix $Q_{b\sigma_k} \in \mathbb{R}^{(q+p+n)\times(q+p+n)}$. Then writing

$$Q_{b\sigma_k} w_o[k] = \begin{bmatrix} w_{a\sigma_k}[k] \\ w_{b\sigma_k}[k] \end{bmatrix} \tag{4.43}$$

where

$$\begin{aligned} w_{a\sigma_k}[k] &\sim \mathcal{N}(0, I_{q+p}) \\ w_{b\sigma_k}[k] &\sim \mathcal{N}(0, I_n) \end{aligned} \tag{4.44}$$

the equation (4.40) can be rewritten as

$$\begin{aligned} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} x[k+1] + \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] \\ + R_{b\sigma_k} w_{a\sigma_k}[k] \end{aligned} \tag{4.45}$$

Hence we get

$$\begin{aligned} R_{b\sigma_k}^{-1} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = R_{b\sigma_k}^{-1} \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} x[k+1] \\ + R_{b\sigma_k}^{-1} \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + R_{b\sigma_k}^{-1} \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] + w_{a\sigma_k}[k] \end{aligned} \tag{4.46}$$

Applying the following QR decomposition

$$R_{b\sigma_k}^{-1} \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} \tag{4.47}$$

gives an invertible triangular matrix $R_{a\sigma_k} \in \mathbb{R}^{n\times n}$ and an orthogonal matrix $Q_{a\sigma_k} \in \mathbb{R}^{(q+p)\times(q+p)}$. So replacing equation (4.47) in (4.46) yields

$$\begin{aligned} R_{b\sigma_k}^{-1} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] \\ y[k+1] \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} x[k+1] \\ + R_{b\sigma_k}^{-1} \begin{bmatrix} -a_{\sigma_k} \\ b_{\sigma_k} \end{bmatrix} + R_{b\sigma_k}^{-1} \begin{bmatrix} -B_{\sigma_k} \\ 0 \end{bmatrix} u[k] + w_{a\sigma_k}[k] \end{aligned} \tag{4.48}$$

which can be rewritten as

$$\begin{aligned} \begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k} u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix} = R_{b\sigma_k} w_{a\sigma_k}[k] \\ + R_{b\sigma_k} Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} x[k+1] \end{aligned} \tag{4.49}$$

If matrix (4.41) is full column rank, then the unknown vector of the obsevation (4.49) can be estimated (note that the rank of the product of a few matrices is equal to the minimum rank). So treating (4.49) as an observation of the form (2.1) with

$$H = R_{b\sigma_k} Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix} \tag{4.50}$$

and noise covariance matrix

$$R = R_{b\sigma_k} R_{b\sigma_k}^T \tag{4.51}$$

and applying relations (2.6) and (2.8), $\hat{x}[k+1]$ and $P_e[k+1]$ are found as

$$\hat{x}[k+1] = \Xi_k \begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k}u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix} \tag{4.52}$$

and

$$P_e[k+1] = S[k+1]S^T[k+1] = (R_{a\sigma_k}^T R_{a\sigma_k})^{-1} \tag{4.53}$$

where

$$\Xi_k = \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}^{\dagger} Q_{a\sigma_k}^T R_{b\sigma_k}^{-1} \tag{4.54}$$

Hence $\hat{x}[k]$ and $S[k]$ can be computed recursively. Assuming the unknown vector of the singular system (4.35) can be estimated, the complete operation of the proposed estimation method is summarized in Algorithm 2.

Problem 4.3.1. *Given $\hat{x}[0]$, $S[0]$ and $y[k]$, estimate the state of a PWA singular system of the form (4.35).*

Solution Algorithm 2

- Initialization

  $\hat{x}[0] = \hat{x}_0$

$S[0] = S_0$

- Iterative Step

  - Estimate active operation mode $\sigma_k \in \{1, ..., M\}$ based on the region to which $\hat{x}[k]$ belongs. The regions are defined as

$$\mathscr{R}_{\sigma_k} = \left\{\hat{x} \mid \overline{L}_{\sigma_k}\overline{\hat{x}} > 0\right\}$$

    where $\overline{L}_{\sigma_k} = \begin{bmatrix} L_{\sigma_k} & l_{\sigma_k} \end{bmatrix}, \overline{\hat{x}} = \begin{bmatrix} \hat{x} \\ 1 \end{bmatrix}$.

  - Given $S[k]$, find $R_{b\sigma_k}$ and $Q_{b\sigma_k}$ using

$$\begin{bmatrix} A_{\sigma_k}S[k] & -W_{\sigma_k} & -N_{\sigma_k} \\ 0 & N_{\sigma_k}^T & V_{\sigma_k} \end{bmatrix} = \begin{bmatrix} R_{b\sigma_k} & 0 \end{bmatrix} Q_{b\sigma_k}$$

  - Given $R_{b\sigma_k}$, find $Q_{a\sigma_k}$ and $R_{a\sigma_k}$ using

$$R_{b\sigma_k}^{-1} \begin{bmatrix} E_{\sigma_k} \\ C_{\sigma_k} \end{bmatrix} = Q_{a\sigma_k} \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}$$

  - Given $\hat{x}[k]$, compute $\hat{x}[k+1]$ and $S[k+1]$

$$\Xi_k = \begin{bmatrix} R_{a\sigma_k} \\ 0 \end{bmatrix}^{\dagger} Q_{a\sigma_k}^T R_{b\sigma_k}^{-1}$$

$$\hat{x}[k+1] = \Xi_k \begin{bmatrix} A_{\sigma_k}\hat{x}[k] + a_{\sigma_k} + B_{\sigma_k}u[k] \\ y[k+1] - b_{\sigma_k} \end{bmatrix}$$

$$S[k+1] = R_{a\sigma_k}^{-1}$$

Remark 4.3.1. *It is noted that in addition to addressing the correlated process and measurement noise, the derived Algorithm* 2 *is different from Algorithm* 1 *even if we set the correlation matrix* N_{σ_k} *to zero. The difference comes from the fact that the condition that the matrix* V_{σ_k} *be invertible is released here. Moreover, the QR decompositions are performed differently in these two algorithms. However, the computational burden of the latter algorithm is more than the former since the QR decompositions are performed on larger matrices.*

4.3.3 Numerical Example

This section presents some simulation results obtained by applying the proposed algorithm 2 on an unknown-input system modeled as a singular system. Consider the following bimodal discrete-time PWA system taken from [40] and modified by adding an unknown input signal $d[k]$ and also by applying correlated process and measurement noise,

$$\begin{cases} x[k+1] = A_{\sigma_k} x[k] + a_{\sigma_k} + D_1 d[k] + W w[k] + N v[k] \\ y[k+1] = C_{\sigma_k} x[k+1] + b_{\sigma_k} + D_2 d[k] + N^T w[k] + V v[k] \\ \qquad \sigma_k \in I = \{1,2\} \end{cases} \tag{4.55}$$

with

$$A_1 = \begin{bmatrix} 0.7969 & -0.2247 \\ 0.1798 & 0.9767 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4969 & -0.2247 \\ 0.0798 & 0.9767 \end{bmatrix} \tag{4.56}$$

$$a_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix} \tag{4.57}$$

$$C_1 = C_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.58}$$

$$b_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}, D_2 = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \tag{4.59}$$

$$W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, V = \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix}, N = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \tag{4.60}$$

$$w[k] \sim \mathcal{N}(0, I_2), \quad v[k] \sim \mathcal{N}(0, I_2) \tag{4.61}$$

$$\sigma_k = \begin{cases} 1 & if\, x_1 \leq 1.4 \\ 2 & if\, x_1 > 1.4 \end{cases} \tag{4.62}$$

We want to use the algorithm 2 to estimate the value of the unknown input signal $d[k]$. Therefore, defining the augmented state $\bar{x}[k] = \begin{bmatrix} x^T[k] & d^T[k] \end{bmatrix}^T$ then the previous system may be transformed into the following singular form

$$\begin{cases} E\bar{x}[k+1] = \breve{A}_{\sigma_k}\bar{x}[k] + a_{\sigma_k} + Ww[k] + Nv[k] \\ y[k+1] = \breve{C}_{\sigma_k}\bar{x}[k+1] + b_{\sigma_k} + N^T w[k] + Vv[k] \\ \quad \sigma_k \in I = \{1, 2\} \end{cases} \tag{4.63}$$

where $E = \begin{bmatrix} I_2 & 0_{2\times1} \end{bmatrix}$, $\breve{A}_{\sigma_k} = \begin{bmatrix} A_{\sigma_k} & D_1 \end{bmatrix}$ and $\breve{C}_{\sigma_k} = \begin{bmatrix} C_{\sigma_k} & D_2 \end{bmatrix}$. Note that since the matrices $\begin{bmatrix} E \\ \breve{C}_{\sigma_k} \end{bmatrix}$ have full column rank, then the state of the singular system (4.63) can be estimated. If we choose $\bar{x}[0] = \begin{bmatrix} 4.8 & -9 & 2.5 \end{bmatrix}^T$ as the initial condition for the real system together with $\hat{\bar{x}}[0] = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$ and $S[0] = diag[0.1, 0.1, 10]$ as the initial conditions for the estimator, then the simulation results based on the proposed algorithm are shown in *Figures*4.5–4.8 which also verify the good performance of the proposed algorithm. In particular, the unknown input signal $d[k]$ with the following expression

$$d[k] = \begin{cases} sin(0.1k) & k \leq 70 \\ 1.3cos(0.1k) & 70 < k \leq 150 \\ 2.5 & 150 < k \end{cases} \tag{4.64}$$

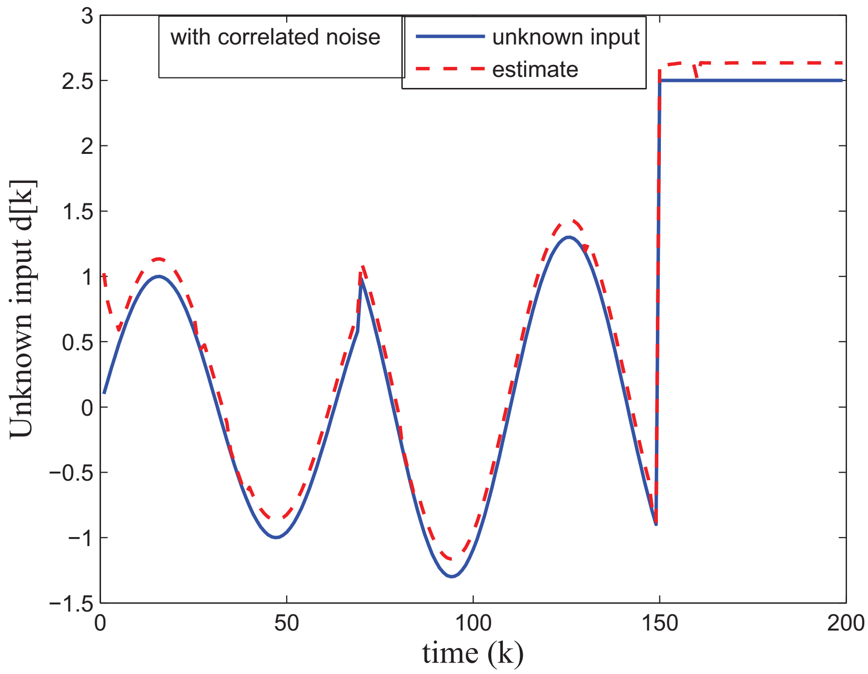

Figure 4.5: Unknown input and its estimate

is well estimated as depicted in *Figure* 4.5.

4.4 Summary

In this chapter two recursive estimation algorithms were proposed for discrete-time piecewise affine stochastic singular systems with simulation evidence that the idea works. The first algorithm was derived for systems with uncorrelated process noise and measurement noise, while the second one was derived for systems with correlated process and measurement noise. A maximum-likelihood (ML)-based estimation method was derived employing successive QR decompositions. As an important feature of this method, the covariance matrix is computed iteratively from its square root which always leads to nonnegative values. Finally, numerical simulations were presented on the case of a discrete-time PWA stochastic system with an unknown input, modeled as a singular system.

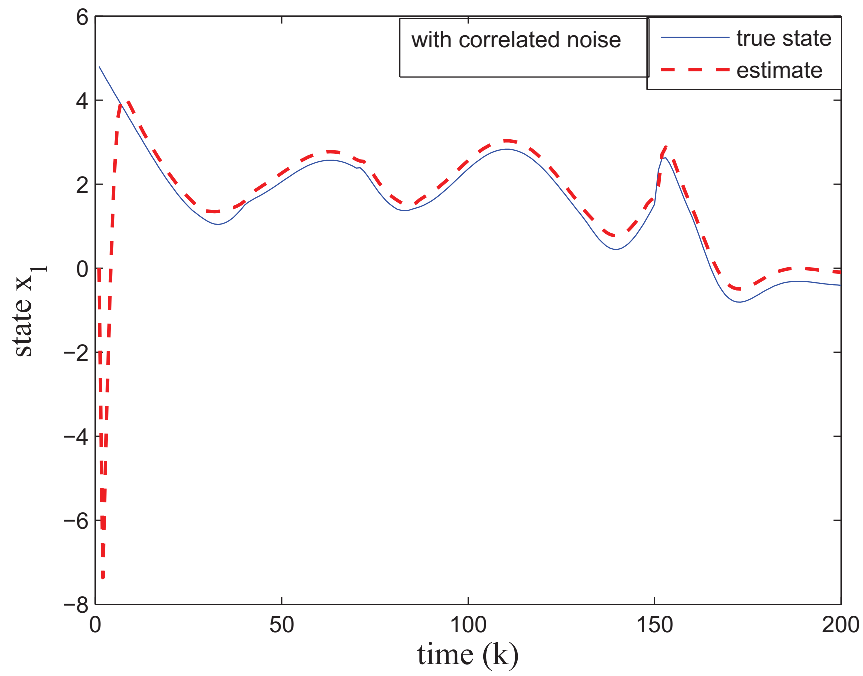

Figure 4.6: The true value of state $x_1[k]$ and its estimate

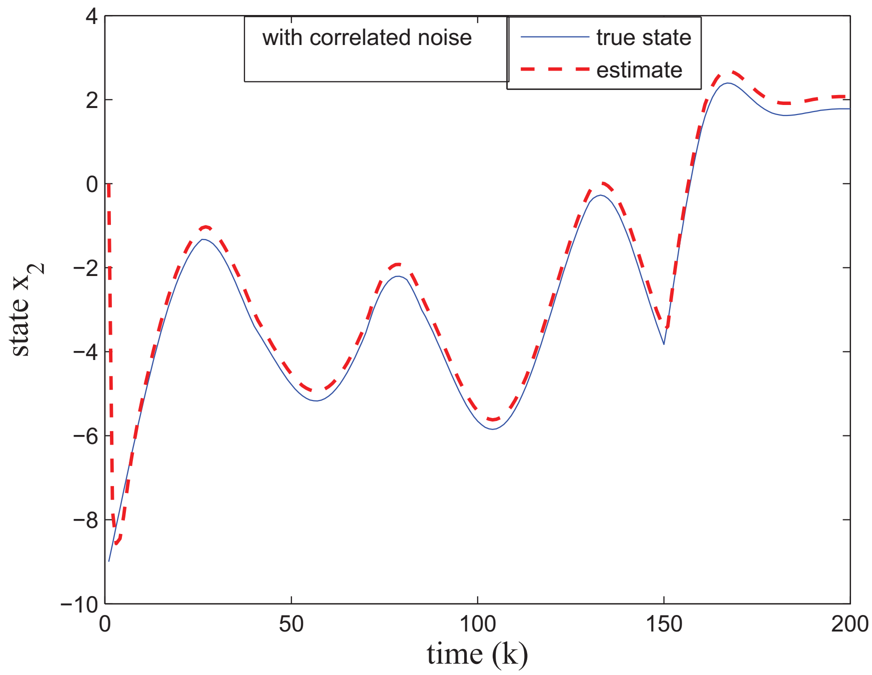

Figure 4.7: The true value of state $x_2[k]$ and its estimate

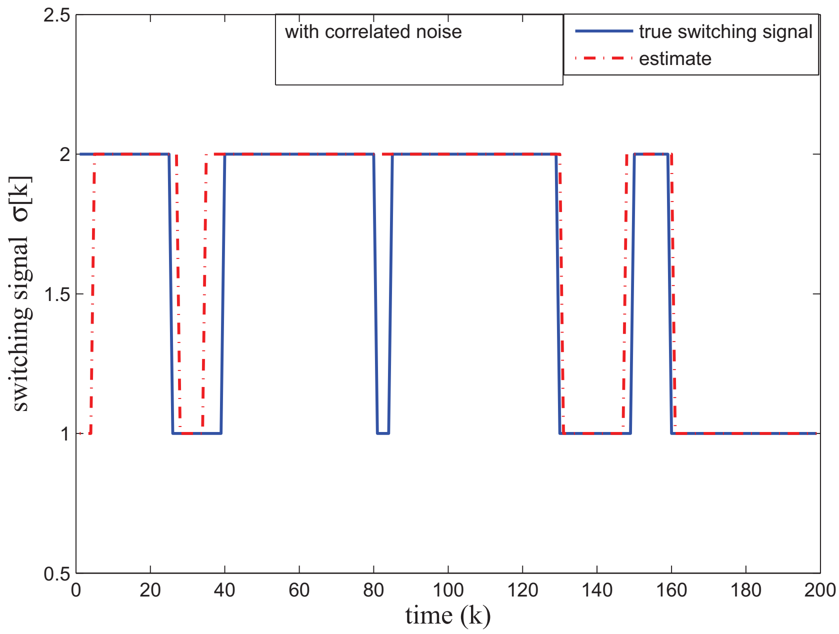

Figure 4.8: The true switching signal σ_k and its estimate

Chapter 5

Conclusions

In this chapter the contributions of the thesis are summarized and the conclusions from this research and potential future work are discussed. Chapter 2 reviews some preliminaries to be used in the following chapters. First, the maximum likelihood estimation problem for a linear observation is studied , and then the algorithmic details of the Continuous-Discrete Extended Kalman Filter (CDEKF) are described.

In Chapter 3, the solution to a third order nonlinear optimal control problem has been presented using the concept of inverse optimality. In this method, the Hamilton-Jacobi-Bellman equation is analytically solved for a class of third order nonlinear optimal control problems for which the dynamics are affine and the cost is quadratic in the input. One important feature of this approach is that the analytical solution for the control input is obtained directly without needing to assume or compute a coordinate transformation, value function or Lyapunov function. The value function and a Lyapunov function can however be computed after the control input has been found. The controller was applied to a path following problem of a Wheeled Mobile Robot (WMR). A Continuous-Discrete Extended Kalman Filter (CDEKF) was also designed to provide the estimated states from noisy measurements for constructing the control law. The simulation results verified the good performance of the combined observer-controller synthesis. Noting that the necessary sensors for full-state feedback may not always be available, for example, the angular velocity ω in the path following problem of WMR might not be available, motivates us to investigate output feedback inverse optimal controllers as another remedy to this problem. Furthermore, the proposed method can potentially be extended to higher order systems assuming that the dynamics are affine and the cost is quadratic in the input.

In Chapter 4, two recursive estimation algorithms were proposed for discrete-time piecewise affine stochastic singular systems with simulation evidence that the idea works. The first algorithm was derived for systems with uncorrelated process noise and measurement noise, and then the second one was derived for systems with correlated process and measurement noise. A maximum-likelihood (ML)-based estimation method was derived employing successive QR decompositions. As an important feature of this method, the covariance matrix is computed iteratively from its square root which always leads to non-negative values. Finally, numerical simulations were presented for the case of a discrete-time PWA stochastic system with an unknown input, modeled as a singular system. In this research work the active operation mode is determined based on the estimated state $\hat{x}[k]$, where the condition governing the jump is deterministic. Extending the current work to consider stochastic jump from one mode to another (such as with multiple model filters) can be addressed in the future.

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