Frobenius structures, integrable systems and Hurwitz spaces

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A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy at
Concordia University.
Montréal, Québec, Canada.

October 2004

Your file Votre référence
ISBN: 0-494-04045-9
Our file Notre référence
ISBN: 0-494-04045-9

Abstract

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This thesis consists of two main parts. In the first part a new family of integrable systems related to Hurwitz spaces of elliptic coverings with simple branch points is constructed. The integrable systems are closely related to Takasaki's version of the Schlesinger system on an elliptic surface. A trigonometric degeneration of the integrable systems is presented. The trigonometric version of an auxiliary system of differential equations for the images of branch points of the covering under a uniformization map with respect to branch points is derived. This system is applied to solving the Boyer-Finley equation (self-dual Einstein equation with a rotating Killing vector). Thereby, a class of implicit solutions to the Boyer-Finley equation is found in terms of objects related to the Hurwitz spaces.

The second part presents two classes of new semisimple Frobenius structures on Hurwitz spaces (spaces of ramified coverings of $\mathbb{C}P^1$). The original construction of Hurwitz Frobenius manifolds by Dubrovin is described in terms of the normalized meromorphic bidifferential W of the second kind on a Riemann surface. In Dubrovin's construction, the branch points $\{\lambda_m\}$ of the covering play the role of canonical coordinates on the Hurwitz Frobenius manifolds. We find new Frobenius structures on Hurwitz spaces with coordinates $\{\lambda_m; \bar{\lambda}_m\}$ in terms of the Schiffer and Bergman kernels (bidifferentials) on a Riemann surface. We call these structures the "real doubles" of the Hurwitz Frobenius manifolds of Dubrovin.

To construct another class of new Frobenius structures on Hurwitz spaces, we introduce a $g(g+1)/2$-parametric deformation of the bidifferential W, where g is the genus of the corresponding Riemann surface. Analogously to the bidifferential W, its deformation defines Frobenius structures on Hurwitz spaces; these structures give a $g(g+1)/2$-parametric deformation of Dubrovin's Hurwitz Frobenius manifolds. Similarly, we introduce the deformations of the Schiffer and Bergman kernels which define Frobenius structures on the Hurwitz spaces with coordinates $\{\lambda_m; \bar{\lambda}_m\}$. Thereby we obtain deformations of the real doubles of the Hurwitz Frobenius manifolds of Dubrovin. Each new Frobenius structure gives a new solution to the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) system. For the simplest Hurwitz space in genus one, the corresponding solutions are found explicitly, together with the corresponding G-function.

Acknowledgments

I am grateful to professor D. Korotkin for guidance, help and constant attention to my work.

Contents

Introduction

Frobenius manifold theory and the theory of integrable systems are wide areas related to each other and to many other branches of mathematics.

The theory of integrable systems deals with nonlinear differential equations arising in physics. During the last 40 years, a number of effective methods were developed for solving partial differential equations that are represented as a compatibility condition of some auxiliary linear system (the U-V pair) of matrix differential equations. Such U-V pairs were found for many well-known differential equations, such as Korteweg-de Vries, nonlinear Shrödinger, sin-Gordon, Ernst, Boussinesq equations and so on.

In the early 90's the fundamental role of the Korteweg-de Vries equation (KdV) in matrix models of two-dimensional gravity and in the topology of moduli spaces of algebraic curves was established [76, 46]. Further development of two-dimensional topological field theory led to the appearance of the geometric and analytic theory of Frobenius manifolds. Frobenius manifolds were introduced by Dubrovin as a geometrical, coordinate free setting for the so-called WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations of associativity. The theory of Frobenius manifolds turns out to be related to rather distant branches of mathematics, such as Gromov - Witten invariants of algebraic varieties, singularity theory, isomonodromy deformations, and the theory of reflection groups and their extensions.

Deformations of Riemann surfaces appear in different aspects of the theory of integrable systems: in the theory of systems of hydrodynamic type [14, 50], theory of algebro-geometric solutions of equations with variable spectral parameter [47], and in the theory of Frobenius manifolds [15].

In particular, in 1978, Maison [55] and Belinskii, Zakharov [3] discovered the integrability of the Ernst equation (stationary axially symmetric Einstein equation); the U-V pair they found depends on a variable spectral parameter γ which is a function of a hidden constant spectral parameter λ and two variables x and y . The function $\gamma(\lambda; x, y)$ coincides with a uniformization map of a genus zero Riemann surface represented as a two-fold covering of the λ-sphere with two branch points at $\lambda = x$ and $\lambda = y$.

In 1987, Burtzev, Mikhailov and Zakharov [9] studied possible deformations of a generic U-V pair and derived a system of differential equations which the variable spectral parameter should satisfy in order for the U-V pair to define a compatible linear system. These equations for the variable spectral parameter were recently solved in [42] in terms of spaces of rational functions of one variable. The coordinates on these spaces are given by the critical values $\{\lambda_m\}$ of the functions. A rational function defines a genus zero covering of $\mathbb{C}P^1$ with branch points given by its critical values $\{\lambda_m\}$. Thus, the deformations of U-V pairs were associated with the simplest Hurwitz spaces – spaces of rational functions of a given degree. In the case of the space of rational functions of degree two these deformations lead to the Ernst equation.

The deformed linear system associated with the space of rational functions of given degree has the following form with respect to the coordinates $\{\lambda_m\}$ [42]:

$$\frac{d\Psi}{d\lambda_m} = U_m \Psi \ , \tag{0.0.1}$$

where the matrix $U_m(P, \{\lambda_n\})$ (P is a point on the covering) has only one simple pole at the ramification point P_m corresponding[1] to the branch point λ_m and does not have any other singularities. The compatibility condition of the linear system (0.0.1) gives new integrable nonlinear PDEs for each degree of the rational function.

Analogously to the link [49] between the Ernst equation and the Schlesinger system, each PDE of this new class has a subset of solutions corresponding to isomonodromic deformations of ordinary differential equation with meromorphic matrix coefficients on the Riemann sphere:

$$\frac{d\Phi}{d\gamma} = A(\gamma)\Phi \ , \qquad A(\gamma) = \sum_j \frac{A_j}{\gamma - z_j} \ , \tag{0.0.2}$$

where A_j are matrices independent of γ and such that $\sum_j A_j = 0$.

A solution $\Phi(\gamma)$ to the equation (0.0.2) can be normalized by $\Phi(\gamma_0) = I$ at some fixed point γ_0 ; it has regular singularities at the points $\gamma = z_j$ and is generically non-single valued on the γ-sphere. The isomonodromic deformations of the system (0.0.2) are induced by variations of the positions of the regular singular points $\{z_j\}$ which preserve the monodromies of the function Φ around $\{z_j\}$. The isomonodromy assumption implies, in the generic case, the following equations for the function Φ with respect to positions of singularities:

$$\frac{d\Phi}{dz_n} = \left(\frac{A_n}{\gamma_0 - z_n} - \frac{A_n}{\gamma - z_n} \right) \Phi \ , \tag{0.0.3}$$

[1] A point on the covering common for different sheets is called a ramification point, and its projection on the base of the covering is called a branch point.

and the Schlesinger system on the matrices A_j $(j \neq k)$:

$$\frac{\partial A_j}{\partial z_k} = \frac{[A_j, A_k]}{z_j - z_k} - \frac{[A_j, A_k]}{\gamma_0 - z_k}\,, \qquad \frac{\partial A_j}{\partial z_j} = -\sum_{k \neq j} \left(\frac{[A_k, A_j]}{z_k - z_j} - \frac{[A_k, A_j]}{z_k - \gamma_0} \right)\,. \tag{0.0.4}$$

Consider now a rational map $\nu(P)$ of degree N with simple critical points and critical values $\{\lambda_m\}_{m=1}^{2N-2}$. Then, the function $\Phi(\nu(P))$, where Φ is a solution to differential equation (0.0.2) on the Riemann sphere, satisfies the linear system (0.0.1) with respect to the variables $\{\lambda_m\}$; the meromorphic functions U_m (and therefore, solutions to the associated nonlinear integrable PDE) can be expressed in terms of the solutions A_j to the Schlesinger system (0.0.4) [42].

The system (0.0.4) appeared in the classical work of Schlesinger [67]. In the early 80's a major contribution to the theory of isomonodromic deformations of system (0.0.4) was made in the works [38, 39] where, in particular, the tau-function of the Schlesinger system was introduced.

The natural question of generalizing the theory of isomonodromic deformations on the sphere to higher genus Riemann surfaces was addressed, for example, in [37, 61, 62]. The most naive generalization of the Schlesinger system (0.0.4) to higher genus meets a fundamental difficulty: a function with just a single pole which could appear in (0.0.3) does not exist on a surface of higher genus. Therefore, the framework of isomonodromic deformations has to be modified. In [62], for example, the isomonodromic deformations of the equation $d_\gamma \Phi = A(\gamma)\Phi$ on a torus with meromorphic connection $A(\gamma)$ having poles of order higher than one were considered. Another alternative is to allow $A(\gamma)$ to be non-single valued.

An analog of the isomonodromic deformations on a torus with a non-single valued matrix $A(\gamma)$ was proposed in [70]. We use a similar idea of introducing non-single valued analogs of matrices U_m on a torus to find an elliptic counterpart of the rational linear system (0.0.1). Namely, we consider the following linear system for a matrix function Ψ on the torus:

$$\frac{d \overset{1}{\Psi}(P)}{d\lambda_m} = \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (\nu(P) - \gamma_m) \overset{2}{J_m} \right) \overset{1}{\Psi}(P) , \tag{0.0.5}$$

where $\overset{12}{r}$ is the classical elliptic r-matrix, J_m are matrices depending only on the variables $\{\lambda_m\}$; $\nu(P)$ is the uniformization map of the elliptic covering to the fundamental parallelogram. The compatibility condition of the linear system (0.0.5) gives a family of new integrable systems on the space of elliptic coverings with simple branch points. The role of the variable spectral parameter is played by the uniformization map $\nu(P)$ of the torus. We find differential equations which describe the dependence of the uniformization map on the branch points of the elliptic covering. In particular, we obtain the following system on the images $\{\gamma_m\}$ of the ramification points under the uniformization map as functions of the branch points $\{\lambda_m\}$ of the covering:

$$\frac{\partial \gamma_n}{\partial \lambda_m} = -\alpha_m[\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)] , \qquad \frac{\partial \gamma_m}{\partial \lambda_m} = \sum_{k=1,\, k\neq m}^{2N} \alpha_k[\rho(\gamma_m - \gamma_k) + \rho(\gamma_k)]. \tag{0.0.6}$$

where $m \neq n$; and $\rho(\gamma)$ is the logarithmic derivative of the odd Jacobi theta function: $\rho(\gamma) = d_\gamma \log \theta_1(\gamma)$. The coefficients $\{\alpha_m\}$ obey the following equations:

$$\frac{\partial \alpha_n}{\partial \lambda_m} = -2\, \alpha_n \alpha_m \rho'(\gamma_n - \gamma_m) \; ; \qquad \frac{\partial \alpha_m}{\partial \lambda_m} = \sum_{k=1,\, k\neq m}^{2N} 2\, \alpha_k \alpha_m \rho'(\gamma_k - \gamma_m) \, . \tag{0.0.7}$$

The system (0.0.6)-(0.0.7) is an elliptic analog of the following systems on the critical points $\{\gamma_m\}$ of the rational maps with respect to its critical values $\{\lambda_m\}$ a part of which appeared in the works by Kupershmidt, Manin [53] and Gibbons, Tsarev [29]:

$$\frac{\partial\gamma_n}{\partial\lambda_m} = \frac{\alpha_m}{\gamma_m - \gamma_n}\,, \qquad \frac{\partial\gamma_m}{\partial\lambda_m} = 1 + \sum_{k=1,\,k\neq m}^{2N-2} \frac{\alpha_k}{\gamma_m - \gamma_k}\,;$$

$$\frac{\partial\alpha_n}{\partial\lambda_m} = 2\,\frac{\alpha_n\alpha_m}{(\gamma_n - \gamma_m)^2}\,, \qquad \frac{\partial\alpha_m}{\partial\lambda_m} = -\sum_{k=1,\,k\neq m}^{2N-2} \frac{2\,\alpha_k\alpha_m}{(\gamma_k - \gamma_m)^2}\,;$$

for $m \neq n$.

Similarly to the case of the Riemann sphere, we establish a correspondence between the proposed linear system on the torus (0.0.5) and Takasaki's version of isomonodromy deformation equations on elliptic surface [70].

The construction of a new class of integrable systems associated with Hurwitz spaces in genus one, together with the link to the elliptic Schlesinger system, is the first main result of this thesis.

Let us now turn to a description of the second main result. Elliptic functions on a genus one covering transform into trigonometric ones when two simple branch points of the covering tend to each other so that the torus degenerates to an infinite cylinder. A part of the corresponding trigonometric degeneration of the system (0.0.6)-(0.0.7) of equations for the images γ_m of ramification points under the uniformization map $\nu(P)$ with respect to simple branch points λ_n of the covering has the form (for $m \neq n$):

$$\frac{\partial\gamma_m}{\partial\lambda_n} = -\pi\alpha_n\Big(\cot\pi(\gamma_m - \gamma_n) + \cot\pi\gamma_n\Big)\,; \qquad \frac{\partial\alpha_n}{\partial\lambda_m} = 2\pi^2\frac{\alpha_n\alpha_m}{\sin^2\pi(\gamma_n - \gamma_m)}. \tag{0.0.8}$$

We apply the system (0.0.8) to a construction of solutions to the Boyer-Finley equation (Chapter 3). The Boyer-Finley equation [7]

$$U_{xy} = (e^U)_{tt} \tag{0.0.9}$$

is a 3-dimensional integrable system which has been much studied by many authors, see [10, 74] and references therein. It arises as a reduction of the self-dual Einstein equations (the equations governing any metric with self-dual Weyl tensor and vanishing Ricci tensor) for Euclidean space-times with rotational symmetry. The Boyer-Finley equation can also be obtained as the dispersionless limit of the Toda lattice equation. In the works [51, 52] this relation to the Toda lattice equation was used to construct a class of solutions to the Boyer-Finley equation by averaging an appropriate two-point Baker-Akhiezer function.

The idea of construction of solutions to the Boyer-Finley equation proposed in our paper [24] is to reduce the Boyer-Finley equation to two systems: an integrable system of hydrodynamic type and to the trigonometric system (0.0.8). The system of hydrodynamic type is solved by the generalized hodograph method (the theory of integrable systems of hydrodynamic type is systematically presented in the review [73]). Combining solutions to the system (0.0.8) in terms of spaces of trigonometric functions with generalized hodograph method we obtain a wide class of implicit solutions to the Boyer-Finley equation (0.0.9).

Let us now turn to a description of the last, and, probably, the most important, results of this thesis – the construction of new classes of Frobenius manifolds associated with Hurwitz spaces.

The structure of a Frobenius manifold [15] can be defined in terms of one quasihomogeneous function of n variables which satisfies the WDVV equation

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i \;, \quad i,j = 1,\ldots,n \,. \tag{0.0.10}$$

Here F_i is the matrix of third derivatives $(F_i)_{mn} = \partial^3_{t^i t^m t^n} F$, and the matrix F_1 is assumed to be constant and nondegenerate. The matrix F_1 determines the Darboux-Egoroff metric on the Frobenius manifold. Conversely, for a Frobenius manifold there exists a quasihomogeneous function, called the prepotential, which satisfies equations (0.0.10).

The Frobenius structures play an important role in algebraic geometry and two-dimensional topological field theory. From the point of view of physics, some solutions to the WDVV system describe the moduli space of topological conformal field theories. In mathematics, solutions to the WDVV system (0.0.10) possessing appropriate analytic properties define generating functions for the Gromov-Witten invariants of some compact algebraic varieties [76]. Frobenius structures also appear in the study of the mirror symmetry of Calabi-Yau manifolds [77], singularity theory [57, 65], Coxeter and generalized Jacobi groups [4].

Locally, any Frobenius manifold, under some genericity assumptions, can be described by a Frobenius manifold structure on a Hurwitz space ([15], App.I). Originally, Frobenius structures on Hurwitz spaces were found by Dubrovin [15], see also [57, 64]. The prepotential of Dubrovin's Hurwitz Frobenius manifold, i.e. solution to the WDVV system, for the simplest three-dimensional Hurwitz space in genus one (consisting of meromorphic functions of degree 2 on a torus), for example, has the form:

$$F = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32}t_2^4 \gamma\left(2\pi i t_3\right) . \tag{0.0.11}$$

Here $\gamma(\mu) \;=\; 4 d_\mu \log \eta(\mu)$, where $\eta(\mu)$ is the Dedekind eta-function given by $\eta(\mu) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ for $q = e^{2\pi i \mu}$. The function (0.0.11) is quasihomogeneous with coefficients of quasihomogeneity given by 1, 1/2, 0 : the relation

$$F(\kappa t_1, \kappa^{1/2} t_2, \kappa^0 t_3) = \kappa^2 F(t_1, t_2, t_3)$$

holds for any $\kappa \neq 0$.

As we have mentioned above, a straightforward generalization of the linear system (0.0.1) from the sphere to higher genus Riemann surfaces runs into difficulties due to the absence of a function with a single first order pole on a surface of positive genus. For a surface of genus one it was still possible to find a linear system (0.0.5) and explicitly develop a scheme leading to integrable systems analogously to the genus zero case. On a surface of genus greater than one an analogous procedure becomes complicated since the transformations of the matrices U_m also depend on the variables $\{\lambda_n\}$ of the system. However, for the scalar systems (0.0.1) there exists [42] a natural analog on higher genus surfaces. For a scalar system in arbitrary genus the functions U_m are proportional to integrals of the canonical meromorphic bidifferential $W(P,Q)$ (see Chapter 1, (1.2.19)) evaluated with respect to one of the arguments at the ramification point P_m.

Solutions to the integrable systems which appear in the scalar case define the Darboux-Egoroff metrics (see Section 1.6.1 for definition) on Hurwitz spaces with coordinates given by the branch points $\{\lambda_m\}$ of the corresponding coverings. These Darboux-Egoroff metrics can be explicitely constructed in terms of the canonical meromorphic bidifferential W as follows:

$$\mathbf{ds^2} = \sum_{m=1}^{L} \left(\oint_l h(Q) W(Q, P_m) \right)^2 (d\lambda_m)^2 \,. \qquad (0.0.12)$$

Here l is an arbitrary smooth contour on the Riemann surface $\mathcal{L}$ not passing through ramification points P_m ; $h(Q)$ is an arbitrary independent of $\{\lambda_m\}$ function defined in a neighbourhood of the contour l.

The Darboux-Egoroff metrics play an important role in the theory of Frobenius manifolds.

It turns out (Chapter 4) that the Darboux-Egoroff metrics of the Hurwitz Frobenius structures of Dubrovin are contained in the family of metrics (0.0.12). Thereby, the whole construction of Hurwitz Frobenius manifolds in [15] can be described in terms of the canonical meromorphic bidifferential W. The local coordinates $\{\lambda_m\}$ on the Hurwitz space give the canonical coordinates on the Frobenius manifolds. The Frobenius structures of Dubrovin, as well as the bidifferential W, do not depend on the complex conjugates $\{\bar{\lambda}_m\}$.

In this thesis (Section 4.5) we find new Frobenius structures on Hurwitz spaces with coordinates $\{\lambda_m; \bar{\lambda}_m\}$. These structures are built in terms of the Schiffer and Bergman kernels [23].

The Schiffer kernel $\Omega(P,Q)$ is the bidifferential with a singularity of the form $(x(P)-x(Q))^{-2}dx(P)dx(Q)$ along the diagonal $P=Q$ such that $p.v.\iint_{\mathcal{L}}\Omega(P,Q)\ \overline{\omega(P)}$ vanishes for any holomorphic differential ω on the surface $\mathcal{L}$. The Bergman kernel $B(P,\bar{Q})$ is a regular bidifferential on $\mathcal{L}$ holomorphic with respect to its first argument and anti-holomorphic with respect to the second one which (up to a factor of $-2\pi i$) is a kernel of an integral operator acting in the space $L_2^{(1,0)}(\mathcal{L})$ of (1,0)-forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic (1,0)-forms. In particular, the following holds for any holomorphic differential ω on the surface $\mathcal{L}$: $\iint_{\mathcal{L}} B(P,\bar{Q})\ \omega(Q) = -2\pi i\omega(P)$. Both kernels, $\Omega(P,Q)$ and $B(P,\bar{Q})$, are independent of the choice of canonical basis of cycles on $\mathcal{L}$. For a surface of genus zero, the Schiffer kernel coincides with the bidifferential W; the Bergman kernel in genus zero vanishes. The two kernels depend on the coordinates $\{\lambda_m\}$ on the Hurwitz space as well as on their complex conjugates $\{\bar{\lambda}_m\}$.

In our framework, the Schiffer and Bergman kernels play a role similar to the role of the bidifferential W in the construction of Dubrovin's Hurwitz Frobenius manifolds.

We call the new Frobenius structures the "real doubles" of Dubrovin's Hurwitz Frobenius manifolds, since they depend on $\{\lambda_m\}$ and $\{\bar{\lambda}_m\}$ and, therefore, have a dimension twice as large as the dimension of the Hurwitz Frobenius structures of Dubrovin.

For example, for the simplest Hurwitz space in genus one (which in Dubrovin's construction leads to the prepotential (0.0.11)), we obtain the following new quasihomogeneous solution to the WDVV system depending on six variables:

$$
\begin{aligned}
F = & -\frac{1}{4}t_1t_2^2 - \frac{1}{4}t_1t_5^2 + \frac{1}{2}t_1^2t_3 - \frac{1}{2}t_1t_4(2t_6 - \frac{1}{2\pi i}) \\
& + t_3^{-1}\left(\frac{1}{4}t_2^2t_4(t_6 - \frac{1}{2\pi i}) + \frac{1}{4}t_4t_5^2t_6 + \frac{1}{2}t_4^2t_6(t_6 - \frac{1}{2\pi i}) + \frac{1}{16}t_2^2t_5^2\right) \\
& + \frac{1}{32}t_2^4\left(-\frac{1}{4\pi i}t_6^{-2}\,\gamma\left(\frac{t_3}{t_6}\right) + t_3^{-1} - \frac{1}{2\pi i}t_3^{-1}t_6^{-1}\right) \\
& + \frac{1}{32}t_5^4\left(-\frac{\pi i}{(2\pi i t_6 - 1)^2}\,\gamma\left(\frac{2\pi i t_3}{1 - 2\pi i t_6}\right) + t_3^{-1} + t_3^{-1}(2\pi i t_6 - 1)^{-1}\right).
\end{aligned}
\tag{0.0.13}
$$

The coefficients of quasihomogeneity of this solution are given by $1, 1/2, 0, 1, 1/2, 0$; and the following relation holds for any $\kappa \neq 0$:

$$
F(\kappa t_1, \kappa^{1/2}t_2, t_3, \kappa t_4, \kappa^{1/2}t_5, t_6) = \kappa^2 F(t_1, t_2, t_3, t_4, t_5, t_6)\ .
$$

The function (0.0.13) is the prepotential of the Frobenius manifold which gives the "real double" of the Frobenius manifold corresponding to the prepotential (0.0.11).

In [26] Getzler derived a system of linear differential equations for a generating function of the genus one Gromov-Witten invariants of smooth projective varieties. This system is defined on any semisimple Frobenius manifold. In [17] Getzler's system was proven to have a unique quasihomogeneous solution, the G-function, given by the formula $G = \log(\tau_I/J^{1/24})$ where τ_I is the isomonodromic tau-function of the Frobenius manifold (see (1.8.21)) and J is the Jacobian of the transformation from canonical coordinates $\{\lambda_m\}$ on the Frobenius manifolds to the flat coordinates $\{t_i\}$. In [44] an expression for the isomonodromic tau-function and, therefore, for the G-function of Dubrovin's Hurwitz Frobenius manifods was obtained in terms of classical objects on a Riemann surface. For the structures of real doubles, the G-function is given up to an additive constant by the following formula:

$$G = -\frac{1}{2}\log\left\{\, |\tau_W|^2 \det(\mathrm{Im}\mathbb{B})\right\} - \frac{1}{24}\log\left\{\prod_{i=1}^{L}\Phi_{(1,0)}(P_i)\Phi_{(0,1)}(P_i)\right\}\,, \tag{0.0.14}$$

where τ_W is the Bergman tau-function computed in [44], (see Section 1.3.4 of this thesis); $\mathbb{B}$ is the matrix of b-periods of the Riemann surface and $\Phi(P) = \Phi_{(1,0)}(P) + \Phi_{(0,1)}(P)$ is the primary differential which corresponds to the Frobenius manifold. (The primary differentials Φ decompose into a sum of their holomorphic $\Phi_{(1,0)}$ and antiholomorphic $\Phi_{(0,1)}$ parts.)

For the Frobenius manifold corresponding to the prepotential (0.0.13), the solution (0.0.14) to Getzler's system as a function of flat coordinates is given by

$$G = -\log\left\{\eta\left(\frac{t_3}{t_6}\right)\eta\left(\frac{2\pi i t_3}{1-2\pi i t_6}\right)(t_2t_5)^{\frac{1}{8}}\left(\frac{2\pi i t_3}{t_6(2\pi i t_6 - 1)}\right)^{\frac{1}{2}}\right\}\,,$$

where η is again the Dedekind function.

The last main result of this thesis (Chapter 5) is a construction of $g(g+1)/2$-parametric deformations of Hurwitz Frobenius manifolds in genus $g \geq 1$. We introduce the following deformation $W_{\mathbf{q}}$ of the canonical meromorphic bidifferential W :

$$W_{\mathbf{q}}(P,Q) := W(P,Q) - 2\pi i\sum_{k,l=1}^{g}(\mathbb{B}+\mathbf{q})^{-1}_{kl}\omega_k(P)\omega_l(Q)\,, \tag{0.0.15}$$

where g is the genus of Riemann surface; $\omega_k(Q)$, $k = 1,\ldots,g$, are the holomorphic normalized differentials; $\mathbb{B}$ is the matrix of b-periods; and $\mathbf{q}$ is a symmetric matrix of parameters which is constant with respect to λ and $\{\lambda_j\}$. The matrix $\mathbf{q}$ must be chosen such that the sum $(\mathbb{B}+\mathbf{q})$ is nondegenerate. The bidifferential $W_{\mathbf{q}}$ tends to W when, for example, all diagonal entries of the matrix $\mathbf{q}$ tend to infinity whereas the off-diagonal ones remain finite.

The properties of the bidifferential $W_{\mathbf{q}}$ (in particular, variational formulas) are similar to those of W; therefore, similarly to the construction of Hurwitz Frobenius manifolds in terms of W, we can find new Frobenius structures by making use of the bidifferential $W_{\mathbf{q}}$. These structures are defined on the Hurwitz space outside the divisor given by the equation $\det(\mathbb{B}+\mathbf{q})=0$. The new structures depend on $g(g+1)/2$ parameters (entries of the symmetric matrix $\mathbf{q}$) which enter the definition of $W_{\mathbf{q}}$ (0.0.15); they coincide with the structures of Dubrovin in the limit when $W_{\mathbf{q}}$ coincide with W. Therefore, we call them the deformations of Dubrovin's Hurwitz Frobenius manifolds.

For the simplest Hurwitz space in genus one, the deformed Frobenius structures turn out to be related to the Schlesinger system on the Riemann sphere. Namely, consider a Schlesinger system (0.0.4) with four singular points and matrix dimension equal to two. Three singular points can, by a Möbius transformation, be mapped to the points $0, 1$ and ∞; the remaining one is denoted by x. If the integrals of motion $\mathrm{tr}A_j^2$ of the Schlesinger system are fixed to equal $1/8$, the functions

$$\Omega_1^2=-(\frac{1}{8}+\mathrm{tr}A_2A_3),\qquad \Omega_2^2=-(\frac{1}{8}+\mathrm{tr}A_1A_3),\qquad \Omega_3^2=-(\frac{1}{8}+\mathrm{tr}A_1A_2),$$

where A_j are solutions to the Schlesinger system, give a solution to the following system:

$$\frac{d\Omega_1}{dx}=\frac{1}{x}\Omega_2\Omega_3\,,\qquad \frac{d\Omega_2}{dx}=-\frac{1}{x-1}\Omega_1\Omega_3\,,\qquad \frac{d\Omega_3}{dx}=\frac{1}{x(x-1)}\Omega_1\Omega_2 \tag{0.0.16}$$

under the constraint

$$\Omega_1^2+\Omega_2^2+\Omega_3^2=-1/4\,. \tag{0.0.17}$$

The four-point Schlesinger system with integrals of motion fixed to be $1/8$ (and therefore, the system (0.0.16)-(0.0.17)) is equivalent to the Painlevé-VI equation with coefficients $(1/8,-1/8,1/8,3/8)$ (see, for example [15, 35])

Any solution to the system (0.0.16) defines rotation coefficients of a Darboux-Egoroff metric which corresponds to a Frobenius structure on a space with coordinates u_1, u_2, u_3 such that $x = (u_1 - u_3)/(u_1 - u_2)$ (see [15]).

The general solution to system (0.0.16)-(0.0.17) depending on two parameters was found by Hitchin [34]. In [2] a nice explicit form of this solution was given in terms of modular functions. In the settings of [2], the variable x defines a position of the movable branch point of the elliptic curve $\nu^2 = \lambda(\lambda - 1)(\lambda - x)$. The solutions [34, 2] define Darboux-Egoroff metrics on the Hurwitz space of genus one coverings with three finite branch points $\lambda_1, \lambda_2, \lambda_3$ for which the variable x is given by $x = (\lambda_1 - \lambda_3)/(\lambda_1 - \lambda_2)$.

In [2] a special limit was also found for the two-parametric family which gives a one-parameter family of solutions to the system (0.0.16)-(0.0.17):

$$\Omega_1 = -\frac{1}{\pi\theta_2^2\theta_3^2}\left(2d_\mu \log\theta_4 + \frac{1}{\mu + \mathbf{q}}\right) , \qquad \Omega_2 = -\frac{1}{\pi\theta_3^2\theta_4^2}\left(2d_\mu \log\theta_2 + \frac{1}{\mu + \mathbf{q}}\right) ,$$
$$\Omega_3 = -\frac{1}{\pi i\theta_2^2\theta_4^2}\left(2d_\mu \log\theta_3 + \frac{1}{\mu + \mathbf{q}}\right) \tag{0.0.18}$$

The Darboux-Egoroff metrics on the simplest (three-coordinate) Hurwitz space in genus one defined by the one-parameter family of solutions (0.0.18) coincide with the Darboux-Egoroff metrics corresponding to our deformations of Dubrovin's Frobenius structures on the Hurwitz space. Therefore, our general construction of deformed Hurwitz Frobenius manifolds gives a natural generalization of the solution (0.0.18) to the system (0.0.16)-(0.0.17) to an arbitrary Hurwitz space.

In genus one, the prepotential of the deformed Frobenius manifold corresponding to the prepotential (0.0.11) has the form:

$$F = -\frac{1}{4}t_1t_2^2 + \frac{1}{2}t_1^2t_3 - \frac{\pi i}{32}t_2^4\left(\frac{1}{(1-2\pi it_3/\mathbf{q})^2}\gamma\left(\frac{2\pi it_3}{1-2\pi it_3/\mathbf{q}}\right) + \frac{2}{\mathbf{q}(1-2\pi it_3/\mathbf{q})}\right) ,$$

where, as before, γ is the logarithmic derivative of the Dedekind eta-function. The corresponding G-function is (up to a constant) given by:

$$G = -\log\left\{\eta\left(\frac{2\pi it_3}{1-2\pi it_3/\mathbf{q}}\right)(t_2)^{\frac{1}{8}}\left(2\pi it_3/\mathbf{q}-1\right)^{-\frac{1}{2}}\right\} ,$$

where η is the Dedekind eta-function.

Analogously to the deformation of the bidifferential $W(P,Q)$, we also introduce the following deformations $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,Q)$ of the Schiffer and Bergman kernels

$$\Omega_{\mathbf{q}}(P,Q) := \Omega(P,Q) - 2\pi i\textstyle\sum_{k,l=1}^{g}(\mathbb{B}^{\Omega}+\mathbf{q})^{-1}_{kl}v_k(P)v_l(Q) ,$$

$$B_{\mathbf{q}}(P,\bar{Q}) := B(P,\bar{Q}) - 2\pi i\textstyle\sum_{k,l=1}^{g}(\mathbb{B}^{\Omega}+\mathbf{q})^{-1}_{kl}v_k(P)\overline{v_l(Q)} .$$

Here v_k, $k=1,\dots,g$ are holomorphic differentials given by the b-periods of the Schiffer kernel: $v_k(P) = \oint_{b_k}\Omega(P,Q)/(2\pi i)$. The differential v_k is normalized by the condition that all its a- and b-periods are purely imaginary except the a_k-period: $\mathrm{Re}\{\oint_{a_j}v_k\} = \delta_{jk}/2$ and $\mathrm{Re}\{\oint_{b_j}v_k\} = 0$ for $j,k=1,\dots,g$. The matrix $\mathbb{B}^{\Omega}$ is the symmetric matrix of their b-periods: $(\mathbb{B}^{\Omega})_{kn} := \oint_{b_k}\mathbf{v}_n$; and $\mathbf{q}$ is a constant matrix such that $\mathbf{q}=\mathbf{q}^T$, $\bar{\mathbf{q}} = -\mathbf{q}$, and the matrix $(\mathbb{B}^{\Omega}+\mathbf{q})$ is invertible. The bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ depend on $g(g+1)/2$ real parameters.

The definition of the deformed kernels $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ is motivated by the idea of finding bidifferentials which satisfy variational formulas analogous to the formulas for Schiffer and Bergman kernels and have other similar properties. In terms of $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,Q)$ we construct the "real doubles" of the deformed Hurwitz Frobenius manifolds; they are defined on the Hurwitz space outside the subspace given by the equation $\det(\mathbb{B}^{\Omega}+\mathbf{q}) = 0$.

This thesis is prepared in the manuscript-based format; it is laid out as follows. In Chapter 1 we present the background material which is used in the rest of the thesis. Chapter 2 contains the paper "Integrable systems related to elliptic branched coverings". Chapter 3 presents a joint work with E. Ferapontov and D. Korotkin "Boyer-Finley equation and systems of hydrodynamic type". Chapter 4 contains the paper "Real doubles" of Hurwitz Frobenius manifolds". Chapter 5 contains the paper "Deformations of Frobenius structures on Hurwitz spaces". The Summary and Conclusion are given at the end.

Contributions of authors

Chapter 3 presents the paper "Boyer-Finley equation and systems of hydrodynamic type" written jointly with E. Ferapontov and D. Korotkin. The idea for this paper appeared during the conference "Nonlinear Evolution Equations and Dynamical Systems" in Cádiz, Spain. E. Ferapontov recognized an auxiliary system derived by the author of the thesis (a system giving the dependence of the images of ramification points of a trigonometric covering under a uniformization map on the branch points of the covering) as a system arising in the context of reduction of the Boyer-Finley equation to a pair of systems of hydrodynamic type. Further technical work was done by the present author under the supervision of D. Korotkin.

Chapter 1

Background material

1.1 Riemann surfaces

Here we collect some classical facts from the theory of Riemann surfaces, for more details see [15, 22, 23, 31, 59]

1.1.1 Definitions. Differentials. Jacobi torus.

A *Riemann surface* is a complex orientable manifold of complex dimension 1 . We consider compact Riemann surfaces. It is a classical result that any compact orientable 1-dimensional complex manifold is diffeomorphic to a sphere with handles. The number of handles is called the *genus* of Riemann surface; it is denoted by g . The basis of the first homology group on a sphere with g handles consists of $2g$ cycles. For two oriented contours intersecting each other at one point the *intersection index* equals either $+1$ or -1 depending on the mutual orientation of the contours. If two contours intersect each other in more than one point, the index is equal to the sum of intersection indices of each single intersection. The *canonical basis of cycles* is the set of $2g$ cycles $\{a_k; b_k\}_{k=1}^g$ with intersection indices given by $a_j \circ a_k = 0$, $b_j \circ b_k = 0$ and $a_j \circ b_k = \delta_{jk}$, where δ_{jk} is the Kronecker symbol. Two arbitrary canonical bases of cycles are related by a transformation defined by an integer symplectic matrix. Namely, let $\{a_k; b_k\}_{k=1}^g$ be a canonical basis of cycles on a Riemann surface. Consider the $2g \times 2g$ matrix $J = \begin{pmatrix} \mathbf{0} & -I \\ I & \mathbf{0} \end{pmatrix}$, where $\mathbf{0}$ and I are, respectively, the zero and identity $g \times g$ matrices. Let an integer $2g \times 2g$ matrix

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{1.1.1}$$

be symplectic, i.e. such that $MJM^T = J$. Then the basis of cycles $\{\tilde{a}_k; \tilde{b}_k\}_{k=1}^g$ given by $\tilde{a}_j = \sum_{k=1}^g (P_{jl}a_l + Q_{jl}b_l)$, $\tilde{b}_j = \sum_{k=1}^g (R_{jl}a_l + S_{jl}b_l)$ is also canonical.

Suppose that all cycles of a canonical basis pass through one point. If we cut the surface $\mathcal{L}$ along these cycles, we obtain a connected domain $\widehat{\mathcal{L}}$ called the *fundamental polygon* of the Riemann surface. For example, for a torus (Riemann surface of genus one) the fundamental domain is a parallelogram. The sides of the fundamental polygon coincide with the positions of a- and b-cycles; each cycle corresponds to two sides of the polygon.

Differentials on Riemann surface. Any 1-form w in a local parameter on a manifold can be written as $w = p(z,\bar{z})dz + q(z,\bar{z})d\bar{z}$. A 1-form is called *holomorphic* if in a neighbourhood of any point it has the form $w = f(z)dz$ for some function f ; a 1-form is called *antiholomorphic* if it has the form $w = f(\bar{z})d\bar{z}$. For a closed 1-form the integral over a path between two points does not depend on the choice of the path within given homology class and therefore is well-defined. The integrals of a closed 1-form ω over the basis cycles of a Riemann surface are called the a- and *b-periods* of the 1-form and denoted by $A_k := \oint_{a_k} \omega$ and $B_k := \oint_{b_k} \omega$. For two closed 1-forms ω and $\tilde{\omega}$ on the surface $\mathcal{L}$ the Riemann bilinear relations hold:

$$\iint_{\mathcal{L}} \omega \wedge \tilde{\omega} = \int_{\partial\widehat{\mathcal{L}}} \tilde{\omega} \int_{P_0}^{P} \omega = \sum_{k=1}^{g} (A_k \tilde{B}_k - \tilde{A}_k B_k) \,, \tag{1.1.2}$$

$\tilde{A}_k$ and $\tilde{B}_k$ are, respectively, the a- and b-periods of $\tilde{\omega}$. The first equality follows from the Stokes theorem and the second one is proved by the integration over the boundary of the fundamental polygon.

If one chooses $\tilde{\omega} = \bar{\omega}$, then the Riemann bilinear relations (1.1.2) imply $\iint_{\mathcal{L}} \omega \wedge \bar{\omega} = 2\,i\,\mathrm{Im}\left\{\sum_{k=1}^g A_k \bar{B}_k\right\}$, and therefore a holomorphic differential for which all a-periods vanish is identically zero. On a Riemann surface of genus g there exist exactly g linearly independent holomorphic 1-forms (Abelian differentials of the first kind). One can always choose a basis of g differentials $\omega_1, \dots, \omega_g$ normalized by $\oint_{a_j} \omega_k = \delta_{jk}$. Integrating these differentials over b-cycles, we get the matrix $\mathbb{B}$ with entries $\mathbb{B}_{kj} = \oint_{b_j} \omega_k$, which is called *the matrix of b-periods* of the Riemann surface. The matrix $\mathbb{B}$ is symmetric, and its imaginary part is positive definite. For the canonical basis of cycles $\{\tilde{a}_k; \tilde{b}_k\}$ related to the basis $\{a_k; b_k\}$ by a symplectic matrix (1.1.1) the transformed matrix $\tilde{\mathbb{B}}$ of b-periods is given by $\tilde{\mathbb{B}} = (R + S\mathbb{B})(Q\mathbb{B} + P)^{-1}$.

Any meromorphic 1-form can be normalized to have zero a-periods by adding a linear combination of the differentials ω_j . An Abelian differential of the *second kind* is a differential $W_Q^{(N)}(P)$ for some integer N which has zero a-periods and is holomorphic everywhere outside $P = Q$, where it has a pole of order $N + 1$. An Abelian differential of the *third kind* is a differential $W_{RS}(P)$ normalized to have zero a-periods and holomorphic outside the points R and S , where it has simple poles with residues $+1$ and -1 , respectively.

The *divisor* (Ω) of a differential Ω is the formal sum of the points D_n where the differential has poles or zeros: $(\Omega) = \sum d_n D_n$, the numbers d_n being the orders of corresponding zeros and poles, taken with the minus sign in the case of poles. The sum $\sum d_n$ is called the *degree* of the divisor.

Jacobi torus. Consider an equivalence relation in $\mathbb{C}^g$: $z_1 \sim z_2$ iff there exist $n, m \in \mathbb{Z}^g$ such that $z_1 - z_2 = n + \mathbb{B}\, m$. The *Jacobi torus* J, or *Jacobian*, of a Riemann surface $\mathcal{L}$ is a factor $J(\mathcal{L}) := \mathbb{C}^g / \sim$. The *Abel map* $\mathcal{A} : \mathcal{L} \to J(\mathcal{L})$ is defined by $\mathcal{A}_k(P) := \int_{P_{\mathcal{A}}}^{P} \omega_k$, where $P_{\mathcal{A}}$ is some fixed starting point for the Abel map.

The Abel map of the divisor $(\Omega) = \sum d_n D_n$ is assumed to be $\mathcal{A}(\Omega) := \sum d_n \mathcal{A}(D_n)$. Two divisors (Ω_1) and (Ω_2) are called equivalent if the degree of their difference is zero, $\deg((\Omega_1) - (\Omega_2)) = 0$, and the Abel maps coincide on $J(\mathcal{L})$: $\mathcal{A}(\Omega_1 - \Omega_2) \equiv 0$. Divisors of all meromorphic $(1,0)$-forms are equivalent. This divisor class is called *canonical* and denoted by $\mathbf{C}$. The degree of the canonical divisor is $\deg \mathbf{C} = 2g - 2$.

1.1.2 Meromorphic functions on Riemann surface and coverings of $\mathbb{C}P^1$

Any meromorphic function on a Riemann surface has equal number of poles and zeros counting their multiplicities. A function with poles at $Q_1, \dots, Q_n$ and zeros at the points $R_1, \dots, R_n$ (if some zeros or poles are multiple, not all Q_j and not all R_j are distinct) exists on a Riemann surface $\mathcal{L}$ if and only if the Abel maps of these two sets of points give the same point on the Jacobian $J(\mathcal{L})$, i.e. $\sum_{j=1}^{n} (\mathcal{A}(R_j) - \mathcal{A}(Q_j)) \equiv 0 \, (\mathrm{mod}\, J(\mathcal{L}))$.

Consider a meromorphic function $\lambda(P) : \mathcal{L} \to \mathbb{C}P^1$ of degree N . The points P_j where derivative of the function vanishes are called the *critical points.* The corresponding critical values are denoted by $\lambda_j := \lambda(P_j)$. The function $\lambda(P)$ represents the Riemann surface as an N-fold ramified covering over $\mathbb{C}P^1$. The points $\{P_j\}$ are called the *ramification points,* their projections on the base of the covering are given by $\{\lambda_j\}$, which are called the *branch points.* The covering is a collection of N copies of $\mathbb{C}P^1$, which are glued together along branch cuts to form a connected manifold. The order of vanishing of the derivative λ' at the point P_j is called the ramification index; this number equals the number of sheets of the covering glued together at the corresponding branch point minus 1 . Denote by ∞^i , $i = 0, \dots, m$ the points of the surface where the function $\lambda(P)$ has poles. The order $n_i + 1$ of the pole at ∞^i gives the number of sheets glued together at the corresponding point of the covering. The numbers $n_0, \ldots, n_m \in \mathbb{N}$ (the ramification indices over infinity) are such that $\sum_{i=0}^{m}(n_i+1) = N$.

We shall only consider the coverings defined by the functions $\lambda(P)$ which have simple critical points P_j . Then, the local parameter on $\mathcal{L}$ near a ramification point $P_j \in \mathcal{L}$ (which is not a pole of λ) can be chosen to be $x_j(P) = \sqrt{\lambda(P) - \lambda_j}$ and in a neighbourhood ∞^i the local parameter z_i is such that $z_i^{-n_i-1}(P) = \lambda(P)$.

The *Riemann-Hurwitz formula* connects the genus g of the surface $\mathcal{L}$, degree N of the function λ , the number L of simple finite branch points and the ramification indices $\{n_i\}$ over infinity:

$$2g - 2 = -2N + L + \sum_{i=0}^{m} n_i \, . \tag{1.1.3}$$

1.2 Classical objects related to a compact Riemann surface

This section contains only a few scattered facts from the theory of Riemann surfaces; the systematic treatment of this topic is contained in numerous existing literature, see, for example, [15, 22, 23, 25, 59].

1.2.1 Theta-functions

For a Riemann surface $\mathcal{L}$ with the matrix of b-periods $\mathbb{B}$ the Jacobi theta-functions with characteristics $p, q \in \mathbb{C}^g$ are defined for $z \in \mathbb{C}^g$ by

$$\theta[p,q](z|\mathbb{B}) := \sum_{m \in \mathbb{Z}^g} \exp\{\pi i \langle \mathbb{B}(m+p), m+p\rangle + 2\pi i \langle p+m, q+z\rangle\} \, . \tag{1.2.1}$$

The theta-functions have the following (quasi) periodicity properties:

$$\theta[p,q](z+e_j) = e^{2\pi i p_j}\theta[p,q](z) \, , \quad \theta[p,q](z+\mathbb{B}e_j) = e^{-\pi i \mathbb{B}_{jj} - 2\pi i (z_j+q_j)}\theta[p,q](z) \, , \tag{1.2.2}$$

where $\{e_j\}_{j=1}^g$ is the standard orthogonal basis in $\mathbb{R}^g$.

From the definition it is easy to see that the theta-functions satisfy the *heat equation*:

$$\frac{\partial^2\theta[p,q](z|\mathbb{B})}{\partial z_i \partial z_j} = 4\pi i \frac{\partial\,\theta[p,q](z|\mathbb{B})}{\partial \mathbb{B}_{ij}} \,. \tag{1.2.3}$$

Theta-functions with and without characteristics are related as follows:

$$\theta[p,q](z) = e^{\pi i \langle \mathbb{B}p,p\rangle + 2\pi i \langle p,(z+q)\rangle}\theta(z+\mathbb{B}p+q) \,. \tag{1.2.4}$$

Consider the theta-function $\theta(\mathcal{A}(P)-d|\mathbb{B})$ where $\mathcal{A}(P)$ is the Abel map. If this function does not vanish identically on $\mathcal{L}$, then it has g zeros (counting multiplicities). The sketch of the proof is as follows. The number of zeros of the function $F(P) = \theta(\mathcal{A}(P)-d|\mathbb{B})$ is given by the integral $\oint_{\partial\widehat{\mathcal{L}}} d\log F(P)/(2\pi i)$. Since the boundary $\partial\widehat{\mathcal{L}}$ of the fundamental polygon of the surface consists of the a- and b- cycles, this integral can be evaluated using the transformation rules (1.2.2) for the theta-functions; the result gives g . Denote the g zeros of the function F by $D_1,\dots,D_g$. Then, by the residue theorem, we have $\oint_{\partial\widehat{\mathcal{L}}} \mathcal{A}_j(P) d\log F(P) = 2\pi i \sum_{k=1}^g \mathcal{A}_j(D_k)$. The integral computed as the sum of integrals over the basis cycles gives $\sum_{k=1}^g \mathcal{A}_j(D_k) = (K^{P_\mathcal{A}} - d)_j$ where $K^{P_\mathcal{A}}$ is the *vector of Riemann constants*:

$$K_j^{P_\mathcal{A}} := \frac{\mathbb{B}_{jj}+1}{2} + \sum_{k\neq j}\oint_{a_k}\mathcal{A}(P)\omega_k(P) \,, \tag{1.2.5}$$

$P_\mathcal{A}$ being the starting point for the Abel map. The vector $K^{P_\mathcal{A}}$ satisfies the relation:

$$2\,K^{P_\mathcal{A}} \equiv -\mathcal{A}(\mathbf{C})\,(\operatorname{mod} J(\mathcal{L}))\,, \tag{1.2.6}$$

where $\mathbf{C}$ is any divisor from canonical class.

For half-integer characteristics $[\alpha,\beta]$, when $\alpha,\beta\in\frac{1}{2}\mathbb{Z}$, the theta-functions are either even or odd: $\theta[\alpha,\beta](-z) = (-1)^{4\langle\alpha,\beta\rangle}\theta[\alpha,\beta](z)$. A half-integer characteristic $[\alpha,\beta]$ is called *odd* if $4\langle\alpha,\beta\rangle$ is odd; it is called *even* otherwise. For genus g there exist $(4^g+2^g)/2$ even and $(4^g-2^g)/2$ odd characteristics.

For $g=1$ there exist three even theta-functions: $\theta_2(z) := \theta[1/2,0](z)$, $\theta_3(z) := \theta[0,0](z)$, $\theta_4(z) := \theta[0,1/2](z)$ and one odd theta-function: $\theta_1(z) := -\theta[1/2,1/2](z)$.

Dedekind η-function. The Dedekind eta-function is a function defined on the complex upper half plane. For any complex number μ with positive imaginary part, we set $q=e^{2\pi i\mu}$. Then, the eta-function is defined by

$$\eta(\mu) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) . \tag{1.2.7}$$

The η-function is holomorphic on the upper half plane.

The η-function satisfies the functional equations

$$\eta(\mu+1) = e^{2\pi i/24}\eta(\mu) , \qquad \eta(-1/\mu) = \sqrt{-i\mu}\,\eta(\mu) , \qquad \overline{\eta(\mu)} = \eta(-\bar{\mu}) .$$

The Dedekind eta-function is related to the Jacobi theta-function θ_1 by:

$$\eta(\mu) = (\theta_1'(0,\mu))^{1/3} . \tag{1.2.8}$$

Weierstrass $\wp$-function. The $\wp$-function is defined on the torus $T=\mathbb{C}/\{2w,2w'\}$ (i.e. $\wp(z+2w)=\wp(z+2w')=\wp(z)$) by the series:

$$\wp(z) = \frac{1}{z^2} + \sum_{m,m'\in\mathbb{Z}}\left[\frac{1}{(z+2mw+2m'w')^2} - \frac{1}{(2mw+2m'w')^2}\right] , \tag{1.2.9}$$

which is uniformly convergent for any z .

The $\wp$-function is even, $\wp(-z) = \wp(z)$, and has a single pole of order two in $T = \mathbb{C}/\{2w, 2w'\}$ with the principal part $\wp(z) \simeq 1/z^2 + \mathcal{O}(z^2)$. In terms of theta-functions it can be written as follows:

$$\wp(z) = -\frac{\partial^2}{\partial z^2} \log \theta_1 \left(\frac{z}{2w}\right) + \frac{1}{12w^2} \frac{\theta_1'''}{\theta_1'} \ .$$

From this formula we find the values of integrals of $\wp$-function over the periods of the torus T :

$$\int_x^{x+2w} \wp(z) dz = \frac{1}{6w} \frac{\theta_1'''}{\theta_1'} \ , \qquad \int_x^{x+2w'} \wp(z) dz = \frac{\pi i}{w} + \frac{w'}{6w^2} \frac{\theta_1'''}{\theta_1'} \tag{1.2.10}$$

where x is any complex number. The following relation on the function $\wp$ can be proven by examining the singularities of both sides at $z = 0$:

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z) - \wp(w)\right)\left(\wp(z) - \wp(w')\right)\left(\wp(z) - \wp(w+w')\right) \ . \tag{1.2.11}$$

The definition of the Weierstrass $\wp$-function implies $\wp(w) + \wp(w') + \wp(w+w') = 0$.

Multivalued differential $\mathcal{C}(P)$. Let us choose the fundamental domain $\widehat{\mathcal{L}}$ such that the equality holds in the relation (1.2.6), i.e. $\mathcal{A}(\mathbf{C}) = -2K^P$; here the Abel map is taken along the path which does not intersect the boundary of $\widehat{\mathcal{L}}$. Introduce the following object on the Riemann surface $\mathcal{L}$:

$$\mathcal{C}(P) = \frac{1}{\det_{1\le\alpha,\beta\le g} \|\omega_\beta^{(\alpha-1)}(P)\|} \sum_{\alpha_1,\dots,\alpha_g=1}^{g} \frac{\partial^g \theta(K^P)}{\partial z_{\alpha_1} \dots \partial z_{\alpha_g}} \omega_{\alpha_1}(P) \dots \omega_{\alpha_g}(P) \ . \tag{1.2.12}$$

The differential $\mathcal{C}(P)$ is a holomorphic multivalued $g(1-g)/2$-differential on $\mathcal{L}$; it is single valued with respect to tracing along a-cycles and gains simple exponential factors under tracing along b-cycles.

1.2.2 Hyperelliptic curves. Thomae formulas

A surface $\mathcal{L}$ of genus g is called *hyperelliptic* if there exists a meromorphic function λ on the surface with exactly two poles. Then, the function $\lambda(P)$ can be used to represent the surface as a two-fold covering of the Riemann sphere $\mathbb{C}P^1$. The covering $(\mathcal{L}, \lambda)$, according to the Riemann-Hurwitz formula (1.1.3), has $2g+2$ simple branch points $\{\lambda_j\}_{j=1}^{2g+2}$. This covering can be considered as an algebraic curve defined by the equation

$$\nu^2 = \prod_{j=1}^{2g+2} (\zeta - \lambda_j) \,. \tag{1.2.13}$$

The holomorphic (non-normalized) differentials on this curve are given by

$$\tilde{\omega}_j = \frac{\zeta^{j-1}}{\nu} d\zeta \,, \qquad j = 1, \ldots, g \,. \tag{1.2.14}$$

Let us choose the branch cuts on $\mathcal{L}$ to connect the points λ_{2n+1} and λ_{2n+2} . Choose the basis cycles a_n to encircle the points λ_{2n+1} and λ_{2n+2} and the cycles b_n to encircle the points λ_2 and λ_{2n+1} . Let us choose the starting point $P_{\mathcal{A}}$ for the Abel map $\mathcal{A}$ (see Section 1.1) to coincide with P_1 . Then, the Abel map between two ramification points is equal to a linear combination with half-integer coefficients of the columns of the matrix $\mathbb{B}$ and the vectors $\{e_j\}_{j=1}^g$ (the standard orthonormal basis in $\mathbb{R}^g$). For example,

$$\int_{P_1}^{P_4} \omega_j = \frac{1}{2} \oint_{b_1} \omega_j \qquad \text{and} \qquad \int_{P_1}^{P_3} \omega_j = \frac{1}{2} \oint_{b_1} \omega_j + \frac{1}{2} \oint_{a_1} \omega_j \,. \tag{1.2.15}$$

Relations (1.2.15) can be proven as follows. The cycle b_1 can be deformed to a homotopically equivalent one whose two parts γ_1 and γ_2 (the path γ_1 from P_1 to P_4 and γ_2 from P_4 to P_1) have the same projection on the base $\mathbb{C}P^1$ (the two parts lie on different sheets of the covering). The holomorphic normalized differentials $\{\omega_j\}$ are given by linear combinations of the non-normalized differentials (1.2.14), therefore the values of a differential

ω_k at two points which lie above each other on the covering differ by a sign (values of ν , the denominator in (1.2.14), at these two points correspond to the different branches of the square root $\sqrt{(\zeta-\lambda_1)\dots(\zeta-\lambda_{2g+2})}$). Thus, the integrals $\int_{\gamma_1}\omega_j$ and $\int_{\gamma_2}\omega_j$ are equal and their sum gives the b_1-period of ω_j . The analogous reasoning holds for the a-periods.

This gives a way to construct half-integer characteristics for the theta-functions (1.2.1). Let us consider the partition of the set of branch points $\{\lambda_j\}_{j=1}^{g}$ consisting of two sets of $g+1$ points: $T := \{\lambda_{i_1},\dots,\lambda_{i_{g+1}}\}$ and $S := \{\lambda_{j_1},\dots,\lambda_{j_{g+1}}\}$. Denote by $\mathcal{A}(T)$ the sum of values of Abel maps of the points from the set T , i.e. $\mathcal{A}(T) := \sum_{k=1}^{g+1}\mathcal{A}(P_{i_k})$. Then the characteristic $[p^T,q^T]$ is defined by the equality: $\mathcal{A}(T)+K^{P_1} = \mathbb{B}p^T+q^T$, where the j-th component of the vector of Riemann constants, for the case of this covering and $P_{\mathcal{A}} = P_1$, has the form:

$$K_j^{P_1} := \frac{j}{2}+\sum_{i=1}^{g}\frac{\mathbb{B}_{ij}}{2}\,, \qquad j=1,\dots,g\,.$$

The Thomae formulas [71] allow to express the theta-constants of the hyperelliptic curve as follows:

$$\theta^4[p^T,q^T](0|\mathbb{B}) = \pm(\det A)^2\prod_{j,k\in T}(\lambda_j-\lambda_k)\prod_{j,k\in S}(\lambda_j-\lambda_k)\,, \tag{1.2.16}$$

where A is the matrix of a-periods of the non-normalized holomorphic differentials (1.2.14): $A_{kl} := \oint_{a_k}(\lambda^{l-1}/\nu)d\lambda$ for $k,l=1,\dots,g$. Taking two different partitions of the set of branch points, we can obtain a relation in which the matrix A is absent; this relation allows to express a ratio of theta-constants only in terms of the branch points.

1.2.3 Prime form

The *prime form* E is the following $(-\frac{1}{2},-\frac{1}{2})$-differential on $\mathcal{L}\times\mathcal{L}$:

$$E(P,Q) := \frac{\theta[\alpha,\beta](\mathcal{A}(P)-\mathcal{A}(Q))}{h_\Delta(P)h_\Delta(Q)}\,, \tag{1.2.17}$$

where $h_\Delta^2(P) := \sum_{j=1}^g \partial_{z_j}\theta[\alpha,\beta](0)\,\omega_j(P)$, and $[\alpha,\beta]$ is an odd characteristics. This definition does not depend on the choice of an odd characteristics $[\alpha,\beta]$. The holomorphic differential h_Δ^2 has $g-1$ zeros of order two; therefore the square root h_Δ is a holomorphic *spinor*, i.e. an object of the form $g(z)\sqrt{dz}$ in a local coordinate z . The differential $h_\Delta^2(P)$ is single valued on the surface; however, its square root may change sign when the argument P goes along topologically nontrivial closed loops on the Riemann surface.These signs along basis cycles are given by the numbers $e^{2\pi i\,\alpha_j}$ and $e^{-2\pi i\,\beta_j}$; therefore the prime form satisfies the relation $E(P^{a_j},Q) = E(P,Q)$, where $E(P^{a_j},Q)$ stands for the analytic continuation of the prime form along the cycle a_j with respect to the first argument. Then, under the analytic continuation along the b-cycles, the prime form changes as follows: $E(P^{b_j},Q) = \exp\{-\pi i\mathbb{B}_{jj} - 2\pi i(\mathcal{A}_j(P)-\mathcal{A}_j(Q))\}E(P,Q)$. The prime form is antisymmetric, i.e. $E(P,Q) = -E(Q,P)$. With respect to the argument P , the prime form has on the surface $\mathcal{L}$ only one zero at $P=Q$ and does not have poles. It has the following local behaviour near the diagonal $P\sim Q$:

$$E(P,Q) \underset{P\sim Q}{=} \frac{x(P)-x(Q)}{\sqrt{dx(P)}\sqrt{dx(Q)}}\left(1-\frac{1}{2}S(P)\,(x(P)-x(Q)) + \mathcal{O}\left((x(P)-x(Q))^2\right)\right)\,. \tag{1.2.18}$$

The quantity $6S(P)$ is a holomorphic *projective connection* (it is called the *Bergman projective connection*): it transforms as follows with respect to the change of local coordinate

$x \to f(x)$:

$$6S(x) \mapsto 6S(f(x))(f'(x))^2 + \{f(x), x\} ,$$

where

$$\{f(x), x\} := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

is the *Schwarzian derivative.* The prime form can be used for constructing meromorphic functions and differentials with given poles on the surface: for example, the function with simple poles at the points $\{Q_l\}$ and zeros at $\{P_l\}$ is given by $f(P) = \prod_l E(P, P_l)/E(P, Q_l)$, where the points $\{P_l\}$ are related to $\{Q_l\}$ according to the Abel theorem, i.e. by $\mathcal{A}(\sum_l P_l) \equiv \mathcal{A}(\sum_l Q_l)(\mathrm{mod}(J(\mathcal{L})))$. The Abelian differential of the third kind W_{RS} can be expressed in terms of the prime form as follows: $W_{RS}(P) = d_P \log\{E(P, R)/E(P, S)\}$.

1.2.4 Kernels on Riemann surface

The kernel $W(P, Q)$ defined by

$$W(P, Q) = d_P d_Q \log E(P, Q) \tag{1.2.19}$$

is the symmetric differential on $\mathcal{L} \times \mathcal{L}$ with the second order pole with biresidue 1 at the diagonal $P = Q$ and the properties:

$$\oint_{a_k} W(P, Q) = 0 ; \qquad \oint_{b_k} W(P, Q) = 2\pi i\, \omega_k(P) ; \qquad k = 1, \ldots, g . \tag{1.2.20}$$

$W(P, Q)$ is called the canonical meromorphic bidifferential [22].

The *Schiffer kernel* $\Omega(P, Q)$ is the symmetric differential on $\mathcal{L} \times \mathcal{L}$ defined by:

$$\Omega(P, Q) = W(P, Q) - \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl} \omega_k(P)\omega_l(Q) \tag{1.2.21}$$

(Im$\mathbb{B}$ is positive definite). For a surface of genus zero, the Schiffer kernel coincides with the kernel W . It can be equivalently defined [23] as the unique bidifferential with a singularity of the form $(x(P)-x(Q))^{-2}dx(P)dx(Q)$ along $P=Q$ and such that the equality

$$\text{p.v.}\iint\limits_{\mathcal{L}}\Omega(P,Q)\overline{\omega(P)}=0 \tag{1.2.22}$$

holds for any holomorphic differential ω .

The *Bergman kernel* $B(P,\bar{Q})$ is defined by:

$$B(P,\bar{Q})=\pi\sum_{k,l=1}^{g}(\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P)\overline{\omega_l(Q)}\ ; \tag{1.2.23}$$

it vanishes for a surface of genus zero. Alternatively, the Bergman kernel can be defined as a regular bidifferential on $\mathcal{L}$ holomorphic with respect to its first argument and antiholomorphic with respect to the second one which (up to the factor of $2\pi i$) is a kernel of an integral operator which acts in the space $L_2^{(1,0)}(\mathcal{L})$ of $(1,0)$-forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic $(1,0)$-forms. In particular, the following holds for any holomorphic differential ω on the surface $\mathcal{L}$:

$$\frac{1}{2\pi i}\iint\limits_{\mathcal{L}}B(P,\bar{Q})\omega(Q)=\omega(P)\ . \tag{1.2.24}$$

From the definitions (1.2.22) and (1.2.24) one can see that the Schiffer and Bergman kernels do not depend on the choice of a canonical basis of cycles $\{a_k,b_k\}_{k=1}^{g}$ on the Riemann surface. For the Bergman kernel, this independence can also be seen directly from (1.2.23) using the relation $(\mathrm{Im}\mathbb{B})_{kl}=\frac{i}{2}\iint_{\mathcal{L}}\omega_k(P)\overline{\omega_l(P)}$, which is equivalent to the Riemann bilinear relations (1.1.2) if one chooses $\omega:=\omega_k$ and $\tilde{\omega}:=\bar{\omega}_l$.

The periods of the Schiffer and Bergman kernels are related to each other by:

$$\oint_{a_k}\Omega(P,Q)=-\oint_{a_k}B(\bar{P},Q)\ ,\qquad \oint_{b_k}\Omega(P,Q)=-\oint_{b_k}B(\bar{P},Q) \tag{1.2.25}$$

where the integrals are taken with respect to the first arguments.

1.2.5 Variation of complex structure

Consider the space of Riemann surfaces of genus g modulo the conformal equivalence. The real dimension of the moduli space of a Riemann surface is $6g-6$.

For a function $f(z,\bar{z})$ holomorphic in some domain, the Cauchy-Riemann equation $\partial_{\bar{z}} f = 0$ holds in the domain. Let us denote by $\mathcal{L}_\mu$ the Riemann surface $\mathcal{L}$ with the following new complex structure: a function f on $\mathcal{L}_\mu$ is called holomorphic in any local coordinate $(z,\bar{z})$ if it satisfies the equation

$$\partial_{\bar{z}} f - \mu(z,\bar{z})\partial_z f = 0 , \tag{1.2.26}$$

where $\mu(z,\bar{z})$ is a form called the *Beltrami differential.* In order for this equation to be well-defined (invariant with respect to a coordinate change), μ must be a -1-differential with respect to z and $+1$-differential with respect to $\bar{z}$. Two Beltrami differentials μ_1 and μ_2 correspond to the same variation of complex structure if

$$\iint_{\mathcal{L}} (\mu_1 - \mu_2)\mathbf{w} = 0$$

holds for any holomorphic quadratic differential $\mathbf{w}$ on the surface.

Consider a smooth infinitesimal deformation $\mathcal{L}^{\varepsilon\mu}$ of the complex structure on the surface $\mathcal{L}$. The local coordinate z changes holomorphically with respect to the parameter ε :

$$z^\varepsilon = z + \varepsilon\, q(z,\bar{z}) + o(\varepsilon) . \tag{1.2.27}$$

Then, the Beltrami differential μ is given by $\mu(z,\bar{z}) = \partial_{\bar{z}} q(z,\bar{z})$. All the defined objects on the surface vary according to this deformation. In particular, the variation of the bidifferential $W(P,Q)$ (1.2.19) is described by the following Rauch formulas ([23],p.57):

$$\delta_\mu W(P,Q) = -\frac{1}{\pi}\iint_{\mathcal{L}} \mu(R)W(P,R)W(R,Q)\,, \qquad \bar{\delta}_\mu W(P,Q) = 0\,. \tag{1.2.28}$$

where $\delta_\mu := \partial_\varepsilon|_{\varepsilon=0}$ and $\bar{\delta}_\mu := \partial_{\bar{\varepsilon}}|_{\varepsilon=0}$. Integration of these formulas over b-periods of the surface gives the Rauch formulas for holomorphic normalized differentials ω_k and the matrix $\mathbb{B}$:

$$\delta_\mu \omega_k(P) = -\frac{1}{\pi}\iint_{\mathcal{L}} \mu(Q)\omega_k(Q)W(P,Q)\,, \qquad \bar{\delta}_\mu \omega_k(Q) = 0. \tag{1.2.29}$$

$$\delta_\mu \mathbb{B}_{ij} = -2i\iint_{\mathcal{L}} \mu(Q)\omega_i(Q)\omega_j(Q)\,, \qquad \bar{\delta}_\mu \mathbb{B}_{ij} = 0. \tag{1.2.30}$$

The variational formulas ([23], p.56) for the Schiffer and Bergman kernels can be obtained using (1.2.28) - (1.2.30).

1.3 Hurwitz spaces

1.3.1 Definition and variational formulas

Two coverings (see Section 1.1.2) are called equivalent if they can be obtained from one another by permutation of sheets. The Hurwitz space is the space of genus g coverings with fixed number of sheets and fixed number and type of ramification points modulo the equivalence relation. The branch points $\{\lambda_j\}$ give the set of local coordinates on the Hurwitz space.

Let us consider the Hurwitz spaces of the following type. Assume that the function $\lambda : \mathcal{L} \to \mathbb{C}P^1$ which defines the covering $(\mathcal{L}, \lambda)$ has $m+1$ poles at the points $\infty^0, \ldots, \infty^m \in \mathcal{L}$ of orders $n_0+1, \ldots, n_m+1$ (i.e. the covering has $m+1$ points which project to $\zeta = \infty$ on the base with the ramification order n_i at the i-th point). The remaining ramification points

$\{P_j\}_{j=1}^L$, which have finite projections $\{\lambda_j\}_{j=1}^L$ on the base of the covering, are assumed to be simple (the ramification index equals one) and distinct. The Hurwitz space of such coverings is denoted by $M_{g;n_0,\dots,n_m}$, where g is the genus of the Riemann surface $\mathcal{L}$ and the numbers $n_0,\dots,n_m$ correspond to the ramification type over the point at infinity (in the case when all numbers n_i equal zero, we shall denote the Hurwitz space by $H_{g;N}$).

Keeping the numbers g , N , n_i and L fixed and moving the positions $\{\lambda_j\}$ of branch points, one changes the local parameters $x_j = \sqrt{\lambda - \lambda_j}$ near simple ramification points of the covering. Therefore this variation changes the complex structure on the Riemann surface $\mathcal{L}$ (see Section 1.2.5). The Taylor series (1.2.27) turns into $\sqrt{\lambda - \lambda_j - \varepsilon} = \sqrt{\lambda - \lambda_j} - \varepsilon(2\sqrt{\lambda - \lambda_j})^{-1} + o(\varepsilon)$. Hence, the Beltrami differential which describes the infinitesimal variation of the complex structure under variation of position of the branch point λ_j is given by

$$\mu_j(P) = -\frac{1}{2}\partial_{\bar{x}_j}\left(\frac{1}{x_j}\right) = -\frac{\pi}{2}\delta(x_j) \; , \tag{1.3.1}$$

where $\delta(z)$ is the two-dimensional delta-function. Since the infinitesimal change of the branch point λ_j is given by ε , we have $\delta_{\mu_j} = \partial_{\lambda_j}$ (see [43] for a detailed discussion of this relation and rigorous proof). Substutution of the Beltrami differential μ_j (1.3.1) into the Rauch variational formula (1.2.28) gives

$$\frac{dW(P,Q)}{d\lambda_j} = \frac{1}{2}W(P,P_j)W(Q,P_j) \; , \qquad \frac{dW(P,Q)}{d\bar{\lambda}_j} = 0 \; , \tag{1.3.2}$$

where $W(P,P_j) := (W(P,Q)/dx_j(Q))\,|_{Q=P_j}$. Formula (1.3.2) can be alternatively proven as follows. The behaviour of $W(P,Q)$ considered as a differential with respect to P at $P \sim P_j$ is given by

$$W(P,Q) \underset{P\sim P_j}{=} \left\{W(P_j,Q) + W_{,1}(P_j,Q)\,x_j(P) + \mathcal{O}(x_j^2(P))\right\} dx_j(P)dx(Q) \; ,$$

where $x_j(P) = \sqrt{\lambda(P) - \lambda_j}$ and $dx_j(P) = d\lambda/(2\sqrt{\lambda(P) - \lambda_j})$; $W_{,1}(P,Q)$ denotes the derivative with respect to the first argument. Differentiating this equality with respect to branch points, we see that the derivative $d_{\lambda_k} W(P,Q)$, considered as a differential with respect to P , has the only singularity at the point P_k of the form:

$$d_{\lambda_k} W(P,Q) \underset{P \sim P_k}{=} \left(\frac{W(P_k, Q)}{2\, x_k^2(P)} + \mathcal{O}(1) \right) dx_k(P) dx(Q).$$

The a-periods of $d_{\lambda_k} W(P,Q)$ vanish since they vanish for $W(P,Q)$. Thus, the derivative $d_{\lambda_k} W(P,Q)$ is proportional to $W(P, P_k)$ with a proportionality coeffiecient depending only on Q . Taking into account the value of the biresidue of $d_{\lambda_k} W(P,Q)$ at P_k and the symmetry of the bidifferential W , we get (1.3.2).

The formulas (1.3.2) integrated over b-cycles of the surface (see (1.2.20) for the b-periods of W) give the variational formulas for holomorphic differentials and the matrix $\mathbb{B}$ of b-periods:

$$\frac{d\,\omega_k(P)}{d\,\lambda_j} = \frac{1}{2} \omega_k(P_j) W(P, P_j)\,, \qquad \frac{d\,\mathbb{B}_{kl}}{d\,\lambda_j} = \pi i\, \omega_k(P_j) \omega_l(P_j)\,. \tag{1.3.3}$$

Using (1.3.2) and (1.3.3), one finds the variational formulas for the Schiffer and Bergman kernels:

$$\begin{aligned} \frac{d\Omega(P,Q)}{d\lambda_j} &= \frac{1}{2} \Omega(P, P_j) \Omega(Q, P_j)\,, & \frac{d\Omega(P,Q)}{d\bar{\lambda}_j} &= \frac{1}{2} B(P, \bar{P}_j) B(Q, \bar{P}_j)\,, \\ \frac{dB(P, \bar{Q})}{d\lambda_j} &= \frac{1}{2} \Omega(P, P_j) B(P_j, \bar{Q})\,, & \frac{dB(P, \bar{Q})}{d\bar{\lambda}_j} &= \frac{1}{2} B(P, \bar{P}_j) \overline{\Omega(Q, P_j)}\,. \end{aligned} \tag{1.3.4}$$

The notation here is analogous to that in (1.3.2), i.e. $\Omega(P, P_j)$ stands for $(\Omega(P,Q)/dx_j(Q))\Big|_{Q=P_j}$ and $B(P, \bar{P}_j) := \left(B(P, \bar{Q}) / \overline{dx_j(Q)} \right) \Big|_{Q=P_j}$.

1.3.2 Spaces of rational functions

A meromorphic function of degree N on a Riemann sphere is a rational function $R(\gamma) = P_N(\gamma)/Q_N(\gamma)$, where P_N and Q_N are polynomials of degree N or less, such that either P_N or Q_N has the degree equal to N . Thus, the Hurwitz space in genus zero is the space of rational functions of fixed degree and fixed orders of poles (the latter gives the type of ramification over the point at infinity). The general form of a rational function,

$$R(\gamma) = \frac{a_N\gamma^N + a_{N-1}\gamma^{N-1} + \cdots + a_0}{(\gamma - b_N)(\gamma - b_{N-1})\ldots(\gamma - b_1)}\ , \tag{1.3.5}$$

is determined by $2N + 1$ parameters. However, the number of critical points (assuming that all of them are simple) equals $2N - 2$, and therefore the corresponding Hurwitz space has $2N - 2$ coordinates. This happens because the Möbius transformations $\gamma \to (a\gamma + b)/(c\gamma + d)$, $ad - bc = 1$ (which has three parameters) leaves the critical values of the rational function unchanged. When two zeros of the denominator in (1.3.5) coincide, the point at infinity becomes a branch point and there is $2N - 3$ finite branch points (local coordinates on the Hurwitz space) and $2N$ parameters left in (1.3.5). If a rational function has a pole at infinity, it can be written in the following form:

$$R(\gamma) = \beta\gamma + \delta + \sum_{k=1}^{N-1} \frac{a_k}{\gamma - b_k}\ .$$

Rational functions of degree two. Consider, for example, the function

$$R(\gamma) = \gamma + \frac{a}{\gamma - b}\ .$$

The covering $\mathcal{L}_R$ defined by the equation $\lambda = R(\gamma)$ has two sheets and two branch points $\lambda_1,\ \lambda_2$. The equation $\lambda = R(\gamma)$ can be written as $\gamma^2 - \gamma(b + \lambda) + a + \lambda b = 0$, and the inverse function $\nu(\lambda)$ such that $R(\nu(\gamma)) \equiv \lambda$ has the form

$$\nu(\lambda) = \frac{2}{\lambda_2 - \lambda_1}\Big(\frac{\lambda_1 + \lambda_2}{2} - \lambda + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\Big) , \tag{1.3.6}$$

where λ_1 and λ_2 are some functions of the parameters a and b . This function gives a map from the two-fold covering $\mathcal{L}_R$ onto the γ-sphere $\mathbb{C}P^1$: points lying on the different sheets of the covering and having equal projections on the λ-sphere correspond to the different branches of the square root in (1.3.6). Thus, one of the sheets is mapped inside the unit disc in the γ-sphere and the other one is mapped outside the unit disc. The inverse map $\nu : \mathcal{L}_R \to \mathbb{C}P^1$ is called the *uniformization map* of the covering $\mathcal{L}_R$.

1.3.3 Spaces of elliptic functions of degree two

A genus one Riemann surface (a torus) can be biholomorphically mapped to a quotient of the complex plane: $T := \mathbb{C}/\{2w, 2w'\}$ where w and w' are two complex numbers. The parallelogram built on the sides $2w$ and $2w'$ in the complex ς-plane is the fundamental domain of the torus. The pairwise identified sides play the role of two basis contours a and b , respectively. The unique normalized holomorphic differential ω is given by $\omega(\varsigma) = d\varsigma/(2w)$. Then, the b-period of the torus is $\mu := \int_0^{2w'} d\varsigma/(2w) = w'/w$.

The simplest genus one covering has two sheets and four simple branch points. It corresponds to the hyperelliptic curve

$$\nu^2 = (\zeta - \lambda_1)(\zeta - \lambda_2)(\zeta - \lambda_3)(\zeta - \lambda_4) , \tag{1.3.7}$$

where λ_j , $j = 1, \ldots, 4$ are the branch points. The normalized holomorphic differential ω is given by

$$\omega = \frac{d\zeta}{\nu}\left[\oint_a \frac{d\zeta}{\nu}\right]^{-1} . \tag{1.3.8}$$

If the basis cycles a and b are chosen as before (see (1.2.15)), i.e. the cycle a encircles the points P_3 and P_4 and the cycle b encircles the points P_2 and P_3 , then the Abel map $\mathcal{A}(P)=\int_{P_1}^{P}\omega$ has the following values at the ramification points:

$$\mathcal{A}(P_1)=0\,,\qquad \mathcal{A}(P_2)=\frac{1}{2}\,,\qquad \mathcal{A}(P_3)=\frac{1}{2}+\frac{\mu}{2}\,,\qquad \mathcal{A}(P_4)=\frac{\mu}{2}\,. \tag{1.3.9}$$

Let us now consider the covering (1.3.7), put $\lambda_1:=\infty$ to be the starting point of the Abel map, and denote λ_4 by λ_1 . This two-fold covering with four branch points, one of them being at infinity, can be defined by the equation $\zeta=\lambda(\varsigma)$ with the following function λ from the torus $T=\mathbb{C}/\{2w,2w'\}$ to $\mathbb{C}P^1$:

$$\lambda(\varsigma)=\wp(\varsigma)+c\,, \tag{1.3.10}$$

where c is a constant with respect to ς , and $\wp$ is the Weierstrass $\wp$-function (1.2.9). The pair (T,λ) has three parameters: w , w' , and c . Therefore there are three coordinates on the Hurwitz space of such coverings – the branch points λ_1 , λ_2 , λ_3 . Relation (1.2.11) implies that the branch points of the covering are given by $\lambda_1=\wp(w)+c$; $\lambda_2=\wp(w')+c$; $\lambda_3=\wp(w+w')+c$; they are related by $\lambda_1+\lambda_2+\lambda_3=3\,c$. Let us denote the corresponding ramification points on the torus T by P_1 , P_2 , P_3 (values of the ς-coordinate at these points are given respectively by $\varsigma_1=w$, $\varsigma_2=w'$ and $\varsigma_3=w+w'$). The fourth ramification point projects to $\zeta=\infty$ and in ς-plane has the coordinate $\varsigma=0$; let us denote this ramification point by $\infty^0\in T$.

From the equation $\zeta=\lambda(\varsigma)$ we have $d\zeta=\wp'(\varsigma)d\varsigma$. And therefore the normalized holomorphic differential $\omega(\varsigma)=d\varsigma/(2w)$ has the form (see (1.2.11)):

$$\omega(\zeta)=\frac{1}{4w}\frac{d\zeta}{\sqrt{(\zeta-\lambda_1)(\zeta-\lambda_2)(\zeta-\lambda_3)}}\,. \tag{1.3.11}$$

Its values at the ramification points (with respect to the standard local parameters) are given by

$$\begin{aligned} \omega(P_1) &= \frac{1}{2w\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}\,, \qquad \omega(P_2) = \frac{1}{2w\sqrt{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}}\,, \\ \omega(P_3) &= \frac{1}{2w\sqrt{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}}\,, \qquad \omega(\infty^0) = \frac{1}{2w}\,. \end{aligned} \tag{1.3.12}$$

Comparing relations (1.3.11) - (1.3.12) to (1.3.8), we see that the corresponding hyperelliptic curve is defined by the equation

$$\nu^2 = 4(\zeta-\lambda_1)(\zeta-\lambda_2)(\zeta-\lambda_3)\,. \tag{1.3.13}$$

The Thomae formulas (1.2.16) in this case have a simple form:

$$\pi^2\theta_2^4 = (2\omega)^2(\lambda_3-\lambda_1)\,, \quad \pi^2\theta_4^4 = (2\omega)^2(\lambda_2-\lambda_3)\,, \quad \pi^2\theta_3^4 = (2\omega)^2(\lambda_2-\lambda_1), \tag{1.3.14}$$

where $\theta_i := \theta_i(0)$, $i = 2,3,4$ are the standard theta-constants. From (1.3.14) we see that $\theta_2^4 + \theta_4^4 = \theta_3^4$.

As before, see (1.3.9), we have

$$\int_{\infty^0}^{P_1} \omega = \frac{\mu}{2}\,, \qquad \int_{\infty^0}^{P_2} \omega = \frac{1}{2}\,, \qquad \int_{\infty^0}^{P_3} \omega = \frac{\mu}{2} + \frac{1}{2}\,. \tag{1.3.15}$$

Using these relations and the Thomae formulas, one can find, for example, the expressions for the meromorphic differential W evaluated at the ramification points P_j with respect to the standard local parameters $x_j = \sqrt{\lambda - \lambda_j}$. Namely, the bidifferential W , by definition (1.2.19), (1.2.17), is given by $W(P,Q) := d_P d_Q \log\theta_1(\mathcal{A}(P) - \mathcal{A}(Q))$, or, equivalently,

$$W(P,Q) = -\,\omega(P)\,\omega(Q)\left(\frac{\theta_1''(\int_Q^P \omega)}{\theta_1'(\int_Q^P \omega)} - \left(\frac{\theta_1'(\int_Q^P \omega)}{\theta_1'(\int_Q^P \omega)}\right)^2\right)\,. \tag{1.3.16}$$

To evaluate expression (1.3.16) at the ramification points, one needs to use (1.3.15) and the following relations on theta-functions, which follow from the definition (1.2.1):

$$\theta_1(z\pm\frac{1}{2})=\pm\,\theta_2(z)\,,\qquad \theta_1(z\pm\frac{\mu}{2})=\pm i\theta_4(z)e^{-\pi i\mu/4\mp\pi iz}\,,$$
$$\theta_1(z+\frac{1}{2}\pm\frac{\mu}{2})=\theta_3(z)e^{-\pi i\mu/4\mp\pi iz}\,.$$

Using also the Thomae formulas (1.3.14), we find:

$$W(P_1,P_2)=\frac{1}{\pi^2\theta_2^2\theta_4^2(\lambda_1-\lambda_2)}\frac{\theta_3''}{\theta_3}\,,\qquad W(P_1,P_3)=\frac{i}{\pi^2\theta_3^2\theta_4^2(\lambda_3-\lambda_1)}\frac{\theta_2''}{\theta_2}\,,$$
$$W(P_2,P_3)=\frac{i}{\pi^2\theta_2^2\theta_3^2(\lambda_2-\lambda_3)}\frac{\theta_4''}{\theta_4}\,.$$

1.3.4 Bergman tau-function

This section presents a summary of main results of the paper [44].

The expansion of the bidifferential W (1.2.19) near the diagonal $P\sim Q$ has the form:

$$W(P,Q)\underset{Q\sim P}{=}\left(\frac{1}{(x(P)-x(Q))^2}+S(x(P))+o(1)\right)dx(P)dx(Q)\,,$$

where the quantity $6S$ is the Bergman projective connection (see the expansion (1.2.18) of the prime form at $P\sim Q$).

Denote by S_i the value of S at the ramification point P_i taken with respect to the local parameter $x_i(P)=\sqrt{\lambda-\lambda_i}$:

$$S_i=S(x_i)|_{x_i=0}\,. \tag{1.3.17}$$

Since the singular part of the bidifferential W in a neighbourhood of the point P_i does not depend on coordinates $\{\lambda_j\}$, the Rauch variational formulas (1.3.2) imply

$$\frac{\partial S_i}{\partial\lambda_j}=\frac{1}{2}W^2(P_i,P_j)\,.$$

The symmetry of this expression provides compatibility for the following system of differential equations which defines the *Bergman tau-function* τ_W :

$$\frac{\partial \log \tau_W}{\partial \lambda_i} = -\frac{1}{2} S_i \,, \qquad i = 1, \ldots, L \,. \tag{1.3.18}$$

The following theorem, proven in [44], gives an explicit formula for the Bergman tau-function in terms of holomorphic objects associated with the Riemann surface $\mathcal{L}$.

Theorem 1.1 *The Bergman tau-function is given by the following expression independent of the points P and Q ([44]):*

$$\tau_W = \mathcal{Q}^{2/3} \prod_{k,n=1\; k<n}^{L+m+1} [E(D_k, D_n)]^{d_k d_n/6} \tag{1.3.19}$$

where $\mathcal{Q}$ is given by

$$\mathcal{Q} = [d\lambda(P)]^{\frac{g-1}{2}}\, \mathcal{C}(P) \prod_{k=1}^{L+m+1} [E(P, D_k)]^{\frac{(1-g)d_k}{2}} \;;$$

$\mathcal{C}(P)$ is the multivalued differential defined by (1.2.12), $E(P,Q)$ is the prime form (1.2.17); $E(D_k, P)$ stands for $E(Q,P)\sqrt{dx_k(Q)}|_{Q=D_k}$; D_k and d_k are points and coefficients of the divisor $(d\lambda) := \sum_{k=1}^{L+m+1} d_k D_k$ of differential $d\lambda(P)$, i.e. $D_j = P_j$, $d_j = 1$ for $j = 1, \ldots, L$ and $D_{L+i+1} = \infty^i$, $d_{L+i+1} = -(n_i + 1)$, $i = 0, \ldots, m$. As before, differentials are evaluated at the points of divisor $(d\lambda)$ with respect to the standard local parameters: $x_j = \sqrt{\lambda - \lambda_j}$ for $j = 1, \ldots, L$ and $x_{L+1+i} = \lambda^{-1/(n_i+1)}$ for $i = 0, \ldots, m$.

In the case of a genus one surface, the Bergman tau-function is given by [44]:

$$\tau_W = \eta^2(\mu) \prod_{D_k \in (d\lambda)} (\omega(D_k))^{-d_k/12} \,,$$

d_k are the coefficients of the divisor $(d\lambda)$: $(d\lambda) = \sum d_k D_k$, and $\eta(\mu)$ is the Dedekind eta-function (1.2.7)-(1.2.8). For the space of coverings (1.3.10), the tau-function has the form:

$$\tau_W = \eta^2(\mu) \left(\prod_{i=1}^{3} \omega(P_j) \right)^{-\frac{1}{12}} (\omega(\infty^0))^{\frac{1}{4}} .$$

1.4 Integrable systems

1.4.1 U-V pairs

One of the broadly used definitions of integrable systems is that these are the nonlinear differential systems which can be represented as a compatibility condition of two linear differential equations:

$$\Phi_x = U\Phi , \qquad \Phi_y = V\Phi , \tag{1.4.1}$$

where U, V and Φ are matrix functions of independent variables (x, y) and a spectral parameter $\gamma \in \mathbb{C}$. The pair of matrices U and V such that the compatibility condition of the system (1.4.1) is equivalent to the nonlinear system is called the *U-V pair* of the nonlinear system.

The compatibility condition for the system (1.4.1) has the form: $U_y - V_x + [U, V] = 0$. It is also called the *zero curvature condition* as it expresses the vanishing of the curvature of the connection corresponding to the 1-form $Udx + Vdy$.

The well-known integrable systems such as Korteweg - de Vries, sin-Gordon, Boussinesq equations have U-V pairs with a constant, i.e. independent of x and y , spectral parameter.

Example 1. For the Korteveg - de Vries equation

$$u_t = u_{xxx} + 6uu_x , \tag{1.4.2}$$

the U-V pair is given by

$$U = \begin{pmatrix} 0 & u+\gamma \\ -1 & 0 \end{pmatrix} , \qquad V = \begin{pmatrix} -u_x & u_{xx} + 2u^2 - 2\gamma u - 4\gamma^2 \\ -2u + 4\gamma & u_x \end{pmatrix} ,$$

where γ is a constant with respect to x and t . The equation (1.4.2) is equivalent to the compatibility condition $U_t - V_x + [U, V] = 0$.

Example 2. The Ernst equation

$$((x-y)G_xG^{-1})_y + ((x-y)G_yG^{-1})_x = 0 , \tag{1.4.3}$$

where $G \in SU(1,1)/U(1)$, is equivalent to the stationary axially symmetric vacuum Einstein equation. The integrability of the Ernst equation was discovered [3, 55] in 1978. The U-V pair for the equation (1.4.3) has a spectral parameter γ which depends on x and y . Namely, the Ernst equation is the compatibility condition of the system (1.4.1) with matrices U and V of the form:

$$U = \frac{G_xG^{-1}}{1-\gamma} , \qquad V = \frac{G_yG^{-1}}{1+\gamma} , \tag{1.4.4}$$

where the spectral parameter γ is "variable", i.e. it is a function of x , y and a "hidden" ("constant") spectral parameter λ :

$$\gamma(\lambda, x, y) = \frac{2}{y-x}\Big(\frac{x+y}{2} - \lambda + \sqrt{(\lambda-x)(\lambda-y)}\Big) . \tag{1.4.5}$$

In this way, the Ernst equation can be viewed as a "deformation" of the principal chiral model equations. For this model, the matrices U and V have the form (1.4.4) where γ is a constant (independent of (x, y)) spectral parameter.

The function $\gamma(\lambda,x,y)$ (1.4.5) is nothing but the uniformization map of the genus zero two-fold covering of the λ-sphere $\mathbb{C}P^1$ with two branch points at $\lambda = x$ and $\lambda = y$ (see Section 1.3.2).

1.4.2 Deformation scheme of Burtzev-Zakharov-Mikhailov

The possibility to construct a class of "deformed" integrable systems, or integrable systems with variable spectral parameter, different from the Ernst equation was first noticed by Burtzev, Mikhailov and Zakharov [9]. They proposed to consider U-V pairs of the form (1.4.1) where x and y are independent variables; matrices U and V depend on (x,y) and a variable spectral parameter γ (which in turn depends on (x,y) and the hidden spectral parameter λ). Namely,

$$\begin{aligned} U(x,y,\gamma) &= u_0(x,y) + \sum_{n=1}^{N_1} \frac{u_n(x,y)}{\gamma(x,y)-\gamma_n(x,y)} \,, \\ V(x,y,\gamma) &= v_0(x,y) + \sum_{n=1}^{N_2} \frac{v_n(x,y)}{\gamma(x,y)-\tilde{\gamma}_n(x,y)} \,. \end{aligned} \tag{1.4.6}$$

As a part of compatibility conditions of the linear system (1.4.1), after an appropriate fractional-linear transformation in the γ-plane, the following system of equations for $\gamma(x,y,\lambda)$ must be satisfied:

$$\frac{\partial\gamma}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\gamma-\tilde{\gamma}_m} = 0 \,, \qquad \frac{\partial\gamma}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\gamma-\gamma_m} = 0 \,, \tag{1.4.7}$$

where b_n and c_n are certain functions of (x,y) . The compatibility condition of the system (1.4.7) gives the following system for $\gamma_n(x,y)$ and $\tilde{\gamma}_n(x,y)$:

$$\frac{\partial\gamma_n}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\gamma_n-\tilde{\gamma}_m} = 0 \,, \qquad \frac{\partial\tilde{\gamma}_n}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\tilde{\gamma}_n-\gamma_m} = 0 \,, \tag{1.4.8}$$

$$\frac{\partial c_n}{\partial y} - 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\gamma_n-\tilde{\gamma}_m)^2} = 0 \,, \qquad \frac{\partial b_n}{\partial x} - 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\tilde{\gamma}_n-\gamma_m)^2} = 0 \,. \tag{1.4.9}$$

A solution to this system in terms of spaces of rational functions was recently found in [42].

1.4.3 Classical elliptic r-matrix

The classical r-matrices (see [21]) were introduced to conveniently describe the Hamiltonian aspects of integrable systems. Namely, it turns out that, in terms of an appropriately defined so-called transition matrix $T(\gamma)$, the fundamental Poisson brackets of many integrable systems can be rewritten in a simple form:

$$\left\{\overset{1}{T}(\gamma)\,,\overset{2}{T}(\mu)\right\} = \left[\overset{12}{r}(\gamma-\mu)\,,\overset{1}{T}(\gamma)\,\overset{2}{T}(\mu)\right]\;,$$

where $\overset{12}{r}(\gamma)$ is a $K^2\times K^2$ matrix acting in the tensor product $\mathbb{C}^K\otimes\mathbb{C}^K$, independent of the physical fields; $\overset{1}{T}:=T\otimes I$; $\overset{2}{T}:=I\otimes T$. As a corollary of the Jacobi identity, the matrix $r(\gamma)$ (called the classical r-matrix) satisfies the so-called classical Yang-Baxter equation:

$$[\overset{12}{r}(\lambda-\mu)\,,\overset{13}{r}(\lambda)+\overset{23}{r}(\mu)]+[\overset{13}{r}(\lambda)\,,\overset{23}{r}(\mu)]=0\,. \tag{1.4.10}$$

The time dynamics (depending on the model, time variable may coincide with either x , y , their sum or difference) is generated by a Hamiltonian which can be found from invariant polynomials of the matrix T (for example, a Hamiltonian can be equal to the residue of $\mathrm{tr}T^2(\gamma)$ at $\gamma=\infty$).

There exist 3 classes of solutions to the classical Yang-Baxter equation: rational, trigonometric and elliptic. In the sequel we make use of the elliptic r-matrices. The rational and trigonometric r-matrices can be considered as an appropriate degenerations of the elliptic ones.

Consider the following combinations of theta-functions (1.2.1) for $(A,B)\neq(0,0)$:

$$w_{AB}(\gamma)=\frac{\theta_{[AB]}(\gamma)\theta'_{[00]}(0)}{\theta_{[AB]}(0)\theta_{[00]}(\gamma)}\;; \tag{1.4.11}$$

where we denote

$$\theta_{[AB]}(\gamma) \equiv \theta_{[AB]}(\gamma;\mu) := \theta\left[\tfrac{A}{K}-\tfrac{1}{2},\tfrac{1}{2}-\tfrac{B}{K}\right](\gamma;\mu) \; .$$

All the w_{AB}'s have a simple pole with unit residue at $\gamma = 0$ and the following twist properties:

$$w_{AB}(\gamma+1) = \epsilon^{A} w_{AB}(\gamma) \; , \quad w_{AB}(\gamma+\mu) = \epsilon^{B} w_{AB}(\gamma) \; , \tag{1.4.12}$$

where $\epsilon = e^{2\pi i/K}$.

The Pauli matrices σ_1, σ_2, σ_3 given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.4.13}$$

form a basis of $sl(2,\mathbb{C})$. The following matrices σ_{AB} for $A, B = 0, \ldots, K-1$, $(A,B) \neq (0,0)$ give the higher rank analogs of the Pauli matrices. They form a basis of $sl(K,\mathbb{C})$ and are defined by

$$\sigma_{AB} := H^{A} F^{B} \; , \tag{1.4.14}$$

where F is the diagonal matrix

$$F := \mathrm{diag}\{1, \epsilon, \epsilon^2, \ldots, \epsilon^{K-1}\} \; , \qquad \epsilon = e^{2\pi i/K} \; . \tag{1.4.15}$$

and H is the following permutation $[K \times K]$ matrix:

$$H := \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & . & . & . & 0 \end{pmatrix} . \tag{1.4.16}$$

The matrices F and H satisfy the relations: $\epsilon FH = HF$, and $F^K = H^K = I$. Together with σ_{AB} , we consider the dual basis σ^{AB} :

$$\sigma^{AB} := \frac{\epsilon^{-AB}}{K}\sigma_{-A,-B} ,$$

such that

$$\mathrm{tr}\,(\sigma_{AB}\sigma^{CD}) = \delta_A^C\delta_B^D .$$

In the case of 2×2 matrices, $K = 2$, the matrices σ_{AB} and σ^{AB} become:

$$\sigma_{10} = \sigma_1 , \qquad \sigma_{11} = i\,\sigma_2 , \qquad \sigma_{01} = \sigma_3 ;$$
$$\sigma^{10} = \frac{1}{2}\sigma_1 , \quad \sigma^{11} = \frac{i}{2}\sigma_2 , \quad \sigma^{01} = \frac{1}{2}\sigma_3 .$$

The classical elliptic r-matrix is given by:

$$\overset{12}{r}(\gamma) = \sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K-1} w_{AB}(\gamma)\,\overset{1}{\sigma}_{AB}\,\overset{2}{\sigma}{}^{AB} ; \tag{1.4.17}$$

it satisfies the classical Yang-Baxter equation (1.4.10). The properties of matrices F and H and the periodicity (1.4.12) of w_{AB} imply the following periodicity properties for the elliptic r-matrix (1.4.17):

$$\begin{aligned} \overset{12}{r}(\gamma+1) &= \overset{1}{F}{}^{-1}\,\overset{12}{r}(\gamma)\,\overset{1}{F} , \\ \overset{12}{r}(\gamma+\mu) &= \overset{1}{H}\,\overset{12}{r}(\gamma)\,\overset{1}{H}{}^{-1} . \end{aligned} \tag{1.4.18}$$

1.5 Isomonodromic deformations

Here we collect some basic facts from the theory of isomonodromic deformations, see [6, 39] for details and references.

1.5.1 Schlesinger system on the Riemann sphere

Consider the following linear differential equation for a matrix-valued function $\Psi(z) \in SL(K,\mathbb{C})$:

$$\frac{d\Psi}{d\lambda} = A(z)\Psi , \qquad A(z) = \sum_{j=1}^{M} \frac{A_j}{z - z_j} , \tag{1.5.1}$$

where the residues $A_j \in sl(K,\mathbb{C})$ are independent of $z \in \mathbb{C}P^1$. The requirement of regularity of the function Ψ at $z = \infty$ gives $\sum_{j=1}^{M} A_j = 0$. Let the solution to (1.5.1) satisfy the initial condition $\Psi(z_0) = I$ at some point $z_0 \in \mathbb{C}P^1$. The matrix Ψ defined in this way has regular singularities at the points $\{z_j\}_{j=1}^{M}$ and is generically non-single valued in $\mathbb{C}P^1$, i.e. it is defined on the universal covering of $\mathbb{C}P^1 \setminus \{z_1, \ldots, z_M\}$. Under analytic continuation around $z = z_j$ it gains the right multipliers M_j which are called the *monodromy matrices*:

$$\Psi(z) \to \Psi(z)M_j .$$

The asymptotical expansion of the function Ψ near the singularities z_j has the form:

$$\Psi(z) = G_j \left(I + \mathcal{O}(z - z_j)\right) (z - z_j)^{T_j} C_j \tag{1.5.2}$$

with constant matrices $G_j,\ C_j \in SL(K,\mathbb{C})$, and a traceless diagonal matrix T_j . In terms of these matrices, the residues A_j and the monodromy matrices are given by

$$A_j = G_j T_j G_j^{-1} , \qquad M_j = C_j^{-1} e^{2\pi i T_j} C_j .$$

The *isomonodromic deformation* of the system (1.5.1) is a deformation which changes positions of singularities $\{z_j\}$ while keeping the monodromy matrices $\{M_j\}$ unchanged. If none of the eigenvalues of the matrix A_j differ by an integer number, then the isomonodromy condition allows to differentiate the expansion (1.5.2) to get the following dependence of Ψ on $\{z_j\}$:

$$\frac{d\Psi}{dz_j} = \left(\frac{A_j}{z_0 - z_j} - \frac{A_j}{z - z_j}\right) \Psi \,. \tag{1.5.3}$$

The compatibility of equations (1.5.1) and (1.5.3) is given by the Schlesinger system for the functions $A_j(\{z_k\})$:

$$\begin{aligned} \frac{\partial A_j}{\partial z_k} &= \frac{[A_j, A_k]}{z_j - z_k} - \frac{[A_j, A_k]}{z_0 - z_k} \,, \qquad j \neq k \,; \\ \frac{\partial A_j}{\partial z_j} &= -\sum_{k \neq j} \left(\frac{[A_k, A_j]}{z_k - z_j} - \frac{[A_k, A_j]}{z_k - z_0}\right) . \end{aligned} \tag{1.5.4}$$

It is easy to check that the quantities $\mathrm{tr} A_j^2$ are integrals of motion for the Schlesinger system.

The tau-function τ_{JM} of the Schlesinger system was introduced in [39]. It is defined by

$$\frac{\partial}{\partial z_j} \log \tau_{JM} = \frac{1}{2} \operatorname*{res}_{z=z_j} \mathrm{tr} \left(\Psi_z \Psi^{-1}\right)^2 \equiv \sum_{j \neq i} \frac{\mathrm{tr} A_i A_j}{z_j - z_i} \,; \qquad \frac{\partial \tau_{JM}}{\partial \bar{z}_j} = 0 \,.$$

If the number of singularities M equals 4 , one can fix by a Möbius transformation three singular points $z_1,\ z_2,\ z_4$ to be $0,\ 1,\ \infty$, respectively; the remaining pole $x := z_3$ is the variable in the Schlesinger system. The system (1.5.1) on the function Ψ in this case has the form:

$$\frac{d\Psi}{d\lambda} = A(z)\Psi \,, \qquad A(z) = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-x} \,, \qquad A_4 \equiv A_\infty = A_1 + A_2 + A_3 \,.$$

Let the normalization point be at infinity: $z_0 := \infty$, $\Psi(\infty) = I$. Then the Schlesinger system (1.5.4) has the form:

$$\frac{dA_1}{dx} = \frac{[A_3, A_1]}{x} \,, \qquad \frac{dA_2}{dx} = \frac{[A_3, A_2]}{x-1} \,, \qquad \frac{dA_3}{dx} = -\frac{[A_3, A_1]}{x} - \frac{[A_3, A_2]}{x-1} \,. \tag{1.5.5}$$

Let us denote by $t_j^2 := \mathrm{tr} A_j^2/2$ the eigenvalues of the matrices A_j , $j = 1, \ldots, 4$, where we put $A_4 = A_1 + A_2 + A_3$.

If the matrix dimension equals 2 , the four-point Schlesinger system (1.5.5) may be equivalently rewritten in terms of a function of one variable, the position $y(x)$ of the zero of the off-diagonal term A_{12} of $A(z)$. The equation for $y(x)$ coincides with the Painlevé-VI equation:

$$\begin{aligned}\frac{d^2y}{dx^2} =& \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{y}{y-x}\right)\left(\frac{dy}{dx}\right)^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)\frac{dy}{dx}\\ &+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha+\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}+\delta\frac{x(x-1)}{(y-x)^2}\right)\end{aligned}$$

with constants $\alpha = (2t_4-1)^2/2$, $\beta = -2t_1^2$, $\gamma = 2t_2^2$, $\delta = (1-4t_3^2)/2$ (see for example [35]).

If the integrals of motion $\mathrm{tr}A_j^2$ are fixed to equal $1/8$, the functions

$$\Omega_1^2 = -(\frac{1}{8}+\mathrm{tr}A_2A_3), \qquad \Omega_2^2 = -(\frac{1}{8}+\mathrm{tr}A_1A_3), \qquad \Omega_3^2 = -(\frac{1}{8}+\mathrm{tr}A_1A_2)$$

give a solution to the following system:

$$\frac{d\Omega_1}{dx} = \frac{1}{x}\Omega_2\Omega_3\,, \qquad \frac{d\Omega_2}{dx} = -\frac{1}{x-1}\Omega_1\Omega_3\,, \qquad \frac{d\Omega_3}{dx} = \frac{1}{x(x-1)}\Omega_1\Omega_2 \tag{1.5.6}$$

with the integral of motion $\Omega_1^2+\Omega_2^2+\Omega_3^2 = -1/4$. In this case, the coefficients of the Painlevé-VI equation are given by $(\alpha,\beta,\gamma,\delta) = (1/8,-1/8,1/8,3/8)$.

1.5.2 Schlesinger system on the torus

Here, following [70], we describe one possible generalization of the Schlesinger system to the case of an elliptic surface.

Consider the elliptic (genus one) surface T with periods 1 and μ , i.e. $T = \mathbb{C}/\{1;\mu\}$. The isomonodromic condition of the form (1.5.1) cannot be written on a genus one surface since a function with a single pole does not exist there. This means that an independent variation of simple poles of $\Psi\Psi^{-1}$ which preserves the monodromies around the singularities and basis cycles of the torus is impossible. Therefore, the notion of isomonodromic deformation has to be modified. For example, one may consider the case where not all the poles are varied independently or assume ([62]) that some of the poles are of order higher than one. Another possibility is to allow the matrix $A = \Psi\Psi^{-1}$ to be non-single valued, i.e. to have "twists" under analytical continuation along the basis cycles a and b of the torus: $A(z+1) = QA(z)Q^{-1}$, $A(z+\mu) = RA(z)R^{-1}$, the matrices Q and R being independent of z . The following isomonodromic deformation on the torus with $Q = F^{-1}$ and $R = H$ (the matrices F and H are defined by (1.4.15), (1.4.16)) was proposed in [70]. For the case of 2×2 matrix Ψ , the twists are given by the Pauli matrices (1.4.13): $Q = \sigma_3$ and $R = \sigma_1$. This choice of the twists results in studying isomonodromic deformations of the system

$$\frac{d\Psi}{d\gamma} = A(\gamma)\Psi , \qquad \overset{1}{A}(\gamma) = \sum_{j=1}^{M} \overset{2}{\mathrm{tr}} \left(\overset{12}{r}(\gamma - z_j)\, \overset{2}{A_j} \right); \tag{1.5.7}$$

where γ is a coordinate on the torus $T = \mathbb{C}/\{1,\mu\}$; $r(\gamma)$ is the elliptic r-matrix (1.4.17); $z_j \in T$, $j = 1, \ldots, M$; M is some integer. The matrix $A(\gamma)$ has only simple poles at the points $\{z_j\}$ with residues A_j . The residues are parameterized as follows:

$$A_j = \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} \sigma_{AB} ,$$

where matrices σ_{AB} are given by (1.4.14); $A_j^{AB} \in \mathbb{C}$. The matrix $A(\gamma)$ has the following periodicity properties:

$$A(\gamma+1) = F^{-1}A(\gamma)F , \qquad A(\gamma+\mu) = HA(\gamma)H^{-1} ,$$

the matrices F and H are defined by (1.4.15), (1.4.16). As in the case of the Riemann sphere the function Ψ has regular singularities at the points $\gamma = z_j$ and the asymptotical expansion is assumed to have the same form:

$$\Psi(\gamma) = (G_j + O(\gamma - z_j))\,(\gamma - z_j)^{T_j} C_j\ , \tag{1.5.8}$$

where matrices G_j , C_j , T_j do not depend on γ; C_j , $G_j \in SL(K,\mathbb{C})$ and T_j are diagonal traceless matrices such that any two entries of T_j do not differ by an integer number. The system (1.5.7) and the periodicity properties of the matrix A imply that the matrix Ψ transforms with respect to periods 1 and μ of the torus T as follows:

$$\Psi(\gamma+1) = F^{-1}\Psi(\gamma)M_a\ , \qquad \Psi(\gamma+\mu) = H\Psi(\gamma)M_b\ ,$$

and that, being analytically continued along a contour l_j surrounding the point z_j , the function Ψ gains a right multiplier:

$$\Psi(\gamma^{l_j}) \quad = \quad \Psi(\gamma)M_j\ .$$

The matrices M_a , M_b and M_j satisfy one relation: $M_a M_b M_a^{-1} M_b^{-1} M_M M_{M-1} \dots M_1 = I$. The matrices M_a , M_b together with M_j are called the monodromy matrices. The monodromy condition is now the assumption of independence of all monodromy matrices of the positions of singularities $\{z_j\}$ and the b-period μ of the elliptic Riemann surface. Like on the Riemann sphere, this condition together with expansion (1.5.8) imply that the function $\partial_{z_j}\Psi\Psi^{-1}$ has the only simple pole at $\gamma = z_j$ with the residue $-A_j$. In addition, it has the following twist properties:

$$\frac{\partial\Psi}{\partial z_j}\Psi^{-1}(\gamma+1) = F^{-1}\frac{\partial\Psi}{\partial z_j}(\gamma)F\ ; \qquad \frac{\partial\Psi}{\partial z_j}\Psi^{-1}(\gamma+\mu) = H\frac{\partial\Psi}{\partial z_j}\Psi^{-1}(\gamma)H^{-1}\ .$$

A holomorphic single valued function does not exist on a torus; therefore two functions coincide if they have the same singularities and twists with respect to the basis cycles.

Hence,

$$\overset{1}{\Psi}_{z_i}\overset{1}{\Psi}{}^{-1} = -\overset{2}{\mathrm{tr}}\left(\overset{12}{r}(\gamma - z_i)\overset{2}{A}_i\right). \tag{1.5.9}$$

The derivative $\Psi_\mu\Psi^{-1}$ with respect to the module μ is holomorphic at $\gamma = z_j$ (but not at $\gamma = z_j + \mu$) and has the following twist properties:

$$\begin{aligned}
&\frac{\partial\Psi}{\partial\mu}\Psi^{-1}(\gamma+1) = F^{-1}\frac{\partial\Psi}{\partial\mu}\Psi(\gamma)F\ ;\\
&\frac{\partial\Psi}{\partial\mu}\Psi^{-1}(\gamma+\mu) = H\left(\frac{\partial\Psi}{\partial\mu}\Psi^{-1}(\gamma) - \frac{\partial\Psi}{\partial\gamma}\Psi^{-1}(g)\right)H^{-1}\ .
\end{aligned} \tag{1.5.10}$$

Consider the following combinations of elliptic theta-functions:

$$\mathcal{Z}_{AB}(\gamma) = \frac{w_{AB}(\gamma)}{2\pi i}\left(\frac{\theta'_{[AB]}(\gamma)}{\theta_{[AB]}(\gamma)} - \frac{\theta'_{[AB]}(0)}{\theta_{[AB]}(0)}\right), \qquad (A,B) \neq (0,0)\ . \tag{1.5.11}$$

The functions $\mathcal{Z}_{AB}$ have no singularities and satisfy the following periodicity properties:

$$\mathcal{Z}_{AB}(\gamma+1) = \epsilon^A\mathcal{Z}_{AB}\ , \qquad \mathcal{Z}_{AB}(\gamma+\mu) = \epsilon^B\left(\mathcal{Z}_{AB}(\gamma) - w_{AB}(\gamma)\right)\ . \tag{1.5.12}$$

The heat equation (1.2.3) for the theta-functions implies the following relation between $\mathcal{Z}_{AB}$ and w_{AB} :

$$\partial_\mu w_{AB}(\gamma;\mu) = \partial_\gamma\mathcal{Z}_{AB}(\gamma;\mu)\ . \tag{1.5.13}$$

The properties (1.5.12), (1.5.13) give the expression for the μ-derivative of Ψ :

$$\Psi_\mu\Psi^{-1} = \sum_{j=1}^{L}\sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K-1} A_j^{AB}\mathcal{Z}_{AB}(\gamma - z_i)\sigma_{AB}\ . \tag{1.5.14}$$

The compatibility condition of (1.5.7), (1.5.9) and (1.5.14) gives the Schlesinger system on the elliptic surface:

$$
\begin{aligned}
\frac{\partial \overset{1}{A}_i}{\partial z_j} &= \left[\overset{1}{A}_i, \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (z_i - z_j) \overset{2}{A}_j \right) \right] , \quad i \neq j , \\
\frac{\partial \overset{1}{A}_i}{\partial z_i} &= - \sum_{j=1, j\neq i}^{L} \left[\overset{1}{A}_i, \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (z_i - z_j) \overset{2}{A}_j \right) \right] , \\
\frac{\partial \overset{1}{A}_i}{\partial \mu} &= - \sum_{j=1}^{L} \left[\overset{1}{A}_i, \overset{2}{\mathrm{tr}} \left(\overset{2}{A}_j \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} \mathcal{Z}_{AB}(z_i - z_j) \, \overset{1}{\sigma}_{AB} \overset{2}{\sigma}{}^{AB} \right) \right] .
\end{aligned}
\tag{1.5.15}
$$

The tau-function of this system is defined as the generating function of the following Hamiltonians:

$$
\begin{aligned}
H_i &= \frac{1}{4\pi i} \oint_{z_i} tr A^2(\gamma) d\gamma = \sum_{j=1, j\neq i}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} w_{AB}(z_i - z_j) , \\
H_\mu &= -\frac{1}{2\pi i} \oint_a tr A^2(\gamma) d\gamma = \frac{1}{2} \sum_{i,j=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} \mathcal{Z}_{AB}(z_i - z_j) .
\end{aligned}
$$

$$
\frac{\partial \log \tau_{\mathrm{S}ch}}{\partial z_i} = H_i , \qquad \frac{\partial \log \tau_{\mathrm{S}ch}}{\partial \mu} = H_\mu .
$$

1.6 Systems of hydrodynamic type

1.6.1 Darboux-Egoroff metrics

The Christoffel symbols Γ^i_{jk} for the Levi-Civita connection ∇ of the metric $\mathbf{ds^2} = \sum_{ij} g_{ij} d\lambda_i d\lambda_j$ are given by

$$
\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \left(\frac{\partial g_{jl}}{\partial \lambda_k} + \frac{\partial g_{kl}}{\partial \lambda_j} - \frac{\partial g_{jk}}{\partial \lambda_l} \right) . \tag{1.6.1}
$$

Here g^{il} are the entries of the matrix g^{-1} , i.e. the following holds: $\sum_k g^{ik} g_{kj} = \delta_{ij}$.

The Riemann curvature tensor R of the Levi-Civita connection ∇ maps three vector fields X , Y and Z to the vector field $R(X,Y)Z$, where

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \; ; \tag{1.6.2}$$

∇_X is the covariant derivative with respect to the vector field X .

A metric is called *flat* if its Riemann curvature tensor vanishes.

A diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ is called potential if there exists a function $U(\{\lambda_k\})$ such that $g_{ii} = \partial^2_{\lambda_i} U$. A diagonal flat potential metric is called a *Darboux-Egoroff* metric. In this thesis, we use the term "metric" for a bilinear quadratic form which is not necessary real and positive definite.

For a diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$, the *rotation coefficients* β_{ij} are defined for $i \neq j$ by

$$\beta_{ij} = \frac{\partial_{\lambda_j} \sqrt{g_{ii}}}{\sqrt{g_{jj}}} \, . \tag{1.6.3}$$

A diagonal metric is potential if and only if its rotation coefficients are symmetric with respect to indices: $\beta_{ij} = \beta_{ji}$. A sufficient condition for flatness of a diagonal metric is provided (due to Darboux and Egoroff) by the following system of equations for rotation coefficients β_{ij} :

$$\partial_{\lambda_k} \beta_{ij} = \beta_{ik}\beta_{kj} \, , \qquad i,j,k \text{ are distinct,} \tag{1.6.4}$$

$$\sum_k \partial_{\lambda_k} \beta_{ij} = 0 \qquad \text{for all } \beta_{ij} \, . \tag{1.6.5}$$

For a flat metric there exists a set of *flat coordinates* in which coefficients of the metric are constant. In flat coordinates the Christoffel symbols (1.6.1) vanish. Hence, the covariant derivative ∇_{t^A} coincides with the usual partial derivative ∂_{t^A} . Therefore, flat coordinates can be found from the equation $\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}t = 0$ ($\mathbf{x}$ and $\mathbf{y}$ are arbitrary vector fields on the manifold). In coordinates $\{\lambda_j\}$ this equation has the form:

$$\partial_{\lambda_i}\partial_{\lambda_j}t = \sum_k \Gamma_{ij}^k \partial_{\lambda_k} t\,. \tag{1.6.6}$$

The Christoffel symbols for a diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ are given by :

$$\Gamma_{ii}^k = -\frac{1}{2}\frac{\partial_{\lambda_k} g_{ii}}{g_{kk}}\,, \quad \Gamma_{ii}^i = \frac{1}{2}\frac{\partial_{\lambda_i} g_{ii}}{g_{ii}}\,, \qquad \Gamma_{ij}^i = \frac{1}{2}\frac{\partial_{\lambda_j} g_{ii}}{g_{ii}}\,, \qquad \Gamma_{ij}^k = 0\,, \quad i,j,k \text{ are distinct}\,.$$

1.6.2 Systems of hydrodynamic type

Here we give a brief introduction into the theory of systems of hydrodynamic type. A rigorous and detailed description of this subject can be found, for example, in the review [73].

A system of differential equations for functions $u_i(x,t)$, $i = 1, \ldots, n$ of the form

$$\partial_t u_i = \sum_{j=1}^n V_{ij}(u)\partial_x u_i\,, \qquad i = 1, \ldots, n\,, \tag{1.6.7}$$

is called a *system of hydrodynamic type*. The coefficients V_{ij} are called the *characteristic speeds*. A system (1.6.7) is called *hamiltonian* [16, 73] if there exist a function $h(u)$ which does not depend on derivatives of u and on a flat nondegenerate metric (g_{ij}) in the space with coordinates $\{u_i\}$ such that

$$V_{ij}(u) = \sum_{k=1}^n g_{ik}\nabla_k\nabla_j h\,. \tag{1.6.8}$$

Here ∇ is the Levi-Civita connection of the metric (g_{ij}) . As easily follows from the flatness of the metric and (1.6.8), a system (1.6.7) with characteristic speeds (V_{ij}) is hamiltonian if and only if there exists a nondegenerate flat metric $(g_{ij}(u))$ such that the following two conditions hold:

$$\sum_{k=1}^{n} g_{ik} V_{jk} = \sum_{k=1}^{n} g_{jk} V_{ik} \; ; \tag{1.6.9}$$

$$\nabla_i V_{jk} = \nabla_j V_{ik} \; . \tag{1.6.10}$$

We shall consider the diagonal systems of hydrodynamic type:

$$\partial_t u_i = V_i(u) \partial_x u_i \; , \qquad i = 1, \ldots, n \; . \tag{1.6.11}$$

Some of the systems (1.6.7) can be diagonalized by an appropriate change of variables u_i . Conditions (1.6.9) imply that for the system (1.6.11) the metric (g_{ij}) is diagonal: $0 = g_{ij} V_j - g_{ij} V_i = g_{ij}(V_j - V_i)$. For a diagonal metric, the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection ∇ vanish if the indices i, j, k are distinct, and (1.6.10) becomes:

$$\partial_i V_k = \Gamma^k_{ik}(V_i - V_k) \; , \qquad i \neq k \; ; \qquad i, k = 1, \ldots, n \; . \tag{1.6.12}$$

Relation (1.6.12) guarantees that the diagonal system (1.6.11) with the matrix (V_i) is hamiltonian and corresponds to the diagonal flat metric $(g_{ii}(u))$. The compatibility of (1.6.12) follows from the vanishing of the Riemann curvature tensor (1.6.2).

For a fixed metric g (and, therefore, the Christoffel symbols), the system (1.6.12) has infinitely many solutions $\{V_k\}$. The set of solutions is parameterized by n functions of one variable. Any solution $\{W_k\}_{k=1}^n$ (which might in particular coincide with $\{V_k\}_{k=1}^n$) gives a diagonal hamiltonian system of hydrodynamic type compatible with (1.6.11):

$$\partial_y u_i = W_i(u) \partial_x u_i \; , \qquad i = 1, \ldots, n \; .$$

Generalized hodograph method For any solution $\{W_k\}_{k=1}^n$ of the system (1.6.12) consider the following system of n equations for n functions u_i ($u = (u_1, \ldots, u_n)$):

$$W_k(u) = V_k(u)t + x \;, \qquad k = 1, \ldots, n, \tag{1.6.13}$$

where x and t are parameters. Differentiating (1.6.13) with respect to x and t and using (1.6.11) and (1.6.12), it is easy to see that equations (1.6.13) define implicitly a solution $\{u_i(x,t)\}$ to the system of hydrodynamic type (1.6.11). The converse is also true, namely, the following theorem holds.

Theorem 1.2 *A smooth solution $\{u_i(x,t)\}$ to the system (1.6.13) satisfies the diagonal hamiltonian system (1.6.11). Conversely, any solution $\{u_i(x,t)\}$ to the system (1.6.11) (locally in a neighbourhood of the point (x_0, t_0) such that $\partial_x u_i(x_0, t_0) \neq 0$ for any i) can be obtained as a solution to the system (1.6.13) for some $\{W_k\}$ which satisfy (1.6.12).*

1.7 Integrable systems associated with the space of rational functions

This section presents a brief review of the work [42].

1.7.1 Differential equations for critical points of rational maps

Consider a rational map $R(\gamma)$ of degree N of the form:

$$R(\gamma) = \gamma + \sum_{m=1}^{N-1} \frac{r_m}{\gamma - \mu_m} \;. \tag{1.7.1}$$

The critical points $\gamma_1 \;, \ldots, \gamma_{2N-2}$ of the function R , which are such that $R'(\gamma_j) = 0$, can be considered as functions of the critical values $\lambda_1, \ldots, \lambda_{2N-2}$, defined by $\lambda_j := R(\gamma_j)$.

The equation

$$\lambda = R(\gamma), \qquad \gamma \in \mathbb{C}P^1, \ \ \lambda \in \mathbb{C}P^1$$

defines a covering $\mathcal{L}$ of the λ-sphere $\mathbb{C}P^1$. The projection from the covering to the base is denoted by π . The inverse $\nu(P)$ to the rational map $R(\gamma)$ is defined such that $R(\nu(P)) = \pi(P)$. The function $\nu(P)$ depends on the variables $\lambda_1, \dots, \lambda_{2N-2}$ as parameters. It has only one pole at the point at infinity of some (number one) sheet of the covering $\mathcal{L}$; therefore,

$$\nu(P) = \lambda + o(1) \qquad \text{as} \ \ P \to \infty^{(1)}. \tag{1.7.2}$$

As is shown in [42] the function $\nu(P)$, considered locally as function of λ and depending on branch points $\lambda_1, \dots, \lambda_{2N-2}$ as parameters, satisfies the following equations:

$$\frac{\partial \nu}{\partial \lambda} = 1 + \sum_{n=1}^{2N-2} \frac{\alpha_n}{\nu - \gamma_n}\,, \qquad \frac{\partial \nu}{\partial \lambda_n} = -\frac{\alpha_n}{\nu - \gamma_n}\,. \tag{1.7.3}$$

where α_m are some functions of the branch points. The compatibility condition of equations (1.7.3) imply the following equations for the critical points γ_m of the rational function (1.7.1) and residues α_m from (1.7.3) considered as functions of critical values λ_m :

$$\frac{\partial \gamma_m}{\partial \lambda_n} = \frac{\alpha_n}{\gamma_n - \gamma_m}\,, \qquad m \neq n\,; \qquad \frac{\partial \gamma_m}{\partial \lambda_m} = 1 + \sum_{n=1,\, n\neq m}^{2N-2} \frac{\alpha_n}{\gamma_m - \gamma_n}\,, \tag{1.7.4}$$

$$\frac{\partial \alpha_m}{\partial \lambda_n} = \frac{2\alpha_n\alpha_m}{(\gamma_n - \gamma_m)^2}\,, \qquad m \neq n\,; \qquad \frac{\partial \alpha_m}{\partial \lambda_m} = -\sum_{n=1,\, n\neq m}^{2N-2} \frac{2\alpha_n\alpha_m}{(\gamma_n - \gamma_m)^2} \tag{1.7.5}$$

for all $m, n = 1, \dots, 2N-2$.

Rational functions of the form (1.7.1) were introduced by Kupershmidt and Manin [53] in connection with Benney systems. The fact that the critical points of these functions satisfy equations (1.7.4), (1.7.5) follows from the paper by Gibbons and Tsarev [29].

It is easy to establish a relationship between solutions to the system (1.7.3), (1.7.4), (1.7.5) and solutions to the system (1.4.7), (1.4.8), (1.4.9) which appears from the deformation scheme of Burtzev, Zakharov, Mikhailov.

Namely, suppose that function $\nu(\lambda, \{\lambda_m\}_{m=1}^{2N-2})$ satisfies equations (1.7.3) with respect to variables λ_m . Assume that $2N-2 = N_1+N_2$ and split the set of variables $\{\lambda_1, \dots, \lambda_{N_1+N_2}\}$ into two subsets: $\{\lambda_1, \dots, \lambda_{N_1}\}$ and $\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_{N_2}\}$ where $\tilde{\lambda}_n \equiv \lambda_{N_1+n}$, $n = 1, \dots, N_2$. The set $\{\gamma_m\}$ of values of function $\nu(P)$ at these points is splited in the same way:

$$\{\gamma_1, \dots, \gamma_{2N-2}\} = \{\gamma_1, \dots, \gamma_{N_1}\} \cup \{\tilde{\gamma}_1, \dots, \tilde{\gamma}_{N_2}\} ,$$

where $\tilde{\gamma}_n \equiv \gamma_{N_1+n}$, $n = 1, \dots, N_2$.

Now assume that the "untilded" variables $\lambda_1, \dots, \lambda_{N_1}$ are arbitrary functions of a variable x and the "tilded" variables $\tilde{\lambda}_1, \dots, \tilde{\lambda}_{N_2}$ are arbitrary functions of a variable y . Then using (1.7.3) we get the derivative of $\nu(P)$ with respect to x :

$$\frac{\partial \nu}{\partial x} = \sum_{m=1}^{N_1} \frac{\partial \nu}{\partial \lambda_m} \frac{\partial \lambda_m}{\partial x} = - \sum_{m=1}^{N_1} \frac{\partial \lambda_m}{\partial x} \frac{\alpha_m}{\nu - \gamma_m} ; \tag{1.7.6}$$

therefore,

$$\frac{\partial \nu}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\nu - \gamma_m} = 0 , \tag{1.7.7}$$

where $c_m \equiv \alpha_m \frac{\partial \lambda_m}{\partial x}$; this coincides with the second equation in (1.4.7). The first equation in (1.4.7) is obtained in the same way after the identification

$$b_m \equiv \alpha_{N_1+m} \frac{\partial \lambda_{N_1+m}}{\partial y} .$$

Equations (1.4.8) and (1.4.9) for γ_n, b_n and c_n as functions of (x, y) arise as compatibility conditions of equations for ν_x and ν_y .

Therefore, spaces of rational maps of given degree provide solutions of the system (1.4.7), (1.4.8), (1.4.9) if we split the set of the branch points $\{\lambda_m\}$ into two subsets and assume that one subset contains the branch points which are arbitrary functions of x only and another subset contains the branch points which are arbitrary functions of y only.

Thus, the system (1.7.9) provides a realizations of the deformation scheme of [9].

1.7.2 Spaces of rational maps and non-autonomous integrable systems

A hierarchy of integrable systems can be constructed starting from an arbitrary branch N-fold covering $\mathcal{L}$ of genus zero as follows. Fix some point $P_0 \in \mathcal{L}$ such that its projection λ_0 on $\mathbb{C}P^1$ is independent of all $\{\lambda_m\}$; then $\gamma_0 \equiv \nu(P_0)$ depends on $\{\lambda_m\}$ according to the equation

$$R(\gamma_0(\lambda_1, \ldots, \lambda_{2N-2})) = \lambda_0 .$$

Consider the following system of first order differential equations for a matrix-valued function $\Psi(P, \{\lambda_m\})$:

$$\frac{d\Psi}{d\lambda_m} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} J_m \Psi , \tag{1.7.8}$$

where J_m are matrix-valued functions of $\{\lambda_m\}$. The compatibility conditions of the linear system (1.7.8) (derived with the help of equations (1.7.3), (1.7.4) and (1.7.5)) imply that the functions J_m can be expressed in terms of a single function $G := \Psi(P_0)$ such that $J_m = (\partial_{\lambda_m} G)\, G^{-1}$. The function G satisfies the following system of nonlinear partial differential equations:

$$\left((\gamma_0 - \gamma_m) G_{\lambda_m} G^{-1}\right)_{\lambda_n} = \left((\gamma_0 - \gamma_n) G_{\lambda_n} G^{-1}\right)_{\lambda_m} . \tag{1.7.9}$$

A simple calculation using equations (1.7.9) and system (1.7.4), (1.7.5), shows that if

$G(\{\lambda_m\})$ is a solution of nonlinear system (1.7.9), then the following system of equations is consistent:

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{(\gamma_0 - \gamma_m)^2}{2\alpha_m} \mathrm{tr} \left(G_{\lambda_m} G^{-1}\right)^2 \tag{1.7.10}$$

where $\tau(\lambda_1, \ldots, \lambda_M)$ is called the tau-function of integrable system (1.7.9).

The Ernst equation. For $N = 2$, the hierarchy (1.7.9) reduces to a single equation. If the fixed point P_0 on the covering $\mathcal{L}$ is chosen to coincide with $\infty^{(2)}$ [i.e. the point of $\mathcal{L}$ where $\lambda = \infty$ and in a neighbourhood of which $\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)} = -\lambda + (\lambda_1 + \lambda_2)/2 + o(1)$], then

$$\gamma_0 = \nu(P_0) = \frac{\lambda_1 + \lambda_2}{2}\,, \qquad \gamma_0 - \gamma_1 = \frac{\lambda_2 - \lambda_1}{4} \qquad \text{and} \qquad \gamma_0 - \gamma_2 = \frac{\lambda_1 - \lambda_2}{4}\,.$$

If we put $\lambda_1 := x$, $\lambda_2 := y$, then equation (1.7.9) coincides with the Ernst equation (1.4.3).

1.7.3 Systems of rank 1 and Darboux-Egoroff metrics

If the function G in (1.7.9) is scalar, the system (1.7.9) can be written [42] as a system of scalar second order differential equations in terms of the function $f(\{\lambda_m\}) = \log G$:

$$(\gamma_m - \gamma_n)^2 \frac{\partial^2 f}{\partial \lambda_m \partial \lambda_n} - \alpha_n \frac{\gamma_m - \gamma_0}{\gamma_n - \gamma_0} \frac{\partial f}{\partial \lambda_m} - \alpha_m \frac{\gamma_n - \gamma_0}{\gamma_m - \gamma_0} \frac{\partial f}{\partial \lambda_n} = 0\,, \qquad m \neq n\,. \tag{1.7.11}$$

In particular, any solution to the matrix system (1.7.9) gives a solution to the scalar system (1.7.11) of the form $f = \log \det G$.

The linear system (1.7.8) turns in rank 1 into the scalar system

$$\frac{\partial \psi(P)}{\partial \lambda_m} = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} \frac{\partial f}{\partial \lambda_m}\,, \tag{1.7.12}$$

where $\psi(P, \{\lambda_m\}) = \log \Psi$. As well as in the matrix case, the function ψ is generically non-single valued on $\mathcal{L}$.

The linear scalar system (1.7.12) admits the following generalization to the Hurwitz spaces $H_{g,N}$:

$$\frac{d\psi(P)}{d\lambda_m} = \frac{\int_{Q_0}^{P} W(P, P_m)}{\int_{Q_0}^{P_0} W(P, P_m)} \frac{\partial f}{\partial \lambda_m} , \tag{1.7.13}$$

where $W(P, Q)$ is the meromorphic bidifferential defined by (1.2.19); the evaluation $W(P, P_m)$ of this bidifferential at ramification points P_m of the covering is done with respect to the standard local parameter near P_m as in (1.3.2); the points Q_0 , $P_0 \in \mathcal{L}$ are such that their projections on the λ-plane are independent of $\{\lambda_k\}$. At the point P_0 we put $f(\{\lambda_m\}) = \psi(P_0)$.

Let us denote $v_m := \int_{Q_0}^{P_0} W(P, P_m)$. Then the compatibility conditions of the system (1.7.13) are given by the following equations:

$$\frac{\partial^2 f}{\partial \lambda_m \partial \lambda_n} - \frac{1}{2} W(P_m, P_n) \left\{ \frac{v_n}{v_m} \frac{\partial f}{\partial \lambda_m} + \frac{v_m}{v_n} \frac{\partial f}{\partial \lambda_n} \right\} = 0 . \tag{1.7.14}$$

The system (1.7.14) thus gives a generalization of the scalar system (1.7.11) to the Hurwitz space in arbitrary genus.

The tau-function of the system (1.7.14) is defined, in agreement with the definition (1.7.10) of the tau-function of linear systems on Hurwitz spaces in genus zero, by the following compatible system of equations:

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{f_{\lambda_m}^2}{v_m^2} , \qquad \frac{\partial \log \tau}{\partial \bar{\lambda}_m} = 0 . \tag{1.7.15}$$

Solutions to the system (1.7.14) are given in terms of the prime form $E(P, Q)$ (1.2.17) by the following theorem.

Theorem 1.3 *Let l be an arbitrary smooth closed contour on $\mathcal{L}$ such that its projection on the λ-plane is independent of $\{\lambda_m\}$ and $P_m \notin l$ for any m . Consider on l an arbitrary function $h(Q)$ independent of $\{\lambda_m\}$.*

Then the function

$$f = \oint_l h(Q) d_Q \log \frac{E(P_0, Q)}{E(Q_0, Q)} \tag{1.7.16}$$

satisfies the system (1.7.14). Corresponding solution to the linear system (1.7.13) is given by

$$\psi(P) = \oint_l h(Q) d_Q \log \frac{E(P, Q)}{E(Q_0, Q)} \,. \tag{1.7.17}$$

In analogy with the variational formulas (1.3.2) - (1.3.4), the *Cauchy kernel* $d_Q[\log\{E(P,Q)/E(Q_0,Q)\}]$ depends on λ_m as follows (assuming that all points P, Q, Q_0 are λ_m-independent):

$$\frac{\partial}{\partial \lambda_m} \left\{ d_Q \log \frac{E(P,Q)}{E(Q_0,Q)} \right\} = \frac{1}{2} W(Q, P_m) \int_{Q_0}^{P} W(P, P_m) \,. \tag{1.7.18}$$

Using (1.7.18), we find derivatives of the function f in the form:

$$\frac{\partial f}{\partial \lambda_m} = \frac{1}{2} \oint_l h(P) W(P, P_m) \int_{Q_0}^{P_0} W(P, P_m) \,.$$

Then for the tau-function (1.7.15), we have

$$\frac{\partial \ln \tau}{\partial \lambda_m} = \left(\frac{1}{2} \oint_l h(P) W(P, P_m) \right)^2 \,. \tag{1.7.19}$$

Each solution f of the rank 1 system (1.7.14) defines a Darboux-Egoroff metric $\mathbf{ds^2} = \sum_m g_{mm} (d\lambda_m)^2$ as follows:

$$g_{mm} = \frac{\partial \log \tau}{\partial \lambda_m} \,. \tag{1.7.20}$$

Using the Rauch variational formulas (1.3.2) for the bidifferential $W(P,Q)$, one finds from (1.7.19) the rotation coefficients (defined by (1.6.3)) for the metric (1.7.20). They are given by the bidifferential W evaluated at the ramification points P_m of the covering with respect to the standard local parameters $x_m(P) = \sqrt{\lambda(P) - \lambda_m}$ near P_m :

$$\beta_{mn} = \frac{1}{2} W(P_m, P_n) \,. \tag{1.7.21}$$

The rotation coefficients (1.7.21) satisfy the system (1.6.4)-(1.6.5) providing the flatness for the metric. Relations (1.6.4) hold due to the variational formulas (1.3.2) for W . To proof relations (1.6.5) for rotation coefficients (1.7.21) one uses the invariance of the bidifferential $W(P,Q)$ with respect to biholomorphic maps. Namely, let us consider the branched covering $\mathcal{L}^\varepsilon$ which is obtained by an ε-shift of all the ramification points P_m in the λ-plane, i.e. the projections of branch points P_m^ε of $\mathcal{L}^\varepsilon$ on λ-plane are equal to $\lambda_m^\varepsilon = \lambda_m + \varepsilon$. Denote by W^ε the bidifferential W defined on the surface $\mathcal{L}^\varepsilon$. Let us denote the projections of points P and Q on the λ-plane by λ_P and λ_Q , respectively. Define the point P^ε to be the point lying on the same sheet as P and having projection $\lambda_P + \varepsilon$ on the λ-plane. In the same way the point Q^ε belongs to the same sheet as Q and projects to $\lambda_Q + \varepsilon$ on the λ-plane. Since $\mathcal{L}^\varepsilon$ can be holomorphically mapped to $\mathcal{L}$ by the transformation $\lambda \to \lambda + \varepsilon$ on all sheets, we have

$$W^\varepsilon(P^\varepsilon, Q^\varepsilon) = W(P,Q)\ . \tag{1.7.22}$$

Assuming that P belongs to a neighbourhood of the branch point P_m and Q belongs to a neighbourhood of the branch point P_n , we can write down the respective local parameters as $x_m(P) = \sqrt{\lambda_P - \lambda_m}$ and $x_n(Q) = \sqrt{\lambda_Q - \lambda_n}$. These parameters are obviously invariant with respect to simultaneous ε-shifts of all $\{\lambda_m\}$ and λ , i.e. $x_m^\varepsilon(P^\varepsilon) = x_m(P)$ and $x_n^\varepsilon(Q^\varepsilon) = x_n(Q)$. Therefore, assuming that $P = P_m$ and $Q = P_n$ and differentiating (1.7.22) with respect to ε at $\varepsilon = 0$, we get (1.6.5): $\sum_k \partial_{\lambda_k}\beta_{nm} = 0$ for rotation coefficients β_{nm} (1.7.21)

1.8 Frobenius manifolds

This section contains a summary of the theory [15] of Frobenius manifolds associated with Hurwitz spaces. The details of Frobenius manifolds theory can be found also in [18, 33, 57]

1.8.1 WDVV system and Frobenius manifolds. Definitions

The Witten – Dijkgraaf – E.Verlinde – H.Verlinde (WDVV) system is the following nonlinear system of partial differential equations:

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i \,, \quad i, j = 1, \dots, n \,, \tag{1.8.1}$$

where F_i is the matrix

$$(F_i)_{jk} = \frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k} \,,$$

and F is a function of n variables $t^1, \dots, t^n$. The system (1.8.1) is highly under-defined; to turn it into a system of ordinary differential equations, the following two conditions arising in applications are usually added:

- Quasihomogeneity (up to a quadratic polynomial): for any nonzero κ and some numbers $\nu_1, \dots, \nu_n, \nu_F$

$$F(\kappa^{\nu_1} t^1, \dots, \kappa^{\nu_n} t^n) = \kappa^{\nu_F} F(t^1, \dots, t^n) + \text{quadratic terms} \,, \tag{1.8.2}$$

- Normalization: F_1 is a constant nondegenerate matrix.

The condition of quasihomogeneity can be rewritten in terms of the Euler vector field

$$E = \sum_\alpha \nu_\alpha t^\alpha \partial_{t^\alpha} \tag{1.8.3}$$

as

$$\mathrm{Lie}_E F = E(F) = \sum_{\alpha} \nu_\alpha t^\alpha \partial_{t^\alpha} F = \nu_F F + \text{quadratic terms} . \tag{1.8.4}$$

Definition 1.1 *A commutative associative vector algebra over* $\mathbb{C}$ *with a unity* $\mathbf{e}$ *is called a* **Frobenius algebra** *if it is supplied with a* $\mathbb{C}$*-bilinear symmetric nondegenerate inner product* $\langle\cdot,\cdot\rangle$ *which has the property* $\langle \mathbf{x}\cdot\mathbf{y},\mathbf{z}\rangle = \langle \mathbf{x},\mathbf{y}\cdot\mathbf{z}\rangle$ *for arbitrary vectors* $\mathbf{x},\mathbf{y},\mathbf{z}$ *from the algebra.*

Definition 1.2 *M is a* **Frobenius manifold** *of the charge* ν *if a structure of a Frobenius algebra is specified on any tangent plane* T_tM *smoothly depending on the point* $t \in M$ *such that*

F1 *the inner product* $\langle\cdot,\cdot\rangle$ *is a flat metric on* M *(not necessarily positive definite).*

F2 *the unit vector field* $\mathbf{e}$ *is covariantly constant with respect to the Levi-Civita connection* ∇ *for the metric* $\langle\cdot,\cdot\rangle$ *, i.e.* $\nabla_{\mathbf{x}}\mathbf{e} = 0$ *holds for any vector field* $\mathbf{x}$ *on* M *.*

F3 *the tensor* $(\nabla_w \mathbf{c})(\mathbf{x},\mathbf{y},\mathbf{z})$ *is symmetric in four vector fields* $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w} \in T_tM$ *, where* $\mathbf{c}$ *is the following symmetric 3-tensor:* $\mathbf{c}(\mathbf{x},\mathbf{y},\mathbf{z}) = \langle \mathbf{x}\cdot\mathbf{y},\mathbf{z}\rangle$ *.*

F4 *There exists on* M *a vector field* E *(the Euler vector field) such that the following holds for any vector fields* $\mathbf{x}$ *,* $\mathbf{y}$ *on* M *:*

$$\nabla_{\mathbf{x}}(\nabla_{\mathbf{y}}E) = 0 , \tag{1.8.5}$$

$$[E,\mathbf{x}\cdot\mathbf{y}] - [E,\mathbf{x}]\cdot\mathbf{y} - \mathbf{x}\cdot[E,\mathbf{y}] = \mathbf{x}\cdot\mathbf{y} , \tag{1.8.6}$$

$$\mathrm{Lie}_E\langle\mathbf{x},\mathbf{y}\rangle := E\langle\mathbf{x},\mathbf{y}\rangle - \langle[E,\mathbf{x}],\mathbf{y}\rangle - \langle\mathbf{x},[E,\mathbf{y}]\rangle = (2-\nu)\langle\mathbf{x},\mathbf{y}\rangle . \tag{1.8.7}$$

The charge ν of a Frobenius manifold is equal to $\nu_F + 3$, where ν_F is the quasihomogeneity coefficient from (1.8.4).

Theorem 1.4 *Any solution to the WDVV equations defined for* $t \in M$ *determines in* M *a structure of a Frobenius manifold and vice versa.*

Proof. Given a Frobenius manifold, denote by $\{t^\alpha\}$ the flat coordinates of the metric $\langle , \rangle$ and by η the constant matrix $\eta_{\alpha\beta} = \langle \partial_{t^\alpha}, \partial_{t^\beta} \rangle$. Due to the covariant constancy of the unit vector field $\mathbf{e}$, we can by a linear change of coordinates put $\mathbf{e} = \partial_{t^1}$. In these coordinates, **F3** implies the existence of a function F such that its third derivatives coincide with the 3-tensor $\mathbf{c}$:

$$c_{\alpha\beta\gamma} := \mathbf{c}(\partial_{t^\alpha}, \partial_{t^\beta}, \partial_{t^\gamma}) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} .$$

In coordinates $\{t^\alpha\}$, the vector algebra has the structure constants $c^\gamma_{\alpha\beta}$ defined by $\partial_{t^\alpha} \cdot \partial_{t^\beta} = c^\gamma_{\alpha\beta} \partial_{t^\gamma}$. The structure constants can be found in terms of derivatives of the function F from $c^\delta_{\alpha\beta} \eta_{\delta\gamma} = c_{\alpha\beta\gamma}$. The associativity conditions for the vector algebra coincide with the WDVV equations for the function F .

The existence of the vector field E provides the quasihomogeneity for the function F . Indeed, requirements (1.8.6), (1.8.7) for the Euler vector field imply

$$\begin{aligned} \mathrm{Lie}_E \mathbf{c}(x,y,z) &:= E\left(\mathbf{c}(x,y,z)\right) - \mathbf{c}([E,x],y,z) - \mathbf{c}(x,[E,y],z) - \mathbf{c}(x,y,[E,z]) \\ &= (3-\nu)\mathbf{c}(x,y,z) . \quad (1.8.8) \end{aligned}$$

The Lie derivative Lie_E commutes with the covariant derivative ∇ as can easily be checked in flat coordinates, when the Euler vector field (due to $\nabla\nabla E = 0$ (1.8.5)) has the form (1.8.3). Therefore (1.8.8) implies $\mathrm{Lie}_E F = (3-\nu)F +$ quadratic terms .

The converse statement can be proven analogously. For any solution F to the WDVV system, let us put $c_{\alpha\beta\gamma} := \partial^3_{t^\alpha t^\beta t^\gamma} F$ and $\eta_{\alpha\beta} \equiv \langle \partial_{t^\alpha}, \partial_{t^\beta} \rangle := c_{1\alpha\beta} \equiv F_1$. The metric $\eta_{\alpha\beta}$ is flat since it has constant coefficients in the coordinates $\{t^\alpha\}$. Define the vector algebra by $\partial_{t^\alpha} \cdot \partial_{t^\beta} := c^\gamma_{\alpha\beta} \partial_{t^\gamma}$ where $c^\delta_{\alpha\beta} \eta_{\delta\gamma} = c_{\alpha\beta\gamma}$. This definition implies $\mathbf{e} = \partial_{t^1}$ and the validity of requirement $\langle \partial_{t^\alpha} \cdot \partial_{t^\beta}, \partial_{t^\gamma} \rangle = \langle \partial_{t^\alpha}, \partial_{t^\beta} \cdot \partial_{t^\gamma} \rangle$. The WDVV system for the function F provides the associativity for the defined vector algebra. Since covariant derivatives in the flat coordinates $\{t^\alpha\}$ coincide with partial derivatives, the covariant derivatives of the tensor $\mathbf{c}$ are symmetric: $\partial_{t^\delta} c_{\alpha\beta\gamma} = \partial^4_{t^\alpha t^\beta t^\gamma t^\delta} F$. The property **F4** is easy to check for the defined metric and vector multiplication. $\diamond$

The function F , defined up to addition of a quadratic polynomial in $t^1, \ldots, t^n$, is called the *prepotential* of the Frobenius manifold.

Definition 1.3 *A Frobenius manifold is called* **semisimple** *if at any point the Frobenius algebra in the tangent space does not have nilpotents.*

1.8.2 Dubrovin's Frobenius structures on Hurwitz spaces

The most well-studied examples of Frobenius manifolds are the semisimple Frobenius manifolds associated with Hurwitz spaces [15]. The semisimplicity of Hurwitz Frobenius manifolds is equivalent to the assumption of simplicity of the branch points of the corresponding coverings of $\mathbb{C}P^1$.

Consider the Hurwitz space $M = M_{g;n_0,\ldots,n_m}$ and construct the following covering $\widehat{M} = \widehat{M}_{g;n_0,\ldots,n_m}$ of this space. A point of $\widehat{M}$ is a triple $\{\mathcal{L}, \lambda, \{a_k, b_k\}_{k=1}^g\}$, where $\{a_k, b_k\}_{k=1}^g$ is a canonical basis of cycles on $\mathcal{L}$. The branch points $\lambda_1, \ldots, \lambda_L$ of the covering $(\mathcal{L}, \lambda)$ play the role of local coordinates on $\widehat{M}$ viewed as a complex manifold.

The canonical vector algebra is defined in the tangent space $T\widehat{M}$ by

$$\partial_{\lambda_i} \cdot \partial_{\lambda_j} = \delta_{ij} \partial_{\lambda_i} \, . \tag{1.8.9}$$

The unit vector field is given by

$$\mathbf{e} = \sum_{i=1}^{L} \partial_{\lambda_i} \, . \tag{1.8.10}$$

For this multiplication law, any diagonal metric $\langle , \rangle$ (bilinear quadratic form) obviously has the property $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$ required in the definition of a Frobenius algebra. The diagonal metrics which give Frobenius structures on Hurwitz spaces are constructed using the primary differentials listed below. The Euler vector field has the following form in canonical coordinates:

$$E = \sum_{i=1}^{L} \lambda_i \partial_{\lambda_i} \, . \tag{1.8.11}$$

For the Euler field (1.8.11), the vector multiplication defined by (1.8.9) satisfies requirement (1.8.6) from **F4**.

Primary differentials. Consider the following five types of differentials on the Riemann surface $\mathcal{L}$.

Type 1. Differentials $\phi_{t^{i;\alpha}}(P)$, $i = 0, \ldots, m$, $\alpha = 1, \ldots, n_i$. The differential $\phi_{t^{i;\alpha}}$ is the normalized Abelian differential ($\oint_{a_k} \phi_{t^{i;\alpha}} = 0$) of the second kind with a pole of order $\alpha + 1$ at ∞^i : $\phi_{t^{i;\alpha}}(P) \sim z_i^{-\alpha-1}(P) dz_i(P)$ at $P \sim \infty^i$.

Type 2. Differentials $\phi_{v^i}(P)$, $i = 1, \ldots, m$. The differential ϕ_{v^i} is the normalized Abelian differential of the second kind with a pole of order $n_1 + 2$ at the point ∞^i . In other words, it has the principal part of the form $\phi_{v^i}(P) \sim -d\lambda(P)$ at $P \sim \infty^i$.

Type 3. Differentials $\phi_{w^i}(P)$, $i = 1, \ldots, m$. The differential ϕ_{w^i} is the normalized Abelian differential of the third kind W_{∞^i,∞^0} , i.e. it has simple poles at ∞^i and ∞^0 with residues $+1$ and -1 , respectively.

Type 4. Differentials $\phi_{r^k}(P)$, $k = 1, \ldots, g$. The differential ϕ_{r^k} is the normalized $(\oint_{a_l} \phi_{r^k} = 0)$ multivalued differential without poles having the following transformations along the b-cycles: $\phi_{r^k}(P^{b_j}) - \phi_{r^k}(P) = -2\pi i d\lambda$. Here $\phi_{r^k}(P^{b_j})$ stands for the analytic continuation of the differential along the cycle b_j on the Riemann surface.

Type 5. Holomorphic normalized differentials $\phi_{s^k}(P) = \omega_k(P)$, $k = 1, \ldots, g$.

A primary differential ϕ can be either a differential of one of the above types 1.-5. or one of the following linear combinations:

$$\phi = \sum_{i=1}^{m} \rho_i \phi_{v^i} + \sum_{k=1}^{g} \sigma_k \phi_{r^k} \ , \qquad \phi = \sum_{i=1}^{m} \rho_i \phi_{\omega^i} + \sum_{k=1}^{g} \sigma_k \phi_{s^k} \ ,$$

with any constants $\{\rho_i\}$ and $\{\sigma_k\}$.

Flat metrics. For a primary differential ϕ consider the following metric on the Hurwitz space:

$$\mathbf{ds}^2_\phi = \sum_{i=1}^{L} \left(\operatorname*{res}_{P=P_i} \frac{\phi^2(P)}{d\lambda(P)} \right) (d\lambda_i)^2 \equiv \frac{1}{2} \sum_{i=1}^{L} \phi^2(P_i)(d\lambda_i)^2 \ . \tag{1.8.12}$$

The condition $\mathbf{ds}^2_\phi(\mathbf{x} \cdot \mathbf{y}, \mathbf{z}) = \mathbf{ds}^2_\phi(\mathbf{x}, \mathbf{y} \cdot \mathbf{z})$ holds for the metrics (1.8.12) and, therefore, the vector algebra (1.8.9) with the bilinear product $\langle , \rangle := \mathbf{ds}^2_\phi$ is Frobenius.

The next theorem states that the metrics (1.8.12) define a structure of a Frobenius manifold on the Hurwitz space.

Theorem 1.5 *([15]) The metrics (1.8.12) are flat. The rotation coefficients of the metrics (1.8.12) do not depend on the choice of a primary differential ϕ defining the metric.*

Any metrics from the family (1.8.12) satisfies **F2** *and the requirement (1.8.7) from* **F4**.

A set of flat coordinates (see Section 1.6.1) of each metric from the family (1.8.12) is given by the following theorem.

Theorem 1.6 *([15]) Let $P_0 \in \mathcal{L}$ be a point which is mapped to zero by the function λ : $\lambda(P_0) = 0$, and let all basis contours $\{a_k, b_k\}$ on the surface start at this point. Then the following functions give a set of flat coordinates for the metric $\mathbf{ds}^2_\phi$ (1.8.12):*

1. $t^{i;\alpha} := \frac{1}{\alpha} \operatorname*{res}_{\infty^i} \lambda(Q)^{\frac{\alpha}{n_i+1}} \phi(Q)$ $\qquad i = 0, \ldots, m\ ;\ \alpha = 1, \ldots, n_i\ .$
2. $v^i := \operatorname*{res}_{\infty^i} \lambda(Q)\phi(Q)$ $\qquad i = 1, \ldots, m\ .$
3. $w^i := \text{v.p.} \int_{\infty^0}^{\infty^i} \phi(Q)$ $\qquad i = 1, \ldots, m\ .$
4. $r^k := -\oint_{a_k} \lambda(Q)\phi(Q)$ $\qquad k = 1, \ldots, g\ .$
5. $s^k := \frac{1}{2\pi i} \oint_{b_k} \phi(Q))$ $\qquad k = 1, \ldots, g\ .$

Here the principal value near infinity is defined by omitting the divergent part of the integral as a function of the local parameter z_i (such that $\lambda = z_i^{-n_i-1}$) .

The nonvanishing entries of the metric in these coordinates are given by:

$$\mathbf{ds}^2_\phi(\partial_{t^{i;\alpha}}, \partial_{t^{j;\beta}}) = \frac{1}{n_i+1}\delta_{ij}\delta_{\alpha+\beta,n_i+1}\ ,$$

$$\mathbf{ds}^2_\phi(\partial_{v^i}, \partial_{w^j}) = \delta_{ij}\ ,$$

$$\mathbf{ds}^2_\phi(\partial_{r^k}, \partial_{s^l}) = -\delta_{kl}\ .$$

The unit vector field $\mathbf{e}$ *is given by* $\mathbf{e} = -\partial_{t^1}$ *, where the coordinate t^1 is defined to coincide with the type of the primary differential ϕ defining the metric. I.e. the coordinate t^1 is such that $\phi \equiv \phi_{t^1}$.*

For each primary differential ϕ consider the multivalued on $\mathcal{L}$ differential Ψ_ϕ :

$$\Psi_\phi(P) := \left(\text{v.p.} \int_{\infty^0}^{P} \phi\right) d\lambda(P) , \tag{1.8.13}$$

where the principal value near ∞^0 is defined by omitting the divergent part of the integral as a function of the local parameter z_0 . This differential generates the primary differentials, namely, the derivatives of the multivalued differential Ψ_ϕ with respect to the flat coordinates for the corresponding metric $\mathbf{ds}^2_\phi$ are independent of the choice of the primary differential ϕ and are given by:

$$\frac{\partial \Psi_\phi}{\partial t^A} = \phi_{t^A} .$$

Prepotential of Frobenius structure A *prepotential* of the Frobenius manifold is a function F of flat coordinates for the corresponding metric such that its third-order derivatives give the symmetric 3-tensor $\mathbf{c}$:

$$\frac{\partial^3 F(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) \equiv \mathbf{ds}^2_\phi(\partial_{t^A} \cdot \partial_{t^B}, \partial_{t^C}) .$$

This function, according to Theorem 1.4, satisfies the WDVV system. The existence of the function F proves **F2**, the symmetry of $\partial_{t^D}\mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C})$ with respect to the coordinates t^A, t^B, t^C, t^D .

Let $\omega^{(1)}$ and $\omega^{(2)}$ be two differentials on the surface $\mathcal{L}$ holomorphic outside of the points $\infty^0, \dots, \infty^m$ with the following behaviour at ∞^i :

$$\omega^{(\alpha)} = \sum_{n=-n^{(\alpha)}}^{\infty} c_{n,i}^{(\alpha)} z_i^n dz_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r_{n,i}^{(\alpha)} \lambda^n \log\lambda\right) , \quad P \sim \infty^i , \tag{1.8.14}$$

where $n^{(\alpha)} \in \mathbb{Z}$ and $c_{n,i}^{(\alpha)}$, $r_{n,i}^{(\alpha)}$ are some coefficients; $z_i = z_i(P)$ is a local parameter near ∞^i . Denote also for $k = 1, \ldots, g$:

$$\oint_{a_k} \omega^{(\alpha)} = A_k^{(\alpha)} , \tag{1.8.15}$$

$$dp_k^{(\alpha)}(\lambda(P)) := \omega^{(\alpha)}(P^{a_k}) - \omega^{(\alpha)}(P) , \quad p_k^{(\alpha)}(\lambda) = \sum_{s>0} p_{sk}^{(\alpha)} \lambda^s , \tag{1.8.16}$$

$$dq_k^{(\alpha)}(\lambda(P)) := \omega^{(\alpha)}(P^{b_k}) - \omega^{(\alpha)}(P) , \quad q_k^{(\alpha)}(\lambda) = \sum_{s>0} q_{sk}^{(\alpha)} \lambda^s . \tag{1.8.17}$$

Here, as before, $\omega(P^{a_k})$ and $\omega(P^{b_k})$ denote the analytic continuation of $\omega(P)$ along the corresponding cycle on the Riemann surface.

Note that if the differential $\omega^{(\alpha)}$ is one of the primary differentials, then the coefficients $c_{n,i}$, $r_{n,i}$, p_{sk}, q_{sk} and A_k from (1.8.14)-(1.8.17) do not depend on coordinates. For $\omega^{(\alpha)}$ coinciding with the multivalued differential Ψ_ϕ , the coefficients defined by (1.8.14)-(1.8.17) (those which are nonconstant) are given by the flat coordinates for the corresponding metric: $c_{-n_i-2,i} = v^i$; $c_{-\alpha-1,i} = t^{i;\alpha}$ for $\alpha = 1, \ldots, n_i$; $c_{-1,i} = w^i$; $q_{1k} = 2\pi i\, r^k$; and $A_k = s^k$.

Definition 1.4 *For two differentials $\omega^{(\alpha)}$ and $\omega^{(\beta)}$ on the surface $\mathcal{L}$ which do not have singularities other than described by (1.8.14)-(1.8.17), the* **pairing** *$\mathcal{F}[\ ,\ ]$ is defined by:*

$$\mathcal{F}[\omega^{(\alpha)} , \omega^{(\beta)}] = \sum_{i=0}^{m} \left(\sum_{n\geq 0} \frac{c_{-n-2,i}^{(\alpha)}}{n+1} c_{n,i}^{(\beta)} + c_{-1,i}^{(\alpha)} \text{v.p.} \int_{P_0}^{\infty^i} \omega^{(\beta)} - \text{v.p.} \int_{P_0}^{\infty^i} \sum_{n>0} r_{n,i}^{(\alpha)} \lambda^n \omega^{(\beta)} \right)$$
$$+ \frac{1}{2\pi i} \sum_{k=1}^{g} \left(- \oint_{a_k} q_k^{(\alpha)}(\lambda) \omega^{(\beta)} + \oint_{b_k} p_k^{(\alpha)}(\lambda) \omega^{(\beta)} + A_k^{(\alpha)} \oint_{b_k} \omega^{(\beta)} \right) , \tag{1.8.18}$$

where P_0 is a point on the surface such that $\lambda(P_0) = 0$.

In terms of the pairing (1.8.18) one can conveniently express the prepotential of the Frobenius structure defined by the metric $\mathbf{ds}^2_\phi$ as follows:

$$F_\phi = \frac{1}{2}\mathcal{F}[\Psi_\phi\ ,\ \Psi_\phi]\ ; \tag{1.8.19}$$

the third derivatives of the prepotential F_ϕ with respect to flat coordinates are given by the *residue formulas*:

$$\frac{\partial^3 F_\phi(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A},\partial_{t^B},\partial_{t^C}) = -\sum_{i=1}^{L} \operatorname*{res}_{P_i} \frac{\phi_{t^A}\phi_{t^B}\phi_{t^C}}{\phi\, d\lambda} \equiv -\frac{1}{2}\sum_{i=1}^{L} \frac{\phi_{t^A}(P_i)\phi_{t^B}(P_i)\phi_{t^C}(P_i)}{\phi(P_i)}\ .$$

Second derivatives of F_ϕ are given by the pairing of the corresponding primary differentials:

$$\partial_{t^A}\partial_{t^B}F_\phi = \mathcal{F}[\phi_{t^A}\ ,\ \phi_{t^B}]\ .$$

The prepotential (1.8.19) is a quasihomogenous function of flat coordinates $\{t^A\}$ of the metric $\mathbf{ds}^2_\phi$, i.e. (1.8.2) holds for some coefficients $\{\nu_A\}$. The coefficients of quasihomogeneity $\{\nu_A\}$ are coefficients of the Euler vector field written in the flat coordinates (see (1.8.2) - (1.8.4)); they can be computed by finding the action of the Euler vector field $E=\sum_{i=1}^L \lambda_i\partial_{\lambda_i}$ on a flat coordinate t^A , i.e. $E(t^A)=\nu_A t^A$. The coefficients $\nu_F = 3-\nu$ are listed below:

$$\begin{array}{lll}
\text{if}\ \ \phi=\phi_{t^{i;\alpha}}\ , & \nu = 1-\dfrac{2\alpha}{n_i+1} & \nu_F = \dfrac{2\alpha}{n_i+1}+2 \\[2ex]
\text{if}\ \ \phi=\phi_{v^i},\ \ \phi=\phi_{r^k}, & \nu=-1 & \nu_F=4 \\[2ex]
\text{if}\ \ \phi=\phi_{\omega^i},\ \ \phi=\phi_{s^k}, & \nu=1 & \nu_F=2\ .
\end{array}$$

Let us denote by $\widehat{M}^\phi_{g;n_1,\dots,n_m}$ the structure of a Frobenius manifold defined by the metric $\mathbf{ds}^2_\phi$ on the Hurwitz space $\widehat{M}_{g;n_1,\dots,n_m}$.

1.8.3 G-function on Hurwitz Frobenius manifolds

The G-function is a solution to the Getzler system of linear differential equations which was derived in [26] (see also [17]). The system is defined on an arbitrary semisimple Frobenius manifold M (semisimple means that the Frobenius vector algebra $T_t M$ has no nilpotents for any $t \in M$).

The Getzler system has unique [17], up to an additive constant, solution G which satisfies the quasihomogeneity condition

$$E(G) = -\frac{1}{4}\sum_{A=1}^{n}\left(1-\nu_A-\frac{\nu}{2}\right)^2 + \frac{\nu n}{48} ,$$

with a constant in the left hand side: ν is the charge, n is the dimension of the Frobenius manifold; $\{\nu_A\}$ are the quasihomogeneity coefficients (1.8.2). In [17] the following formula (which proves the conjecture of Givental [30]) was derived for this quasihomogeneous solution:

$$G = \log\frac{\tau_{\rm I}}{J^{1/24}} , \tag{1.8.20}$$

where J is the Jacobian of transformation from canonical to the flat coordinates, $J = \det\left(\frac{\partial t^\alpha}{\partial \lambda_i}\right)$; and $\tau_{\rm I}$ is the isomonodromic tau-function of the Frobenius manifold defined by

$$\frac{\partial \log \tau_{\rm I}}{\partial \lambda_i} := \frac{1}{2}\sum_{j\neq i, j=1}^{n} \beta_{ij}^2(\lambda_i - \lambda_j) , \qquad i = 1, \dots, n \tag{1.8.21}$$

(β_{ij} are the rotation coefficients (1.6.3)). The Jacobian J of a Hurwitz Frobenius structure $\widehat{M}^\phi$ (the dimension n of the Frobenius manifold equals L) is given by ([17], p. 36):

$$J = \frac{1}{2^{L/2}}\prod_{i=1}^{L}\phi(P_i) .$$

The function G (1.8.20) for Dubrovin's Frobenius structure $\widehat{M}_{1;1}^{\phi_s}$ on Hurwitz space in genus one was computed in [17]:

$$G = -\log\left\{\eta(2\pi i t_3)(t_2)^{\frac{1}{8}}\right\} + \text{const}\,,$$

where $\eta(\mu)$ is the Dedekind eta-function (1.2.8) given by $\eta(\mu) = (\theta_1'(0))^{1/3}$; $\{t_1, t_2, t_3\}$ are the flat coordinates on the Frobenius manifold. In [44] the G-function was computed for Frobenius structures on Hurwitz spaces in arbitrary genus. In terms of the function τ_W (1.3.18) the G-function (1.8.20) for the Frobenius structure $\widehat{M}_{g;n_1,\dots,n_m}^{\phi}$ is given by [44]:

$$G = -\frac{1}{2}\log\tau_W - \frac{1}{24}\log\prod_{i=1}^{L}\phi(P_i) + \text{const}\,; \tag{1.8.22}$$

the expression for the Bergman tau-function τ_W is given in Theorem 1.1 (Section 1.3.4).

The isomonodromic tau-function $\tau_{\rm I}$ (1.8.21) for a Frobenius structure $\widehat{M}^{\phi}$ is related to the Bergman tau-function τ_W by the formula ([45]):

$$\tau_{\rm I} = (\tau_W)^{-\frac{1}{2}}\,. \tag{1.8.23}$$

1.8.4 Genus one case and the Chazy equation

Consider the torus $T = \mathbb{C}/\{2w, 2w'\}$ with b-period $\mu = w'/w$ and the Hurwitz space of coverings (T, λ) (1.3.10) defined by the equation

$$\zeta = \lambda(\varsigma)\,, \qquad \lambda(\varsigma) = \wp(\varsigma) + c\,, \tag{1.8.24}$$

where c is a constant with respect to ς , and ζ is a coordinate on $\mathbb{C}P^1$, the base of the covering.

Consider the Frobenius structure $\widehat{M}_{1;1}^{\phi_s}$. The primary differential ϕ_s is the normalized holomorphic differential $\omega = d\varsigma/(2w)$ in the coordinate ζ given by (1.3.11). In a neighbourhood of the point ∞^0 (at ∞^0 we have $\zeta = \infty$ and $\varsigma = 0$) the local coordinate is $z_0 = 1/\zeta$ and ϕ_s has the form:

$$\phi_s(P) = -\frac{1}{2w}\left(1 + \frac{3\,c}{2} z_0^2(P) + o(z_0^2(P))\right) dz_0(P)\,. \tag{1.8.25}$$

The metric $\mathbf{ds}^2_\phi = \frac{1}{2}\sum_{i=1}^L \phi^2(P_i)(d\lambda_i)^2$ in canonical coordinates $\{\lambda_i\}$ has the form (see (1.3.12) for $\phi(P_i)$):

$$\mathbf{ds}^2_{\phi_s} = \frac{1}{8\omega^2}\left\{\frac{(d\lambda_1)^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(d\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(d\lambda_3)^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}\,.$$

The flat coordinates of this metric are given by

$$t_1 := s = -\oint_a \lambda(Q)\phi_s(Q)\,; \qquad t_2 := t^{0;1} = -2\operatorname*{res}_{\infty^0}\phi_s(Q)/z_0(Q)\,;$$
$$t_3 := r = \frac{1}{2\pi i}\oint_b \phi_s(Q)\,.$$

For the coordinate $t^{0;1}$ from (1.8.25) we have $t^{0;1} = 1/w$, r is the b-period of the torus ($\mu = w/w'$): $r = \mu/(2\pi i)$. The coordinate s can be computed using (1.2.10):

$$s = -\int_x^{x+2w} (\wp(\varsigma) + c)\frac{d\varsigma}{2w} = -\frac{1}{12w^2}\frac{\theta_1'''}{\theta_1'} - c\,.$$

The metric $\mathbf{ds}^2_{\phi_s}$ in flat coordinates, according to the Theorem 1.6, is given by $\mathbf{ds}^2_{\phi_s} = 1/2(dt_2)^2 - 2dt_1 dt_3$.

The multivalued differential $\Psi_{\phi_s}(P) = \left(\int_{\infty^0}^P \omega\right) d\lambda(P)$ at the point $P = \infty^0$ has the singularity of the form:

$$\Psi_{\phi_s}(P) \simeq \frac{1}{2w}\left(\frac{2}{z_0^2(P)} + c\right) dz_0(P) \equiv \left(\frac{t_2}{z_0^2(P)} + \frac{ct_2}{2}\right) dz_0(P)\,.$$

For the prepotential (1.8.19) $F_{\phi_s} = (1/2)\mathcal{F}[\Psi_{\phi_s}, \Psi_{\phi_s}]$ we have

$$F_{\phi_s} = \frac{1}{2}\left(\frac{t_2^2}{2}c + \frac{1}{2\pi i}\left(-2\pi i\, t_3 \oint_a \lambda\Psi_{\phi_s} + \oint_b \lambda\Psi_{\phi_s} + t_1\oint_b \Psi_{\phi_s}\right)\right)\,.$$

To write this expression in terms of the flat coordinates we use the relations

$$\oint_a \lambda \Psi_{\phi_s} = -\oint_a \frac{\lambda^2}{2}\phi_s = -\frac{1}{4w}\int_x^{x+2w} (\wp(\varsigma)+c)^2 d\varsigma \ ,$$
$$\oint_b \lambda \Psi_{\phi_s} = -\oint_b \frac{\lambda^2}{2}\phi_s = -\frac{1}{4w}\int_x^{x+2w'} (\wp(\varsigma)+c)^2 d\varsigma \ ,$$
$$\oint_b \Psi_{\phi_s} = -\oint_b \lambda\phi_s = -\frac{1}{2w}\int_x^{2w'} (\wp(\varsigma)+c) d\varsigma \ ,$$

where x is any complex number and the integrals $\int_x^{2w'} \wp(\varsigma)$ and $\int_x^{2w} \wp(\varsigma)$ are given by (1.2.10). For the integrals of $\wp^2(\varsigma)$ it suffices to have the following relation. Consider an auxiliary function $f(z) = \int^z \wp^2(\varsigma) d\varsigma$, $z \in \mathbb{C}$. This function has zero residue at the point $z = 0$. Therefore the integral $\oint_l f(z)$ is zero, where l is the boundary of the parallelogram with vertices x , $x + 2w$, $x + 2w + 2w'$, $x + 2w'$ which contains the point $z = 0$ inside. Computing this integral alternatively, using the relations $f(z+2w) = f(z) + \int_z^{2w} \wp^2(\varsigma)$ and $f(z+2w') = f(z) + \int_z^{2w'} \wp^2(\varsigma)$, we get $w\int_x^{x+2w'} \wp^2(\varsigma) = w'\int_x^{x+2w} \wp^2(\varsigma)$.

Finally, we obtain the following expression for the prepotential:

$$F_{\phi_s} = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32}\, t_2^4\, \gamma(2\pi i t_3) \ , \tag{1.8.26}$$

where γ is the following function of the period $\mu = 2\pi i\, t_3$ of the torus:

$$\gamma(\mu) := 4 d_\mu \log \eta(\mu) \equiv \frac{1}{3\pi i}\frac{\theta_1'''(0;\mu)}{\theta_1'(0;\mu)} \tag{1.8.27}$$

($\eta(\mu)$ is the Dedekind eta-function (1.2.7)). The prepotential is a quasihomogeneous function: for any $\kappa \neq 0$ we have $F_{\phi_s}(\kappa t_1, \kappa^{1/2} t_2, \kappa^0 t_3) = \kappa^2 F_{\phi_s}(t_1, t_2, t_3)$. The Euler vector field $E = \sum_{i=1}^3 \lambda_i \partial_{\lambda_i}$ in flat coordinates has the form: $E = t_1\partial_{t_1} + \frac{1}{2}t_2\partial_{t_2}$.

The function γ (1.8.27) satisfies the Chazy equation:

$$f''' = 6ff'' - 9f'^{\,2}. \tag{1.8.28}$$

Actually, as can be verified by a direct calculation, any function of the form (1.8.26) with γ being any solution to the Chazy equation satisfies the WDVV system [15]. The general solution to (1.8.28) has the form:

$$f(\mu) = \gamma\left(\frac{a\mu + b}{c\mu + d}\right)\frac{1}{(c\mu + d)^2} - \frac{2c}{c\mu + d}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ and γ is given by (1.8.27). Therefore, for any $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$, the function

$$F_{\phi_s} = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32}\, t_2^4 \left(\gamma\left(\frac{a\mu + b}{c\mu + d}\right)\frac{1}{(c\mu + d)^2} - \frac{2c}{c\mu + d}\right)$$

is a solution to the WDVV system (1.8.1)-(1.8.2).

The main result of the following work is a construction of new integrable systems related to the Hurwitz space in genus one. These systems can be considered as "deformations" of some integrable systems with spectral parameter living on an elliptic surface.

In 1989 Burtzev, Zakharov and Mikhailov [9] proposed a way to "deform" an abstract U-V pair,

$$\Phi_x = U\Phi \ , \qquad \Phi_y = V\Phi \ ,$$

(see Section 1.4.2). In the traditional framework of inverse scattering method, the matrices U and V are functions of x and y and a constant spectral parameter γ . With respect to γ , U and V are either rational, trigonometric or elliptic functions. The "deformation" means that γ is allowed to depend on x and y and a "hidden" spectral parameter λ . The most well-known and probably most important example of such deformation is the Ernst equation. In 1978, by Maison and Belinskii – Zakharov, the Ernst equation was found to be a compatibility condition of a linear system with a non-constant spectral parameter γ . The expression for γ in this case coincides with the uniformization map of the genus zero two-fold covering of $\mathbb{C}P^1$; therefore the Ernst equation is naturally related to the Hurwitz space $H_{0;2}$ (space of rational functions of degree 2). The integrability of the Ernst equation becomes a partial case of the general deformation scheme proposed in [9]. In this work, the authors considered a generic U-V pair and derived a system of equations which must be satisfied by the spectral parameter γ in order for the matrices U and V to satisfy the zero curvature condition. However, for a long time an effective procedure of solving the system for the spectral parameter γ was not known. Only recently, in [42], this system was solved for the case of rational matrices $U(\gamma)$ and $V(\gamma)$ in terms of spaces of rational functions; a solution γ was given by the uniformization map of a genus zero surface. In this way a new hierarchy of nonlinear integrable systems related to the Hurwitz spaces $H_{0,N}$ was obtained. In the case $N = 2$, the hierarchy reduces to the Ernst equation.

In the framework of [42], the matrices U and V depend on x and y through the coordinates $\{\lambda_m\}$ (critical values of a rational function) on the Hurwitz space $H_{0,N}$ (space of rational functions of degree N). The corresponding linear system with respect to $\{\lambda_m\}$ has the form:

$$\frac{d\Phi(P)}{d\lambda_m} = U_m(P)\Phi(P) , \qquad m = 1, \ldots, 2N-2 ,$$

where P is a point on the covering. Here the rational matrix $U_m(P, \{\lambda_j\})$ has the only singularity, which is a simple pole at the ramification point $P = P_m$ of the covering (in addition to a simple pole at infinity). Such a function exists on a genus zero surface. However, for an analog of this linear system on a surface of genus greater or equal to one, any function $U_m(P)$ with single first-order pole must be non-single valued, i.e. it must gain some multiplicative and (or) additive "twists" under analytical continuation along topologically non-trivial cycles of the surface. These transformations for genus greater than one also depend on branch points of the covering, which makes the corresponding integrable system transcendently nonlinear.

In genus one, however, one can find the functions $U_m(P)$ with one simple pole such that under analytical continuation along the basic cycles they are conjugated by matrices which remain constant under the deformation of the Riemann surface (these matrices define the so-called rigid vector bundles over an elliptic curve [1]). The independence of the matrices of transformations of the coordinates $\{\lambda_j\}$ on the Hurwitz space allows us to develop a scheme of deformation of integrable systems on an elliptic surface analogous to the genus zero case [42]. The result is a new family of deformed integrable systems. These new integrable systems turn out to be closely related to Takasaki's version of the isomonodromy deformation equations on an elliptic surface.

Chapter 2

Integrable systems related to elliptic branched coverings

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J. Phys. A: Math. Gen. 36, 10585-10605 (2003)

Abstract. The new integrable systems associated with the space of elliptic branched coverings are constructed. The relationship of these systems with elliptic Schlesinger's system [70] is described. For the standard two-fold elliptic coverings the integrable system is written explicitly. The trigonometric degeneration of our construction is presented.

2.1 Introduction

The most well-studied integrable systems like Korteweg-de Vries, non-linear Schrödinger, sin-Gordon [60] appear as compatibility conditions of the auxiliary linear system

$$\Phi_x = U\Phi \ , \quad \Phi_y = V\Phi \ , \tag{2.1.1}$$

where U , V and Ψ are matrix functions of (x, y) and a constant (i.e. independent of x and y) spectral parameter $\gamma \in \mathbb{C}$. Matrices U and V for these systems are meromorphic functions of γ with (x, y)-independent positions of poles.

In 1978 Belinskii and Zakharov [3] and Maison [55] discovered integrability of the Ernst equation

$$((x-y)G_xG^{-1})_y + ((x-y)G_yG^{-1})_x = 0 \ , \tag{2.1.2}$$

where $G \in SU(1,1)/U(1)$, which does not fit into this framework. Namely, the Ernst equation is a compatibility condition of the system (2.1.1) with matrices U and V of the form:

$$U = \frac{G_xG^{-1}}{1-\gamma} \ , \quad V = \frac{G_yG^{-1}}{1+\gamma} \ , \tag{2.1.3}$$

where the spectral parameter γ is the function of x , y and a "hidden" ("constant") spectral parameter λ :

$$\gamma(\lambda, x, y) = \frac{2}{y-x}\Big(\frac{x+y}{2} - \lambda + \sqrt{(\lambda - x)(\lambda - y)}\Big) . \tag{2.1.4}$$

Therefore the Ernst equation can be viewed as a "deformation" of the principal chiral model (PCM) equations. For this model the matrices U and V have the same form (2.1.3) but γ is a constant (independent of (x, y)) spectral parameter.

The same equation (2.1.2) for $G \in SU(2)/U(1)$ plays the role of the Gauss-Weingarten system for the so-called Bianchi surfaces in $\mathbb{R}^3$ (surfaces of negative Gaussian curvature of special form [5, 48]).

The general deformation scheme of linear systems of the type (2.1.1) was proposed in 1989 by Burtsev, Mikhailov and Zakharov [9]. Assuming that the spectral parameter γ in (2.1.1) depends on x and y , they derived a system of differential equations on γ which provide a part of compatibility condition of the linear system (2.1.1). Solutions of the system for γ was found in the recent work [42]; in this work γ is given by the inverse map to the uniformization map of a rational (genus zero) N-fold branched covering of the Riemann sphere when the branch points of the covering are chosen to be independent variables. In other words, a deformation of the linear system (2.1.1) was associated with the space of rational functions of degree N with simple critical points. In the case of two-fold rational covering, if the matrix dimension equals 2 , this scheme leads to the Ernst equation.

In [42] it was also shown how to generalize this approach to the Hurwitz spaces of genus $g \geq 2$ (spaces of meromorphic functions on the Riemann surface of genus g) for matrix systems. However, for the genus grater or equal to two it is difficult to present any explicit equations. The linear system associated with a genus g branched covering $\mathcal{L}$ has the following form [42]:

$$\frac{d\Psi}{d\lambda_m} = U_m \Psi \,, \tag{2.1.5}$$

where the matrix $U_m(P, \{\lambda_m\})$, $P \in \mathcal{L}$ has only one simple pole at the ramification point P_m of the covering $\mathcal{L}$ and does not have any other singularities. Such a function exists on a genus zero surface, but for the higher genus it must be non-single valued. This means that for genus greater than one the matrices U_m get some multiplicative and (or) additive transformations under tracing along topologically non-trivial cycles of the surface. These transformations depend on branch points of the covering, which makes the corresponding integrable system transcendently nonlinear.

In genus one, however, it is possible to develop in detail a scheme analogous to the genus zero case and this is the purpose of the present paper.

Consider the Hurwitz space $H_{1,N}$, the space of N-fold genus one coverings of the Riemann sphere with simple ramification points (coverings consisting of N copies of $\mathbb{C}P^1$ with $2N$ ramification points). Projections of the ramification points on the base of the covering are called the branch points; we assume them to be distinct and denote by $\lambda_1, \ldots, \lambda_{2N}$. Consider the Abel map $\nu : \mathcal{L} \to \mathbb{C}$ from the genus one covering $\mathcal{L}$ onto its fundamental domain in the complex γ-plane. We denote by $\gamma_1, \ldots, \gamma_{2N}$ the images of the ramification points under this map. They satisfy the following equations as functions of the branch points:

$$\begin{aligned}
\frac{\partial \gamma_n}{\partial \lambda_m} &= -\alpha_m[\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)] \,, \qquad m \neq n \,, \\
\frac{\partial \gamma_m}{\partial \lambda_m} &= \sum_{n=1,\, n\neq m}^{2N} \alpha_n[\rho(\gamma_m - \gamma_n) + \rho(\gamma_n)] \,,
\end{aligned}$$

where ρ denotes the logarithmic derivative of the Jacobi theta function θ_1 ; α_m are some coefficients subject to the differential equations:

$$
\begin{aligned}
\frac{\partial \alpha_n}{\partial \lambda_m} &= -2\,\alpha_n \alpha_m \rho'(\gamma_n - \gamma_m) \;, \\
\frac{\partial \alpha_m}{\partial \lambda_m} &= \sum_{n=1,\, n\neq m}^{2N} 2\,\alpha_n \alpha_m \rho'(\gamma_n - \gamma_m) \;.
\end{aligned}
$$

On a covering of genus one the linear system (2.1.5) can be written in terms of the elliptic r-matrix, whose transformations under tracing along non-trivial contours of the covering are given by similarity transformations independent of the branch points. Namely, in this paper we consider the linear system (2.1.5) where matrices U_m look as follows:

$$
\overset{1}{U}_m (P) = \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (\nu(P) - \gamma_m)\, \overset{2}{J}_m \right) \tag{2.1.6}
$$

with some matrices $J_m(\{\lambda_k\})$, $P \in \mathcal{L}$. Here we consider all matrices as operators in the tensor product of two copies of $\mathbb{C}^K$ $:\overset{1}{A}= A\otimes I$, $\overset{2}{A}= I\otimes A$; the elliptic r-matrix $\overset{12}{r}$ is a linear operator in $\mathbb{C}^K \otimes \mathbb{C}^K$. The main result of this paper is the integrability of the following system:

$$
\begin{aligned}
\frac{\partial \overset{1}{J}_m}{\partial \lambda_n} &= -\; \alpha_n \overset{1}{J}_m \rho'(\gamma_m - \gamma_n) \;-\; \alpha_m \overset{2}{\mathrm{tr}} \left(\overset{12}{r}{}'(\gamma_m - \gamma_n)\, \overset{2}{J}_n \right) \\
&\quad -\; \left[\overset{1}{J}_m, \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (\gamma_m - \gamma_n)\, \overset{2}{J}_n \right) \right] .
\end{aligned} \tag{2.1.7}
$$

It appears as compatibility condition of the linear system (2.1.5), (2.1.6). The systems (2.1.7) are a genus one analogs of the integrable systems constructed in [42]; they give elliptic generalizations of the Ernst equation (2.1.2).

We define the τ-function for the integrable system (2.1.7) as follows:

$$
\frac{\partial \log \tau}{\partial \lambda_m} = \frac{1}{2\alpha_m} \mathrm{tr}(J_m^2) \;. \tag{2.1.8}
$$

This system is compatible as a corollary of (2.1.7). For the genus zero two-fold coverings this definition gives rise to one of the metric coefficients on the corresponding space-time [42].

The non-linear integrable system (2.1.7), together with the associated linear system (2.1.5), (2.1.6), turns out to be closely related with the elliptic Schlesinger system proposed by Takasaki [70]. Namely, from each solution of the elliptic Schlesinger system we can obtain a solution of the system (2.1.7). For these solutions there is a simple link between τ-function (2.1.8) and τ-function of the elliptic Schlesinger system:

$$\tau(\{\lambda_m\}) = \prod_{j=1}^{L} \left(\frac{\partial \nu}{\partial \lambda}(Q_j)\right)^{\mathrm{tr}A_j^2/2} \tau_{\mathrm{S}ch}\left(\{z_k\}\right)\big|_{z_k=\gamma(Q_k)} , \qquad (2.1.9)$$

where $\{z_1, \ldots, z_L\}$ is a set of points in the γ-plane which forms a part of monodromy data for the elliptic Schlesinger system; $Q_1, \ldots, Q_L$ are points on the covering whose images under the Abel map ν are given by $z_1, \ldots, z_L$ and whose projection on the λ-sphere do not depend on the branch points $\{\lambda_m\}$; matrices $A_1, \ldots, A_L$ solve the Schlesinger system; the variables $\mathrm{tr}A_j^2$ are integrals of the elliptic Schlesinger system.

The paper is organized as follows. In section 2.2 we discuss the genus zero case and present a slight generalization of the scheme proposed in [42]. In section 2.3 we derive auxiliary differential equations describing the dependence of the Abel map ν of the genus one covering on the branch points. Further, we introduce the linear system (2.1.5), (2.1.6) and derive the integrable system (2.1.7) as its compatibility condition. Then we define the tau-function of the integrable system. Finally, we write explicitly the system (2.1.7) in the case of the simplest elliptic covering. Section 2.4 is devoted to a description of the link of the integrable systems constructed in section 2.3 with the elliptic Schlesinger system proposed by Takasaki [70]. In section 2.5 we describe the trigonometric degeneration of the constructed integrable systems (2.1.7).

2.2 Integrable systems related to space of rational functions.

The goal of this section is to describe integrable systems related to the space of rational functions. We present a different version of the construction proposed in [42]. Consider the space of rational functions of degree N with $2N-2$ critical points which have the following form:

$$R(\gamma) = \frac{a_N\gamma^N + a_{N-1}\gamma^{N-1} + \cdots + a_0}{\gamma^N + b_{N-1}\gamma^{N-1} + \cdots + b_0}\ . \tag{2.2.1}$$

The genus zero algebraic curve

$$\lambda = R(\gamma)$$

can be realized as an N-fold branched covering $\mathcal{L}$ of the λ-sphere $\mathbb{C}P^1$; a point P of the covering is a pair (λ, γ) . We denote by π the projection operator from the covering onto the underlying λ-sphere: $\pi(P) = \lambda$ Functions (2.2.1) have $2N-2$ critical points counting multiplicities; according to the Riemann-Hurwitz formula, the genus of the corresponding covering $\mathcal{L}$ is zero. We assume the ramification points of the covering to be simple and finite; denote them by $P_1, \ldots, P_{2N-2}$. Their projections $\pi(P_m) = \lambda_m$ on the λ-sphere (the branch points) are critical values of the rational function $R(\gamma)$: $\lambda_m = R(\gamma_m)$, where $\{\gamma_m\}$ are critical points of the function R , i.e. solutions of the equation $R'(\gamma) = 0$. We assume all branch points $\lambda_1, \ldots, \lambda_{2N-2}$ to be distinct.

To each element l of the fundamental group $\pi_1(\mathbb{C} \setminus \{\lambda_1, \ldots, \lambda_{2N-2}\})$ one can assign an element σ_l of the symmetric group S_N , which describes how the sheets of the covering permute when λ goes along the contour l . In this way we can assign to the covering $\mathcal{L}$ a representation of $\pi_1(\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_{2N-2}\})$ in S_N . For the fixed number of sheets, type of branch points and the assigned representation, the covering is determined by positions of the branch points, i.e. $\{\lambda_m\}_{m=1}^{2N-2}$ gives a set of local coordinates on the Hurwitz space. Observe that this set has $2N - 2$ elements whereas the corresponding rational function (2.2.1) is defined by $2N + 1$ parameters. This is because any Möbius transformation in the γ-sphere (determined by three parameters),

$$\gamma \mapsto \frac{a\gamma + b}{c\gamma + d}\,, \qquad ad - bc = 1\,, \tag{2.2.2}$$

leaves positions of the branch points $\{\lambda_m\}$ invariant.

For our purposes we fix the coefficient a_N in the nominator of the rational function to be a constant, say $a_N = 1$; then the rational function (2.2.1) becomes:

$$R(\gamma) = \frac{\gamma^N + a_{N-1}\gamma^{N-1} + \cdots + a_0}{\gamma^N + b_{N-1}\gamma^{N-1} + \cdots + b_0}\,; \tag{2.2.3}$$

and at infinity the following asymptotics takes place:

$$\lambda = 1 + \frac{\beta}{\gamma} + o(\frac{1}{\gamma})\,, \qquad \text{as} \quad \gamma \sim \infty\,, \tag{2.2.4}$$

where we denoted $\beta = a_{N-1} - b_{N-1}$.

We shall consider the critical points $\{\gamma_n\}$ of the rational function (2.2.3) as functions of its critical values $\{\lambda_n\}$. First, note that on the covering $\mathcal{L}$ there defined a one-to-one function $\nu : \mathcal{L} \to \mathbb{C}P^1$ such that $R(\nu(P)) = \pi(P)$; in particular, the images of ramification points are the critical points of the rational function: $\nu(P_m) = \gamma_m$. The function $\nu(P)$ takes every value only once, thus $\nu(P)$ is holomorphic everywhere except the point which is mapped to infinity; then we can write the expansion of $\nu(P)$ with respect to the local parameter

$\sqrt{\lambda-\lambda_m}$ in a neighbourhood of the ramification point P_m (for any $m=1,\dots,2N-2$) as follows:

$$\nu(P)=\gamma_m+v_m\sqrt{\lambda-\lambda_m}+O(\lambda-\lambda_m)\ ,\qquad P\to P_m\ . \tag{2.2.5}$$

Let us differentiate these expansions with respect to λ_n and rewrite the result in terms of ν , using the relation $\sqrt{\lambda-\lambda_n}=(\nu-\gamma_n)/v_n+O((\nu-\gamma_n)^2)$ which follows from (2.2.5). We see that the function $\partial\nu/\partial\lambda_n$ is a meromorphic function of ν which has a first order pole at the point γ_n and is regular at all other critical points, i.e.

$$\frac{\partial\nu}{\partial\lambda_n}=\frac{\alpha_n}{\gamma_n-\nu}+f(\nu)\ , \tag{2.2.6}$$

where $\alpha_n=(v_n/2)^2$, and $f(\nu)$ is a function regular everywhere except the point at infinity. We find the behavior of this function at infinity differentiating the asymptotics (2.2.4) (which holds for $\gamma=\nu(P)$ since locally, in a neighbourhood of the preimage of infinity $P\sim\nu^{-1}(\infty)$, the function $\gamma(\lambda)=\nu(P)$ gives the inverse to $R(\gamma)$, $\gamma\sim\infty$) with respect to λ_n :

$$0=\frac{\beta_{\lambda_n}}{\nu}-\nu_{\lambda_n}\frac{\beta}{\nu^2}+o(\frac{1}{\nu})\ ,\quad \text{as}\quad \nu\sim\infty\ . \tag{2.2.7}$$

This implies the following equations describing the dependence of the function ν on the critical values of the corresponding rational function (2.2.3):

$$\frac{\partial\nu}{\partial\lambda_n}=\frac{\alpha_n}{\gamma_n-\nu}+\frac{\beta_{\lambda_n}}{\beta}\nu+c_n\ ,\qquad n=1,\dots,2N-2\ , \tag{2.2.8}$$

with some functions $c_n=c_n(\{\lambda_k\})$.

The compatibility condition of the system (2.2.8) gives the following system of differential equations for the critical points $\{\gamma_m\}$ of the rational function (2.2.3):

$$\frac{\partial\gamma_m}{\partial\lambda_n}=\frac{\alpha_n}{\gamma_n-\gamma_m}+\frac{\beta_{\lambda_n}}{\beta}\gamma_m+c_n\ ,\qquad n\neq m\ . \tag{2.2.9}$$

Remark 2.1 We get the same equations if instead of the rational function (2.2.3) consider the one of the form:

$$R(\gamma) = \beta\gamma + \delta + \sum_{k=1}^{N-1} \frac{a_k}{\gamma - b_k} , \tag{2.2.10}$$

which can be obtained from (2.2.3) by a Möbius transformation.

Consider now the following system of linear differential equations for a matrix-valued function $\Psi(P, \{\lambda_m\})$ $(m = 1, \ldots, 2N-2)$:

$$\frac{d\Psi}{d\lambda_m}(P) = \frac{\gamma_0 - \gamma_m}{\nu(P) - \gamma_m} G_{\lambda_m} G^{-1} \Psi(P) , \tag{2.2.11}$$

where $\gamma_o = \nu(P_0)$, the projection $\pi(P_0) = \lambda_0 \in \mathbb{C}P^1$ of the point P_0 is independent of all $\{\lambda_m\}$; $G(\{\lambda_m\})$ is a matrix-valued function. The compatibility condition for (2.2.11) is given by the following system of non-autonomous (since all γ_m and γ_0 are non-trivial algebraic functions of $\{\lambda_m\}$) coupled PDE's:

$$\left(\frac{\gamma_0 - \gamma_m}{\beta} G_{\lambda_m} G^{-1}\right)_{\lambda_n} = \left(\frac{\gamma_0 - \gamma_n}{\beta} G_{\lambda_n} G^{-1}\right)_{\lambda_m} . \tag{2.2.12}$$

The described construction of the integrable systems gives a realization of the scheme of Burtsev, Mikhailov, Zakharov [9] who derived the compatibility conditions for the deformed linear system of the type (2.1.1). They obtained differential equations on the variable spectral parameter of the linear system which form a part of the compatibility condition. It was shown in [42] that the function $\nu(P)$ is a solution of these differential equations.

In the case of the two-fold coverings ($N = 2$) corresponding to the rational function of the form (2.2.10) with $\beta = 1$, $\delta = 0$ (the normalization considered in [42]) system (2.2.12) coincides with the Ernst equation (1.4.3) after the identification $\lambda_1 = x$, $\lambda_2 = y$ (see [42]).

There exists a well-known relationship between these rational two-fold coverings and the surface theory: the Gauss-Weingarten equation for a surface in $\mathbb{R}^3$ with the Gaussian curvature $K = -[\rho(x,y)]^{-2}$ can be written in the following form [48]:

$$(\rho G_x G^{-1})_y + (\rho G_y G^{-1})_x = 0 , \tag{2.2.13}$$

for $G \in SU(2)/U(1)$, which for the case of the Bianchi surfaces ($\rho(x,y) = x - y$) formally coincides with equation (1.4.3).

Here the natural question arises: are there other coverings for which the system (2.2.12) takes the form of the Gauss-Weingarten equation for some surfaces? (Then it would be a new integrable case in surface theory.) This occurs if the system (2.2.12) has the property $\gamma_0 - \gamma_m = -(\gamma_0 - \gamma_n)$ for some pair of indeces m , n ; that is

$$\frac{\gamma_m + \gamma_n}{2} = \gamma_0 , \tag{2.2.14}$$

where $\gamma_0 = \nu(P_0)$ is the image of the point $P_0 \in \mathcal{L}$ whose projection λ_0 on the λ-sphere does not depend on $\{\lambda_k\}$. Existence of such systems is an open question. Since the covering is locally defined by $2N-2$ independent variables $\{\lambda_m\}$, two additional parameters of the rational function (2.2.3) could be used to impose some relations on $\{\gamma_m\}$. As it was already noted the freedom to choose these parameters corresponds to two Möbius transformations in the γ-sphere: $\gamma \to a\gamma$ and $\gamma \to \gamma + b$. But the condition (2.2.14) is invariant with respect to both of these transformations, which means that for the given degree N of a rational function we do not have any freedom to impose condition (2.2.14) for any pair of m and n . However, there is still a possibility that (2.2.14) holds for some rational coverings as in the case of $N = 2$.

2.3 Integrable systems related to elliptic branched coverings

In this section we construct an elliptic analog of the integrable system (2.2.12).

2.3.1 Differential equations for images of ramification points of elliptic coverings in fundamental domain

The Hurwitz space $H_{1,N}$ is the space of meromorphic functions of degree N on Riemann surfaces of genus one. Consider a meromorphic double-periodic function R of $\gamma \in \mathbb{C}$ with periods 1 and μ and N simple poles within the fundamental domain $T = \mathbb{C}/\{1, \mu\}$. As a function on T , $R(\gamma)$ has degree N . The equation

$$\lambda = R(\gamma) \tag{2.3.1}$$

defines an N-fold branched covering (we again call it $\mathcal{L}$) of the Riemann sphere. A point P of the covering is a pair: $P = (\lambda, \gamma)$. According to the Riemann-Hurwitz formula, this covering has $2N$ ramification points counting multiplicities; we assume them to be simple and finite and denote by $P_1, \dots, P_{2N}$. Projections $\{\pi(P_m)\}$ of the ramification points onto the λ-sphere (the base of the covering) are called the branch points. They are given by critical values $\lambda_1, \dots, \lambda_{2N}$ of the meromorphic function $R(\gamma)$: $\lambda_m = R(\hat{\gamma}_m)$, where $\hat{\gamma}_1, \dots, \hat{\gamma}_{2N}$ are critical points of $R(\gamma)$, solutions of the equation $R'(\gamma) = 0$. We assume the branch points to be distinct: $\lambda_m \neq \lambda_n$ for $m \neq n$. Our choice of the local parameters on $\mathcal{L}$ is standard: in a neighborhood of a ramification point P_m we take $x(P) = \sqrt{\lambda - \lambda_m}$, $P \in \mathcal{L}$, $P \sim P_m$; in a neighborhood of a point at infinity on any sheet we take $x = 1/\lambda$; at any other point variable λ itself is used as a local coordinate. To the covering $\mathcal{L}$ it is assigned a representation of the fundamental group $\pi_1 (\mathbb{C} \setminus \{\lambda_1, \dots, \lambda_{2N}\})$ in the following way. To each element l of the fundamental group one can assign an element σ_l of the symmetric group S_N , which describes how the sheets permute when λ goes along the contour l on the base of the covering. We fix this representation of $\pi_1(\mathbb{C}\setminus\{\lambda_1,\dots,\lambda_{2N}\})$. Then for the fixed number of sheets and type of ramification points (we fix them to be simple) the covering is determined by positions of the branch points; thus we consider $\{\lambda_m\}_{m=1}^{2N}$ as a set of local coordinates on the space of elliptic coverings.

Introduce on $\mathcal{L}$ some canonical basis of cycles (a,b) . Denote by $\mathbf{v}(P)$, $P \in \mathcal{L}$ the holomorphic Abelian differential with normalized a-period:

$$\oint_a \mathbf{v} = 1\,; \tag{2.3.2}$$

for our covering it has the form:

$$\mathbf{v}(P) = \frac{d\lambda}{R'(\gamma)} = d\gamma\,. \tag{2.3.3}$$

The integral over b-cycle gives the module μ of the elliptic Riemann surface $\mathcal{L}$:

$$\mu = \oint_b \mathbf{v}\,. \tag{2.3.4}$$

The function $\nu(P)$ which maps $\mathcal{L}$ onto the fundamental domain $T = \mathbb{C}/\{1,\mu\}$ is given by the Abel map

$$\nu(P) = \int_{\infty^{(0)}}^{P} \mathbf{v}\,, \tag{2.3.5}$$

where we choose the initial point of integration to coincide with the point at infinity on some (the "zero"th) sheet of the covering $\mathcal{L}$. We denote the images of the ramification points under this map by γ_m . They differ from the critical points $\{\hat{\gamma}_m\}$ of the function R by a shift (corresponding to the choice of initial point of integration in (2.3.5)) modulo the period lattice $\{k\mu + l\ ;\ l\,, k \in \mathbb{N}\}$:

$$\gamma_m \equiv \hat{\gamma}_m - c\ , \quad m = 1, \ldots, 2N\ , \tag{2.3.6}$$

where c is the second coordinate of the point $\infty^{(0)} \in \mathcal{L}\ :\ \infty^{(0)} = (\infty, c)$.

The Jacobi theta functions are given by

$$\theta[p,q](\gamma;\mu) = \sum_{m\in\mathbb{Z}} \exp\{\pi i\mu(m+p)^2 + 2\pi i\,(m+p)(\gamma+q)\}\ . \tag{2.3.7}$$

We denote by $\rho(\gamma)$ the logarithmic derivative of theta-function $\theta_1(\gamma) = -\theta[1/2,1/2](\gamma)$:

$$\rho(\gamma) = \frac{d}{d\gamma}\log\theta_1(\gamma)\ ; \tag{2.3.8}$$

it has the following periodicity properties:

$$\rho(\gamma+1) = \rho(\gamma)\ , \qquad \rho(\gamma+\mu) = \rho(\gamma) - 2\pi i\ . \tag{2.3.9}$$

The derivative $\rho'(\gamma)$ coincides with the Weierstrass $\mathcal{P}$-function up to a rescaling of the argument and an additive constant.

The following theorem describes the dependence of the map $\nu(P)$ (2.3.5) on λ and the branch points $\{\lambda_m\}$; it provides an elliptic version of equations (2.2.8).

Theorem 2.1 *The function $\nu(\lambda, \{\lambda_m\})$ defined by (2.3.5) satisfies the following system of differential equations:*

$$\frac{\partial\nu}{\partial\lambda} = \sum_{k=1}^{2N} \alpha_k[\rho(\nu-\gamma_k) + \rho(\gamma_k)]\ , \tag{2.3.10}$$

$$\frac{\partial\nu}{\partial\lambda_m} = -\alpha_m[\rho(\nu-\gamma_m) + \rho(\gamma_m)]\ , \quad m = 1, \ldots, 2N\ , \tag{2.3.11}$$

where we denoted

$$\alpha_m = \frac{1}{2}v_m^2 = \frac{1}{2}\Big[\frac{\mathbf{v}(P)}{d\sqrt{\lambda-\lambda_m}}\Big|_{P=P_m}\Big]^2\ ; \tag{2.3.12}$$

and $\{\gamma_m\}$ are the images of the ramification points under the map $\nu\ :\ \gamma_m = \nu(P_m)$.

Remark 2.2 The form (2.3.3) of the holomorphic normalized differential implies that $\alpha_m = [R''(\hat{\gamma}_m)]^{-1}$.

Proof of theorem 2.1. From (2.3.5) we see that the function $\nu(P)$ is holomorphic in a neighborhood of the ramification point P_m and behaves as follows:

$$\nu(P) = \gamma_m + v_m\sqrt{\lambda - \lambda_m} + \mathcal{O}(\lambda - \lambda_m) \quad \text{as} \quad P \to P_m \, , \tag{2.3.13}$$

where $\sqrt{\lambda - \lambda_m}$ is the local coordinate in a neighborhood of P_m , and v_m is defined by (2.3.12). Therefore, in this neighborhood

$$\begin{aligned} \frac{\partial \nu}{\partial \lambda}(P) &= \frac{v_m}{2\sqrt{\lambda - \lambda_m}} + \mathcal{O}(1) \, , && (2.3.14) \\ \frac{\partial \nu}{\partial \lambda_m}(P) &= -\frac{v_m}{2\sqrt{\lambda - \lambda_m}} + \mathcal{O}(1) \, , && (2.3.15) \\ \frac{\partial \nu}{\partial \lambda_n}(P) &= \mathcal{O}(1) \, , \qquad n \neq m \, . && (2.3.16) \end{aligned}$$

We rewrite these expansions in terms of the coordinate ν taking into account definition (2.3.12) of α_m and the correspondence between local parameters $\nu - \gamma_m$ and $\sqrt{\lambda - \lambda_m}$ given by (2.3.13):

$$\frac{\partial \nu}{\partial \lambda}(P) = \frac{\alpha_m}{\nu - \gamma_m} + \mathcal{O}(1) \, , \qquad \frac{\partial \nu}{\partial \lambda_n}(P) = -\delta_{mn}\frac{\alpha_m}{\nu - \gamma_m} + \mathcal{O}(1) \, , \tag{2.3.17}$$

as $P \to P_m$.

The function $\nu(P)$ transforms as follows under the tracing along basic cycles on $\mathcal{L}$:

$$\nu(P^a) = \nu(P) + 1 \, , \qquad \nu(P^b) = \nu(P) + \mu \, , \tag{2.3.18}$$

where $\nu(P^a)$, $\nu(P^b)$ denote the analytic continuation of $\nu(P)$ along a- and b-cycles, respectively. Therefore the derivative ν_λ is periodic with respect to tracing along the basic cycles.

Then the function ν_λ has periods 1 and μ in the γ-plane. Its local behavior at the points γ_m , $m = 1, \ldots, 2N$ is given by (2.3.17). Hence, we conclude that $\sum_{k=1}^{2N} \alpha_k = 0$ as sum of residues, and the derivative ν_λ can be expressed as follows in terms of function ρ :

$$\frac{\partial \nu}{\partial \lambda}(\nu) = \sum_{k=1}^{2N} \alpha_k \rho(\nu - \gamma_k) + const\,. \tag{2.3.19}$$

To determine the constant in (2.3.19) consider a neighborhood of $P = \infty^{(0)}$. The Abel map (2.3.5) is zero at this point, $\nu(\infty^{(0)}) = 0$, and we can write its behavior there as follows:

$$\nu(\lambda) = \frac{\alpha}{\lambda} + \mathcal{O}\Big(\frac{1}{\lambda^2}\Big) \quad \text{as} \quad \lambda \to \infty\,. \tag{2.3.20}$$

(Note that $\alpha \neq 0$ since we assume $\infty^{(0)}$ not to be a ramification point.) Therefore, for the λ-derivative we have in a neighbourhood of $P = \infty^{(0)}$:

$$\frac{\partial \nu}{\partial \lambda}(\lambda) = -\frac{\alpha}{\lambda^2} + \mathcal{O}\Big(\frac{1}{\lambda^3}\Big)\,, \quad \lambda \to \infty\,.$$

Rewriting as before this expansion in terms of the coordinate ν ((2.3.20) implies $\lambda \sim \alpha/\nu$) we see that $\nu_\lambda(\nu = 0) = 0$:

$$\frac{\partial \nu}{\partial \lambda}(\nu) = -\frac{\nu^2}{\alpha} + \mathcal{O}\left(\nu^3\right)\,, \quad \text{as} \quad \nu(P) \to 0\,.$$

Therefore, (2.3.19) turns into (2.3.10).

Consider now ν_{λ_m} . In the γ-plane it has only one simple pole at $\nu = \gamma_m$ as it follows from (2.3.17). The periodicity properties (2.3.18) of the Abel map imply that

$$\frac{\partial \nu}{\partial \lambda_m}(\nu + 1) = \frac{\partial \nu}{\partial \lambda_m}(\nu) \quad \text{and} \quad \frac{\partial \nu}{\partial \lambda_m}(\nu + \mu) = \frac{\partial \nu}{\partial \lambda_m}(\nu) + \frac{\partial \mu}{\partial \lambda_m}\,. \tag{2.3.21}$$

The function $-\alpha_m \rho(\nu - \gamma_m) + const$ satisfies the periodicity condition (2.3.21) since, due to the Rauch variational formulas [63], we have:

$$\frac{\partial \mu}{\partial \lambda_m} = \pi i v_m^2 = 2\pi i \alpha_m\,. \tag{2.3.22}$$

To find the constant term we again put $\nu = 0$, i.e. $P = \infty^{(0)}$. Then from the asymptotics (2.3.20) we see that $\nu_{\lambda_m}(\nu = 0) = 0$, which leads to (2.3.11).

$\diamond$

Remark 2.3 Equations (2.3.11) can also be deduced from the Rauch variational formulae for the differential $\mathbf{v}$ [63].

Compatibility conditions of the system (2.3.10), (2.3.11) imply the system of differential equations describing the dependence of $\{\gamma_m\}$ on the branch points $\{\lambda_m\}$ (the indeces run through the set $\{1, \ldots, 2N\}$):

$$\frac{\partial \gamma_n}{\partial \lambda_m} = -\alpha_m[\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)] \,, \qquad m \neq n \,, \tag{2.3.23}$$

$$\frac{\partial \gamma_m}{\partial \lambda_m} = \sum_{k=1,\, k\neq m}^{2N} \alpha_k[\rho(\gamma_m - \gamma_k) + \rho(\gamma_k)] \,. \tag{2.3.24}$$

The equations for residues α_m which also follow from the compatibility of (2.3.10) and (2.3.11) look as follows:

$$\frac{\partial \alpha_n}{\partial \lambda_m} = -2\,\alpha_n\alpha_m\rho'(\gamma_n - \gamma_m) \,, \qquad m \neq n \,, \tag{2.3.25}$$

$$\frac{\partial \alpha_m}{\partial \lambda_m} = \sum_{k=1,\, k\neq m}^{2N} 2\,\alpha_k\alpha_m\rho'(\gamma_k - \gamma_m) \,. \tag{2.3.26}$$

In fact, equations (2.3.25) and (2.3.26) are nothing but the Rauch variational formulas [63] for the holomorphic differential $\mathbf{v}$.

2.3.2 Integrable systems

Denote the matrix dimension of our system by K . The classical elliptic r-matrix is the following linear operator in the tensor product of two copies of $\mathbb{C}^K$:

$$\overset{12}{r}(\gamma) = \sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K-1} w_{AB}(\gamma)\, \overset{1}{\sigma}_{AB}\, \overset{2}{\sigma}^{AB} , \tag{2.3.27}$$

where w_{AB} are given by the combinations of Jacobi's theta functions (2.3.7) ($(A,B) \neq (0,0)$):

$$w_{AB}(\gamma) = \frac{\theta_{[AB]}(\gamma)\theta'_{[00]}(0)}{\theta_{[AB]}(0)\theta_{[00]}(\gamma)} ; \tag{2.3.28}$$

where we denote

$$\theta_{[AB]}(\gamma) = \theta_{[AB]}(\gamma;\mu) = \theta\left[\tfrac{A}{K}-\tfrac{1}{2},\tfrac{1}{2}-\tfrac{B}{K}\right](\gamma;\mu) .$$

All the w_{AB} have a simple pole with unit residue at $\gamma = 0$ and the following twist properties:

$$w_{AB}(\gamma+1) = \epsilon^A w_{AB}(\gamma) , \qquad w_{AB}(\gamma+\mu) = \epsilon^B w_{AB}(\gamma) , \tag{2.3.29}$$

where $\epsilon = e^{2\pi i/K}$. The matrices σ_{AB} are the higher rank analogs of the Pauli matrices; they form a basis of $sl(K,\mathbb{C})$ and are defined as follows (for $(A,B) \neq (0,0)$):

$$\sigma_{AB} = H^A F^B , \tag{2.3.30}$$

where F is the diagonal matrix

$$F = \mathrm{diag}\{1, \epsilon, \epsilon^2, \ldots, \epsilon^{K-1}\} ,$$

and H is the permutation matrix

$$H = \begin{pmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1 \\ 1 & 0 & . & . & . & 0 \end{pmatrix} .$$

These matrices satisfy the relations $\epsilon FH = HF$, and $F^K = H^K = I$.

Together with σ_{AB} we introduce the dual basis σ^{AB} :

$$\sigma^{AB} = \frac{\epsilon^{-AB}}{K} \sigma_{-A,-B} , \tag{2.3.31}$$

such that

$$\mathrm{tr}\,(\sigma_{AB}\sigma^{CD}) = \delta^C_A \delta^D_B . \tag{2.3.32}$$

From (2.3.29) and properties of matrices F and H we derive the following periodicity properties of the elliptic r-matrix (2.3.27):

$$\begin{aligned} \overset{12}{r}(\gamma + 1) &= \overset{1}{F}{}^{-1} \overset{12}{r}(\gamma) \overset{1}{F} , \\ \overset{12}{r}(\gamma + \mu) &= \overset{1}{H} \overset{12}{r}(\gamma) \overset{1}{H}{}^{-1} . \end{aligned} \tag{2.3.33}$$

In the sequel we shall also need the functions

$$\mathcal{Z}_{AB}(\gamma) = \frac{w_{AB}(\gamma)}{2\pi i} \Big(\frac{\theta'_{[AB]}(\gamma)}{\theta_{[AB]}(\gamma)} - \frac{\theta'_{[AB]}(0)}{\theta_{[AB]}(0)} \Big) , \qquad (A, B) \neq (0, 0) , \tag{2.3.34}$$

which have no singularities and transform as follows:

$$\mathcal{Z}_{AB}(\gamma + 1) = \epsilon^A \mathcal{Z}_{AB} , \qquad \mathcal{Z}_{AB}(\gamma + \mu) = \epsilon^B \left(\mathcal{Z}_{AB}(\gamma) - w_{AB}(\gamma) \right) . \tag{2.3.35}$$

Using the fact that theta functions satisfy the heat equation,

$$\frac{\partial^2\theta[p,q](\gamma;\mu)}{\partial\gamma^2} = 4\pi i\frac{\partial\theta[p,q](\gamma,\mu)}{\partial\mu}\,, \tag{2.3.36}$$

we get the the following relation between $\mathcal{Z}_{AB}$ and w_{AB} :

$$\partial_\mu w_{AB}(\gamma;\mu) = \partial_\gamma \mathcal{Z}_{AB}(\gamma;\mu)\,. \tag{2.3.37}$$

Now we are in a position to write down an "elliptic" counterpart of the linear system (2.2.11):

$$\frac{d\stackrel{1}{\Psi}(P)}{d\lambda_m} = \stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\nu(P)-\gamma_m)\stackrel{2}{J_m}\right)\stackrel{1}{\Psi}(P)\,, \tag{2.3.38}$$

where

$$J_m = \sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K-1} J_m^{AB}\sigma_{AB}$$

with scalars J_m^{AB} . Here $m=1,\ldots,2N$, $\Psi=\Psi(P,\{\lambda_m\})$ is a matrix-valued function; as before, $\nu(P)$ is the Abel map (2.3.5) from the covering $\mathcal{L}$ onto its fundamental domain $T=\mathbb{C}/\{1,\mu\}$; $\gamma_m=\nu(P_m)$. The compatibility condition of this system

$$\begin{aligned}&\left(\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\nu(P)-\gamma_m)\stackrel{2}{J_m}\right)\right)_{\lambda_n} - \left(\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\nu(P)-\gamma_n)\stackrel{2}{J_n}\right)\right)_{\lambda_m}\\ &\quad+\left[\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\nu(P)-\gamma_m)\stackrel{2}{J_m}\right),\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\nu(P)-\gamma_n)\stackrel{2}{J_n}\right)\right]=0\end{aligned} \tag{2.3.39}$$

gives the system of differential equations for matrices J_m as functions of the branch points λ_m :

$$\begin{aligned}\frac{\partial\stackrel{1}{J_m}}{\partial\lambda_n} &= -\ \alpha_n\stackrel{1}{J_m}\rho'(\gamma_m-\gamma_n)\ -\ \alpha_m\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}{}'(\gamma_m-\gamma_n)\stackrel{2}{J_n}\right)\\ &\quad -\ \left[\stackrel{1}{J_m},\stackrel{2}{\mathrm{tr}}\left(\stackrel{12}{r}(\gamma_m-\gamma_n)\stackrel{2}{J_n}\right)\right],\quad m\neq n\,,\end{aligned} \tag{2.3.40}$$

where r' stands for the derivative of the r-matrix with respect to its argument. To prove that the compatibility condition reduces to (2.3.40) we, first, compute the derivatives in (2.3.39) using the chain rule:

$$r_{\lambda_n}(\gamma) = r_\mu(\gamma)\mu_{\lambda_n} + r'(\gamma)\gamma_{\lambda_n} \; .$$

The derivative of the period μ is given by (2.3.22); for differentiation of ν and $\{\gamma_m\}$ one uses the equations (2.3.11) and (2.3.23) respectively. Then we note that the vector bundle χ over the Riemann surface $\mathcal{L}$, whose monodromy matrices along the cycles a and b are given by F^{-1} and H respectively, is stable [36]. Checking the periodicity properties of the left hand side of (2.3.39) we see that it is a section of the adjoint bundle $\mathrm{ad}\chi$. Due to the stability of χ the bundle $\mathrm{ad}\chi$ does not have holomorphic sections (see for example [72]). Therefore, for condition (2.3.39) to hold it suffices that the left hand side has no singularities; this is equivalent to the system (2.3.40).

Equations (2.3.40) form the non-autonomous non-linear integrable system associated with the space of elliptic coverings which gives an elliptic analog of the integrable system (2.2.12).

2.3.3 Tau-function

Let us introduce an object which we shall call the tau-function of the system (2.3.40):

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{1}{2\alpha_m} \mathrm{tr}(J_m^2) \; . \tag{2.3.41}$$

To prove consistency of the definition we compute the derivatives of the right hand side, $\frac{\partial}{\partial \lambda_n}(\frac{1}{2\alpha_m}\mathrm{tr}(J_m^2))$, using (2.3.40). Then we get:

$$\frac{\partial^2 \log \tau}{\partial \lambda_m \partial \lambda_n} = -\overset{1}{\mathrm{tr}}\,\overset{2}{\mathrm{tr}}\Big(\overset{1}{J_m}\overset{2}{J_n}\overset{12}{r}{}'(\gamma_m - \gamma_n)\Big) \; ;$$

this expression is symmetric in m and n , due to the following properties of r-matrix:

$$\overset{12}{r}(\gamma) = -\overset{21}{r}(-\gamma)\,, \qquad \text{and} \qquad \overset{12}{r}{}'(\gamma) = \overset{21}{r}{}'(-\gamma)\,.$$

This proves compatibility of the equations (2.3.41).

An alternative definition of the tau-function (2.3.41) can be given in terms of the one form $d\Psi\Psi^{-1} = \Psi_\nu\Psi^{-1}d\nu$:

$$\frac{\partial \log\tau}{\partial\lambda_m} = \frac{1}{2}\mathrm{res}|_{P_m}\Big\{\frac{\mathrm{tr}(d\Psi\Psi^{-1})^2}{d\lambda}\Big\}\,. \tag{2.3.42}$$

To prove the equivalence of the two definitions, first note that we can write:

$$d\lambda = \frac{\partial\lambda}{\partial\nu}d\nu\,. \tag{2.3.43}$$

Therefore using (2.3.10) for $\partial\nu/\partial\lambda$ we get

$$\frac{\mathrm{tr}(d\Psi\Psi^{-1})^2}{d\lambda} = \frac{\partial\nu}{\partial\lambda}\mathrm{tr}\left(\Psi_\nu\Psi^{-1}\right)^2 d\nu = \Big(\sum_{k=1}^{2N}\alpha_k\left(\rho(\nu-\gamma_k)+\rho(\gamma_k)\right)\Big)\mathrm{tr}(\Psi_\nu\Psi^{-1})^2 d\nu\,. \tag{2.3.44}$$

Further, we write the "full" derivative of Ψ with respect to λ_m as follows:

$$\frac{d\Psi}{d\lambda_m} = \frac{\partial\Psi}{\partial\lambda_m} + \frac{\partial\nu}{\partial\lambda_m}\frac{\partial\Psi}{\partial\nu}\,, \tag{2.3.45}$$

then using the form of the linear system (2.3.38) and formula (2.3.11) for the derivative of ν , we rewrite (2.3.45) in the form:

$$\overset{2}{\mathrm{tr}}\Big(\overset{12}{r}(\nu-\gamma_m)\,\overset{2}{J_m}\Big) = \frac{\partial\overset{1}{\Psi}}{\partial\lambda_m}\overset{1}{\Psi}{}^{-1} - \alpha_m\Big(\rho(\nu-\gamma_m)+\rho(\gamma_m)\Big)\frac{\partial\overset{1}{\Psi}}{\partial\nu}\overset{1}{\Psi}{}^{-1}\,, \tag{2.3.46}$$

from which one can find $\mathrm{tr}(\Psi_\nu\Psi^{-1})^2$ and see that (2.3.42) is equivalent to (2.3.41).

2.3.4 Integrable system in the case of two-fold elliptic coverings

The simplest elliptic covering $\mathcal{L}$ has two sheets and four ramification points. It corresponds to the hyperelliptic curve given by the following equation:

$$\omega^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4) ;$$

λ_m , $m = 1, \dots, 4$ are branch points. On the covering we choose the basic cycle a to encircle ramification points P_1 , P_2 , and b-cycle to encircle points P_2 and P_3 . For this Riemann surface the normalized holomorphic differential $\mathbf{v}$ is given by

$$\mathbf{v} = \frac{d\lambda}{\omega} \left[\oint_a \frac{d\lambda}{\omega} \right]^{-1} . \tag{2.3.47}$$

As before, μ is the b-period of the surface $\mathcal{L}$: $\mu = \oint_b \mathbf{v}(P)$. Consider the map $\tilde{\nu}$ from the covering $\mathcal{L}$ onto its fundamental domain $T = \mathbb{C}/\{1, \mu\}$:

$$\tilde{\nu}(P) = \int_{P_1}^{P} \mathbf{v}(P) ;$$

this map differs from the map $\nu(P)$ (2.3.5) by a function of branch points:

$$\nu(P) = \tilde{\nu}(P) + h(\{\lambda_m\}) ,$$

where $h(\{\lambda_m\}) = \int_{\infty^{(0)}}^{P_1} \mathbf{v}(P)$. For our choice of basic cycles the images $\tilde{\gamma}_m$ of ramification points under the map $\tilde{\nu}$ are given by:

$$\tilde{\gamma}_1 = \tilde{\nu}(P_1) = 0 ; \qquad \tilde{\gamma}_2 = \tilde{\nu}(P_2) = \frac{1}{2} ;$$
$$\tilde{\gamma}_3 = \tilde{\nu}(P_3) = \frac{1}{2} + \frac{\mu}{2} ; \qquad \tilde{\gamma}_4 = \tilde{\nu}(P_4) = \frac{\mu}{2} ;$$

Since $\gamma_m - \gamma_n = \tilde{\gamma}_m - \tilde{\gamma}_n$ (where $\gamma_m = \nu(P_m)$, $m = 1, \dots, 4$ are as before the images of ramification points under the map ν (2.3.5)), we can use these values of $\{\tilde{\gamma}_m\}$ to write explicitly the system (2.3.40) for the simplest covering. To do this we also calculate the coefficients $\{\alpha_m\}_{m=1}^4$ defined by (2.3.12). The form (2.3.47) of the normalized holomorphic differential $\mathbf{v}$ implies:

$$\alpha_1 = \frac{v_1^2}{2} = \frac{2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)A^2} ,$$

where $A = \oint_a \frac{d\lambda}{\omega}$. From the Thomae formulas [23] we see that

$$A^2 = \frac{4\pi^2\theta_4^4}{(\lambda_1 - \lambda_4)(\lambda_3 - \lambda_2)} ,$$

and therefore we have the following expressions for the coefficients α_m :

$$\alpha_1 = \frac{\lambda_3 - \lambda_2}{2\pi^2\theta_4^4(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} ; \qquad \alpha_2 = -\frac{\lambda_1 - \lambda_4}{2\pi^2\theta_4^4(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} ;$$
$$\alpha_3 = \frac{\lambda_1 - \lambda_4}{2\pi^2\theta_4^4(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_4)} ; \qquad \alpha_4 = -\frac{\lambda_3 - \lambda_2}{2\pi^2\theta_4^4(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)} .$$

Now we can write down the integrable system (2.3.40) explicitly for $K = 2$ (K is the matrix dimension of the system). In this case we use the standard Pauli basis $\{\sigma_1 , \sigma_2 , \sigma_3\}$ related to the matrices σ_{AB} as follows:

$$\begin{gathered} \sigma_{10} = \sigma_1 , \qquad \sigma_{11} = i\,\sigma_2 , \qquad \sigma_{01} = \sigma_3 ; \\ \sigma^{10} = \frac{1}{2}\sigma_1 , \qquad \sigma^{11} = \frac{i}{2}\sigma_2 , \qquad \sigma^{01} = \frac{1}{2}\sigma_3 . \end{gathered} \tag{2.3.48}$$

The corresponding notation for components of J_m is:

$$J_m^1 = J_m^{10} , \qquad J_m^2 = iJ_m^{11} , \qquad J_m^3 = J_m^{01} . \tag{2.3.49}$$

We shall write the equations for $(J_1)_{\lambda_2}$ $(J_1 = J_1^1\sigma_1 + J_1^2\sigma_2 + J_1^3\sigma_3)$. The remaining equations for $(J_m)_{\lambda_n}$ in the case of two-fold elliptic covering have a similar form.

$$\frac{\partial J_1^1}{\partial \lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^1 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} + 2\pi i J_1^3 J_2^2 \theta_4^2 \,,$$

$$\frac{\partial J_1^2}{\partial \lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^2 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} - 2\pi i J_1^3 J_2^1 \theta_3^2 \,,$$

$$\frac{\partial J_1^3}{\partial \lambda_2} = \frac{\lambda_1 - \lambda_4}{2\pi^2(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_4)} J_1^3 \frac{1}{\theta_4^4}\frac{\theta_2''}{\theta_2} + \frac{\lambda_3 - \lambda_2}{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} J_2^3 \frac{\theta_3^2}{\theta_4^2}$$
$$+ 2\pi i \left(J_1^2 J_2^1 \theta_3^2 - J_1^1 J_2^2 \theta_4^2 \right) \,.$$

Here $\theta_2 = \theta[\frac{1}{2}, 0](0)$; $\theta_3 = \theta[0,0](0)$; $\theta_4 = \theta[0, \frac{1}{2}](0)$ and $\theta_2'' = \theta_2''(0)$ are the standard theta-constants.

2.4 Relationship to the Schlesinger system

The elliptic Schlesinger system [70] describes isomonodromic deformations of solutions $\Psi(\gamma, \{z_i\})$ of the following matrix linear differential equation:

$$\frac{d\Psi}{d\gamma} = A(\gamma)\Psi, \tag{2.4.1}$$

where γ is a coordinate on the torus $T = \mathbb{C}/\{1, \mu\}$; $A(\gamma)$ is a meromorphic $sl(K, \mathbb{C})$-valued matrix:

$$\overset{1}{A}(\gamma) = \sum_{j=1}^{L} \overset{2}{\mathrm{tr}} \left(\overset{12}{r}(\gamma - z_j) \overset{2}{A}_j \right);$$

$r(\gamma)$ is the elliptic r-matrix (2.3.27); $z_j \in T$, $j = 1, \ldots, L$; L is some integer. At the points $\{z_j\}$ the matrix $A(\gamma)$ has simple poles with residues A_j . The residues are, in turn, parameterized as follows:

$$A_j = \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} \sigma_{AB} \,, \tag{2.4.2}$$

where matrices σ_{AB} are given by (2.3.30); $A_j^{AB} \in \mathbb{C}$. The matrix $A(\gamma)$ has the following periodicity properties:

$$A(\gamma+1) = F^{-1}A(\gamma)F\ , \qquad A(\gamma+\mu) = HA(\gamma)H^{-1}\ .$$

It is assumed that Ψ has asymptotical expansion near z_j , $j = 1, \ldots, L$, of the form:

$$\Psi(\gamma) = (G_j + O(\gamma - z_j))\,(\gamma - z_j)^{T_j} C_j\ , \tag{2.4.3}$$

where matrices G_j , C_j , T_j do not depend on γ ; C_j , $G_j \in SL(K, \mathbb{C})$ and T_j are diagonal traceless matrices such that any two entries of T_j do not differ by an integer number. The function Ψ transforms as follows with respect to periods 1 and μ of the torus T :

$$\begin{aligned} \Psi(\gamma+1) &= F^{-1}\Psi(\gamma)M_a\ , \\ \Psi(\gamma+\mu) &= H\Psi(\gamma)M_b\ , \end{aligned}$$

and being analytically continued along a contour l_j surrounding the point z_j the function Ψ gains a right multiplier:

$$\Psi(\gamma^{l_j}) = \Psi(\gamma)M_j\ ,$$

where M_a , M_b , M_j are called the monodromy matrices. The assumption of independence of all monodromy matrices of the positions of singularities $\{z_j\}$ and the b-period μ of the elliptic Riemann surface is called the isomonodromy condition. This condition together with expansion (2.4.3) gives the following dependence of Ψ on μ and $\{z_j\}_{j=1}^{L}$:

$$\overset{1}{\Psi}_{z_i}\overset{1}{\Psi}{}^{-1} = -\overset{2}{\mathrm{tr}}\left(\overset{12}{r}(\gamma - z_i)\,\overset{2}{A}_i\right), \tag{2.4.4}$$

$$\Psi_\mu \Psi^{-1} = \sum_{j=1}^{L}\ \sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K-1} A_j^{AB}\mathcal{Z}_{AB}(\gamma - z_i)\sigma_{AB}\ , \tag{2.4.5}$$

the functions $\mathcal{Z}_{AB}$ were defined by (2.3.34). The compatibility condition of (2.4.4), (2.4.5) and (2.4.1) gives the Schlesinger system on the elliptic surface:

$$\begin{aligned}
\frac{\partial \overset{1}{A_i}}{\partial z_j} &= \left[\overset{1}{A_i}, \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (z_i - z_j) \overset{2}{A_j} \right) \right] , \quad i \neq j , \\
\frac{\partial \overset{1}{A_i}}{\partial z_i} &= - \sum_{j=1, j\neq i}^{L} \left[\overset{1}{A_i}, \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (z_i - z_j) \overset{2}{A_j} \right) \right] , \\
\frac{\partial \overset{1}{A_i}}{\partial \mu} &= - \sum_{j=1}^{L} \left[\overset{1}{A_i}, \overset{2}{\mathrm{tr}} \left(\overset{2}{A_j} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} \mathcal{Z}_{AB}(z_i - z_j) \, \overset{1}{\sigma}_{AB} \overset{2}{\sigma}{}^{AB} \right) \right] .
\end{aligned} \tag{2.4.6}$$

The tau-function of this system is defined as the generating function of the following Hamiltonians:

$$H_i = \frac{1}{4\pi i} \oint_{z_i} trA^2(\gamma) d\gamma = \sum_{j=1, j\neq i}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} w_{AB}(z_i - z_j) , \tag{2.4.7}$$

$$H_\mu = -\frac{1}{2\pi i} \oint_{a} trA^2(\gamma) d\gamma = \frac{1}{2} \sum_{i,j=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} A_j^{AB} A_{iAB} \mathcal{Z}_{AB}(z_i - z_j) . \tag{2.4.8}$$

$$\frac{\partial \log \tau_{\mathrm{Sch}}}{\partial z_i} = H_i , \quad \frac{\partial \log \tau_{\mathrm{Sch}}}{\partial \mu} = H_\mu . \tag{2.4.9}$$

The following theorem shows how (analogously to the rational case [42]) solutions of the elliptic Schlesinger system (2.4.6) induce solutions of system (2.3.38) and (2.3.40).

Theorem 2.2 *Let $\mathcal{L}$ be a genus 1 covering of the λ-sphere with simple ramification points $P_1, \dots, P_{2N}$, which have different λ-projections $\lambda_1, \dots, \lambda_{2N}$. Consider a set of L points $\{Q_1, \dots, Q_L\}$ on $\mathcal{L}$ such that their projections $\pi(Q_i)$ are independent of $\{\lambda_m\}$. Let ν be the Abel map (2.3.5) onto the fundamental domain of the covering, $\nu : \mathcal{L} \to T$. Consider the Schlesinger system (2.4.6) with $z_i = \nu(Q_i)$ and its solution $\{A_j(\{z_i\})\}_{j=1}^{L}$. Let $\Psi(\gamma, \{z_i\})$ be the corresponding solution of system (2.4.1). We can consider Ψ as a function on the* *covering $\mathcal{L}$ via the Abel map:*

$$\Psi(P) = \Psi(\nu(P), \{\nu(Q_i)\}). \tag{2.4.10}$$

Then

1. *the function $\Psi(P)$ satisfies the linear system (2.3.38) with J_m defined by*

$$\overset{1}{J}_m = -\alpha_m \sum_{j=1}^{L} \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (\gamma_m - z_j) \overset{2}{A}_j \right), \tag{2.4.11}$$

   and, hence, J_m's solve the system (2.3.40);

2. *the tau-function τ (2.3.41) of the system (2.3.40) is related to the tau-function τ_{Sch} (2.4.9) of the elliptic Schlesinger system according to:*

$$\tau(\lambda_m) = \prod_{j=1}^{L} \left(\frac{\partial \nu}{\partial \lambda}(Q_j) \right)^{\mathrm{tr} A_j^2/2} \tau_{\mathrm{Sch}} \Big(\{z_k\}|_{z_k = \nu(Q_k)} \Big) . \tag{2.4.12}$$

Remark 2.4 Formula (2.4.12) coincides with the one relating the tau-function of the rational system (2.2.11) and the tau-function of the Schlesinger system on the Riemann sphere, see [42].

Proof. Since the solution Ψ of (2.4.1) is defined on the space of branch coverings as in (2.4.10), we can differentiate it with respect to λ_m according to the chain rule:

$$\frac{d\Psi}{d\lambda_m} = \frac{\partial \Psi}{\partial \nu} \frac{\partial \nu}{\partial \lambda_m} + \sum_{j=1}^{L} \frac{\partial \Psi}{\partial z_j} \frac{\partial z_j}{\partial \lambda_m} + \frac{\partial \Psi}{\partial \mu} \frac{\partial \mu}{\partial \lambda_m} .$$

(Recall that μ is the b-period of the elliptic Riemann surface.) We differentiate $z_i = \nu(Q_i)$ according to the formula (2.3.11) for derivatives of ν and use also formulas (2.3.22), (2.4.1), (2.4.4), (2.4.5). Using the relation

$$w_{AB}(\gamma - z_j)\Big(\rho(z_j - \gamma_m) - \rho(\gamma - \gamma_m)\Big) + 2\pi i \mathcal{Z}_{AB}(\gamma - z_j)$$
$$= -w_{AB}(\gamma_m - z_j) w_{AB}(\gamma - \gamma_m) ,$$

which can be proved by checking periodicity properties of both sides as $\gamma \to \gamma+1$, $\gamma \to \gamma+\mu$ and behavior at the pole $\gamma = \gamma_m$, we obtain:

$$\frac{d \overset{1}{\Psi}}{d\lambda_m} = -\alpha_m \overset{2}{\mathrm{tr}} \left(\overset{12}{r} (\nu - \gamma_m) \sum_{j=1}^{L} \overset{3}{\mathrm{tr}} \left(\overset{23}{r} (\gamma_m - z_j) \overset{3}{A_j} \right) \right) \overset{1}{\Psi} \ . \tag{2.4.13}$$

We single out the ν-dependent term and denote the rest by J_m :

$$\overset{2}{J_m} := -\alpha_m \sum_{j=1}^{L} \overset{3}{\mathrm{tr}} \left(\overset{23}{r} (\gamma_m - z_j) \overset{3}{A_j} \right) .$$

This leads to the system (2.3.38) and proves the first part of the theorem. For the second part, equality (2.4.12), we shall prove the following relation between the two tau-functions:

$$\frac{\partial \log \tau}{\partial \lambda_m} = \frac{\partial \log \tau_{Sch}}{\partial \lambda_m} + \sum_{j=1}^{L} \frac{\mathrm{tr} A_j^2}{2} \frac{\partial \log \frac{\partial \nu}{\partial \lambda}}{\partial \lambda_m} |_{\nu = z_j} \ . \tag{2.4.14}$$

This leads to (2.4.12) if one observes that

$$\frac{\partial \ \mathrm{tr} A_j^2}{\partial \lambda_m} = 0 \ ,$$

which follows from the Schlesinger system (2.4.6). To show (2.4.14) let us first note two auxiliary relations. The first one is:

$$\begin{aligned} & w_{AB}(z_i - z_j)\Big(\rho(z_j - \gamma) - \rho(z_i - \gamma)\Big) \\ = \quad & w_{AB}(\gamma - z_j) w_{-A-B}(\gamma - z_i) - 2\pi i \ \mathcal{Z}_{AB}(z_i - z_j) \end{aligned} \tag{2.4.15}$$

for any pair of non equal indeces i , j . This relation can be verified examining singularities and periodicity properties and then noting that at the point $\gamma = \frac{1}{2}(z_i + z_j)$ both sides are equal due to the equality

$$2 w_{AB}(2\gamma) \rho(\gamma) = w_{AB}^2(\gamma) + 2\pi i \mathcal{Z}_{AB}(2\gamma),$$

which, in turn, can be proved by the same method. One can apply the similar considerations to verify the second identity which we shall use:

$$w_{AB}(\gamma)w_{-A-B}(\gamma) = 2\pi i \mathcal{Z}_{AB}(0) - \rho'(\gamma)\ . \tag{2.4.16}$$

To show (2.4.14) we differentiate the tau-function of the elliptic Schlesinger system τ_{Sch} with respect to λ_m :

$$\frac{\partial \log \tau_{Sch}}{\partial \lambda_m} = \sum_{i=1}^{L} \frac{\partial \log \tau_{Sch}}{\partial z_i}\frac{\partial z_i}{\partial \lambda_m} + \frac{\partial \log \tau_{Sch}}{\partial \mu}\frac{\partial \mu}{\partial \lambda_m}\ .$$

Then we rewrite all the terms explicitly using (2.4.9), (2.4.7), (2.4.8), (2.3.11), (2.3.22) and simplify the obtained expression applying the auxiliary identity (2.4.15). Noting also that

$$\mathcal{Z}_{-A-B}(-\gamma) = \mathcal{Z}_{AB}(\gamma)\ , \tag{2.4.17}$$

one arrives at the following expression:

$$\begin{aligned}\frac{\partial \log \tau_{Sch}}{\partial \lambda_m} = {}& \alpha_m K\Big(\sum_{\substack{i,j=1\\ i<j}}^{L} \sum_{\substack{A,B=0\\ (A,B)\neq(0,0)}}^{K-1} \epsilon^{AB} A_j^{AB} A_i^{-A-B} w_{AB}(\gamma_m - z_j) w_{-A-B}(\gamma_m - z_i) \\ &+ \pi i \sum_{i=1}^{L} \sum_{\substack{A,B=0\\ (A,B)\neq(0,0)}}^{K-1} \epsilon^{AB} A_i^{AB} A_i^{-A-B} \mathcal{Z}_{AB}(0)\Big).\end{aligned} \tag{2.4.18}$$

The derivative in the second term of the right hand side of (2.4.14) can be obtained using (2.3.11) as follows:

$$\frac{\partial \log \frac{\partial \nu}{\partial \lambda}}{\partial \lambda_m} = \frac{\partial}{\partial \lambda}\Big(\frac{\partial \nu}{\partial \lambda_m}\Big)\Big/\frac{\partial \nu}{\partial \lambda} = \frac{\partial}{\partial \nu}\Big(\frac{\partial \nu}{\partial \lambda_m}\Big)\ ,$$

hence

$$\frac{\partial \log \frac{\partial \nu}{\partial \lambda}}{\partial \lambda_m}\Big|_{\nu = z_j} = -\alpha_m \rho'(z_j - \gamma_m)\ . \tag{2.4.19}$$

A certain simplification using the second auxiliary identity (2.4.16) leads to the following expression for the right hand side of (2.4.14):

$$\frac{\partial \log \tau_{\mathrm{Sch}}}{\partial \lambda_m} + \sum_{i=1}^{L} \frac{\mathrm{tr} A_i^2}{2} \frac{\partial \log \frac{\partial \nu}{\partial \lambda}}{\partial \lambda_m}|_{\nu = z_i}$$

$$= \frac{\alpha_m}{2} \Big(\sum_{i,j=1}^{L} \sum_{\substack{A,B=0 \\ (A,B)\neq(0,0)}}^{K-1} K \epsilon^{AB} A_j^{AB} A_i^{-A-B} w_{AB}(\gamma_m - z_j) w_{-A-B}(\gamma_m - z_i) \Big) , \qquad (2.4.20)$$

which is nothing but $\mathrm{tr}(J_m^2)/2\alpha_m$, where J_m are given by (2.4.11). Thus the right hand side of (2.4.14) is equal to $(\log \tau)_{\lambda_m}$ where the tau-function τ is defined by (2.3.41).

◇

2.5 Trigonometric degeneration of the elliptic coverings and corresponding integrable systems.

Here we describe the trigonometric version of system (2.3.40), obtained by a degeneration of the covering $\mathcal{L}$. Further, as an illustration, we consider the two-fold covering when all coefficients of the obtained system can be computed explicitely.

Set the matrix dimension K of the system to be 2 . An elliptic N-fold covering has, according to the Riemann-Hurwitz formula, $2N$ branch points (recall that we assume them to be simple and distinct). If we let one branch cut to degenerate (i.e. we let two ramification points connected by a branch cut to tend to each other), the elliptic covering turns into a rational one with $2N-2$ ramification points and a double point remaining from the degenerated branch cut.

Assume that the points P_{2N-1} and P_{2N} are connected by a branch cut $[P_{2N-1}, P_{2N}]$. Moreover, choose the basic a-cycle on $\mathcal{L}$ to surround this branch cut. Consider $\{\lambda_m\}_{m=1}^{2N-2}$ as independent variables and λ_{2N-1} and λ_{2N} as fixed parameters. Take the limit $\lambda_{2N-1}, \lambda_{2N} \to \lambda_Q$ with λ_Q independent of $\{\lambda_m\}_{m=1}^{2N-2}$. Then the branch cut $[P_{2N-1}, P_{2N}]$ degenerates and the elliptic curve $\mathcal{L}$ turns into the rational curve $\mathcal{L}_0$ with two marked points Q_1 and Q_2 (a double point) which lie on different sheets of $\mathcal{L}_0$ and have the same projection on the λ-plane:

$$\pi(Q_1) = \pi(Q_2) = \lambda_Q \, .$$

The basic a-cycle on $\mathcal{L}$ turns into a contour on $\mathcal{L}_0$ surrounding one of the points Q_1 or Q_2 . Suppose that it surrounds Q_1 in the positive direction. Denote by $\zeta(P)$, $P \in \mathcal{L}_0$ the one-to-one map from the genus zero covering $\mathcal{L}_0$ with ramification points $P_1, \ldots, P_{2N-2}$ to the Riemann sphere; for simplicity we fix this map by requirement $\zeta : \infty^{(0)} \to \infty$ such that in a neighborhood of $\infty^{(0)}$

$$\zeta(\lambda) = \lambda + o(1) \, . \tag{2.5.1}$$

Denote the images of points Q_1 and Q_2 on the Riemann sphere by κ_1 and κ_2 respectively:

$$\kappa_1 = \zeta(Q_1) \, ; \qquad \kappa_2 = \zeta(Q_2) \, . \tag{2.5.2}$$

The holomorphic differential $\mathbf{v}(P)$ degenerates to the meromorphic on $\mathcal{L}_0$ differential $\mathbf{v}_0$ with the simple poles at Q_1 and Q_2 and residues $1/2\pi i$ and $-1/2\pi i$ respectively. This differential can be written in terms of the coordinate ζ as follows:

$$\mathbf{v}_0(\zeta) = \frac{1}{2\pi i}\Big(\frac{1}{\zeta - \kappa_1} - \frac{1}{\zeta - \kappa_2}\Big) d\zeta \, . \tag{2.5.3}$$

The b-period μ of the Riemann surface $\mathcal{L}$ in the limit $P_{2N-1} \to P_{2N}$ has the following behavior:

$$\mu = \frac{1}{\pi i} \log |\lambda_{2N-1} - \lambda_{2N}| + O(1) \, , \tag{2.5.4}$$

i.e. $\mu \to +i\infty$ in this limit, and the fundamental domain $T = \mathbb{C}/\{1, \mu\}$ of the covering $\mathcal{L}$ turns into a cylinder. The map ν (2.3.5) now maps the degenerated covering $\mathcal{L}_0$ onto the cylinder in γ-sphere:

$$\nu(P) = \int_{\infty^{(0)}}^{P} \mathbf{v}_0 = \frac{1}{2\pi i}\int_{\infty}^{\zeta}\left(\frac{1}{\zeta-\kappa_1} - \frac{1}{\zeta-\kappa_2}\right) d\zeta = \frac{1}{2\pi i}\log\frac{\zeta-\kappa_1}{\zeta-\kappa_2}\,. \tag{2.5.5}$$

From the definition (2.3.7) of the Jacobi theta-functions, we deduce the behavior of logarithmic derivative $\rho(\gamma)$ of $\theta_1 = \theta[\frac{1}{2}, \frac{1}{2}]$ as $\mu \to +i\infty$:

$$\rho(\gamma) \to \pi\cot\pi\gamma\,, \tag{2.5.6}$$

and therefore,

$$\rho'(\gamma) \to -\frac{\pi^2}{\sin^2\pi\gamma}\,. \tag{2.5.7}$$

Similarly, the r-matrix becomes in this limit (for the matrix dimension $K = 2$):

$$\overset{12}{r}(\gamma) \to \overset{12}{r}_0(\gamma) = \frac{1}{2}\frac{\pi}{\sin\pi\gamma}\,\overset{1}{\sigma}_1\overset{2}{\sigma}_1 + \frac{1}{2}\frac{\pi}{\sin\pi\gamma}\,\overset{1}{\sigma}_2\,\overset{2}{\sigma}_2 + \frac{1}{2}\pi\cot\pi\gamma\,\overset{1}{\sigma}_3\overset{2}{\sigma}_3\,, \tag{2.5.8}$$

where we use the Pauli basis $\{\sigma_i\}_{i=1}^3$ (2.3.48); r_0 is the so-called trigonometric r-matrix. Differential equations (2.3.23)–(2.3.26) for $\{\gamma_m\}_{m=1}^{2N-2}$ (images of non-degenerated ramification points $P_1, \dots, P_{2N-2}$ under the map ν (2.5.5)) take the form (for $m \neq n$):

$$\frac{\partial\gamma_n}{\partial\lambda_m} = -\pi\alpha_m^0\left[\cot\pi(\gamma_n-\gamma_m) + \cot\pi\gamma_m\right]\,, \tag{2.5.9}$$

$$\begin{aligned}
\frac{\partial\gamma_m}{\partial\lambda_m} &= \pi\sum_{n=1,\,n\neq m}^{2N-2}\alpha_n^0\left[\cot\pi(\gamma_m-\gamma_n) + \cot\pi\gamma_n\right]\,,\\
\frac{\partial\alpha_n^0}{\partial\lambda_m} &= 2\,\pi^2\frac{\alpha_n^0\alpha_m^0}{\sin^2\pi(\gamma_n-\gamma_m)}\,,\\
\frac{\partial\alpha_m^0}{\partial\lambda_m} &= -2\,\pi^2\sum_{n=1,\,n\neq m}^{2N-2}\frac{\alpha_n^0\alpha_m^0}{\sin^2\pi(\gamma_n-\gamma_m)}\,,
\end{aligned} \tag{2.5.10}$$

where by α_m^0 we denoted the analog of the coefficient α_m in the degenerated case:

$$\alpha_m^0 = \frac{1}{2} v_{0m}^2 = \frac{1}{2} \left[\frac{\mathbf{v}_0(P)}{d\sqrt{\lambda - \lambda_m}} \Big|_{P=P_m} \right]^2 , \quad m = 1, \ldots, 2N-2 . \tag{2.5.11}$$

Remark 2.5 Differential equations (2.5.9) can be obtained directly from the form (2.5.5) of the map ν using the fact that the map ζ satisfies equations (2.2.9) with $\beta = 1$, $c_n = 0$.

Remark 2.6 The system (2.5.9), (2.5.10) after a simple change of variables coincides with equations for characteristic speeds of the system of hydrodynamic type to which the Boyer-Finley equation (self-dual Einstein equation with one Killing vector) $U_{xy} = (e^U)_{tt}$ reduces [24].

The linear system for the matrix Ψ is written now via the trigonometric r-matrix r_0 :

$$\frac{d \overset{1}{\Psi} (P)}{d\lambda_m} = \overset{2}{\mathrm{tr}} \left(\overset{12}{r}_0 \left(\nu(P) - \gamma_m\right) \overset{2}{J_m} \right) \overset{1}{\Psi} (P) , \tag{2.5.12}$$

$m = 1, \ldots, 2N-2$. Then, the trigonometric version of system (2.3.40) for $J_m = J_m^1 \sigma_1 + J_m^2 \sigma_2 + J_m^3 \sigma_3$ (for notation see (2.3.49)) gives the compatibility condition of the above linear system:

$$\begin{aligned}
\frac{\partial J_m^1}{\partial \lambda_n} &= \frac{\alpha_n^0 \pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)} J_m^1 + \frac{\alpha_m^0 \pi^2 \cos \pi(\gamma_m - \gamma_n)}{\sin^2 \pi(\gamma_m - \gamma_n)} J_n^1 \\
&+ \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)} \left(J_m^2 J_n^3 \cos \pi(\gamma_m - \gamma_n) - J_m^3 J_n^2 \right) , \\
\frac{\partial J_m^2}{\partial \lambda_n} &= \frac{\alpha_n^0 \pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)} J_m^2 + \frac{\alpha_m^0 \pi^2 \cos \pi(\gamma_m - \gamma_n)}{\sin^2 \pi(\gamma_m - \gamma_n)} J_n^2 \\
&+ \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)} \left(J_m^3 J_n^1 - J_m^1 J_n^3 \cos \pi(\gamma_m - \gamma_n) \right) , \\
\frac{\partial J_m^3}{\partial \lambda_n} &= \frac{\alpha_n^0 \pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)} J_m^3 + \frac{\alpha_m^0 \pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)} J_n^3 \\
&+ \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)} \left(J_m^1 J_n^2 - J_m^2 J_n^1 \right) ,
\end{aligned} \tag{2.5.13}$$

the indeces m and n are different and range in the set $\{1, \ldots, 2N-2\}$.

All the involved coefficients can be explicitly computed if we start with the two-fold elliptic covering. After the degeneration we get a rational covering $\mathcal{L}_0$ with two ramification points P_1 and P_2 (with the λ-projections λ_1 and λ_2) and the marked points with the projection λ_Q independent of λ_1 and λ_2 . The one-to-one map ζ from this covering to the Riemann sphere which satisfies condition (2.5.1) has the following form:

$$\zeta(P) = \frac{1}{2}\Big(\lambda + \frac{\lambda_1 + \lambda_2}{2} + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\Big) , \tag{2.5.14}$$

where $\lambda = \pi(P)$, the projection of the point P on the base of the covering. Knowing the expression for the map ζ allows us to find the images γ_1 , γ_2 of the non-degenerate ramification points P_1 , P_2 under the map $\nu(P)$ since it can be explicitly integrated (see(2.5.5)):

$$\gamma_m = \frac{1}{2\pi i} \log \frac{\zeta_m - \kappa_1}{\zeta_m - \kappa_2} , \tag{2.5.15}$$

where $\zeta_m = \zeta(P_m)$, $m = 1, 2$ are the images on the Riemann sphere of ramification points. One can find them from the form (2.5.14) of the map ζ :

$$\zeta_1 = \zeta(\lambda_1) = \frac{3\lambda_1 + \lambda_2}{4} , \qquad \zeta_2 = \zeta(\lambda_2) = \frac{\lambda_1 + 3\lambda_2}{4} ;$$

for the ζ-images $\kappa_{1,2}$ of points Q_1 and Q_2 (2.5.2) we have:

$$\kappa_{1,2} = \frac{1}{2}\Big(\lambda_Q + \frac{\lambda_1 + \lambda_2}{2} \pm \sqrt{(\lambda_Q - \lambda_1)(\lambda_Q - \lambda_2)}\Big) .$$

Now one can easily see from (2.5.15) that

$$e^{2\pi i \gamma_1} = \frac{\sqrt{\lambda_1 - \lambda_Q} - \sqrt{\lambda_2 - \lambda_Q}}{\sqrt{\lambda_1 - \lambda_Q} + \sqrt{\lambda_2 - \lambda_Q}} = -e^{2\pi i \gamma_2} , \tag{2.5.16}$$

and, therefore, $\gamma_1 - \gamma_2 = \pm 1/2$. The same conclusion can be made if one observes that $\gamma_1 - \gamma_2$ is equal to one half of the integral over a-period of the differential $\mathbf{v}$ (see definition

(2.3.5) of the map ν). The sign of the difference $\gamma_1 - \gamma_2$ is determined by a choice of direction of the a-cycle.

It remains to calculate one more ingredient of system (2.3.40) for J_m , namely, the coefficients $\alpha^0_{1,2}$ (2.5.11). Denoting by $v_0(x)$ a locally defined function such that $\mathbf{v_0} = v_0(x)dx$ (x being a local parameter on the covering), from the relation

$$v_0(x)dx = \frac{1}{2\pi i}\Big(\frac{1}{\zeta - \kappa_1} - \frac{1}{\zeta - \kappa_2}\Big)d\zeta$$

we deduce that

$$\frac{d\zeta}{dx}(\lambda_m) = 2\pi i\ v_{0m}\frac{(\zeta_m - \kappa_1)(\zeta_m - \kappa_2)}{\kappa_1 - \kappa_2}\ , \qquad m = 1, 2\ .$$

From the explicit form (2.5.14) of the map $\zeta(P)$ one can compute the coefficients $(d\zeta/dx)(\lambda_m)$ of expansion of $\zeta(P)$ in neighbourhoods of ramification points P_1 , P_2 . Then we obtain the expressions for $\alpha^0_m = \frac{1}{2}v^2_{0m}$ $(m = 1, 2)$:

$$\alpha^0_1 = -\frac{1}{2\pi^2}\frac{\lambda_2 - \lambda_Q}{\lambda_1 - \lambda_Q}\frac{1}{\lambda_1 - \lambda_2}\ , \qquad \alpha^0_2 = -\frac{1}{2\pi^2}\frac{\lambda_1 - \lambda_Q}{\lambda_2 - \lambda_Q}\frac{1}{\lambda_2 - \lambda_1}\ .$$

In the limit $\lambda_Q \to \infty$, summarizing all the above calculations, we get from (2.5.13) the system of equations for $J_1 = J^1_1\sigma_1 + J^2_1\sigma_2 + J^3_1\sigma_3$:

$$\begin{aligned}
\frac{\partial J^1_1}{\partial \lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J^1_1 - 2\pi i\ J^3_1 J^2_2\ ,\\
\frac{\partial J^2_1}{\partial \lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J^2_1 + 2\pi i\ J^3_1 J^1_2\ ,\\
\frac{\partial J^3_1}{\partial \lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}(J^3_1 - J^3_2) + 2\pi i\ (J^1_1 J^2_2 - J^2_1 J^1_2)\ ,
\end{aligned}$$

and the similar system for J_2 :

$$\begin{aligned}
\frac{\partial J_2^1}{\partial \lambda_1} &= \frac{1}{2}\frac{1}{\lambda_2-\lambda_1}J_2^1 + 2\pi i\; J_2^3 J_1^2 \;, \\
\frac{\partial J_2^2}{\partial \lambda_1} &= \frac{1}{2}\frac{1}{\lambda_2-\lambda_1}J_2^2 - 2\pi i\; J_2^3 J_1^1 \;, \\
\frac{\partial J_2^3}{\partial \lambda_1} &= \frac{1}{2}\frac{1}{\lambda_1-\lambda_2}(J_1^3 - J_2^3) + 2\pi i\; (J_1^1 J_2^2 - J_1^2 J_2^1) \;.
\end{aligned}$$

Acknowledgments. I thank D.Korotkin for advising me in the course of the work and careful reading of the manuscript. I also thank E. Ferapontov, A. Kokotov and A. Mikhailov for useful discussions.

In the previous work the variable spectral parameter in the U-V pair of the integrable system on the torus was given by the Abel map $\nu(P)$ on the genus one surface (which in genus one gives a uniformization map of the Riemann surface). The map ν was considered as a function of the coordinates $\{\lambda_k\}$ on the corresponding Hurwitz space. In particular, the system of differential equations describing the dependence of the images $\{\gamma_m\}$ of ramification points of the associated covering under the map $\nu(P)$ on $\{\lambda_k\}$ was derived. Under a certain procedure of degeneration of the genus one covering, the system for $\{\gamma_m\}$ transforms into a trigonometric version. This trigonometric degeneration of the system for $\{\gamma_m\}$ can be used to find a wide class of solutions to the Boyer-Finley equation

$$U_{xy} = (e^U)_{tt} \; ,$$

which is equivalent to the self-dual Einstein equation with a Killing vector. Namely, assume the function U to depend on the variables $\{\lambda_k\}$, which, in turn, are functions of x, y and t. The functions $\{\lambda_m(x, y, t)\}$ are assumed to satisfy a system of hydrodynamic type:

$$\frac{\partial \lambda_m}{\partial x} = V_m(\{\lambda_k\})\frac{\partial \lambda_m}{\partial t} \; , \qquad \frac{\partial \lambda_m}{\partial y} = W_m(\{\lambda_k\})\frac{\partial \lambda_m}{\partial t} \; .$$

The compatibility conditions of this system together with the Boyer-Finley equation for the function $U(\{\lambda_k(x, y, t)\})$ imply a set of equations which allow us to parameterize the characteristic speeds $\{V_m\}$ and $\{W_m\}$ by a family of functions which satisfy the trigonometric system of equations derived in Chapter 2. Thereby we obtain a class of solutions to the Boyer-Finley equation in terms of objects associated with the spaces of trigonometric functions of arbitrary degree.

Chapter 3

Boyer-Finley equation and systems of hydrodynamic type

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Classical and Quantum Gravity 19, L205-L210 (2002)

Abstract. We reduce Boyer-Finley equation to a family of compatible systems of hydrodynamic type, with characteristic speeds expressed in terms of spaces of rational functions. The systems of hydrodynamic type are then solved by the generalized hodograph method, providing solutions of the Boyer-Finley equation including functional parameters.

In this paper we construct solutions of the dispersionless non-linear PDE – the Boyer-Finley equation (self-dual Einstein equation with a Killing vector),

$$U_{xy} = (e^U)_{tt}, \tag{3.0.1}$$

via reduction to a family of compatible systems of hydrodynamic type.

This equation was actively studied during last twenty years by many authors; we just mention works [7, 10, 74, 66, 58, 19, 8, 51, 52, 56, 32]. So far the most general scheme of the construction of its solutions was developed in [51, 52]. In these works solutions of the Boyer-Finley equation were derived by averaging an appropriate two-point Baker-Akhiezer function in genus zero which corresponds to the two-dimensional Toda lattice equations, the underlying Riemann surface deforming according to the Whitham equations. Some particular solutions of the Boyer-Finley equation were constructed in [10, 74, 19, 58]; their relationship to the solutions of [51, 52] remains unclear.

The goal of the present paper is to give an alternative scheme of solving the Boyer-Finley equation. Namely, we consider reductions of this equation to multi-component systems of hydrodynamic type in the spirit of [28, 29], see also [54, 56, 32]; the equations for characteristic speeds of these systems are solved in terms of rational branched coverings. The systems of hydrodynamic type are then solved by the generalized hodograph method [73].

Now we describe a way to find solutions of the Boyer-Finley equation. First we ignore the condition of reality of the function U and construct complex-valued solutions of equation (3.0.1); then we formulate restrictions on the parameters which guarantee the reality of U.

Let us assume U to be a function of L variables $\lambda_1, \ldots, \lambda_L$, where $\lambda_m(x, y, t)$ ("Riemann invariants") satisfy a pair of systems of hydrodynamic type

$$\partial_x \lambda_m = V_m(\{\lambda_k\}) \partial_t \lambda_m , \qquad \partial_y \lambda_m = W_m(\{\lambda_k\}) \partial_t \lambda_m . \tag{3.0.2}$$

A direct substitution of the function $U(\lambda_1, \ldots, \lambda_L)$ into the Boyer-Finley equation implies the algebraic relation among the functions U , V_m and W_m,

$$V_m W_m = e^U, \tag{3.0.3}$$

along with the following differential equations:

$$\partial_m \partial_n U (V_m W_n + V_n W_m) = 2 \partial_n \partial_m (e^U) , \qquad m \neq n, \tag{3.0.4}$$

$$\frac{\partial_m V_n}{V_m - V_n} = \frac{\partial_m W_n}{W_m - W_n} , \qquad m \neq n, \tag{3.0.5}$$

where $\partial_m \equiv \partial / \partial \lambda_m$.

Relations (3.0.3) allow one to parameterize the functions V_m and W_m by a new set of variables $\varphi_m(\{\lambda_k\})$ as follows:

$$V_m = \exp\left\{ 2i\varphi_m + \frac{U}{2} \right\} , \qquad W_m = \exp\left\{ -2i\varphi_m + \frac{U}{2} \right\} . \tag{3.0.6}$$

In terms of these variables, equations (3.0.4) and (3.0.5) take the form

$$\partial_m \partial_n U = -\frac{\partial_m U \partial_n U}{2 \sin^2(\varphi_m - \varphi_n)} , \qquad \partial_m \varphi_n = \frac{1}{4} \cot(\varphi_n - \varphi_m) \partial_m U , \tag{3.0.7}$$

where $m \neq n$.

A class of solutions of this system is related to the space of rational functions in the following way. Consider a rational function

$$R(\mu) = \mu + \sum_{k=1}^{N-1} \frac{a_k}{\mu - b_k}\,, \quad \mu \in \mathbb{CP}^1. \tag{3.0.8}$$

In application to Benney's hierarchy, functions of this form were first introduced in [53]. The equation

$$\lambda = R(\mu) \tag{3.0.9}$$

defines an N−sheeted covering $\mathcal{L}$ of the λ−sphere. A point $P \in \mathcal{L}$ is a pair of complex numbers, $P = (\lambda, \mu)$. We consider the generic case when the function $R(\mu)$ has $2N-2$ non-coinciding finite critical points, i.e., the equation

$$R'(\mu) = 0$$

has $2N-2$ distinct roots $\mu_1, \ldots, \mu_{2N-2}$. The corresponding critical values,

$$\lambda_n = R(\mu_n), \qquad n = 1, \ldots, 2N-2, \tag{3.0.10}$$

are projections onto the λ−sphere of the branch points of the covering $\mathcal{L}$ (we denote branch points by $P_n = (\lambda_n, \mu_n)$; all of them are simple as a corollary of non-coincidence of μ_n for different n). An additional condition we impose on the function $R(\mu)$ is that all λ_n are different. Now, observe that the number of parameters of the rational function (3.0.8) is equal to the number of branch points; therefore, we can take $\lambda_1, \ldots, \lambda_{2N-2}$ as local coordinates on the space of rational functions. It was shown in [42] that the critical points $\{\mu_m\}$ of the rational function $R(\mu)$ depend on $\{\lambda_n\}$ in the following way:

$$\frac{\partial \mu_m}{\partial \lambda_n} = \frac{\beta_n}{\mu_n - \mu_m}\,; \qquad \frac{\partial \beta_m}{\partial \lambda_n} = \frac{2\beta_n\beta_m}{(\mu_n - \mu_m)^2}\,, \qquad m \neq n\,. \tag{3.0.11}$$

These equations appeared also in [28, 29] in the theory of hydrodynamic reductions of Benney's moment equations. The inverse function $\mu(P) = R^{-1}(P)$ is defined on the covering $\mathcal{L}$. As a function of $\{\lambda_n\}$, it satisfies the system of differential equations [42]:

$$\frac{\partial \mu}{\partial \lambda_n} = \frac{\beta_n}{\mu_n - \mu} . \tag{3.0.12}$$

Let us now choose two points Q_1 and Q_2 on the covering $\mathcal{L}$ such that their projections $\lambda(Q_1)$ and $\lambda(Q_2)$ onto the $\lambda-$sphere do not depend on $\{\lambda_n\}$. Then, consider the following function:

$$\gamma(P) = \frac{1}{2\pi i} \log \frac{\mu(P) - \mu(Q_1)}{\mu(P) - \mu(Q_2)} , \tag{3.0.13}$$

which maps $\mathcal{L}$ onto a cylinder. We shall be interested in the images $\{\gamma_m\}$ of branch points under this map as functions of $\{\lambda_n\}$. In the sequel we denote $\mu(Q_1)$ by κ_1 and $\mu(Q_2)$ by κ_2 . According to (3.0.12), they satisfy the equations

$$\frac{\partial \kappa_j}{\partial \lambda_n} = \frac{\beta_n}{\mu_n - \kappa_j} , \qquad j = 1, 2 . \tag{3.0.14}$$

From the expression (3.0.13) for $\gamma(P_m)$ we have

$$\gamma_m = \frac{1}{2\pi i} \log \frac{\mu_m - \kappa_1}{\mu_m - \kappa_2} . \tag{3.0.15}$$

Differentiation of this relation with respect to λ_n using (3.0.11), (3.0.13) and (3.0.14) gives

$$\frac{\partial \gamma_m}{\partial \lambda_n} = \frac{1}{2\pi i} \frac{\beta_n}{\mu_n - \mu_m} \left[\frac{1}{\mu_m - \kappa_1} - \frac{1}{\mu_m - \kappa_2} \right] . \tag{3.0.16}$$

If we now express μ_n and μ_m in terms of γ_n , γ_m from (3.0.15) and set

$$\alpha_n = -\frac{\beta_n}{4\pi^2} \left[\frac{1}{\mu_m - \kappa_1} - \frac{1}{\mu_m - \kappa_2} \right]^2 , \tag{3.0.17}$$

we get the following system of differential equation for the functions $\gamma_m(\{\lambda_n\})$:

$$\frac{\partial \gamma_m}{\partial \lambda_n} = -\pi\alpha_n\Big(\cot\pi(\gamma_m - \gamma_n) + \cot\pi\gamma_n\Big) , \quad m \neq n . \tag{3.0.18}$$

Similarly, the functions α_n satisfy the following equations:

$$\frac{\partial \alpha_n}{\partial \lambda_m} = 2\pi^2 \frac{\alpha_n \alpha_m}{\sin^2 \pi(\gamma_n - \gamma_m)} , \quad m \neq n . \tag{3.0.19}$$

It turns out that a simple transformation allows one to construct solutions of system (3.0.7) from the set of functions γ_m and α_m . Namely, the system of equations (3.0.18), (3.0.19) coincides with system (3.0.7) if rewritten in terms of the new variables U and φ_n such that

$$\frac{\partial U}{\partial \lambda_m} = -4\pi^2 \alpha_m \tag{3.0.20}$$

and

$$\varphi_n = \pi(\gamma_n + \psi) , \tag{3.0.21}$$

where

$$\frac{\partial \psi}{\partial \lambda_m} = \pi\alpha_m \cot\pi\gamma_m . \tag{3.0.22}$$

The existence of functions U and ψ is provided by the compatibility conditions,

$$\frac{\partial}{\partial \lambda_n}\alpha_m = \frac{\partial}{\partial \lambda_m}\alpha_n \tag{3.0.23}$$

and

$$\frac{\partial}{\partial \lambda_n}(\alpha_m \cot\pi\gamma_m) = \frac{\partial}{\partial \lambda_m}(\alpha_n \cot\pi\gamma_n) , \tag{3.0.24}$$

which follow from (3.0.19) and (3.0.18).

Ultimately, formulas (3.0.6) determine $\{V_m\}$ and $\{W_m\}$ as functions of $\{\lambda_k\}$. In order to obtain a solution of the Boyer-Finley equation (3.0.1) we need U as an explicit function of x, y and t, that is, we need to solve the system of hydrodynamic type (3.0.2). The tool which is usually used for this purpose is the generalized hodograph method [73]. Instead of solving (3.0.2), we find a smooth solution $\Lambda(x, y, t) = (\lambda_1, \dots, \lambda_L)$ of the following system

$$\varphi_m(\Lambda) = t + V_m(\Lambda)x + W_m(\Lambda)y, \tag{3.0.25}$$

where the functions $\{\varphi_m\}$ satisfy the linear system

$$\frac{\partial_m \varphi_n}{\varphi_m - \varphi_n} = \frac{\partial_m V_n}{V_m - V_n} = \frac{\partial_m W_n}{W_m - W_n}\,, \quad m \neq n\,. \tag{3.0.26}$$

To see that an implicit solution (3.0.25) for $\{\lambda_m(x, y, t)\}$ indeed satisfies (3.0.2), one needs to differentiate (3.0.25) with respect to x , y and t [73].

To be able to use this method we need to construct functions φ_m , i.e. we need to solve for φ_m the system

$$\frac{\partial_m \varphi_n}{\varphi_m - \varphi_n} = \frac{\partial_m V_n}{V_m - V_n}\,, \quad m \neq n\,. \tag{3.0.27}$$

Observe that for $m \neq n$

$$\frac{\partial_n V_m}{V_m - V_n} = -\frac{\pi^2 \alpha_n}{\sin^2 \pi(\gamma_m - \gamma_n)}\,; \tag{3.0.28}$$

this is a simple corollary of definitions of V_m and W_m and equations (3.0.20), (3.0.7).

Then the following functions satisfy equations (3.0.27):

$$\varphi_m = \pi^2 \oint_l \frac{H(\lambda)d\gamma}{\sin^2 \pi(\gamma - \gamma_m)}\,, \tag{3.0.29}$$

where l is an arbitrary closed contour on the branched covering $\mathcal{L}$ such that its projection on the λ-plane does not depend on the branch points $\{\lambda_n\}$ and such that $P_m \notin l$ for all m ;

$H(\lambda)$ is an arbitrary function on l independent of $\{\lambda_n\}$. The proof of this fact is a simple calculation using equations (3.0.18), (3.0.12) and the link (3.0.13) between γ and μ.

Note that in this framework we can fix positions of branch points $\{\lambda_{L+1}, \dots, \lambda_{2N-2}\}$ and consider the dependence of all functions on the remaining set of variables $\{\lambda_1, \dots, \lambda_L\}$, where $L \leq 2N - 2$.

The following theorem summarizes our construction of solutions of the Boyer-Finley equation.

Theorem 3.1 *Let functions $\alpha_m(\{\lambda_n\})$ and $\gamma_m(\{\lambda_n\})$, $m, n = 1, \dots, L$, $L \leq 2N - 2$, be associated with an N-sheeted branched covering as described above. Define the potentials $U(\{\lambda_n\})$ and $\psi(\{\lambda_n\})$ to be solutions of the following system of equations:*

$$\frac{\partial U}{\partial \lambda_m} = -4\pi^2 \alpha_m \,, \qquad m = 1, \dots, L \,; \tag{3.0.30}$$

$$\frac{\partial \psi}{\partial \lambda_m} = \pi \alpha_m \cot \pi \gamma_m \,, \qquad m = 1, \dots, L \,. \tag{3.0.31}$$

Let the $(x, y, t)-$dependence of branch points λ_n, $n = 1, \dots, L$, be governed by the following system of L equations,

$$\pi^2 \oint_l \frac{H(\lambda) d\gamma}{\sin^2 \pi(\gamma - \gamma_m)} = t + x\, V_m + y\, W_m \,, \qquad m = 1, \dots, L \,, \tag{3.0.32}$$

where

$$V_m = e^{2\pi i(\gamma_m + \psi) + U/2} \,, \qquad W_m = e^{-2\pi i(\gamma_m + \psi) + U/2} \,; \tag{3.0.33}$$

l is an arbitrary $\{\lambda_n\}$-independent contour on $\mathcal{L}$ such that all $P_m \notin l$; $H(\lambda)$ is an arbitrary summable $\{\lambda_n\}$-independent function on l.

Then the function $U(\{\lambda_n(x, y, t)\})$ satisfies the Boyer-Finley equation (3.0.1).

Remark 3.1 If an N-sheeted rational branched covering $\mathcal{L}$ with two marked points Q_1 , Q_2 is fixed, the solution of the Boyer-Finley equation constructed according to this theorem is defined by

(a) a functional parameter $H(\lambda)$ and

(b) a number $L \leq 2N - 2$, which has a meaning of the number of components λ_m satisfying systems of hydrodynamic type (3.0.2) with characteristic speeds (3.0.33).

The application of theorem 3.1 in practice requires calculation of quadratures (3.0.30) and (3.0.31); besides that, one needs to resolve implicit relations (3.0.32) to find the dependence of λ_m on (x, y, t).

So far we were dealing with complex solutions of the Boyer-Finley equation (3.0.1); it is easy to formulate conditions on the parameters of our solutions which provide the reality of the function U.

Let us assume the function $R(\gamma)$ to satisfy the "reality condition"

$$\overline{R(\bar{\gamma})} = R(\gamma). \tag{3.0.34}$$

Then the branch covering $\mathcal{L}$ is invariant with respect to the antiholomorphic involution τ , which acts on the points (λ, μ) of the covering $\mathcal{L}$ as follows:

$$\tau : \ (\lambda, \mu) \to (\bar{\lambda}, \bar{\mu}). \tag{3.0.35}$$

Assume also that both points Q_1, Q_2 are invariant with respect to τ, i.e., $\kappa_{1,2} \in \mathbb{R}$. Let also the contour l be invariant with respect to the involution and the function $H(P)$ satisfy the relation $H(P) = -\overline{H(P^\tau)}$. Then one can choose the constants of integration in (3.0.30) and (3.0.31) such that the solution $U(x, y, t)$ of the Boyer-Finley equation given by theorem 3.1 is real.

Indeed, the invariance of the covering $\mathcal{L}$ with respect to τ means that all λ_m are either real or form conjugate pairs; the same holds for the set $\{\mu_m\}$. The expression (3.0.13) for the map γ implies

$$\overline{\gamma(P^\tau)} = -\gamma(P) , \tag{3.0.36}$$

therefore, all $\gamma_m \equiv \gamma(P_m)$ are either imaginary, $\overline{\gamma_m} = -\gamma_m$, or form anti-conjugate pairs. Applying complex conjugation to both sides of equation (3.0.18), we find that α_m are either real ($\alpha_m \in \mathbb{R}$ if $\lambda_m \in \mathbb{R}$) or form conjugate pairs, $\alpha_m = \overline{\alpha_n}$, if $\lambda_m = \overline{\lambda_n}$. This readily implies that one can choose the integration constant in the definition (3.0.30) of potential U in such a way that U is a real function of $\{\lambda_m\}$.

For completeness, we should also check that the reality condition does not contradict the solvability of system (3.0.32). Assume, for simplicity, that all λ_m are real, i.e., all γ_m are imaginary and all α_m are real. Then the potential function ψ solving system (3.0.31) can be chosen to be imaginary, and both V_m and W_m (3.0.33) are real. Together with $\overline{H(P^\tau)} = -H(P)$ and (3.0.36), it implies that both sides of equations (3.0.32) are real. Therefore, system (3.0.32) gives L real equations for L real variables $\{\lambda_m\}$, and generically has solutions.

Similar consideration applies when some λ_m's form conjugate pairs; in this case the corresponding equations (3.0.32) will be conjugate to each other, and the number of real equations will again coincide with the number of real variables.

Acknowledgments We thank Tamara Grava and Yavuz Nutku for useful discussions.

The scalar version of integrable systems found in Chapter 2 can be generalized [42] to Hurwitz spaces in arbitrary genus (see Section 1.7.3). Any solution to the scalar system in arbitrary genus corresponds to a Darboux-Egoroff metric on the Hurwitz space. The rotation coefficients of those Darboux-Egoroff metrics are given by the canonical meromorphic bidifferential W evaluated at the ramification points $\{P_m\}$ of the covering with respect to the standard local parameters.

On the other hand, the rotation coefficients of flat metrics of Frobenius structures on Hurwitz spaces found by Dubrovin [15] are also given by the bidifferential W evaluated at the ramification points of the corresponding covering.

In the next paper it is shown that the flat metrics of Hurwitz Frobenius manifolds of [15] belong to the family of metrics defined by solutions to the scalar systems from [42], and that the construction of Frobenius manifolds on Hurwitz spaces [15] can be naturally described in terms of the meromorphic bidifferential W . According to the Rauch variational formulas, the bidifferential W and therefore the structures of a Frobenius manifold of Dubrovin are holomorphic on the Hurwitz space.

In this paper, we construct families of Darboux-Egoroff metrics on Hurwitz spaces in terms of the Schiffer and Bergman kernels on Riemann surfaces. These metrics are metrics on a Hurwitz space considered as a real manifold. It turns out that the constructed family of Darboux-Egoroff metrics also contains metrics which correspond to Frobenius manifolds. The dimension of these manifolds is double with respect to the dimension of Dubrovin's structures. We call them the "real doubles" of Hurwitz Frobenius manifolds of Dubrovin.

Chapter 4

"Real doubles" of Hurwitz Frobenius manifolds

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math-ph/0402015,

to appear in Commun. Math. Phys.

Abstract. New Frobenius structures on Hurwitz spaces are found. A Hurwitz space is considered as a real manifold; therefore the number of coordinates is twice as large as the number of coordinates on Hurwitz Frobenius manifolds of Dubrovin. Simple branch points of a ramified covering and their complex conjugates play the role of canonical coordinates on the constructed Frobenius manifolds. Corresponding solutions to WDVV equations and G-functions are obtained.

4.1 Introduction

Frobenius manifolds were introduced by B. Dubrovin [13] as a geometric interpretation of the Witten - Dijkgraaf - E.Verlinde - H.Verlinde (WDVV) equations from two-dimensional topological field theory [12, 75].

The theory of Frobenius manifolds is related to various branches of mathematics: the theory of singularities – some ingredients of a Frobenius manifold had long existed on the base space of the universal unfolding of a hypersurface singularity. Besides singularity theory, Frobenius manifold structures have been found on cohomology spaces of smooth projective varieties (the theory of Gromov-Witten invariants); on extended moduli spaces of Calabi-Yau manifolds; on orbit spaces of Coxeter groups, extended affine Weil groups and Jacobi groups; and on Hurwitz spaces (see the references in [15, 57]).

The aim of the present work is to construct a new class of semisimple (vector algebra on any tangent space has no nilpotents) Frobenius manifolds associated with Hurwitz spaces. The dimension of Dubrovin's Frobenius manifolds on Hurwitz spaces is equal to the complex dimension of the Hurwitz space. In this paper we build Frobenius structures of a double dimension on the real Hurwitz space. We consider the Hurwitz space as a real manifold, i.e. we complement the set of its usual local coordinates by the set of their complex conjugates. We call new Frobenius manifolds the "real doubles" of Hurwitz Frobenius manifolds of Dubrovin (in some cases the prepotential of a "real double" is real-valued, however this is not always the case).

We start with a construction of a family of Darboux-Egoroff (flat potential diagonal) metrics on a real Hurwitz space in genus greater than zero. The Hurwitz space we consider is the space of coverings $(\mathcal{L},\lambda)$ of $\mathbb{C}P^1$, where $\mathcal{L}$ is a Riemann surface of genus $g \geq 1$, λ is a meromorphic function on $\mathcal{L}$ with simple finite critical points $P_1,\dots,P_L$ and possibly with critical points at infinity. The real Hurwitz space has local coordinates $\{\lambda_1,\dots,\lambda_L;\bar\lambda_1,\dots,\bar\lambda_L\}$, where $\lambda_i = \lambda(P_i)$. The Darboux-Egoroff metrics on this space are written in terms of the Schiffer $\Omega(P,Q)$ and Bergman $B(P,\bar Q)$ kernels on a Riemann surface of genus $g \geq 1$. These kernels are defined by [23]:

$$\Omega(P,Q) = W(P,Q) - \pi \sum_{i,j=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\omega_j(Q)\ ,$$

$$B(P,\bar Q) = \pi \sum_{i,j=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{ij}\omega_i(P)\overline{\omega_j(Q)}\ ,$$

where $W(P,Q) = d_P d_Q \log E(P,Q)$ is the canonical bidifferential of the second kind on $\mathcal{L}$; $E(P,Q)$ is the prime form; $\{\omega_i\}_{i=1}^{g}$ are holomorphic differentials on $\mathcal{L}$ normalized with respect to a given canonical basis of cycles by $\oint_{a_i}\omega_j = \delta_{ij}$; and $\mathbb{B}$ is the symmetric matrix of their b-periods: $\mathbb{B}_{ij} = \oint_{b_i}\omega_j$.

The kernels can equivalently be characterized as follows [23]. The Schiffer kernel is the bidifferential with a singularity of the form $(x(P)-x(Q))^{-2}dx(P)dx(Q)$ along the diagonal $P=Q$ such that $p.v.\iint_{\mathcal{L}}\Omega(P,Q)\ \overline{\omega(P)} = 0$ holds for any holomorphic differential ω on the surface. The Bergman kernel is a regular bidifferential on $\mathcal{L}$ holomorphic with respect to its first argument and antiholomorphic with respect to the second one. It is (up to a factor of $2\pi i$) a kernel of an integral operator acting in the space $L_2^{(1,0)}(\mathcal{L})$ of (1,0)-forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic (1,0)-forms. In particular, for any holomorphic differential ω on the surface $\mathcal{L}$ the following relation holds: $\iint_{\mathcal{L}} B(P,\bar{Q})\,\omega(Q) = 2\pi i\omega(P)$. Both kernels, $\Omega(P,Q)$ and $B(P,\bar{Q})$, are independent of the choice of a canonical basis of cycles $\{a_k, b_k\}$.

We consider the following family of metrics on the real Hurwitz space:

$$\mathbf{ds^2} = \sum_{j=1}^{L}\left(\oint_l h(Q)\Omega(Q,P_j)\right)^2 (d\lambda_j)^2 + \sum_{j=1}^{L}\left(\oint_l h(Q)B(Q,\bar{P}_j)\right)^2 (d\bar{\lambda}_j)^2 . \qquad (4.1.1)$$

Here l is an arbitrary contour on the surface not passing through ramification points and such that its projection on the base of the covering does not depend on coordinates $\{\lambda_i;\bar{\lambda}_i\}$; h is an arbitrary function defined in a neighbourhood of the contour. The rotation coefficients β_{ij} of the metrics (4.1.1) are given by the Schiffer and Bergman kernels evaluated at the ramification points of the covering with respect to the local parameters given by $\sqrt{\lambda(P)-\lambda_i}$:

$$\beta_{ij} = \Omega(P_i,P_j)\,, \qquad \beta_{i\bar{j}} = B(P_i,\bar{P}_j)\,, \qquad \beta_{\bar{i}\bar{j}} = \overline{\Omega(P_i,P_j)}\,.$$

As a consequence of Rauch variational formulas for the Schiffer and Bergman kernels, we have relations $\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{kj}$ for the rotation coefficients for distinct indices i,j,k from the set $\{m;\bar{m}\}_{m=1}^{L}$. These relations provide main conditions for the flatness of metrics (4.1.1).

Some of the metrics (4.1.1) correspond to Frobenius structures on the Hurwitz space. We describe these structures and find their prepotentials and flat coordinates of the corresponding flat metric. A prepotential as a function of flat coordinates satisfies the WDVV system.

Since for the surface of genus zero the Bergman kernel vanishes and the Schiffer kernel coincides with $W(P,Q)$, the metrics (4.1.1) and therefore the construction of Frobenius manifolds suggested here is only new for a Hurwitz space in genus ≥ 1 . For the Riemann sphere, our construction coincides with that of Dubrovin. For the simplest Hurwitz space in genus one, which has the real dimension 6, we compute explicitly prepotentials of three new Frobenius manifolds. One of these prepotentials has the form:

$$\begin{aligned} F = & -\frac{1}{4}t_1t_2^2 - \frac{1}{4}t_1t_5^2 + \frac{1}{2}t_1t_4(2t_3 - \frac{1}{2\pi i}) - \frac{1}{2}t_1^2t_6 - \frac{1}{2}t_3(t_3 - \frac{1}{2\pi i})\frac{t_4^2}{t_6} - \frac{1}{16}\frac{t_2^2t_5^2}{t_6} \\ & - \frac{t_2^4}{32t_6} - \frac{1}{128\pi i}\frac{t_2^4}{t_6^2}\gamma\left(\frac{t_3}{t_6}\right) + \frac{t_3t_4t_5^2}{4t_6} \\ & - \frac{t_5^4}{32t_6} - \frac{1}{128\pi i}\frac{t_5^4}{t_6^2}\gamma\left(\frac{1-2\pi i t_3}{2\pi i t_6}\right) + \frac{(t_3 - \frac{1}{2\pi i})t_4t_2^2}{4t_6}, \end{aligned} \tag{4.1.2}$$

where $\gamma(\mu) = 4\partial_\mu \log \eta(\mu)$ for η being the Dedekind η-function. The function F is quasihomogeneous, i.e. it satisfies

$$F(\kappa t_1, \kappa^{1/2}t_2, t_3, \kappa t_4, \kappa^{1/2}t_5, t_6) = \kappa^2 F(t_1, t_2, t_3, t_4, t_5, t_6)$$

for any nonzero constant κ . The matrix F_1 formed by third derivatives $F_{t_1t_it_j}$ is constant and invertible; it gives the flat metric (written in flat coordinates) from the family of metrics (4.1.1) which corresponds to the Frobenius structure (4.1.2). The functions

$$c_{ij}^k = \sum_n (F_1^{-1})_{kn}\frac{\partial^3 F}{\partial t_i \partial t_j \partial t_n}$$

define an associative commutative algebra in the tangent space to the underlying Hurwitz space: $\partial_{t_i} \cdot \partial_{t_j} = c_{ij}^k \partial_{t_k}$. (This is equivalent [15] to the WDVV system for the function F.)

Associated with any semisimple Frobenius manifold is the G-function, the solution to Getzler's system of linear differential equations derived in [26] within the study of recursion relations for the genus one Gromov-Witten invariants of smooth projective varieties. This system may be written for any semisimple Frobenius manifold. In [17] it was proven that, for an arbitrary semisimple Frobenius manifold, the Getzler system has a unique quasihomogeneous solution given by

$$G = \log \frac{\tau_{\rm I}}{J^{1/24}} \,. \tag{4.1.3}$$

Here J is the Jacobian of the transformation between canonical and flat coordinates on the Frobenius manifold; $\tau_{\rm I}$ is the isomonodromic tau-function associated to the Frobenius manifold. For the Frobenius structures described here the ingredients of the formula (4.1.3) can be computed using results of papers [44, 45]. For example, the isomonodromic tau-function $\tau_{\rm I}$ of the new Frobenius manifolds is related to the isomonodromic tau-function $\tau_{\rm I}^0$ of Dubrovin's Hurwitz Frobenius manifolds by the formula:

$$\tau_{\rm I} = |\tau_{\rm I}^0|^2 \, (\det {\rm Im}\mathbb{B})^{-\frac{1}{2}} \ ,$$

where $\mathbb{B}$ is the matrix of b-periods of the underlying Riemann surface. The function $\tau_{\rm I}^{-2}$ coincides with an appropriately regularized ratio of the determinant of Laplacian on the Riemann surface and the surface volume in the singular metric $|d\lambda|^2$, see [11, 44, 69].

For the Frobenius manifold corresponding to the prepotential (4.1.2), the G-function is expressed in terms of the Dedekind eta-function as follows:

$$G = -\log \left\{ \eta\left(\frac{t_3}{t_6}\right) \eta\left(\frac{1-2\pi i t_3}{2\pi i t_6}\right) (t_2 t_5)^{\frac{1}{8}} \, t_6^{-\frac{3}{4}} \right\} + \text{const} \,.$$

We hope that in the future the construction of a "real double" can be extended to arbitrary Frobenius manifolds. Presumably this extension can be done on the level of the Riemann-Hilbert problem associated with a Frobenius manifold. The most intriguing case would then be the Frobenius manifolds related to quantum cohomologies; we hope that their "real doubles" might find an interesting geometrical application.

We notice that a class of solutions to the WDVV system related to real Hurwitz spaces was previously constructed in the work [20]. However, the full structure of a Frobenius manifold was not discussed in [20], and an explicit relationship of prepotentials of [20] and solutions to WDVV equations constructed in this work remains unclear.

The paper is organized as follows. In the next section we give definitions of the WDVV system and Frobenius manifold and discuss the one-to-one correspondence between them. In Section 4.3 we describe the Hurwitz space we shall build Frobenius structures on, the W-bidifferential and the Schiffer and Bergman kernels on a Riemann surface and introduce flat metrics on Hurwitz spaces in terms of the kernels. In Section 4.4 we reformulate the structures of Frobenius manifolds on Hurwitz spaces introduced by Dubrovin in terms of the W-bidifferential. Section 4.5 contains the main result of the paper, the Frobenius structures on Hurwitz spaces considered as real manifolds. Section 4.6 is devoted to calculation of the G-function for the new Frobenius structures. In Section 4.7 we consider the simplest Hurwitz space in genus one and present explicit expressions for prepotentials and G-functions of the corresponding Frobenius manifolds.

4.2 Frobenius manifolds and WDVV equations

The Witten - Dijkgraaf - E.Verlinde - H.Verlinde (WDVV) system looks as follows:

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i\,, \quad i,j = 1,\ldots,n\,,$$

where F_i is the $n \times n$ matrix

$$(F_i)_{lm} = \frac{\partial^3 F}{\partial t^i \partial t^l \partial t^m} ,$$

and F is a scalar function of n variables $t^1, \ldots, t^n$. In the theory of Frobenius manifolds one imposes the following two conditions on the function F :

- Quasihomogeneity (up to a quadratic polynomial): for any nonzero κ and some numbers $\nu_1, \ldots, \nu_n, \nu_F$

$$F(\kappa^{\nu_1} t^1, \ldots, \kappa^{\nu_n} t^n) = \kappa^{\nu_F} F(t^1, \ldots, t^n) + \text{quadratic terms} , \tag{4.2.1}$$

- Normalization: F_1 is a constant nondegenerate matrix.

The condition of quasihomogeneity can be rewritten in terms of the Euler vector field

$$E := \sum_\alpha \nu_\alpha t^\alpha \partial_{t^\alpha} \tag{4.2.2}$$

as follows:

$$\mathrm{Lie}_E F = E(F) = \sum_\alpha \nu_\alpha t^\alpha \partial_{t^\alpha} F = \nu_F F + \text{quadratic terms} . \tag{4.2.3}$$

Definition 4.1 *An algebra A over $\mathbb{C}$ is called a (commutative)* **Frobenius algebra** *if:*

- *it is a commutative associative $\mathbb{C}$-algebra with a unity* $\mathbf{e}$.
- *it is supplied with a $\mathbb{C}$-bilinear symmetric nondegenerate inner product $\langle \cdot, \cdot \rangle$ having the property $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$ for arbitrary vectors x, y, z from A .*

Definition 4.2 *M is a* **Frobenius manifold** *of the charge ν if a structure of a Frobenius algebra smoothly depending on the point $t \in M$ is specified on any tangent plane $T_t M$ such that*

F1 *the inner product* $\langle\cdot,\cdot\rangle$ *is a flat metric on* M *(not necessarily positive definite).*

F2 *the unit vector field* $\mathbf{e}$ *is covariantly constant with respect to the Levi-Civita connection* ∇ *for the metric* $\langle\cdot,\cdot\rangle$, *i.e.* $\nabla_{\mathbf{x}}\mathbf{e} = 0$ *for any vector field* $\mathbf{x}$ *on* M .

F3 *the tensor* $(\nabla_{\mathbf{w}}\mathbf{c})(\mathbf{x},\mathbf{y},\mathbf{z})$ *is symmetric in four vector fields* $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w} \in T_tM$, *where* $\mathbf{c}$ *is the following symmetric 3-tensor:* $\mathbf{c}(\mathbf{x},\mathbf{y},\mathbf{z}) = \langle\mathbf{x}\cdot\mathbf{y},\mathbf{z}\rangle$.

F4 *there exists on* M *a vector field* E *(the Euler field) such that the following conditions hold for any vector fields* $\mathbf{x}$, $\mathbf{y}$ *on* M

$$\nabla_{\mathbf{x}}(\nabla_{\mathbf{y}}E) = 0 \,, \tag{4.2.4}$$

$$[E,\mathbf{x}\cdot\mathbf{y}] - [E,\mathbf{x}]\cdot\mathbf{y} - \mathbf{x}\cdot[E,\mathbf{y}] = \mathbf{x}\cdot\mathbf{y} \,, \tag{4.2.5}$$

$$\mathrm{Lie}_E\langle\mathbf{x},\mathbf{y}\rangle := E\langle\mathbf{x},\mathbf{y}\rangle - \langle[E,\mathbf{x}],\mathbf{y}\rangle - \langle\mathbf{x},[E,\mathbf{y}]\rangle = (2-\nu)\langle\mathbf{x},\mathbf{y}\rangle \,. \tag{4.2.6}$$

The charge ν of a Frobenius manifold is equal to $\nu_F + 3$, where ν_F is the quasihomogeneity coefficient from (4.2.3).

Theorem 4.1 *([15]) Any solution* $F(t)$ *of the WDVV equations with* $\nu_1 \neq 0$ *defined for* $t \in M$ *determines on* M *a structure of a Frobenius manifold and vice versa.*

Proof (see [15]). Given a Frobenius manifold, denote by $\{t^\alpha\}$ the flat coordinates of the metric $\langle\cdot,\cdot\rangle$ and by η the constant matrix $\eta_{\alpha\beta} = \langle\partial_{t^\alpha},\partial_{t^\beta}\rangle$. Due to the covariant constancy of the unit vector field $\mathbf{e}$, we can by a linear change of coordinates put $\mathbf{e} = \partial_{t^1}$. In this coordinates, the condition **F3** of Definition 4.2 implies the existence of a function F whose third derivatives give the 3-tensor $\mathbf{c}$:

$$\mathbf{c}_{\alpha\beta\gamma} = \mathbf{c}(\partial_{t^\alpha},\partial_{t^\beta},\partial_{t^\gamma}) = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \,.$$

The WDVV equations for the function F provide the associativity condition for the Frobenius algebra defined by relations $\partial_{t^\alpha} \cdot \partial_{t^\beta} = \mathbf{c}_{\alpha\beta}^{\gamma} \partial_{t^\gamma}$, where the structure constants $\mathbf{c}_{\alpha\beta}^{\gamma}$ are found from $\mathbf{c}_{\alpha\beta}^{\delta} \eta_{\delta\gamma} = \mathbf{c}_{\alpha\beta\gamma}$. The existence of the vector field E implies the quasihomogeneity of the function F . Indeed, requirements (4.2.5), (4.2.6) on the Euler vector field imply

$$\begin{aligned} \mathrm{Lie}_E \mathbf{c}(x,y,z) &:= E\left(\mathbf{c}(x,y,z)\right) - \mathbf{c}([E,x],y,z) - \mathbf{c}(x,[E,y],z) - \mathbf{c}(x,y,[E,z]) \\ &= (3-\nu)\mathbf{c}(x,y,z) \,. \end{aligned} \quad (4.2.7)$$

The Lie derivative Lie_E commutes with the covariant derivative ∇ as can easily be checked in flat coordinates when the Euler vector field (due to (4.2.4)) has the form (4.2.2). Therefore, (4.2.7) implies $\mathrm{Lie}_E F = (3-\nu)F +$ quadratic terms .

The converse statement can be proven analogously. $\diamond$

The function F , defined up to an addition of an arbitrary quadratic polynomial in $t^1, \ldots, t^n$, is called the *prepotential* of the Frobenius manifold.

Definition 4.3 *A Frobenius manifold M is called* **semisimple** *if the Frobenius algebra in the tangent space at each point of M does not have nilpotents.*

In this paper we only consider semisimple Frobenius structures.

4.3 Kernels on Riemann surfaces and Darboux-Egoroff metrics

4.3.1 Hurwitz spaces

Hurwitz space is the moduli space of pairs $(\mathcal{L}, \lambda)$ where $\mathcal{L}$ is a compact Riemann surface of genus g and $\lambda : \mathcal{L} \to \mathbb{C}P^1$ is a meromorphic function on $\mathcal{L}$ of degree N . The pair $(\mathcal{L}, \lambda)$

represents the surface as an N-fold ramified covering $\mathcal{L}_\lambda$ of $\mathbb{C}P^1$ defined by the equation

$$\zeta = \lambda(P)\,, \qquad P \in \mathcal{L}$$

(ζ is a coordinate on $\mathbb{C}P^1$). In this way the surface $\mathcal{L}$ can be viewed as a collection of N copies of $\mathbb{C}P^1$ which are glued together along branch cuts. Critical points P_j of the function $\lambda(P)$ correspond to ramification points of the covering. The projections λ_j of ramification points on the base of the covering ($\mathbb{C}P^1$ with coordinate ζ) are the images of critical points P_j of the function $\lambda(P)$ (λ_j are called the branch points): $\lambda'(P_j) = 0$; $\lambda_j = \lambda(P_j)$.

We assume that all finite branch points $\{\lambda_j \,|\lambda_j < \infty\}$ are simple (i.e. there are exactly two sheets glued together at the corresponding point) and denote their number by L . We also assume that the function λ has $m+1$ poles at the points of $\mathcal{L}$ denoted by $\infty^0, \dots, \infty^m$; the pole at ∞^i has the order $n_i + 1$. In terms of sheets of the covering, there are $m+1$ points which project to $\zeta = \infty$ on the base; the numbers $\{n_i + 1\}$ give the number of sheets glued at each of these points ($n_0, \dots, n_m \in \mathbb{N}$ are such that $\sum_{i=0}^m (n_i + 1) = N$, they are called the ramification indices).

The local parameter near a simple ramification point $P_j \in \mathcal{L}$ (which is not a pole of λ) is $x_j(P) = \sqrt{\lambda(P) - \lambda_j}$; and in a neighbourhood $P \sim \infty^i$ the local parameter z_i is given by $z_i(P) = (\lambda(P))^{-1/(n_i+1)}$.

The Riemann-Hurwitz formula connects the genus g of the surface, degree N of the function λ , the number L of simple finite branch points, and the ramification indices n_i over infinity:

$$2g - 2 = -2N + L + \sum_{i=0}^{m} n_i\,. \tag{4.3.1}$$

Two coverings are said to be equivalent if one can be obtained from the other by a permutation of sheets. The set of equivalence classes of described coverings will be denoted by $M = M_{g;n_0,\dots,n_m}$. We shall work with a covering $\widehat{M} = \widehat{M}_{g;n_0,\dots,n_m}$ of this space. A point of the space $\widehat{M}$ is a triple $\{\mathcal{L}, \lambda, \{a_k, b_k\}_{k=1}^g\}$, where $\{a_k, b_k\}_{k=1}^g$ is a canonical basis of cycles on $\mathcal{L}$. The branch points $\lambda_1, \dots, \lambda_L$ play the role of local coordinates on $\widehat{M}$, viewed as a complex manifold.

4.3.2 Bidifferential W , Bergman and Schiffer kernels

First, we summarize properties of three well-known symmetric bidifferentials on Riemann surfaces. Being suitably evaluated at the ramification points $\{P_j\}$, these kernels will play the role of rotation coefficients of flat metrics on Hurwitz spaces.

The meromorphic bidifferential $W(P,Q)$ defined by

$$W(P,Q) := d_P d_Q \log E(P,Q) \tag{4.3.2}$$

is the symmetric differential on $\mathcal{L} \times \mathcal{L}$ with the second order pole at the diagonal $P = Q$ with biresidue 1 and the properties:

$$\oint_{a_k} W(P,Q) = 0 \; ; \qquad \oint_{b_k} W(P,Q) = 2\pi i \, \omega_k(P) \; ; \qquad k = 1, \dots, g \; . \tag{4.3.3}$$

Here $\{a_k, b_k\}_{k=1}^g$ is the canonical basis of cycles on $\mathcal{L}$; $\{\omega_k(P)\}_{k=1}^g$ is the corresponding set of holomorphic differentials normalized by $\oint_{a_l} \omega_k = \delta_{kl}$; and $E(P,Q)$ is the prime form on the surface $\mathcal{L}$. The dependence of the bidifferential W on branch points of the Riemann surface is given by the Rauch variational formulas [42, 63]:

$$\frac{\partial W(P,Q)}{\partial \lambda_j} = \frac{1}{2} W(P,P_j) W(Q,P_j) \; , \tag{4.3.4}$$

where $W(P,P_j)$ denotes the evaluation of the bidifferential $W(P,Q)$ at $Q = P_j$ with respect to the standard local parameter $x_j(Q) = \sqrt{\lambda(Q) - \lambda_j}$ near the ramification point P_j :

$$W(P, P_j) := \frac{W(P,Q)}{dx_j(Q)}|_{Q=P_j} \ . \tag{4.3.5}$$

The bidifferential $W(P,Q)$ depends holomorphically on the branch points $\{\lambda_j\}$ in contrast to the following two bidifferentials [23].

The *Schiffer kernel* $\Omega(P,Q)$ is the symmetric differential on $\mathcal{L} \times \mathcal{L}$ defined by:

$$\Omega(P,Q) := W(P,Q) - \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P)\omega_l(Q) \ , \tag{4.3.6}$$

where $\mathbb{B}$ is the symmetric matrix of b-periods of holomorphic normalized differentials $\{\omega_k\}$: $\mathbb{B}_{kl} = \oint_{b_k} \omega_l$, which depends holomorphically on the branch points $\{\lambda_j\}$. This kernel has the same singularity structure as the bidifferential W , it depends on $\{\bar{\lambda}_j\}$ due to the terms added to W , since $\mathrm{Im}\mathbb{B} = (\mathbb{B} - \bar{\mathbb{B}})/(2i)$ and $\bar{\mathbb{B}}$ is a function of $\{\bar{\lambda}_j\}$. For a surface of genus zero the Schiffer kernel coincides with W .

The *Bergman kernel* $B(P, \bar{Q})$ is defined by:

$$B(P, \bar{Q}) = \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P)\overline{\omega_l(Q)} \ . \tag{4.3.7}$$

It vanishes for a surface of genus zero.

An important property of the Schiffer and Bergman kernels is independence of the choice of a canonical basis of cycles $\{a_k, b_k\}_{k=1}^{g}$ on the Riemann surface. This can be seen, for example, from the following definitions (see Fay [23]) equivalent to (4.3.6) and (4.3.7).

The Schiffer kernel is the unique symmetric bidifferential with a singularity of the form $(x(P) - x(Q))^{-2}dx(P)dx(Q)$ along $P = Q$ and such that

$$\text{p.v.} \iint_{\mathcal{L}} \Omega(P,Q)\overline{\omega(P)} = 0 \tag{4.3.8}$$

holds for any holomorphic differential ω .

The Bergman kernel is (up to the multiplier $2\pi i$) a kernel of an integral operator which acts in the space $L_2^{(1,0)}(\mathcal{L})$ of $(1,0)$-forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic $(1,0)$-forms. In particular, the following holds for any holomorphic differential ω on the surface $\mathcal{L}$:

$$\frac{1}{2\pi i}\iint\limits_{\mathcal{L}} B(P,\bar{Q})\omega(Q) = \omega(P) . \tag{4.3.9}$$

For the Bergman kernel the independence of the choice of a canonical basis of cycles can also be seen directly from (4.3.7) using $(\mathrm{Im}\mathbb{B})_{kl} = \frac{i}{2}\iint_{\mathcal{L}}\omega_k(P)\overline{\omega_l(P)}$.

The periods of Schiffer and Bergman kernels are related to each other as follows:

$$\oint_{a_k} \Omega(P,Q) = -\oint_{a_k} B(\bar{P},Q) , \qquad \oint_{b_k} \Omega(P,Q) = -\oint_{b_k} B(\bar{P},Q) \tag{4.3.10}$$

where the integrals are taken with respect to the first argument. Their derivatives with respect to branch points and their complex conjugates are given by:

$$\begin{aligned} \frac{\partial\Omega(P,Q)}{\partial\lambda_j} &= \frac{1}{2}\Omega(P,P_j)\Omega(Q,P_j) , \qquad & \frac{\partial\Omega(P,Q)}{\partial\bar{\lambda}_j} &= \frac{1}{2}B(P,\bar{P}_j)B(Q,\bar{P}_j) , \\ \frac{\partial B(P,\bar{Q})}{\partial\lambda_j} &= \frac{1}{2}\Omega(P,P_j)B(P_j,\bar{Q}) , \qquad & \frac{\partial B(P,\bar{Q})}{\partial\bar{\lambda}_j} &= \frac{1}{2}B(P,\bar{P}_j)\overline{\Omega(Q,P_j)} . \end{aligned} \tag{4.3.11}$$

The notation here is analogous to that in (4.3.5), i.e. $\Omega(P,P_j)$ stands for $(\Omega(P,Q)/dx_j(Q))\,|_{Q=P_j}$ and $B(P,\bar{P}_j) := \left(B(P,\bar{Q})/\overline{dx_j(Q)}\right)|_{Q=P_j}$. To prove (4.3.11) one uses the variational formulas (4.3.4) for $W(P,Q)$, and the following Rauch variational foumulas for holomorphic normalized differentials $\{\omega_k\}$ and for the matrix of b-periods [63]:

$$\frac{\partial\,\omega_k(P)}{\partial\lambda_j} = \frac{1}{2}\omega_k(P_j)W(P,P_j) , \qquad \frac{\partial\,\mathbb{B}_{kl}}{\partial\lambda_j} = \pi i\,\omega_k(P_j)\omega_l(P_j) , \tag{4.3.12}$$

where we write $\omega_k(P_j)$ for $(\omega_k(P)/dx_j(P))|_{P=P_j}$. Derivatives of ω_k and $\mathbb{B}$ with respect to $\{\bar{\lambda}_j\}$ vanish.

4.3.3 Darboux-Egoroff metrics

Now we are in a position to introduce two families of Darboux-Egoroff (flat potential diagonal) metrics on Hurwitz spaces written in terms of the described bidifferentials. Following the terminology of Dubrovin, we call a bilinear quadratic form a metric even if it is not positive definite.

A diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ is called potential if there exists a function U such that $\partial_{\lambda_i} U = g_{ii}$ for all i . A potential diagonal metric is flat (Riemann curvature tensor vanishes) if its *rotation coefficients* β_{ij} defined for $i \neq j$ by

$$\beta_{ij} := \frac{\partial_{\lambda_j}\sqrt{g_{ii}}}{\sqrt{g_{jj}}} \tag{4.3.13}$$

satisfy the system of equations:

$$\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{kj} \ , \qquad i,j,k \ \text{ are distinct,} \tag{4.3.14}$$

$$\sum_k \partial_{\lambda_k}\beta_{ij} = 0 \qquad \text{for all } \ \beta_{ij} \ . \tag{4.3.15}$$

Darboux-Egoroff metrics in terms of the bidifferential W

The following family of diagonal metrics (bilinear quadratic forms) on the Hurwitz space first appeared in [42] where it was realized that the corresponding rotation coefficients are given by the bidifferential W (see (4.3.17)) and that the metrics are flat:

$$\mathbf{ds^2} = \sum_{j=1}^{L}\left(\oint_l h(Q)W(Q,P_j)\right)^2 (d\lambda_j)^2 \ . \tag{4.3.16}$$

Here l is an arbitrary smooth contour on the Riemann surface $\mathcal{L}$ such that $P_j \notin l$ for any j , and its image $\lambda(l)$ in $\mathbb{C}P^1$ is independent of the branch points $\{\lambda_j\}$; $h(Q)$ is an arbitrary independent of $\{\lambda_j\}$ function defined in a neighbourhood of the contour l .

Using variational formulas (4.3.4), we find that rotation coefficients of the metric (4.3.16) are given by the bidifferential $W(P,Q)$ evaluated at the ramification points of the surface $\mathcal{L}$ with respect to the standard local parameters $x_j = \sqrt{\lambda - \lambda_j}$ near P_j :

$$\beta_{ij} = \frac{1}{2} W(P_i, P_j) , \quad i,j = 1, \ldots, L , \; i \neq j . \tag{4.3.17}$$

Here $W(P_i, P_j)$, similarly to (4.3.5), stands for $(W(P,Q)/(dx_i(P)dx_j(Q)))\,|_{P=P_i,Q=P_j}$. Note that rotation coefficients β_{ij} (4.3.17) are symmetric with respect to indices, therefore the metrics (4.3.16) are potential. The next proposition shows that they are Darboux-Egoroff metrics.

Proposition 4.1 *[42] Rotation coefficients (4.3.17) satisfy equations (4.3.14), (4.3.15) and therefore metrics (4.3.16) are flat.*

Proof. Variational formulas (4.3.4) with $P = P_i$, $Q = P_k$, for different i, j, k imply relations (4.3.14) for rotation coefficients (4.3.17). Equations (4.3.15) hold for the coefficients due to the invariance of $W(P,Q)$ with respect to biholomorphic maps of the Riemann surface. Namely, consider the covering $\mathcal{L}_\lambda^\delta$ obtained from $\mathcal{L}_\lambda$ by a simultaneous δ-shift $\lambda \to \lambda + \delta$ on all sheets. The surface $\mathcal{L}$ is mapped by this transformation to $\mathcal{L}^\delta$ so that the point $P \in \mathcal{L}$ goes to $P^\delta \in \mathcal{L}^\delta$ which belongs to the same sheet of the covering as P and is such that $\lambda(P^\delta) = \lambda(P) + \delta$. Denote by W^δ the bidifferential W on the surface $\mathcal{L}^\delta$. Since the transformation $\lambda \to \lambda + \delta$ is biholomorphic, we have $W^\delta(P^\delta, Q^\delta) = W(P,Q)$. The same relation is true for $W(P,Q)/(dx_i(P)dx_j(Q))$ when points P and Q are in neighbourhoods of ramification points P_i and P_j , respectively:

$$\frac{W^\delta(P^\delta, Q^\delta)}{dx_i^\delta(P^\delta)dx_j^\delta(Q^\delta)} = \frac{W(P,Q)}{dx_i(P)dx_j(Q)} . \tag{4.3.18}$$

Note that $x_i(P) = \sqrt{\lambda(P) - \lambda_i}$ does not change under a simultaneous shift of all branch points and λ . After the substitution $P = P_i$, $Q = P_j$ in (4.3.18) the differentiation with respect to δ at $\delta = 0$ gives the sum of derivatives with respect to branch points: $\sum_{k=1}^{L} \partial_{\lambda_k} W(P_i, P_j) = 0$. Thus, the rotation coefficients (4.3.17) satisfy also (4.3.15). Therefore the metrics (4.3.16) are flat. $\diamond$

Darboux-Egoroff metrics in terms of Schiffer and Bergman kernels

Now let us consider the Hurwitz space M as a real manifold, i.e. a manifold with a set of local coordinates formed by the branch points and their complex conjugates. As an analogue of the family of metrics (4.3.16) on the space of coverings $M = M_{g;n_0,\dots,n_m}$ with the local coordinates $\{\lambda_1, \dots, \lambda_L; \bar\lambda_1, \dots, \bar\lambda_L\}$ we consider the following two families of metrics:

$$\mathbf{ds_1^2} = \sum_{j=1}^{L} \left(\oint_l h(Q)\Omega(Q, P_j) \right)^2 (d\lambda_j)^2 + \sum_{j=1}^{L} \left(\oint_l h(Q) B(Q, \bar P_j) \right)^2 (d\bar\lambda_j)^2 \tag{4.3.19}$$

and

$$\mathbf{ds_2^2} = \mathrm{Re} \left\{ \sum_{j=1}^{L} \left(\oint_l h(Q)\Omega(Q, P_j) + \oint_l \overline{h(Q)} B(\bar Q, P_j) \right)^2 (d\lambda_j)^2 \right\} . \tag{4.3.20}$$

Here, as before, l is an arbitrary contour on the surface not passing through $\{P_j\}$ and such that its image $\lambda(l)$ in ζ-plane is independent of branch points $\{\lambda_j\}$; h is an arbitrary function independent of $\{\lambda_j\}$ defined in some neighbourhood of the contour.

From variational formulas (4.3.11) for the Schiffer and Bergman kernels we see that these metrics are potential and their rotation coefficients are given by the kernels evaluated at ramification points of $\mathcal{L}$:

$$\beta_{ij} = \frac{1}{2}\Omega(P_i, P_j) , \qquad \beta_{i\bar j} = \frac{1}{2} B(P_i, \bar P_j) , \qquad \beta_{\bar i \bar j} = \overline{\beta_{ij}} . \tag{4.3.21}$$

Here $i, j = 1, \dots, L$ and the index $\bar{j}$ corresponds to differentiation with respect to $\bar{\lambda}_j$. Similarly to the notation in (4.3.17), we understand $\Omega(P_i, P_j)$ and $B(P_i, \bar{P}_j)$ as follows:

$$\Omega(P_i, P_j) := \frac{\Omega(P, Q)}{dx_i(P) dx_j(Q)}\Big|_{P=P_i,\, Q=P_j}, \qquad B(P_i, \bar{P}_j) := \frac{B(P, \bar{Q})}{dx_i(P)\overline{dx_j(Q)}}\Big|_{P=P_i,\, Q=P_j}.$$

Remark 4.1 Note that rotation coefficients of the metrics (4.3.19), (4.3.20) are defined on the space $M_{g;n_0,\dots,n_m}$, in contrast to rotation coefficients (4.3.17). The coefficients (4.3.17) are given by the bidifferential W , which depends on the choice of a canonical basis of cycles $\{a_i, b_i\}$, and therefore are defined on the covering $\widehat{M}_{g;n_0,\dots,n_m}$ (see Section 4.3.1). However, the metrics of the type (4.3.19), (4.3.20) which will be used in Section 4.5 still depend on the choice of cycles $\{a_i, b_i\}$ through the choice of contours l .

Proposition 4.2 *Rotation coefficients (4.3.21) satisfy equations (4.3.14), (4.3.15) and therefore metrics (4.3.19), (4.3.20) are flat.*

The proof is analogous to that of Proposition 4.1. Here δ should be taken real, $\delta \in \mathbb{R}$.

Note that in equations (4.3.14), (4.3.15) i, j, k run through the set of all possible indices which in this case is $\{1, \dots, L; \bar{1}, \dots, \bar{L}\}$, where we put $\lambda_{\bar{k}} := \bar{\lambda}_k$.

4.4 Dubrovin's Frobenius structures on Hurwitz spaces

We start with a description of Dubrovin's construction [15] of Frobenius manifolds on the space $\widehat{M} = \widehat{M}_{g;n_0,\dots,n_m}$ using the bidifferential $W(P, Q)$. The branch points $\lambda_1, \dots, \lambda_L$ are the local coordinates on $\widehat{M}$.

To introduce a structure of a Frobenius algebra on the tangent space $T_t\widehat{M}$ for some point $t \in \widehat{M}$ we take coordinates $\lambda_1, \ldots, \lambda_L$ to be canonical for multiplication, i.e we define

$$\partial_{\lambda_i} \cdot \partial_{\lambda_j} = \delta_{ij} \partial_{\lambda_i} \; . \tag{4.4.1}$$

Then, the unit vector field is given by

$$\mathbf{e} = \sum_{i=1}^{L} \partial_{\lambda_i} \; . \tag{4.4.2}$$

For this multiplication law, the diagonal metrics (4.3.16) obviously have the property $\langle \mathbf{x} \cdot \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \cdot \mathbf{z} \rangle$ required in the definition of a Frobenius algebra. Therefore together with the multiplication (4.4.1) the metrics (4.3.16) define a family of Frobenius algebras on $T_t\widehat{M}$.

Among the family of metrics (4.3.16) (and Frobenius algebras) we are going to isolate those corresponding to Frobenius manifolds.

The Euler vector field has the following form in canonical coordinates [15]:

$$E = \sum_{i=1}^{L} \lambda_i \partial_{\lambda_i} \; . \tag{4.4.3}$$

4.4.1 Primary differentials

As is easy to see, with the Euler field (4.4.3), the multiplication (4.4.1) satisfies requirement (4.2.5) from **F4**. Condition (4.2.6) then reduces to

$$E\left(\langle \partial_{\lambda_i}, \partial_{\lambda_i} \rangle\right) = -\nu \langle \partial_{\lambda_i}, \partial_{\lambda_i} \rangle \; . \tag{4.4.4}$$

The following proposition describes the metrics from family (4.3.16) which satisfy this condition.

Proposition 4.3 *Let the contour l in (4.3.16) be either a closed contour on $\mathcal{L}$ or a contour connecting points ∞^i and ∞^j for some i and j . In the latter case we regularize the integral* *by omitting its divergent part as a function of the corresponding local parameter near* ∞^i *. Choose a function* $h(Q)$ *in (4.3.16) to be* $h(Q) = C\,\lambda^n(Q)$ *(where* C *is a constant). Then the Euler vector field (4.4.3) acts on metrics (4.3.16) according to (4.4.4) with* $\nu = 1 - 2n$ *.*

Proof. Let us again use the invariance of the bidifferential W under biholomorphic mappings of the Riemann surface $\mathcal{L}$. Consider the mapping $\mathcal{L}_\lambda \to \mathcal{L}_\lambda^\epsilon$ when the transformation $\lambda \to (1+\epsilon)\lambda$ is performed on every sheet of the covering $\mathcal{L}_\lambda$. A point P of the surface is then mapped to the point P^ϵ of the same sheet such that $\lambda(P^\epsilon) = (1+\epsilon)\lambda(P)$. If W^ϵ is the bidifferential W on $\mathcal{L}_\lambda^\epsilon$, then $W^\epsilon(P^\epsilon, Q^\epsilon) = W(P, Q)$. For the local parameter $x_i = \sqrt{\lambda - \lambda_i}$ in a neighbourhood of a ramification point P_i , we have $dx_i^\epsilon = \sqrt{1+\epsilon}\; dx_i$. A contour l of the specified type is invariant as a path of integration in (4.3.16) with respect to this transformation. Therefore we have

$$\left(\oint_{l^\epsilon} \lambda^n(Q^\epsilon) \frac{W^\epsilon(Q^\epsilon, P^\epsilon)}{dx_j^\epsilon(P^\epsilon)} \right)^2 = (1+\epsilon)^{2n-1} \left(\oint_l \lambda^n(Q) \frac{W(Q,P)}{dx_j(P)} \right)^2 . \tag{4.4.5}$$

Putting $P = P_j$, we differentiate (4.4.5) with respect to ϵ at $\epsilon = 0$. This yields the action of the vector field E on the metric coefficient in the left-hand side and proves the proposition. $\diamond$

Proposition 4.4 *Rotation coefficients (4.3.17) given by the bidifferential* W *satisfy* $E\,(\beta_{ij}) = -\beta_{ij}$ *.*

Proof. This is a corollary of Proposition 4.3 and can be proven by a straightforward calculation using (4.4.4) and the definition of rotation coefficients (4.3.13). Alternatively, it can be proven directly by the method used in the proof of Proposition 4.3. $\diamond$

So far we have restricted the family of flat metrics to those of the form (4.3.16) with $h = C\,\lambda^n$ and the contour l being either closed or connecting points ∞^i , ∞^j :

$$\mathbf{ds^2} = \sum_{j=1}^{L} \left(C \oint_l \lambda^n(Q) W(Q, P_j) \right)^2 (d\lambda_j)^2 \,. \tag{4.4.6}$$

An additional restriction comes from **F2**, the requirement of covariant constancy of the unit vector field (4.4.2) with respect to the Levi-Civita connection.

Lemma 4.1 *If a diagonal metric* $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ *is potential (i.e.* $\partial_{\lambda_i} g_{jj} = \partial_{\lambda_j} g_{ii}$ *holds) and its coefficients* g_{ii} *are annihilated by the unit vector field (4.4.2)* ($\mathbf{e}(g_{ii}) = 0$)*, then the vector field* $\mathbf{e}$ *is covariantly constant with respect to the Levi-Civita connection of the metric* $\mathbf{ds^2}$.

The proof is a simple calculation using the following expression for the Christoffel symbols via coefficients of a diagonal metric. For distinct i, j, k we have:

$$\Gamma^k_{ii} = -\frac{1}{2} \frac{\partial_{\lambda_k} g_{ii}}{g_{kk}} \,, \qquad \Gamma^i_{ii} = \frac{1}{2} \frac{\partial_{\lambda_i} g_{ii}}{g_{ii}} \,, \qquad \Gamma^i_{ij} = \frac{1}{2} \frac{\partial_{\lambda_j} g_{ii}}{g_{ii}} \,, \qquad \Gamma^k_{ij} = 0 \,. \tag{4.4.7}$$

Thus, we need to find the metrics of the form (4.4.6) such that the unit vector field $\mathbf{e}$ annihilates their coefficients. These metrics can be written as

$$\mathbf{ds}^2_\phi = \sum_{i=1}^{L} \left(\operatorname*{res}_{P=P_i} \frac{\phi^2(P)}{d\lambda(P)} \right) (d\lambda_i)^2 \equiv \frac{1}{2} \sum_{i=1}^{L} \phi^2(P_i)(d\lambda_i)^2 \,, \tag{4.4.8}$$

where ϕ is a differential of one of the five types listed below in Theorem 4.2. These differentials are called primary and all have the form $\phi(P) = C \oint_l \lambda^n(Q) W(Q, P)$ with some specific choice of a contour l and function $C\lambda^n$. In other words, we shall consider five types of combinations of a contour and a function $C\lambda^n$. Let us write these combinations in the form of operations of integration over the contour with the weight function. The operations, applied to a 1-form f , have the following form:

1. $\mathrm{I}_{t^{i;\alpha}}[f(Q)] := \frac{1}{\alpha} \operatorname*{res}_{\infty^i} \lambda(Q)^{\frac{\alpha}{n_i+1}} f(Q)$ $\qquad i = 0, \dots, m \; ; \; \alpha = 1, \dots, n_i$.

2. $\mathrm{I}_{v^i}[f(Q)] := \operatorname*{res}_{\infty^i} \lambda(Q) f(Q)$ $\qquad i = 1, \dots, m$.

3. $\mathrm{I}_{w^i}[f(Q)] := \text{v.p.} \int_{\infty^0}^{\infty^i} f(Q)$ $\qquad i = 1, \dots, m$.

4. $\mathrm{I}_{r^k}[f(Q)] := -\oint_{a_k} \lambda(Q) f(Q)$ $\qquad k = 1, \dots, g$.

5. $\mathrm{I}_{s^k}[f(Q)] := \frac{1}{2\pi i} \oint_{b_k} f(Q)$ $\qquad k = 1, \dots, g$.

Here the principal value near infinity is defined by omitting the divergent part of the integral as a function of the local parameter z_i (such that $\lambda = z_i^{-n_i-1}$) .

Theorem 4.2 *Let us choose a point $P_0 \in \mathcal{L}$ which is mapped to zero by the function λ , i.e. $\lambda(P_0) = 0$, and let all basis contours $\{a_k, b_k\}$ start at this point. Then, the defined operations 1.-5. applied to the bidifferential W give a set of L differentials, called primary, with the following singularities (characteristic properties). By z_i we denote the local parameter near ∞^i such that $z_i^{-n_i-1} = \lambda$, n_i being the ramification index at ∞^i .*

1. $\phi_{t^{i;\alpha}}(P) := \mathrm{I}_{t^{i;\alpha}}[W(P,Q)]$ $\quad \sim z_i^{-\alpha-1}(P) dz_i(P) \, , \; P \sim \infty^i \; ;$ $\quad i = 0,\dots,m \; ; \; \alpha = 1, \dots, n_i$.

2. $\phi_{v^i}(P) := \mathrm{I}_{v^i}[W(P,Q)]$ $\quad \sim -d\lambda(P) \, , \; P \sim \infty^i \; ;$ $\quad i = 1, \dots, m$.

3. $\phi_{w^i}(P) := \mathrm{I}_{w^i}[W(P,Q)] :$ $\quad \operatorname*{res}_{\infty^i} \phi_{w^i} = 1 \; ; \; \operatorname*{res}_{\infty^0} \phi_{w^i} = -1 \; ;$ $\quad i = 1, \dots, m$.

4. $\phi_{r^k}(P) := \mathrm{I}_{r^k}[W(P,Q)] :$ $\quad \phi_{r^k}(P^{b_k}) - \phi_{r^k}(P) = 2\pi i d\lambda(P) \; ;$ $\quad k = 1, \dots, g$.

5. $\phi_{s^k}(P) := \mathrm{I}_{s^k}[W(P,Q)] :$ $\quad$ *holomorphic differential* $\quad k = 1, \dots, g$.

Here $\phi_{r^k}(P^{b_k}) - \phi_{r^k}(P)$ denotes the transformation of the differential under analytic continuation along the cycle b_k on the Riemann surface.

All above differentials have zero a-periods except ϕ_{s^l} which satisfy: $\oint_{a_k} \phi_{s^l} = \delta_{kl}$.

Proof. Let us prove that

$$\phi_{t^i;\alpha}(P) \underset{P\sim\infty^i}{\sim} z_i^{-\alpha-1}(P)dz_i(P) . \tag{4.4.9}$$

It is easy to see that the differential $\phi_{t^i;\alpha}(P)$ has a singularity only at $P = \infty^i$. Let us consider the expansion of the bidifferential W at $Q \sim \infty^i$:

$$W(P,Q) \underset{Q\sim\infty^i}{\simeq} W(P,\infty^i) + W'_{,2}(P,\infty^i)z_i(Q) + \frac{1}{2}W''_{,2}(P,\infty^i)z_i^2(Q) + \dots . \tag{4.4.10}$$

Since $W(P,Q) \simeq \left((z_i(P) - z_i(Q))^{-2} + \mathcal{O}(1)\right) dz_i(P)dz_i(Q)$ when $P \sim Q \sim \infty^i$ then we have for the $(\alpha-1)$-th coefficient of the expansion (4.4.10)

$$\frac{1}{\alpha!}W_{,2}^{(\alpha-1)}(P,\infty^i) \underset{P\sim\infty^i}{\sim} \frac{dz_i(P)}{z_i^{\alpha+1}(P)} ,$$

which proves (4.4.9). The case $\alpha = n_i + 1$ proves $\phi_{v^i}(P) \underset{P\sim\infty^i}{\sim} -d\lambda(P)$.

For the differentials ϕ_{w^i} the theorem can be proven analogously.

The differential $\phi_{r^k}(P)$ is not defined at the points of the contour a_k , however it has certain limits as P approaches the contour from different sides; thus $\phi_{r^k}(P)$ is defined and single valued on the fundamental polygon $\widehat{\mathcal{L}}$ of the surface. (The fundamental polygon $\widehat{\mathcal{L}}$ is obtained by cutting the surface along all basis cycles a_k and b_k provided they all start at one point.) Let us denote $dq_k^i(P) := \phi_{r^k}(P^{b_i}) - \phi_{r^k}(P)$ (as we shall see below, dq_k^i is indeed an exact differential) and consider the differential $\phi_{r^k}(P)\int_{P_0}^P \omega_k$ (ω_k is one of the normalized holomorphic differentials such that $\oint_{a_j} \omega_k = \delta_{jk}$). This differential has no poles inside $\widehat{\mathcal{L}}$. Therefore its integral over the boundary of $\widehat{\mathcal{L}}$ equals zero. On the other hand, since the boundary $\partial\widehat{\mathcal{L}}$ consists of cycles $\{a_j\}$ and $\{b_j\}$ the integral can be rewritten via periods of the differentials as follows:

$$0=\oint_{\partial\tilde{\mathcal{L}}}\phi_{r^k}(P)\int_{P_0}^{P}\omega_k=\oint_{b_k}\phi_{r^k}-\sum_j\oint_{a_j}\phi_{r^k}\mathbb{B}_{jk}+\sum_j\oint_{a_j}q_k^j\omega_k \tag{4.4.11}$$

($\mathbb{B}_{jk}=\oint_{b_j}\omega_k$). Due to the choice of the point P_0 where all basis cycles start, we can change the order of integration in expressions $\oint_{b_k}\oint_{a_k}\lambda(Q)W(P,Q)$ as can be checked by a local (near the point P_0) calculation of the integral. Therefore we have

$$\oint_{a_j}\phi_{r^k}(P)=0 \quad \text{for all } j \qquad \text{and} \qquad \oint_{b_k}\phi_{r^k}(P)=-2\pi i\oint_{a_k}\lambda(Q)\omega_k(Q)\,.$$

Then, the relation (4.4.11) takes the form

$$0=-2\pi i\oint_{a_k}\lambda(Q)\omega_k(Q)+\sum_j\oint_{a_j}q_k^j(Q)\omega_k(Q)\,,$$

and we conclude that $q_k^j(Q)=2\pi i\lambda(Q)\delta_{jk}$.

For differentials ϕ_{s^k} the statement of the theorem follows from properties (4.3.3) of the bidifferential W . For all primary differentials (except ϕ_{s^k}) a-periods are zero since they are zero for W . $\diamond$

4.4.2 Flat coordinates

For a flat metric there exists a set of coordinates in which coefficients of the metric are constant. These coordinates are called the *flat coordinates* of the metric. In flat coordinates the Christoffel symbols vanish and the covariant derivative ∇_{t^A} is the usual partial derivative ∂_{t^A} . Therefore flat coordinates can be found from the equation $\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}t=0$ ($\mathbf{x}$ and $\mathbf{y}$ are arbitrary vector fields on the manifold). In canonical coordinates this equation has the form:

$$\partial_{\lambda_i}\partial_{\lambda_j}t=\sum_k\Gamma_{ij}^k\partial_{\lambda_k}t\,, \tag{4.4.12}$$

where the Christoffel symbols are given by (4.4.7). For different i, j, k , the Christoffel symbols of the metrics $\mathbf{ds}^2_\phi$ (4.4.8) have the form:

$$\Gamma^k_{ii} = -\beta_{ik}\frac{\phi(P_i)}{\phi(P_k)}, \qquad \Gamma^i_{ii} = -\sum_{j,\, j\neq i}\Gamma^i_{ij}, \qquad \Gamma^i_{ij} = \beta_{ij}\frac{\phi(P_j)}{\phi(P_i)}, \qquad \Gamma^k_{ij} = 0 .$$

Theorem 4.3 *([15]) The following functions give a set of flat coordinates of the metric* $\mathbf{ds}^2_\phi$ *(4.4.8):*

$$\begin{aligned}
t^{i;\alpha} &= -(n_i+1)\mathrm{I}_{t^{i;1+n_i-\alpha}}[\phi] && i=0,\dots,m\,;\ \alpha = 1,\dots,n_i\\
v^i &= -\mathrm{I}_{w^i}[\phi] && i=1,\dots,m\\
w^i &= -\mathrm{I}_{v^i}[\phi] && i=1,\dots,m\\
r^k &= \mathrm{I}_{s^k}[\phi] && k=1,\dots,g\\
s^k &= \mathrm{I}_{r^k}[\phi] && k=1,\dots,g\,.
\end{aligned}$$

Nonzero entries of the constant matrix of the metric in these coordinates are:

$$\begin{aligned}
\mathbf{ds}^2_\phi(\partial_{t^{i;\alpha}}, \partial_{t^{j;\beta}}) &= \frac{1}{n_i+1}\delta_{ij}\delta_{\alpha+\beta, n_i+1}\,,\\
\mathbf{ds}^2_\phi(\partial_{v^i}, \partial_{w^j}) &= \delta_{ij}\,,\\
\mathbf{ds}^2_\phi(\partial_{r^k}, \partial_{s^l}) &= -\delta_{kl}\,.
\end{aligned}$$

For notational convenience we denote an arbitrary flat coordinate by t^A , and a primary differential by ϕ_{t^A} , i.e.

$$t^A \in \{t^{i;\alpha}\,;\ v^i, w^i\,;\ r^k, s^k\ |i=0,\dots,m\,;\ \alpha=1,\dots,n_i\,; k=1,\dots,g\}\,.$$

Proposition 4.5 *In flat coordinates* $\{t^A\}$ *of the metric* $\mathbf{ds}^2_\phi$ *, the Euler vector field (4.4.3) has the form (4.2.2) with coefficients* $\{\nu_A\}$ *depending on the choice of a primary differential* ϕ *:*

- *if* $\phi = \phi_{t^{i_o;\alpha}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}\left(1 + \frac{\alpha}{n_{i_o}+1} - \frac{\alpha}{n_i+1}\right) t^{i;\alpha}\partial_{t^{i;\alpha}} + \sum_{i=1}^{m}\left(\frac{\alpha}{n_{i_o}+1}v^i\partial_{v^i} + (1+\frac{\alpha}{n_{i_o}+1})\omega^i\partial_{\omega^i}\right)$$
$$+\sum_{k=1}^{g}\left(\frac{\alpha}{n_{i_o}+1}r^k\partial_{r^k} + (1+\frac{\alpha}{n_{i_o}+1})s^k\partial_{s^k}\right) ,$$

- *if* $\phi = \phi_{v^{i_o}}$ *or* $\phi = \phi_{r^{k_o}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}(2 - \frac{\alpha}{n_i+1})t^{i;\alpha}\partial_{t^{i;\alpha}} + \sum_{i=1}^{m}\left(v^i\partial_{v^i} + 2\omega^i\partial_{\omega^i}\right) + \sum_{k=1}^{g}\left(r^k\partial_{r^k} + 2s^k\partial_{s^k}\right) ,$$

- *if* $\phi = \phi_{\omega^{i_o}}$ *or* $\phi = \phi_{s^{k_o}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}(1 - \frac{\alpha}{n_i+1})t^{i;\alpha}\partial_{t^{i;\alpha}} + \sum_{i=1}^{m}\omega^i\partial_{\omega^i} + \sum_{k=1}^{g}s^k\partial_{s^k} .$$

Proposition 4.6 *(see [15]) The unit vector field* $\mathbf{e}$ *(4.4.2) in the flat coordinates of the metric* $\mathbf{ds}^2_{\phi_{t^{A_0}}}$ *has the form:* $\mathbf{e} = -\partial_{t^{A_0}}$.

Thus, the coordinate t^{A_0} is naturally marked. Let us denote it by t^1 so that $\mathbf{e} = -\partial_{t^1}$.

In flat coordinates the Christoffel symbols of the Levi-Civita connection vanish. Therefore the proposition implies that the unit vector field is covariantly constant (**F2**).

4.4.3 Prepotentials of Frobenius structures

Definition 4.4 *A* **prepotential** *of a Frobenius manifold is a function* F *of flat coordinates of the corresponding metric such that its third derivatives are given by the symmetric 3-tensor* $\mathbf{c}$ *from the definition of a Frobenius manifold (***F3***):*

$$\frac{\partial^3 F(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \mathbf{ds}^2_{\phi}(\partial_{t^A}\cdot\partial_{t^B}, \partial_{t^C}) . \qquad (4.4.13)$$

By presenting this function (defined up to a quadratic polynomial in flat coordinates) for each metric $\mathbf{ds}^2_\phi$ we shall prove the symmetry in four indices (A,B,C,D) of the tensor $(\nabla_{\partial_{t^D}}\mathbf{c})(\partial_{t^A},\partial_{t^B},\partial_{t^C})$ and therefore complete the construction of the Frobenius manifold.

We shall denote the Frobenius manifold corresponding to the metric $\mathbf{ds}^2_\phi$ by $\widehat{M}^\phi = \widehat{M}^\phi_{g;n_0,\dots,n_m}$ and its prepotential by F_ϕ .

Remark 4.2 Proposition 4.6 implies that the third order derivatives (4.4.13) are constant if one of the derivatives is taken with respect to the coordinate t^1

$$\frac{\partial^3 F}{\partial t^1 \partial t^A \partial t^B} = -\mathbf{ds}^2_{\phi_{t^1}}(\partial_{t^A},\partial_{t^B}) .$$

Before writing a formula for the prepotential we shall define a pairing of differentials. Let $\omega^{(1)}$ and $\omega^{(2)}$ be two differentials on the surface $\mathcal{L}$ holomorphic outside of the points $\infty^0,\dots,\infty^m$ with the following behaviour at ∞^i :

$$\omega^{(\alpha)} = \sum_{n=-n^{(\alpha)}}^{\infty} c^{(\alpha)}_{n,i} z_i^n dz_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r^{(\alpha)}_{n,i}\lambda^n \log\lambda\right) , \quad P\sim\infty^i , \tag{4.4.14}$$

where $n^{(\alpha)}\in\mathbb{Z}$ and $c^{(\alpha)}_{n,i}$, $r^{(\alpha)}_{n,i}$ are some coefficients; $z_i = z_i(P)$ is a local parameter near ∞^i . Denote also for $k=1,\dots,g$:

$$\oint_{a_k}\omega^{(\alpha)} = A^{(\alpha)}_k , \tag{4.4.15}$$

$$\omega^{(\alpha)}(P^{a_k}) - \omega^{(\alpha)}(P) = dp^{(\alpha)}_k(\lambda(P)) , \quad p^{(\alpha)}_k(\lambda) = \sum_{s>0} p^{(\alpha)}_{sk}\lambda^s , \tag{4.4.16}$$

$$\omega^{(\alpha)}(P^{b_k}) - \omega^{(\alpha)}(P) = dq^{(\alpha)}_k(\lambda(P)) , \quad q^{(\alpha)}_k(\lambda) = \sum_{s>0} q^{(\alpha)}_{sk}\lambda^s . \tag{4.4.17}$$

Here, as before, $\omega(P^{a_k})$ and $\omega(P^{b_k})$ denote the analytic continuation of $\omega(P)$ along the corresponding cycle on the Riemann surface.

Note that if $\omega^{(\alpha)}$ is one of the primary differentials (defined in Theorem 4.2), then the coefficients $c_{n,i}$, $r_{n,i}$, p_{sk}, q_{sk} and A_k do not depend on coordinates.

Definition 4.5 *For two differentials whose singularity structures are given by (4.4.14) - (4.4.17) define a* **pairing** $\mathcal{F}[\ ,\ ]$ *as follows:*

$$\mathcal{F}[\omega^{(\alpha)}\ ,\ \omega^{(\beta)}] = \sum_{i=0}^{m}\left(\sum_{n\geq 0}\frac{c^{(\alpha)}_{-n-2,i}}{n+1}c^{(\beta)}_{n,i} + c^{(\alpha)}_{-1,i}\text{v.p.}\int_{P_0}^{\infty^i}\omega^{(\beta)} - \text{v.p.}\int_{P_0}^{\infty^i}\sum_{n>0} r^{(\alpha)}_{n,i}\lambda^n\omega^{(\beta)}\right)$$
$$+\frac{1}{2\pi i}\sum_{k=1}^{g}\left(-\oint_{a_k} q^{(\alpha)}_k(\lambda)\omega^{(\beta)} + \oint_{b_k} p^{(\alpha)}_k(\lambda)\omega^{(\beta)} + A^{(\alpha)}_k\oint_{b_k}\omega^{(\beta)}\right)\ ,$$

where P_0 is a marked point on the surface such that $\lambda(P_0) = 0$.

For any primary differential ϕ we consider a (multivalued on $\mathcal{L}$) function ∂ :

$$\partial(P) = \text{v.p.}\int_{\infty^0}^{P}\phi\,. \tag{4.4.18}$$

One can see that singularities of the differential $\partial d\lambda$ can be described by formulas similar to (4.4.14) - (4.4.17). The corresponding coefficients $c_{n,i}$, $r_{n,i}$, p_{sk}, q_{sk} and A_k for $\omega = \partial d\lambda$ depend on coordinates $\{\lambda_k\}$ in contrast to those for primary differentials.

Theorem 4.4 *([15]) The following function gives a prepotential of the Frobenius manifold* $\widehat{M}^{\phi}$:

$$F_\phi = \frac{1}{2}\mathcal{F}[\partial d\lambda\ ,\ \partial d\lambda]\ , \tag{4.4.19}$$

where p is the multivalued function (4.4.18). The third derivatives of F_ϕ are given by

$$\begin{aligned}\frac{\partial^3 F_\phi(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} &= \mathbf{c}(\partial_{t^A},\partial_{t^B},\partial_{t^C}) = -\sum_{i=1}^{L}\operatorname*{res}_{P_i}\frac{\phi_{t^A}\phi_{t^B}\phi_{t^C}}{d\partial d\lambda}\\ &\equiv -\frac{1}{2}\sum_{i=1}^{L}\frac{\phi_{t^A}(P_i)\phi_{t^B}(P_i)\phi_{t^C}(P_i)}{\phi(P_i)}\,.\end{aligned} \tag{4.4.20}$$

Theorem 4.5 *([15]) The second derivatives of the prepotential F_ϕ are given by the pairing of the corresponding primary differentials:*

$$\partial_{t^A}\partial_{t^B}F_\phi = \mathcal{F}[\phi_{t^A}\ ,\ \phi_{t^B}]\ .$$

For the described Frobenius manifold $\widehat{M}^\phi$, the prepotential (4.4.19) is a quasihomogenous function of flat coordinates $\{t^A\}$ of the metric $\mathbf{ds}^2_\phi$, i.e. the following holds for some numbers $\{\nu_A\}$ and ν_F and any nonzero constant κ :

$$F_\phi(\kappa^{\nu_1}t^1,\ldots,\kappa^{\nu_n}t^n) = \kappa^{\nu_F}F_\phi(t^1,\ldots,t^n) + \text{quadratic terms}\ .$$

This follows from the existence of the Euler vector field satisfying (4.2.4) - (4.2.6) (see the proof of Theorem 4.1).

The coefficients of quasihomogeneity $\{\nu_A\}$ are coefficients of the Euler vector field written in flat coordinates (see (4.2.1) - (4.2.3)); they are given by Proposition 4.5. The coefficient $\nu_F = 3-\nu$ can be computed for each Frobenius structure $\widehat{M}^\phi$ using Proposition 4.3:

$$\text{if}\ \ \phi=\phi_{t^{i;\alpha}}\ ,\quad \text{then}\qquad \nu = 1-\frac{2\alpha}{n_i+1}\qquad \nu_F = \frac{2\alpha}{n_i+1}+2$$

$$\text{if}\ \ \phi=\phi_{v^i}\ \text{ or }\ \phi=\phi_{r^k}\ ,\ \text{then}\qquad \nu=-1\qquad \nu_F=4$$

$$\text{if}\ \ \phi=\phi_{\omega^i}\ \text{ or }\ \phi=\phi_{s^k}\ ,\ \text{then}\qquad \nu=1\qquad \nu_F=2\ .$$

Remark 4.3 A linear combination of primary differentials corresponding to the same charge ν also gives a Frobenius structure. Namely, the above construction works for

$$\phi = \sum_{i=1}^m \kappa_i\phi_{v^i} + \sum_{k=1}^g \sigma_k\phi_{r^k} \qquad\text{and}\qquad \phi = \sum_{i=1}^m \kappa_i\phi_{\omega^i} + \sum_{k=1}^g \sigma_k\phi_{s^k}\ ,$$

with any constants $\{\kappa_i\}$ and $\{\sigma_k\}$. The unit vector field in these cases, respectively, is given by

$$\mathbf{e} = -\left(\sum_{i=1}^m \kappa_i\partial_{v^i} + \sum_{k=1}^g \sigma_k\partial_{r^k}\right) \qquad\text{and}\qquad \mathbf{e} = -\left(\sum_{i=1}^m \kappa_i\partial_{\omega^i} + \sum_{k=1}^g \sigma_k\partial_{s^k}\right)\ .$$

After a linear change of variables, the unit field can be written as $\mathbf{e} = -\partial_{\xi^1}$ for a new variable ξ^1 , since the coordinates $\{v^i\}$ and $\{r^k\}$ ($\{\omega^i\}$ and $\{s^k\}$) have equal quasihomogeneity coefficients.

4.5 "Real doubles" of Dubrovin's Frobenius structures on Hurwitz spaces

In this section we consider the moduli space $\widehat{M} = \widehat{M}_{g;n_0,\dots,n_m}$ as a real manifold. The set of local coordinates is given by the set of branch points of the covering $\mathcal{L}_\lambda$ and their complex conjugates: $\{\lambda_1, \dots, \lambda_L \; ; \; \bar{\lambda}_1, \dots, \bar{\lambda}_L\}$. On the space $\widehat{M}$ with coordinates $\{\lambda_i; \bar{\lambda}_i\}$ we shall build a Frobenius structure in a way analogous to the one described in Section 4.4. The construction will be based on a family of flat metrics on $\widehat{M}(\{\lambda_i; \bar{\lambda}_i\})$ of the type (4.3.19), (4.3.20) with rotation coefficients given by the Schiffer and Bergman kernels. Since in genus zero the Schiffer kernel coincides with the bidifferential W and the Bergman kernel vanishes, we only get essentially new metrics (and therefore new Frobenius structures) for Hurwitz spaces in genus greater than zero.

We start with a description of a Frobenius algebra in the tangent space. The coordinates $\{\lambda_1, \dots, \lambda_L \; ; \; \bar{\lambda}_1, \dots, \bar{\lambda}_L\}$ are taken to be canonical for multiplication:

$$\partial_{\lambda_i} \cdot \partial_{\lambda_j} = \delta_{ij} \partial_{\lambda_i} \; , \tag{4.5.1}$$

where indices i, j range now in the set of all indices, i.e. $i, j \in \{1, \dots, L; \bar{1}, \dots, \bar{L}\}$, and we put $\lambda_{\bar{i}} := \bar{\lambda}_i$. The unit vector field of the algebra is given by

$$\mathbf{e} = \sum_{i=1}^{L} \left(\partial_{\lambda_i} + \partial_{\bar{\lambda}_i} \right) . \tag{4.5.2}$$

The role of an inner product of the Frobenius algebra is played by one of the metrics (4.3.19), (4.3.20). The new vector field E , analogously, is

$$E := \sum_{i=1}^{L} \left(\lambda_i \partial_{\lambda_i} + \bar{\lambda}_i \partial_{\bar{\lambda}_i} \right) \ . \tag{4.5.3}$$

4.5.1 Primary differentials

Together with the multiplication (4.5.1), the Euler field (4.5.3) satisfies relation (4.2.5) of **F4**. Its action (4.2.6) on a diagonal metric takes the form:

$$E\left(\langle \partial_{\lambda_k}, \partial_{\lambda_k} \rangle\right) = -\nu \langle \partial_{\lambda_k}, \partial_{\lambda_k} \rangle \ , \quad k \in \{1, \ldots, L; \bar{1}, \ldots, \bar{L}\} \ . \tag{4.5.4}$$

Among the metrics (4.3.19), (4.3.20) we choose, similarly to Proposition 4.3, those for which this condition holds.

Proposition 4.7 *Let the contour l in (4.3.19), (4.3.20) be either closed or connecting points ∞^i and ∞^j for some i, j . In the latter case we regularize the integral by omitting its divergent part as a function of the local parameter z_i (or as a function of $\bar{z}_i$) near ∞^i . Then the metrics (4.3.19), (4.3.20) with $h(Q) = C\lambda^n(Q)$ (where C is a constant) satisfy (4.5.4) with $\nu = 1 - 2n$ and the Euler field (4.5.3).*

Proof. The proof is the same as for Proposition 4.3: we use the fact that Bergman and Schiffer kernels are invariant under biholomorphic mappings of the Riemann surface. The biholomorphic map to be taken in this case is $\lambda \to (1+\epsilon)\lambda$, where ϵ is real. $\diamond$

Proposition 4.8 *Rotation coefficients (4.3.21) given by the Schiffer and Bergman kernels satisfy $E\left(\beta_{ij}\right) = -\beta_{ij}$, $i, j \in \{1, \ldots, L; \bar{1}, \ldots, \bar{L}\}$, where the Euler field E is given by (4.5.3).*

Proof. This statement is a corollary of Proposition 4.7; it can also be proven directly by using the invariance of the kernels under the mapping of Riemann surfaces $\mathcal{L}_\lambda \to \mathcal{L}_\lambda^\epsilon$, $\lambda \to (1+\epsilon)\lambda$, for $\epsilon \in \mathbb{R}$. $\diamond$

Among the metrics $\mathbf{ds^2} = \sum_i \left(g_{ii}(d\lambda_i)^2 + g_{\bar{i}\bar{i}}(d\bar{\lambda}_i)^2\right)$ of the form (4.3.19), (4.3.20) with $h = C\lambda^n$ and a contour l of the type required in Proposition 4.7 only those ones correspond to Frobenius manifolds whose coefficients satisfy $\mathbf{e}(g_{ii}) = \mathbf{e}(g_{\bar{i}\bar{i}}) = 0$ ($\mathbf{e}$ is the unit vector field (4.5.2)). This follows from **F2** and Lemma 4.1, which is obviously valid for the unit vector field (4.5.2) and diagonal potential metrics (4.3.19), (4.3.20). Therefore we need to find the combinations of a contour l and a function $h = C\lambda^n$ such that formulas (4.3.19), (4.3.20) give metrics whose coefficients are annihilated by the vector field $\mathbf{e}$. We list those combinations in the form of operations $\mathrm{I}[f(Q)] = \oint_l C\lambda^n f(Q)$ applied to a differential f of the form $f = f_{(1,0)} + f_{(0,1)}$. We say that a differential is of the $(1,0)$-type if in a local coordinate z it can be represented as $f_{(1,0)} = f_1(z)dz$, and is of the $(0,1)$-type if in a local coordinate it has a form $f_{(0,1)} = f_2(\bar{z})d\bar{z}$. We shall also call $f_{(1,0)}$ and $f_{(0,1)}$ the holomorphic and antiholomorphic parts of a differential f , respectively. We denote by $\tilde{\mathrm{res}}$ the coefficient in front of $d\bar{z}/\bar{z}$ in the Laurent expansion of a differential. As before, z_i is the local parameter in a neighbourhood of ∞^i such that $z_i^{-n_i-1}(Q) = \lambda(Q)$, $Q \sim \infty^i$.

For $i = 0, \ldots, m$; $\alpha = 1, \ldots, n_i$ we define:

$$\mathbf{1.}\ \mathrm{I}_{t^{i;\alpha}}[f(Q)] := \frac{1}{\alpha} \operatorname*{res}_{\infty^i} z_i^{-\alpha}(Q) f_{(1,0)}(Q) \qquad \mathbf{2.}\ \mathrm{I}_{t^{\bar{i};\bar{\alpha}}}[f(Q)] := \frac{1}{\alpha} \operatorname*{\tilde{res}}_{\infty^i} \bar{z}_i^{-\alpha}(Q) f_{(0,1)}(Q)$$

$$\mathbf{3.}\ \mathrm{I}_{v^i}[f(Q)] := \operatorname*{res}_{\infty^i} \lambda(Q) f_{(1,0)}(Q) \qquad \mathbf{4.}\ \mathrm{I}_{v^{\bar{i}}}[f(Q)] := \operatorname*{\tilde{res}}_{\infty^i} \bar{\lambda}(Q) f_{(0,1)}(Q)\ .$$

For $i = 1, \ldots, m$ we define:

$$\mathbf{5.}\ \mathrm{I}_{w^i}[f(Q)] := \mathrm{v.p.} \int_{\infty^0}^{\infty^i} f_{(1,0)}(Q) \qquad \mathbf{6.}\ \mathrm{I}_{w^{\bar{i}}}[f(Q)] := \mathrm{v.p.} \int_{\infty^0}^{\infty^i} f_{(0,1)}(Q)\ .$$

As before, the principal value near infinity is defined by omitting the divergent part of an integral as a function of the corresponding local parameter.

For $k = 1, \dots, g$ we define:

$$\mathbf{7.}\ \mathrm{I}_{r^k}[f(Q)] := -\oint_{a_k} \lambda(Q) f_{(1,0)}(Q) - \oint_{a_k} \bar{\lambda}(Q) f_{(0,1)}(Q)$$

$$\mathbf{8.}\ \mathrm{I}_{u^k}[f(Q)] := \oint_{b_k} \lambda(Q) f_{(1,0)}(Q) + \oint_{b_k} \bar{\lambda}(Q) f_{(0,1)}(Q)$$

$$\mathbf{9.}\ \mathrm{I}_{s^k}[f(Q)] := \frac{1}{2\pi i} \oint_{b_k} f_{(1,0)}(Q)$$

$$\mathbf{10.}\ \mathrm{I}_{t^k}[f(Q)] := -\frac{1}{2\pi i} \oint_{a_k} f_{(1,0)}(Q)\ .$$

Applying these operations to the sum of Schiffer and Bergman kernels, we shall obtain a set of primary differentials Φ , each of which gives a Darboux-Egoroff metric and a corresponding Frobenius structure. These differentials, listed below, decompose into a sum of holomorphic and antiholomorphic parts. The a-periods vanish for all primary differentials except for the differentials labeled by the index s^k ; the b-periods do not vanish only for the differentials having the index t^k . This normalization and a given type of singularity characterize a primary differential completely due to the following lemma.

Lemma 4.2 *If a single valued differential on a Riemann surface of the form $w = w_{(1,0)} + w_{(0,1)}$ has zero a- and b-periods and its parts $w_{(1,0)}$ and $w_{(0,1)}$ are everywhere analytic with respect to local parameters z and $\bar{z}$, respectively, then the differential w is zero.*

Proof. Since the holomorphic and antiholomorphic parts of the differential must be regular and single valued on the surface, we can write w in the form: $w = \sum_{k=1}^{g} \alpha_k \omega_k + \sum_{k=1}^{g} \beta_k \overline{\omega_k}$, where $\{\omega_i\}$ are holomorphic normalized differentials. The vanishing of a-periods gives $\alpha_k = -\beta_k$ and vanishing of b-periods implies that all α_k should be zero. ◇

We list primary differentials together with their characteristic properties. A proof that the differentials have the given properties is essentially contained in the proof of Theorem 4.2.

Let us fix a point P_0 on $\mathcal{L}$ such that $\lambda(P_0) = 0$, and let all the basic cycles $\{a_k, b_k\}_{k=1}^g$ on the surface start at this point. This enables us to change the order of integration in expressions of the type $\oint_{b_k}\oint_{a_k}\lambda(P)\Omega(P,Q)$ (this can be checked by a local calculation of the integral near the point P_0) and compute a- and b-periods of the following primary differentials.

For $i = 0, \dots, m$; $\alpha = 1, \dots, n_i$:

1. $\Phi_{t^{i;\alpha}}(P) = \mathrm{I}_{t^{i;\alpha}}\left[\Omega(P,Q) + B(\bar{P},Q)\right] \quad \sim (z_i^{-\alpha-1} + \mathcal{O}(1))dz_i + \mathcal{O}(1)d\bar{z}_i \ , \quad P \sim \infty^i \ .$

2. $\Phi_{t^{\overline{i;\alpha}}}(P) = \overline{\Phi_{t^{i;\alpha}}(P)}$.

For $i = 1, \dots, m$:

3. $\Phi_{v^i}(P) = \mathrm{I}_{v^i}\left[\Omega(P,Q) + B(\bar{P},Q)\right] \qquad \sim -\,d\lambda + \mathcal{O}(1)\,(dz_i + d\bar{z}_i) \ , \quad P \sim \infty^i \ .$

4. $\Phi_{v^{\bar{i}}}(P) = \overline{\Phi_{v^i}(P)}$.

5. $\Phi_{w^i}(P) = \mathrm{I}_{w^i}\left[\Omega(P,Q) + B(\bar{P},Q)\right]$; $\qquad \operatorname*{res}_{\infty^i} \Phi_{w^i} = 1$; $\operatorname*{res}_{\infty^0} \Phi_{w^i} = -1$.

6. $\Phi_{w^{\bar{i}}}(P) = \overline{\Phi_{w^i}(P)}$.

For $k = 1, \dots, g$:

7. $\Phi_{r^k}(P) = \mathrm{I}_{r^k}\left[2\mathrm{Re}\left\{\Omega(P,Q) + B(\bar{P},Q)\right\}\right]$; $\quad \Phi_{r^k}(P^{b_k}) - \Phi_{r^k}(P) = 2\pi i d\lambda - 2\pi i d\bar{\lambda}$.

8. $\Phi_{u^k}(P) = \mathrm{I}_{u^k}\left[2\mathrm{Re}\left\{\Omega(P,Q) + B(\bar{P},Q)\right\}\right]$; $\quad \Phi_{u^k}(P^{a_k}) - \Phi_{u^k}(P) = 2\pi i d\lambda - 2\pi i d\bar{\lambda}$.

9. $\Phi_{s^k}(P) = \mathrm{I}_{s^k}\left[\Omega(P,Q) + B(\bar{P},Q)\right]$; no singularities.

10. $\Phi_{t^k}(P) = \mathrm{I}_{t^k}\left[\Omega(P,Q) + B(\bar{P},Q)\right]$; no singularities.

Here, as before, $\lambda = \lambda(P)$ and $z_i = z_i(P)$ is the local parameter at $P \sim \infty^i$ such that $\lambda = z_i^{-n_i-1}$.

Note that due to properties (4.3.10) of the Schiffer and Bergman kernels and the choice of the point P_0 (see the proof of Theorem 4.2), only the primary differentials of the last two types have nonzero a- and b-periods. Let us denote an arbitrary differential from the list by Φ_{ξ^A} ; then the following holds:

$$\oint_{a_\alpha} \Phi_{\xi^A} = \delta_{\xi^A, s^\alpha} \; ; \qquad \oint_{b_\alpha} \Phi_{\xi^A} = \delta_{\xi^A, t^\alpha}$$

(δ is the Kronecker symbol). The number of primary differentials is $2L$ by virtue of the Riemann-Hurwitz formula (4.3.1).

Each of the primary differentials Φ defines a metric of the type (4.3.19), (4.3.20) by the formula:

$$\mathbf{ds}^2_\Phi = \frac{1}{2}\sum_{i=1}^{L} \Phi^2_{(1,0)}(P_i)(d\lambda_i)^2 + \frac{1}{2}\sum_{i=1}^{L} \Phi^2_{(0,1)}(P_i)(d\bar{\lambda}_i)^2 \; , \tag{4.5.5}$$

where $\Phi_{(1,0)}$ and $\Phi_{(0,1)}$ are, respectively, the holomorphic and antiholomorphic parts of the differential Φ . The evaluation of differentials at a ramification point P_i is done with respect to the standard local parameter $x_i = \sqrt{\lambda - \lambda_i}$, i.e. $\Phi_{(1,0)}(P_i) = (\Phi_{(1,0)}(P)/dx_i(P))\,|_{P=P_i}$. As is easy to see, metrics of the type (4.3.20) correspond to differentials $\Phi = \Phi_{u^k}$ and $\Phi = \Phi_{r^k}$.

Proposition 4.9 *Primary differentials satisfy the following relations:*

$$\mathbf{e}\left(\Phi_{(1,0)}(P_i)\right) = 0 \; , \qquad \mathbf{e}\left(\Phi_{(0,1)}(P_i)\right) = 0 \; , \tag{4.5.6}$$

for any ramification point P_i .

The proposition implies that the unit vector field $\mathbf{e}$ (4.5.2) annihilates coefficients of the metric $\mathbf{ds}^2_\Phi$ (4.5.5).

Proof. Consider the covering $\mathcal{L}_\lambda^\delta$ obtained from $\mathcal{L}_\lambda$ by a δ-shift of the points of every sheet, choosing $\delta \in \mathbb{R}$; this shift maps the point P of the surface to the point P^δ which belongs to the same sheet and for which $\lambda(P^\delta) = \lambda(P) + \delta$. Denote by Ω^δ and B^δ the corresponding kernels on $\mathcal{L}_\lambda^\delta$. They are invariant with respect to biholomorphic mappings of the Riemann surface, i.e. $\Omega^\delta(P^\delta, Q^\delta) = \Omega(P,Q)$, and $B^\delta(P^\delta, Q^\delta) = B(P,Q)$. The local parameters near ramification points also do not change: $x_i(P) = x_i^\delta(P) = \sqrt{\lambda(P) - \lambda_i}$. Therefore for differentials Φ_{ω^i} , $\Phi_{\omega^{\bar{i}}}$, Φ_{s^k} , and Φ_{t^k} , the statement of proposition follows immediately from this invariance. For them we have, for example,

$$\Phi^\delta_{\omega^i{}_{(1,0)}}(P_j^\delta) = \Phi_{\omega^i{}_{(1,0)}}(P_j) \ ; \qquad \Phi^\delta_{\omega^i{}_{(0,1)}}(P_j^\delta) = \Phi_{\omega^i{}_{(0,1)}}(P_j) \ .$$

Differentiation of these equalities with respect to δ at $\delta = 0$ gives the action of the unit vector field $\mathbf{e}$ (4.5.2) on the differential in the left and zero in the right side.

Consider now the differential $\tilde{\Phi}(P) = -\oint_{a_k} \lambda(Q)\Omega(P,Q) - \oint_{a_k} \bar{\lambda}(Q)B(P,\bar{Q})$, which is related to the differential Φ_{r^k} as follows: $\Phi_{r^k}(P) = 2\mathrm{Re}\{\tilde{\Phi}(P)\}$. On the shifted covering $\mathcal{L}_\lambda^\delta$ we have

$$\begin{aligned} \tilde{\Phi}^\delta(P_i^\delta) &= -\oint_{a_k^\delta} \lambda(Q^\delta)\Omega^\delta(P_i^\delta, Q^\delta) - \oint_{a_k^\delta} \bar{\lambda}(Q^\delta)B^\delta(P_i^\delta, \bar{Q^\delta}) \\ &= -\oint_{a_k} (\lambda(Q) + \delta)\Omega(P_i, Q) - \oint_{a_k} (\bar{\lambda}(Q) + \delta)B(P_i, \bar{Q}) \ . \end{aligned} \tag{4.5.7}$$

Differentiating both sides of this equality with respect to δ at $\delta = 0$ and using the property (4.3.10) of the Schiffer and Bergman kernels, we prove formulas (4.5.6) for the differentials

Φ_{r^k}; the proof for Φ_{u^k} is analogous.

To prove (4.5.6) for the remaining differentials consider the local parameter z_i near infinity ∞^i ; under the δ-shift it transforms as follows:

$$z_i^{-\alpha}(P^\delta) \underset{\delta\sim 0}{=} (\lambda(P)+\delta)^{\frac{\alpha}{n_i+1}} = z_i^{-\alpha}(P) + \frac{\alpha}{n_i+1}(z_i(P))^{-\alpha+n_i+1}\delta + \mathcal{O}(\delta^2) \, .$$

Therefore $\Phi_{t^{i;\alpha}}(P_j)$ on the covering $\mathcal{L}_\lambda^\delta$ is given by

$$\Phi^\delta_{t^{i;\alpha}}(P_j^\delta) = \frac{1}{\alpha}\operatorname*{res}_{\infty^i}\left(z_i^{-\alpha}(P) + \frac{\alpha}{n_i+1}(z_i(P))^{-\alpha+n_i+1}\delta + \mathcal{O}(\delta^2)\right)\left(\Omega(P,P_j) + B(P,\bar{P}_j)\right) \, .$$

Differentiating both sides with respect to δ at $\delta=0$, we get

$$\mathbf{e}\left(\Phi_{t^{i;\alpha}(1,0)}(P_j)\right) = \frac{1}{n_i+1}\operatorname*{res}_{\infty^i}(z_i(P))^{-\alpha+n_i+1}\Omega(P,P_j) \, ,$$

$$\mathbf{e}\left(\Phi_{t^{i;\alpha}(0,1)}(P_j)\right) = \frac{1}{n_i+1}\operatorname*{res}_{\infty^i}(z_i(P))^{-\alpha+n_i+1}B(P,\bar{P}_j) \, .$$

The right sides are zero for non-negative powers of z_i , i.e. for $\alpha = 1,\ldots,n_i+1$. This proves the statement of the proposition for differentials $\Phi_{t^{i;\alpha}}$ and Φ_{v^i} ($\alpha = n_i+1$ corresponds up to a constant to the case of differential Φ_{v^i}). $\diamond$

Remark 4.4 This calculation also shows that differentials $\Phi_{t^{i;\alpha}}$ for $i = 0,\ldots,m$; $\alpha = 1,\ldots,n_i$ and Φ_{v^i} for $i = 1,\ldots,m$ give the full set of primary differentials of the type $\oint_l C\lambda^n(\Omega(P,Q)+B(P,Q))$ for l being a small contour encircling one of the infinities.

Note that we cannot consider $\Phi_{v^0}(P)$ as an independent differential due to the relation $\sum_{i=0}^m \Phi_{v^i}(P) = -(m+1)d\lambda(P)$, where $d\lambda(P) = d\zeta$ is a differential on $\mathbb{C}P^1$, the base of the covering.

Thus, we have constructed $2L$ differentials (see the Riemann-Hurwitz formula (4.3.1)); each of them gives by formula (4.5.5) a Darboux-Egoroff metric which satisfies **F2** ($\nabla\mathbf{e} = 0$), and on which the Euler field acts according to (4.2.6) from **F4**.

Our next goal is to find a set of flat coordinates for each of the metrics (4.5.5).

4.5.2 Flat coordinates

Let us write the Christoffel symbols of the metric $\mathbf{ds}^2_\Phi$ (4.5.5) in terms of the corresponding primary differential Φ . We shall use the following lemma which can be proven by a simple calculation using the definition of primary differentials and variational formulas (4.3.11) for the Schiffer and Bergman kernels.

Lemma 4.3 *The derivatives of primary differentials with respect to canonical coordinates are given by*

$$\frac{\partial \Phi_{\xi^A}(P)}{\partial \lambda_k} = \frac{1}{2}\Phi_{\xi^A(1,0)}(P_k)\left(\Omega(P,P_k) + B(\bar{P},P_k)\right) \tag{4.5.8}$$

$$\frac{\partial \Phi_{\xi^A}(P)}{\partial \bar{\lambda}_k} = \frac{1}{2}\Phi_{\xi^A(0,1)}(P_k)\left(B(P,\bar{P}_k) + \overline{\Omega(P,P_k)}\right) . \tag{4.5.9}$$

Then non-vanishing Christoffel symbols of the metric $\mathbf{ds}^2_\Phi$ can be expressed as follows in terms of the primary differential Φ and rotation coefficients β_{ij} (4.3.21):

$$\Gamma^j_{jk} = \beta_{jk}\frac{\Phi_{(1,0)}(P_k)}{\Phi_{(1,0)}(P_j)} = -\Gamma^i_{kk}\ ; \qquad \Gamma^j_{j\bar{k}} = \beta_{j\bar{k}}\frac{\Phi_{(0,1)}(P_k)}{\Phi_{(1,0)}(P_j)} = -\Gamma^j_{\bar{k}\bar{k}}\ ; \qquad \Gamma^j_{jj} = -\sum_{l\neq j}\Gamma^j_{jl}\ ;$$

$$\Gamma^{\bar{j}}_{\bar{j}k} = \beta_{\bar{j}k}\frac{\Phi_{(1,0)}(P_k)}{\Phi_{(0,1)}(P_j)} = -\Gamma^{\bar{j}}_{kk}\ ; \qquad \Gamma^{\bar{j}}_{\bar{j}\bar{k}} = \beta_{\bar{j}\bar{k}}\frac{\Phi_{(0,1)}(P_k)}{\Phi_{(0,1)}(P_j)} = -\Gamma^{\bar{j}}_{\bar{k}\bar{k}}\ . \tag{4.5.10}$$

Note that the index of summation l runs through the set $\{1,\dots,L;\bar{1},\dots,\bar{L}\}$.

Flat coordinates can be found from the system of differential equation (4.4.12). Due to formulas (4.5.10), this system can be rewritten as follows:

$$\partial_{\lambda_j}\partial_{\lambda_k}t = \Gamma^j_{jk}\partial_{\lambda_j}t + \Gamma^k_{jk}\partial_{\lambda_k}t\ , \quad j\neq k\in\{1,\dots,L,\bar{1},\dots,\bar{L}\} \tag{4.5.11}$$

$$\mathbf{e}(t) = \mathrm{const}\ . \tag{4.5.12}$$

Substituting expressions (4.5.10) for Christoffel symbols into system (4.5.11) and using Lemma 4.3, one proves the next theorem by a straightforward computation.

Theorem 4.6 *The following functions (and their linear combinations) satisfy the system (4.5.11):*

$$t_1 = \oint_{l_1} h_1(\lambda(P))\Phi_{(1,0)}(P) \quad \textit{and} \quad t_2 = \oint_{l_2} h_2(\bar{\lambda}(P))\Phi_{(0,1)}(P)\ , \tag{4.5.13}$$

where l_1 , l_2 are two arbitrary contours on the surface $\mathcal{L}$ which do not pass through ramification points and are such that their images $\lambda(l_1)$ and $\lambda(l_2)$ in ζ-plane do not depend on $\{\lambda_k; \bar{\lambda}_k\}$; arbitrary functions h_1 , h_2 are defined in some neighbourhoods of l_1 and l_2 , respectively, and are also independent of the coordinates $\{\lambda_k; \bar{\lambda}_k\}$. The integration is regularized by omitting the divergent part where needed.

Among solutions (4.5.13) we need to isolate those which satisfy equation (4.5.12), the second part of the system identifying flat coordinates. The operations I_{ξ^A} applied to the differential $\Phi(P)$ give functions of the form (4.5.13), and it turns out that flat coordinates can be obtained in this way. Namely, the following theorem holds.

Theorem 4.7 *Let P_0 be a marked point on $\mathcal{L}$ such that $\lambda(P_0) = 0$. Let all the basic cycles $\{a_k, b_k\}_{k=1}^g$ start at the point P_0 . Then the following functions give a set of flat coordinates of the metric $\mathbf{ds}^2_\Phi$ (4.5.5).*

For $i = 0, \dots, m$; $\alpha = 1, \dots, n_i$:

$$t^{i;\alpha} := -(n_i+1)\mathrm{I}_{t^{i;1+n_i-\alpha}}[\Phi] = \frac{n_i+1}{\alpha-n_i-1}\,\underset{\infty^i}{\mathrm{res}}\, z_i^{\alpha-n_i-1}\Phi_{(1,0)}\ ;$$

$$t^{\overline{i;\alpha}} := -(n_i+1)\mathrm{I}_{\overline{t^{i;1+n_i-\alpha}}}[\Phi] = \frac{n_i+1}{\alpha-n_i-1}\,\underset{\infty^i}{\tilde{\mathrm{res}}}\, \bar{z}_i^{\alpha-n_i-1}\Phi_{(0,1)}\ .$$

For $i=1,\dots,m$:

$$v^i := -\mathrm{I}_{w^i}[\Phi] = -\mathrm{v.p.}\int_{\infty^0}^{\infty^i} \Phi_{(1,0)} \;; \qquad v^{\bar i} := -\mathrm{I}_{w^{\bar i}}[\Phi] = -\mathrm{v.p.}\int_{\infty^0}^{\infty^i} \Phi_{(0,1)} \;;$$

$$w^i := -\mathrm{I}_{v^i}[\Phi] = -\operatorname*{res}_{\infty^i} \lambda\Phi_{(1,0)} \;; \qquad w^{\bar i} := -\mathrm{I}_{v^{\bar i}}[\Phi] = -\operatorname*{r\bar{e}s}_{\infty^i} \bar\lambda\Phi_{(0,1)} \;.$$

For $k=1,\dots,g$:

$$r^k := \mathrm{I}_{s^k}[\Phi] = \frac{1}{2\pi i}\oint_{b_k} \Phi_{(1,0)} \;; \qquad u^k := -\mathrm{I}_{t^k}[\Phi] = \frac{1}{2\pi i}\oint_{a_k} \Phi_{(1,0)} \;;$$

$$s^k := \mathrm{I}_{r^k}[\Phi] = -\oint_{a_k} \left(\lambda\Phi_{(1,0)} + \bar\lambda\Phi_{(0,1)}\right) \;; \qquad t^k := -\mathrm{I}_{u^k}[\Phi] = -\oint_{b_k} \left(\lambda\Phi_{(1,0)} + \bar\lambda\Phi_{(0,1)}\right) \;.$$

As before, we use the notation $\mathrm{r\bar{e}s} f := \overline{\mathrm{res}\bar f}$.

Let us denote the flat coordinates by ξ^A *, i.e. we assume*

$$\xi^A \in \{t^{i;\alpha}, t^{\overline{i;\alpha}} \;;\; v^i \;,\; v^{\bar i} \;,\; w^i \;,\; w^{\bar i} \;;\; r^k \;,\; u^k \;,\; s^k \;,\; t^k\}$$

for $i=0,\dots,m$, $\alpha=1,\dots,n_i$; $k=1,\dots,g$ *(except* v^0, $v^{\bar 0}$ *and* w^0, $w^{\bar 0}$ *, which do not exist).*

Proof. Theorem 4.6 implies that these functions satisfy equations (4.5.11). The remaining equations (4.5.12), $\mathbf{e}(\xi^A)=\mathrm{const}$, can be proven by the same reasoning as in the proof of Proposition 4.9. ◇

Note that the action of the unit vector field $\mathbf{e}$ (4.5.2) on a coordinate ξ^A is nonzero if and only if the type of the coordinate coincides with the type of the primary differential which defines the metric. I.e. for the metric $\mathbf{ds}^2_\Phi$ with $\Phi=\Phi_{\xi^{A_0}}$ the coordinate ξ^{A_0} is naturally marked and we shall denote it by ξ^1 . One can prove that, for any choice of Φ , the corresponding coordinate ξ^1 is such that relations $\mathbf{e}(\xi^1)=-1$ and $\mathbf{e}(\xi^A)=0$ hold for $\xi^A\neq\xi^1$. Therefore we have $\mathbf{e}=-\partial_{\xi^1}$ (see also Proposition 4.10 below).

Remark 4.5 By virtue of the Riemann-Hurwitz formula (see Section 4.3.1), the number of functions listed in the theorem equals $2L$, i.e. coincides with the number of canonical coordinates $\{\lambda_i; \bar{\lambda}_i\}$.

The next theorem gives an expression of the metric $\mathbf{ds}^2_\Phi$ in coordinates $\{\xi^A\}$ and by that shows again that functions $\{\xi^A\left(\{\lambda_k; \bar{\lambda}_k\}\right)\}$ are independent and play the role of flat coordinates for the metric.

Theorem 4.8 *In coordinates $\{\xi^A\}$ from Theorem 4.7 the metric $\mathbf{ds}^2_\Phi$ (4.5.5) is given by a constant matrix whose nonzero entries are the following:*

$$\mathbf{ds}^2_\Phi\left(\partial_{t^{i;\alpha}}, \partial_{t^{j;\beta}}\right) = \mathbf{ds}^2_\Phi\left(\partial_{t^{\overline{i;\alpha}}}, \partial_{t^{\overline{j;\beta}}}\right) = \frac{1}{n_i+1}\delta_{ij}\delta_{\alpha+\beta,n_i+1}\ ,$$

$$\mathbf{ds}^2_\Phi\left(\partial_{v^i}, \partial_{\omega^j}\right) = \mathbf{ds}^2_\Phi\left(\partial_{v^{\bar{i}}}, \partial_{\omega^{\bar{j}}}\right) = \delta_{ij}\ ,$$

$$\mathbf{ds}^2_\Phi\left(\partial_{r^i}, \partial_{s^j}\right) = -\delta_{ij}\ ,$$

$$\mathbf{ds}^2_\Phi\left(\partial_{u^i}, \partial_{t^j}\right) = \delta_{ij}\ .$$

We shall prove this theorem later, after introducing a pairing of differentials (4.5.24).

To further investigate properties of the flat coordinates let us choose one of the primary differentials Φ and build a multivalued differential on the surface $\mathcal{L}$ as follows:

$$\Psi(P) = \left(\text{v.p.}\int_{\infty^0}^P \Phi_{(1,0)}\right)d\lambda + \left(\text{v.p.}\int_{\infty^0}^P \Phi_{(0,1)}\right)d\bar{\lambda}\ . \tag{4.5.14}$$

This differential will play a role similar to the role of the differential $\partial d\lambda$ in the construction of Dubrovin (see formula (4.4.19) for prepotential). Note that $\Psi(P)$ decomposes into a sum of holomorphic and antiholomorphic differentials: $\Psi = \Psi_{(1,0)} + \Psi_{(0,1)}$.

Theorem 4.9 *The derivatives of the multivalued differential Ψ (4.5.14) with respect to flat coordinates $\{\xi^A\}$ are given by the corresponding primary differentials:*

$$\frac{\partial \Psi}{\partial \xi^A} = \Phi_{\xi^A} \, .$$

Proof. Consider an expansion of the differential Ψ in a neighbourhood of one of the infinities ∞^i on the surface. We omit the singular part which does not depend on coordinates. As before, z_i is a local coordinate in a neighbourhood of the i-th infinity, n_i is the corresponding ramification index. For $i \neq 0$ we have

$$\Psi(P) \underset{P\sim\infty^i}{=} \text{singular part} + \left(v^i(n_i+1) z_i^{-n_i-2} + \sum_{\alpha=1}^{n_i} t^{i;\alpha} z_i^{-\alpha-1} + w^i z_i^{-1} + \mathcal{O}(1) \right) dz_i$$
$$+ \left(v^{\bar{i}}(n_i+1) \bar{z}_i^{-n_i-2} + \sum_{\alpha=1}^{n_i} t^{\overline{i;\alpha}} \bar{z}_i^{-\alpha-1} + w^{\bar{i}} \bar{z}_i^{-1} + \mathcal{O}(1) \right) d\bar{z}_i \, . \quad (4.5.15)$$

We see that the expansion coefficients of the singular part are exactly the flat coordinates of the metric $\mathbf{ds^2_\Phi}$. The coordinates $t^{0;\alpha}$, $\alpha = 1, \ldots, n_0$ appear similarly in expansion at the infinity ∞^0 . The remaining coordinates ξ^A correspond to other characteristics of the multivalued differential Ψ . Namely, we have

$$\oint_{a_k} \Psi = s^k \, , \qquad \oint_{b_k} \Psi = t^k \, ; \quad (4.5.16)$$

$$\Psi(P^{a_k}) - \Psi(P) = 2\pi i u^k d\lambda - 2\pi i u^k d\bar{\lambda} + \delta_{\Phi,\Phi_{s^k}} d\bar{\lambda} + \delta_{\Phi,\Phi_{u^k}} (2\pi i d\lambda - 2\pi i d\bar{\lambda}) \, , \quad (4.5.17)$$

$$\Psi(P^{b_k}) - \Psi(P) = 2\pi i r^k d\lambda - 2\pi i r^k d\bar{\lambda} + \delta_{\Phi,\Phi_{t^k}} d\bar{\lambda} + \delta_{\Phi,\Phi_{r^k}} (2\pi i d\lambda - 2\pi i d\bar{\lambda}) \, , \quad (4.5.18)$$

where $\Psi(P^{a_k})$, $\Psi(P^{b_k})$ stand for the analytic continuation of $\Psi(P)$ along the corresponding cycles of the Riemann surface.

This parameterization of the differential Ψ by the flat coordinates, together with Lemma 4.2, proves the theorem. $\diamond$

As a corollary we get the following lemma.

Lemma 4.4 *The derivatives of canonical coordinates* $\{\lambda_i\,;\bar{\lambda}_i\}$ *with respect to flat coordinates* $\{\xi^A\}$ *of the metric* $\mathbf{ds}^2_\Phi$ *are as follows*

$$\frac{\partial\lambda_i}{\partial\xi^A} = -\frac{\Phi_{\xi^A(1,0)}(P_i)}{\Phi_{(1,0)}(P_i)}\,, \qquad \frac{\partial\bar{\lambda}_i}{\partial\xi^A} = -\frac{\Phi_{\xi^A(0,1)}(P_i)}{\Phi_{(0,1)}(P_i)}\,,$$

where $\Phi(P)$ *is the primary differential which defines the metric* $\mathbf{ds}^2_\Phi$.

Proof. Theorem 4.9 implies the following relations:

$$\partial_{\xi^A}\left\{\left(\int_{\infty^0}^{P}\Phi_{(1,0)}\right)d\lambda\right\} = \Phi_{\xi^A(1,0)}\,, \qquad \partial_{\xi^A}\left\{\left(\int_{\infty^0}^{P}\Phi_{(0,1)}\right)d\bar{\lambda}\right\} = \Phi_{\xi^A(0,1)}\,. \tag{4.5.19}$$

(The divergent terms which we omit by taking the principal value of the integrals in a neighbourhood of ∞^0 do not depend on $\{\xi^A\}$.) We shall use the so-called thermodynamical identity

$$\partial_\alpha(fdg)_{g=const} = -\partial_\alpha(gdf)_{f=const} \tag{4.5.20}$$

for f being a function of another function g and some parameters $\{p_\alpha\}$, i.e. $f = f(g;p_1,\dots,p_n)$, where g can be expressed locally as a function of f , i.e. $g = g(f;p_1,\dots,p_n)$; ∂_α denotes the derivative with respect to one of the parameters $p = \{p_\alpha\}$. Relation (4.5.20) can be proven by differentiation of the identity $f(g(f;p);p) \equiv f$ with respect to a parameter p_α , which gives $\partial_\alpha g df/dg + \partial_\alpha f = 0$. We use the thermodynamical identity (4.5.20) for functions $f(P) = \int_{\infty^0}^P \Phi_{(1,0)}$ and $g(P) = \lambda(P)$ to get

$$\partial_{\xi^A}\left\{\int_{\infty^0}^{P}\Phi_{(1,0)}\right\}d\lambda = -\partial_{\xi^A}\left\{\lambda(P)\right\}\Phi_{(1,0)}(P)\,,$$

and similarly,

$$\partial_{\xi^A}\left\{\int_{\infty^0}^{P}\Phi_{(0,1)}\right\}d\bar{\lambda} = -\partial_{\xi^A}\left\{\bar{\lambda}(P)\right\}\Phi_{(0,1)}(P)\,.$$

Evaluating these relations at the critical points $P = P_i$, using that $\lambda'(P_i) = 0$ and equalities (4.5.19), we prove the lemma. $\diamond$

Proposition 4.10 *The unit vector field (4.5.2) is a tangent vector field in the direction of one of the flat coordinates. Namely, in flat coordinates of the metric* $\mathbf{ds}^2_\Phi$ *(4.5.5) corresponding to the primary differential* $\Phi = \Phi_{\xi^{A_0}}$ *, the unit vector of the Frobenius algebra is given by* $\mathbf{e} = -\partial_{\xi^{A_0}}$.

Let us denote the marked coordinate by ξ^1 so that $\mathbf{e} = -\partial_{\xi^1}$.

Proof. This can be verified by a simple calculation using the chain rule $\partial_{\xi^1} = \sum_{i=1}^L \left(\frac{\partial \lambda_i}{\partial \xi^1}\partial_{\lambda_i} + \frac{\partial \bar\lambda_i}{\partial \xi^1}\partial_{\bar\lambda_i}\right)$ and expressions for $\partial\lambda_i/\partial\xi^1$ provided by Lemma 4.4. $\diamond$

4.5.3 Prepotentials of new Frobenius structures

A prepotential of the Frobenius structure which corresponds to a primary differential Φ is a function $F_\Phi(\{\xi^A\})$ of flat coordinates of the metric $\mathbf{ds}^2_\Phi$ such that its third derivatives are given by the tensor $\mathbf{c}$ from **F3**:

$$\frac{\partial^3 F_\Phi(\xi)}{\partial\xi^A\partial\xi^B\partial\xi^C} = \mathbf{c}(\partial_{\xi^A}, \partial_{\xi^B}, \partial_{\xi^C}) = \mathbf{ds}^2_\Phi\left(\partial_{\xi^A}\cdot\partial_{\xi^B}, \partial_{\xi^C}\right) . \tag{4.5.21}$$

We shall construct a prepotential F_Φ for each primary differential Φ . This will prove that **F3** (symmetry of the tensor $(\nabla_{\xi^A}\mathbf{c})(\partial_{\xi^B}, \partial_{\xi^C}, \partial_{\xi^D})$) holds in our construction. In order to write an expression for prepotential we define a new pairing of multivalued differentials as follows.

Let $\omega^{(\alpha)}(P)$, $\alpha = 1, 2\dots$ be a differential on $\mathcal{L}$ which can be decomposed into a sum of holomorphic ($\omega^{(\alpha)}_{(1,0)}$) and antiholomorphic ($\omega^{(\alpha)}_{(0,1)}$) parts, $\omega^{(\alpha)} = \omega^{(\alpha)}_{(1,0)} + \omega^{(\alpha)}_{(0,1)}$, which are analytic outside infinities and have the following behaviour at $P \sim \infty^i$ (we write λ for

$\lambda(P)$, and $z_i = z_i(P)$ for a local parameter $z_i^{-n_i-1} = \lambda$ at $P \sim \infty^i$):

$$\begin{aligned} \omega^{(\alpha)}_{(1,0)}(P) &= \sum_{n=-n_1^{(\alpha)}}^{\infty} c^{(\alpha)}_{n,i} z_i^n dz_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r^{(\alpha)}_{n,i}\lambda^n \log\lambda\right) , \\ \omega^{(\alpha)}_{(0,1)}(P) &= \sum_{n=-n_2^{(\alpha)}}^{\infty} c^{(\alpha)}_{\bar{n},i} \bar{z}_i^n d\bar{z}_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r^{(\alpha)}_{\bar{n},i}\bar{\lambda}^n \log\bar{\lambda}\right) , \end{aligned} \tag{4.5.22}$$

where $n_1^{(\alpha)}, n_2^{(\alpha)} \in \mathbb{Z}$; and $c^{(\alpha)}_{n,i}$, $r^{(\alpha)}_{n,i}$, $c^{(\alpha)}_{\bar{n},i}$, $r^{(\alpha)}_{\bar{n},i}$ are some coefficients. Denote also for $k = 1, \ldots, g$:

$$A_k^{(\alpha)} := \oint_{a_k} \omega^{(\alpha)} , \qquad B_k^{(\alpha)} := \oint_{b_k} \omega^{(\alpha)} ,$$

$$\begin{aligned} dp_k^{(\alpha)}(\lambda(P)) &:= \omega^{(\alpha)}_{(1,0)}(P^{a_k}) - \omega^{(\alpha)}_{(1,0)}(P) , \qquad p_k^{(\alpha)}(\lambda) = \sum_{s>0} p^{(\alpha)}_{sk}\lambda^s , \\ dp_{\bar{k}}^{(\alpha)}(\bar{\lambda}(P)) &:= \omega^{(\alpha)}_{(0,1)}(P^{a_k}) - \omega^{(\alpha)}_{(0,1)}(P) , \qquad p_{\bar{k}}^{(\alpha)}(\bar{\lambda}) = \sum_{s>0} p^{(\alpha)}_{\bar{s}\bar{k}}\bar{\lambda}^s , \\ dq_k^{(\alpha)}(\lambda(P)) &:= \omega^{(\alpha)}_{(1,0)}(P^{b_k}) - \omega^{(\alpha)}_{(1,0)}(P) , \qquad q_k^{(\alpha)}(\lambda) = \sum_{s>0} q^{(\alpha)}_{sk}\lambda^s , \\ dq_{\bar{k}}^{(\alpha)}(\bar{\lambda}(P)) &:= \omega^{(\alpha)}_{(0,1)}(P^{b_k}) - \omega^{(\alpha)}_{(0,1)}(P) , \qquad q_{\bar{k}}^{(\alpha)}(\bar{\lambda}) = \sum_{s>0} q^{(\alpha)}_{\bar{s}\bar{k}}\bar{\lambda}^s . \end{aligned} \tag{4.5.23}$$

Note that all primary differentials and the differential $\Psi(P)$ have singularity structures which are described by (4.5.22) - (4.5.23). For $\omega^{(\alpha)}$ being one of the primary differentials, the coefficients $c_{n,i}$, $r_{n,i}$, $c_{\bar{n},i}$, $r_{\bar{n},i}$, A_k , B_k , p_{sk} , q_{sk} , $p_{\bar{s}\bar{k}}$, $q_{\bar{s}\bar{k}}$ do not depend on coordinates on the Hurwitz space.

Definition 4.6 *For two differentials $\omega^{(\alpha)}$, $\omega^{(\beta)}$ having singularities of the type (4.5.22), (4.5.23), we define the* **pairing** $\mathcal{F}[\ ,\ ]$ *as follows:*

$$\begin{aligned}
\mathcal{F}[\omega^{(\alpha)}\ ,\ \omega^{(\beta)}] = \sum_{i=0}^{m} \Bigg(& \sum_{n\geq 0} \frac{c^{(\alpha)}_{-n-2,i}}{n+1} c^{(\beta)}_{n,i} + c^{(\alpha)}_{-1,i} \text{v.p.} \int_{P_0}^{\infty^i} \omega^{(\beta)}_{(1,0)} - \text{v.p.} \int_{P_0}^{\infty^i} \sum_{n>0} r^{(\alpha)}_{n,i} \lambda^n \omega^{(\beta)}_{(1,0)} \\
& + \sum_{n\geq 0} \frac{c^{(\alpha)}_{-\bar{n}-2,i}}{n+1} c^{(\beta)}_{\bar{n},i} + c^{(\alpha)}_{-\bar{1},i} \text{v.p.} \int_{P_0}^{\infty^i} \omega^{(\beta)}_{(0,1)} - \text{v.p.} \int_{P_0}^{\infty^i} \sum_{n>0} r^{(\alpha)}_{\bar{n},i} \bar{\lambda}^n \omega^{(\beta)}_{(0,1)} \Bigg) \\
+ \frac{1}{2\pi i} \sum_{k=1}^{g} \Bigg(& - \oint_{a_k} q^{(\alpha)}_k(\lambda) \omega^{(\beta)}_{(1,0)} + \oint_{a_k} q^{(\alpha)}_{\bar{k}}(\bar{\lambda}) \omega^{(\beta)}_{(0,1)} + \oint_{b_k} p^{(\alpha)}_k(\lambda) \omega^{(\beta)}_{(1,0)} \\
& - \oint_{b_k} p^{(\alpha)}_{\bar{k}}(\bar{\lambda}) \omega^{(\beta)}_{(0,1)} + A^{(\alpha)}_k \oint_{b_k} \omega^{(\beta)}_{(1,0)} - B^{(\alpha)}_k \oint_{a_k} \omega^{(\beta)}_{(1,0)} \Bigg)\ . \qquad (4.5.24)
\end{aligned}$$

As before, P_0 is the marked point on $\mathcal{L}$ such that $\lambda(P_0) = 0$, and the cycles $\{a_k, b_k\}$ all pass through P_0.

From this definition one can see that if the first differential in the pairing is one of the primary differentials $\Phi_{\xi A}$ then this pairing gives the corresponding operation $\mathrm{I}_{\xi A}$ applied to the second differential:

$$\mathcal{F}[\Phi_{\xi A}, \omega] = \mathrm{I}_{\xi A}[\omega]\ . \qquad (4.5.25)$$

Theorem 4.10 *The pairing (4.5.24) is commutative for all primary differentials except for differentials Φ_{t^k} and Φ_{s^k}, $k = 1, \ldots, g$ which commute up to a constant:*

$$\mathcal{F}[\Phi_{s^k}, \Phi_{t^k}] = \mathcal{F}[\Phi_{t^k}, \Phi_{s^k}] - \frac{1}{2\pi i}\ . \qquad (4.5.26)$$

Proof. Due to the relation (4.5.25) we should compare the action of superpositions of operations $\mathrm{I}_{\xi A}\mathrm{I}_{\xi B}$ and $\mathrm{I}_{\xi B}\mathrm{I}_{\xi A}$ on the sum of Schiffer and Bergman kernels. This sum is only singular when the points P and Q coincide. Therefore among the operations I_{ω^i}, $\mathrm{I}_{\omega^{\bar{i}}}$, I_{r^k}, I_{u^k}, I_{s^k}, I_{t^k} those ones commute, being applied to $\Omega(P,Q) + B(P,Q)$, which are given by integrals over non-intersecting contours on the surface. In the set of contours used in the definition of the operations I_{ξ^A} , the only contours that intersect each other are the basis cycles a_k and b_k . A simple local calculation in a neighbourhood of the intersection point P_0 shows that the order of integration can be changed in the integral $\oint_{a_k}\oint_{b_k}\lambda(P)\Omega(P,Q)$ due to the assumption $\lambda(P_0)=0$. Therefore the only non-commuting operations, among the mentioned above, are I_{s^k} and I_{t^k} . The difference in (4.5.26) can be computed using formulas (4.3.3) for integrals of the bidifferential $W(P,Q)$ over a- and b-cycles.

By a similar reasoning one can see that operations of the type $\mathrm{I}_{t^{i;\alpha}}$, $\mathrm{I}_{t^{\bar{i};\alpha}}$, I_{v^i} and $\mathrm{I}_{v^{\bar{i}}}$ for $i=0,\dots,m$, $\alpha=1,\dots,n_i$ commute with the previous ones. They commute with each other due to the symmetry properties of the kernels. $\diamond$

Now we are in a position to prove Theorem 4.8, which gives the metric $\mathbf{ds}^2_\Phi$ in flat coordinates.

Proof of Theorem 4.8. For computation of the metric on vectors ∂_{ξ^A} we shall use the relation

$$\mathbf{ds}^2_\Phi(\partial_{\xi^A},\partial_{\xi^B})=\mathbf{e}\left(\mathcal{F}[\Phi_{\xi^A},\Phi_{\xi^B}]\right) \ , \tag{4.5.27}$$

which we prove first.

Using Lemma 4.4, we express the vectors ∂_{ξ^A} via canonical tangent vectors:

$$\partial_{\xi^A}=-\sum_{i=1}^{L}\left(\frac{\Phi_{\xi^A(1,0)}(P_i)}{\Phi_{(1,0)}(P_i)}\partial_{\lambda_i}+\frac{\Phi_{\xi^A(0,1)}(P_i)}{\Phi_{(0,1)}(P_i)}\partial_{\bar{\lambda}_i}\right) \ . \tag{4.5.28}$$

Therefore for the metric (4.5.5) we obtain:

$$\mathbf{ds}^2_\Phi(\partial_{\xi^A},\partial_{\xi^B})=\frac{1}{2}\sum_{i=1}^{L}\left(\Phi_{\xi^A(1,0)}(P_i)\Phi_{\xi^B(1,0)}(P_i)+\Phi_{\xi^A(0,1)}(P_i)\Phi_{\xi^B(0,1)}(P_i)\right) \ . \tag{4.5.29}$$

For computation of the right-hand side of (4.5.27) we note that, in the pairing of two primary differentials, only contribution of the second one depends on coordinates, therefore we have

$$\mathbf{e}\left(\mathcal{F}[\Phi_{\xi^A}, \Phi_{\xi^B}]\right) = \mathcal{F}[\Phi_{\xi^A}, \mathbf{e}\left(\Phi_{\xi^B}\right)] \ . \tag{4.5.30}$$

The action of the vector field $\mathbf{e}$ on primary differentials is provided by Lemma 4.3. From (4.5.25) we know that the pairing in the right side of (4.5.30) is just the operation I_{ξ^A} applied to $\mathbf{e}(\Phi_{\xi^B})$. Therefore in the right-hand side of (4.5.27) we have

$$\frac{1}{2}\sum_{i=1}^{L}\Big(\Phi_{\xi^B{}_{(1,0)}}(P_i)\mathrm{I}_{\xi^A}\left[\Omega(P,P_i)+B(\bar{P},P_i)\right]+\Phi_{\xi^B{}_{(0,1)}}(P_i)\mathrm{I}_{\xi^A}\left[B(P,\bar{P}_i)+\overline{\Omega(P,P_i)}\right]\Big)$$

$$= \frac{1}{2}\sum_{i=1}^{L}\left(\Phi_{\xi^A{}_{(1,0)}}(P_i)\Phi_{\xi^B{}_{(1,0)}}(P_i)+\Phi_{\xi^A{}_{(0,1)}}(P_i)\Phi_{\xi^B{}_{(0,1)}}(P_i)\right) \ . \tag{4.5.31}$$

Together with (4.5.29), this proves (4.5.27).

Now let us compute $\mathbf{ds}^2_\Phi\left(\partial_{r^i}, \partial_{\xi^A}\right)$. According to (4.5.27) we need to compute the action of the unit field $\mathbf{e}$ on the following quantity

$$\mathcal{F}[\Phi_{r^i}, \Phi_{\xi^A}] \equiv \mathrm{I}_{r^i}[\Phi_{\xi^A}] = -\oint_{a_i}\lambda(P)\Phi_{\xi^A{}_{(1,0)}}(P) - \oint_{a_i}\bar{\lambda}(P)\Phi_{\xi^A{}_{(0,1)}}(P) \ .$$

Let's again consider the biholomorphic map of the covering $\mathcal{L}_\lambda \to {\mathcal{L}_\lambda}^\delta$ performed by a simultaneous δ-shift ($\delta \in \mathbb{R}$) of the points on all sheets (see proof of Proposition 4.9). Since

$$\Phi^\delta_{\xi^A}(P^\delta) = \Phi_{\xi^A}(P) \tag{4.5.32}$$

we get

$$\mathbf{e}\left(\mathcal{F}[\Phi_{r^i}, \Phi_{\xi^A}]\right) = \frac{d}{d\delta}|_{\delta=0}\Big(-\oint_{a_i}(\lambda(P)+\delta)\Phi_{\xi^A{}_{(1,0)}}(P) - \oint_{a_i}(\bar{\lambda}(P)+\delta)\Phi_{\xi^A{}_{(0,1)}}(P)\Big)$$

$$= -\oint_{a_i}\Phi_{\xi^A}(P) = -\delta_{\xi^A, s^i} \ .$$

Therefore $\mathbf{ds^2_\Phi}\left(\partial_{r^i},\partial_{\xi^A}\right)=-\delta_{\xi^A,s^i}$. Analogously we prove that $\mathbf{ds^2_\Phi}\left(\partial_{u^i},\partial_{\xi^A}\right)=\delta_{\xi^A,t^i}$. To compute the remaining coefficients of the metric consider the operator $\mathcal{D}_\mathbf{e}=\frac{\partial}{\partial\lambda}+\mathbf{e}$. It annihilates any primary differential:

$$\mathcal{D}_\mathbf{e}\left(\Phi_{\xi^A}(P)\right)=0 \tag{4.5.33}$$

as can be proven by differentiation $d/d\delta|_{\delta=0}$ of the equality (4.5.32). Therefore, applying the operator $\mathcal{D}_\mathbf{e}$ to the expansion of the multivalued differential Ψ_{ξ^A} near the point ∞^i , we obtain the following relation for the corresponding (see (4.5.22)) coefficients $c_{l,i}$:

$$\mathbf{e}\left(\overset{\Phi_{\xi^A}}{c_{l,i}}\right)=\frac{l+1}{n_i+1}\overset{\Phi_{\xi^A}}{c_{l-n_i-1,i}}\,.$$

Therefore we have

$$\mathbf{ds^2_\Phi}\left(\partial_{t^{i;\alpha}},\partial_{\xi^A}\right)=\mathbf{e}\left(\mathrm{I}_{t^{i;\alpha}}[\Phi_{\xi^A}]\right)=\mathbf{e}\left(\frac{1}{\alpha}\overset{\Phi_{\xi^A}}{c_{\alpha-1,i}}\right)=\frac{1}{n_i+1}\delta_{\xi^A,t^{i;1+n_i-\alpha}}\,,$$

and $\mathbf{ds^2_\Phi}\left(\partial_{v^i},\partial_{\xi^A}\right)=\mathbf{e}\left(\overset{\Phi_{\xi^A}}{c_{n_i,i}}\right)=\overset{\Phi_{\xi^A}}{c_{-1,i}}=\delta_{\xi^A,\omega^i}$. Thus, we computed the entries of the matrix listed in the theorem and proved that they are the only nonzero ones. ◇

Formulas (4.5.28) and (4.5.29) yield the following expression for the tensor $\mathbf{c}=\mathbf{ds^2_\Phi}(\partial_{\xi^A}\cdot\partial_{\xi^B},\partial_{\xi^C})$ (compare with expression (4.4.20) for the tensor $\mathbf{c}$ of Dubrovin's construction):

$$\mathbf{c}(\partial_{\xi^A},\partial_{\xi^B},\partial_{\xi^C})=-\frac{1}{2}\sum_{i=1}^{L}\left(\frac{\Phi_{\xi^A(1,0)}(P_i)\Phi_{\xi^B(1,0)}(P_i)\Phi_{\xi^C(1,0)}(P_i)}{\Phi_{(1,0)}(P_i)}\right.$$
$$\left.+\frac{\Phi_{\xi^A(0,1)}(P_i)\Phi_{\xi^B(0,1)}(P_i)\Phi_{\xi^C(0,1)}(P_i)}{\Phi_{(0,1)}(P_i)}\right)\,. \tag{4.5.34}$$

The next theorem gives a prepotential of the Frobenius manifold, a function of flat coordinates $\{\xi^A\}$, which, according to Theorem 4.1, solves the WDVV system.

Theorem 4.11 *For each primary differential Φ consider the differential $\Psi(P)$ (4.5.14), multivalued on the surface $\mathcal{L}$. For the Frobenius structure defined on the manifold $\widehat{M}_{g;n_0,\dots,n_m}(\{\lambda_i;\bar{\lambda}_i\})$ by the metric $\mathbf{ds}^2_\Phi$ (4.5.5), multiplication law (4.5.1), and Euler field (4.5.3), the prepotential F_Φ is given by the pairing (4.5.24) of the differential Ψ with itself:*

$$F_\Phi = \frac{1}{2}\mathcal{F}[\Psi\ ,\ \Psi]\ . \tag{4.5.35}$$

The second order derivatives of the prepotential are given by

$$\partial_{\xi^A}\partial_{\xi^B}F_\Phi = \mathcal{F}[\Phi_{\xi^A}\ ,\ \Phi_{\xi^B}] - \frac{1}{4\pi i}\delta_{\xi^A,s^k}\delta_{\xi^B,t^k} + \frac{1}{4\pi i}\delta_{\xi^A,t^k}\delta_{\xi^B,s^k}\ , \tag{4.5.36}$$

where δ is the Kronecker symbol.

Proof. To prove that the function F_Φ is a prepotential we need to check that its third order derivatives coincide with the tensor $\mathbf{c}$ (4.5.34). We shall first prove that the second derivatives have the form (4.5.36) and then differentiate them with respect to a flat coordinate ξ^C .

The first differentiation of F_Φ with respect to a flat coordinate gives:

$$\partial_{\xi^A}F_\Phi = \frac{1}{2}\mathcal{F}[\Phi_{\xi^A},\Psi] + \frac{1}{2}\mathcal{F}[\Psi,\Phi_{\xi^A}]\ . \tag{4.5.37}$$

The first term in the right side of (4.5.37) equals $\frac{1}{2}\mathrm{I}_{\xi^A}[\Psi]$ (see (4.5.25)). Consider the second term. From expansions (4.5.15) of the multivalued differential Ψ and its integrals and transformations (4.5.16)-(4.5.18) over basis cycles we know that the coefficients for Ψ which enter formula (4.5.24) for the pairing are nothing but the flat coordinates of $\mathbf{ds}^2_\Phi$. Therefore, writing explicitly the singular part in expansions (4.5.15) and using also (4.5.16)

- (4.5.18), we have for the second term in (4.5.37):

$$
\begin{aligned}
\mathcal{F}[\Psi,\Phi_{\xi^A}] = \sum_{i=0}^{m} \Bigg(& v^i(1-\delta_{i0})\mathrm{I}_{v^i}[\Phi_{\xi^A}] + \sum_{\alpha=1}^{n_i} t^{i;\alpha}\mathrm{I}_{t^{i;\alpha}}[\Phi_{\xi^A}] + \omega^i \mathrm{I}_{\omega^i}[\Phi_{\xi^A}] \\
& + \delta_{\Phi,\Phi_{\omega^i}}\left(\frac{\mathrm{I}_{v^0}[\Phi_{\xi^A}]}{n_0+1} - \frac{\mathrm{I}_{v^i}[\Phi_{\xi^A}]}{n_i+1}\right) + \delta_{\Phi,\Phi_{\omega^i}}\mathrm{I}_{\omega^i}[\lambda\Phi_{\xi^A(1,0)}] - \frac{1}{2}\delta_{\Phi,\Phi_{v^i}}\mathrm{I}_{v^i}[\lambda\Phi_{\xi^A(1,0)}] \\
& - \sum_{\alpha=1}^{n_i}\frac{\alpha(n_i+1)}{n_i+1+\alpha}\delta_{\Phi,\Phi_{t^{i;\alpha}}}\mathrm{I}_{t^{i;\alpha}}[\lambda\Phi_{\xi^A(1,0)}]\Bigg) \\
+ \sum_{i=0}^{m} \Bigg(& v^{\bar{i}}(1-\delta_{i0})\mathrm{I}_{v^{\bar{i}}}[\Phi_{\xi^A}] + \sum_{\alpha=1}^{n_i} t^{\overline{i;\alpha}}\mathrm{I}_{t^{\overline{i;\alpha}}}[\Phi_{\xi^A}] + \omega^{\bar{i}} \mathrm{I}_{\omega^{\bar{i}}}[\Phi_{\xi^A}] \\
& + \delta_{\Phi,\Phi_{\omega^{\bar{i}}}}\left(\frac{\mathrm{I}_{v^{\bar{0}}}[\Phi_{\xi^A}]}{n_0+1} - \frac{\mathrm{I}_{v^{\bar{i}}}[\Phi_{\xi^A}]}{n_i+1}\right) + \delta_{\Phi,\Phi_{\omega^{\bar{i}}}}\mathrm{I}_{\omega^{\bar{i}}}[\bar{\lambda}\Phi_{\xi^A(0,1)}] - \frac{1}{2}\delta_{\Phi,\Phi_{v^{\bar{i}}}}\mathrm{I}_{v^{\bar{i}}}[\bar{\lambda}\Phi_{\xi^A(0,1)}] \\
& - \sum_{\alpha=1}^{n_i}\frac{\alpha(n_i+1)}{n_i+1+\alpha}\delta_{\Phi,\Phi_{t^{i;\alpha}}}\mathrm{I}_{t^{\overline{i;\alpha}}}[\bar{\lambda}\Phi_{\xi^A(0,1)}]\Bigg) \\
+ \sum_{k=1} \Bigg(& r^k\mathrm{I}_{r^k}[\Phi_{\xi^A}] + u^k\mathrm{I}_{u^k}[\Phi_{\xi^A}] + s^k\mathrm{I}_{s^k}[\Phi_{\xi^A}] + t^k\mathrm{I}_{t^k}[\Phi_{\xi^A}] + \frac{1}{2}\delta_{\Phi,\Phi_{r^k}}\mathrm{I}_{r^k}[\lambda\Phi_{\xi^A(1,0)} + \bar{\lambda}\Phi_{\xi^A(0,1)}] \\
& + \frac{1}{2}\delta_{\Phi,\Phi_{u^k}}\mathrm{I}_{u^k}[\lambda\Phi_{\xi^A(1,0)} + \bar{\lambda}\Phi_{\xi^A(0,1)}] + \frac{1}{2\pi i}\delta_{\Phi,\Phi_{s^k}}\oint_{b_k}\bar{\lambda}\Phi_{\xi^A(0,1)} + \frac{1}{2\pi i}\delta_{\Phi,\Phi_{t^k}}\oint_{a_k}\bar{\lambda}\Phi_{\xi^A(0,1)}\Bigg) .
\end{aligned}
\tag{4.5.38}
$$

Here the Kronecker symbol, for example, $\delta_{\Phi,\Phi_{\omega^i}}$ is equal to one if the primary differential Φ (which defines the metric $\mathbf{ds}^2_\Phi$ and the differential Ψ) is Φ_{ω^i} .

Suppose the primary differential Φ_{ξ^A} is of the types 1, 3, 5, i.e. suppose $\xi^A \in \{t^{i;\alpha}, v^i, \omega^i\}$. Then $\Phi_{\xi^A}(P) = \mathrm{I}_{\xi^A}[\Omega(P,Q) + B(\bar{P},Q)]$. In this case the operation I_{ξ^A} commutes with all the others (see Theorem 4.10). Therefore we can rewrite (4.5.38) as an action of I_{ξ^A} on some differential which depends on $\lambda(Q)$ only (and does not depend on $\bar{\lambda}(Q)$): $\mathcal{F}[\Psi,\Phi_{\xi^A}] = \mathrm{I}_{\xi^A}[\tilde{\Psi}_{(1,0)}(Q)]$. Analogously, we find that for primary differentials of the types 2, 4, 6 , when $\xi^A \in \{t^{\overline{i;\alpha}},\ v^{\bar{i}},\ \omega^{\bar{i}}\}$, the right-hand side in (4.5.38) is equal to the action of I_{ξ^A} on a differential depending only on $\bar{\lambda}(Q)$, i.e. $\mathcal{F}[\Psi,\Phi_{\xi^A}] = \mathrm{I}_{\xi^A}[\tilde{\Psi}_{(0,1)}(Q)]$. Examining the properties of the differential $\tilde{\Psi}_{(1,0)}(Q) + \tilde{\Psi}_{(0,1)}(Q)$ such as singularities, behaviour under analytic continuation along cycles $\{a_k, b_k\}$ and integrals over these cycles, we obtain with the help of Lemma 4.2: $\Psi(Q) = \tilde{\Psi}_{(1,0)}(Q) + \tilde{\Psi}_{(0,1)}(Q)$, and therefore $\Psi_{(1,0)}(Q) = \tilde{\Psi}_{(1,0)}(Q)$, $\Psi_{(0,1)}(Q) = \tilde{\Psi}_{(0,1)}(Q)$. Hence, for primary differentials of the types $1-6$ we have

$$\mathcal{F}[\Psi, \Phi_{\xi^A}] = \mathrm{I}_{\xi^A}[\Psi] \, . \tag{4.5.39}$$

Similarly, for differentials Φ_{r^k} and Φ_{u^k} , we get

$$\mathcal{F}[\Psi, \Phi_{r^k}] = -\oint_{a_k} \lambda(Q)\tilde{\Psi}_{(1,0)}(Q) - \oint_{a_k} \bar{\lambda}(Q)\tilde{\Psi}_{(0,1)}(Q) \, ,$$
$$\mathcal{F}[\Psi, \Phi_{u^k}] = -\oint_{b_k} \lambda(Q)\tilde{\Psi}_{(1,0)}(Q) - \oint_{b_k} \bar{\lambda}(Q)\tilde{\Psi}_{(0,1)}(Q) \, ,$$

which proves that (4.5.39) also holds for $\xi^A \in \{r^k,\ u^k\}$.

Formula (4.5.39) changes for the primary differentials Φ_{s^k} and Φ_{t^k} : the additional terms appear due to non-commutativity of the corresponding operations (Theorem 4.10):

$$\mathcal{F}[\Psi, \Phi_{s^k}] = \mathrm{I}_{s^k}[\Psi] - \frac{t^k}{2\pi i} \, ; \qquad \mathcal{F}[\Psi, \Phi_{t^k}] = \mathrm{I}_{t^k}[\Psi] + \frac{s^k}{2\pi i} \, .$$

Coming back to the differentiation (4.5.37) of the function F_Φ , we have

$$\partial_{\xi^A} F_\Phi = \mathcal{F}[\Phi_{\xi^A}, \Psi] - \delta_{\xi^A, s^k}\frac{t^k}{4\pi i} + \delta_{\xi^A, t^k}\frac{s^k}{4\pi i} \, . \tag{4.5.40}$$

Note that the contribution of the primary differential Φ_{ξ^A} into the pairing $\mathcal{F}[\Phi_{\xi^A}, \Psi]$ does not depend on coordinates. Therefore, by virtue of Theorem 4.9, the differentiation of (4.5.40) with respect to ξ^B gives the expression (4.5.36) for second derivatives of the function F_Φ .

To find third derivatives of F_Φ we differentiate (4.5.36) with respect to a flat coordinate ξ^C :

$$\partial_{\xi^C}\partial_{\xi^B}\partial_{\xi^A} F_\Phi = \mathcal{F}[\Phi_{\xi^A}\ ,\ \partial_{\xi^C}\Phi_{\xi^B}] = \mathrm{I}_{\xi^A}[\partial_{\xi^C}\Phi_{\xi^B}] \, . \tag{4.5.41}$$

Then we express the vector ∂_{ξ^C} via canonical tangent vectors $\{\partial_{\lambda_i}\}$ as in (4.5.28) and use formulas from Lemma 4.3 for derivatives of primary differentials. Analogously to the computation (4.5.31) we find that derivatives (4.5.41) are given by the right-hand side of (4.5.34), i.e. equal to the 3-tensor $\mathbf{c}(\partial_{\xi^C}, \partial_{\xi^B}, \partial_{\xi^A})$. $\diamond$

Thus, by proving that the function F_Φ given by (4.5.35) is a prepotential (see Definition 4.4) we completed the construction of Frobenius manifold corresponding to the primary differential Φ on the space $\widehat{M}_{g;n_0,\dots,n_m}$. Let us denote this manifold by $\widehat{M}^\Phi = \widehat{M}^\Phi_{g;n_0,\dots,n_m}$.

4.5.4 Quasihomogeneity

Now we shall show that the prepotential F_Φ (4.5.35) is a quasihomogeneous function of flat coordinates (see (4.2.1)). According to Theorem 4.1, the prepotential satisfies

$$E(F_\Phi) = \nu_F F_\Phi + \text{quadratic terms} . \tag{4.5.42}$$

In the next proposition we prove that the vector field E has the form (4.2.2), i.e.

$$E = \sum_A \nu_A \xi^A \partial_{\xi^A} , \tag{4.5.43}$$

and compute the coefficients $\{\nu_A\}$.

Proposition 4.11 *In flat coordinates $\{\xi^A\}$ of the metric $\mathbf{ds}^2_\Phi$, the Euler vector field (4.5.3) has the form (4.5.43) (and therefore is covariantly linear) with coefficients $\{\nu_A\}$ depending on the choice of a primary differential Φ as follows:*

- *if* $\Phi = \Phi_{t^{i_o;\alpha}}$ *or* $\Phi = \Phi_{\overline{t^{i_o;\alpha}}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}\left(t^{i;\alpha}\partial_{t^{i;\alpha}} + t^{\overline{i;\alpha}}\partial_{\overline{t^{i;\alpha}}}\right)\left(1+\frac{\alpha}{n_{i_o}+1}-\frac{\alpha}{n_i+1}\right)$$
$$+\sum_{i=1}^{m}\left(\frac{\alpha}{n_{i_o}+1}(v^i\partial_{v^i}+v^{\bar{i}}\partial_{v^{\bar{i}}})+(1+\frac{\alpha}{n_{i_o}+1})(\omega^i\partial_{\omega^i}+\omega^{\bar{i}}\partial_{\omega^{\bar{i}}})\right)$$
$$+\sum_{k=1}^{g}\left(\frac{\alpha}{n_{i_o}+1}(r^k\partial_{r^k}+u^k\partial_{u^k})+(1+\frac{\alpha}{n_{i_o}+1})(s^k\partial_{s^k}+t^k\partial_{t^k})\right)$$

- *if* $\Phi = \Phi_{v^{i_o}}$, $\Phi = \Phi_{v^{\bar{i}_o}}$, $\Phi = \Phi_{r^{k_o}}$ *or* $\Phi = \Phi_{u^{k_o}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}(2-\frac{\alpha}{n_i+1})(t^{i;\alpha}\partial_{t^{i;\alpha}} + t^{\overline{i;\alpha}}\partial_{\overline{t^{i;\alpha}}})+\sum_{i=1}^{m}\left(v^i\partial_{v^i}+v^{\bar{i}}\partial_{v^{\bar{i}}}+2(\omega^i\partial_{\omega^i}+\omega^{\bar{i}}\partial_{\omega^{\bar{i}}})t\right)$$
$$+\sum_{k=1}^{g}\left(r^k\partial_{r^k}+u^k\partial_{u^k}+2(s^k\partial_{s^k}+t^k\partial_{t^k})\right)$$

- *if* $\Phi = \Phi_{\omega^{i_o}}$, $\Phi = \Phi_{\omega^{\bar{i}_o}}$, $\Phi = \Phi_{s^{k_o}}$ *or* $\Phi = \Phi_{t^{k_o}}$ *then*

$$E = \sum_{i=0}^{m}\sum_{\alpha=1}^{n_i}(1-\frac{\alpha}{n_i+1})(t^{i;\alpha}\partial_{t^{i;\alpha}} + t^{\overline{i;\alpha}}\partial_{\overline{t^{i;\alpha}}})+\sum_{i=1}^{m}(\omega^i\partial_{\omega^i}+\omega^{\bar{i}}\partial_{\omega^{\bar{i}}})+\sum_{k=1}^{g}(s^k\partial_{s^k}+t^k\partial_{t^k})\ .$$

Proof. Let us compute the action of the Euler vector field on a flat coordinate ξ^A . Consider again the biholomorphic map $\mathcal{L}_\lambda \to \mathcal{L}_\lambda^\epsilon$ defined by the transformation $P \mapsto P^\epsilon$ on $\mathcal{L}$ such that $\lambda(P^\epsilon) = \lambda(P)(1+\epsilon)$, $\epsilon \in \mathbb{R}$, performed on every sheet of the covering $\mathcal{L}_\lambda$. Since the kernels Ω and B are invariant under this map, the primary differentials transform as follows:

$$\text{for } \Phi = \Phi_{t^{i;\alpha}} \text{ or } \Phi = \Phi_{\overline{t^{i;\alpha}}} : \qquad \Phi^\epsilon(P^\epsilon) = (1+\epsilon)^{\frac{\alpha}{n_i+1}}\Phi(P)$$
$$\text{for } \Phi = \Phi_{v^i},\ \Phi = \Phi_{v^{\bar{i}}},\ \Phi = \Phi_{r^k} \text{ or } \Phi = \Phi_{u^k} : \qquad \Phi^\epsilon(P^\epsilon) = (1+\epsilon)\Phi(P)$$
$$\text{for } \Phi = \Phi_{\omega^i},\ \Phi = \Phi_{\omega^{\bar{i}}},\ \Phi = \Phi_{s^k} \text{ or } \Phi = \Phi_{t^k} : \qquad \Phi^\epsilon(P^\epsilon) = \Phi(P)\ ,$$

where Φ^ϵ is the corresponding differential on the covering $\mathcal{L}_\lambda^\epsilon$.

Let us choose, for example, the primary differential $\Phi_{t^{i_o;\alpha}}$. Flat coordinates of the metric $\mathbf{ds}^2_{\Phi_{t^{i_o;\alpha}}}$ are functions of $\{\lambda_j\}$ and $\{\bar\lambda_j\}$ only. If we consider corresponding functions on $\mathcal{L}^\epsilon$ and differentiate them with respect to ϵ at $\epsilon = 0$, we get the action of the vector field E (4.5.3) on the flat coordinates:

$$E(t^{i;\alpha}) = \frac{d}{d\epsilon}|_{\epsilon=0} \frac{n_i+1}{\alpha - n_i - 1} \operatorname*{res}_{\infty^i} (\lambda(P^\epsilon))^{\frac{n_i+1-\alpha}{n_i+1}} \Phi^\epsilon_{t^{i_o;\alpha}(1,0)}(P^\epsilon)$$

$$= \frac{d}{d\epsilon}|_{\epsilon=0}(1+\epsilon)^{\frac{n_i+1-\alpha}{n_i+1}+\frac{\alpha}{n_{i_o}+1}} t^{i;\alpha} = (1 - \frac{\alpha}{n_i+1} + \frac{\alpha}{n_{i_o}+1}) t^{i;\alpha}.$$

Therefore the vector field E depends on the coordinate $t^{i;\alpha}$ as $E = (1 - \frac{\alpha}{n_i+1} + \frac{\alpha}{n_{i_o}+1}) t^{i;\alpha} \partial_{t^{i;\alpha}} + \ldots$. Similarly we compute the dependence on the other flat coordinates. $\diamond$

The action (4.5.42) of the Euler field (4.5.43) on the prepotential F_Φ is equivalent to the condition of quasihomogeneity for F_Φ , i.e. $F_\Phi(\kappa^{\nu_1}\xi^1, \ldots, \kappa^{\nu_{2L}}\xi^{2L}) = \kappa^{\nu_F} F_\Phi(\xi^1, \ldots, \xi^{2L}) +$ *quadratic terms* with the coefficients of quasihomogeneity $\{\nu_A\}$ computed in Proposition 4.11. As for the coefficient ν_F , the proof of Theorem 4.1 implies that $\nu_F = 3 - \nu$, where the charge ν of a Frobenius manifold was computed in Proposition 4.7. Thus, we have

for $\Phi = \Phi_{t^{i;\alpha}}$ or $\Phi = \Phi_{\overline{t^{i;\alpha}}}$: $\quad \nu = 1 - \dfrac{2\alpha}{n_i+1} \quad \nu_F = \dfrac{2\alpha}{n_i+1} + 2$

for $\Phi = \Phi_{v^i}$, $\Phi = \Phi_{v^{\bar i}}$, $\Phi = \Phi_{r^k}$ or $\Phi = \Phi_{u^k}$: $\quad \nu = -1 \quad \nu_F = 4$

for $\Phi = \Phi_{\omega^i}$, $\Phi = \Phi_{\omega^{\bar i}}$, $\Phi = \Phi_{s^k}$ or $\Phi = \Phi_{t^k}$: $\quad \nu = 1 \quad \nu_F = 2$.

Remark 4.6 The described construction also holds for the differential Φ being a linear combination of the primary differentials which correspond to the same charge ν . In other words, the differential Φ which defines a Frobenius structure can be one of the following:

1. $\Phi = c_{i;\alpha}\Phi_{t^{i;\alpha}} + c_{\overline{i;\alpha}}\Phi_{t^{\overline{i;\alpha}}}$ for some pair $(i;\alpha): \; i \in \{0,\dots,m\}, \; \alpha \in \{1,\dots,n_i-1\}$,

2. $\Phi = \sum_{i=1}^{m}\left(\kappa_i\Phi_{v^i} + \kappa_{\bar{i}}\Phi_{v^{\bar{i}}}\right) + \sum_{k=1}^{g}\left(\sigma_k\Phi_{r^k} + \rho_k\Phi_{u^k}\right)$,

3. $\Phi = \sum_{i=1}^{m}\left(\kappa_i\Phi_{\omega^i} + \kappa_{\bar{i}}\Phi_{\omega^{\bar{i}}}\right) + \sum_{k=1}^{g}\left(\sigma_k\Phi_{s^k} + \rho_k\Phi_{t^k}\right)$,

where the coefficients do not depend on a point of the Hurwitz space. The unit vector fields for the structures defined by these combinations, respectively, are given by:

1. $\mathbf{e} = -c_{i;\alpha}\partial_{t^{i;\alpha}} - c_{\overline{i;\alpha}}\partial_{t^{\overline{i;\alpha}}}$ for some pair $(i;\alpha): \; i = 0,\dots,m, \; \alpha = 1,\dots,n_i-1$

2. $\mathbf{e} = -\sum_{i=1}^{m}\left(\kappa_i\partial_{v^i} + \kappa_{\bar{i}}\partial_{v^{\bar{i}}}\right) - \sum_{k=1}^{g}\left(\sigma_k\partial_{r^k} + \rho_k\partial_{u^k}\right)$

3. $\mathbf{e} = -\sum_{i=1}^{m}\left(\kappa_i\partial_{\omega^i} + \kappa_{\bar{i}}\partial_{\omega^{\bar{i}}}\right) - \sum_{k=1}^{g}\left(\sigma_k\partial_{s^k} + \rho_k\partial_{t^k}\right)$.

In each case, by a linear change of variables, the field $\mathbf{e}$ can be made equal to ∂_{ξ^1} for some new variable ξ^1. This change of variables does not affect the quasihomogeneity of the prepotential since the flat coordinates which enter each of the three combinations have equal coefficients of quasihomogeneity (see Proposition 4.11).

4.6 G-function of Hurwitz Frobenius manifolds

The G-function is a solution to the Getzler system of linear differential equations, which was derived in [26] (see also [17]). The system is defined on an arbitrary semisimple Frobenius manifold M.

It was proven in [17] that the Getzler system has unique, up to an additive constant, solution G which satisfies the quasihomogeneity condition

$$E(G) = -\frac{1}{4}\sum_{A=1}^{n}\left(1-\nu_A-\frac{\nu}{2}\right)^2 + \frac{\nu n}{48} ,$$

with a constant in the left side: ν is the charge, n is the dimension of the Frobenius manifold; $\{\nu_A\}$ are the quasihomogeneity coefficients (4.2.1). In [17] the following formula (which proves the conjecture of A. Givental [30]) for this quasihomogeneous solution was derived:

$$G = \log \frac{\tau_I}{J^{1/24}} , \tag{4.6.1}$$

where J is the Jacobian of transformation from canonical to the flat coordinates, $J = \det\left(\frac{\partial t^\alpha}{\partial \lambda_i}\right)$; and τ_I is the isomonodromic tau-function of the Frobenius manifold defined by

$$\frac{\partial \log \tau_I}{\partial \lambda_i} = H_i := \frac{1}{2}\sum_{j\neq i, j=1}^{n} \beta_{ij}^2(\lambda_i - \lambda_j) , \qquad i = 1, \dots, n . \tag{4.6.2}$$

The function G (4.6.1) for the Frobenius manifold $\widehat{M}_{1;1}^{\phi_s}$ was computed in [17]. In [44, 45] expression (4.6.1) was computed for Dubrovin's Frobenius structures on Hurwitz spaces in arbitrary genus. Theorem 4.12 below summarizes the main results of papers [44] and [45].

Denote by S the following term in the asymptotics of the bidifferential $W(P,Q)$ (4.3.2) near the diagonal $P \sim Q$:

$$W(P,Q) \underset{Q\sim P}{=} \left(\frac{1}{(x(P)-x(Q))^2} + S(x(P)) + o(1)\right) dx(P)dx(Q)$$

($6S(x(P))$ is called the Bergman projective connection [23]). By S_i we denote the value of S at the ramification point P_i taken with respect to the local parameter $x_i(P) = \sqrt{\lambda - \lambda_i}$:

$$S_i = S(x_i)|_{x_i=0} . \tag{4.6.3}$$

Since the singular part of the bidifferential W in a neighbourhood of the point P_i does not depend on coordinates $\{\lambda_j\}$, the Rauch variational formulas (4.3.4) imply

$$\frac{\partial S_i}{\partial \lambda_j} = \frac{1}{2} W^2(P_i, P_j) \; .$$

The symmetry of this expression provides compatibility for the following system of differential equations which defines the Bergman tau-function τ_W :

$$\frac{\partial \log \tau_W}{\partial \lambda_i} = -\frac{1}{2} S_i \; , \qquad i = 1, \ldots, L \; .$$

Theorem 4.12 *The isomonodromic tau-function $\tau_{\rm I}$ (4.6.2) for a holomorphic Frobenius structure $\widehat{M}^{\phi}$ is related to the Bergman tau-function τ_W as follows ([45]):*

$$\tau_{\rm I} = (\tau_W)^{-\frac{1}{2}} \; , \tag{4.6.4}$$

where τ_W is given by the following expression independent of the points P and Q ([44]):

$$\tau_W = \mathcal{Q}^{2/3} \prod_{k,l=1 \; k<l}^{L+m+1} [E(D_k, D_l)]^{d_k d_l / 6} \tag{4.6.5}$$

and

- *$\mathcal{Q}$ is given by*

$$\mathcal{Q} = [d\lambda(P)]^{\frac{g-1}{2}} \, \mathcal{C}(P) \prod_{k=1}^{L+m+1} [E(P, D_k)]^{\frac{(1-g)d_k}{2}}$$

where $\mathcal{C}(P)$ is the following multivalued $g(1-g)/2$-differential on $\mathcal{L}$

$$\mathcal{C}(P) = \frac{1}{\det_{1 \le \alpha, \beta \le g} \|\omega_\beta^{(\alpha-1)}(P)\|} \sum_{\alpha_1, \ldots, \alpha_g = 1}^{g} \frac{\partial^g \theta(K^P)}{\partial z_{\alpha_1} \ldots \partial z_{\alpha_g}} \omega_{\alpha_1}(P) \ldots \omega_{\alpha_g}(P)$$

- *$\sum_{k=1}^{L+m+1} d_k D_k$ is the divisor $(d\lambda)$ of the differential $d\lambda(P)$, i.e. $D_l = P_l$, $d_l = 1$ for $l = 1, \ldots, L$ and $D_{L+i+1} = \infty^i$, $d_{L+i+1} = -(n_i + 1)$, $i = 0, \ldots, m$. As before, we evaluate a differential at the points of the divisor $(d\lambda)$ with respect to the standard local parameters: $x_j = \sqrt{\lambda - \lambda_j}$ for $j = 1, \ldots, L$ and $x_{L+1+i} = \lambda^{-1/(n_i+1)}$ for $i = 0, \ldots, m$*

- $\theta(z|\mathbb{B})$, $z \in \mathbb{C}^g$ *is the theta-function;* $E(P,Q)$ *is the prime form;* $E(D_k, P)$ *stands for* $E(Q,P)\sqrt{dx_k(Q)}|_{Q=D_k}$

- K^P *is the vector of Riemann constants; the fundamental domain* $\widehat{\mathcal{L}}$ *is chosen so that the Abel map of the divisor* $(d\lambda)$ *is given by* $\mathcal{A}((d\lambda)) = -2K^P$.

4.6.1 G-function for manifolds $\widehat{M}^\phi$

Theorem 4.12 gives the numerator of expression (4.6.1) for the G-function of holomorphic Frobenius structures $\widehat{M}^\phi$ on Hurwitz spaces described in Section 4.4. For the denominator we have (see [17], [45])

$$J = \frac{1}{2^{L/2}} \prod_{i=1}^{L} \phi(P_i) ,$$

where ϕ is the primary differential from the list of Theorem 4.2 which corresponds to the Frobenius structure $\widehat{M}^\phi$.

Summarizing above formulas, we get the following expression for the G-function of the Frobenius manifold $\widehat{M}^\phi$:

$$G = -\frac{1}{2}\log \tau_W - \frac{1}{24}\log \prod_{i=1}^{L} \phi(P_i) + \text{const} , \qquad (4.6.6)$$

τ_W is given by (4.6.5).

4.6.2 G-function for "real doubles" $\widehat{M}^\Phi$

For the Frobenius structures with canonical coordinates $\{\lambda_1, \dots, \lambda_L; \bar{\lambda}_1, \dots, \bar{\lambda}_L\}$, corresponding to the primary differentials Φ from Section 4.5, the Jacobian of transformation between canonical and flat coordinates is given by

$$J = \det\left(\frac{\partial \xi^A}{\partial \lambda_i} \Big| \frac{\partial \xi^A}{\partial \bar{\lambda}_i}\right) = \frac{1}{2^L} \prod_{i=1}^{L} \Phi_{(1,0)}(P_i)\Phi_{(0,1)}(P_i) . \qquad (4.6.7)$$

The definition (4.6.2) of the isomonodromic tau-function in this case becomes:

$$\begin{aligned}\frac{\partial \log \tau_{\mathrm{I}}}{\partial \lambda_i} &= H_i := \frac{1}{2}\sum_{j\neq i, j=1}^{L} \beta_{ij}^2(\lambda_i-\lambda_j) + \frac{1}{2}\sum_{j=1}^{L}\beta_{i\bar{j}}^2(\lambda_i-\bar{\lambda}_j)\\ \frac{\partial \log \tau_{\mathrm{I}}}{\partial \bar{\lambda}_i} &= H_{\bar{i}} := \frac{1}{2}\sum_{j=1}^{L}\beta_{\bar{i}j}^2(\bar{\lambda}_i-\lambda_j) + \frac{1}{2}\sum_{j\neq i,j=1}^{L}\beta_{\bar{i}\bar{j}}^2(\bar{\lambda}_i-\bar{\lambda}_j)\ .\end{aligned} \tag{4.6.8}$$

Analogously to relation (4.6.4) one can prove (see [45] and Proposition 4.12 below) that the function τ_{I} is $-1/2$ power of the function τ_Ω , which is defined by the Schiffer kernel $\Omega(P,Q)$ (4.3.6) as follows. The asymptotics of the kernel $\Omega(P,Q)$ near the diagonal is

$$\Omega(P,Q) \underset{Q\sim P}{=} \left(\frac{1}{(x(P)-x(Q))^2} + S_\Omega(x(P)) + o(1)\right) dx(P)dx(Q)\ .$$

Denote by Ω_i the evaluation of the term $S_\Omega(x)$ at the ramification point P_i with respect to the local parameter $x_i = \sqrt{\lambda-\lambda_i}$:

$$\Omega_i = (S_\Omega(x_i))\,|_{x_i=0} = S_i + \Sigma_i\ ,$$

where S_i is the same as in (4.6.3) and Σ_i is given by $\Sigma_i = -\pi\sum_{k,l=1}^{g}(\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P_i)\omega_l(P_i)$. The differentiation formulas (4.3.11) for the kernels Ω and B imply

$$\begin{aligned}\frac{\partial\Omega_i}{\partial\lambda_j} &= \frac{1}{2}\Omega^2(P_i,P_j) = 2\beta_{ij}^2\ , & \frac{\partial\Omega_i}{\partial\bar{\lambda}_j} &= \frac{1}{2}B^2(\bar{P}_j,P_i) = 2\beta_{i\bar{j}}^2\ ,\\ \frac{\partial\overline{\Omega}_i}{\partial\lambda_j} &= \frac{1}{2}B^2(\bar{P}_i,P_j) = 2\beta_{\bar{i}j}^2\ , & \frac{\partial\overline{\Omega}_i}{\partial\bar{\lambda}_j} &= \frac{1}{2}\overline{\Omega^2(P_i,P_j)} = 2\beta_{\bar{i}\bar{j}}^2\ ,\end{aligned} \tag{4.6.9}$$

which allows the following definition of the tau-function τ_Ω :

$$\frac{\partial\log\tau_\Omega}{\partial\lambda_i} = -\frac{1}{2}\Omega_i\ , \qquad \frac{\partial\log\tau_\Omega}{\partial\bar{\lambda}_i} = -\frac{1}{2}\overline{\Omega}_i\ . \tag{4.6.10}$$

From Rauch variational formulas (4.3.12) we find

$$\frac{\partial\log\det(\mathrm{Im}\mathbb{B})}{\partial\lambda_i} = -\frac{1}{2}\Sigma_i\ , \qquad \frac{\partial\log\det(\mathrm{Im}\mathbb{B})}{\partial\bar{\lambda}_i} = -\frac{1}{2}\overline{\Sigma}_i\ ,$$

and therefore

$$\tau_\Omega = \mathrm{const}\ |\tau_W|^2\det(\mathrm{Im}\mathbb{B})\ . \tag{4.6.11}$$

Remark 4.7 This tau-function coincides with an appropriately regularized ratio of determinant of Laplacian on $\mathcal{L}$ and the volume of $\mathcal{L}$ in the singular metric $|d\lambda|^2$ (see [11, 44, 69]).

Now we are able to compute the function $\tau_{\rm I}$ (4.6.8) by proving the following proposition.

Proposition 4.12 *The isomonodromic tau-function $\tau_{\rm I}$ for a Frobenius structure with canonical coordinates $\{\lambda_1, \dots, \lambda_L; \bar\lambda_1, \dots, \bar\lambda_L\}$ on the Hurwitz space is related to the function τ_Ω (4.6.10) by*

$$\tau_{\rm I} = (\tau_\Omega)^{-1/2}\,. \tag{4.6.12}$$

Proof. Using the relation (4.6.9) between derivatives of Ω_i and rotation coefficients β_{ij} , we write for the Hamiltonians H_i (4.6.8):

$$H_i = \frac{1}{4}\lambda_i \left(\sum_{j\neq i, j=1}^{L} \partial_{\lambda_j}\Omega_i + \sum_{j=1}^{L} \partial_{\bar\lambda_j}\Omega_i \right) - \frac{1}{4}\sum_{j\neq i, j=1}^{L} \lambda_j \partial_{\lambda_j}\Omega_i - \frac{1}{4}\sum_{j=1}^{L} \bar\lambda_j \partial_{\bar\lambda_j}\Omega_i\,. \tag{4.6.13}$$

For the quantities Ω_i one can prove the relations

$$\sum_{j=1}^{L}\left(\frac{\partial}{\partial\lambda_j} + \frac{\partial}{\partial\bar\lambda_j}\right)\Omega_i = 0\,, \qquad \sum_{j=1}^{L}\left(\lambda_j\frac{\partial}{\partial\lambda_j} + \bar\lambda_j\frac{\partial}{\partial\bar\lambda_j}\right)\Omega_i = -\Omega_i\,. \tag{4.6.14}$$

To prove (4.6.14) we use the invariance of the Schiffer kernel $\Omega(P,Q)$ under two biholomorphic maps of the Riemann surface $\mathcal{L} \mapsto \mathcal{L}^\delta$ and $\mathcal{L} \mapsto \mathcal{L}^\epsilon$ given by transformations $\lambda \to \lambda + \delta$ and $\lambda \to \lambda(1+\epsilon)$ performed simultaneously on all sheets of the covering $\mathcal{L}_\lambda$ (see proofs of Propositions 4.2 and 4.3).

Substitution of (4.6.14) into (4.6.13) yields

$$H_i = -\frac{1}{4}\sum_{j=1}^{L}\left(\lambda_j\partial_{\lambda_j}\Omega_i + \bar\lambda_j\partial_{\bar\lambda_j}\Omega_i\right) = \frac{1}{4}\Omega_i\,.$$

Similarly, we get for $H_{\bar i}$ the relation: $H_{\bar i} = \frac{1}{4}\overline{\Omega}_i$. ◇

Formulas (4.6.7), (4.6.11) and (4.6.12) give the expression for the function G (4.6.1), i.e. we have proven the following theorem.

Theorem 4.13 *The G-function of the Frobenius manifold $\widehat{M}^{\Phi}$ is given by*

$$G = -\frac{1}{2}\log\left\{\, |\tau_W|^2 \det(\mathrm{Im}\mathbb{B})\right\} - \frac{1}{24}\log\left\{\prod_{i=1}^{L}\Phi_{(1,0)}(P_i)\Phi_{(0,1)}(P_i)\right\} + \mathrm{const}\,, \qquad (4.6.15)$$

where the Bergman tau-function τ_W is given by (4.6.5).

4.7 Examples in genus one

Since the described construction in the case of genus zero does not lead to new structures, the simplest examples we can compute are the Frobenius structures in genus one. The simplest Hurwitz space in genus one is $M_{1;1}$. We shall compute the prepotentials of Frobenius manifolds $\widehat{M}_{1;1}^{\phi_s}$ and $\widehat{M}_{1;1}^{\Phi_s}$, $\widehat{M}_{1;1}^{\Phi_t}$, $\widehat{M}_{1;1}^{\Phi_s+\sigma\Phi_t}$ (for a nonzero constant $\sigma \in \mathbb{C}$) given by formulas (4.4.19) and (4.5.35), respectively, and the corresponding G-functions (4.6.6) and (4.6.15).

The Riemann surface of genus one can be represented as a quotient $\mathcal{L} = \mathbb{C}/\{2\omega, 2\omega'\}$, where $\omega, \omega' \in \mathbb{C}$. The space $M_{1;1}$ consists of the genus one two-fold coverings of $\mathbb{C}P^1$ with simple branch points, one of them being at infinity. These coverings can be defined by the function

$$\lambda(\varsigma) = \wp(\varsigma) + c\,, \qquad (4.7.1)$$

where $\wp$ is the Weierstrass elliptic function $\wp : \mathcal{L} \to \mathbb{C}P^1$ and c is a constant with respect to ς .

We denote by $\lambda_1, \lambda_2, \lambda_3$ the finite branch points of the coverings (4.7.1) and consider them as local coordinates on the space $\widehat{M}_{1;1}$.

4.7.1 Holomorphic Frobenius structure $\widehat{M}_{1;1}^{\phi_s}$

The primary differential ϕ_s is the holomorphic normalized differential (see (4.3.3)):

$$\phi(\varsigma)=\phi_s(\varsigma)=\frac{1}{2\pi i}\oint_b W(\varsigma,\tilde{\varsigma}). \tag{4.7.2}$$

It can be expressed as follows via λ and ς :

$$\phi(\lambda(\varsigma))=\frac{1}{4\omega}\frac{d\lambda}{\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}} \qquad\qquad \phi(\varsigma)=\frac{d\varsigma}{2\omega}\,. \tag{4.7.3}$$

The expansion of multivalued differential $pd\lambda=\left(\int_0^\varsigma\phi\right)d\lambda$ at infinity with respect to the local parameter $z=\lambda^{-1/2}$ is given by

$$pd\lambda=\frac{1}{2\omega}\left(\frac{2}{z^2}+c+\mathcal{O}(z)\right)dz\,.$$

The Darboux-Egoroff metric (4.4.8) corresponding to our choice of primary differential ϕ has in canonical coordinates $\{\lambda_i\}$ the form

$$\mathbf{ds}^2_{\phi_s}=\frac{1}{8\omega^2}\left\{\frac{(d\lambda_1)^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{(d\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{(d\lambda_3)^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}\,. \tag{4.7.4}$$

The set of flat coordinates of this metric is

$$\begin{aligned}
t_1 &:= s=-\oint_a\lambda\phi_s=-\frac{1}{2\omega}\int_x^{x+2\omega}(\wp(\varsigma)+c)d\varsigma=-\frac{\pi i}{4\omega^2}\gamma-c\\
t_2 &:= t^{0;1}=\operatorname*{res}_{\varsigma=0}\frac{1}{\sqrt{\lambda}}pd\lambda=\frac{1}{\omega}\\
t_3 &:= r=\frac{1}{2\pi i}\oint_b\phi=\frac{1}{2\pi i}\frac{\omega'}{\omega}\,,
\end{aligned} \tag{4.7.5}$$

where we denote by γ the following function of period $\mu=2\pi i t_3$ of the torus $\mathcal{L}$:

$$\gamma(\mu)=\frac{1}{3\pi i}\frac{\theta_1'''(0;\mu)}{\theta_1'(0;\mu)}\,. \tag{4.7.6}$$

This function satisfies the Chazy equation (see for example [15]):

$$\gamma'''=6\gamma\gamma''-9\gamma'^{\,2}\,. \tag{4.7.7}$$

The metric (4.7.4) in coordinates (4.7.5) is constant and has the form:

$$\mathbf{ds}^2_{\phi_s} = \frac{1}{2}(dt_2)^2 - 2dt_1 dt_3 \ .$$

The prepotential (4.4.19) (it was computed in [4, 15]) of the Frobenius structure $\widehat{M}^{\phi_s}_{1;1}$ is given by

$$F_{\phi_s} = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32}\, t_2^4\, \gamma(2\pi i t_3) \ .$$

This function is quasihomogeneous, i.e. the following relation

$$F_{\phi_s}(\kappa^{\nu_1}t_1, \kappa^{\nu_2}t_2, \kappa^{\nu_3}t_3) = \kappa^{\nu_F} F_{\phi_s}(t_1, t_2, t_3) \tag{4.7.8}$$

holds for any $\kappa \neq 0$ and the quasihomogeneity factors

$$\nu_1 = 1\ , \qquad \nu_2 = \frac{1}{2}\ , \qquad \nu_3 = 0 \qquad \text{and} \qquad \nu_F = 2\ . \tag{4.7.9}$$

The Euler vector field $E = \sum_{i=1}^{3} \lambda_i \partial_{\lambda_i}$ in flat coordinates has the form:

$$E = \sum_{k=1}^{3} \nu_\alpha t_\alpha \partial_{t_\alpha} = t_1 \partial_{t_1} + \frac{1}{2} t_2 \partial_{t_2} \ ;$$

and the quasihomogeneity (4.7.8), (4.7.9) can be written as $E(F_{\phi_s}(t_1,t_2,t_3)) = 2F_{\phi_s}(t_1,t_2,t_3)$.

The corresponding G-function was computed in [17] :

$$G = -\log\left\{\eta(2\pi i t_3)(t_2)^{\frac{1}{8}}\right\} + \text{const}\ ,$$

where $\eta(\mu)$ is the Dedekind eta-function: $\eta(\mu) = (\theta_1'(0))^{1/3}$. (See [44] for the function τ_W in genus one.)

4.7.2 "Real doubles" in genus one

We consider the same coverings $(\mathcal{L}, \lambda)$ with $\mathcal{L} = \mathbb{C}/\{2\omega, 2\omega'\}$, and the function λ given by (4.7.1) . The coverings have simple branch points $\lambda_1, \lambda_2, \lambda_3$ and ∞ . The set of such coverings is considered now as a space with local coordinates $\{\lambda_1, \lambda_2, \lambda_3; \bar\lambda_1, \bar\lambda_2, \bar\lambda_3\}$.

The manifold $\widehat{M}_{1;1}^{\Phi_s}$

The primary differential $\Phi = \Phi_s$ has the form ($\mu = \omega'/\omega$ is the period of the torus $\mathcal{L}$):

$$\Phi(\varsigma) = \Phi_s(\varsigma) = \frac{\bar{\mu}}{\bar{\mu}-\mu}\frac{d\varsigma}{2\omega} + \frac{\mu}{\mu-\bar{\mu}}\frac{\overline{d\varsigma}}{2\bar{\omega}} \,. \tag{4.7.10}$$

The corresponding Darboux-Egoroff metric (4.5.5) is given by

$$\mathbf{ds}^2_{\Phi_s} = \mathrm{Re}\left\{ \frac{1}{4\omega^2}\left(\frac{\bar{\mu}}{\bar{\mu}-\mu}\right)^2 \left(\frac{(d\lambda_1)^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(d\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(d\lambda_3)^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \right) \right\} \,. \tag{4.7.11}$$

The flat coordinates of this metric are

$$\begin{aligned}
&t_1 := s = \mathrm{Re}\left\{ \frac{\bar{\mu}}{\mu-\bar{\mu}} \int_x^{x+2\omega} (\wp(\varsigma)+c)\frac{d\varsigma}{\omega} \right\} &\quad& t_4 := t = \mathrm{Re}\left\{ \frac{\bar{\mu}}{\mu-\bar{\mu}} \int_x^{x+2\omega'} (\wp(\varsigma)+c)\frac{d\varsigma}{\omega} \right\} \\
&t_2 := t^{0;1} = \frac{\bar{\mu}}{\bar{\mu}-\mu}\frac{1}{\omega} && t_5 := t^{\overline{0;1}} = \bar{t}_2 \\
&t_3 := r = \frac{1}{2\pi i}\frac{\mu\bar{\mu}}{\bar{\mu}-\mu} && t_6 := u = \frac{1}{2\pi i}\frac{\bar{\mu}}{\bar{\mu}-\mu} \,.
\end{aligned} \tag{4.7.12}$$

Note that $\mu = t_3/t_6$, $\bar{\mu} = 2\pi i t_3/(2\pi i t_6 - 1)$ and for the solution (4.7.6) to the Chazy equation we have $\overline{\gamma(\mu)} = -\gamma(-\bar{\mu})$.

The metric (4.7.11) in the flat coordinates has the form

$$\mathbf{ds}^2_{\Phi_s} = \frac{1}{2}(dt_2)^2 + \frac{1}{2}(dt_5)^2 - 2dt_1dt_3 + 2dt_4dt_6 \,.$$

The corresponding prepotential (4.5.35) is

$$\begin{aligned}
F_{\Phi_s} = & -\frac{1}{4}t_1t_2^2 - \frac{1}{4}t_1t_5^2 + \frac{1}{2}t_1^2t_3 - \frac{1}{2}t_1t_4(2t_6 - \frac{1}{2\pi i}) \\
& + t_3^{-1}\left(\frac{1}{4}t_2^2t_4(t_6 - \frac{1}{2\pi i}) + \frac{1}{4}t_4t_5^2t_6 + \frac{1}{2}t_4^2t_6(t_6 - \frac{1}{2\pi i}) + \frac{1}{16}t_2^2t_5^2 \right) \\
& + \frac{1}{32}t_2^4\left(-\frac{1}{4\pi i}t_6^{-2}\,\gamma\left(\frac{t_3}{t_6}\right) + t_3^{-1} - \frac{1}{2\pi i}t_3^{-1}t_6^{-1} \right) \\
& + \frac{1}{32}t_5^4\left(-\frac{\pi i}{(2\pi i t_6 - 1)^2}\,\gamma\left(\frac{2\pi i t_3}{1-2\pi i t_6}\right) + t_3^{-1} + t_3^{-1}(2\pi i t_6 - 1)^{-1} \right) .
\end{aligned} \tag{4.7.13}$$

Note that the coordinates $t_1,\ t_3,\ t_4$ are real, t_2 and t_5 are complex conjugates of each other and t_6 has a constant imaginary part, $\bar{t}_6 = t_6 - 1/2\pi i$. In these coordinates, the prepotential F_{Φ_s} is a real-valued function. However, F_{Φ_s} also satisfies the WDVV system when considered as a function of six complex coordinates; in that case, F_{Φ_s} is not real.

This function is quasihomogeneous: the relation $F_{\Phi_s}(\kappa^{\nu_1}t_1,\ldots,\kappa^{\nu_6}t_6)=\kappa^{\nu_F}F_{\Phi_s}(t_1,\ldots,t_6)$ holds for any $\kappa\neq 0$ and the quasihomogeneity factors

$$\begin{aligned} &\nu_1 = 1\ , \qquad \nu_2 = \frac{1}{2}\ , \qquad \nu_3 = 0\ , \\ &\nu_4 = 1\ , \qquad \nu_5 = \frac{1}{2}\ , \qquad \nu_6 = 0\ , \qquad \nu_F = 2\ . \end{aligned} \tag{4.7.14}$$

The Euler vector field $E=\sum_{i=1}^{3}(\lambda_i\partial_{\lambda_i}+\bar{\lambda}_i\partial_{\bar{\lambda}_i})$ has the following form in the flat coordinates:

$$E = \sum_{\alpha=1}^{6}\nu_\alpha t_\alpha \partial_{t_\alpha} = t_1\partial_{t_1} + \frac{1}{2}t_2\partial_{t_2} + t_4\partial_{t_4} + \frac{1}{2}t_5\partial_{t_5}\ ,$$

and the quasihomogeneity of F_{Φ_s} can be written as $E(F_{\Phi_s}(t_1,\ldots,t_6)) = 2F_{\Phi_s}(t_1,\ldots,t_6)$.

The corresponding G-function (4.6.15) (real-valued as a function of coordinates (4.7.12)) is given by

$$G = -\log\left\{\eta\left(\frac{t_3}{t_6}\right)\eta\left(\frac{2\pi i t_3}{1-2\pi i t_6}\right)(t_2t_5)^{\frac{1}{8}}\left(\frac{2\pi i t_3}{t_6(2\pi i t_6 - 1)}\right)^{\frac{1}{2}}\right\} + \text{const}\ .$$

Here we use the relation $\overline{\eta(\mu)} = \eta(-\bar{\mu})$ for the Dedekind η-function.

The manifold $\widehat{M}_{1;1}^{\Phi_t}$

The primary differential $\Phi = \Phi_t$ has the form ($\mu = \omega'/\omega$ is the period of torus):

$$\Phi(\varsigma) = \Phi_t(\varsigma) = \frac{1}{\mu - \bar{\mu}}\frac{d\varsigma}{2\omega} - \frac{1}{\mu-\bar{\mu}}\frac{\overline{d\varsigma}}{2\bar{\omega}}\ . \tag{4.7.15}$$

The corresponding Darboux-Egoroff metric (4.5.5) is given by

$$\mathbf{ds}^2_{\Phi_t} = \mathrm{Re}\left\{ \frac{1}{4\omega^2}\left(\frac{1}{\bar{\mu}-\mu}\right)^2 \left(\frac{(d\lambda_1)^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(d\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{(d\lambda_3)^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \right) \right\} . \tag{4.7.16}$$

The flat coordinates of this metric are

$$\begin{aligned}
t_1 &:= t = \mathrm{Re}\left\{ \frac{1}{\bar{\mu}-\mu}\int_x^{x+2\omega'} (\wp(\varsigma)+c)\frac{d\varsigma}{\omega} \right\} & t_4 &:= s = \mathrm{Re}\left\{ \frac{1}{\bar{\mu}-\mu}\int_x^{x+2\omega} (\wp(\varsigma)+c)\frac{d\varsigma}{\omega} \right\} \\
t_2 &:= t^{0;1} = \frac{1}{\mu-\bar{\mu}}\frac{1}{\omega} & t_5 &:= t^{\overline{0;1}} = \bar{t}_2 \\
t_3 &:= r = \frac{1}{2\pi i}\frac{\mu}{\mu-\bar{\mu}} & t_6 &:= u = \frac{1}{2\pi i}\frac{1}{\mu-\bar{\mu}}
\end{aligned} \tag{4.7.17}$$

In terms of these coordinates, the period of the torus and its conjugate can be expressed as: $\mu = t_3/t_6$ and $\bar{\mu} = (2\pi i t_3 - 1)/2\pi i t_6$.

The metric (4.7.16) in flat coordinates has the form:

$$\mathbf{ds}^2_{\Phi_t} = \frac{1}{2}(dt_2)^2 + \frac{1}{2}(dt_5)^2 + 2dt_1dt_6 - 2dt_3dt_4 .$$

The corresponding prepotential (4.5.35) is given by

$$\begin{aligned}
F_{\Phi_t} = &-\frac{1}{4}t_1t_2^2 - \frac{1}{4}t_1t_5^2 + \frac{1}{2}t_1t_4(2t_3 - \frac{1}{2\pi i}) - \frac{1}{2}t_1^2t_6 - \frac{1}{2}t_3(t_3 - \frac{1}{2\pi i})\frac{t_4^2}{t_6} - \frac{1}{16}\frac{t_2^2t_5^2}{t_6} \\
&- \frac{t_2^4}{32t_6} - \frac{1}{128\pi i}\frac{t_2^4}{t_6^2}\gamma\left(\frac{t_3}{t_6}\right) + \frac{t_3t_4t_5^2}{4t_6} \\
&- \frac{t_5^4}{32t_6} - \frac{1}{128\pi i}\frac{t_5^4}{t_6^2}\gamma\left(\frac{1-2\pi i t_3}{2\pi i t_6}\right) + \frac{(t_3 - \frac{1}{2\pi i})t_4t_2^2}{4t_6} .
\end{aligned} \tag{4.7.18}$$

This function is also real if the coordinates are of the form (4.7.17): in this case t_1, t_4, t_6 are real, $t_2 = \bar{t}_5$, and t_3 has a constant imaginary part, namely, we have $\bar{t}_3 = t_3 - \frac{1}{2\pi i}$. Last two lines in (4.7.18) are complex conjugates of each other since for the function γ (4.7.6) we have $\overline{\gamma(\mu)} = -\gamma(-\bar{\mu})$.

The function F_{Φ_t} (4.7.18) is quasihomogeneous. The quasihomogeneity factors $\{\nu_i\}$ and ν_F are the same as for the above example (the function F_{Φ_s}), they are given by (4.7.14).

The G-function for $\widehat{M}_{1;1}^{\Phi_t}$ (it is also real-valued as a function of coordinates (4.7.17)) is given by

$$G = -\log\left\{\eta\left(\frac{t_3}{t_6}\right)\eta\left(\frac{1-2\pi i t_3}{2\pi i t_6}\right)(t_2 t_5)^{\frac{1}{8}}\, t_6^{-\frac{1}{2}}\right\} + \text{const}\,,$$

where, again, η is the Dedekind eta-function.

The manifold $\widehat{M}_{1;1}^{\Phi_s+\sigma\Phi_t}$

According to Remark 4.6 in the end of Section 4.5, there exists a Frobenius structure built from a linear combination of two primary differentials Φ_s and Φ_t . Here, we compute a prepotential which corresponds to the differential $\Phi = \Phi_s + \sigma\Phi_t$ for σ being a nonzero parameter.

We start with the differential

$$\Phi(\varsigma) = \Phi_s(\varsigma) + \sigma\Phi_t(\varsigma) = \frac{\bar\mu - \sigma}{\bar\mu - \mu}\frac{d\varsigma}{2\omega} + \frac{\sigma - \mu}{\bar\mu - \mu}\frac{\overline{d\varsigma}}{2\bar\omega}\,.$$

The corresponding Darboux-Egoroff metric (4.5.5) is given by

$$\begin{aligned}
\mathbf{ds}^2_\Phi = \frac{1}{8\omega^2}\left(\frac{\bar\mu-\sigma}{\bar\mu-\mu}\right)^2 \Bigg(&\frac{(d\lambda_1)^2}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{(d\lambda_2)^2}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} \\
&+ \frac{(d\lambda_3)^2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\Bigg) + \frac{1}{8\bar\omega^2}\left(\frac{\sigma-\mu}{\bar\mu-\mu}\right)^2\Bigg(\frac{(d\bar\lambda_1)^2}{(\bar\lambda_1-\bar\lambda_2)(\bar\lambda_1-\bar\lambda_3)} \\
&+ + \frac{(d\bar\lambda_2)^2}{(\bar\lambda_2-\bar\lambda_1)(\bar\lambda_2-\bar\lambda_3)} + \frac{(d\bar\lambda_3)^2}{(\bar\lambda_3-\bar\lambda_1)(\bar\lambda_3-\bar\lambda_2)}\Bigg)\,. \qquad (4.7.19)
\end{aligned}$$

The flat coordinates t and s of the metric (4.7.19) are

$$\begin{aligned}
t &= \frac{\bar\mu-\sigma}{\bar\mu-\mu}\int_x^{x+2\omega'}(\wp(\varsigma)+c)\frac{d\varsigma}{2\omega} + \frac{\sigma-\mu}{\bar\mu-\mu}\int_x^{x+2\omega'}\overline{(\wp(\varsigma)+c)}\frac{d\bar\varsigma}{2\bar\omega} \\
s &= \frac{\bar\mu-\sigma}{\bar\mu-\mu}\int_x^{x+2\omega}(\wp(\varsigma)+c)\frac{d\varsigma}{2\omega} + \frac{\sigma-\mu}{\bar\mu-\mu}\int_x^{x+2\omega}\overline{(\wp(\varsigma)+c)}\frac{d\bar\varsigma}{2\bar\omega}\,.
\end{aligned} \qquad (4.7.20)$$

We need to perform a linear change of variables in order to have the unit field $\mathbf{e}$ in the form $\mathbf{e} = -\partial_{t^1}$. After this change of variables, we get the following set of flat coordinates for the metric (4.7.19):

$$\begin{aligned} t_1 &:= s + \sigma^{-1} t & t_4 &:= s - \sigma^{-1} t \\ t_2 &:= t^{0;1} = \frac{1}{\omega} \frac{\bar{\mu} - \sigma}{\bar{\mu} - \mu} & t_5 &:= t^{\overline{0;1}} = \frac{1}{\bar{\omega}} \frac{\sigma - \mu}{\bar{\mu} - \mu} \\ t_3 &:= r = \frac{1}{2\pi i} \frac{(\bar{\mu} - \sigma)\mu}{\bar{\mu} - \mu} & t_6 &:= u = \frac{1}{2\pi i} \frac{1}{\mu - \bar{\mu}} \ . \end{aligned} \tag{4.7.21}$$

In the coordinates (4.7.21), the metric has the form:

$$\mathbf{ds}^2_\Phi = \frac{1}{2}(dt_2)^2 + \frac{1}{2}(dt_5)^2 - dt_1 dt_3 + \sigma dt_1 dt_6 - dt_3 dt_4 - \sigma dt_4 dt_6 \ . \tag{4.7.22}$$

The period of the torus and its complex conjugate can be expressed in terms of the coordinates (4.7.21) as follows: $\mu = t_3/t_6$ and $\bar{\mu} = (\sigma - 2\pi i t_3)/(1 - 2\pi i t_6)$, respectively.

Then, the prepotential (4.5.35) is the following function of 6 variables:

$$\begin{aligned} F_{\Phi_s + \sigma\Phi_t} = &-\frac{1}{64\pi i} \frac{t_2^4}{t_6^2} \gamma\left(\frac{t_3}{t_6}\right) - \frac{\pi i}{16} \frac{t_5^4}{(2\pi i t_6 - 1)^2} \gamma\left(\frac{2\pi i t_3 - \sigma}{1 - 2\pi i t_6}\right) - \frac{1}{8\pi i} \frac{t_2^2}{t_6}(t_1 + t_4) \\ &- \frac{\sigma}{8\pi i}(t_1^2 - t_4^2) + \frac{1}{8\pi i} \frac{t_3}{t_6}(t_1 + t_4)^2 + \frac{\pi i}{2t_6(t_3 - \sigma t_6)} \frac{1}{(2\pi i t_6 - 1)} \times \\ &\times \left(\frac{(t_2^2 + t_5^2) t_6}{2} - \frac{t_2^2}{4\pi i} - (t_1 + t_4) t_3 t_6 + \frac{(t_1 + t_4) t_3}{2\pi i} + \sigma (t_1 - t_4) t_6^2 - \frac{\sigma (t_1 - t_4) t_6}{2\pi i} \right)^2 . \end{aligned} \tag{4.7.23}$$

In the limit $\sigma \to 0$, the metric (4.7.22) becomes singular and the function (4.7.23) does not satisfy the WDVV system. To obtain from (4.7.23) the prepotential F_{Φ_s} , corresponding to the case $\sigma = 0$, one has to rewrite $F_{\Phi_s + \sigma\Phi_t}$ in terms of the original variables (4.7.20) and then put $\sigma = 0$.

The function $F_{\Phi_s + \sigma\Phi_t}$ is quasihomogeneous. The quasihomogeneity factors $\{\nu_i\}$ and ν_F are given by (4.7.14).

The G-function for $\widehat{M}_{1;1}^{\Phi_s+\sigma\Phi_t}$ is given by

$$G = -\log\left\{\eta\left(\frac{t_3}{t_6}\right)\eta\left(\frac{2\pi i t_3-\sigma}{1-2\pi i t_6}\right)(t_2t_5)^{\frac{1}{8}}\left(\frac{t_3-\sigma t_6}{t_6(1-2\pi i t_6)}\right)^{\frac{1}{2}}\right\} + \text{const}\ .$$

A computer check shows that functions F_{Φ_s} (4.7.13), F_{Φ_t} (4.7.18), and $F_{\Phi_s+\sigma\Phi_t}$ (4.7.23) indeed satisfy the WDVV system.

Acknowledgments I am grateful to D. Korotkin, A. Kokotov, M. Bertola and S. Natanzon for many useful discussions and to B. Dubrovin for important comments and pointing out some mistakes in an earlier version of this paper.

The construction of the previous work was based on the properties of the canonical meromorphic bidifferential $W(P,Q)$ and the Schiffer and Bergman kernels. Namely, the Rauch variational formulas and the invariance of the bidifferential W under biholomorphic maps of the Riemann surface imply the flatness of diagonal potential metrics on the Hurwitz space constructed in terms of W. The type of singularity of $W(P,Q)$ (the second order pole on the diagonal) and the vanishing of its $a-$periods allow us to find the subfamily of these Darboux-Egoroff metrics which corresponds to Dubrovin's Frobenius structures on the Hurwitz space. Similarly, the properties of the Schiffer and Bergman kernels allow us to construct the "real doubles" of Frobenius structures from [15].

In the next work we introduce a $g(g+1)/2-$parametric deformation $W_{\mathbf{q}}$ of the bidifferential W . The deformed bidifferential $W_{\mathbf{q}}$ defines, analogously to W, a family of Darboux-Egoroff metrics on the Hurwitz space. This family, again, turns out to contain metrics which correspond to Frobenius structures on Hurwitz spaces; those structures can be considered as natural deformations of Dubrovin's Hurwitz Frobenius manifolds. We also introduce analogous deformations of the Schiffer and Bergman kernels which therefore define deformations of the real doubles of the Hurwitz Frobenius structures from [15].

Chapter 5

Deformations of Hurwitz Frobenius structures

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math-ph/0408026, to appear in Internat. Math. Res. Notices

Abstract. Deformations of Dubrovin's Hurwitz Frobenius manifolds are constructed. The deformations depend on $g(g+1)/2$ complex parameters where g is the genus of the corresponding Riemann surface. In genus one, the flat metric of the deformed Frobenius manifold coincides with a metric associated with a one-parameter family of solutions to the Painlevé-VI equation with coefficients $(1/8, -1/8, 1/8, 3/8)$. Analogous deformations of real doubles of the Hurwitz Frobenius manifolds are also found; these deformations depend on $g(g+1)/2$ real parameters.

5.1 Introduction

The structure of a Frobenius manifold was introduced in [15] (see also [57]) to give a geometric reformulation of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) system of differential equations on the function F of n variables ([12, 75]):

$$F_i F_1^{-1} F_j = F_j F_1^{-1} F_i \,, \quad i,j = 1,\ldots,n \,, \tag{5.1.1}$$

where F_i is the matrix

$$(F_i)_{mn} = \frac{\partial^3 F}{\partial t^i \partial t^m \partial t^n} \,, \tag{5.1.2}$$

and the function F is such that F_1 is a constant nondegenerate matrix, and there exist constants $\nu_1, \ldots, \nu_n, \nu_F$ such that for any nonzero constant κ the following relation (quasi-homogeneity) holds:

$$F(\kappa^{\nu_1} t^1, \ldots, \kappa^{\nu_n} t^n) = \kappa^{\nu_F} F(t^1, \ldots, t^n) + \text{quadratic terms} \,. \tag{5.1.3}$$

The function F is called the *prepotential* of the corresponding Frobenius manifold.

Here we consider the so-called semisimple Frobenius structures on Hurwitz spaces (a Frobenius manifold is called *semisimple* if the associated algebra in the tangent space does not have nilpotents). The Hurwitz space is the space of pairs $(\mathcal{L}, \lambda)$ modulo an equivalence relation (see Section 5.2.1) where $\mathcal{L}$ is a Riemann surface of genus g and λ is a function on the surface, $\lambda : \mathcal{L} \to \mathbb{C}P^1$, of a fixed degree. The finite critical values of the function λ (semisimplicity implies they are all simple) serve as local coordinates on the Hurwitz space. Frobenius structures on Hurwitz spaces in any genus were originally found in [15]. Local coordinates on the Hurwitz space become canonical coordinates on the Frobenius manifold. In [15], App. I, it is shown that any Frobenius manifold, under some genericity assumption, can be locally described in terms of Hurwitz spaces: for any Frobenius manifold there exists a function of one complex variable (called the *superpotential*) meromorphic in some domain in $\mathbb{C}$ and such that canonical coordinates on the Frobenius manifold are given by critical values of this function. If the superpotential can be analytically continued to a meromorphic function on a compact Riemann surface then the corresponding Frobenius manifold is isomorphic to a Hurwitz Frobenius manifold; in this case the Hurwitz space is the space of coverings defined by the superpotential. Therefore, one might expect that any natural result concerning Hurwitz Frobenius manifods can be extended to an arbitrary Frobenius manifold. In [68] new semisimple Frobenius structures which can be considered as *real doubles* of the semisimple Hurwitz Frobenius manifolds of Dubrovin [15] were found. Those Frobenius structures are built on Hurwitz spaces considered as real manifolds.

For the simplest Hurwitz space in genus one the Frobenius structure of [15] gives the following solution to the WDVV system:

$$F = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32} t_2^4 \gamma(2\pi i t_3) \ , \tag{5.1.4}$$

where $\gamma(\mu) = \theta_1'''/(3\pi i \theta_1')$; and $\theta_1(z) = -\theta[\frac{1}{2}, \frac{1}{2}](z)$ is the odd elliptic Jacobi theta function. The function $\gamma(\mu)$ satisfies the Chazy equation

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2 \ . \tag{5.1.5}$$

It is known ([15], App. C) that the function of the form (5.1.4) will still satisfy the WDVV system if the function γ in (5.1.4) is replaced by an arbitrary solution to the Chazy equation (5.1.5). The general solution to the Chazy equation has the form:

$$f(\mu) = \gamma\left(\frac{a\mu + b}{c\mu + d}\right)\frac{1}{(c\mu + d)^2} - \frac{2c}{c\mu + d} \tag{5.1.6}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$.

In particular, in the case of $SL(2, \mathbb{C})$-transformations of the form $\begin{pmatrix} 1 & 0 \\ -1/\mathbf{q} & 1 \end{pmatrix}$ we get the following solution to WDVV equations:

$$F = -\frac{1}{4}t_1 t_2^2 + \frac{1}{2}t_1^2 t_3 - \frac{\pi i}{32} t_2^4 \left(\frac{1}{(1 - 2\pi i t_3/\mathbf{q})^2}\gamma\left(\frac{2\pi i t_3}{1 - 2\pi i t_3/\mathbf{q}}\right) + \frac{2}{\mathbf{q}(1 - 2\pi i t_3/\mathbf{q})}\right) \ , \tag{5.1.7}$$

This function is obtained from (5.1.4) by replacing the function $\gamma(2\pi i t_3)$ by $f(2\pi i t_3)$ from (5.1.6) with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/\mathbf{q} & 1 \end{pmatrix}$. If $(1/\mathbf{q}) \in \mathbb{Z}$ then the solutions (5.1.4) and (5.1.7) coincide due to the modular invariance of the function γ; for $(1/\mathbf{q}) \notin \mathbb{Z}$ we obtain a one-parameter deformation of the solution (5.1.4).

The main result of this paper is a generalization of this deformation procedure to semi-simple Hurwitz Frobenius manifolds in any genus. Namely, we construct a $g(g+1)/2$-parametric deformation of Dubrovin's Frobenius structures [15] on Hurwitz spaces. For the simplest Hurwitz space in genus one our deformation coincides with the deformation (5.1.7) of the prepotential (and corresponding Frobenius manifold) (5.1.4).

The idea of the construction is the following. All ingredients of semisimple Hurwitz Frobenius manifolds of Dubrovin can be conveniently described in terms of the canonical meromorphic bidifferential W on a Riemann surface $\mathcal{L}$. The bidifferential W is defined as follows. Introduce on $\mathcal{L}$ a canonical basis of cycles $\{a_k; b_k\}$. Then $W(P,Q)$ is a symmetric bidifferential which has a second order pole with biresidue 1 on the diagonal $P \sim Q$ and has vanishing a-periods; it can be expressed in terms of the prime form $E(P,Q)$ as follows $W(P,Q) := d_P d_Q \log E(P,Q)$. For a Hurwitz space of coverings $(\mathcal{L}, \lambda)$ with simple ramification points $\{P_j\}$, the dependence of the bidifferential W on the branch points $\{\lambda_j\}$ is given by the Rauch variational formulas [42, 63]:

$$\frac{\partial W(P,Q)}{\partial \lambda_j} = \frac{1}{2} W(P,P_j) W(Q,P_j) \,, \tag{5.1.8}$$

where $W(P,P_j) := (W(P,Q)/dx_j(Q))\,|_{Q=P_j}$.

The main ingredient of Frobenius structures is a *Darboux-Egoroff* metric. A diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ is called a Darboux-Egoroff metric if it is flat (its curvature tensor vanishes) and potential (there exists a function U such that $g_{ii} = \partial_{\lambda_i} U$ holds for any i). The Darboux-Egoroff lemma states that a diagonal metric is potential and flat if its *rotation coefficients* β_{ij} defined for $i \neq j$ by $\beta_{ij} = (\partial_{\lambda_j} \sqrt{g_{ii}})/\sqrt{g_{jj}}$ are symmetric, $\beta_{ij} = \beta_{ji}$, and satisfy the system of equations:

$$\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{kj}\ , \qquad i,j,k \text{ are distinct,} \tag{5.1.9}$$

$$\sum_k \partial_{\lambda_k}\beta_{ij} = 0 \qquad \text{for all } \beta_{ij}\ . \tag{5.1.10}$$

For the family of Hurwitz Frobenius manifolds introduced in [15], the rotation coefficients of the corresponding Darboux-Egoroff metrics are given by $\beta_{ij} = W(P_i,P_j)/2$. These rotation coefficients satisfy equations (5.1.9) due to the Rauch formulas (5.1.8).

In this work we introduce the following deformation of the bidifferential W :

$$W_{\mathbf{q}}(P,Q) := W(P,Q) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}+\mathbf{q})^{-1}_{kl}\omega_k(P)\omega_l(Q)\ ,$$

where g is the genus of the Riemann surface; $\omega_l(Q) := \oint_{b_l} W(P,Q)/(2\pi i)$ form the basis of holomorphic differentials normalized by $\oint_{a_k}\omega_l = \delta_{kl}$; $\mathbb{B}_{kl} := \oint_{b_k}\omega_l$ is the matrix of b-periods; and $\mathbf{q}$ is a symmetric matrix of parameters constant with respect to $\{\lambda_j\}$ and λ . The matrix $\mathbf{q}$ must be chosen such that the sum $(\mathbb{B}+\mathbf{q})$ is not degenerate.

The bidifferential $W_{\mathbf{q}}(P,Q)$ turns out to satisfy the following variational formulas which look identical to the variational formulas (5.1.8) for W :

$$\partial_{\lambda_j} W_{\mathbf{q}}(P,Q) = \frac{1}{2}\, W_{\mathbf{q}}(P,P_j)\, W_{\mathbf{q}}(Q,P_j)\ . \tag{5.1.11}$$

Therefore, the quantities $W_{\mathbf{q}}(P_i,P_j)/2$ give rotation coefficients of some Darboux-Egoroff metric: the bidifferential W is symmetric, i.e. $W(P,Q) = W(Q,P)$; the variational formulas (5.1.11) imply relations (5.1.9) for the rotation coefficients; the equations (5.1.10) can be proven analogously to the case of rotation coefficients given by the bidifferential W . The variational formulas (5.1.11) hold for the points of the Hurwitz space which do not belong to the divisor defined by the equation $\det(\mathbb{B}+\mathbf{q}) = 0$. The corresponding Darboux-Egoroff metrics are also defined outside this divisor. Analogously to [15] (see also [68]) we find a family of Darboux-Egoroff metrics on Hurwitz spaces with rotation coefficients $W_{\mathbf{q}}(P_i, P_j)/2$ and build corresponding Frobenius structures. In the limit as some entries of the matrix $\mathbf{q}$ tend to infinity so that all entries of the matrix $(\mathbb{B}+\mathbf{q})^{-1}$ tend to zero (in particular this condition holds if all diagonal entries of the matrix $\mathbf{q}$ tend to infinity and nondiagonal entries remain finite) the bidifferential $W_{\mathbf{q}}$ turns into W and our Frobenius structures coincide with those of [15].

The second result of the paper is a construction of real doubles [68] of the deformed semi-simple Hurwitz Frobenius manifolds. This is done by introducing deformations of the Schiffer and Bergman kernels. The Schiffer and Bergman kernels were used in the construction of real doubles in [68]; they are defined by the following formulas:

$$\Omega(P,Q) := W(P,Q) - \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P)\omega_l(Q) \, ,$$

$$B(P,\bar{Q}) := \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl}\omega_k(P)\overline{\omega_l(Q)} \, ,$$

respectively. In the case of a genus zero Riemann surface the Schiffer kernel coincides with W and the Bergman kernel vanishes.

The following alternative definitions [23] independent of the choice of a canonical basis of cycles $\{a_k; b_k\}_{k=1}^g$ on the Riemann surface can be given for the two kernels. The Schiffer kernel is the symmetric bidifferential which has a second order pole along the diagonal $P=Q$ and is such that $p.v.\iint_{\mathcal{L}}\Omega(P,Q)\,\overline{\omega(P)} = 0$ holds for any holomorphic differential ω on the surface. The Bergman kernel is a regular bidifferential on $\mathcal{L}$ holomorphic with respect to its first argument and antiholomorphic with respect to the second one which (up to a factor of $2\pi i$) is a kernel of an integral operator acting in the space $L_2^{(1,0)}(\mathcal{L})$ of (1,0)-

forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic (1,0)-forms. In particular, for any holomorphic differential ω on the surface $\mathcal{L}$ the following relation holds: $\iint_{\mathcal{L}} B(P,\bar{Q})\,\omega(Q) = 2\pi i\,\omega(P)$.

In contrast to the bidifferential W , which is holomorphic with respect to the moduli coordinates $\{\lambda_k\}$, the Schiffer and Bergman kernels depend on the complex structure of the Riemann surface through the branch points $\{\lambda_k\}$ of the covering $(\mathcal{L}\,,\lambda)$ *and* their complex conjugates $\{\bar{\lambda}_k\}$. Therefore, in [68] the Hurwitz space was considered as a real manifold, i.e. a manifold with coordinates $\{\lambda_k;\bar{\lambda}_k\}$. A family of Darboux-Egoroff metrics on this real space was found; the rotation coefficients of those metrics are given by the Schiffer and Bergman kernels suitably evaluated at ramification points of the covering. The flatness for the metrics is provided by variational formulas for the kernels Ω and B . Some of the Darboux-Egoroff metrics proved to correspond to Frobenius structures on the Hurwitz space with coordinates $\{\lambda_k;\bar{\lambda}_k\}$. Those Frobenius structures were called the real doubles of Dubrovin's Hurwitz Frobenius structures.

We introduce the following deformations $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,Q)$ of the Schiffer and Bergman kernels. Consider the holomorphic differentials $v_k(P) := \oint_{b_k}\Omega(P,Q)/(2\pi i)$. The differential v_k is normalized by the condition that all its a- and b-periods are purely imaginary except the a_k-period: $\mathrm{Re}\{\oint_{a_j} v_k\} = \delta_{jk}/2$ and $\mathrm{Re}\{\oint_{b_j} v_k\} = 0$ for $j,k = 1,\dots,g$. The matrix $\mathbb{B}^{\Omega}$ of b-periods of differentials v_k (which is symmetric and imaginary) is given by: $\mathbb{B}^{\Omega} := \bar{\mathbb{B}}(\bar{\mathbb{B}} - \mathbb{B})^{-1}\mathbb{B}$; it is the matrix of pairwise scalar products of differentials v_k in the space $L_2^{(1,0)}(\mathcal{L})$, i.e. $\iint_{\mathcal{L}} \overline{v_k(P)} \wedge v_l(P) = {\mathbb{B}^{\Omega}}_{kl}$. Then, if a constant matrix $\mathbf{q}$ is such that $\mathbf{q} = \mathbf{q}^T$, $\bar{\mathbf{q}} = -\mathbf{q}$ and the matrix $(\mathbb{B}^{\Omega} + \mathbf{q})$ is invertible, we can define the deformed Schiffer and Bergman kernels by:

$$\begin{aligned}\Omega_{\mathbf{q}}(P,Q) &:= \Omega(P,Q) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}^{\Omega} + \mathbf{q})^{-1}_{kl} v_k(P) v_l(Q) \,, \\ B_{\mathbf{q}}(P,\bar{Q}) &:= B(P,\bar{Q}) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}^{\Omega} + \mathbf{q})^{-1}_{kl} v_k(P) \overline{v_l(Q)} \,,\end{aligned} \tag{5.1.12}$$

respectively. The integral operator with the deformed kernel $B_{\mathbf{q}}(P,\bar{Q})/(2\pi i)$ maps the space $L_2^{(1,0)}(\mathcal{L})$ onto the space $\mathcal{H}^{(1,0)}(\mathcal{L})$ and acts in the space $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic differentials as a linear operator which in the basis $\{v_k\}$ is given by the matrix $\mathbf{q}(\mathbb{B}^{\Omega} + \mathbf{q})^{-1}$. Similarly, the action of the integral operator with the kernel $\Omega_{\mathbf{q}}(P,Q)/(2\pi i)$ in the space $\mathcal{H}^{(1,0)}(\mathcal{L})$ is defined by the matrix $-\mathbb{B}^{\Omega}(\mathbb{B}^{\Omega} + \mathbf{q})^{-1}$ (see formula (5.2.21)).

The motivation for the definition (5.1.12) is that variational formulas for the bidifferentials $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,Q)$ defined in this way are similar to variational formulas for the Schiffer and Bergman kernels. Therefore, the deformations $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$, analogously to the kernels Ω and B, define rotation coefficients of some Darboux-Egoroff metrics on the Hurwitz space with coordinates $\{\lambda_k; \bar{\lambda}_k\}$. We find a family of such metrics; they are defined on the Hurwitz space outside the subspace of codimension one given by the equation $\det(\mathbb{B}^{\Omega} + \mathbf{q}) = 0$. It turns out that this family also contains a class of metrics which correspond to new Frobenius structures. We call these structures the real doubles of the deformed Frobenius manifolds.

The paper is organized as follows. In the next section we define the Hurwitz spaces and several families of Darboux-Egoroff metrics on them constructed using the bidifferentials introduced above. In Section 5.3 we give a definition of Frobenius structures and construct deformations of Dubrovin's Hurwitz Frobenius manifolds [15]. In Section 5.4 we construct the real doubles of the deformations. In Section 5.5 we compute expressions for the G-

function on each constructed Frobenius manifold. Section 5.6 is devoted to a calculation of prepotentials and G-functions of the deformations of Frobenius manifolds and their real doubles in the case of the simplest Hurwitz space in genus one. In that section we also describe the relationship of the example of prepotential, the Chazy equation and isomonodromic deformations related to the Painlevé-VI equation. We show that in genus one the constructed one-parameter deformations have a two-parametric generalization which can be possibly extended to Hurwitz spaces in any genus which we hope to address in the future.

5.2 Darboux-Egoroff metrics on Hurwitz spaces

5.2.1 Hurwitz spaces

Consider a compact Riemann surface $\mathcal{L}$ of genus g and a meromorphic function $\lambda : \mathcal{L} \to \mathbb{C}P^1$ of degree N. The equation

$$\zeta = \lambda(P) , \qquad P \in \mathcal{L}$$

(ζ is a coordinate on $\mathbb{C}P^1$) represents the surface as an N-fold ramified covering of $\mathbb{C}P^1$. The covering is a collection of N copies of $\mathbb{C}P^1$ which are glued together along the cuts connecting the ramification points to form a connected manifold. The ramification points $P_j \in \mathcal{L}$ are the critical points of the function $\lambda(P)$, i.e. they satisfy $\lambda'(P_j) = 0$; their projections $\lambda_j = \lambda(P_j)$ on the base of the covering $\mathbb{C}P^1$ are called the branch points.

We assume that the function λ has $m+1$ poles at some points $\infty^0, \dots, \infty^m \in \mathcal{L}$, and we denote by $n_i + 1$ the order of the pole at ∞^i. In other words, there are $m+1$ points on the covering which project to $\zeta = \infty$ on the base; in the point ∞^i there are $\{n_i + 1\}$ sheets glued together ($n_0, \dots, n_m \in \mathbb{N}$ are such that $\sum_{i=0}^{m}(n_i + 1) = N$). The numbers $\{n_i\}$ are called the ramification indices. We assume the remaining ramification points which have finite projections on the base, $\lambda_j < \infty$, to be simple (i.e. there are exactly two sheets glued together at the corresponding point on the covering) and denote their number by L.

The local parameter near a simple ramification point $P_j \in \mathcal{L}$ (which is not a pole of λ) is $x_j(P) = \sqrt{\lambda(P) - \lambda_j}$ and in a neighbourhood $P \sim \infty^i$ the local parameter z_i is such that $z_i^{-n_i-1}(P) = \lambda(P)$.

For each genus g of the surface, the Riemann-Hurwitz formula gives the possible values of degree N of the function λ, number L of simple finite branch points and the ramification indices n_i over infinity:

$$2g - 2 = -2N + L + \sum_{i=0}^{m} n_i \,. \tag{5.2.1}$$

Two coverings are called equivalent if one of them can be obtained from the other by a permutation of sheets. The space of equivalence classes of described coverings is the Hurwitz space; we denote it by $M = M_{g;n_0,\ldots,n_m}$. We shall work with the following covering $\widehat{M} = \widehat{M}_{g;n_0,\ldots,n_m}$ of the Hurwitz space. A point of the space $\widehat{M}$ is a triple $\{\mathcal{L}, \lambda, \{a_k, b_k\}_{k=1}^{g}\}$, where $\{a_k, b_k\}_{k=1}^{g}$ is a canonical basis of cycles on $\mathcal{L}$. The branch points $\{\lambda_i\}$ give a set of local coordinates on the space $\widehat{M}$.

5.2.2 Symmetric bidifferentials on Riemann surfaces

On a Riemann surface $\mathcal{L}$ of genus g with a canonical basis of cycles $\{a_k; b_k\}_{k=1}^{g}$, let $\{\omega_k(P)\}_{k=1}^{g}$ be the set of holomorphic differentials normalized by $\oint_{a_k} \omega_j = \delta_{jk}$. The symmetric matrix $\mathbb{B}$ of b-periods of the surface is defined by $\mathbb{B}_{kj} = \oint_{b_k} \omega_j$; its imaginary part is positive definite.

Now we shall introduce the following bidifferentials on the Riemann surface $\mathcal{L}$.

1. The canonical meromorphic bidifferential $W(P,Q)$ is defined by

$$W(P,Q) := d_P d_Q \log E(P,Q) \, , \tag{5.2.2}$$

where $E(P,Q)$ is the prime form on the surface. The bidifferential can be uniquely characterized by the following properties: it is symmetric; it has a second-order pole on the diagonal $P=Q$ with biresidue 1 ; and its a-periods vanish:

$$\oint_{a_k} W(P,Q) = 0 \, , \qquad k = 1, \ldots, g \, . \tag{5.2.3}$$

The b-periods of $W(P,Q)$ are given by the holomorphic normalized differentials: $\oint_{b_k} W(P,Q)$ $= 2\pi i \, \omega_k(P)$, $k = 1, \ldots, g$. For a covering $(\mathcal{L}, \lambda)$ the bidifferential W depends on the simple branch points $\{\lambda_j\}$ of the covering according to the Rauch variational formulas [42, 63]:

$$\frac{\partial W(P,Q)}{\partial \lambda_j} = \frac{1}{2} W(P,P_j) W(Q,P_j) \, , \tag{5.2.4}$$

where $W(P,P_j)$ denotes the evaluation of $W(P,Q)$ at $Q=P_j$ with respect to the standard local parameter $x_j(Q) = \sqrt{\lambda(Q) - \lambda_j}$ near a ramification point P_j :

$$W(P,P_j) = \left. \frac{W(P,Q)}{dx_j(Q)} \right|_{Q=P_j} \, . \tag{5.2.5}$$

Being integrated over b-cycles of the surface, the Rauch formulas (5.2.4) give the variational formulas for holomorphic differentials and the matrix $\mathbb{B}$ of b-periods:

$$\frac{\partial \omega_k(P)}{\partial \lambda_j} = \frac{1}{2} \omega_k(P_j) W(P,P_j) \, , \qquad \frac{\partial \mathbb{B}_{kl}}{\partial \lambda_j} = \pi i \, \omega_k(P_j) \omega_l(P_j) \, . \tag{5.2.6}$$

2. For a covering $(\mathcal{L}, \lambda)$ of genus $g \geq 1$ consider a symmetric nondegenerate matrix $\mathbf{q}$ which is independent of the branch points $\{\lambda_j\}$ and such that the inverse $(\mathbb{B} + \mathbf{q})^{-1}$ exists.

Then, we define a symmetric bidifferential $W_{\mathbf{q}}(P,Q)$ which is the following deformation of the bidifferential $W(P,Q)$:

$$W_{\mathbf{q}}(P,Q) := W(P,Q) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}+\mathbf{q})^{-1}_{kl}\omega_k(P)\omega_l(Q) \,. \tag{5.2.7}$$

This bidifferential has the same singularity structure as the W-bidifferential and satisfies the normalization condition:

$$\oint_{a_k} W_{\mathbf{q}}(P,Q) + \sum_{j=1}^{g}(\mathbf{q}^{-1})_{jk} \oint_{b_j} W_{\mathbf{q}}(P,Q) = 0 \,. \tag{5.2.8}$$

The bidifferential $W_{\mathbf{q}}$ turns into W, for example, in the limit when all diagonal entries $\mathbf{q}_{ii}$ of the matrix $\mathbf{q}$ tend to infinity while the off-diagonal entries remain finite. In this limit, the matrix $(\mathbb{B}+\mathbf{q})^{-1}$ tends to the zero matrix.

Consider now the Hurwitz space $M_{g;n_0,\dots,n_m}$ of pairs $(\mathcal{L},\lambda)$. The equation

$$\det(\mathbb{B}+\mathbf{q}) = 0 \tag{5.2.9}$$

defines a divisor in $M_{g;n_0,\dots,n_m}$, which we denote by $\mathcal{D}_{\mathbf{q}}$. A simple computation shows that $W_{\mathbf{q}}(P,Q)$ satisfies the variational formulas which formally look exactly as variational formulas (5.2.4) for $W(P,Q)$:

$$\frac{\partial W_{\mathbf{q}}(P,Q)}{\partial \lambda_j} = \frac{1}{2} W_{\mathbf{q}}(P,P_j) W_{\mathbf{q}}(Q,P_j) \,. \tag{5.2.10}$$

These formulas hold at the points of the Hurwitz space where the bidifferential $W_{\mathbf{q}}$ is well defined, i.e. outside of divisor $\mathcal{D}_{\mathbf{q}}$ (5.2.9).

Note that both bidifferentials W and $W_{\mathbf{q}}$, as well as the differentials ω_k and the matrix $\mathbb{B}$, are holomorphic with respect to branch points $\{\lambda_k\}$, i.e. they do not depend on $\bar{\lambda}_k$ ([23], p. 54).

3. The Schiffer and Bergman bidifferentials (kernels) are defined on a Riemann surface of genus $g \geq 1$ by

$$\Omega(P, Q) := W(P, Q) - \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl} \omega_k(P) \omega_l(Q) \,, \tag{5.2.11}$$

$$B(P, \bar{Q}) := \pi \sum_{k,l=1}^{g} (\mathrm{Im}\mathbb{B})^{-1}_{kl} \omega_k(P) \overline{\omega_l(Q)} \,, \tag{5.2.12}$$

respectively. The following equivalent definitions can be given for these bidifferentials, which, in particular, show that the bidifferentials are independent of the choice of a canonical basis of cycles $\{a_k; b_k\}$. Namely, $\Omega(P, Q)$ can be defined as a symmetric bidifferential having a second order pole with biresidue 1 at the diagonal $P \sim Q$, such that for any holomorphic differential ω the following holds: $\iint_{\mathcal{L}} \Omega(P, Q)\overline{\omega(Q)} = 0$. The bidifferential $B(P, \bar{Q})$ is a regular bidifferential holomorphic with respect to one of its arguments and antiholomorphic with respect to the other one. The integral operator with the kernel $B(P, \bar{Q})/(2\pi i)$ acts in the space $L_2^{(1,0)}(\mathcal{L})$ of $(1,0)$-forms as an orthogonal projector onto the subspace $\mathcal{H}^{(1,0)}(\mathcal{L})$ of holomorphic $(1,0)$-forms [23]. In particular, in the space $\mathcal{H}^{(1,0)}(\mathcal{L})$ it acts as the identity operator, i.e. $\iint_{\mathcal{L}} B(P, \bar{Q})\omega(Q)/(2\pi i) = \omega(P)$.

The periods of bidifferentials (5.2.11) and (5.2.12) are related to each other as follows:

$$\oint_{a_k} \Omega(P, Q) = -\oint_{a_k} B(\bar{P}, Q) \,, \qquad \oint_{b_k} \Omega(P, Q) = -\oint_{b_k} B(\bar{P}, Q) \,, \tag{5.2.13}$$

where the integrals are taken with respect to the first argument. The variational formulas for the Schiffer and Bergman kernels have the form:

$$\begin{aligned} \frac{\partial \Omega(P, Q)}{\partial \lambda_j} &= \frac{1}{2}\Omega(P, P_j)\Omega(Q, P_j) \,, & \frac{\partial \Omega(P, Q)}{\partial \bar{\lambda}_j} &= \frac{1}{2}B(P, \bar{P}_j)B(Q, \bar{P}_j) \,, \\ \frac{\partial B(P, \bar{Q})}{\partial \lambda_j} &= \frac{1}{2}\Omega(P, P_j)B(P_j, \bar{Q}) \,, & \frac{\partial B(P, \bar{Q})}{\partial \bar{\lambda}_j} &= \frac{1}{2}B(P, \bar{P}_j)\overline{\Omega(Q, P_j)} \,. \end{aligned} \tag{5.2.14}$$

The notation here is analogous to that in (5.2.5), i.e. $\Omega(P, P_j)$ stands for $(\Omega(P,Q)/dx_j(Q))\Big|_{Q=P_j}$ and $B(P,\bar{P}_j) := \left(B(P,\bar{Q})/\overline{dx_j(Q)}\right)\Big|_{Q=P_j}$.

Note that the Schiffer and Bergman kernels depend on both, $\{\lambda_k\}$ and $\{\bar{\lambda}_k\}$ (holomorphic and anti-holomorphic coordinates on the Hurwitz space), in contrast to bidifferentials W (5.2.2) and $W_{\mathbf{q}}$ (5.2.7), which depend only on holomorphic coordinates $\{\lambda_k\}$.

4. As an analogue of the deformation $W_{\mathbf{q}}$ (5.2.7) of the bidifferential W , we shall define deformations of the Schiffer and Bergman kernels, the bidifferentials $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,Q)$. Let us first introduce the holomorphic differentials

$$v_k(P) := \frac{1}{2\pi i}\oint_{b_k}\Omega(P,Q) \tag{5.2.15}$$

and the matrix $\mathbb{B}^{\Omega}$ of their b-periods ${\mathbb{B}^{\Omega}}_{kj} = \oint_{b_k} v_j$:

$$\mathbb{B}^{\Omega} := \bar{\mathbb{B}}(\bar{\mathbb{B}} - \mathbb{B})^{-1}\mathbb{B}\ .$$

This matrix is symmetric as can be seen from the following representation of $\mathbb{B}^{\Omega}$ as a sum of two symmetric matrices: $\mathbb{B}^{\Omega} = \mathbb{B}(\bar{\mathbb{B}} - \mathbb{B})^{-1}\mathbb{B} + \mathbb{B}$. Therefore, since $\mathbb{B}^{\Omega}$ is also anti-Hermitian, it is a purely imaginary matrix.

The differentials v_k can be characterized as holomorphic differentials on the Riemann surface of genus g whose all a- and b-periods are purely imaginary except one. Namely, for the differentials (5.2.15) we have $\mathrm{Re}\{\oint_{b_j} v_k\} = 0$ and $\mathrm{Re}\{\oint_{a_j} v_k\} = \delta_{jk}/2$ for $j,k = 1,\dots,g$. (Recall that by virtue of the Riemann bilinear relations, a holomorphic differential whose all periods are imaginary is zero.)

Remark 5.1 The differentials given by a-periods of the Schiffer kernel, $u_k(P) := -\oint_{a_k}\Omega(P,Q)/(2\pi i)$, can be described as holomorphic differentials satisfying the condition $\mathrm{Re}\{\oint_{b_j} u_k\} = \delta_{jk}/2$ and $\mathrm{Re}\{\oint_{a_j} u_k\} = 0$.

The matrix $\mathbb{B}^\Omega$ can also be expressed as the scalar product of the differentials v_k in the space $L_2^{(1,0)}(\mathcal{L})$ of $(1,0)$-forms, i.e. $\mathbb{B}^\Omega{}_{kl} = \iint_{\mathcal{L}} \overline{v_k(P)} \wedge v_l(P)$.

The variational formulas for the differentials v_k and the matrix $\mathbb{B}^\Omega$ are analogous to the Rauch formulas (5.2.6):

$$\frac{\partial v_k(P)}{\partial \lambda_j} = \frac{1}{2}\,\Omega(P,P_j)v_k(P_j)\,, \qquad \frac{\partial v_k(P)}{\partial \bar{\lambda}_j} = \frac{1}{2}\,B(P,\bar{P}_j)\overline{v_k(P_j)}\,, \tag{5.2.16}$$

$$\frac{\partial \mathbb{B}^\Omega{}_{kl}}{\partial \lambda_j} = \pi i\; v_k(P_j)\; v_l(P_j)\,, \qquad \frac{\partial \mathbb{B}^\Omega{}_{kl}}{\partial \bar{\lambda}_j} = \pi i\; \overline{v_k(P_j)}\;\overline{v_l(P_j)}\,. \tag{5.2.17}$$

The following deformed differentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ satisfy variational formulas which are similar to those for the kernels Ω and B (5.2.14). Consider a constant nondegenerate matrix $\mathbf{q}$ such that $\mathbf{q} = \mathbf{q}^T$, $\bar{\mathbf{q}} = -\mathbf{q}$ and the inverse $(\mathbb{B}^\Omega + \mathbf{q})^{-1}$ exists. Then, we define

$$\Omega_{\mathbf{q}}(P,Q) := \Omega(P,Q) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}^\Omega + \mathbf{q})^{-1}_{kl} v_k(P) v_l(Q)\,, \tag{5.2.18}$$

$$B_{\mathbf{q}}(P,\bar{Q}) := B(P,\bar{Q}) - 2\pi i \sum_{k,l=1}^{g} (\mathbb{B}^\Omega + \mathbf{q})^{-1}_{kl} v_k(P) \overline{v_l(Q)}\,. \tag{5.2.19}$$

The bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ turn into the Schiffer and Bergman kernels, respectively, when all entries of the matrix $(\mathbb{B}^\Omega + \mathbf{q})^{-1}$ tend to zero. This happens, for example, if all diagonal entries of the matrix $\mathbf{q}$ tend to infinity, and all off-diagonal entries remain finite.

The bidifferentials (5.2.18) and (5.2.19) are defined for the points of the Hurwitz space which do not belong to the subspace $\mathcal{D}^\Omega_{\mathbf{q}}$ of real codimension one given by the equation

$$\det(\mathbb{B}^\Omega + \mathbf{q}) = 0\,. \tag{5.2.20}$$

Similarly to the integral operator with the kernel $B(P,\bar{Q})/(2\pi i)$, the integral operator with the deformed kernel $B_{\mathbf{q}}(P,\bar{Q})/(2\pi i)$ also maps the space $L_2^{(1,0)}(\mathcal{L})$ onto $\mathcal{H}^{(1,0)}(\mathcal{L})$. In the space $\mathcal{H}^{(1,0)}(\mathcal{L})$ it acts as a linear operator which in the basis $\{v_k\}$ (5.2.15) is represented by the matrix $\mathbf{q}(\mathbb{B}^\Omega+\mathbf{q})^{-1}$. Namely, if we denote by $\mathbf{v}(P)$ the vector of differentials whose k-th component is the differential $v_k(P)$, then the following holds:

$$\frac{1}{2\pi i}\iint_{\mathcal{L}} B_{\mathbf{q}}(\bar{P},Q)\mathbf{v}(P)=\mathbf{q}(\mathbb{B}^\Omega+\mathbf{q})^{-1}\mathbf{v}(Q)\,.$$

The integral operator with the kernel $\Omega_{\mathbf{q}}(P,Q)/(2\pi i)$ acts in $\mathcal{H}^{(1,0)}(\mathcal{L})$ as follows:

$$\frac{1}{2\pi i}\iint_{\mathcal{L}} \Omega_{\mathbf{q}}(P,Q)\overline{\mathbf{v}(P)}=-\mathbb{B}^\Omega(\mathbb{B}^\Omega+\mathbf{q})^{-1}\mathbf{v}(Q)\,. \tag{5.2.21}$$

Note that when the matrix of parameters $\mathbf{q}$ tends to infinity so that the deformed bidifferential $\Omega_{\mathbf{q}}$ tends to the Schiffer kernel, the right hand side of (5.2.21) vanishes and this formula turns into the characteristic property of the Schiffer kernel Ω.

Periods of the bidifferentials (5.2.18) and (5.2.19) are related as follows: for any $k=1,\dots,g$

$$\begin{gathered}\oint_{a_k}\left(\Omega_{\mathbf{q}}(P,Q)+B_{\mathbf{q}}(P,\bar{Q})\right)+\sum_{j=1}^{g}(\mathbf{q}^{-1})_{kj}\oint_{b_j}\Omega_{\mathbf{q}}(P,Q)=0\,,\\ \oint_{b_k}\left(\Omega_{\mathbf{q}}(P,Q)+B_{\mathbf{q}}(P,\bar{Q})\right)=0\,,\end{gathered} \tag{5.2.22}$$

where the integrals are taken with respect to Q. The following variational formulas for $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ can be derived from (5.2.18), (5.2.19) by a straightforward computation using variational formulas (5.2.14), (5.2.16) and (5.2.17). They hold outside the subspace $\mathcal{D}^\Omega_{\mathbf{q}}$ (5.2.20):

$$\begin{aligned}\frac{\partial\Omega_{\mathbf{q}}(P,Q)}{\partial\lambda_j}&=\frac{1}{2}\Omega_{\mathbf{q}}(P,P_j)\Omega_{\mathbf{q}}(Q,P_j)\,, &\quad \frac{\partial\Omega_{\mathbf{q}}(P,Q)}{\partial\bar{\lambda}_j}&=\frac{1}{2}B_{\mathbf{q}}(P,\bar{P}_j)B_{\mathbf{q}}(Q,\bar{P}_j)\,,\\ \frac{\partial B_{\mathbf{q}}(P,\bar{Q})}{\partial\lambda_j}&=\frac{1}{2}\Omega_{\mathbf{q}}(P,P_j)B_{\mathbf{q}}(P_j,\bar{Q})\,, &\quad \frac{\partial B_{\mathbf{q}}(P,\bar{Q})}{\partial\bar{\lambda}_j}&=\frac{1}{2}B_{\mathbf{q}}(P,\bar{P}_j)\overline{\Omega_{\mathbf{q}}(Q,P_j)}\,.\end{aligned} \tag{5.2.23}$$

Remark 5.2 All defined bidifferentials except the Schiffer and Bergman kernels Ω and B, depend on the choice of a canonical basis of cycles $\{a_k;b_k\}$.

5.2.3 Darboux-Egoroff metrics defined by the bidifferentials

A diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ is called *potential* if there exists a function $U(\{\lambda_j\})$ whose derivatives give the metric coefficients: $g_{ii} = \partial_{\lambda_i} U$ for any i. A metric is called *flat* if its curvature tensor vanishes. A diagonal potential flat metric is called a *Darboux-Egoroff* metric. The Darboux-Egoroff lemma states that a diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ is Darboux-Egoroff if its *rotation coefficients* β_{ij} defined for $i \neq j$ by

$$\beta_{ij} = \frac{\partial_{\lambda_j}\sqrt{g_{ii}}}{\sqrt{g_{jj}}} \tag{5.2.24}$$

are symmetric, $\beta_{ij} = \beta_{ji}$, (this implies $\mathbf{ds^2}$ is potential) and satisfy the system of equations:

$$\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{kj}\ , \quad i, j, k \ \text{ are distinct,} \tag{5.2.25}$$

$$\sum_k \partial_{\lambda_k}\beta_{ij} = 0 \qquad \text{for all } \ \beta_{ij}\ . \tag{5.2.26}$$

Consider the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ of coverings $(\mathcal{L}, \lambda)$ described in Section 5.2.1. Let us fix an arbitrary contour l on the surface $\mathcal{L}$ which does not pass through ramification points $\{P_j\}$ of the covering and whose projection on $\mathbb{C}P^1$ does not change under small variations of the branch points $\{\lambda_j\}$. Let us also fix a function $h(P)$ defined in a neighbourhood of the contour l; asume this function to be independent of $\{\lambda_j\}$. Then the following formula defines (see [42]) a family of Darboux-Egoroff metrics on the Hurwitz space:

$$\mathbf{ds^2} = \sum_{j=1}^{L}\left(\oint_l h(Q)W(Q,P_j)\right)^2 (d\lambda_j)^2\ . \tag{5.2.27}$$

Following [15], we use the word "metric" for a bilinear quadratic (not necessary real and positive) form.

The variational formulas (5.2.4) for the W-bidifferential immediately imply that rotation coefficients for the metrics (5.2.27) are given by

$$\beta_{ij} = \frac{1}{2} W(P_i, P_j) \,, \tag{5.2.28}$$

where, as usual, the W-bidifferential is evaluated at ramification points with respect to the standard local parameter $x_j(P) = \sqrt{\lambda(P) - \lambda_j}$.

The following proposition was proven in [42]. Here we reproduce the proof given in [42] since an analogous procedure will be used in our present context.

Proposition 5.1 *[42] Rotation coefficients (5.2.28) are symmetric and satisfy equations (5.2.25), (5.2.26) and therefore metrics (5.2.27) are the Darboux-Egoroff metrics.*

Proof. The symmetry of the rotation coefficients follows from the symmetry of the bidifferential $W(P, Q)$ with respect to the arguments P and Q . Variational formulas (5.2.4) with $P = P_i$, $Q = P_k$, for different i, j, k imply relations (5.2.25) for rotation coefficients (5.2.28).

To verify relations (5.2.26) let us note that the differential operator $\sum_k \partial_{\lambda_k}$ in (5.2.26) can be represented as follows. Consider a biholomorphic map $(\mathcal{L}, \lambda) \to (\mathcal{L}, \lambda + \delta)$ of the covering which takes a point P to the point P^δ belonging to the same sheet and having projection $\lambda + \delta$ on the base of the covering. Then, for a function of branch points $f(\{\lambda_k\})$ we have $\sum_k \partial_{\lambda_k} f = (df^\delta/d\delta)|_{\delta=0}$, where f^δ is the analog of the function f on the covering $(\mathcal{L}, \lambda + \delta)$.

Note also that the definition of $W(P, Q)$ implies its invariance with respect to the map $(\mathcal{L}, \lambda) \to (\mathcal{L}, \lambda + \delta)$: if W^δ is the bidifferential W defined on the covering $(\mathcal{L}, \lambda + \delta)$ we have $W(P, Q) = W^\delta(P^\delta, Q^\delta)$. Since the local parameters $x_i(P) = \sqrt{\lambda(P) - \lambda_i}$ in neighbourhoods of ramification points also do not change under a simultaneous shift of all branch points and λ , we have $\sum_k \partial_{\lambda_k} W(P_i, P_j) = 0$. $\diamond$

Analogously, there exist families of Darboux-Egoroff metrics whose rotation coefficients are given by the other bidifferentials defined above.

Theorem 5.1 *Let a contour l and a function h be as in (5.2.27). Then the following formulas define the Darboux-Egoroff metrics on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ outside the divisor $\mathcal{D}_{\mathbf{q}}$ (5.2.9):*

$$\mathbf{ds^2} = \sum_{j=1}^{L} \left(\oint_l h(Q) W_{\mathbf{q}}(Q, P_j) \right)^2 (d\lambda_j)^2 . \tag{5.2.29}$$

The rotation coefficients of metrics (5.2.29) are given by $\beta_{ij} = W_{\mathbf{q}}(P_i, P_j)/2$ for $i \neq j$.

Proof. The proof of Proposition 5.1 obviously holds for $\beta_{ij} = W_{\mathbf{q}}(P_i, P_j)/2$, therefore the metrics (5.2.29) are Darboux-Egoroff. $\diamond$

The following bilinear quadratic forms were introduced in [68]; they can be considered as metrics on the real Hurwitz space, i.e. the moduli space of coverings with local coordinates $\{\lambda_k; \bar{\lambda}_k\}$. We shall denote the real Hurwitz space by $M^{\mathrm{real}}_{g;n_0,\dots,n_m}$. Let now the function h and the projection of the contour l onto the λ-sphere be independent of the coordinates $\{\lambda_k; \bar{\lambda}_k\}$. Consider the following two metrics:

$$\mathbf{ds^2} = \sum_{j=1}^{L} \left(\oint_l h(Q) \Omega(Q, P_j) \right)^2 (d\lambda_j)^2 + \sum_{j=1}^{L} \left(\oint_l h(Q) B(Q, \bar{P}_j) \right)^2 (d\bar{\lambda}_j)^2 \tag{5.2.30}$$

and

$$\mathbf{ds^2} = \mathrm{Re} \left\{ \sum_{j=1}^{L} \left(\oint_l h(Q) \Omega(Q, P_j) + \oint_l \overline{h(Q)} B(\bar{Q}, P_j) \right)^2 (d\lambda_j)^2 \right\} . \tag{5.2.31}$$

Both families, (5.2.30) and (5.2.31), have rotation coefficients given by

$$\beta_{ij} = \frac{1}{2}\Omega(P_i, P_j)\ , \qquad \beta_{i\bar{j}} = \frac{1}{2}B(P_i, \bar{P}_j)\ , \qquad \beta_{\bar{i}\bar{j}} = \overline{\beta_{ij}}\ , \tag{5.2.32}$$

where $i, j = 1, \ldots, L$ and the index $\bar{j}$ corresponds to differentiation with respect to $\bar{\lambda}_j$.

The proof of the flatness of these metrics is analogous to the proof of Proposition 5.1. The variational formulas (5.2.14) give relations (5.2.25) for rotation coefficients. To prove relations (5.2.26) we note that all bidifferentials are invariant with respect to the biholomorphic map $(\mathcal{L}, \lambda) \to (\mathcal{L}, \lambda + \delta)$ since all of them can be written in terms of $W(P, Q)$ (for example $2\pi i\, \omega_k = \oint_{b_k} W$). On the space M^{real} (we skip the indices for brevity), equations (5.2.26) read $\sum_{k=1}^{L}(\partial_{\lambda_k} + \partial_{\bar{\lambda}_k})\beta_{ij} = 0$; to prove them we apply the method of the proof of Proposition 5.1 with $\delta \in \mathbb{R}$ to the kernels Ω and B .

Since for finding rotation coefficients and proving the flatness of the metrics (5.2.30)-(5.2.31) we only used variational formulas for the Schiffer and Bergman kernels, which look identical to those for the bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$, the similar metrics can be written in terms of $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$. Therefore, we have the following theorem.

Theorem 5.2 *Let a contour l and a function h be as in (5.2.30), (5.2.31). Then the following formulas define Darboux-Egoroff metrics on the Hurwitz space $\widehat{M}^{\text{real}}_{g;n_0,\ldots,n_m}$ outside the subspace $\mathcal{D}^{\Omega}_{\mathbf{q}}$ (5.2.20):*

$$\mathbf{ds^2} = \sum_{j=1}^{L}\left(\oint_l h(Q)\Omega_{\mathbf{q}}(Q, P_j)\right)^2 (d\lambda_j)^2 + \sum_{j=1}^{L}\left(\oint_l h(Q)B_{\mathbf{q}}(Q, \bar{P}_j)\right)^2 (d\bar{\lambda}_j)^2 \tag{5.2.33}$$

and

$$\mathbf{ds^2} = \operatorname{Re}\left\{\sum_{j=1}^{L}\left(\oint_l h(Q)\Omega_{\mathbf{q}}(Q, P_j) + \oint_l \overline{h(Q)}B_{\mathbf{q}}(\bar{Q}, P_j)\right)^2 (d\lambda_j)^2\right\}\ . \tag{5.2.34}$$

The rotation coefficients of metrics of both families (5.2.33) and (5.2.34) are given by

$$\beta_{ij} = \frac{1}{2}\Omega_{\mathbf{q}}(P_i, P_j) , \qquad \beta_{i\bar{j}} = \frac{1}{2}B_{\mathbf{q}}(P_i, \bar{P}_j) , \qquad \beta_{\bar{i}\bar{j}} = \overline{\beta_{ij}} . \tag{5.2.35}$$

Note that coefficients of metrics (5.2.30), (5.2.31), written in terms of the Schiffer and Bergman kernels, do not depend on the choice of basis of cycles $\{a_k; b_k\}$. Therefore, those metrics are defined on the Hurwitz space M^{real}, whereas the metrics (5.2.33), (5.2.34) are defined on the covering $\widehat{M}^{\text{real}}$ of the Hurwitz space, i.e. in order to define metrics (5.2.33), (5.2.34) one must specify the choice of a canonical basis of cycles.

Each family of metrics (5.2.27), (5.2.29), (5.2.30)-(5.2.31) and (5.2.33)-(5.2.34) contains a class of metrics which correspond to Frobenius structures on the Hurwitz space. Such structures for metrics (5.2.27) were found in [15] (see also [68]). For the family (5.2.30)-(5.2.31) Frobenius structures were described in [68]. In this paper we shall construct Frobenius manifolds corresponding to the metrics (5.2.29) and (5.2.33)-(5.2.34). Thereby we shall construct deformations of the Hurwitz Frobenius manifolds of [15] and [68].

5.2.4 Systems of hydrodynamic type

A Darboux-Egoroff metric defines (see, for example, [73]) an integrable system of hydrodynamic type for the branch points $\{\lambda_k\}$ considered as functions of two independent coordinates x and t :

$$\partial_x \lambda_m = V_m(\{\lambda_k\}) \partial_t \lambda_m . \tag{5.2.36}$$

where the functions $\{V_m\}$, called the *characteristic speeds*, are related to the Christoffel symbols Γ^k_{nm} of the metric by:

$$\partial_{\lambda_m} V_n = \Gamma^n_{nm}(V_m - V_n) , \qquad m \neq n . \tag{5.2.37}$$

The nonvanishing Christoffel symbols for a diagonal metric $\mathbf{ds^2} = \sum_j g_{jj}(d\lambda_j)^2$ are given by:

$$\Gamma^k_{ii} = -\frac{1}{2}\frac{\partial_{\lambda_k} g_{ii}}{g_{kk}}\,, \qquad \Gamma^i_{ii} = \frac{1}{2}\frac{\partial_{\lambda_i} g_{ii}}{g_{ii}}\,, \qquad \Gamma^i_{ij} = \frac{1}{2}\frac{\partial_{\lambda_j} g_{ii}}{g_{ii}} \qquad i,j,k \text{ are distinct}\,. \tag{5.2.38}$$

If the metric $\mathbf{ds^2}$ is Darboux-Egoroff, then the equations (5.2.37) for characteristic speeds are compatible. In particular, for the metrics (5.2.27) the systems of hydrodynamic type (5.2.36) were constructed and solved in [42]. Here we note that analogous systems are associated with the Darboux-Egoroff metrics (5.2.29). Namely, if the metric $\mathbf{ds^2}$ belongs to the family (5.2.29), corresponding to the bidifferential $W_{\mathbf{q}}$, then the system (5.2.37) is defined on the Hurwitz space outside the divisor $\mathcal{D}_{\mathbf{q}}$ (5.2.9). Solutions to (5.2.37) are given by

$$V_m(\{\lambda_k\}) = \frac{\oint_{l_1} h_1(Q) W_{\mathbf{q}}(Q, P_m)}{\oint_l h(Q) W_{\mathbf{q}}(Q, P_m)}\;; \tag{5.2.39}$$

where the contour l and function h are those which define the metric $\mathbf{ds^2}$ as in (5.2.27); and l_1 and h_1 are such that the projection of the contour l_1 on the base of the covering and the function h_1 are independent of branch points $\{\lambda_j\}$. Relations (5.2.37) for the functions (5.2.39) can be verified by a simple calculation using the variational formulas (5.2.10) for the bidifferential $W_{\mathbf{q}}$.

Solutions to the system of hydrodynamic type (5.2.36) are constructed by the generalized *hodograph method* [73]. Namely, for the functions $V_m(\{\lambda_m\})$ which satisfy equations (5.2.37) consider an arbitrary solution $\{U_m(\{\lambda_k\})\}$ to the system

$$\frac{\partial_{\lambda_n} U_m}{U_m - U_n} = \frac{\partial_{\lambda_n} V_m}{V_m - V_n}\,, \qquad m,n = 1,\dots L\,. \tag{5.2.40}$$

Then, the system of equations

$$U_m(\{\lambda_k\}) = t + V_m(\{\lambda_k\})\, x \tag{5.2.41}$$

defines an implicit solution $\{\lambda_m(x,t)\}$ to the system of hydrodynamic type (5.2.36). A solution to the system (5.2.40) is obviously given by formulas (5.2.39) with some other pair $(l_2\,, h_2)$ instead of $(l_1\,, h_1)$.

Let us assume $l_1 = l_2 = l$. Then, the hodograph method for the system (5.2.36), (5.2.39) is summarized in the following theorem.

Theorem 5.3 *Let us fix a contour l on the covering which does not pass through ramification points. Consider functions h, h_1, h_2 defined in a neighbourhood of the contour. Assume that the functions and the projection of the contour l on $\mathbb{C}P^1$ are independent of the branch points $\{\lambda_j\}$. Then, a solution $\{\lambda_m(x,t)\}$ to the system of hydrodynamic type (5.2.36), (5.2.39) can be implicitly defined on $M_{g;n_0,\dots,n_m} \setminus \mathcal{D}_{\mathbf{q}}$ (where $\mathcal{D}_{\mathbf{q}}$ is the divisor (5.2.9)) by the following system:*

$$\oint_l (h_2(Q) - h(Q)t - h_1(Q)x)\, W_{\mathbf{q}}(Q, P_m) = 0\,, \qquad m = 1, \dots, L\,.$$

For families (5.2.30)-(5.2.31), (5.2.33)-(5.2.34) of Darboux-Egoroff metrics a naive definition – analogous to (5.2.36) – of systems of hydrodynamic type does not lead to a compatible system on variables $\{\lambda_k\}$ and $\{\bar\lambda_k\}$. In this case the equations on λ_k and $\bar\lambda_k$ are not complex conjugate to each other. However, an analogous procedure may work in the sense of analytic continuation, if λ_k and $\bar\lambda_k$ are considered as independent complex variables.

5.3 Deformations of Hurwitz Frobenius structures

5.3.1 Definition of Frobenius manifold

Definition 5.1 *A commutative associative algebra over* $\mathbb{C}$ *with a unity* $\mathbf{e}$ *is called a* **Frobenius algebra** *if it is supplied with a* $\mathbb{C}$*-bilinear symmetric nondegenerate inner product* $\langle\cdot,\cdot\rangle$ *which has the property* $\langle \mathbf{x}\cdot\mathbf{y},\mathbf{z}\rangle = \langle \mathbf{x},\mathbf{y}\cdot\mathbf{z}\rangle$ *for arbitrary elements* $\mathbf{x},\mathbf{y},\mathbf{z}$ *of the algebra.*

Definition 5.2 M *is a* **Frobenius manifold** *of charge* ν *if a structure of a Frobenius algebra is defined in any tangent plane* $T_\chi M$ *; this structure should smoothly depend on the point* $\chi\in M$ *and be such that*

F1 *the inner product* $\langle\cdot,\cdot\rangle$ *is a flat metric on* M *(not necessarily real positive definite);*

F2 *the unit vector field* $\mathbf{e}$ *is covariantly constant with respect to the Levi-Civita connection* ∇ *of the metric* $\langle\cdot,\cdot\rangle$ *, i.e. the covariant derivative in the direction of any vector field* $\mathbf{x}$ *on* M *vanishes:* $\nabla_{\mathbf{x}}\mathbf{e}=0$ *;*

F3 *the tensor* $(\nabla_{\mathbf{w}}\mathbf{c})(\mathbf{x},\mathbf{y},\mathbf{z})$ *is symmetric in four vector fields* $\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}$ *on* M *, where* $\mathbf{c}$ *is the following symmetric* 3*-tensor:* $\mathbf{c}(\mathbf{x},\mathbf{y},\mathbf{z})=\langle\mathbf{x}\cdot\mathbf{y},\mathbf{z}\rangle$ *;*

F4 *there exists a vector field* E *(the Euler vector field) such that for any pair of vector fields* $\mathbf{x}$ *and* $\mathbf{y}$ *on* M

$$\nabla_{\mathbf{x}}(\nabla_{\mathbf{y}}E)=0\ , \tag{5.3.1}$$

$$[E,\mathbf{x}\cdot\mathbf{y}]-[E,\mathbf{x}]\cdot\mathbf{y}-\mathbf{x}\cdot[E,\mathbf{y}]=\mathbf{x}\cdot\mathbf{y}\ , \tag{5.3.2}$$

$$\mathrm{Lie}_E\langle\mathbf{x},\mathbf{y}\rangle := E\langle\mathbf{x},\mathbf{y}\rangle-\langle[E,\mathbf{x}],\mathbf{y}\rangle-\langle\mathbf{x},[E,\mathbf{y}]\rangle=(2-\nu)\langle\mathbf{x},\mathbf{y}\rangle\ . \tag{5.3.3}$$

The structure described in Definition 5.2 is equivalent to the WDVV system (5.1.1)-(5.1.3). Requirement **F3** implies the existence of a function F depending on flat coordinates $t = \{t^A\}$ of the metric from **F1** whose third order derivatives give the tensor $\mathbf{c}$:

$$\frac{\partial^3 F(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \langle \partial_{t^A} \cdot \partial_{t^B}, \partial_{t^C} \rangle \ . \tag{5.3.4}$$

The associativity conditions of the Frobenius algebra are equivalent to the equations (5.1.1) and the existence of the vector field E from **F4** provides the quasihomogeneity (5.1.3) for the function F.

The function F defined by (5.3.4) up to a quadratic polynomial in flat coordinates is called the *prepotential* of the Frobenius manifold M.

Definition 5.3 *A Frobenius manifold M is called* **semisimple** *if for any point $\chi \in M$ the Frobenius algebra in the tangent space $T_\chi M$ has no nilpotents.*

For semisimple Frobenius manifolds, the flat metric in the definition of a Frobenius manifold is also diagonal and potential ([15], Lemmas 3.6-3.7), hence it is in fact a Darboux-Egoroff metric.

In this paper we only consider semisimple Frobenius manifolds.

5.3.2 Flat metrics

The Frobenius structures on Hurwitz spaces which correspond to the Darboux-Egoroff metrics of the type (5.2.27) were found by Dubrovin [15]. In [68] the construction of [15] was reformulated in terms of the bidifferential $W(P,Q)$ (5.2.2). Analyzing this construction one can see that it is essentially based on the following properties of $W(P,Q)$.

- The variational formulas (5.2.4) for $W(P,Q)$ which provide the flatness for the metrics (5.2.27).

- Invariance of $W(P,Q)$ with respect to two maps of coverings: $(\mathcal{L},\lambda) \to (\mathcal{L},\lambda+\delta)$ and $(\mathcal{L},\lambda) \to (\mathcal{L},(1+\epsilon)\lambda)$ which take a point P of the surface to the points P^δ and P^ϵ which lie on the same sheet of the covering and have projections $\lambda+\delta$ and $(1+\epsilon)\lambda$ on $\mathbb{C}P^1$, respectively. The bidifferential $W(P,Q)$ is invariant under the action of these two maps, i.e. we have $W^\delta(P^\delta,Q^\delta) = W(P,Q)$ and $W^\epsilon(P^\epsilon,Q^\epsilon) = W(P,Q)$, where W^δ and W^ϵ are the bidifferentials W defined on the corresponding coverings.

  These properties provide the validity of conditions (**F2**) and (**F4**) for a certain class of the metrics (5.2.27).

- The type of singularity of $W(P,Q)$ at $P \simeq Q$ (quadratic pole with biresidue 1).

- The normalization $\oint_{a_k} W(P,Q) = 0$ for all $k = 1,\dots,g$.

Let us notice that the bidifferential $W_{\mathbf{q}}(P,Q)$ (5.2.7) possesses a similar set of properties. The variational formulas (5.2.10) for $W_{\mathbf{q}}(P,Q)$ are identical to those for $W(P,Q)$. Furthermore, $W_{\mathbf{q}}(P,Q)$ is invariant with respect to the maps $(\mathcal{L},\lambda) \to (\mathcal{L},\lambda+\delta)$ and $(\mathcal{L},\lambda) \to (\mathcal{L},(1+\epsilon)\lambda)$ since it is expressed in terms of the bidifferential $W(P,Q)$, holomorphic normalized 1-forms $\{\omega_k\}_{k=1}^g$ and the matrix of b-periods $\mathbb{B}$. Finally, $W_{\mathbf{q}}(P,Q)$ has the same singularity structure as $W(P,Q)$ at $P \simeq Q$ and is normalized by (5.2.8).

Therefore, we conclude that in analogy with the construction of [15] it should be possible to find Frobenius structures for Darboux-Egoroff metrics from the family (5.2.29). Then $g(g+1)/2$ parameters contained in the bidifferential $W_{\mathbf{q}}(P,Q)$ will be inherited by the corresponding Frobenius manifolds.

Consider now the limit in which some of the entries of the matrix $\mathbf{q}$ tend to infinity in such a way that for *any* matrix $\mathbb{B}$ independent of $\mathbf{q}$ the matrix $(\mathbb{B}+\mathbf{q})^{-1}$ tends to the zero matrix (for example, let $\mathbf{q}_{ii} \to \infty$ for any i and $\mathbf{q}_{ij}$ be finite for $i \neq j$). In this limit the bidifferential $W_{\mathbf{q}}$ turns into W, and our construction coincides with that of [15]. For a finite constant symmetric matrix $\mathbf{q}$ it gives a $g(g+1)/2$-parametric deformation of Frobenius manifolds of [15].

Each matrix $\mathbf{q}$ defines by the equation (5.2.9) the divisor $\mathcal{D}_{\mathbf{q}}$ on the Hurwitz space $\widehat{M} = \widehat{M}_{g;n_0,\dots,n_m}$. We shall describe structures of the Frobenius manifolds corresponding to some metrics of the type (5.2.29). These Frobenius structures are defined on the Hurwitz space outside the divisor $\mathcal{D}_{\mathbf{q}}$.

The associative algebra is defined on each tangent space by

$$\partial_{\lambda_i} \cdot \partial_{\lambda_j} := \delta_{ij} \partial_{\lambda_i} \; ; \tag{5.3.5}$$

the coordinates $\{\lambda_j\}$ are thus canonical for multiplication. As is easy to see, the algebra (5.3.5) does not have nilpotents. The unit vector field is given by

$$\mathbf{e} = \sum_{i=1}^{L} \partial_{\lambda_i} \; . \tag{5.3.6}$$

For this multiplication a bilinear quadratic form $\langle , \rangle$ has the property $\langle x \cdot y, z \rangle = \langle x, y \cdot z \rangle$ if it is diagonal in the coordinates $\{\lambda_j\}$. Therefore the metrics (5.2.29) define a Frobenius algebra in the tangent space at each point of the Hurwitz space outside the divisor $\mathcal{D}_{\mathbf{q}}$.

The Euler vector field has the following standard form:

$$E := \sum_{i=1}^{L} \lambda_i \partial_{\lambda_i} \; . \tag{5.3.7}$$

It is easy to see that condition (5.3.2) is satisfied for the multiplication (5.3.5). The condition (5.3.3) for a diagonal metric $\mathbf{ds^2} = \sum_i g_{ii}(d\lambda_i)^2$ reduces to $E(g_{jj}) = -\nu g_{jj}$. To verify the requirement $\nabla_{\mathbf{x}}\mathbf{e} = 0$ (**F2**) we note that the metrics (5.2.29) are potential, i.e. $\partial_{\lambda_j} g_{ii} = \partial_{\lambda_i} g_{jj}$, and therefore, as is easy to check by a straightforward calculation, $\nabla_{\mathbf{x}}\mathbf{e} = 0$ holds if $\mathbf{e}(g_{jj}) = 0$. Thus, among the metrics (5.2.29), we need to find those which for some constant ν satisfy

$$E(g_{jj}) = -\nu g_{jj} \qquad \text{and} \qquad \mathbf{e}(g_{jj}) = 0\,. \tag{5.3.8}$$

The action of the vector fields $\mathbf{e}$ and E on a function of the canonical coordinates $\{\lambda_j\}$ only can be represented via the maps of coverings: $(\mathcal{L},\lambda) \to (\mathcal{L},\lambda+\delta)$ and $(\mathcal{L},\lambda) \to (\mathcal{L},(1+\epsilon)\lambda)$, respectively. These maps take a point P of the surface to the points P^δ and P^ϵ which lie on the same sheet of the covering and have projections $\lambda+\delta$ and $(1+\epsilon)\lambda$ on $\mathbb{C}P^1$ (i.e. $\lambda(Q^\delta) = \lambda(Q)+\delta$ and $\lambda(Q^\epsilon) = (1+\epsilon)\lambda(Q)$). The bidifferential $W_{\mathbf{q}}(P,Q)$ is invariant under the action of these two maps, i.e. we have $W^\delta_{\mathbf{q}}(P^\delta,Q^\delta) = W_{\mathbf{q}}(P,Q)$ and $W^\epsilon_{\mathbf{q}}(P^\epsilon,Q^\epsilon) = W_{\mathbf{q}}(P,Q)$, where $W^\delta_{\mathbf{q}}$ and $W^\epsilon_{\mathbf{q}}$ are the bidifferentials $W_{\mathbf{q}}$ defined on the corresponding coverings. For the evaluation of $W_{\mathbf{q}}(P,Q)$ at $P=P_j$ we have to take into account transformations of the standard local parameter near a ramification point: $x^\delta_j(P^\delta) = x_j(P)$ and $x^\epsilon_j(P^\epsilon) = \sqrt{1+\epsilon}\, x_j(P)$.

Then it is easy to see that the requirement $E(g_{jj}) = -\nu g_{jj}$ is satisfied for a metric of the type (5.2.29) if $h(Q) = const\ \lambda^n(Q)$ and the contour l is invariant under the map

$\lambda \to (1+\epsilon)\lambda$, i.e. if it is either a closed contour or a contour connecting points ∞^i and ∞^j :

$$E(g_{jj}) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(\oint_{l^\epsilon} \lambda^n(Q^\epsilon) \frac{W^\epsilon_{\mathbf{q}}(Q^\epsilon, P^\epsilon)}{dx^\epsilon_j(P^\epsilon)}\Big|_{P=P_j} \right)^2$$
$$= \frac{d}{d\epsilon}\Big|_{\epsilon=0} (1+\epsilon)^{2n-1} \left(\oint_l \lambda^n(Q) \frac{W_{\mathbf{q}}(Q,P)}{dx_j(P)}\Big|_{P=P_j} \right)^2 = (2n-1) g_{jj} \; . \quad (5.3.9)$$

The condition $\mathbf{e}(g_{jj}) = 0$ holds if the combination of a contour l and a function $h(Q) = const\ \lambda(Q)$ in (5.2.29) is one of the following combinations. Let us write these combinations in the form of integral operations applied to some $(1,0)$-form $f(Q)$ on the surface:

1. $\mathrm{I}_{t^{i;\alpha}}[f(Q)] := \frac{1}{\alpha} \operatorname{res}_{\infty^i} \lambda(Q)^{\frac{\alpha}{n_i+1}} f(Q)$ $\qquad i = 0, \ldots, m\ ;\ \alpha = 1, \ldots, n_i$.

2. $\mathrm{I}_{v^i}[f(Q)] := \operatorname{res}_{\infty^i} \lambda(Q) f(Q)$ $\qquad i = 1, \ldots, m$.

3. $\mathrm{I}_{w^i}[f(Q)] := \text{v.p.} \int_{\infty^0}^{\infty^i} f(Q)$ $\qquad i = 1, \ldots, m$.

4. $\mathrm{I}_{r^k}[f(Q)] := -\oint_{a_k} \lambda(Q) f(Q) - \sum_{n=1}^{g} (\mathbf{q}^{-1})_{nk} \oint_{b_n} \lambda(Q) f(Q)$ $\qquad k = 1, \ldots, g$.

5. $\mathrm{I}_{s^k}[f(Q)] := \frac{1}{2\pi i} \oint_{b_k} f(Q)$ $\qquad k = 1, \ldots, g$.

Here the principal value near infinity is defined by omitting the divergent part of an integral as a function of the local parameter z_i (such that $\lambda = z_i^{-n_i-1}$). The number of operations is $L = \sum_{i=0}^m n_i + 2m + 2g$, where $\sum_{i=0}^m (n_i + 1) = N$, according to the Riemann-Hurwitz formula (5.2.1).

We shall denote the set of operations 1.-5. by $\{\mathrm{I}_{t^A}\}$, i.e. we define the index t^A by $t^A \in \{t^{i;\alpha}\ ;\ v^i, w^i\ ;\ r^k, s^k\}$. Here, t^A is used as a formal index, however, later it will denote a flat coordinate of the flat metric of a Frobenius manifold.

Theorem 5.4 *Let us choose a point P_0 on the surface which is mapped to zero by the function λ, i.e. $\lambda(P_0) = 0$, and let all basic contours $\{a_k, b_k\}$ on the surface start at this* *point. Let the constant matrix* $\mathbf{q}$ *be symmetric nondegenerate and such that* $\det(\mathbb{B}+\mathbf{q}) \neq 0$ *. Then, the operations* I_{t^A} *applied to* $f_P(Q) := W_{\mathbf{q}}(P,Q)$ *give a set of* L *differentials, called primary, whose characteristic properties are listed below.*

Primary differential	**Characteristic property**	
1. $\phi_{t^{i;\alpha}}(P) := \mathrm{I}_{t^{i;\alpha}}[W_{\mathbf{q}}(P,Q)]$	$\sim z_i^{-\alpha-1}(P)dz_i(P)$, $P \sim \infty^i$;	$i = 0, ..., m$;
	single valued on $\mathcal{L}$;	$\alpha = 1, ..., n_i$.
2. $\phi_{v^i}(P) := \mathrm{I}_{v^i}[W_{\mathbf{q}}(P,Q)]$	$\sim -d\lambda(P)$, $P \sim \infty^i$;	$i = 1, \ldots, m$.
	single valued on $\mathcal{L}$;	
3. $\phi_{w^i}(P) := \mathrm{I}_{w^i}[W_{\mathbf{q}}(P,Q)]$:	$\operatorname{res}_{\infty^i} \phi_{w^i} = 1$; $\operatorname{res}_{\infty^0} \phi_{w^i} = -1$;	$i = 1, \ldots, m$.
	single valued on $\mathcal{L}$;	
4. $\phi_{r^k}(P) := \mathrm{I}_{r^k}[W_{\mathbf{q}}(P,Q)]$:	*has no poles,*	$k = 1, \ldots, g$.
	$\phi_{r^k}(P^{a_j}) - \phi_{r^k}(P) = -2\pi i(\mathbf{q}^{-1})_{kj}d\lambda(P)$,	
	$\phi_{r^k}(P^{b_j}) - \phi_{r^k}(P) = \delta_{kj}2\pi i d\lambda(P)$;	
5. $\phi_{s^k}(P) := \mathrm{I}_{s^k}[W_{\mathbf{q}}(P,Q)]$:	*holomorphic differential on* $\mathcal{L}$.	$k = 1, \ldots, g$.

Here z_i *is the local parameter near* ∞^i *such that* $z_i^{-n_i-1} = \lambda$; n_i *is the ramification index at* ∞^i ; $\phi(P^{a_j}) - \phi(P)$ *and* $\phi(P^{b_j}) - \phi(P)$ *denote the transformations of a differential* ϕ *under analytic continuation along cycles* a_j *and* b_j *, respectively.*

The primary differentials **1.-5.** *satisfy the following normalization condition (*δ *is the Kronecker symbol):*

$$\oint_{a_k} \phi_{t^A} + \sum_{n=1}^{g} (\mathbf{q}^{-1})_{nk} \oint_{b_n} \phi_{t^A} = \delta_{t^A, s^k} \ . \tag{5.3.10}$$

Let ϕ be one of the primary differentials. Then, the following metrics

$$\mathbf{ds}^2_\phi = \frac{1}{2}\sum_{i=1}^{L}\phi^2(P_i)(d\lambda_i)^2 \tag{5.3.11}$$

belong to the family (5.2.29). Their entries $g_{ii} = \phi^2(P_i)/2$ satisfy the relation $\mathbf{e}(g_{ii}) = 0$.

We shall denote the set of differentials by $\{\phi_{t^A}\}$, i.e. we assume that the index t^A belongs to the set of indices $\{t^{i;\alpha}\ ;\ v^i\ , w^i\ ;\ r^k\ , s^k\}$.

Proof. According to the assumption made in the theorem, the cycles a_k and b_k intersect each other at the point P_0 such that $\lambda(P_0) = 0$. Therefore, as can be verified by a simple local calculation in a neighbourhood of the point P_0, the order of integration can be changed in the integral $\oint_{a_k}\oint_{b_k}\lambda(P)\Omega(P,Q)$. Similarly, one can prove that the following change of order of integration is valid:

$$\begin{aligned}
&\oint_{a_k}\mathrm{I}_{t^A}[W_{\mathbf{q}}(P,Q)] = \mathrm{I}_{t^A}\left[\oint_{a_k}W_{\mathbf{q}}(P,Q)\right] + \delta_{t^A,s^k}\ ;\\
&\oint_{b_k}\mathrm{I}_{t^A}[W_{\mathbf{q}}(P,Q)] = \mathrm{I}_{t^A}\left[\oint_{b_k}W_{\mathbf{q}}(P,Q)\right]\ .
\end{aligned} \tag{5.3.12}$$

Therefore, the normalization (5.2.8) of $W_{\mathbf{q}}(P,Q)$ implies the normalization (5.3.10) of the primary differentials.

Now we shall use the invariance of the bidifferential W under the biholomorphic map of coverings $(\mathcal{L}, \lambda) \to (\mathcal{L}, \lambda+\delta)$ to prove that the unit vector field annihilates coefficients of the metrics (5.3.11). The action of the vector field $\mathbf{e}$ on $\phi(P_j)$ is given by the derivative $(d/d\delta)\phi^\delta(P_j^\delta)|_{\delta=0}$, where ϕ^δ and P^δ are the corresponding objects on the covering $(\mathcal{L}, \lambda+\delta)$.

For the primary differential $\phi_{t^{i;\alpha}}$ we have:

$$\begin{aligned}
\mathbf{e}(\phi_{t^{i;\alpha}}(P_j)) &= \frac{d}{d\delta}\Big|_{\delta=0}\left\{\phi^{\delta}_{t^{i;\alpha}}(P^{\delta}_j)\right\} = \frac{d}{d\delta}\Big|_{\delta=0}\left\{\frac{1}{\alpha}\operatorname*{res}_{\infty^i}(\lambda(P)+\delta)^{\frac{\alpha}{n_i+1}}W^{\delta}_{\mathbf{q}}(P^{\delta},P^{\delta}_j)\right\} \\
&= \frac{d}{d\delta}\Big|_{\delta=0}\left\{\frac{1}{\alpha}\operatorname*{res}_{\infty^i}\left(z_i^{-\alpha}(P)+\frac{\alpha}{n_i+1}(z_i(P))^{-\alpha+n_i+1}\delta+\mathcal{O}(\delta^2)\right)W_{\mathbf{q}}(P,P_j)\right\} \\
&= \frac{1}{n_i+1}(z_i(P))^{-\alpha+n_i+1}W_{\mathbf{q}}(P,P_j)\ ,
\end{aligned}$$

which is zero for $\alpha=1,\dots,n_i+1$ (for $\alpha=n_i+1$ this computation shows that $\mathbf{e}(\phi_{v^i}(P_j))$ $=0$). For primary differentials ϕ_{w^i} and ϕ_{s^k} , the relation $\mathbf{e}(\phi(P_j))=0$ follows from the invariance of $W_{\mathbf{q}}(P,Q)$ and the path of integration under the map $\lambda\to\lambda+\delta$. For $\phi=\phi_{r^k}$, this relation easily follows from the vanishing of the combination of periods (5.2.8) for $W_{\mathbf{q}}(P,Q)$. $\diamond$

Thus, we have L (see the Riemann-Hurwitz formula (5.2.1)) Darboux-Egoroff metrics (5.3.11) which satisfy the requirements of the definition of a Frobenius manifold.

The next lemma shows that one uniquely specifies a holomorphic differential by fixing the values of combinations of its periods which appear in the right hand side in (5.3.10) .

Lemma 5.1 *Let $\mathcal{L}$ be a Riemann surface and $\mathbb{B}$ be its matrix of b-periods. Consider a constant symmetric nondegenerate matrix $\mathbf{q}$ such that the sum $(\mathbb{B}+\mathbf{q})$ is also nondegenerate. Then a holomorphic differential ω on the surface $\mathcal{L}$ vanishes if for every $k=1,\dots,g$*

$$\oint_{a_k}\omega+\sum_{n=1}^{g}(\mathbf{q}^{-1})_{nk}\oint_{b_n}\omega=0\ . \tag{5.3.13}$$

Proof. A holomorphic differential ω can be represented as a linear combination of the holomorphic differentials ω_k normalized by the condition $\oint_{a_j}\omega_k=\delta_{jk}$. Then, the lemma can be proved by a simple calculation using the well-known fact that a holomorphic differential vanishes if all its a-periods vanish. $\diamond$

Proposition 5.2 *Let w be a differential on the Riemann surface having only poles with a given singular part and (or) a given non-singlevaluedness of additive type along basic cycles. Then the differential w can be uniquely fixed by specifying the values of the combinations $\oint_{a_k} w + \sum_{n=1}^{g} (\mathbf{q}^{-1})_{nk} \oint_{b_n} w$ of its periods for each $k = 1, \ldots, g$, where the constant symmetric matrix $\mathbf{q}$ is such that* $\det(\mathbb{B} + \mathbf{q}) \neq 0$.

Proof. Suppose there exist two differentials with identical singularity structures of the type described in the proposition. Then, their difference is zero by virtue of Lemma 5.1. $\diamond$

5.3.3 Flat coordinates

For a flat metric there exists a set of *flat coordinates.* These are coordinates in which coefficients of the metric are constant. The Christoffel symbols in flat coordinates vanish and the covariant derivative ∇_{t^A} along the vector field in the direction of the flat coordinate t^A coincides with the usual partial derivative ∂_{t^A} . Therefore, flat coordinates can be found from equations $\nabla_{\mathbf{x}} \nabla_{\mathbf{y}}\, t = 0$ where $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vector fields on the manifold. The next theorem shows that flat coordinates of the metric (5.3.11) can be found by applying the operations I_{t^A} to the primary differential ϕ which defines the metric.

Theorem 5.5 *The following functions form a set of flat coordinates of the metric* $\mathbf{ds}^2_\phi$ *(5.3.11):*

$$
\begin{aligned}
t^{i;\alpha} &:= -\,(n_i+1)\;\; \mathrm{I}_{t^{i;1+n_i-\alpha}}[\phi] = \frac{n_i+1}{\alpha-n_i-1} \operatorname*{res}_{\infty^i} z_i^{\alpha-n_i-1}\phi && i=0,\ldots,m\,;\ \alpha=1,\ldots,n_i \\
v^i &:= -\,\mathrm{I}_{w^i}[\phi] \quad = -\,\text{v.p.} \int_{\infty^0}^{\infty^i} \phi && i=1,\ldots,m \\
w^i &:= -\,\mathrm{I}_{v^i}[\phi] \quad = -\operatorname*{res}_{\infty^i} \lambda\phi && i=1,\ldots,m
\end{aligned}
$$

$$r^k := \mathrm{I}_{s^k}[\phi] = \frac{1}{2\pi i}\oint_{b_k} \phi \qquad\qquad k = 1, \ldots, g$$

$$s^k := \mathrm{I}_{r^k}[\phi] = -\oint_{a_k} \lambda\phi - \sum_{n=1}^{g} (\mathbf{q}^{-1})_{kn} \oint_{b_n} \lambda\phi \qquad\qquad k = 1, \ldots, g\ .$$

As before, we denote the above functions by $\{t^A\}$ *, i.e. we assume* $t^A \in \{t^{i;\alpha}\ ;\ v^i\ , w^i\ ;\ r^k\ , s^k\}$.

Proof. Let us verify that the functions $\{t^A\}$ satisfy equations $\nabla_{\mathbf{x}}\nabla_{\mathbf{y}}\, t = 0$ defining flat coordinates of the metric $\mathbf{ds}^2_\phi$. These equations can be rewritten for the basis vector fields $\mathbf{x}, \mathbf{y} \in \{\partial_{\lambda_j}\}$ in canonical coordinates $\{\lambda_k\}$ as follows:

$$\partial_{\lambda_i}\partial_{\lambda_j} t = \sum_{k=1}^{L} \Gamma^k_{ij} \partial_{\lambda_k} t\ , \qquad i, j = 1, \ldots, L\ , \tag{5.3.14}$$

where Γ^k_{ij} denote the Christoffel symbols for the Levi-Civita connection of the metric $\mathbf{ds}^2_\phi$.

The variational formulas (5.2.10) for $W_{\mathbf{q}}(P,Q)$ imply the following expressions for derivatives of primary differentials:

$$\frac{\partial \phi_{t^A}(P)}{\partial \lambda_j} = \frac{1}{2}\phi_{t^A}(P_j) W_{\mathbf{q}}(P, P_j)\ . \tag{5.3.15}$$

Using (5.3.15) we find the nonzero Christoffel symbols for the diagonal metric $\mathbf{ds}^2_\phi$ in terms of the primary differential ϕ :

$$\Gamma^k_{ik} = \beta_{ik}\frac{\phi(P_i)}{\phi(P_k)} = -\Gamma^k_{ii} \qquad \text{for } k \neq i\ ; \quad \text{and} \qquad \Gamma^k_{kk} = -\sum_{j, j\neq k} \Gamma^k_{kj}\ .$$

To prove the last equality one uses the fact that the unit vector field annihilates coefficients of the metric $\mathbf{ds}^2_\phi$ (5.3.11). Then, system (5.3.14) takes the form:

$$\partial_{\lambda_i}\partial_{\lambda_j} t = \beta_{ij}\left(\frac{\phi(P_j)}{\phi(P_i)}\partial_i t + \frac{\phi(P_i)}{\phi(P_j)}\partial_j t\right)\ , \qquad i \neq j\ , \tag{5.3.16}$$

$$\mathbf{e}(t) = const\ . \tag{5.3.17}$$

To show that the system (5.3.16)-(5.3.17) is equivalent to (5.3.14) we differentiate (5.3.17) with respect to λ_j and use the expressions for Christoffel symbols in terms of the primary differential ϕ.

A straightforward differentiation using (5.3.15) shows that the functions listed in the theorem satisfy (5.3.16). To prove that (5.3.17) holds for the functions $\{t^A\}$ we again consider the transformations $\{t^{A\delta}\}$ of these functions under the map $\lambda \to \lambda + \delta$. Then, we find the action of the unit vector field on $\{t^A\}$ using the relation $\mathbf{e}(t^A) = (d/d\delta)t^{A\delta}|_{\delta=0}$ (see the proof of Theorem 5.4). $\diamond$

The constant in (5.3.17) can be found by the method described in the proof of Theorem 5.5; it is nonzero (equals -1) only if t is the flat coordinate of the same type as the primary differential ϕ which defines the metric $\mathbf{ds}^2_\phi$. Therefore we have the following corollary which shows again that the unit vector field is covariantly constant (**F2**).

Corollary 5.1 *The unit vector field* $\mathbf{e}$ *(5.3.6) in the flat coordinates* $\{t^A\}$ *of the metric* $\mathbf{ds}^2_\phi$ *defined by the primary differential* $\phi = \phi_{t^{A_0}}$ *has the form:* $\mathbf{e} = -\partial_{t^{A_0}}$.

Let us denote by t^1 the flat coordinate t^{A_0} of the metric defined by the primary differential $\phi_{t^{A_0}}$ so that $\mathbf{e} = -\partial_{t^1}$.

For each primary differential ϕ it is convenient to consider a multivalued differential Ψ_ϕ defined by:

$$\Psi_\phi(P) := \left(\text{v.p.} \int_{\infty^0}^{P} \phi\right) d\lambda \, , \tag{5.3.18}$$

where the principal value near ∞^0 is defined by omitting the divergent part as a function of the local parameter z_0. This differential

- is singular at the points ∞^i. The nonconstant coefficients in expansions near ∞^i are given by the flat coordinates $\{t^A\}$ of the corresponding metric $\mathbf{ds}^2_\phi$. For $i \neq 0$ we have

$$\Psi_\phi(P) \underset{P\sim\infty^i}{=} \text{singular part} + \left(v^i(n_i+1)z_i^{-n_i-2} + \sum_{\alpha=1}^{n_i} t^{i;\alpha} z_i^{-\alpha-1} + w^i z_i^{-1} + \mathcal{O}(1) \right) dz_i \,. \quad (5.3.19)$$

The coordinates $t^{0;\alpha}$ appear similarly in expansion in a neighbourhood of ∞^0.

- transforms as follows under analytic continuation along the cycles $\{a_k; b_k\}$:

$$\Psi_\phi(P^{a_k}) - \Psi_\phi(P) = -2\pi i \sum_{n=1}^{g} (\mathbf{q}^{-1})_{kn} r^n d\lambda + \delta_{\phi,\phi_{s^k}} d\lambda - 2\pi i (\mathbf{q}^{-1})_{kk} \delta_{\phi,\phi_{r^k}} d\lambda \,, \quad (5.3.20)$$

$$\Psi_\phi(P^{b_k}) - \Psi_\phi(P) = 2\pi i r^k d\lambda + \delta_{\phi,\phi_{r^k}} 2\pi i d\lambda \,. \quad (5.3.21)$$

- is such that the combinations from (5.3.10) of its a- and b-periods are given by coordinates s^k :

$$\oint_{a_k} \Psi_\phi + \sum_{n=1}^{g} (\mathbf{q}^{-1})_{kn} \oint_{b_n} \Psi_\phi = s^k \,. \quad (5.3.22)$$

For each ϕ the multivalued differential Ψ_ϕ generates the set of primary differentials according to the following theorem.

Theorem 5.6 *Derivatives of the multivalued differential Ψ_ϕ (5.3.18) with respect to the flat coordinates t^A from Theorem 5.5 of the metric $\mathbf{ds}^2_\phi$ are given by the corresponding primary differentials:*

$$\frac{\partial \Psi_\phi}{\partial t^A} = \phi_{t^A} \quad (5.3.23)$$

(we notice the independence of this derivative of the choice of a primary differential ϕ).

Proof. Consider the differential $\partial_{t^A}\Psi_\phi$. From formulas (5.3.19) - (5.3.22) we see that its properties (expansions near the points ∞^i, transformations along the cycles $\{a_k; b_k\}$ and the normalization (5.3.10)) coincide with analogous properties of the primary differential ϕ_{t^A} . Thus, the differentials $\partial_{t^A}\Psi_\phi$ and ϕ_{t^A} are equal by virtue of Proposition 5.2. $\diamond$

Corollary 5.2 *The derivatives of canonical coordinates $\{\lambda_j\}$ with respect to the flat coordinates $\{t^A\}$ of the metric $\mathbf{ds}^2_\phi$ are given by*

$$\frac{\partial \lambda_j}{\partial t^A} = -\frac{\phi_{t^A}(P_j)}{\phi(P_j)} \,. \tag{5.3.24}$$

Proof. We shall use the reciprocity identity $\partial_\alpha(fdg)_{g=const} = -\partial_\alpha(gdf)_{f=const}$. It holds for two functions f and g which can be locally expressed as functions of each other and some parameters $\{p_\alpha\}$, i.e. $f = f(g; p_1, \dots, p_n)$ and $g = g(f; p_1, \dots, p_n)$; where ∂_α stands for a derivative with respect to the parameter p_α . The reciprocity identity can be proven by differentiation of the identity $f(g(f;p);p) \equiv f$ with respect to p_α , i.e. $\partial_\alpha g df/dg + \partial_\alpha f = 0$. For $f(P) = \int_{\infty^0}^P \phi$ and $g(P) = \lambda(P)$ we have

$$(\partial_{t^A} \int_{\infty^0}^P \phi) d\lambda = -(\partial_{t^A}\lambda(P))\phi(P) \,. \tag{5.3.25}$$

Using (5.3.23) and $\lambda'(P_j) = 0$ we evaluate (5.3.25) at the critical points $P = P_j$ to obtain (5.3.24). $\diamond$

5.3.4 Prepotential of Frobenius structures

To complete the construction of Frobenius manifolds we need to show that requirement **F3** holds. This can be done by constructing a prepotential, i.e. a function F of flat coordinates $\{t^A\}$ of the corresponding metric $\mathbf{ds}_\phi$ such that

$$\frac{\partial^3 F_\phi}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = \mathbf{ds}^2_\phi(\partial_{t^A}\cdot\partial_{t^B}, \partial_{t^C}) \,. \tag{5.3.26}$$

First, we need to define a pairing of differentials. Let $\omega^{(1)}$ and $\omega^{(2)}$ be two differentials on the surface $\mathcal{L}$ holomorphic outside of the points $\infty^0, \ldots, \infty^m$ with the following behaviour at ∞^i :

$$\omega^{(\alpha)} = \sum_{n=-n^{(\alpha)}}^{\infty} c_{n,i}^{(\alpha)} z_i^n dz_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r_{n,i}^{(\alpha)} \lambda^n \log \lambda\right) , \quad P \sim \infty^i , \tag{5.3.27}$$

where $n^{(\alpha)} \in \mathbb{Z}$ and $c_{n,i}^{(\alpha)}$, $r_{n,i}^{(\alpha)}$ are some coefficients; $z_i = z_i(P)$ is a local parameter near ∞^i. Denote also for $k = 1, \ldots, g$ the coefficients $A_k^{(\alpha)}$ to be

$$A_k^{(\alpha)} := \oint_{a_k} \omega^{(\alpha)} + \sum_{n=1}^{g} (\mathbf{q}^{-1})_{kn} \oint_{b_n} \omega^{(\alpha)} . \tag{5.3.28}$$

Again for $k = 1, \ldots, g$, denote the transformations of differentials under analytic continuation along the cycles $\{a_k; b_k\}$ of the Riemann surface by:

$$dp_k^{(\alpha)}(\lambda(P)) := \omega^{(\alpha)}(P^{a_k}) - \omega^{(\alpha)}(P) , \quad p_k^{(\alpha)}(\lambda) = \sum_{s>0} p_{sk}^{(\alpha)} \lambda^s , \tag{5.3.29}$$

$$dq_k^{(\alpha)}(\lambda(P)) := \omega^{(\alpha)}(P^{b_k}) - \omega^{(\alpha)}(P) , \quad q_k^{(\alpha)}(\lambda) = \sum_{s>0} q_{sk}^{(\alpha)} \lambda^s . \tag{5.3.30}$$

Note that the coefficients defined by (5.3.27) - (5.3.30) for the primary differentials do not depend on coordinates $\{t^A\}$ in contrast to the analogous coefficients for the differential Ψ_ϕ .

Definition 5.4 *Let $\omega^{(\alpha)}$ and $\omega^{(\beta)}$ be differentials which do not have singularities other than those described by (5.3.27)-(5.3.30). The* **pairing** *$\mathcal{F}[\ ,\ ]$ of such differentials is defined by:*

$$\mathcal{F}[\omega^{(\alpha)} , \omega^{(\beta)}] := \sum_{i=0}^{m} \left(\sum_{n\geq 0} \frac{c_{-n-2,i}^{(\alpha)}}{n+1} c_{n,i}^{(\beta)} + c_{-1,i}^{(\alpha)} \text{v.p.} \int_{P_0}^{\infty^i} \omega^{(\beta)} - \text{v.p.} \int_{P_0}^{\infty^i} \sum_{n>0} r_{n,i}^{(\alpha)} \lambda^n \omega^{(\beta)} \right)$$
$$+ \frac{1}{2\pi i} \sum_{k=1}^{g} \left(-\oint_{a_k} q_k^{(\alpha)}(\lambda) \omega^{(\beta)} + \oint_{b_k} p_k^{(\alpha)}(\lambda) \omega^{(\beta)} + A_k^{(\alpha)} \oint_{b_k} \omega^{(\beta)} \right) , \tag{5.3.31}$$

where P_0 is a point on the surface such that $\lambda(P_0) = 0$.

Note that the pairing is defined so that the following holds:

$$\mathcal{F}[\phi_{t^A},\omega^{(\beta)}]=\mathrm{I}_{t^A}[\omega^{(\beta)}] \qquad \text{and} \qquad \mathcal{F}[\Psi_\phi,\phi_{t^A}]=\mathrm{I}_{t^A}[\Psi_\phi]\,. \tag{5.3.32}$$

Here $\omega^{(\beta)}$ is any differential for which the pairing is defined.

The last relation can be checked by a straightforward computation using (5.3.12) and Proposition 5.2. Now it is easy to prove the next theorem.

Theorem 5.7 *Let us choose one of the primary differentials ϕ given by Theorem 5.4 and build the multivalued differential Ψ_ϕ (5.3.18). The following function gives a prepotential of the Frobenius structure defined by the metric $\mathbf{ds}^2_\phi$, multiplication (5.3.5) and the Euler field (5.3.7) on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ outside the divisor $\mathcal{D}_\mathbf{q}$ (5.2.9):*

$$F_\phi=\frac{1}{2}\mathcal{F}[\Psi_\phi\ ,\ \Psi_\phi]\,. \tag{5.3.33}$$

The second derivatives of the prepotential F_ϕ with respect to flat coordinates are given by the pairing of the corresponding primary differentials:

$$\partial_{t^A}\partial_{t^B}F_\phi=\mathcal{F}[\phi_{t^A}\ ,\ \phi_{t^B}]\,. \tag{5.3.34}$$

Proof. Differentiating the function F_ϕ (5.3.33) using (5.3.32) we obtain:

$$\partial_{t^A}F_\phi=\frac{1}{2}\mathcal{F}[\phi_{t^A},\Psi_\phi]+\frac{1}{2}\mathcal{F}[\Psi_\phi,\phi_{t^A}]=\mathcal{F}[\phi_{t^A},\Psi_\phi]\,. \tag{5.3.35}$$

Since the coefficients defined by (5.3.27) - (5.3.30) for a primary differential do not depend on coordinates, the differentiation of both sides in (5.3.35) gives (5.3.34). To find the third order derivatives of the function F_ϕ let us write the vector ∂_{t^A} (using Corollary 5.2) in the form:

$$\partial_{t^A}=-\sum_{i=1}^{L}\frac{\phi_{t^A}(P_i)}{\phi(P_i)}\partial_{\lambda_i}\,. \tag{5.3.36}$$

A straightforward computation using (5.3.34), (5.3.36) and the expression (5.3.15) for derivatives of primary differentials with respect to canonical coordinates shows that the third derivatives coincide with the tensor $\mathbf{c}$:

$$\frac{\partial^3 F_\phi(t)}{\partial_{t^A}\partial_{t^B}\partial_{t^C}} = \mathbf{c}(\partial_{t^A}, \partial_{t^B}, \partial_{t^C}) = -\frac{1}{2}\sum_{i=1}^{L} \frac{\phi_{t^A}(P_i)\phi_{t^B}(P_i)\phi_{t^C}(P_i)}{\phi(P_i)} \ .$$

$\diamond$

The prepotential F_ϕ satisfies the WDVV system (5.1.1) with respect to the flat coordinates $\{t^A\}$. Corollary 5.1 implies that $\partial^3_{t^1 t^A t^B}(F_{\phi_{t^1}}) = -\mathbf{ds}^2_{\phi_{t^1}}(\partial_{t^A}, \partial_{t^B})$. Therefore, the matrix F_1 (5.1.2) is constant since metric coefficients are constant in flat coordinates of the metric.

Let us denote by $\widehat{M}^{\phi,\mathbf{q}}$ the Frobenius structure on the Hurwitz space $\widehat{M}$ defined by the metric $\mathbf{ds}^2_\phi$, where ϕ is one of the primary differentials from Theorem 5.4.

Theorem 5.8 *Consider the flat metric $\mathbf{ds}^2_\phi$ given by (5.3.11). The nonvanishing matrix entries of this metric in the flat coordinates given by Theorem 5.5 are the following:*

$$\mathbf{ds}^2_\phi(\partial_{t^{i;\alpha}}, \partial_{t^{j;\beta}}) = \frac{1}{n_i+1}\delta_{ij}\delta_{\alpha+\beta,n_i+1} \ ; \qquad \mathbf{ds}^2_\phi(\partial_{v^i}, \partial_{w^j}) = \delta_{ij} \ ; \qquad \mathbf{ds}^2_\phi(\partial_{r^k}, \partial_{s^l}) = -\delta_{kl} \ .$$

Proof. The proof is given in [15], p. 163 (see also [68]); it uses the relation $\mathbf{ds}^2_\phi(\partial_{t^A}, \partial_{t^B}) = \mathbf{e}\,(\mathcal{F}[\phi_{t^A}, \phi_{t^B}])$ and the representation of the unit vector field $\mathbf{e}$ via the action of the map $(\mathcal{L}, \lambda) \to (\mathcal{L}, \lambda + \delta)$ as in Proposition 5.1. $\diamond$

The existence of the Euler vector field E provides the quasihomogeneity (5.1.3) for the prepotential. Coefficients ν_A of quasihomogeneity coincide with those of the Frobenius structures in [15] (see also [68]); these coefficients are the coefficients of the Euler vector field written in the flat coordinates: $E = \sum_A \nu_A t^A \partial_{t^A}$. They can be found by computing the action of E on the flat coordinates $\{t^A\}$ as in (5.3.9). The charges ν of the constructed Frobenius manifolds can be computed from (5.3.9); they are given by $\nu = 1 - 2\alpha/(n_i+1)$ for $\phi = \phi_{t^{i;\alpha}}$; $\nu = -1$ for $\phi = \phi_{v^i}$ and $\phi = \phi_{r^k}$; $\nu = 1$ for $\phi = \phi_{\omega^i}$ and $\phi = \phi_{s^k}$. A linear combination of the differentials corresponding to the same charge ν can be taken as a new primary differential for which a Frobenius structure can be built in the described way.

5.4 Real doubles of the deformed Frobenius structures

Here we shall construct real doubles of the deformed semisimple Hurwitz Frobenius structures found in Section 5.3. We use ideas of the work [68] where the real doubles were found for the nondeformed Hurwitz Frobenius structures of [15]. The construction of [68] is based on the properties of the Schiffer and Bergman kernels $\Omega(P,Q)$ and $B(P,Q)$ given by (5.2.11), (5.2.12). Analogous structures for the deformations of Frobenius manifolds are obtained using the "deformed" kernels Ω and B, i.e. the bidifferentials $\Omega_{\mathbf{q}}(P,Q)$ and $B_{\mathbf{q}}(P,\bar{Q})$ (5.2.18), (5.2.19). Here we simply state main theorems; an essential part of the proofs can be found in [68] and Section 5.3.

In this section we consider the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ as a space with local coordinates $\{\lambda_1,\dots,\lambda_L;\bar{\lambda}_1,\dots,\bar{\lambda}_L\}$. We shall denote it by $\widehat{M}^{\mathrm{real}}_{g;n_0,\dots,n_m}$. The multiplication in the tangent space is again defined by $\partial_{\lambda_i}\cdot\partial_{\lambda_j} := \delta_{ij}\partial_{\lambda_j}$. The indices i and j range now in the set $\{1,\dots,L;\bar{1},\dots,\bar{L}\}$ and we define $\lambda_{\bar{i}} := \bar{\lambda}_i$. This algebra obviously does not have nilpotents. The Euler vector field has the standard form: $E := \sum_{i=1}^{L}\left(\lambda_i\partial_{\lambda_i} + \bar{\lambda}_i\partial_{\bar{\lambda}_i}\right)$.

Let us fix a point P_0 on the surface $\mathcal{L}$ such that $\lambda(P_0) = 0$, and let all basis cycles $\{a_k,b_k\}_{k=1}^{g}$ start at this point. Let us fix a constant symmetric nondegenerate matrix $\mathbf{q}$

such that the matrix $\mathbb{B}^{\Omega} + \mathbf{q}$ is invertible (see (5.2.20)). Denote by $f_{(1,0)}$ and $f_{(0,1)}$ the holomorphic and antiholomorphic parts of a differential f which can be represented in the form $f = f_{(1,0)} + f_{(0,1)}$. We say that a differential is of the $(1,0)$-type if in a local coordinate z it has the form $f_{(1,0)} = f_1(z)dz$, and a differential is of the $(0,1)$-type if in a local coordinate it has the form $f_{(0,1)} = f_2(\bar{z})d\bar{z}$.

Consider the following set of operations. Let $\bar{\text{res}}$ stand for the coefficient in front of $d\bar{z}/\bar{z}$ in the Laurent expansion of a differential. As before, z_i is the local parameter in a neighbourhood of ∞^i defined by $z_i^{-n_i-1}(Q) = \lambda(Q)$, $Q \sim \infty^i$.

For $i = 0, \dots, m$; $\alpha = 1, \dots, n_i$ we define:

$$\mathbf{1}.\ \mathrm{I}_{t^{i;\alpha}}[f(Q)] := \frac{1}{\alpha} \operatorname*{res}_{\infty^i} z_i^{-\alpha}(Q) f_{(1,0)}(Q) \qquad \mathbf{2}.\ \mathrm{I}_{t^{\bar{i};\bar{\alpha}}}[f(Q)] := \frac{1}{\alpha} \operatorname*{\bar{res}}_{\infty^i} \bar{z}_i^{-\alpha}(Q) f_{(0,1)}(Q).$$

For $i = 1, \dots, m$ we define:

$$\mathbf{3}.\ \mathrm{I}_{v^i}[f(Q)] := \operatorname*{res}_{\infty^i} \lambda(Q) f_{(1,0)}(Q) \qquad \mathbf{4}.\ \mathrm{I}_{v^{\bar{i}}}[f(Q)] := \operatorname*{\bar{res}}_{\infty^i} \bar{\lambda}(Q) f_{(0,1)}(Q)$$

$$\mathbf{5}.\ \mathrm{I}_{w^i}[f(Q)] := \text{v.p.} \int_{\infty^0}^{\infty^i} f_{(1,0)}(Q) \qquad \mathbf{6}.\ \mathrm{I}_{w^{\bar{i}}}[f(Q)] := \text{v.p.} \int_{\infty^0}^{\infty^i} f_{(0,1)}(Q).$$

As before, the principal value near infinity is defined by omitting the divergent part of an integral as a function of the corresponding local parameter.

For $k = 1, \dots, g$ we define:

$$\mathbf{7}.\ \mathrm{I}_{r^k}[f(Q)] := -\oint_{a_k} \lambda(Q) f_{(1,0)}(Q) - \oint_{a_k} \bar{\lambda}(Q) f_{(0,1)}(Q) - \sum_{j=1}^{L} (\mathbf{q}^{-1})_{kj} \oint_{b_j} \lambda(Q) f_{(1,0)}(Q)$$

$$\mathbf{8}.\ \mathrm{I}_{u^k}[f(Q)] := \oint_{b_k} \lambda(Q) f_{(1,0)}(Q) + \oint_{b_k} \bar{\lambda}(Q) f_{(0,1)}(Q)$$

$$\mathbf{9}.\ \mathrm{I}_{s^k}[f(Q)] := \frac{1}{2\pi i} \oint_{b_k} f_{(1,0)}(Q)$$

$$\mathbf{10}.\ \mathrm{I}_{t^k}[f(Q)] := -\frac{1}{2\pi i} \oint_{a_k} f_{(1,0)}(Q) - \frac{1}{2\pi i} \sum_{j=1}^{L} (\mathbf{q}^{-1})_{kj} \oint_{b_j} f_{(1,0)}(Q).$$

Let us denote the set of operations by $\{\mathrm{I}_{\xi^A}\}$, i.e. assume the index ξ^A to belong to the set $\{t^{i;\alpha}, t^{\overline{i;\alpha}};\ v^i, v^{\bar{i}}, w^i, w^{\bar{i}};\ r^k, u^k, s^k, t^k\}$. Here we use ξ^A as a formal index; later by ξ^A we shall denote a flat coordinate on the Frobenius manifold.

The operations I_{ξ^A} define primary differentials Φ_{ξ^A} as follows.

$$
\begin{aligned}
\Phi_{\xi^A}(P) &:= \mathrm{I}_{\xi^A}\left[\Omega_{\mathbf{q}}(P,Q) + B_{\mathbf{q}}(\bar{P},Q)\right] \qquad &&\text{for} \qquad \xi^A \notin \{r^k, u^k\}\ ; \\
\Phi_{\xi^A}(P) &:= \mathrm{I}_{\xi^A}\left[2\mathrm{Re}\left\{\Omega_{\mathbf{q}}(P,Q) + B_{\mathbf{q}}(\bar{P},Q)\right\}\right] \qquad &&\text{for} \qquad \xi^A \in \{r^k, u^k\}\ .
\end{aligned}
\tag{5.4.1}
$$

Variational formulas (5.2.23) imply the dependence of primary differentials (5.4.1) on canonical coordinates:

$$
\begin{aligned}
\frac{\partial \Phi_{\xi^A}(P)}{\partial \lambda_k} &= \frac{1}{2}\Phi_{\xi^A{}_{(1,0)}}(P_k)\left(\Omega_{\mathbf{q}}(P,P_k) + B_{\mathbf{q}}(\bar{P},P_k)\right)\ ; \\
\frac{\partial \Phi_{\xi^A}(P)}{\partial \bar{\lambda}_k} &= \frac{1}{2}\Phi_{\xi^A{}_{(0,1)}}(P_k)\left(B_{\mathbf{q}}(P,\bar{P}_k) + \overline{\Omega_{\mathbf{q}}(P,P_k)}\right)\ .
\end{aligned}
\tag{5.4.2}
$$

Here $\Phi_{\xi^A{}_{(1,0)}}$ and $\Phi_{\xi^A{}_{(0,1)}}$ are holomorphic and antiholomorphic parts of the differential Φ_{ξ^A}, respectively. Relations (5.2.22) for the periods of bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ imply similar relations on periods of differentials Φ_{ξ^A} (δ is the Kronecker symbol):

$$
\oint_{a_k} \Phi_{\xi^A} + \sum_{j=1}^{g} (\mathbf{q}^{-1})_{kj} \oint_{b_j} \Phi_{\xi^A{}_{(1,0)}} = \delta_{\xi^A, s^k} \qquad \text{and} \qquad \oint_{b_k} \Phi_{\xi^A} = \delta_{\xi^A, t^k}\ . \tag{5.4.3}
$$

To prove relations (5.4.3) we integrate both sides of equalities (5.4.1) over a- and b-cycles. Due to the choice of the point P_0, one can interchange integration and the operations I_{ξ^A} according to the rule (5.3.12) (note that $W_{\mathbf{q}}(P,Q)$ and the sum $\Omega_{\mathbf{q}}(P,Q) + B_{\mathbf{q}}(\bar{P},Q)$ have the same singularity structure).

The primary differentials $\{\Phi_{\xi^A}\}$ (5.4.1) are alternatively specified as follows. They are differentials of the form $\Phi_{\xi^A} = \Phi_{\xi^A{}_{(1,0)}} + \Phi_{\xi^A{}_{(0,1)}}$ which are normalized by relations (5.4.3) and possess the following properties (for proof see Theorem 2 of [68]):

1. $\Phi_{t^{i;\alpha}}(P) \quad \sim (z_i^{-\alpha-1} + \mathcal{O}(1))dz_i + \mathcal{O}(1)d\bar{z}_i\,, \quad P \sim \infty^i\,; \ \ \Phi_{t^{i;\alpha}}$ is single valued on $\mathcal{L}$.

2. $\Phi_{t^{\overline{i;\alpha}}}(P) \quad = \overline{\Phi_{t^{i;\alpha}}(P)}\,.$

3. $\Phi_{v^i}(P) \quad \sim -\,d\lambda + \mathcal{O}(1)\,(dz_i + d\bar{z}_i)\ \ ,\ \ P \sim \infty^i\,; \ \ \Phi_{v^{\bar{i}}}$ is single valued on $\mathcal{L}$.

4. $\Phi_{v^{\bar{i}}}(P) \quad = \overline{\Phi_{v^i}(P)}\,.$

5. $\Phi_{w^i}(P):$ $\quad \operatorname*{res}_{\infty^i} \Phi_{w^i} = 1\,; \ \ \operatorname*{res}_{\infty^0} \Phi_{w^i} = -1\,.$ Φ_{w^i} is single valued on $\mathcal{L}$.

6. $\Phi_{w^{\bar{i}}}(P) \quad = \overline{\Phi_{w^i}(P)}\,.$

7. $\Phi_{r^k}(P):$ has no poles;

$$\Phi_{r^k}(P^{b_j}) - \Phi_{r^k}(P) = 2\pi i\,\delta_{kj}(d\lambda - d\bar{\lambda});$$

$$\Phi_{r^k}(P^{a_j}) - \Phi_{r^k}(P) = -2\pi i\,(\mathbf{q}^{-1})_{kj} d\lambda\,.$$

8. $\Phi_{u^k}(P):$ has no poles;

$$\Phi_{u^k}(P^{a_j}) - \Phi_{u^k}(P) = 2\pi i\,\delta_{kj}(d\lambda - d\bar{\lambda})\,.$$

9. $\Phi_{s^k}(P):$ single valued on $\mathcal{L}$ and has no poles.

10. $\Phi_{t^k}(P):$ single valued on $\mathcal{L}$ and has no poles.

Here, as before, $\lambda = \lambda(P)$, and $z_i = z_i(P)$ is the local parameter at $P \sim \infty^i$ such that $\lambda = z_i^{-n_i-1}$. The indices i, k and $i;\alpha$ take values specified in the definition of operations I_{ξ^A}.

The next theorem gives the Darboux-Egoroff metrics which satisfy requirements **F2** and **F4** (for proof see (5.3.9), Theorem 5.4, and [68] Propositions 7 and 9).

Theorem 5.9 *The metrics of the form*

$$\mathbf{ds}^2_\Phi = \frac{1}{2}\sum_{i=1}^{L}\Phi^2_{(1,0)}(P_i)(d\lambda_i)^2 + \frac{1}{2}\sum_{i=1}^{L}\Phi^2_{(0,1)}(P_i)(d\bar{\lambda}_i)^2 , \tag{5.4.4}$$

belong to the family (5.2.33), (5.2.34) of Darboux-Egoroff metrics. Here $\Phi_{(1,0)}$ *and* $\Phi_{(0,1)}$ *are respectively the holomorphic and antiholomorphic parts of one of the primary differentials:* $\Phi(P) = \Phi_{(1,0)}(P) + \Phi_{(0,1)}(P)$. *The metric coefficients satisfy* $\mathbf{e}(\Phi^2_{(1,0)}(P_i)) = 0$, $\mathbf{e}(\Phi^2_{(0,1)}(P_i)) = 0$ *and* $E(\Phi^2_{(1,0)}(P_i)) = -\nu\Phi^2_{(1,0)}(P_i)$, $E(\Phi^2_{(0,1)}(P_i)) = -\nu\Phi^2_{(0,1)}(P_i)$ *for some constant* ν.

A set of flat coordinates $\{\xi^A\} := \{t^{i;\alpha}, t^{\overline{i;\alpha}};\ v^i, v^{\bar{i}}, w^i, w^{\bar{i}};\ r^k, u^k, s^k, t^k\}$ (see Section 5.3.3) of the metrics $\mathbf{ds}^2_\Phi$ (5.4.4) is given by operations I_{ξ^A} applied to the primary differential Φ which defines the metric (see Theorem 5.5, and [68] Theorem 7). Namely, the flat coordinates of $\mathbf{ds}^2_\Phi$ are given by:

for $i = 0, \dots, m;\ \alpha = 1, \dots, n_i:\ t^{i;\alpha} := -(n_i+1)\mathrm{I}_{t^{i;1+n_i-\alpha}}[\Phi];\ t^{\overline{i;\alpha}} := -(n_i+1)\mathrm{I}_{\overline{t^{i;1+n_i-\alpha}}}[\Phi]$;

for $i = 1, \dots, m:\quad v^i := -\mathrm{I}_{w^i}[\Phi];\quad v^{\bar{i}} := -\mathrm{I}_{w^{\bar{i}}}[\Phi];\quad w^i := -\mathrm{I}_{v^i}[\Phi];\quad w^{\bar{i}} := -\mathrm{I}_{v^{\bar{i}}}[\Phi]$;

for $k = 1, \dots, g:\quad r^k := \mathrm{I}_{s^k}[\Phi];\quad u^k := -\mathrm{I}_{t^k}[\Phi];\quad s^k := \mathrm{I}_{r^k}[\Phi];\quad t^k := -\mathrm{I}_{u^k}[\Phi]$.

As before, the unit vector field $\mathbf{e}$ is a vector field in the direction of the flat coordinate which has the same type as the differential defining the metric. Namely, in the flat coordinates of the metric $\mathbf{ds}^2_\Phi$ with $\Phi = \Phi_{\xi^{A_0}}$, the unit field is given by $\mathbf{e} = -\partial_{\xi^{A_0}}$. We shall denote this coordinate by ξ^1 so that $\mathbf{e} = -\partial_{\xi^1}$.

Lemma 5.2 *In the Hurwitz space outside the submanifold* $\mathcal{D}^\Omega_{\mathbf{q}}$ *defined by (5.2.20), the derivatives of canonical coordinates* $\{\lambda_i\,;\bar{\lambda}_i\}$ *with respect to flat coordinates* $\{\xi^A\}$ *of the metric*

$\mathbf{ds^2_\Phi}$ *are given by*

$$\frac{\partial\lambda_i}{\partial\xi^A} = -\frac{\Phi_{\xi^A(1,0)}(P_i)}{\Phi_{(1,0)}(P_i)}\,, \qquad\qquad \frac{\partial\bar\lambda_i}{\partial\xi^A} = -\frac{\Phi_{\xi^A(0,1)}(P_i)}{\Phi_{(0,1)}(P_i)}\,,$$

where $\Phi(P)$ is the primary differential which defines the metric.

The proof of this lemma repeats the proof of Lemma 4 in [68].

The analog of the multivalued differential (5.3.18) in the construction of real doubles is

$$\Psi_\Phi(P) = \left(\text{v.p.}\int_{\infty^0}^P \Phi_{(1,0)}\right) d\lambda + \left(\text{v.p.}\int_{\infty^0}^P \Phi_{(0,1)}\right) d\bar\lambda\,. \tag{5.4.5}$$

The multivalued differential Ψ_Φ again generates the set of primary differentials Φ_{ξ^A} according to the relation $\partial_{\xi^A}\Psi_\Phi = \Phi_{\xi^A}$.

A prepotential of the Frobenius structure can be found with the help of the pairing of differentials which we shall define now.

Let $\omega^{(\alpha)}(P)$, $\alpha = 1, 2\dots$ be a differential on $\mathcal{L}$ which can be written as a sum of holomorphic and antiholomorphic differentials, $\omega^{(\alpha)} = \omega^{(\alpha)}_{(1,0)} + \omega^{(\alpha)}_{(0,1)}$, which are analytic outside of infinities and have the following behaviour at $P \sim \infty^i$ ($z_i = z_i(P)$ is a local parameter at $P \sim \infty^i$ such that $z_i^{-n_i-1} = \lambda$):

$$\begin{aligned} \omega^{(\alpha)}_{(1,0)}(P) &= \sum_{n=-n_1^{(\alpha)}}^{\infty} c^{(\alpha)}_{n,i} z_i^n dz_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r^{(\alpha)}_{n,i}\lambda^n \log\lambda\right)\,, \\ \omega^{(\alpha)}_{(0,1)}(P) &= \sum_{n=-n_2^{(\alpha)}}^{\infty} c^{(\alpha)}_{\bar n,i} \bar z_i^n d\bar z_i + \frac{1}{n_i+1} d\left(\sum_{n>0} r^{(\alpha)}_{\bar n,i}\bar\lambda^n \log\bar\lambda\right)\,, \end{aligned} \tag{5.4.6}$$

where $n_1^{(\alpha)}, n_2^{(\alpha)} \in \mathbb{Z}$; and $c^{(\alpha)}_{n,i}$, $r^{(\alpha)}_{n,i}$, $c^{(\alpha)}_{\bar n,i}$, $r^{(\alpha)}_{\bar n,i}$ are some complex numbers. Denote also for $k = 1, \dots, g$ the combinations of periods:

$$A_k^{(\alpha)} := \oint_{a_k}\omega^{(\alpha)} + \sum_{j=1}^{g}(\mathbf{q}^{-1})_{kj}\oint_{b_j}\omega(\alpha)_{(1,0)}\,, \quad B_k^{(\alpha)} := \oint_{b_k}\omega^{(\alpha)}\,; \tag{5.4.7}$$

and the transformations along basis cycles:

$$
\begin{aligned}
dp_k^{(\alpha)}(\lambda(P)) &:= \omega_{(1,0)}^{(\alpha)}(P^{a_k}) - \omega_{(1,0)}^{(\alpha)}(P)\,, \quad & p_k^{(\alpha)}(\lambda) &= \sum_{s>0} p_{sk}^{(\alpha)}\lambda^s\,,\\
dp_{\bar{k}}^{(\alpha)}(\bar{\lambda}(P)) &:= \omega_{(0,1)}^{(\alpha)}(P^{a_k}) - \omega_{(0,1)}^{(\alpha)}(P)\,, \quad & p_{\bar{k}}^{(\alpha)}(\bar{\lambda}) &= \sum_{s>0} p_{\bar{s}\bar{k}}^{(\alpha)}\bar{\lambda}^s\,,\\
dq_k^{(\alpha)}(\lambda(P)) &:= \omega_{(1,0)}^{(\alpha)}(P^{b_k}) - \omega_{(1,0)}^{(\alpha)}(P)\,, \quad & q_k^{(\alpha)}(\lambda) &= \sum_{s>0} q_{sk}^{(\alpha)}\lambda^s\,,\\
dq_{\bar{k}}^{(\alpha)}(\bar{\lambda}(P)) &:= \omega_{(0,1)}^{(\alpha)}(P^{b_k}) - \omega_{(0,1)}^{(\alpha)}(P)\,, \quad & q_{\bar{k}}^{(\alpha)}(\bar{\lambda}) &= \sum_{s>0} q_{\bar{s}\bar{k}}^{(\alpha)}\bar{\lambda}^s\,.
\end{aligned}
\tag{5.4.8}
$$

Note that if the differential $\omega^{(\alpha)}$ is one of the primary differentials Φ_{ξ^A} (5.4.1), the coefficients defined by (5.4.6) - (5.4.8) do not depend on coordinates $\{\xi^A\}$.

Definition 5.5 *For any two differentials $\omega^{(\alpha)}$ and $\omega^{(\beta)}$ which can be represented as a sum of holomorphic and antiholomorphic differentials, $\omega^{(\alpha)} = \omega_{(1,0)}^{(\alpha)} + \omega_{(0,1)}^{(\alpha)}$, and have only singularities of the type (5.4.6), (5.4.8), the* **pairing** *$\mathcal{F}[\ ,\ ]$ is defined by:*

$$
\begin{aligned}
\mathcal{F}[\omega^{(\alpha)}\,,\,\omega^{(\beta)}] := \sum_{i=0}^{m}\Bigg(&\sum_{n\geq 0}\frac{c_{-n-2,i}^{(\alpha)}}{n+1}c_{n,i}^{(\beta)} + c_{-1,i}^{(\alpha)}\text{v.p.}\int_{P_0}^{\infty^i}\omega_{(1,0)}^{(\beta)} - \text{v.p.}\int_{P_0}^{\infty^i}\sum_{n>0} r_{n,i}^{(\alpha)}\lambda^n\omega_{(1,0)}^{(\beta)}\\
&+\sum_{n\geq 0}\frac{c_{-\bar{n}-\bar{2},i}^{(\alpha)}}{n+1}c_{\bar{n},i}^{(\beta)} + c_{-\bar{1},i}^{(\alpha)}\text{v.p.}\int_{P_0}^{\infty^i}\omega_{(0,1)}^{(\beta)} - \text{v.p.}\int_{P_0}^{\infty^i}\sum_{n>0} r_{\bar{n},i}^{(\alpha)}\bar{\lambda}^n\omega_{(0,1)}^{(\beta)}\Bigg)\\
+\frac{1}{2\pi i}\sum_{k=1}^{g}\Bigg(&-\oint_{a_k} q_k^{(\alpha)}(\lambda)\omega_{(1,0)}^{(\beta)} + \oint_{a_k} q_{\bar{k}}^{(\alpha)}(\bar{\lambda})\omega_{(0,1)}^{(\beta)} + \oint_{b_k} p_k^{(\alpha)}(\lambda)\omega_{(1,0)}^{(\beta)}\\
-\oint_{b_k} p_{\bar{k}}^{(\alpha)}(\bar{\lambda})\omega_{(0,1)}^{(\beta)} &+ A_k^{(\alpha)}\oint_{b_k}\omega_{(1,0)}^{(\beta)} - B_k^{(\alpha)}\left(\oint_{a_k}\omega_{(1,0)}^{(\beta)} + \sum_{j=1}^{g}(\mathbf{q}^{-1})_{kj}\oint_{b_j}\omega_{(1,0)}^{(\beta)}\right)\Bigg)\,.
\end{aligned}
\tag{5.4.9}
$$

As before, P_0 is a point on $\mathcal{L}$ such that $\lambda(P_0) = 0$. The cycles $\{a_k, b_k\}$ all pass through P_0 .

The next theorem gives a prepotential of the Frobenius manifold, i.e. a function of flat coordinates $\{\xi^A\}$ which satisfies the WDVV system.

Theorem 5.10 *For each primary differential Φ consider the differential $\Psi_\Phi(P)$ (5.4.5), multivalued on the surface $\mathcal{L}$. Consider the Frobenius structure defined by the metric $\mathbf{ds}^2_\Phi$*

(5.4.4), multiplication law $\partial_{\lambda_i}\cdot\partial_{\lambda_j} = \delta_{ij}\partial_{\lambda_j}\,;\quad i,j\in\{1,\dots,L;\bar 1,\dots,\bar L\}\,,\quad \lambda_{\bar i}:=\bar\lambda_i$ *, and the Euler vector field* $E=\sum_{i=1}^{L}\left(\lambda_i\partial_{\lambda_i}+\bar\lambda_i\partial_{\bar\lambda_i}\right)$ *. This Frobenius structure is defined on the manifold* $\widehat{M}^{\rm real}_{g;n_0,\dots,n_m}$ *outside the submanifold* $\mathcal{D}^{\Omega}_{\mathbf{q}}$ *of codimension one given by the equation* $\det\left(\mathbb{B}^{\Omega}+\mathbf{q}\right)=0$ *. The prepotential* F_Φ *for this Frobenius manifold is given by the pairing (5.4.9) of the differential* Ψ_Φ *with itself:*

$$F_\Phi=\frac{1}{2}\mathcal{F}[\Psi_\Phi\ ,\ \Psi_\Phi]\,. \tag{5.4.10}$$

The second order derivatives of the prepotential are given by:

$$\partial_{\xi^A}\partial_{\xi^B}F_\Phi=\mathcal{F}[\Phi_{\xi^A}\ ,\ \Phi_{\xi^B}]-\frac{1}{4\pi i}\delta_{\xi^A,s^k}\delta_{\xi^B,t^k}+\frac{1}{4\pi i}\delta_{\xi^A,t^k}\delta_{\xi^B,s^k}\,. \tag{5.4.11}$$

Two last terms in (5.4.11) do not vanish only for the primary differentials Φ_{s^k} *and* Φ_{t^k} *when the pairing* $\mathcal{F}$ *is not commutative. The third order derivatives coincide with the tensor* $\mathbf{c}$ *:* $\partial_{\xi^A}\partial_{\xi^B}\partial_{\xi^C}F_\Phi=\mathbf{c}(\partial_{\xi^A},\partial_{\xi^B},\partial_{\xi^C}):=\mathbf{ds}^2_\Phi\left(\partial_{\xi^A}\cdot\partial_{\xi^B},\partial_{\xi^C}\right)$ *.*

The proof of this theorem is analogous to proofs of Theorem 5.7 and [68], Theorem 11.

The quasihomogeneity factors $\{\nu_A\}$ (5.1.3) for the constructed deformations of real doubles of Frobenius manifolds coincide with those for the undeformed real doubles ([68], Proposition 11).

Let us denote the constructed deformations of real doubles of Frobenius structures by $\widehat{M}^{{\rm real}\,\Phi\,,\,\mathbf{q}}_{g;n_0,\dots,n_m}$.

The charges ν (see Definition 5.2) of the manifolds $\widehat{M}^{{\rm real}\,\Phi\,,\,\mathbf{q}}_{g;n_0,\dots,n_m}$ are as follows: if one chooses $\Phi:=\Phi_{t^{i;\alpha}}$ or $\Phi:=\Phi_{t^{\overline{i;\alpha}}}$ the charge is $\nu=1-2\alpha/(n_i+1)$; for $\Phi:=\Phi_{v^i}$, $\Phi:=\Phi_{v^{\bar i}}$, $\Phi:=\Phi_{r^k}$ or $\Phi:=\Phi_{u^k}$ the charge is $\nu=-1$; for $\Phi:=\Phi_{\omega^i}$, $\Phi:=\Phi_{\omega^{\bar i}}$, $\Phi:=\Phi_{s^k}$ or $\Phi:=\Phi_{t^k}$ the charge is $\nu=1$.

5.5 G-function of the deformed Frobenius manifolds

The G-function is a solution to the Getzler system introduced in [26]. The system is defined on an arbitrary semisimple Frobenius manifold. It was shown in [17] that the Getzler system has a unique quasihomogeneous solution and that this solution has the form:

$$G = \log \frac{\tau_{\mathrm{I}}}{J^{1/24}} \, . \tag{5.5.1}$$

Here J is the Jacobian of transformation from canonical to the flat coordinates, $J = \det(\partial_{\lambda_i} t^\alpha)$; and τ_{I} is the isomonodromic tau-function of the n-dimensional Frobenius manifold defined by

$$\frac{\partial \log \tau_{\mathrm{I}}}{\partial \lambda_i} = \frac{1}{2} \sum_{j\neq i, j=1}^{n} \beta_{ij}^2 (\lambda_i - \lambda_j) \, , \qquad i = 1, \ldots, n \, . \tag{5.5.2}$$

The G-function (5.5.1) of Dubrovin's Hurwitz Frobenius manifolds [15] for the space of two-fold genus one coverings was computed in [17]. In [44] the G-function was computed for an arbitrary Hurwitz Frobenius manifold of [15]. As it was proven in [45], the isomonodromic tau-function τ_{I} for Hurwitz Frobenius manifolds can be expressed in terms of the so-called Bergman tau-function τ_W on Hurwitz spaces: $\tau_W = \tau_{\mathrm{I}}^{-2}$, where the Bergman tau-function is defined as follows. Denote by S^W the following term in the asymptotics of the kernel $W(P,Q)$ (5.2.2) near the diagonal $P \sim Q$:

$$W(P,Q) \underset{Q\sim P}{=} \left(\frac{1}{(x(P)-x(Q))^2} + S^W(x(P)) + o(1) \right) dx(P) dx(Q)$$

(the quantity $6S^W(x(P))$ is called the Bergman projective connection [23]). Choosing the local parameter to be $x_i(P) = \sqrt{\lambda - \lambda_i}$, we denote by S_i^W the value of S^W at a ramification point P_i :

$$S_i^W = S^W(x_i) \Big|_{x_i=0} \, . \tag{5.5.3}$$

Since the singular part of the W-kernel in a neighbourhood of the point P_i does not depend on coordinates $\{\lambda_j\}$, the Rauch variational formulas (5.2.4) imply $\partial_{\lambda_j} S_i^W = W^2(P_i, P_j)/2$. The symmetry of this expression with respect to the indices i and j provides the compatibility for the system of differential equations which defines the tau-function τ_W :

$$\frac{\partial \log \tau_W}{\partial \lambda_i} = -\frac{1}{2} S_i^W , \qquad i = 1, \ldots, L . \tag{5.5.4}$$

The G-function of the deformed Hurwitz Frobenius manifolds can be computed analogously to the method of [45].

Theorem 5.11 *The G-function (5.5.1) of the deformed Hurwitz Frobenius structures $\widehat{M}^{\phi,\mathbf{q}}$ is given by*

$$G = -\frac{1}{2} \log \left\{ \tau_W \frac{\det(\mathbb{B} + \mathbf{q})}{\det \mathbf{q}} \right\} - \frac{1}{24} \log \prod_{i=1}^{L} \phi(P_i) + \text{const} , \tag{5.5.5}$$

the Bergman tau-function τ_W on the Hurwitz space is given by formula (1.5) from the paper [44]. The G-function (5.5.5) is defined on the Hurwitz space $\widehat{M}$ outside the divisor $\mathcal{D}_{\mathbf{q}}$ given by the equation $\det(\mathbb{B} + \mathbf{q}) = 0$.

The constant $\frac{1}{2} \log \{\det \mathbf{q}\}$ is added in the right hand side to normalize the G-function so that it coincides with the G-function of [15] as $\mathbf{q}$ tends to infinity in such a way that $W_{\mathbf{q}}$ tends to W (the function G is defined up to an additive constant).

Proof. According to the general formula ([17], p.36) the Jacobian of a Frobenius manifold is up to a constant given by the product of square roots of all nonvanishing coefficients of the Darboux-Egoroff metric $\mathbf{ds^2}$. Therefore, the Jacobian J for the Hurwitz Frobenius manifold $\widehat{M}^{\phi,\mathbf{q}}$ is given by $J = 2^{-L/2} \left(\prod_{i=1}^{L} \phi(P_i) \right)$.

To compute the isomonodromic tau-function (5.5.2) for deformed Hurwitz Frobenius manifolds $\widehat{M}^{\phi,\mathbf{q}}$ we introduce a deformed Bergman tau-function $\tau_{W_{\mathbf{q}}}$. The analogous to

S^W coefficient $S^{W_{\mathbf{q}}}(x(P))$ in the expansion of $W_{\mathbf{q}}(P,Q)$ near $P \simeq Q \simeq P_i$ is given by $S_i^{W_{\mathbf{q}}} = S_i^W - 2\pi i \sum_{k,l=1}^{L} (\mathbb{B}+\mathbf{q})^{-1}_{kl}\omega_k(P_i)\omega_l(P_i)$. As a corollary of the variational formulas (5.2.10), we have

$$\partial_{\lambda_j} S_i^{W_{\mathbf{q}}} = \frac{1}{2} W_{\mathbf{q}}^2(P_i, P_j) = 2\beta_{ij}^2 , \tag{5.5.6}$$

which allows to consistently define the tau-function $\tau_{W_{\mathbf{q}}}$ as follows:

$$\frac{\partial \log \tau_{W_{\mathbf{q}}}}{\partial \lambda_i} = -\frac{1}{2} S_i^{W_{\mathbf{q}}} , \qquad i = 1, \dots, L . \tag{5.5.7}$$

As is easy to verify using the definitions (5.5.4) and (5.5.7) of τ_W and $\tau_{W_{\mathbf{q}}}$, the "deformed" and "undeformed" tau-functions are related as follows:

$$\tau_{W_{\mathbf{q}}} = \tau_W \det(\mathbb{B}+\mathbf{q}) .$$

Indeed, differentiation of the logarithm of this expression with respect to a branch point gives: $\partial_{\lambda_j} \log\{\tau_W \det(\mathbb{B}+\mathbf{q})\} = -S_i^W/2 + \mathrm{tr}\{(\mathbb{B}+\mathbf{q})^{-1}\partial_{\lambda_i}(\mathbb{B}+\mathbf{q})\}$. The matrix $\mathbf{q}$ is independent of the branch points; using the derivatives of the matrix $\mathbb{B}$ given by the Rauch variational formulas (5.2.6) we prove that $\partial_{\lambda_j} \log\{\tau_W \det(\mathbb{B}+\mathbf{q})\} = -S_i^{W_{\mathbf{q}}}/2$.

Now, let us prove that the isomonodromic tau-function $\tau_{\mathrm{I},\mathbf{q}}$ defined by (5.5.2) for the manifolds $\widehat{M}^{\phi,\mathbf{q}}$ is given by $\tau_{\mathrm{I},\mathbf{q}} = (\tau_{W_{\mathbf{q}}})^{-1/2}$. First, we use relations (5.5.6) to rewrite the definition (5.5.2) of $\tau_{\mathrm{I},\mathbf{q}}$ in terms of the quantities $S_i^{W_{\mathbf{q}}}$. To complete the proof it remains to use the equations $\mathbf{e}(S_i^{W_{\mathbf{q}}}) \equiv \sum_{j=1}^{L} \partial_{\lambda_j} S_i^{W_{\mathbf{q}}} = 0$ and $E(S_i^{W_{\mathbf{q}}}) \equiv \sum_{j=1}^{L} \lambda_j \partial_{\lambda_j} S_i^{W_{\mathbf{q}}} = -S_i^{W_{\mathbf{q}}}$ which can be proven analogously to the similar relations (5.3.8) for coefficients of metrics (5.3.11). $\diamond$

The following theorem gives an expression for the G-function of the manifolds $\widehat{M}^{\mathrm{real}\,\Phi,\mathbf{q}}$.

Theorem 5.12 *The G-function of the deformations $\widehat{M}^{\mathrm{real}\,\Phi,\mathbf{q}}$ of real doubles of Hurwitz Frobenius manifolds has the form:*

$$G = -\frac{1}{2}\log\left\{|\tau_W|^2 \det\left(\mathrm{Im}\mathbb{B}(\mathbb{B}^\Omega + \mathbf{q})\mathbf{q}^{-1}\right)\right\} - \frac{1}{24}\log\left\{\prod_{i=1}^{L}\Phi_{(1,0)}(P_i)\Phi_{(0,1)}(P_i)\right\} + \mathrm{const}\,, \quad (5.5.8)$$

where τ_W is given by formula (1.5) of [44]. The G-function (5.5.8) is defined on the Hurwitz space $\widehat{M}^{\mathrm{real}}$ outside the submanifold given by the equation $\det(\mathbb{B}^\Omega + \mathbf{q}) = 0$.

The constant $\frac{1}{2}\log\{\det\mathbf{q}\}$ is added in the right hand side to make the G-function (5.5.8) coincide with the G-function of real doubles of [68] in the limit when the construction of deformations reduces to that of [68].

Proof. The G-function (5.5.8) can be computed analogously to the G-function (5.5.5) of the Frobenius manifolds $\widehat{M}^{\phi,\mathbf{q}}$ by proving (similarly to [45], see also [68]) the following expression for the isomonodromic tau-function defined by (5.5.2): $\tau_{I,\mathbf{q}} = (\tau_{\Omega_\mathbf{q}})^{-1/2}$. (Note that the dimension of the Frobenius manifold $\widehat{M}^{\mathrm{real}\,\Phi,\mathbf{q}}$ is $2L$.) The function $\tau_{\Omega_\mathbf{q}}$ is another analogue of the Bergman tau-function on Hurwitz spaces; it is defined as follows. Denote by $S_i^{\Omega_\mathbf{q}}$ the analogous to S_i^W coefficient in expansion of the bidifferential $\Omega_\mathbf{q}(P,Q)$ when both arguments are in a neighbourhood of the ramification point P_i. Then, the following differential equations define the function $\tau_{\Omega_\mathbf{q}}$:

$$\frac{\partial \log \tau_{\Omega_\mathbf{q}}}{\partial \lambda_i} = -\frac{1}{2}S_i^{\Omega_\mathbf{q}}\,, \qquad \frac{\partial \log \tau_{\Omega_\mathbf{q}}}{\partial \bar{\lambda}_i} = -\frac{1}{2}\overline{S_i^{\Omega_\mathbf{q}}}\,. \quad (5.5.9)$$

Using differentiation formulas (5.2.6) and (5.2.17) for the matrices $\mathbb{B}$ and $\mathbb{B}^\Omega$, respectively, we prove that

$$\tau_{\Omega_\mathbf{q}} = |\tau_W|^2 \det(\mathrm{Im}\mathbb{B})\det(\mathbb{B}^\Omega + \mathbf{q})\,.$$

According to the general formula ([17], p.36), the Jacobian J for the manifolds $\widehat{M}^{\mathrm{real}\Phi,\mathbf{q}}$ has the form: $J = 2^{-L}\prod_{i=1}^{L}\Phi_{(1,0)}(P_i)\Phi_{(0,1)}(P_i)$. Substitution of this expression and the expression for the isomonodromic tau-function $\tau_{1,\mathbf{q}} = \left(|\tau_W|^2 \det(\mathrm{Im}\mathbb{B}) \det(\mathbb{B}^\Omega + \mathbf{q})\right)^{-1/2}$ into (5.5.1) proves the theorem. $\diamond$

5.6 Examples in genus one

The bidifferential $W_{\mathbf{q}}(P,Q)$ (5.2.7) is only different from $W(P,Q)$ in genus $g \geq 1$, therefore the deformations of Hurwitz Frobenius structures are constructed only in positive genera.

Consider the simplest Hurwitz space of two-fold coverings of genus one. According to the Riemann-Hurwitz formula (5.2.1), such coverings have four ramification points. Let one of them be over the point at infinity and denote the remaining three by P_1, P_2, P_3. These coverings can be defined as the pairs $(\mathcal{L}, \lambda(\zeta))$ where $\mathcal{L}$ is the torus $\mathcal{L} = \mathbb{C}/\{2w, 2w'\}$, $w, w' \in \mathbb{C}$, and $\lambda : \mathcal{L} \to \mathbb{C}P^1$ is the function

$$\lambda(\varsigma) = \wp(\varsigma) + c \,,$$

$\wp$ is the Weierstrass elliptic function and c is a constant with respect to ς. The ratio $\mu = w'/w$ is the period of the torus, it is the b-period of the unique normalized holomorphic differential $\omega(\varsigma) = d\varsigma/(2w)$, i.e. $\mu = \oint_b \omega(\varsigma)$. The pair $(\mathcal{L}, \lambda)$ depends on three parameters: w, w' and c. The branch points $\lambda_1, \ \lambda_2, \ \lambda_3$ of the covering can be expressed in terms of these parameters. The ς-coordinates of ramification points are solutions to the equation $\lambda'(\varsigma) = 0$. This equation has three solutions in the domain $\mathbb{C}/\{2w, 2w'\}$ due to the following relation on the $\wp$-function:

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z) - \wp(w)\right)\left(\wp(z) - \wp(w')\right)\left(\wp(z) - \wp(w + w')\right) \,,$$

where $\wp(w) + \wp(w') + \wp(w + w') = 0$. Hence, the branch points of the covering are given by $\lambda_1 = \wp(w) + c; \ \lambda_2 = \wp(w') + c; \ \lambda_3 = \wp(w + w') + c$. The local parameter in a neighbourhood of a ramification point P_i is $x_i(P) = \sqrt{\lambda(P) - \lambda_i}$. The branch points $\lambda_1, \lambda_2, \lambda_3$ play the role of local coordinates on the space of pairs $(\mathcal{L}, \lambda)$; they are canonical coordinates on Frobenius manifolds.

5.6.1 3-dimensional Frobenius manifold and Chazy equation

Here we give explicit formulas for ingredients of the Frobenius structure $\widehat{M}_{1;1}^{\phi_s,\mathbf{q}}$ on the Hurwitz space $\widehat{M}_{1;1}$ outside the divisor $\mathcal{D}_{\mathbf{q}}$ defined by the equation $\mu = -\mathbf{q}$ for some nonzero constant $\mathbf{q} \in \mathbb{C}$. The differential ϕ_s (see Theorem 5.4) is given by

$$\phi_s(\varsigma) = \frac{1}{2\pi i} \oint_b W_{\mathbf{q}}(\varsigma, \tilde{\varsigma}) = \frac{\mathbf{q}}{\mu + \mathbf{q}} \omega(\varsigma). \tag{5.6.1}$$

The set of flat coordinates from Theorem 5.5 of the metric $\mathbf{ds}^2_{\phi_s}$ (5.3.11) is formed by the following three functions:

$$\begin{aligned} t_1 &:= s = -\oint_a \lambda\phi_s = -\frac{1}{2w}\int_x^{x+2w} (\wp(\varsigma) + c)d\varsigma = -\frac{\pi i}{4w^2}\gamma - c - \frac{\pi i}{\mu + \mathbf{q}}\frac{1}{2w}\,, \\ t_2 &:= t^{0;1} = \operatorname*{res}_{\varsigma=0} \frac{1}{\sqrt{\lambda}}\left(\int_{\infty^0}^P \phi_s\right) d\lambda = \frac{\mathbf{q}}{\mu + \mathbf{q}}\frac{1}{w}\,, \\ t_3 &:= r = \frac{1}{2\pi i}\oint_b \phi_s = \frac{1}{2\pi i}\frac{\mathbf{q}\,\mu}{\mu + \mathbf{q}}\,. \end{aligned} \tag{5.6.2}$$

Here γ is such that $\int_x^{x+2w} \wp(\varsigma)d\varsigma = \pi i\gamma/(2w)$ for any $x \in \mathbb{C}$, i.e.

$$\gamma(\mu) = \frac{1}{3\pi i}\frac{\theta_1'''(0;\mu)}{\theta_1'(0;\mu)}\,. \tag{5.6.3}$$

The metric $\mathbf{ds}^2_{\phi_s}$ in coordinates (5.6.2) is constant: $\mathbf{ds}^2_{\phi_s} = (1/2)(dt_2)^2 - 2dt_1dt_3$. The prepotential (5.3.33) has the form

$$F = -\frac{1}{4}t_1t_2^2 + \frac{1}{2}t_1^2t_3 - \frac{\pi i}{32}t_2^4\left(\frac{1}{(1 - 2\pi i t_3/\mathbf{q})^2}\gamma\left(\frac{2\pi i t_3}{1 - 2\pi i t_3/\mathbf{q}}\right) + \frac{2}{\mathbf{q}(1 - 2\pi i t_3/\mathbf{q})}\right)\,. \tag{5.6.4}$$

This is a quasihomogeneous function: it satisfies $F_{\phi_s}(\kappa t_1, \kappa^{1/2}t_2, \kappa^0 t_3) = \kappa^2 F_{\phi_s}(t_1, t_2, t_3)$ for any nonzero constant κ . The Euler vector field (5.3.7) in coordinates (5.6.2) has the form:

$E = t_1\partial_{t_1} + (1/2)t_2\partial_{t_2}$.

To compute the function G (5.5.5) for the manifold $\widehat{M}_{1;1}^{\phi_s,\mathbf{q}}$ we use the following expression for the function τ_W on the space $\widehat{M}_{1;1}$ (see [44]): $\tau_W = \eta^2(\mu)\,(2w)^{-1/4}\left(\prod_{i=1}^{L}\omega(P_i)\right)^{-1/12}$ where $\eta(\mu)$ is the Dedekind eta-function $\eta(\mu) = (\theta_1'(0))^{1/3}$. Then, we have for the G-function:

$$G = -\log\left\{\eta\left(\frac{2\pi i t_3}{1-2\pi i t_3/\mathbf{q}}\right)(t_2)^{\frac{1}{8}}\,(2\pi i t_3/\mathbf{q}-1)^{-\frac{1}{2}}\right\}.$$

In [15] a relationship was established between the 3-dimensional WDVV system and the Chazy equation

$$f''' = 6ff'' - 9f'^{\,2}. \tag{5.6.5}$$

Namely, the function of the form

$$F = -\frac{1}{4}t_1t_2^2 + \frac{1}{2}t_1^2t_3 - \frac{\pi i}{32}t_2^4 f(2\pi i t_3) \tag{5.6.6}$$

satisfies the WDVV system iff the function f is a solution to the Chazy equation. The function γ (5.6.3) satisfies the Chazy equation, and the Frobenius manifold $\widehat{M}_{1;1}^{\omega}$ of [15] has the prepotential (5.6.6) with $f = \gamma$. We shall call the Frobenius manifold $\widehat{M}_{1;1}^{\omega}$ [15] the *Chazy Frobenius manifold.*

The group $SL(2,\mathbb{C})$ maps one solution $f(\mu)$ of the Chazy equation to another solution $\tilde{f}(\mu)$ as follows:

$$\tilde{f}(\mu) = f\left(\frac{a\mu+b}{c\mu+d}\right)\frac{1}{(c\mu+d)^2} - \frac{2c}{c\mu+d}, \qquad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL(2,\mathbb{C}). \tag{5.6.7}$$

Therefore there exists a 3-parametric family of Frobenius manifolds of the form (5.6.6):

$$F = -\frac{1}{4}t_1t_2^2 + \frac{1}{2}t_1^2t_3 - \frac{\pi i}{32}t_2^4\left(\gamma\left(\frac{a2\pi i t_3+b}{c2\pi i t_3+d}\right)\frac{1}{(c2\pi i t_3+d)^2} - \frac{2c}{c2\pi i t_3+d}\right). \tag{5.6.8}$$

In the case of integer coefficients a, b, c, d, (5.6.8) coincides with (5.6.6) with $f = \gamma$.

The manifold $\widehat{M}_{1;1}^{\phi_s,\mathbf{q}}$ (5.6.4) gives a realization of a one-parameter subfamily of manifolds (5.6.8) for $a = 1$, $b = 0$, $c = -1/\mathbf{q}$, $d = 1$. Thus we call it the *deformed Chazy Frobenius manifold.*

5.6.2 Relationship to isomonodromic deformations

It was shown in [2] that the functions

$$\Omega_1 = -\frac{1}{\pi\theta_2^2\theta_3^2}\left(2d_\mu \log\theta_4 + \frac{1}{\mu+\mathbf{q}}\right), \qquad \Omega_2 = -\frac{1}{\pi\theta_3^2\theta_4^2}\left(2d_\mu \log\theta_2 + \frac{1}{\mu+\mathbf{q}}\right), \tag{5.6.9}$$
$$\Omega_3 = -\frac{1}{\pi i\theta_2^2\theta_4^2}\left(2d_\mu \log\theta_3 + \frac{1}{\mu+\mathbf{q}}\right)$$

satisfy the system of equations

$$\frac{d\Omega_1}{dx} = \frac{1}{x}\Omega_2\Omega_3\,, \qquad \frac{d\Omega_2}{dx} = -\frac{1}{x-1}\Omega_1\Omega_3\,, \qquad \frac{d\Omega_3}{dx} = \frac{1}{x(x-1)}\Omega_1\Omega_2\,, \tag{5.6.10}$$

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = -1/4\,,$$

where

$$x = \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}\,.$$

The correspondence of notation in [2] to the one we use here is as follows: $\Omega_1^{[2]} = \Omega_3$, $\Omega_2^{[2]} = i\Omega_1$, $\Omega_3^{[2]} = -i\Omega_2$, $\lambda_1^{[2]} = \lambda_3$, $\lambda_2^{[2]} = \lambda_2$, $\lambda_3^{[2]} = \lambda_1$ and $i\mu^{[2]} = \mu$. The one-parameter solutions (5.6.9) were obtained as a certain limit of the general two-parametric family of solutions of (5.6.10) found in [34, 2].

For any solution $\{\Omega_1, \Omega_2, \Omega_3\}$ to the system (5.6.10) the formulas

$$\beta_{12} = \frac{\Omega_3}{\lambda_1 - \lambda_2}\,, \qquad \beta_{23} = \frac{\Omega_1}{\lambda_2 - \lambda_3}\,, \qquad \beta_{13} = \frac{\Omega_2}{\lambda_3 - \lambda_1} \tag{5.6.11}$$

give rotation coefficients of some metric on the space $\widehat{M}_{1;1}$ which corresponds to a locally defined Frobenius structure ([15], Proposition 3.5). The above system (5.6.10) implies the flatness of this metric (equations (5.2.25)-(5.2.26)) and the following relation on the rotation coefficients:

$$\sum_{k=1}^{3} \lambda_k \partial_{\lambda_k} \beta_{ij} = -\beta_{ij} \,. \tag{5.6.12}$$

Proposition 5.3 *The rotation coefficients of the deformations $\widehat{M}_{1;1}^{\phi,\mathbf{q}}$ of Frobenius structures $\widehat{M}_{1;1}^{\phi}$ coincide with the coefficients (5.6.11) built from the solutions Ω_i (5.6.9) to system (5.6.10).*

Proof. The space $\widehat{M}_{1;1}$ is the space of coverings of $\mathbb{C}P^1$ which have four simple ramification points P_1, P_2, P_3 and ∞^0. The Frobenius structures $\widehat{M}_{1;1}^{\phi,\mathbf{q}}$ described in Section 5.3 have rotation coefficients $\beta_{12} = W_{\mathbf{q}}(P_1, P_2)/2$, $\beta_{13} = W_{\mathbf{q}}(P_1, P_3)/2$ and $\beta_{23} = W_{\mathbf{q}}(P_2, P_3)/2$. Let us choose the a-cycle to encircle points P_1 and P_3, and the b-cycle to encircle P_2 and P_3. Then we have

$$\int_{\infty^0}^{P_1} \omega = \frac{\mu}{2}\,, \qquad \int_{\infty^0}^{P_2} \omega = \frac{1}{2}\,, \qquad \int_{\infty^0}^{P_3} \omega = \frac{\mu}{2} + \frac{1}{2}\,, \tag{5.6.13}$$

where ω is the normalized holomorphic differential $\omega = d\lambda\left(4w\sqrt{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}\right)^{-1}$. For the bidifferential $W(P,Q) := d_P d_Q \log \theta_1(\int_P^Q \omega)$, using relations (5.6.13), we get

$$W(P_1, P_2) = -\omega(P_1)\omega(P_2)\frac{\theta_3''}{\theta_3}\,, \qquad W(P_1, P_3) = -\omega(P_1)\omega(P_3)\frac{\theta_2''}{\theta_2}\,,$$
$$W(P_2, P_3) = -\omega(P_2)\omega(P_3)\frac{\theta_4''}{\theta_4}\,,$$

where $\theta_1(z) = -\theta[1/2, 1/2](z)$ and $\theta_2 = \theta[1/2, 0](0)$, $\theta_3 = \theta[0, 0](0)$, $\theta_4 = \theta[0, 1/2](0)$ are the standard theta-constants. Then, using the Thomae formulas [23]

$$\pi^2\theta_2^4 = (2\omega)^2(\lambda_3 - \lambda_1)\,, \qquad \pi^2\theta_4^4 = (2\omega)^2(\lambda_2 - \lambda_3)\,, \qquad \pi^2\theta_3^4 = (2\omega)^2(\lambda_2 - \lambda_1)\,,$$

and the heat equation for theta functions, $\partial^2_{zz}\theta[p,q](z) = 4\pi i\partial_\mu\theta[p,q](z)$, we find that the rotation coefficients $\beta_{ij} = W_{\mathbf{q}}(P_i, P_j)/2$ are given by (5.6.9), (5.6.11). $\diamond$

The system (5.6.10) arises in the context of isomonodromic deformations of the matrix differential equation

$$\frac{d\Psi}{\partial\lambda} = \left(\frac{A^0}{\lambda} + \frac{A^1}{\lambda - 1} + \frac{A^x}{\lambda - x}\right)\Psi,$$

where $A^0, A^1, A^x \in sl(2,\mathbb{C})$ and $\Psi \in SL(2,\mathbb{C})$. A solution Ψ to this system has regular singularities at the points $\lambda = 0$, $\lambda = 1$, $\lambda = x$ and $\lambda = \infty$. *Monodromy matrices* M_γ are defined for a closed path $\gamma : [0,1] \to \mathbb{C} \setminus \{0, 1, x\}$ encircling a singularity by

$$\Psi(\gamma(1)) = \Psi(\gamma(0))M_\gamma\,.$$

The *isomonodromy condition* is the requirement for monodromy matrices to remain constant as x varies. This is equivalent to the Schlesinger system for the matrices A :

$$\frac{dA^0}{dx} = \frac{[A^x, A^0]}{x}\,, \qquad \frac{dA^1}{dx} = \frac{[A^x, A^1]}{x-1}\,, \qquad \frac{dA^x}{dx} = -\frac{[A^x, A^0]}{x} - \frac{[A^x, A^1]}{x-1}\,. \tag{5.6.14}$$

This system implies that the functions $\mathrm{tr}(A^0)^2$, $\mathrm{tr}(A^1)^2$, $\mathrm{tr}(A^x)^2$ are constant. If we fix them to be all equal $1/8$ then the functions

$$\Omega_1^2 = -(\frac{1}{8} + \mathrm{tr}A^1A^x), \qquad \Omega_2^2 = -(\frac{1}{8} + \mathrm{tr}A^0A^x), \qquad \Omega_3^2 = -(\frac{1}{8} + \mathrm{tr}A^0A^1)$$

give a solution to the system (5.6.10). The system (5.6.10) is also equivalent to the Painlevé-VI equation with coefficients $(1/8, -1/8, 1/8, 3/8)$, see [15], Appendix E, and [34, 2, 40].

5.6.3 Real double of deformed Chazy Frobenius manifold

Let us fix an imaginary constant $\mathbf{q}$ and consider the real Hurwitz space $\widehat{M}_{1;1}^{\mathrm{real}}$ with coordinates $\{\lambda_k; \bar{\lambda}_k\}$ outside the subspace defined by $\mu^\Omega = -\mathbf{q}$, where $\mu^\Omega := \mu\bar{\mu}/(\bar{\mu} - \mu)$

$= \oint_b \oint_b \Omega(P,Q)$. The construction of a real double $\widehat{M}_{1;1}^{\mathrm{real}\,\Phi_s,\mathbf{q}}$ of the deformed Chazy Frobenius manifold $\widehat{M}_{1;1}^{\phi_s,\mathbf{q}}$ is based on the primary differential Φ_s (see (5.4.1)). The differential Φ_s on a genus one surface is given by

$$\Phi_s(\varsigma) = \frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \left(\frac{\bar\mu}{\bar\mu - \mu}\omega(\varsigma) + \frac{\mu}{\mu - \bar\mu}\overline{\omega(\varsigma)} \right) .$$

The set of flat coordinates $\{\xi^A\}$ of the corresponding metric $\mathbf{ds}^2_{\Phi_s}$ (5.4.4) is given by the following six functions:

$$t_1 := s = -\frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \mathrm{Re}\left\{ \frac{\bar\mu}{\bar\mu - \mu} \int_x^{x+2\omega} (\wp(\varsigma) + c)\frac{d\varsigma}{w} \right\} - \frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \frac{\bar\mu}{\bar\mu - \mu} \oint_b (\wp(\varsigma) + c) ,$$

$$t_2 := t^{0;1} = \frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \frac{\bar\mu}{\bar\mu - \mu} \frac{1}{w} , \qquad t_3 := r = \frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \frac{\mu^\Omega}{2\pi i} ,$$

$$t_4 := t = -\frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \mathrm{Re}\left\{ \frac{\bar\mu}{\bar\mu - \mu} \int_x^{x+2\omega'} (\wp(\varsigma) + c)\frac{d\varsigma}{w} \right\} ,$$

$$t_5 := t^{\overline{0;1}} = \bar t_2 , \qquad t_6 := u = \frac{\mathbf{q}}{\mu^\Omega + \mathbf{q}} \frac{1}{2\pi i} \frac{\bar\mu}{\bar\mu - \mu} .$$

The metric $\mathbf{ds}^2_{\Phi_s}$ in these coordinates is constant: $\mathbf{ds}^2_{\Phi_s} = (dt_2)^2/2 + (dt_5)^2/2 - 2dt_1dt_3 + 2dt_4dt_6$.

The prepotential (5.4.10) has the form:

$$\begin{aligned} F_{\Phi_s} = & -\frac{1}{4}t_1t_2^2 - \frac{1}{4}t_1t_5^2 + \frac{1}{2}t_1^2t_3 - \frac{1}{2}t_1t_4(2t_6 - \frac{1}{2\pi i}) \\ & + t_3^{-1}\left(\frac{1}{4}t_2^2t_4(t_6 - \frac{1}{2\pi i}) + \frac{1}{4}t_4t_5^2t_6 + \frac{1}{2}t_4^2t_6(t_6 - \frac{1}{2\pi i}) + \frac{1}{16}t_2^2t_5^2 \right) \\ & + \frac{1}{32}t_2^4\left(-\frac{1}{4\pi i(t_6 - t_3/\mathbf{q})^2}\,\gamma\left(\frac{t_3}{t_6 - t_3/\mathbf{q}}\right) + t_3^{-1} - \frac{1}{2\pi i t_3(t_6 - t_3/\mathbf{q})} \right) \\ & + \frac{1}{32}t_5^4\left(-\frac{\pi i}{(2\pi i t_6 - 1)^2}\,\gamma\left(\frac{2\pi i t_3}{1 - 2\pi i t_6}\right) + t_3^{-1} + t_3^{-1}(2\pi i t_6 - 1)^{-1} \right). \end{aligned} \tag{5.6.15}$$

The prepotential F_Φ is a quasihomogeneous function: for any nonzero constant κ it satisfies

$$F_{\phi_s}(\kappa t_1, \kappa^{1/2}t_2, \kappa^0 t_3, \kappa t_4, \kappa^{1/2}t_5, \kappa^0 t_6) = \kappa^2 F_{\phi_s}(t_1, \ldots, t_6) .$$

The Euler vector field (5.3.7) in coordinates (5.6.2) is: $E = t_1\partial_{t_1} + t_2/2\partial_{t_2} + t_4\partial_{t_4} + t_5/2\partial_{t_5}$.

The G-function (5.5.8) up to an additive constant has the form:

$$G = -\log\left\{\eta\left(\frac{t_3}{t_6 - t_3/\mathbf{q}}\right)\eta\left(\frac{2\pi i t_3}{1-2\pi i t_6}\right)(t_2 t_5)^{\frac{1}{8}}\left(\frac{t_3}{(t_6 - t_3/\mathbf{q})(2\pi i t_6 - 1)}\right)^{\frac{1}{2}}\right\}, \quad (5.6.16)$$

where we used the following relation for the Dedekind function: $\overline{\eta(\mu)} = \eta(-\bar{\mu})$.

The G-function (5.6.16) and the prepotential (5.6.15) coincide with the corresponding objects of the real double construction of [68] in the limit $\mathbf{q} \to \infty$.

Open problems

Proposition 5.3 of Section 5.6.2 shows that rotation coefficients of the flat metric of the simplest deformed Frobenius manifold $\widehat{M}_{1;1}^{\phi_s,\mathbf{q}}$ are given by formulas (5.6.11) with $\{\Omega_1, \Omega_2, \Omega_3\}$ being a one-parameter family (5.6.9) of solutions to the system (5.6.10). The general solution to the system (5.6.10) which was found in [34, 2] depends on two parameters. For this solution, formulas (5.6.11) define rotation coefficients which also correspond [15] to a Frobenius structure. The natural question is to find those structures which give a two-parametric deformation of Dubrovin's Hurwitz Frobenius manifold in genus one. The second problem will be to possibly generalize such deformations to Hurwitz spaces in arbitrary genus and find real doubles of obtained structures.

Present work provides an indication that the construction of "real doubles" of Dubrovin's Hurwitz Frobenius manifolds, proposed in [68], might have a universal character. To find a natural real double construction for an arbitrary Frobenius manifold and to clarify its meaning in applications to quantum cohomologies and other areas where Frobenius manifolds play a significant role is an interesting direction for further study.

Acknowledgments. I thank D. Korotkin for useful discussions.

Summary

The first main result of this thesis is presented in Chapter 2. It is a construction of new integrable systems related to Hurwitz spaces of genus one coverings of $\mathbb{C}P^1$ with simple branch points. The new integrable systems are deformations in the sense of Burtzev – Mikhailov – Zakharov of integrable systems (with constant spectral parameter) possessing U-V pairs living on elliptic surface. The new integrable systems are closely related to Takasaki's version of elliptic Schlesinger system.

We present a trigonometric degeneration of our construction. We consider the limit when two ramification points of an elliptic (genus one) covering connected by a branch cut tend to each other. Then the elliptic covering degenerates into a genus zero covering with a double point (two marked points which lie on different sheets of the covering and have the same projection on the base). We derive an auxiliary system of equations for the images of ramification points under the Abel map as functions of the branch points $\{\lambda_m\}$. The trigonometric version of this system is used in Chapter 3 to find implicit solutions to the Boyer-Finley equation [7] $U_{xy} = (e^U)_{tt}$ (self-dual Einstein equation with a Killing vector) for a real function $U(x, y, t)$.

We assume the function U to depend on L variables $\lambda_1, \ldots, \lambda_L$, which are, in turn,

functions of three variables x, y and t and satisfy a pair of systems of hydrodynamic type. The differential equations for characteristic speeds of the system of hydrodynamic type and the function U with respect to variables $\{\lambda_m\}$ reduce to the trigonometric system derived in Chapter 2. The solution U to the Boyer-Finley equation is then found in terms of functions on the space of trigonometric coverings. Implicit solutions $\{\lambda_m(x,y,t)\}$ to the systems of hydrodynamic type are constructed by the generalized hodograph method.

We provide conditions for the parameters of the construction (the covering, etc.) which imply the reality of the solution U.

The construction of a class of implicit solutions to the Boyer-Finley equation in terms of functions on the Hurwitz space is the main result of the Chapter 3.

In Chapter 4 we construct "real doubles" of Hurwitz Frobenius manifolds of Dubrovin [15]. These are the new Frobenius structures on the Hurwitz space. Thereby, we find new solutions to the WDVV system (see Section 1.8).

In [15] a family of L semisimple (vector algebra in the tangent space has no nilpotents) Frobenius structures is constructed on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ (see Sections 1.3 and 1.8.2 for definition of $\widehat{M}_{g;n_0,\dots,n_m}$) with local complex coordinates given by the branch points $\{\lambda_m\}_{m=1}^{L}$ of the covering (L is the complex dimension of the Hurwitz space).

First, we notice that the canonical meromorphic bidifferential W (1.2.19) plays the central role in the construction of Dubrovin. Namely, in [15] there were constructed L primary differentials on the genus g Riemann surface (the numbers g and L are related by the Riemann-Hurwitz formula (1.1.3)). Each of the primary differentials defines a Frobenius structure on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$. It turns out that the primary differentials can be obtained by certain L integral operations applied to the bidifferential W. For example, a subset of the set of primary differentials is formed by the normalized holomorphic differentials $\{\omega_k\}_{k=1}^{g}$ on the Riemann surface of genus g. For these primary differentials we have: $\omega_k(P) = \oint_{b_k} W(P,Q)/(2\pi i)$. Similarly, all the objects of Dubrovin's construction (see Section 1.8) are expressed in terms of the bidifferential W.

The rotation coefficients β_{ij} of the Darboux-Egoroff metrics on the Hurwitz Frobenius manifolds of Dubrovin are given by the bidifferential W evaluated at the ramification points of the coverings with respect to the standard local parameter x_j near a simple ramification point P_j :

$$\beta_{ij} = \frac{1}{2} W(P_i, P_j) , \tag{5.6.17}$$

where we put $W(P_i, P_j) = W(P,Q)/(dx_i(P)dx_j(Q))|_{P=P_i, Q=P_j}$.

The Rauch variational formulas (1.3.2) for the bidifferential W provide relations $\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{jk}$ for rotation coefficients (5.6.17) which form a part of sufficient condition for the flatness of the metric with rotation coefficients β_{ij} .

We use the term "real doubles" for new family of $2L$ semisimple Frobenius structures on Hurwitz spaces $\widehat{M}_{g;n_0,\dots,n_m}$ with coordinates $\{\lambda_m; \bar{\lambda}_m\}_{m=1}^{L}$. We notice that the variational formulas (1.3.4) for the Schiffer and Bergman kernels, $\Omega(P,Q)$ and $B(P,\bar{Q})$, defined by (1.2.21) and (1.2.23) respectively, imply relations $\partial_{\lambda_k}\beta_{ij} = \beta_{ik}\beta_{jk}$ and $\partial_{\bar{\lambda}_k}\beta_{ij} = \beta_{i\bar{k}}\beta_{j\bar{k}}$ and $\partial_{\lambda_k}\beta_{i\bar{j}} = \beta_{ik}\beta_{\bar{j}k}$ for the functions β_{ij} (symmetric with respect to indices) defined by

$$\beta_{ij} = \frac{1}{2}\Omega(P_i, P_j) \qquad \beta_{i\bar{j}} = \frac{1}{2}B(P_i, \bar{P}_j) \qquad \beta_{\bar{i}\bar{j}} = \frac{1}{2}\overline{\Omega(P_i, P_j)} . \tag{5.6.18}$$

Using the idea of [42] for constructing the metrics with rotation coefficients (5.6.17) on Hurwitz spaces, we find a family of metrics on Hurwitz spaces $\widehat{M}_{g;n_0,\dots,n_m}$ with local coordinates $\{\lambda_m; \bar{\lambda}_m\}$ which have rotation coefficients given by (5.6.18).

Among this family of metrics we find $2L$ metrics which correspond to Frobenius structures on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ with coordinates $\{\lambda_m; \bar{\lambda}_m\}_{m=1}^{L}$. Similarly to [15], we get a set of $2L$ primary differentials Φ_{ξ^A} expressed in terms of the Schiffer and Bergman kernels.

We denote the Frobenius structure on the Hurwitz space $\widehat{M}_{g;n_0,\dots,n_m}$ corresponding to the primary differential Φ by $\widehat{M}^{\Phi}_{g;n_0,\dots,n_m}$; the Darboux-Egoroff metric on the manifold $\widehat{M}^{\Phi}$ is denoted by $\mathbf{ds^2_\Phi}$.

For each of the $2L$ flat metrics $\mathbf{ds^2_\Phi}$ we find a set of flat coordinates $\{\xi^A\}$ (the coordinates in which the metric coefficients are constant) and a formula for the prepotential F_Φ for each of the $2L$ Frobenius structures $\widehat{M}^{\Phi}_{g;n_0,\dots,n_m}$. The prepotential F_Φ satisfies the WDVV system as a function of flat coordinates $\{\xi^A\}$ of the metric $\mathbf{ds^2_\Phi}$. The prepotential is a quasihomogeneous function of the flat coordinates. We find coefficients of quasihomogeneity for each Frobenius structure.

The G-function is a solution to the Getzler system; it is defined for a semisimple Frobenius manifold by [17]: $G = \log(\tau_{\mathrm{I}}/J^{1/24})$, where J is the Jacobian of transformation from coordinates $\{\lambda_m\}$ to the flat coordinates, and τ_{I} is the isomonodromic tau-function of the Frobenius manifold (see Section 1.8.3).

We compute the G-function for the Frobenius manifolds $\widehat{M}^{\Phi}_{g;n_0,\dots,n_m}$ using the results of [45] and [44] where the Bergman tau-function τ_W related to the isomonodromic tau function τ_{I} by $\tau_{\mathrm{I}} = (\tau_W)^{-1/2}$ was computed. We show that the isomonodromic tau-function for the real doubles $\widehat{M}^{\Phi}_{g;n_0,\dots,n_m}$ can be expressed in terms of the tau-function τ_W and the matrix $\mathbb{B}$ of b-periods of the Riemann surface as follows: $\tau_{\mathrm{I}} = \mathrm{const}\left(|\tau_W|^2 \det(\mathrm{Im}\mathbb{B})\right)^{-1/2}$.

In genus zero our construction of Hurwitz Frobenius manifolds reduces to that of Dubrovin. Therefore, the simplest examples of the new Frobenius structures are the structures on the Hurwitz spaces in genus one. We find explicit expressions for all ingredients of three new Frobenius structures on the Hurwitz space $\widehat{M}_{1;1}$.

In Chapter 5 we construct a $g(g+1)/2$-parametric deformation of the Hurwitz Frobenius manifolds of Dubrovin [15] and an analogous deformation of their real doubles constructed in Chapter 4.

We introduce a deformation $W_{\mathbf{q}}$ of the canonical meromorphic bidifferential W . The deformation depends on $g(g+1)/2$ complex paramters. The deformed bidifferential $W_{\mathbf{q}}$ satisfies the same type variational formulas as the bidifferential W . This allows us to find the Darboux-Egoroff metrics on Hurwitz spaces whose rotation coefficients are given by the bidifferential $W_{\mathbf{q}}$ analogously to formulas (5.6.17).

Similarly to the construction of Hurwitz Frobenius manifolds in terms of the bidifferential W , we find Frobenius structures on Hurwitz spaces built in terms of the bidifferential $W_{\mathbf{q}}$. These Frobenius manifolds naturally give a $g(g+1)/2$-parametric deformation of the Hurwitz Frobenius structures of Dubrovin; they coincide with the manifolds of Dubrovin in the limit when $W_{\mathbf{q}}$ tends to W .

In genus one, the prepotential of the constructed deformed Frobenius manifold is related to the prepotential of Dubrovin's "non-deformed" manifold by a transformation described in [15]. Thus, we obtained a realization of the Frobenius structure corresponding to this transformation of the prepotential on the Hurwitz space and its generalization to Hurwitz spaces in arbitrary genus.

We describe the relationship of the prepotential of the deformed Hurwitz Frobenius structure in genus one and the solution to the Painlevé-VI equation found in [34, 2].

In order to construct the real doubles of the deformed Frobenius structures, we introduce the deformations $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ of the Schiffer and Bergman kernels which depend on $g(g+1)/2$ real parameters. The variational formulas for bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ are similar to those for the Schiffer and Bergman kernels.

We build Frobenius structures on Hurwitz spaces with coordinates $\{\lambda_m; \bar{\lambda}_m\}$ in terms of the bidifferentials $\Omega_{\mathbf{q}}$ and $B_{\mathbf{q}}$ analogously to the construction of the real doubles in Chapter 4. The obtained manifolds give the deformations of real doubles of the Hurwitz Frobenius manifolds of Dubrovin.

We find the corresponding solutions to the Getzler system and compute explicitly the examples of deformed Frobenius structures for the simplest Hurwitz space in genus one.

Conclusion

The main results of this thesis are related to the theory of Frobenius manifolds.

We introduced the "real doubles" of semisimple Frobenius structures on Hurwitz spaces in genus greater than zero. We showed that to each existing Frobenius structure on the Hurwitz space with usual local complex coordinates there correspond two Frobenius structures of a double dimension. These structures are built on the Hurwitz space considered as a real manifold with the set of coordinates formed by the usual coordinates and their complex conjugates. As we know from [15], Hurwitz Frobenius manifolds possess most of the characteristic properties of a generic Frobenius manifold. This suggests the natural question: is it possible to find a "real double" for an arbitrary Frobenius manifold? If such structures exist it will be interesting to understand their meaning in connection with various areas of mathematics which are related to Frobenius manifolds theory.

We also constructed $g(g+1)$-parametric deformations of the Hurwitz Frobenius structures of Dubrovin together with their real doubles. For that purpose, we introduced deformations of the normalized meromorphic bidifferential of the second kind W and the Schiffer and Bergman kernels on the Riemann surface. These are new bidifferentials which satisfy variational formulas similar to the Rauch variational formulas for the non-deformed kernels.

The constructed deformations of Hurwitz Frobenius manifolds might admit two possible generalizations. The first way of generalizing is to explore the relationship of the deformations in genus one and the Chazy equation. Namely, a solution to the Chazy equation enters the expression for the prepotential of Dubrovin's Frobenius structure on the simplest Hurwitz space in genus one. The general solution to the Chazy equation depends on three parameters. It determines a three-parametric family of Hurwitz Frobenius manifolds in genus one. Our deformations belong to this family; they form a one-parameter subfamily. Thus, it can be possible to find the realization on the Hurwitz spaces of the Frobenius structures determined by the general (three-parametric) solution to the Chazy equation and find its generalization to higher genera.

Another possible way to find new deformations is to look at the link with the Painlevé-VI equation with coefficients $(1/8, -1/8, 1/8, 3/8)$. Namely, in genus one, the rotation coefficients of flat metrics of the deformed Frobenius structures are built from a one-parameter family of solutions to this equation. The general family of solutions to the Painlevé-VI equation with given coefficients depends on two parameters. Therefore, it should be possible to find the Hurwitz Frobenius structures which for the simplest Hurwitz space in genus one correspond to the general two-parametric family of solutions to the Painlevé-VI equation.

The other results of this thesis are the construction of new family of integrable systems related to the Hurwitz spaces in genus one, finding an elliptic analog of the systems appeared in the work of Kupershmidt and Manin [53] (which describes the dependence of the critical points of the inverse to the uniformization map on its critical values) and applying its trigonometric degeneration to a construction of implicit solutions to the Boyer-Finley equation. We hope that among the new systems one could find those having physical or geometrical significance.

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