Heuristic Results for Ratio Conjectures of $L_E(1, \chi)$

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ABSTRACT

Heuristic Results for Ratio Conjectures of $L_E(1, \chi)$

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Let $L_E(s, \chi)$ be the Hasse-Weil L-function of an elliptic curve E defined over $\mathbb{Q}$ and twisted by a Dirichlet character χ of order k and of conductor $\mathfrak{f}_\chi$. Keating and Snaith [KS00b] and [KS00a] introduced the way to study L-functions through random matrix theory of certain topological groups. Conrey, Keating, Rubinstein, and Snaith [CKRS02] and David, Fearnley, and Kisilevsky [DFK04] developed their ideas in statistics of families of critical values of $L_E(1, \chi)$ twisted by Dirichlet characters of conductors $\leq X$ and proposed conjectures regarding the number of vanishings in their families and the ratio conjectures of moments and vanishings which are strongly supported by numerical experiments.

In this thesis, we review and develop their works and propose the ratio conjectures of moments and vanishings in the family of $L_E(1, \chi)$ twisted by Dirichlet characters of conductors $\mathfrak{f}_\chi \leq X$ and order of some odd primes, especially 3, 5, and 7 inspired by the connections of L-function theory and random matrix theory. Moreover, we support our result on the ratio conjectures of moments and vanishings of the families for some certain elliptic curves by numerical experiments.

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Chapter 1

Introductions and Notations

In this thesis, we review and develop the work of Conrey, Keating, Rubinstein, and Snaith [CKRS02] and David, Fearnley, and Kisilevsky [DFK04] and propose the ratio conjectures of the t-th moments and vanishings in the family of $L_E(1, \chi)$ twisted by Dirichlet characters of conductors $\mathfrak{f}_\chi \leq X$ and order of some odd primes, especially 3, 5, and 7 inspired by the connections of L-function theory and random matrix theory. Moreover, we support our result on the ratio conjectures of moments and vanishings of the families for some certain elliptic curves by numerical experiments. All the experiments were conducted using PARI/GP [PAR10].

Here are some notations used through this paper:

- $\nu(n)$ - the number of distinct primes dividing an integer n.
- p, k, t, X- globally reserved for the notation for a fixed prime, order of Dirichlet character which is an odd prime or weight of modular form depending on context, order of moments, and the maximum conductor respectively.
- E, E/K, N_E - an elliptic curve over $\mathbb{Q}$, over a number field K, and the conductor of a given elliptic curve E respectively.
- $L_E(s, \chi)$ - L-function of an elliptic curve E twisted by a Dirichlet character χ, where χ is a Dirichlet character of conductor $\mathfrak{f}_\chi$ and of order k.
- $M_k(N) := M_k(\Gamma_0(N))$ - the space of modular forms of weight k for $\Gamma_0(N)$.

- $M_k(N,\psi) := \{f \in M_k(\Gamma_1(N)) \mid f|A = \psi(d)f \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\}$ where ψ is a Dirichlet character mod N.
- $S_k(N) := S_k(\Gamma_0(N))$ - the space of cusp forms of weight k for $\Gamma_0(N)$.
- $S_k(N,\psi) := \{f \in S_k(\Gamma_1(N)) \mid f|A = \psi(d)f \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\}$ where ψ is a Dirichlet character mod N.
- $U(N)$ - the set of $N \times N$ unitary matrices over $\mathbb{C}$, i.e. the set of $N \times N$ matrices A over $\mathbb{C}$ such that $AA^* = A^*A = I_N$ where A^* is the complex conjugate transpose of A and I_N is the $N \times N$ identity matrix.
- $Y_k(X) = \{\chi \mid \chi \text{ is a Dirichlet character of order } k \text{ and } \mathfrak{f}_\chi \leq X\}$.
- $Z_k(X) = \{\chi \in Y_k(X) \mid \mathfrak{f}_\chi \text{ is a prime}\}$.
- $Y_{E,k}(X) = \{\chi \mid \chi \text{ is a character of order } k,\ \mathfrak{f}_\chi \leq X, \text{ and } (\mathfrak{f}_\chi, N_E) = 1\}$.
- $Z_{E,k}(X) = \{\chi \in Y_{E,k}(X) \mid \mathfrak{f}_\chi \text{ is a prime}\}$.
- $V_{E,k}(X) = \{\chi \in Y_{E,k}(X) \mid L_E(1,\chi) = 0\}$.
- $W_{E,k}(X) = \{\chi \in Z_{E,k}(X) \mid L_E(1,\chi) = 0\}$.

Note that $V_{E,k}(X) \subset Y_{E,k}(X) \subset Y_k(X)$ and $W_{E,k}(X) \subset Z_{E,k}(X) \subset Y_{E,k}(X)$ for any E.

The Hasse and Weil theorem [Sil09] followed by the modularity theorem [BCDT01] and [TW95] asserts that for a given E and $\mathrm{Re}(s) > 3/2$,

$$L_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_{p \nmid N_E} \left(1 - \frac{a_p}{p^s} + \frac{p}{p^{2s}}\right)^{-1} \prod_{p \mid N_E} \left(1 - \frac{a_p}{p^s}\right)^{-1}$$

can be extended to an analytic function for all $s \in \mathbb{C}$ and $L_E(s) = L_f(s)$, which is the L-series for some normalized eigenform $f \in S_2(N_E)$. Similarly, a L-function twisted by a Dirichlet character of conductor $\mathfrak{f}_\chi$

$$L_E(s,\chi) = \sum_{n=1}^{\infty} \frac{a_n \chi(n)}{n^s}$$

also can be extended to an analytic function for all s and $L_E(s,\chi) = L_{f_\chi}(s)$, which is the L-series for some normalized eigenform $f_\chi = \sum_{n=1}^{\infty} a_n \chi(n) q^n \in S_2(N_E (\mathfrak{f}_\chi)^2, \chi^2)$

where $q = \exp(2\pi i)$. Note that if $f \in S_k(N, \psi)$, then f_χ is a cusp form of $S_k\left(N(\mathfrak{f}_\chi)^2, \psi\chi^2\right)$, see proposition 17 in [Kob93]. In particular $S_k(N) = S_k\left(N, \chi_1\right)$ where χ_1 is the trivial Dirichlet character of conductor 1. Furthermore, assume $(\mathfrak{f}_\chi, N_E) = 1$ and let

$$\Lambda_E(s, \chi) = \left(\mathfrak{f}_\chi \sqrt{N_E}/2\pi\right)^s \Gamma(s) L_E(s, \chi).$$

Then, the functional equation [Shi76] for $\Lambda_E(s, \chi)$ is

$$\Lambda_E(s, \chi) = \frac{\omega_E \chi(N_E) \tau(1, \chi)^2}{\mathfrak{f}_\chi} \Lambda_E(2 - s, \overline{\chi}). \tag{1.1}$$

where $\tau(1, \chi)$ is the Gauss sum of χ, defined in 2.2, and ω_E is the eigenvalue of the Fricke involution acting on the normalized eigenform $f \in S_2(N)$ corresponding to E, see 2.8 in [Cre97]. The Fricke involution is defined by a vector space isomorphism $\phi : M_k(N, \psi) \to M_k(N, \overline{\psi})$ defined by

$$f \mapsto f\Big| \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

where $|$ is the slash operator, defined in 3.2. For a normalized eigenform $f \in S_2(N)$

$$f\Big| \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = \pm f.$$

Therefore, we have $\omega_E = \pm 1$ [Cre97].

In the following section, we study modular symbols for a $f_\chi \in S_2(N_E \left(\mathfrak{f}_\chi\right)^2, \chi^2)$ and the relation between them and corresponding L-values at $s = 1$.

The idea of studying number theory, especially L-functions, using random matrix theory was introduced by Montgomery and Dyson who noticed the agreement between the statistics of some random matrices, more specifically the pair correlation between their eigenvalues, and the zeroes of the Riemann zeta function [Mon73]. Other aspects of the relation between L-functions and random matrix theory are explained in chapter 4.

Chapter 2

Dirichlet Characters and Gauss sums

Let G be a finite abelian group. Then, we call a group homomorphism $\chi : G \to \mathbb{C}^\times$ a character of G. Furthermore, let $\widehat{G}$ be the set of all characters of G and define the multiplication of two characters χ and χ' as $(\chi\chi')(g) = \chi(g)\chi'(g)$ for all $g \in G$. Then, $\widehat{G}$ is also a finite abelian group with the identity element which is the trivial group homomorphism called the trivial character. We have the following isomorphism:

Proposition 2.1. *For a finite abelian group G,*

$$\widehat{G} \simeq G.$$

Proof. Since G is a finite abelian group, by the decomposition theorem of finite abelian groups, we can write $G = \oplus_{i=1}^{m} G_i$ where m is a positive integer and G_i's are finite cyclic abelian groups. Let g_i be a generator of G_i and $|G_i| = n_i$ for each $i = 1, \ldots, m$. Then, any character χ of G_i can be determined by $\chi(g_i)$ and $\chi(g_i)$ can be any n_i-th root of unity. So, $|G_i| = n_i = |\widehat{G_i}|$. Furthermore, let $\psi \in \widehat{G_i}$ such that $\psi(g_i) = \zeta_{n_i}$ for some primitive n_i-th root of unity. Then, given $\chi \in \widehat{G_i}$, $\chi(g_i) = \zeta_{n_i}^{l}$ for some integer l. So, $\chi(g_i^j) = \chi(g_i)^j = \zeta_{n_i}^{lj} = \psi(g_i)^{lj} = \psi(g_i^j)^l$ for any integer j. Therefore, $\widehat{G_i}$ is cyclic and $\widehat{G_i} \simeq G_i$.

Now, let $A = G_i$ and $B = G_j$ for $i \neq j$ and their orders be n_A and n_B respectively.

We prove $\psi : \widehat{A} \oplus \widehat{B} \to \widehat{A \oplus B}$ is a group isomorphism defined by

$$(\chi_A, \chi_B)(a, b) \mapsto \chi_A \chi_B(a, b) := \chi_A(a)\chi_B(b)$$

for $\chi_A \in \widehat{A}$, $\chi_B \in \widehat{B}$, $a \in A$, and $b \in B$. Then, it is easy to see $\ker \psi = \{(\chi_{A,1}, \chi_{B,1})\}$ where $\chi_{A,1}$ and $\chi_{B,1}$ are the trivial characters of A and B respectively. Furthermore, let $\chi \in \widehat{A \oplus B}$ and set

$$\chi_A(a) = \chi(a, 1_B) \text{ and } \chi_B(b) = \chi(1_A, b) \quad \text{for all } a \in A \text{ and } b \in B.$$

Then, $\chi_A \in \widehat{A}$, $\chi_B \in \widehat{B}$ and $\psi((\chi_A, \chi_B))(a, b) = \chi_A(a)\chi_B(b) = \chi(a, 1_B)\chi(1_A, b) = \chi(a, b)$ for all $a \in A$ and $b \in B$. Therefore, ψ is an isomorphism and the proof is completed by induction . □

Now, consider a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ for a positive integer N. We extend it to an arithmetic function called a Dirichlet character χ mod N as

Definition 2.1.

$$\chi(n) = \begin{cases} \chi(n \bmod N), & \textit{if } (n, N) = 1 \\ 0, & \textit{otherwise} \end{cases},$$

where (n, N) is the greatest common divisor of n and N.

A Dirichlet character χ is completely multiplicative on $\mathbb{Z}$. There is the unique Dirichlet character mod 1 which sends all n to 1. We call it the trivial Dirichlet character and denote it by χ_1. Also, for each N, there exists the Dirichlet character χ mod N such that $\chi(n) = 1$ if $(n, N) = 1$ and $\chi(n) = 0$ otherwise. We denote such character by χ_0 and call it the principal character. Furthermore, for each N and a Dirichlet character χ mod N, there is a Dirichlet character χ' mod d that induces χ for some $d \mid N$. More precisely

$$\chi(n) = \chi'(n \bmod d)\chi_0(n)$$

for all integers n and the principal character χ_0 mod N. In other words, let ϕ be a natural map $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/d\mathbb{Z})^\times$. Then, the following diagram commutes:

$$
\begin{array}{ccc}
(\mathbb{Z}/N\mathbb{Z})^\times & \xrightarrow{\phi} & (\mathbb{Z}/d\mathbb{Z})^\times \\
\Big\downarrow{\scriptstyle \chi} & \swarrow{\scriptstyle \chi'} & \\
\mathbb{C}^\times & &
\end{array}
$$

Thus, we can take the minimum of those d's for each χ mod N and call it the conductor of χ mod N and denote it by $\mathfrak{f}_\chi$. Furthermore, we call χ a primitive Dirichlet character mod N if $\mathfrak{f}_\chi = N$. We can identify a Dirichlet character $\chi \pmod N$ with a unique primitive Dirichlet character χ' of conductor $\mathfrak{f}_\chi$ by $\chi = \chi'\chi_0$. In particular, χ_1 induces χ_0 mod N for all N.

It is more convenient to use primitive Dirichlet characters mod $\mathfrak{f}_\chi$ than Dirichlet characters χ mod N. Consider the set

$$G_N = \{\chi \mid \chi \text{ is a primitive character of conductor } \mathfrak{f}_\chi \mid N\}.$$

Let $\chi', \psi' \in G_N$ and $M = \operatorname{lcm}(\mathfrak{f}_{\chi'}, \mathfrak{f}_{\psi'})$. Then, $M|N$ and χ' and ψ' induce some characters χ and $\psi \in \widehat{(\mathbb{Z}/M\mathbb{Z})}^\times$ respectively. Let $\pi(n) = \chi(n)\psi(n)$. Then, we can find the unique primitive character π' which induces π. Then, clearly $\pi' \in G_N$ and, in this way, we define the multiplication of χ' and ψ' as $\chi'\psi' = \pi'$. It is not hard to prove G_N is an abelian group with the identity χ_1 and we can identify $\widehat{(\mathbb{Z}/N\mathbb{Z})}^\times$ with a subgroup of G_N

$$\{\chi \mid \chi \text{ is a primitive Dirichlet character mod } \mathfrak{f}_\chi \text{ for some } \mathfrak{f}_\chi \text{ where } \mathfrak{f}_\chi \mid N\}.$$

We denote the inverse of χ by $\overline{\chi}$ and $\overline{\chi}(n) = \overline{\chi(n)}$. More importantly, as a special case of proposition 3.8 (a) in [Was82], we can view a Dirichlet character mod N as a homomorphism : $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to \mathbb{C}^\times$ by the following proposition:

Proposition 2.2. *Let $\mathbb{Q}(\zeta_N)$ be the cyclotomic field where ζ_N is a primitive N-th* *root of unity. Then,*

$$\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times .$$

Proof. See theorem 2.5 in [Was82]. □

Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Then, proposition 2.2 implies that $|\widehat{G}| = \varphi(N)$ where φ is the Euler totient function. In summary, we have the following isomorphisms:

$$\widehat{G} \;\simeq\; G \;\simeq\; (\mathbb{Z}/N\mathbb{Z})^\times \;\simeq\; (\widehat{\mathbb{Z}/N\mathbb{Z}})^\times.$$

Define a Gauss sum $\tau(a, \chi)$ of an integer a and a Dirichlet character mod N by

Definition 2.2.

$$\tau(a, \chi) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i a n/N).$$

We need some properties of Gauss sums of a Dirichlet character mod N.

Lemma 2.1. *If* $\chi \bmod N$ *is primitive, then for all integer* a*,*

$$\tau(a, \chi) = \overline{\chi}(a)\tau(1, \chi).$$

Proof. Suppose χ is a primitive character mod N. First, if $(a, N) = 1$, then we have

$$\begin{aligned}
\tau(a, \chi) &= \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i a n/N) \\
&= \overline{\chi}(a) \sum_{n=0}^{N-1} \chi(an) \exp(2\pi i a n/N) \text{ since } \overline{\chi}(a)\chi(a) = 1 \\
&= \overline{\chi}(a)\tau(1, \chi) \text{ by re-arranging the summation over } na.
\end{aligned}$$

Now, let $(a, N) = d > 1$. Then, $\overline{\chi}(a)\tau(1, \chi) = 0$ since $\overline{\chi}(a) = \overline{\chi(a)} = 0$. Furthermore, there are some integers a' and N' such that $(a', N') = 1$ and $a = a'd$ and $N = N'd$. Thus,

$$\tau(a, \chi) = \sum_{n=0}^{N-1} \chi(n) \exp(2\pi i a n/N) = \sum_{j=0}^{N'-1} \left(\sum_{\substack{l \equiv j \bmod N' \\ 0 \le l \le N-1}} \chi(l) \right) \exp(2\pi i a' j/N'). \quad (2.1)$$

Consider a natural homomorphism $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N'\mathbb{Z})^\times$. Then, $\ker\phi = \{n \in (\mathbb{Z}/N\mathbb{Z})^\times \mid n \equiv 1 \bmod N'\}$. So, since χ is primitive and $N \neq 1$, we have $\sum_{n\in\ker\phi} \chi(n) = 0$. Then, for each fixed j mod N', the inner sum in (2.1) is 0. Therefore, $\tau(a, N) = 0$. ◻

Lemma 2.2. *For a primitive Dirichlet character* χ mod N*,*

$$|\tau(1,\chi)|^2 = \chi(-1)\tau(1,\chi)\tau(1,\overline{\chi}) = N.$$

Proof. The first equality follows from

$$\begin{aligned}\overline{\tau(1,\chi)} &= \sum_{n=0}^{N-1} \overline{\chi}(n)\exp(-2\pi i n/N)\\ &= \sum_{m=0}^{N-1} \chi(-1)\overline{\chi}(m)\exp(2\pi i m/N), \text{ by re-arranging the summation}\\ &= \chi(-1)\tau(1,\overline{\chi}).\end{aligned}$$

For the second equality,

$$\begin{aligned}|\tau(1,\chi)|^2 = \tau(1,\chi)\overline{\tau(1,\chi)} &= \sum_{n=0}^{N-1} \overline{\chi}(n)\tau(1,\chi)\exp(-2\pi i n/N)\\ &= \sum_{n=0}^{N-1} \tau(n,\chi)\exp(-2\pi i n/N) \text{ by lemma 2.1}\\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} \chi(m)\exp(2\pi i(m-1)n/N)\\ &= \sum_{m=0}^{N-1} \chi(m)\sum_{n=0}^{N-1} \exp(2\pi i(m-1)n/N) = N,\\ &\quad\text{since } \sum_{n=0}^{N-1} \exp(2\pi i(m-1)n/N) = 0 \text{ for } m \neq 1.\end{aligned}$$

◻

For a Dirichlet character χ of conductor $\mathfrak{f}_\chi$ and of odd prime order, $\chi(-1) = 1$ and lemma 2.2 implies that

$$\overline{\tau(1,\chi)} = \chi(-1)\tau(1,\overline{\chi}) = \tau(1,\overline{\chi}). \tag{2.2}$$

Recall $Y_k(X)$ is the set of Dirichlet characters of order k and conductor $\leq X$. It is useful to get an analytic approximation of $|Y_k(X)|$ for the study of vanishings and moments of families of $L_E(1,\chi)$ in the chapter 4. First, let's consider the number of distinct Dirichlet characters χ of order k and conductor $\mathfrak{f}_\chi$. David, Fearnley, and Kisilevsky [DFK04] showed that there are $2^{\nu(\mathfrak{f}_\chi)}$ distinct cubic characters of conductor $\mathfrak{f}_\chi$. With the same argument, we can generalize this result for higher odd prime conductors as follows.

Lemma 2.3. *Let k be an odd prime positive integer and χ be a Dirichlet character of order k and conductor $\mathfrak{f}_\chi$. Then, there are $(k-1)^{\nu(\mathfrak{f}_\chi)}$ distinct characters of order k and conductor $\mathfrak{f}_\chi$.*

Proof. For a Dirichlet character χ of order k and conductor $\mathfrak{f}_\chi = \prod_{i=1}^{\nu(\mathfrak{f}_\chi)} p_i^{a_i}$ where p_i's are pairwise distinct primes and $a_i \in \mathbb{Z}^{\geq 1}$, χ can be factorized by

$$\chi = \prod_{i=1}^{\nu(\mathfrak{f}_\chi)} \chi_i$$

where χ_i is a character of order k and conductor $p_i^{a_i}$ for $1 \leq i \leq \nu(\mathfrak{f}_\chi)$. However, a_i should be 2 if $p_i = k$ and 1 if $p_i \neq k$ since, otherwise, there is a character of order k and conductor p_i inducing that χ_i for each i. Furthermore, each χ_i is a character of order k and conductor $p_i \neq k$ if and only if $p_i \equiv 1 \bmod k$. It is due to the Lagrange theorem of group theory and the fact that the group of Dirichlet characters of conductor $p_i^{a_i}$ has order $\phi(p_i^{a_i}) = p_i^{a_i-1}(p_i-1)$ for an odd prime k and $a_i > 1$. Therefore, $\mathfrak{f}_\chi = k^{2a}\prod_{i=1}^{\nu(\mathfrak{f}_\chi-1)} p_i$ where $a = 0$ or 1, p_i are distinct primes other than k, and $p_i \equiv 1 \bmod k$. Since for each χ_i there are $k-1$ distinct characters of order k, the proof is completed. □

Now we consider the Dirichlet series $L_k(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where a_n is the number of distinct Dirichlet characters of order k and conductor n. Note that $a_n = (k-1)^{\nu(n)}$,

$a_1 = 1$, and a_n is a multiplicative function from lemma 2.3. Thus, we have

$$
\begin{aligned}
L_k(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} &= \left(1 + \frac{a_{k^2}}{k^{2s}}\right) \prod_{p \equiv 1 \bmod k} \left(1 + \frac{a_p}{p^s}\right) \\
&= \left(1 + \frac{k-1}{k^{2s}}\right) \prod_{p \equiv 1 \bmod k} \left(1 + \frac{k-1}{p^s}\right)
\end{aligned} \tag{2.3}
$$

for $\mathrm{Re}(s) > c$ for some positive constant c. We need to use a Dedekind zeta function over a number field K to investigate some analytic properties of $L_k(s)$, see [IK04] for a reference. The Dedekind zeta function $\zeta_K(s)$ over K is defined by

$$
\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{(N(\mathfrak{a}))^s} = \prod_{\mathfrak{p}} \left(1 - \frac{1}{(N(\mathfrak{p}))^s}\right)^{-1} \tag{2.4}
$$

for $\mathrm{Re}(s) > 1$ and where $N(\mathfrak{a})$ is the norm of a non-zero ideal $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$. When $K = \mathbb{Q}$, the Dedekind zeta function $\zeta_{\mathbb{Q}}(s)$ is just the Riemann zeta function $\zeta(s)$. Furthermore, ζ_K has a simple pole at $s = 1$ with residue

$$
\operatorname*{Res}_{s=1} \zeta_K(s) = \frac{2^{r_1+r_2} \pi^{r_2} R_K h_K}{\omega_K \sqrt{D_K}} \tag{2.5}
$$

where r_1 and r_2 are real and complex embedding respectively into $\mathbb{C}$, ω_K is the number of roots of unity, R_K, h_K, and D_K are the regulator, class number, and the absolute discriminant of K respectively.

Let k be an odd prime and ζ_k be a primitive root of unity. Then, since $\mathbb{Q}(\zeta_k)/\mathbb{Q}$ is a cyclotomic extension, hence a Galois extension, of degree $\phi(k)$, the Euler totient function, we have the prime decomposition for a rational prime p in $\mathcal{O}_K$ as

$$
p\mathcal{O}_K = (\mathfrak{p}_1 \cdots \mathfrak{p}_g)^e
$$

satisfying $\phi(k) = efg$ where e is the ramification index, f is the inertia degree, and g is the decomposition index. We have some remarks regarding cyclotomic extensions $\mathbb{Q}(\zeta_k)/\mathbb{Q}$ and an odd prime k ,

- p is totally ramified if and only if $p = k$.

- $p \equiv 1 \bmod k$ if and only if p splits completely if and only if $g = k-1$.
- the inertia degree f of a rational prime p is the order of p in $(\mathbb{Z}/k\mathbb{Z})^{\times}$.

Based on the above remarks, we can write Euler factors of the Dedekind zeta functions $\zeta_K(s)$ for $K = \mathbb{Q}(\zeta_3), \mathbb{Q}(\zeta_5)$, and $\mathbb{Q}(\zeta_7)$ by investigating f since

$$\prod_{\mathfrak{p}|p}\left(1-\frac{1}{(N(\mathfrak{p}))^s}\right)^{-1} = \left(1-\frac{1}{p^{fs}}\right)^{-g}.$$

Indeed, when $K = \mathbb{Q}(\zeta_3)$, we have

$$\zeta_K(s) = \left(1-\frac{1}{3^s}\right)^{-1}\prod_{p\equiv 1 \bmod 3}\left(1-\frac{1}{p^s}\right)^{-2}\prod_{p\equiv 2 \bmod 3}\left(1-\frac{1}{p^{2s}}\right)^{-1}.$$

When $K = \mathbb{Q}(\zeta_5)$, we have

$$\zeta_K(s) = \left(1-\frac{1}{5^s}\right)^{-1}\prod_{p\equiv 1 \bmod 5}\left(1-\frac{1}{p^s}\right)^{-4}\prod_{\substack{p\equiv 2 \bmod 5\\ p\equiv 3 \bmod 5}}\left(1-\frac{1}{p^{4s}}\right)^{-1}\prod_{p\equiv 4 \bmod 5}\left(1-\frac{1}{p^{2s}}\right)^{-2}. \tag{2.6}$$

When $K = \mathbb{Q}(\zeta_7)$, we have

$$\begin{aligned}\zeta_K(s) = \left(1-\frac{1}{7^s}\right)^{-1}&\prod_{p\equiv 1 \bmod 7}\left(1-\frac{1}{p^s}\right)^{-6}\prod_{\substack{p\equiv 2 \bmod 7\\ p\equiv 4 \bmod 7}}\left(1-\frac{1}{p^{3s}}\right)^{-2}\\ &\times\prod_{\substack{p\equiv 3 \bmod 7\\ p\equiv 5 \bmod 7}}\left(1-\frac{1}{p^{6s}}\right)^{-1}\prod_{p\equiv 6 \bmod 7}\left(1-\frac{1}{p^{2s}}\right)^{-3}.\end{aligned} \tag{2.7}$$

We can calculate the residue of $\zeta_K(s)$ at $s = 1$ for $K = \mathbb{Q}(\zeta_3), \mathbb{Q}(\zeta_5)$, and $\mathbb{Q}(\zeta_7)$. For $K = \mathbb{Q}(\zeta_k)$ and an odd prime k, r_1 and r_2 in equation (2.5) satisfy $k-1 = \phi(k) = [K:\mathbb{Q}] = r_1 + 2r_2$ and the maximal real subfield of $\mathbb{Q}(\zeta_k)$ is $\mathbb{Q}\left(\zeta_k + \zeta_k^{-1}\right)$ so that $\left[K:\mathbb{Q}\left(\zeta_k+\zeta_k^{-1}\right)\right] = 2$ since ζ_k is a root of $X^2 - \left(\zeta_k+\zeta_k^{-1}\right)X + 1$. Therefore, $r_1 = 0$ and $r_2 = (k-1)/2$. Furthermore, all roots of unity of K are of form $\pm\zeta_k^j$ for $j = 0, 1, \ldots, k-1$. Hence, $\omega_K = 2k$. The class numbers h_K for the above cyclotomic fields can be found at page 352 and in the table at page 353 in [Was82]. Moreover, we can find D_K in proposition 2.7 in [Was82] as for an odd prime k,

$$D_K = (-1)^{\phi(k)/2}\frac{k^{\phi(k)}}{\prod_{p|k} p^{\phi(k)/(p-1)}} = k^{(k-2)}.$$

k	r_1	r_2	R_K	h_K	w_K	D_K	$\mathrm{Res}_{s=1}\,\zeta_K(s)$
$k=3$	0	1	1	1	6	3	0.6045998
$k=5$	2	1	0.9624237	1	10	125	0.0688654
$k=7$	4	1	2.1018187	1	14	16807	0.0370571

Table 2.1: Invariants of $K=\mathbb{Q}(\zeta_k)$ for $k=3,5$, and 7

Therefore, finally by the using PARI/GP [PAR10] to compute R_K, we obtain table 2.1. Now using similar arguments of proposition 5.2 for $L_3(s)$ in [DFK04] we investigate analytic properties of $L_5(s)$ and $L_7(s)$.

Lemma 2.4. *$L_5(s)$ and $L_7(s)$ have simple poles at $s=1$.*

Proof. First we prove the result for $L_5(s)$. Let $K=\mathbb{Q}(\zeta_5)$. Recall the Euler product of $L_5(s)$ in (2.3) and then

$$\begin{aligned}
L_5(s) &= \left(1+\frac{4}{25^s}\right)\prod_{p\equiv 1 \bmod 5}\left(1+\frac{4}{p^s}\right)\\
&= \left(1+\frac{4}{25^s}\right)\prod_{p\equiv 1 \bmod 5}\left(1-\frac{1}{p^s}\right)^4\left(1+\frac{4}{p^s}\right)\prod_{p\equiv 1 \bmod 5}\left(1-\frac{1}{p^s}\right)^{-4}\\
&= \left(1-\frac{1}{5^s}+\frac{4}{25^s}-\frac{4}{125^s}\right)\prod_{p\equiv 1 \bmod 5}\left(1-\frac{10}{p^{2s}}+\frac{20}{p^{3s}}-\frac{15}{p^{4s}}+\frac{4}{p^{5s}}\right)\\
&\quad\times\prod_{\substack{p\equiv 2 \bmod 5\\ p\equiv 3 \bmod 5}}\left(1-\frac{1}{p^{4s}}\right)\prod_{p\equiv 4 \bmod 5}\left(1-\frac{1}{p^{2s}}\right)^2\zeta_K(s).
\end{aligned}$$

Set

$$\begin{aligned}
f(s) := &\left(1-\frac{1}{5^s}+\frac{4}{25^s}-\frac{4}{125^s}\right)\prod_{p\equiv 1 \bmod 5}\left(1-\frac{10}{p^{2s}}+\frac{20}{p^{3s}}-\frac{15}{p^{4s}}+\frac{4}{p^{5s}}\right)\\
&\times\prod_{\substack{p\equiv 2 \bmod 5\\ p\equiv 3 \bmod 5}}\left(1-\frac{1}{p^{4s}}\right)\prod_{p\equiv 4 \bmod 5}\left(1-\frac{1}{p^{2s}}\right)^2.
\end{aligned}$$

Then, by considering terms with smallest power of $1/p^s$ where $p\neq 5$ in the factors, $f(s)$ is analytic at $s=1$. Note $\zeta_K(s)$ converges for $\mathrm{Re}(s)>1$. Therefore, $L_5(s)$ has the constant $c=1$ in (2.3) and has a simple pole at $s=1$.

Now, let $K = \mathbb{Q}(\zeta_7)$ and recall the Euler product of $L_7(s)$ in(2.3). Then,

$$\begin{aligned}
L_7(s) &= \left(1+\frac{6}{49^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1+\frac{6}{p^s}\right) \\
&= \left(1+\frac{6}{49^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1-\frac{1}{p^s}\right)^6\left(1+\frac{6}{p^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1-\frac{1}{p^s}\right)^{-6} \\
&= \left(1-\frac{1}{7^s}+\frac{6}{49^s}-\frac{6}{343^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1-\frac{21}{p^{2s}}+\frac{70}{p^{3s}}-\frac{105}{p^{4s}}+\frac{84}{p^{5s}}-\frac{35}{p^{6s}}+\frac{6}{p^{7s}}\right) \\
&\quad \times \prod_{\substack{p\equiv 2 \bmod 7\\ p\equiv 4 \bmod 7}}\left(1-\frac{1}{p^{3s}}\right)^2\prod_{\substack{p\equiv 3 \bmod 7\\ p\equiv 5 \bmod 7}}\left(1-\frac{1}{p^{6s}}\right)\prod_{p\equiv 6 \bmod 7}\left(1-\frac{1}{p^{2s}}\right)^3\zeta_K(s).
\end{aligned}$$

Set

$$\begin{aligned}
g(s) := &\left(1-\frac{1}{7^s}+\frac{6}{49^s}-\frac{6}{343^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1-\frac{21}{p^{2s}}+\frac{70}{p^{3s}}-\frac{105}{p^{4s}}+\frac{84}{p^{5s}}-\frac{35}{p^{6s}}+\frac{6}{p^{7s}}\right) \\
&\times \prod_{\substack{p\equiv 2 \bmod 7\\ p\equiv 4 \bmod 7}}\left(1-\frac{1}{p^{3s}}\right)^2\prod_{\substack{p\equiv 3 \bmod 7\\ p\equiv 5 \bmod 7}}\left(1-\frac{1}{p^{6s}}\right)\prod_{p\equiv 6 \bmod 7}\left(1-\frac{1}{p^{2s}}\right)^3.
\end{aligned}$$

Then, again, we consider the powers in the factors and $g(s)$ is analytic at $s = 1$. Therefore, $L_7(s)$ has the constant $c = 1$ in (2.3) and has a simple pole at $s = 1$. □

David, Fearnley, and Kisilevsky [DFK04] showed that

$$|Y_3(X)| = \sum_{n\leq X} a_n \ \sim \ c_3 X \tag{2.8}$$

where $c_3 = \dfrac{11\sqrt{3}}{18\pi}\displaystyle\prod_{p\equiv 1 \bmod 3}\left(1-\frac{2}{p(p+1)}\right)$ by using $L_3(s) = \sum_{n=1}^{\infty} a_n/n^s$ and the Ikehara-Wiener Tauberian theorem for $|Y_3(X)|$, see [MV07]. Note that for functions $f(x)$ and $g(x)$,

$$f(x) \sim g(x) \iff \lim_{x\to\infty}\frac{f(x)}{g(x)} = 1.$$

Similarly we apply the Tauberian theorem for $Y_k(X)$ for $k = 5$ and 7.

Proposition 2.3 (Ikehara-Wiener Tauberian Theorem). *Let* $f(s) = \displaystyle\sum_{n=1}^{\infty} a_n/n^s$ *for* $\mathrm{Re}(s) > 1$ *such that* $a_n \geq 0$. *Suppose* $f(s)$ *can be extended to be analytic for* $\mathrm{Re}(s) \geq 1$

with a simple pole at $s = 1$ with residue c. Then,

$$\sum_{n \le X} a_n \sim cX.$$

Furthermore, if $f(s)$ has a pole of order r with residue c at $s = 1$, then

$$\sum_{n \le X} a_n \sim cX (\log X)^{r-1}.$$

Theorem 2.1.

$$|Y_5(X)| \sim c_5 X \qquad \textit{and} \qquad |Y_7(X)| \sim c_7 X$$

where c_5 and c_7 are some constants.

Proof. First, note that for an odd prime k, $|Y_k(X)| = \sum_{n \le X} a_n$ where $L_k(s) = \sum_{n=1}^{\infty} a_n/n^s$ and a_n is the number of distinct Dirichlet characters of order k and conductor n. Then, by lemma 2.3, $a_n = (k-1)^{\nu(n)}$ for each positive integer n. From lemma 2.4, $L_5(s)$ and $L_7(s)$ converge for $\mathrm{Re}(s) > 1$ and have a simple pole at $s = 1$ with the residue c_5 and c_7 respectively. Furthermore, for all $n \le 1$, $a_n \ge 0$ for both Dirichlet series. Therefore, by proposition 2.3, we obtain the result.

□

We can also use the work of Cohen, Diaz, and Olivier [CDO02] to obtain asymptotic estimates of $|Y_k(X)|$ in theorem 2.1. They computed the asymptotic estimates for

$$N(k, X) := \#\{L \mid L/\mathbb{Q} \text{ is a cyclic field extension of degree } k \text{ and } |D_L| \le X\}.$$

where $|D_L|$ is the discriminant of a number field L. More specifically, they obtained for an odd prime k,

$$N(k, X) \sim c_k X^{1/(k-1)}$$

for some constant c_k. Moreover, they also computed c_k as

$$c_k = \frac{k^2+k-1}{k^2(k-1)} \prod_{d|k-1} (\zeta_{K_d}(d))^{\mu(d)} \prod_{d|k-1} \left(1-\frac{1}{k^d}\right)^{\mu(d)} \times \prod_{p\equiv 1 \bmod k} \left(\left(1+\frac{k-1}{p}\right) \prod_{d|k-1} \left(1-\frac{1}{p^d}\right)^{\frac{(k-1)}{d}\mu(d)} \right).$$

where K_d is the unique subfield of $K = \mathbb{Q}(\zeta_k)$ such that $[K : K_d] = d$ and $\mu(d)$ is the Möbius function.

Let $G = \mathrm{Gal}(L/\mathbb{Q})$ where $L/\mathbb{Q}$ is a cyclic extension of degree k. Then, G is a cyclic group of order k. So, we can consider the corresponding group of Dirichlet characters $\widehat{G}$. Then, by theorem 3.11 and 4.3 in [Was82] we have

$$\zeta_L(s) = \prod_{\chi\in\widehat{G}} L(s,\chi) \quad \text{and} \quad |D_L| = \prod_{\chi\in\widehat{G}} \mathfrak{f}_\chi.$$

where $L(s,\chi)$ is a Dirichlet L-function. The above relation between D_L and $\mathfrak{f}_\chi$ implies that counting cyclic extensions $L/\mathbb{Q}$ of degree k with $D_L \le X$ is same as counting characters χ of order k and $\mathfrak{f}_\chi \le X^{1/(k-1)}$ since for each $\mathfrak{f}_\chi$ there are exactly $k-1$ distinct characters of conductor $\mathfrak{f}_\chi$. More precisely we can see

$$|Y_k(X)| = N(k, X^{k-1}) \sim c_k X.$$

Corollary 2.1. *Fix an E with conductor N_E. Then,*

$$|Y_{E,5}(X)| \sim c_5' X \qquad \text{and} \qquad |Y_{E,7}(X)| \sim c_7' X \tag{2.9}$$

where c_5' and c_7' are some constants.

Proof. Consider $L_{E,k}(s) = \sum_{n=1}^{\infty} \frac{a_n'}{n^s}$ where $a_n' = a_n$ if $(n, N_E) = 1$ and $a_n' = 0$ otherwise. Then,

$$L_{E,k}(s) = \begin{cases} \prod_{\substack{p\equiv 1 \bmod k \\ p|N_E}} \left(1+\frac{k-1}{p^s}\right)^{-1} L_k(s) & \text{if } k \nmid N_E \\ \left(1+\frac{k-1}{k^{2s}}\right)^{-1} \prod_{\substack{p\equiv 1 \bmod k \\ p|N_E}} \left(1+\frac{k-1}{p^s}\right)^{-1} L_k(s) & \text{if } k \mid N_E \end{cases}.$$

Since each factor of $L_{E,k}(s)$ is analytic at $s = 1$, with the same argument of theorem 2.1 we complete the proof. □

David, Fearnley, and Kisilevsky [DFK04] also showed that

$$\sum_{n \le X} b_n \ \sim \ d_3 X \log^2 X$$

for some constant d_3 which is the residue of the $L(s) = \sum_{n=1}^{\infty} b_n/n^s$ and where

$$b_n = \begin{cases} 6^{\nu(n)} & \text{if } n \text{ is the conductor of a cubic Dirichlet character} \\ 0 & \text{otherwise.} \end{cases}$$

Let k be odd prime. We define a Dirichlet series $B_k(s) = \sum_{n=1}^{\infty} b_n/n^s$ where

$$b_n = \begin{cases} (k(k-1))^{\nu(n)} & \text{if } n \text{ is the conductor of a Dirichlet character of order } k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\sum_{n \le X} b_n = \sum_{\mathfrak{f}_\chi \le X} k^{\nu(\mathfrak{f}_\chi)}$ where the right hand side is the sum over all Dirichlet characters χ of order k and conductors $\mathfrak{f}_\chi \le X$. Now, we use the same argument to show

Proposition 2.4.

$$\sum_{n \le X} 20^{\nu(n)} \ \sim \ d_5 X \log^4 X \qquad \textit{and} \qquad \sum_{n \le X} 42^{\nu(n)} \ \sim \ d_7 X \log^6 X$$

where d_5 and d_7 are some constants.

Proof. Set $K = \mathbb{Q}(\zeta_5)$ and

$$b_n = \begin{cases} (20)^{\nu(n)} & \text{if } n \text{ is the conductor of a quintic Dirichlet character} \\ 0 & \text{otherwise} \end{cases}.$$

Following the Euler products in (2.3), consider the Dirichlet series

$$B_5(s) = \sum_{n=1}^{\infty} b_n/n^s = \left(1 + \frac{20}{25^s}\right) \prod_{p \equiv 1 \bmod 5} \left(1 + \frac{20}{p^s}\right) = f(s)\left(\zeta_K(s)\right)^r.$$

Consider $(1 + 20/p^s)$ and $(1 - 1/p^s)^4$ for $p \equiv 1 \bmod 5$ in the above product and the Euler product of $\zeta_K(s)$. Then, in order to make $f(s)$ analytic at $s = 1$, we can choose the smallest positive integer power r of the Dedekind zeta function $\zeta_K(s)$ to remove

$20/p^s$ terms in the factor of $(1+20/p^s)$. By some computations, we choose $r = 5$. Then,

$$B_5(s) = f(s)\left(\zeta_K(s)\right)^5$$

where

$$f(s) = \left(1+\frac{20}{25^s}\right)\left(1-\frac{1}{5^s}\right)^5 \prod_{p\equiv 1 \bmod 5}\left(1+\frac{20}{p^s}\right)\left(1-\frac{1}{p^s}\right)^{20} \times \prod_{\substack{p\equiv 2 \bmod 5\\ p\equiv 3 \bmod 5}}\left(1-\frac{1}{p^{4s}}\right)^5 \prod_{p\equiv 4 \bmod 5}\left(1-\frac{1}{p^{2s}}\right)^{10}.$$

Since $f(s)$ is analytic at $s = 1$ and $\left(\zeta_K(s)\right)^5$ has a pole of order 5 there, $B_5(s)$ has a pole of order 5 with the residue $d_5 := f(1)\,\mathrm{Res}_{s=1}\left(\zeta_K(s)\right)^5$. Therefore, by the Tauberian theorem 2.3, we obtain $\sum_{n\leq X} 20^{\nu(n)} \sim d_5 X \log^4 X$.

We use the same argument for $\sum_{n\leq X} 42^{\nu(n)}$. Set $K = \mathbb{Q}(\zeta_7)$ and

$$b_n = \begin{cases} (42)^{\nu(n)} & \text{if } n \text{ is the conductor of a 7-th Dirichlet character} \\ 0 & \text{otherwise} \end{cases}.$$

Then, we have

$$B_7(s) = \sum_{n=1}^{\infty} b_n/n^s = \left(1+\frac{42}{49^s}\right)\prod_{p\equiv 1 \bmod 7}\left(1+\frac{42}{p^s}\right) = g(s)\left(\zeta_K(s)\right)^7.$$

Then, $g(s)$ is analytic at $s = 1$ and $\left(\zeta_K(s)\right)^7$ has a pole of order 7 there. Therefore, $B_7(s)$ has a pole of order 7 and, by the Tauberian theorem 2.3, we obtain $\sum_{n\leq X} 42^{\nu(n)} \sim d_7 X \log^6 X$ where $d_7 := g(1)\,\mathrm{Res}_{s=1}\left(\zeta_K(s)\right)^7$. □

Chapter 3

Discretisation of Critical L-values

3.1 Modular Symbols

This section is composed of a review of the first chapter in the Mazur, Tate, and Teitelbaum's article [MTT86] and the introduction of the algebraic parts of $L_E(1,\chi)$. We start from the definition of a modular symbol $\{\alpha,\beta\}$ for a $f \in S_k(N_E,\psi)$ by using modular integral on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$:

Definition 3.1. *For $r \in \mathbb{P}^1(\mathbb{Q})$ and $f \in S_2(N_E,\psi)$, define the modular integral by*

$$MI(f,r) := 2\pi i \int_{i\infty}^{r} f(z)dz = \begin{cases} 2\pi \displaystyle\int_0^{\infty} f(r+it)dt, & \text{if } r \neq \infty \\ 0, & \text{if } r = \infty \end{cases},$$

and for α and $\beta \in \mathbb{Q}$ and $\beta > 0$, define the modular symbol by

$$\{\alpha,\beta\}(f) := MI(f,-\alpha/\beta). \tag{3.1}$$

Note that the integral in the definition of modular symbols has a vertical contour in the complex plane. Since f is a cusp form, it has a q-Fourier expansion and the terms are exponentially decreasing. That implies that modular integrals converge absolutely. Furthermore, since modular integrals are $\mathbb{C}$-linear in f, so are modular symbols. Recall that the slash operator on a modular form $f \in S_2(N_E,\psi)$ is defined as

$$(f|A)(z) := \frac{\det(A)^{k/2}}{(cz+d)^k} f(A(z)) \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}). \tag{3.2}$$

In particular, for $f \in S_2(N_E, \psi)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N_E)$,

$$(f|A)(z) = \psi(d) f(z). \tag{3.3}$$

We denote $\psi(A) := \psi(d)$. One of the most crucial properties of modular integrals is that for $f \in S_2(N_E, \psi)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$, and $r \in \mathbb{P}^1(\mathbb{Q})$,

$$\begin{aligned}
MI(f|A, r) &= 2\pi i \int_\infty^r (f|A)(z) dz \\
&= 2\pi i \int_\infty^r \frac{\det(A)}{(cz+d)^2} f(A(z)) dz \\
&= 2\pi i \int_\infty^r f(A(z)) dA(z) \\
&\quad \text{since } dA(z)/dz = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{\det(A)}{(cz+d)^2} \\
&= 2\pi i \int_{A(\infty)}^{A(r)} f(z) dz \\
&= 2\pi i \int_\infty^{A(r)} f(z) dz - 2\pi i \int_\infty^{A(\infty)} f(z) dz \\
&= MI(f, A(r)) - MI(f, A(\infty)).
\end{aligned} \tag{3.4}$$

Now, consider a coset representation of $\Gamma_0(N_E) \subset SL_2(\mathbb{Z})$. Then, it is easy to see $[SL_2(\mathbb{Z}) : \Gamma_0(N_E)]$ is finite. Let's denote those finite coset representatives of $\Gamma_0(N_E)$ by $A_i \in SL_2(\mathbb{Z})$.

Proposition 3.1. *Let α and β be integers and $\beta > 0$. Then, for a fixed $f \in S_2(N_E, \psi)$, every modular symbol $\{\alpha, \beta\}(f)$ is in a $\mathbb{Z}[\psi]$-module of $\mathbb{C}$ generated by all modular integrals of form of*

$$MI(f, A_i(\infty)) - MI(f, A_i(0)) \tag{3.5}$$

where the A_i's are all of the above coset representatives and where $\mathbb{Z}[\psi]$ is the $\mathbb{Z}$-module generated by the values of ψ. Moreover, $\{\alpha, \beta\}(f)$ depends only on $\alpha \bmod \beta$.

Proof. Assume that $(\alpha, \beta) = 1$. We use induction on β. If $\beta = 1$, then

$$\begin{aligned}\{\alpha, 1\}(f) = MI(f, \alpha) &= MI\left(f, \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}(0)\right)\\ &= MI\left(f, \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}(0)\right) - MI\left(f, \begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}(\infty)\right).\end{aligned}$$

In this case, the coset representative is the identity. Now, suppose $\beta > 1$. We can choose β' such that $0 \leq \beta' < \beta$ and $\alpha\beta' \equiv 1 \bmod \beta$. Set $\alpha' = (\alpha\beta' - 1)/\beta$ and $A = \begin{pmatrix}-\alpha' & -\alpha\\ \beta' & \beta\end{pmatrix} \in SL_2(\mathbb{Z})$. Then, there are $A' \in \Gamma_0(N_E)$ and A_j such that $A = A'A_j$. Therefore,

$$\begin{aligned}\{\alpha', \beta'\}(f) - \{\alpha, \beta\}(f) &= MI(f, A(\infty)) - MI(f, A(0))\\ &= MI(f, A'A_j(\infty)) - MI(f, A'A_j(0))\\ &= \psi(A')\Big(MI(f, A_j(\infty)) - MI(f, A_j(0))\Big) \text{ by (3.3) and (3.4).}\end{aligned} \tag{3.6}$$

Since by the assumption of the induction $\{\alpha', \beta'\}(f)$ is in the $\mathbb{Z}[\psi]$-module, the first part of the proof is done. For the second part, write $\alpha = m\beta + \alpha', 0 \leq \alpha' < \beta$. Then,

$$\begin{aligned}\{\alpha, \beta\}(f) &= \{m\beta + \alpha', \beta\}(f) = MI(f, m - \alpha'/\beta)\\ &= MI\left(f, \begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}(-\alpha'/\beta)\right) = \psi(1)MI(f, \alpha'/\beta)\\ &= MI(f, \alpha'/\beta) = \{\alpha', \beta\}(f).\end{aligned}$$

□

So, in particular, if we take the trivial Dirichlet character mod N_E for ψ, i.e $f \in S_2(N_E)$, then every modular symbols $\{\alpha, \beta\}(f)$ is in a $\mathbb{Z}$-module of $\mathbb{C}$ generated by all the above modular integrals.

Consider $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(N_E)$ and suppose that $|a_n/n^c| < \infty$ for some constant c as $n \to \infty$. Note that for a normalized $f(z) \in S_k(N_E)$ we have $c = 1$ since

$a_n \ll n$ from equation 5.7 in [Iwa97] for $k = 2$. Then, consider

$$\begin{aligned}\int_0^\infty f(it)t^s \frac{dt}{t} &= \int_0^{1/\sqrt{N_E}} f(it)t^s \frac{dt}{t} + \int_{1/\sqrt{N_E}}^\infty f(it)t^s \frac{dt}{t} \\ &= \int_{1/\sqrt{N_E}}^\infty f\left(i\frac{1}{N_E u}\right) \frac{du}{(N_E)^s u^{s+1}} + \int_{1/\sqrt{N_E}}^\infty f(it)t^s \frac{dt}{t} \text{ by } t \to \frac{1}{N_E u}\end{aligned}$$

By the Fricke involution for $f(z) \in S_k(N_E)$, we have

$$f\left(i\frac{1}{N_E u}\right) = f\left(\begin{pmatrix} 0 & -1 \\ N_E & 0 \end{pmatrix}(iu)\right) = -N_E u^2 \left(f\Big| \begin{pmatrix} 0 & -1 \\ N_E & 0 \end{pmatrix}\right)(iu) = \pm N_E u^2 f(iu).$$

Therefore, the above equation becomes

$$\begin{aligned}&= \pm \int_{1/\sqrt{N_E}}^\infty f(iu) \frac{du}{(N_E u)^{s-1}} + \int_{1/\sqrt{N_E}}^\infty f(it)t^s \frac{dt}{t} \\ &= \int_{1/\sqrt{N_E}}^\infty f(it) \left(t^s \pm \frac{t^{2-s}}{(N_E)^{s-1}}\right) \frac{dt}{t}.\end{aligned} \tag{3.7}$$

This integral converges for all $s \in \mathbb{C}$. Moreover, we have

$$\begin{aligned}\int_0^\infty f(it)t^s \frac{dt}{t} &= \int_0^\infty \left(\sum_{n=1}^\infty a_n \exp(-2\pi n t)\right) t^s \frac{dt}{t} \\ &= \sum_{n=1}^\infty a_n \int_0^\infty t^s \exp(-2\pi n t) \frac{dt}{t}, \\ &\quad \text{by the bounded convergence theorem} \\ &= \sum_{n=1}^\infty a_n \left(\frac{1}{2\pi n}\right)^s \int_0^\infty x^s \exp(-x) \frac{dx}{x} \text{ where } x = 2\pi n t \\ &= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n=1}^\infty \frac{a_n}{n^s} = \frac{\Gamma(s)}{(2\pi)^s} L_f(s).\end{aligned} \tag{3.8}$$

Note that since $\Gamma(s)$ has no zero for all s and simples poles at non-positive integers s the above integral representation of $L_f(s)$ extends to entire function on $\mathbb{C}$. In particular, by the definition of modular symbols (3.1) and $\Gamma(1) = 1$, we have for $f \in S_2(N_E)$,

$$\{0, 1\}(f) = 2\pi \int_0^\infty f(it)dt = L_f(1). \tag{3.9}$$

Fix χ to be a Dirichlet character of conductor $\mathfrak{f}_\chi$. Now, we can find the relation between the critical L-value $L_{f_\chi}(1)$ and the corresponding modular symbols.

Lemma 3.1. *Suppose $f \in S_2(N_E)$. Then, for $f_{\overline{\chi}}(z) = \sum_{n=1}^{\infty} \overline{\chi}(n) a_n q^n$,*

$$f_{\overline{\chi}}(z) = \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) f(z + \frac{a}{\mathfrak{f}_\chi}). \tag{3.10}$$

Proof.

$$\begin{aligned}
f_{\overline{\chi}}(z) &= \sum_{n=1}^{\infty} \overline{\chi}(n) a_n q^n = \sum_{n=1}^{\infty} \frac{\tau(n,\chi)}{\tau(1,\chi)} a_n q^n \text{ by lemma 2.1} \\
&= \frac{1}{\tau(1,\chi)} \sum_{n=1}^{\infty} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) a_n \exp(2\pi i n (z + \frac{a}{\mathfrak{f}_\chi})) \\
&= \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) \sum_{n=1}^{\infty} a_n \exp(2\pi i n (z + \frac{a}{\mathfrak{f}_\chi})) \\
&= \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) f(z + \frac{a}{\mathfrak{f}_\chi}).
\end{aligned}$$

□

By $\mathbb{C}$-linearity of modular integrals, hence also of modular symbols, for any $r \in \mathbb{Q}$,

$$\begin{aligned}
MI(f_{\overline{\chi}}(z), r) &= \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) MI(f(z + \frac{a}{\mathfrak{f}_\chi}), r) \\
&= \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) MI(f \Big| \begin{pmatrix} 1 & \frac{a}{\mathfrak{f}_\chi} \\ 0 & 1 \end{pmatrix} (z), r) \qquad (3.11) \\
&= \frac{1}{\tau(1,\chi)} \sum_{a \bmod \mathfrak{f}_\chi} \chi(a) MI(f, r + \frac{a}{\mathfrak{f}_\chi}), \text{ by (3.4).}
\end{aligned}$$

From the above relation of the modular integrals of f_χ and f and that of $L_f(1)$ and the modular symbol $\{0,1\}(f)$ in (3.9), we can deduce the relation of $L_{f_\chi}(1)$ and the corresponding modular symbols.

Theorem 3.1. *For $f \in S_2(N_E)$,*

$$L_{f_\chi}(1) = \frac{\tau(1,\chi)}{\mathfrak{f}_\chi} \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a) \{a, \mathfrak{f}_\chi\}(f). \tag{3.12}$$

Proof. We prove this for $\overline{\chi}$. First, consider $\{b,m\}(f_{\overline{\chi}})$ for some integers b and m with $m>0$. Then,

$$\begin{aligned}\{b,m\}(f_{\overline{\chi}}) &= MI\left(f_{\overline{\chi}}, -\frac{b}{m}\right)\\ &= \frac{1}{\tau(1,\chi)}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)MI\left(f, -\frac{b}{m}+\frac{a}{\mathfrak{f}_\chi}\right) \text{ by (3.11)}\\ &= \frac{1}{\tau(1,\chi)}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)MI\left(f, -\frac{(b\mathfrak{f}_\chi - am)}{m\mathfrak{f}_\chi}\right)\\ &= \frac{1}{\tau(1,\chi)}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)\{b\mathfrak{f}_\chi - am, m\mathfrak{f}_\chi\}(f)\end{aligned} \tag{3.13}$$

Thus, from (3.9) and by taking $b=0$ and $m=1$ in (3.13), we have

$$\begin{aligned}L_{f_{\overline{\chi}}}(1) = \{0,1\}(f_{\overline{\chi}}) &= \frac{1}{\tau(1,\chi)}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)\{-a,\mathfrak{f}_\chi\}(f)\\ &= \frac{\chi(-1)\tau(1,\overline{\chi})}{\mathfrak{f}_\chi}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)\{-a,\mathfrak{f}_\chi\}(f)\\ &= \frac{\tau(1,\overline{\chi})}{\mathfrak{f}_\chi}\sum_{a \bmod \mathfrak{f}_\chi}\chi(-a)\{-a,\mathfrak{f}_\chi\}(f)\\ &= \frac{\tau(1,\overline{\chi})}{\mathfrak{f}_\chi}\sum_{a \bmod \mathfrak{f}_\chi}\chi(a)\{a,\mathfrak{f}_\chi\}(f) \text{ by re-arranging the summation.}\end{aligned}$$

□

The usual Hecke operator acting on $f \in S_2(N_E)$ is $f|T_n := \sum_{j=0}^{n-1} f\Big|\begin{pmatrix}1 & j\\ 0 & n\end{pmatrix} + f\Big|\begin{pmatrix}n & 0\\ 0 & 1\end{pmatrix}$ and the usual U_n operator is $f|U_n := \sum_{j=0}^{n-1} f\Big|\begin{pmatrix}1 & j\\ 0 & n\end{pmatrix}$. Adopting $\delta(n) = \begin{cases}1, & \text{if } n \nmid N_E\\ 0, & \text{if } n \mid N_E\end{cases}$, we extend the definition of Hecke operators T_n by

$$f|T_n = \sum_{j=0}^{n-1} f\Big|\begin{pmatrix}1 & j\\ 0 & n\end{pmatrix} + \delta(n) f\Big|\begin{pmatrix}n & 0\\ 0 & 1\end{pmatrix}. \tag{3.14}$$

The following proposition tells us the action of T_n on modular symbols.

Proposition 3.2. *For $f \in S_2(N_E)$ and a prime p,*

$$\{a, \mathfrak{f}_\chi\}(f|T_p) = \sum_{j=0}^{p-1}\{a - j\mathfrak{f}_\chi, p\mathfrak{f}_\chi\}(f) + \delta(p)\{a, \mathfrak{f}_\chi/p\}(f). \tag{3.15}$$

Proof.

$$\begin{aligned}
\{a, \mathfrak{f}_\chi\}(f|T_p) &= \{a, \mathfrak{f}_\chi\}\left(\sum_{j=0}^{p-1} f\Big|\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix} + \delta(p)f\Big|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}\right)\\
&= MI\left(\sum_{j=0}^{p-1} f\Big|\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix} + \delta(p)f\Big|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, -a/\mathfrak{f}_\chi\right)\\
&= \sum_{j=0}^{p-1} MI\left(f\Big|\begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}, -a/\mathfrak{f}_\chi\right) + \delta(p)MI\left(f\Big|\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, -a/\mathfrak{f}_\chi\right)\\
&\quad \text{by } \mathbb{C}\text{-linearity in } f\\
&= \sum_{j=0}^{p-1}\left\{MI\left(f, \begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}(-a/\mathfrak{f}_\chi)\right) - MI\left(f, \begin{pmatrix}1 & j\\ 0 & p\end{pmatrix}(\infty)\right)\right\}\\
&\quad + \delta(p)\left\{MI\left(f, \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}(-a/\mathfrak{f}_\chi)\right) - MI\left(f, \begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}(\infty)\right)\right\} \text{ by (3.4)}\\
&= \sum_{j=0}^{p-1} MI\left(f, -\frac{(a - j\mathfrak{f}_\chi)}{p\mathfrak{f}_\chi}\right) + \delta(p)MI\left(f, -\frac{a}{(\mathfrak{f}_\chi/p)}\right) \text{ since } MI(f,\infty) = 0\\
&= \sum_{j=0}^{p-1}\{a - j\mathfrak{f}_\chi, p\mathfrak{f}_\chi\}(f) + \delta(p)\{a, \mathfrak{f}_\chi/p\}(f).
\end{aligned}$$

□

The following theorem is the crucial part in the discretisation of critical L-values. The proof relies on the lattice theory of modular curves and modular parametrization. We skip the proof.

Theorem 3.2. *Let $f \in S_2(N_E)$ and χ be a primitive Dirichlet character of conductor $\mathfrak{f}_\chi$. Then, there are two complex numbers Ω^+ and Ω^- such that for $a \in \mathbb{Z}$,*

$$\Lambda^\pm(a, \mathfrak{f}_\chi)(f) := \frac{1}{\Omega^\pm}\Big(\{a, \mathfrak{f}_\chi\}(f) \pm \{-a, \mathfrak{f}_\chi\}(f)\Big)$$

are integers.

Proof. See 2.8 in [Cre97]. Note that $\Omega^{\pm}$ depends only on a given elliptic curve. $\square$

By the definition of modular symbols by modular integrals, we have

$$\begin{aligned}\{a,\mathfrak{f}_\chi\}(f) \pm \{-a,\mathfrak{f}_\chi\}(f) &= MI(f,-a/\mathfrak{f}_\chi) \pm MI(f,a/\mathfrak{f}_\chi)\\ &= 2\pi\left(\int_0^\infty f(-a/\mathfrak{f}_\chi+i\tau)d\tau \pm \int_0^\infty f(a/\mathfrak{f}_\chi+i\tau)d\tau\right).\end{aligned}$$

The q-expansions of $f(-a/\mathfrak{f}_\chi+i\tau)$ and $f(a/\mathfrak{f}_\chi+i\tau)$ for eigenform f corresponding to $L_E(s)$ tells us that they are complex conjugate. Indeed,

$$\begin{aligned}q := \exp(2\pi i(-a/\mathfrak{f}_\chi+i\tau)) &= \exp(2\pi(-\tau - ia/\mathfrak{f}_\chi))\\ &= \overline{\exp(2\pi(-\tau + ia/\mathfrak{f}_\chi))}\\ &= \overline{\exp(2\pi i(a/\mathfrak{f}_\chi+i\tau))}\end{aligned}$$

which implies

$$\begin{aligned}f(-a/\mathfrak{f}_\chi+i\tau) &= \sum_{n=1}^\infty a_n q^n = \overline{\overline{\sum_{n=1}^\infty a_n q^n}}\\ &= \overline{\sum_{n=1}^\infty \overline{a_n q^n}} = \overline{\sum_{n=1}^\infty a_n \overline{q}^n} = \overline{f(a/\mathfrak{f}_\chi+i\tau)} \text{ since all } a_n\in\mathbb{Z}.\end{aligned}$$

Therefore, for the eigenform f for $L_E(s)$,

$$\Lambda^{sign}(a,\mathfrak{f}_\chi)(f) = \begin{cases} 2\,\mathrm{Re}\left(\{a,\mathfrak{f}_\chi\}(f)\right)/\Omega^+, & \text{if } sign = +\\ 2\,\mathrm{Im}\left(\{a,\mathfrak{f}_\chi\}(f)\right)/\Omega^-, & \text{if } sign = -\end{cases}.$$

Let

$$L^{alg}_{f_\chi}(1) := \sum_{a \bmod \mathfrak{f}_\chi} \overline{\chi}(a)\Lambda^{sign(\chi)}(a,\mathfrak{f}_\chi)(f)$$

where $sign(\chi) = \begin{cases} +, & \text{if } \chi(-1)=1\\ -, & \text{if } \chi(-1)=-1\end{cases}$. Then, we have

$$\begin{aligned}L^{alg}_{f_\chi}(1) &= \frac{1}{\Omega^{sign(\chi)}}\sum_{a \bmod \mathfrak{f}_\chi}\left(\overline{\chi}(a)\{a,\mathfrak{f}_\chi\}(f) + \chi(-1)\overline{\chi}(a)\{-a,\mathfrak{f}_\chi\}(f)\right)\\ &= \frac{1}{\Omega^{sign(\chi)}}\sum_{a \bmod \mathfrak{f}_\chi}\left(\overline{\chi}(a)\{a,\mathfrak{f}_\chi\}(f) + \chi(-1)\overline{\chi}(-1)\overline{\chi}(-a)\{-a,\mathfrak{f}_\chi\}(f)\right)\\ &= \frac{2}{\Omega^{sign(\chi)}}\sum_{a \bmod \mathfrak{f}_\chi}\overline{\chi}(a)\{a,\mathfrak{f}_\chi\}(f) = \frac{2\mathfrak{f}_\chi L_{f_\chi}(1)}{\Omega^{sign(\chi)}\tau(1,\chi)}.\end{aligned}$$

Let $\mathbb{Z}[\chi]$ be the $\mathbb{Z}$-module generated by the values of χ which are k-th root of unity. Then, integrality of $\Lambda^{\pm}(a, \mathfrak{f}_\chi)(f)$ implies $L^{alg}_{f_\chi}(1) \in \mathbb{Z}[\chi]$. We let $L^{alg}_E(1,\chi) := L^{alg}_{f_\chi}(1)$. Then, by the modularity theorem we have the algebraic part of $L_E(1,\chi)$:

$$L^{alg}_E(1,\chi) = \frac{2\mathfrak{f}_\chi}{\Omega^{sign(\chi)}\tau(1,\chi)} L_E(1,\chi). \tag{3.16}$$

It is interesting since it gives us the discretisation of some families of $L_E(1,\chi)$ for fixed elliptic curves. We next review the work of David, Fearnley, and Kisilevsky [DFK06] about the discretisation of $L^{alg}_E(1,\chi)$ for odd prime twists.

3.2 Algebraic Parts of $L_E(1,\chi)$

In [DFK06] they consider the cyclotomic extension $\mathbb{Q}(\zeta_k)$ for a primitive k-th root of unity where k is an odd prime and the totally real subfield of $\mathbb{Q}(\zeta_k)$ which is $\mathbb{Q}(\zeta_k + \zeta_k^{-1})$ [Was82]. Then, the extension has the degree $\frac{k-1}{2}$ since the minimal polynomial of ζ_k for the extension $\mathbb{Q}(\zeta_k)$ from $\mathbb{Q}(\zeta_k+\zeta_k^{-1})$ is $X^2-(\zeta_k+\zeta_k^{-1})X+1$ and $[\mathbb{Q}(\zeta_k+\zeta_k^{-1}):\mathbb{Q}] = [\mathbb{Q}(\zeta_k):\mathbb{Q}]/[\mathbb{Q}(\zeta_k):\mathbb{Q}(\zeta_k+\zeta_k^{-1})]$. Furthermore, we can consider the ring of integers of $\mathbb{Q}(\zeta_k+\zeta_k^{-1})$ is $\mathbb{Z}[\zeta_k+\zeta_k^{-1}]$, see proposition 2.16 in [Was82]. Then, we have the following theorem for $L^{alg}_E(1,\chi)$.

Theorem 3.3. *Let $L_E(s,\chi)$ is a L-function of an elliptic curve E over $\mathbb{Q}$ twisted by a Dirichlet character of order k for an odd prime k. Then,*

$$|L^{alg}_E(1,\chi)| = \begin{cases} |n_\chi| & , \text{ if } \omega_E = 1 \\ |\zeta_k - \zeta_k^{-1}|\,|n_\chi| & , \text{ if } \omega_E = -1 \end{cases} \tag{3.17}$$

where ζ_k is a primitive k-th root of unity and $n_\chi \in \mathbb{Z}[\zeta_k+\zeta_k^{-1}]$.

Proof. First, since k is odd, $\chi(-1) = 1$ and $sign(\chi) = +$ and we let $\Omega = \Omega^+$. Then, by the functional equation (1.1) and (3.16) we have

$$\begin{aligned} L_E^{alg}(1,\chi) &= \frac{2\mathfrak{f}_\chi}{\Omega\tau(1,\chi)} L_E(1,\chi) \\ &= \frac{2\omega_E \chi(N_E)\tau(1,\chi)}{\Omega} L_E(1,\overline{\chi}) \\ &= \frac{2\omega_E \chi(N_E)\tau(1,\chi)}{\Omega} \overline{L_E(1,\chi)} \text{ since } E = E/\mathbb{Q} \\ &= \omega_E \chi(N_E) \overline{L_E^{alg}(1,\chi)} \text{ by (2.2) and (3.16)} \end{aligned} \tag{3.18}$$

Since $\omega_E = \pm 1$ and the value of $\chi(N_E)$ is a primitive k-th root of unity, we can use that $\overline{\left(\zeta_k - \zeta_k^{-1}\right)} = -\left(\zeta_k - \zeta_k^{-1}\right)$ and $\chi(N_E)^{(k+1)/2} = \chi(N_E)\chi(N_E)^{(k-1)/2}$. If $\omega_E = 1$, then $L_E^{alg}(1,\chi)$ is a real number or a real multiple of $\chi(N_E)^{(k+1)/2}$, more specifically a multiple of some element $n_\chi \in \mathbb{Z}[\zeta_k + \zeta_k^{-1}]$, and it satisfies equation (3.18). If $\omega_E = -1$, then $L_E^{alg}(1,\chi) = n_\chi \left(\zeta_k - \zeta_k^{-1}\right)\chi(N_E)^{(k+1)/2}$ for some $n_\chi \in \mathbb{Z}[\zeta_k + \zeta_k^{-1}]$ and satisfies equation (3.18). Then, the proof can be completed by taking absolute values.

□

The above theorem implies that $L_E(1,\chi) = 0$ if and only if $L_E^{alg}(1,\chi) = 0$ if and only if $n_\chi = 0$. We show some examples.

- Cubic cases, i.e. $k = 3$, as shown in [DFK04], $\left(\zeta_3 - \zeta_3^{-1}\right) = i\sqrt{3}$ and $\mathbb{Z}[\zeta_3 + \zeta_3^{-1}] = \mathbb{Z}$. Therefore,

$$|L_E^{alg}(1,\chi)| = \begin{cases} n_\chi & \text{, if } \omega_E = 1 \\ n_\chi\sqrt{3} & \text{, if } \omega_E = -1 \end{cases} \tag{3.19}$$

  where n_χ is some non-negative integer.

- Quintic case, i.e. $k = 5$, as shown in [DFK06], we have $\left(\zeta_5 - \zeta_5^{-1}\right) = i\sqrt{\frac{5+\sqrt{5}}{2}}$ and $\mathbb{Z}[\zeta_5 + \zeta_5^{-1}] = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$. Thus,

$$|L_E^{alg}(1,\chi)| = \begin{cases} |n_\chi| & \text{, if } \omega_E = 1 \\ |n_\chi|\sqrt{\frac{5+\sqrt{5}}{2}} & \text{, if } \omega_E = -1 \end{cases}$$

  where $n_\chi \in \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

David, Fearnley, and Kisilevsky proposed the following conjecture regarding n_χ of E over $\mathbb{Q}$ with a k-torsion points which is supported by their numerical data [DFK04] for some cubic twists.

Conjecture 1 (David, Fearnley, and Kisilevsky)**.** *If E is isogenous to a curve with rational 3-torsion and χ is a cubic character of conductor $\mathfrak{f}_\chi$, then n_χ is divisible by $3^{\nu(\mathfrak{f}_\chi)-1}$.*

For an odd prime k, we generalize the conjecture for k-th twists.

Conjecture 2. *If E is isogenous to a curve with rational k-torsion and χ is a character of order k and conductor $\mathfrak{f}_\chi$, then n_χ is divisible by $k^{\nu(\mathfrak{f}_\chi)-1}$.*

Note when E has a 5-torsion point, the conjecture implies $5^{\nu(\mathfrak{f}_\chi)-1}$ divides a and b for $n_\chi = a + b\left(\frac{-1+\sqrt{5}}{2}\right)$ for some a and $b \in \mathbb{Z}$ and when E has a 7-torsion point, $7^{\nu(\mathfrak{f}_\chi)-1}$ divides a, b, and c for $n_\chi = a + b\left(\zeta_7 + \zeta_7^{-1}\right) + c\left(\zeta_7^2 + \zeta_7^{-2}\right)$ for some a, b, and $c \in \mathbb{Z}$.

Fix an elliptic curve E over $\mathbb{Q}$. We can get a bound of $|L_E(1,\chi)|$ to ensure vanishing of it by using the geometry of numbers for n_χ. Let $K = \mathbb{Q}(\zeta_k), K^+ = \mathbb{Q}(\zeta_k + \zeta_k^{-1})$, and $\mathcal{O}_{K^+} = \mathbb{Z}[\zeta_k + \zeta_k^{-1}]$. Then, n_χ is an algebraic integer in $\mathcal{O}_{K^+}$ from theorem 3.3 and there is an embedding $\pi : \mathcal{O}_{K^+} \to \mathbb{R}^{(k-1)/2}$ by sending $\alpha \in \mathcal{O}_{K^+}$ to $\left(\sigma_1(\alpha), \dots, \sigma_{(k-1)/2}(\alpha)\right)$ where $\sigma_i \in \mathrm{Gal}(K^+/\mathbb{Q})$ for $1 \le i \le (k-1)/2$. Let $\alpha_1, \dots, \alpha_{(k-1)/2}$ be an integral basis of $\mathcal{O}_{K^+}$. Then, the image of $\mathcal{O}_{K^+}$ is a lattice in $\mathbb{R}^{(k-1)/2}$ generated by

$$\omega_1 = \pi(\alpha_i), \dots, \omega_{(k-1)/2} = \pi(\alpha_{(k-1)/2}).$$

In other words, $\omega_1, \dots, \omega_{(k-1)/2}$ is a basis for the subspace, the image of $\mathcal{O}_{K^+}$, of the vector space $\mathbb{R}^{(k-1)/2}$. Thus, by the discretisation of n_χ we have

$$n_\chi = 0 \iff \pi\left(n_\chi\right) = (n_\chi^{\sigma_1}, n_\chi^{\sigma_2}, \dots, n_\chi^{\sigma_{(k-1)/2}}) \in R \tag{3.20}$$

where $R = \{r_1\omega_1 + \ldots + r_{(k-1)/2}\omega_{(k-1)/2} \mid -1 < r_i < 1 \text{ for } 1 \leq i \leq (k-1)/2\}$.

Lemma 3.2. *For an odd prime k, a character χ of order k and conductor $\mathfrak{f}_\chi$, and any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$,*

$$|L_E(1, \chi^\sigma)| = \frac{A_{E,k}}{\sqrt{\mathfrak{f}_\chi}} |n_\chi^\sigma| \tag{3.21}$$

where χ^σ and n_χ^σ are the character obtained by acting σ on χ and n_χ respectively and $A_{E,k}$ is a constant depending on E and k.

Proof. Note that χ^σ is a group homomorphism $(\mathbb{Z}/\mathfrak{f}_\chi\mathbb{Z})^\times \to \langle\zeta_k\rangle \in \mathbb{C}^\times$ by sending $g \in (\mathbb{Z}/\mathfrak{f}_\chi\mathbb{Z})^\times$ to $\sigma(\chi(g))$. Thus, χ^σ is a character of order k and conductor $\mathfrak{f}_\chi$. Furthermore, the constant

$$A_{E,k} = \begin{cases} |\Omega/2| & \text{if } \omega_E = 1 \\ |\Omega/2| \left|\zeta_k - \zeta_k^{-1}\right| & \text{if } \omega_E = -1 \end{cases}.$$

See lemma 3.1 in [DFK06] for a detailed proof. □

Now we show some bounds of $|L_E(1, \chi)|$ when $n_\chi = 0$ for $k = 3$ and 5 and $k \geq 7$. For $k = 3$, since n_χ is a rational integer and $\mathrm{Gal}(K^+/\mathbb{Q})$ is trivial,

$$|n_\chi| = 0 \iff |n_\chi| = |\pi(n_\chi)| < 1 \iff |L_E(1, \chi)| < \frac{A_{E,k}}{\sqrt{\mathfrak{f}_\chi}} \text{ by lemma 3.2.}$$

For $k = 5$, $n_\chi \in \mathcal{O}_{K^+} = \mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ and $\mathrm{Gal}(K^+/\mathbb{Q}) =< 1, \sigma >$ where σ is the automorphism sending $\sqrt{5}$ to $-\sqrt{5}$. Thus, the image of $\mathcal{O}_{K^+}$ in $\mathbb{R}^2$ is the lattice generated by

$$\omega_1 = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right) \text{ and } \omega_2 = \left(\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right).$$

Let $R_1 = \{(r_1, r_2) \in \mathbb{R}^2 \mid -1 < r_1, r_2 < 1\}$ and $R_2 = \{(r_1, r_2) \in \mathbb{R}^2 \mid -\sqrt{5} < r_1, r_2 < \sqrt{5}\}$. Then, $R_1 \subset R \subset R_2$ and by lemma 3.2,

$$\begin{aligned} \left(n_\chi, n_\chi^\sigma\right) \in R_1 &\iff |n_\chi|, \left|n_\chi^\sigma\right| < 1 \iff |L_E(1, \chi)|, |L_E(1, \chi^\sigma)| < \frac{A_{E,k}}{\sqrt{\mathfrak{f}_\chi}} \\ \left(n_\chi, n_\chi^\sigma\right) \in R_2 &\iff |n_\chi|, \left|n_\chi^\sigma\right| < \sqrt{5} \iff |L_E(1, \chi)|, |L_E(1, \chi^\sigma)| < \frac{\sqrt{5}A_{E,k}}{\sqrt{\mathfrak{f}_\chi}}. \end{aligned} \tag{3.22}$$

For $k \geq 7$, let $\alpha_1, \dots, \alpha_{(k-1)/2}$ be an integral basis of $\mathcal{O}_{K^+}$ and $\sigma_i \in \mathrm{Gal}(K/\mathbb{Q})$ be a restricted automorphism to $\mathrm{Gal}(K^+/\mathbb{Q})$ for $1 \leq i \leq (k-1)/2$. Consider the region in $\mathbb{R}^{(k-1)/2}$

$$\widetilde{R} = \{(r_1, \dots, r_{(k-1)/2}) \mid -M < r_i < M \text{ for } 1 \leq i \leq (k-1)/2\}$$

where $M = \max_{1 \leq i \leq (k-1)/2} \sum_{j=1}^{(k-1)/2} |\sigma_i(\alpha_i)|$. Then, $R \subset \widetilde{R}$ and

$$n_\chi = 0 \Rightarrow \pi(n_\chi) \in \widetilde{R} \iff |L_E(1, \chi^{\sigma_i})| < \frac{A'_{E,k}}{\sqrt{\mathfrak{f}_\chi}} \qquad \text{for } 1 \leq i \leq (k-1)/2 \tag{3.23}$$

where $A'_{E,k}$ is a constant depending on E and k. We use those bounds to get the probability that $L_E(1, \chi) = 0$ in section 4.3.

Chapter 4

Random Matrix Theory for Critical L-values

4.1 Random Matrix Theory for $U(N)$

Katz and Sarnak [KS99] showed that the distribution of low lying zeroes, which are zeroes on the critical line with imaginary part is less than some height, in some families of L-functions fits that of some parameters of eigenvalues of matrices from one of the classical compact groups. In particular, our family of $L_E(1,\chi)$'s over χ's which are of order k and of conductor $\mathfrak{f}_\chi \leq X$ has a symmetry type of $U(N)$. Moreover, Keating and Snaith [KS00b] and [KS00a] proposed that the moments of the values of $\zeta(1/2+it)$ can be asympototically approximated to some multiple of the moments of the values of the characteristic polynomials of random unitary matrices following GUE(Gaussian unitary ensemble). They extended that the idea for some families of L-functions. Conrey, Keating, Rubinstein, and Snaith [CKRS02] suggested some asymptotic conjectures for the moments and the vanishings of critical L-values of an elliptic curve twisted by some family of quadratic characters inspired by [KS00b] and [KS00a]. David, Fearnley, and Kisilevsky [DFK04] proposed similar conjectures for cubic twists.

In this section, we review the work of Conrey, Keating, Rubinstein, and Snaith [CKRS02] and David, Fearnley, and Kisilevsky [DFK04]. First, we introduce some properties of $U(N)$.

- $U(N)$ is a multiplicative group with identity I_N and the inverse of $A \in U(N)$ is A^*. However, it is not abelian for $N > 1$.

- As linear transformations, $A \in U(N)$ preserves the inner product in the vector space of $\mathbb{C}^N$ and the norm in particular. It implies that the sequence of eigenvalues of $A \in U(N)$ is $\{e^{i\theta_j}\}$ for $0 \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N < 2\pi$. Then, that of A^* is $\{e^{-i\theta_j}\}$ and $\det(A) = \prod_{j=1}^{N} e^{i\theta_j} = \exp\left(i \sum_{j=1}^{N} \theta_j\right)$.

- Let A and $B \in U(N)$. Then, we say A is conjugate to B, denoted by $A \sim B$, if there is $D \in U(N)$ such that $D^*AD = B$. For each eigenvalue of B, say λ_1, $D^*ADx = Bx = \lambda_1 x$ for some eigenvector x. Thus, $ADx = \lambda_1 Dx$. Letting $x' = Dx$, $Ax' = \lambda_1 x'$. Therefore, all eigenvalues of A coincide with those of B. In particular, $A \sim \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and, hence, $A = D \operatorname{diag}(\lambda_1, \ldots, \lambda_N) D^*$ for some $D \in U(N)$. This property says each conjugacy class can be identified with a sequence of eigenvalues $\{e^{i\theta_1}, \ldots, e^{i\theta_N}\}$, more simply $\{\theta_1, \ldots, \theta_N\}$ where $0 \leq \theta_1 \leq \ldots \leq \theta_N < 2\pi$.

The topological structure of $U(N)$ provides us the calculus with respect to the Haar measure.

Definition 4.1. *A group G is a topological group if there are continuous maps $f : G \times G \to G$ and $h : G \to G$ by $(a, b) \mapsto ab$ and $g \mapsto g^{-1}$ respectively. Note that $G \times G$ is a product topological space of G.*

Let $M_N(\mathbb{C})$ be the set of all $N \times N$ complex matrices and consider the map $\psi : M_N(\mathbb{C}) \to \mathbb{R}^{2N^2}$ by $\{a_{jl} + ib_{jl}\} \mapsto (a_{jl}, b_{jl})$ for $1 \leq j, l \leq N$ and a_{jl} and $b_{jl} \in \mathbb{R}$. Then, ψ is a homeomorphism. Thus, since $\mathbb{R}^{2N^2}$ is a Euclidean space, an open set of

$M_N(\mathbb{C})$ is an open set of $\mathbb{R}^{2N^2}$, i.e. $O \in M_N(\mathbb{C})$ is open if and only if for any image of $\{a_{jl} + ib_{jl}\}$ under the map ψ in contained by an open ball of $\mathbb{R}^{2N^2}$. Moreover, the topology of $U(N)$ is inherited by that of $M_N(\mathbb{C})$, i.e. $O \subset U(N)$ is open if there is an open set of $O' \subset M_N(\mathbb{C})$ such that $O = O' \cap U(N)$.
Note that the usual norm of the Euclidean space $\mathbb{R}^{2N^2}$ is defined as for (a_j)

$$\|(a_j)\| = \sqrt{\left(\sum_{j=1}^{2N^2} a_j^2\right)}.$$

The distance between A and $B \in M_N(\mathbb{C})$ is defined by $\|A - B\|$. This is a metric of the space $M_N(\mathbb{C})$. The norm of the image of $U(N)$ is bounded, i.e. for any $\{a_{jl} + ib_{jl}\} \in U(N)$

$$\begin{aligned}\|\psi(\{a_{jl} + ib_{jl}\})\| &= \|(a_{11}, b_{11}, \ldots, a_{NN}, b_{NN})\| \\ &= \sqrt{a_{11}^2 + b_{11}^2 + \ldots + a_{NN}^2 + b_{NN}^2} \\ &\leq N\sqrt{2} \text{ since all } |a_{jl}| \text{ and } |b_{jl}| \leq 1.\end{aligned}$$

Suppose a sequence (U_j) where $U_j \in U(N)$ converges to $U_\infty \in M_N(\mathbb{C})$. Then, the sequence $\left(U_j^*\right)$ converges to $U_\infty^* \in M_N(\mathbb{C})$. Since $U_j U_j^* = I_N$ for all j, $U_\infty U_\infty^* = I_N$. So, U_∞ is unitary. So, $U(N)$ is closed in $M_N(\mathbb{C})$. Therefore, closedness and boundedness of $U(N)$ in $M_N(\mathbb{C})$ imply that $U(N)$ is compact by the Heine-Borel theorem for Euclidean space.

Consider the set of diagonal representatives, $D := \{\mathrm{diag}\,(\theta_1, \ldots, \theta_N)\}$, of the conjugacy classes of $U(N)$. Let a map $f : U(N) \to \mathbb{C}$ such that $f(A) = f(B)$ if $A \sim B$ for A and $B \in U(N)$. We call such f a class function on $U(N)$. Then, from the property above, we have for $A \in U(N)$

$$\begin{aligned}\det(A - \lambda I_N) &= \det\left(U \,\mathrm{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_N}\right) U^* - \lambda I_N\right) \text{ for some } U \in U(N) \\ &= \det\left(U \left(\mathrm{diag}\left(e^{i\theta_1}, \ldots, e^{i\theta_N}\right) - \lambda I_N\right) U^*\right) \\ &= \prod_{j=1}^{N} \left(e^{i\theta_j} - \lambda\right).\end{aligned} \tag{4.1}$$

Therefore, $\det(A-\lambda I_N)$ is a class function. For a measurable set S of $U(N)$, we call a measure μ a left(right) Haar measure if for any compact subset $C \in U(N)$, $\mu(C) < \infty$ and for all $g \in U(N)$, $\mu(S) = \mu(gS)(\mu(S) = \mu(Sg))$. Since $U(N)$ is compact, hence locally compact, by theorem 6.8 and proposition 6.15b in [Kna05], we have

Proposition 4.1. *There is a left and right Haar measure, called a Haar measure, for $U(N)$ which is unique up to a constant.*

Furthermore, the following is Weyl's integration formula, see theorem 8.59 in [Kna96], with the Haar measure for $U(N)$ which give us the integration of the moments of the values of $|\det(A - I_N)|$. For a class function f on $U(N)$ and the Haar measure μ of $U(N)$,

$$\int_{U(N)} f(A)d\mu = \frac{1}{N!(2\pi)^N} \int_{[0,2\pi)^N} f(\theta_1, \ldots, \theta_N) \prod_{1\leq j<l\leq N} \left|e^{i\theta_j} - e^{i\theta_l}\right|^2 d\theta_1 \cdots d\theta_N. \tag{4.2}$$

We define t-th moments of $|\det(A - I_N)|$ as

Definition 4.2.

$$M_U(t, N) = \int_{U(N)} |\det(A - I_N)|^t \, d\mu. \tag{4.3}$$

David, Fearnley, and Kisilevsky [DFK04] proposed some moment conjectures for some families of $L_E(1, \chi)$ over a Dirichlet character χ of order 3 and conductor $\mathfrak{f}_\chi$ by using the idea of Conrey, Keating, Rubinstein, and Snaith [CKRS02] that the distribution of the critical L-values for some families of quadratic twists set by a maximal conductor fits that of values of $|\det(A - I_N)|$ for $U(N)$ set by N related to that maximal conductor. Thus, here we need to study of statistics of values of $|\det(A - I_N)|$, for examples, moments and probability density function for values of $|\det(A - I_N)|$.

Fix the order of $M_U(t,N)$ and N. Then, using Weyl's integration formula for the class function $f = |\det(A - I_N)|^t$ on $U(N)$ in (4.1), we can write

$$M_U(t,N) = \frac{1}{N!(2\pi)^N} \int_{[0,2\pi)^N} \prod_{j=1}^{N} \left|e^{i\theta_j} - 1\right|^t \prod_{1\le j<l\le N} \left|e^{i\theta_j} - e^{i\theta_l}\right|^2 d\theta_1 \cdots d\theta_N \qquad (4.4)$$

By using the double angle formulae for trigonometric functions, we have

$$\begin{aligned}
\prod_{1\le j<l\le N} \left|e^{i\theta_j} - e^{i\theta_l}\right|^2 &= \prod_{1\le j<l\le N} \left|1 - e^{i(\theta_j-\theta_l)}\right|^2 \\
&= \prod_{1\le j<l\le N} \left|1 - \cos(\theta_j - \theta_l) - i\sin(\theta_j - \theta_l)\right|^2 \\
&= \prod_{1\le j<l\le N} \left|1 - \cos^2\left(\frac{\theta_j-\theta_l}{2}\right) + \sin^2\left(\frac{\theta_j-\theta_l}{2}\right) - 2i\sin\left(\frac{\theta_j-\theta_l}{2}\right)\cos\left(\frac{\theta_j-\theta_l}{2}\right)\right|^2 \\
&= 2^{N(N-1)} \prod_{1\le j<l\le N} \left|\sin^2\left(\frac{\theta_j-\theta_l}{2}\right) - i\sin\left(\frac{\theta_j-\theta_l}{2}\right)\cos\left(\frac{\theta_j-\theta_l}{2}\right)\right|^2 \\
&= 2^{N(N-1)} \prod_{1\le j<l\le N} \left|\sin\left(\frac{\theta_j-\theta_l}{2}\right)\right|^2.
\end{aligned}$$

Similarly, we have

$$\prod_{j=1}^{N} \left|e^{i\theta_j} - 1\right|^t = 2^{tN} \prod_{j=1}^{N} \left|\sin\left(\frac{\theta_j}{2}\right)\right|^t.$$

Therefore, the RHS of equation (4.4) can be written as

$$\begin{aligned}
M_U(t,N) &= \frac{2^{N(N-1)}2^{tN}}{N!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j=1}^{N} \left|\sin\left(\frac{\theta_j}{2}\right)\right|^t \prod_{1\le j<l\le N} \left|\sin\left(\frac{\theta_j-\theta_l}{2}\right)\right|^2 d\theta_1 \cdots d\theta_N \\
&\qquad \text{by replacing } \theta_i \mapsto 2\theta_i \text{ for all } 1 \le i \le N \\
&= \frac{2^{N^2}2^{tN}}{N!(2\pi)^N} \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} \left|\sin(\theta_j)\right|^t \prod_{1\le j<l\le N} \left|\sin(\theta_j - \theta_l)\right|^2 d\theta_1 \cdots d\theta_N.
\end{aligned}$$

Applying the double angle formula to the second product, we have

$$\begin{aligned}
M_U(t,N) = \frac{2^{N^2}2^{tN}}{N!(2\pi)^N} \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} \left|\sin(\theta_j)\right|^t \\
\times \prod_{1\le j<l\le N} \left|\sin(\theta_j)\right|^2 \left|\sin(\theta_l)\right|^2 \left|\cot(\theta_j) - \cot(\theta_l)\right|^2 d\theta_1 \cdots d\theta_N.
\end{aligned}$$

Notice that

$$\prod_{1\le j<l\le N} |\sin(\theta_j)|^2 \, |\sin(\theta_l)|^2 = \prod_{j=1}^{N} |\sin(\theta_j)|^{2(N-1)}.$$

Using the above equality, we have

$$\begin{aligned} M_U(t,N) = \frac{2^{N^2} 2^{tN}}{N!(2\pi)^N} \int_0^{\pi} \cdots \int_0^{\pi} \prod_{j=1}^{N} |\sin(\theta_j)|^{2(N-1)+t} \\ \times \prod_{1\le j<l\le N} |\cot(\theta_j) - \cot(\theta_l)|^2 \, d\theta_1 \cdots d\theta_N. \end{aligned} \tag{4.5}$$

By letting $x_j = -\cot(\theta_j)$, we have

$$dx_j = \frac{d\theta_j}{\sin^2(\theta_j)} \quad \text{and} \quad \frac{1}{1+x_j^2} = \sin^2(\theta_j).$$

Using these, we have

$$\begin{aligned} \prod_{j=1}^{N} |\sin(\theta_j)|^{2(N-1)+t} \, d\theta_1 \cdots d\theta_N &= \prod_{j=1}^{N} \left|1+x_j^2\right|^{-N-t/2} dx_1 \cdots dx_N \\ &= \prod_{j=1}^{N} (1+ix_j)^{-N-t/2} (1-ix_j)^{-N-t/2} dx_1 \cdots dx_N. \end{aligned}$$

Therefore, equation (4.5) becomes

$$\begin{aligned} M_U(t,N) = \frac{2^{N^2+tN}}{N!(2\pi)^N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{j=1}^{N} (1+ix_j)^{-N-t/2} (1-ix_j)^{-N-t/2} \\ \times \prod_{1\le j<l\le N} |x_j - x_l|^2 \, dx_1 \cdots dx_N. \end{aligned} \tag{4.6}$$

Now we can evaluate equation (4.5) by using the Selberg's integration formula, see equation 17.5.2 in [Meh91], which is

$$\begin{aligned} &\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le j<l\le N} |x_j - x_l|^{2\gamma} \prod_{j=1}^{N} (a+ix_j)^{-\alpha} (b-ix_j)^{-\beta} dx_1 \cdots dx_N \\ &= \frac{(2\pi)^N}{(a+b)^{(\alpha+\beta)N - \gamma N(N-1) - N}} \prod_{j=0}^{N-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+\beta-(N+j-1)\gamma-1)}{\Gamma(1+\gamma)\Gamma(\alpha-j\gamma)\Gamma(\beta-j\gamma)} \end{aligned}$$

for $\mathrm{Re}(a), \mathrm{Re}(b), \mathrm{Re}(\alpha)$, and $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\alpha+\beta) > 1$, and

$$\frac{-1}{N} < \mathrm{Re}(\gamma) < \min\left\{\frac{\mathrm{Re}(\alpha)}{N-1}, \frac{\mathrm{Re}(\beta)}{N-1}, \frac{\mathrm{Re}(\alpha+\beta+1)}{2(N-1)}\right\}.$$

For our equation (4.5), take $a = b = \gamma = 1$ and $\alpha = \beta = N + t/2$ for $\mathrm{Re}(t) > -2$. Then, we have

$$(a+b)^{(\alpha+\beta)N-\gamma N(N-1)-N} = 2^{N^2+tN} \text{ and } \alpha + \beta - (N+j-1)\gamma - 1 = N + t - j.$$

Note that the functional equation of $\Gamma(z)$ implies that

$$\prod_{j=0}^{N-1} \Gamma(2+j) = \prod_{j=0}^{N-1} (j+1)\Gamma(1+j) = N! \prod_{j=1}^{N} \Gamma(j),\ \Gamma(2) = \Gamma(1) = 1,$$
$$\prod_{j=0}^{N-1} \Gamma(N+t-j) = \prod_{j=1}^{N} \Gamma(j+t),\ \text{and } \prod_{j=0}^{N-1} \Gamma(N+\frac{t}{2}-j) = \prod_{j=1}^{N} \Gamma(j+\frac{t}{2}).$$

These give us

$$M_U(t,N) = \frac{1}{N!} \prod_{j=0}^{N-1} \frac{\Gamma(2+j)\Gamma(N+t-j)}{\Gamma(2)\Gamma(N+\frac{t}{2}-j)^2} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+t)}{\Gamma(j+\frac{t}{2})^2}. \tag{4.7}$$

Note that this formula (4.7) for t-th moments holds for $\mathrm{Re}(t) > -2$ and any positive integer N and is analytic for $\mathrm{Re}(t) > -1$.

Recall the definition 4.2 of t-th moments of $|\det(A - I_N)|$. Let $P_U(x,N)$ be the probability density function for $x = |\det(A - I_N)|$. Then, we can consider the probability theoretic relation between $M_U(t,N)$ and $P_U(x,N)$ by

$$M_U(t,N) = \int_0^\infty x^t P_U(x,N)dx.$$

Then, we can apply the Mellin transform to get $P_U(x,N)$, i.e.

$$\begin{aligned} P_U(x,N) &= \frac{1}{2\pi i x} \int_{c-i\infty}^{c+i\infty} M_U(t,N) x^{-t} dt \ \text{ for some } c > -1. \\ &= \frac{1}{2\pi i x} \int_{c-i\infty}^{c+i\infty} x^{-t} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+t)}{\Gamma(j+\frac{t}{2})^2} dt \end{aligned} \tag{4.8}$$

Moreover, we can evaluate $P_U(x,N)$ by considering the behaviour of the integrand in (4.8) as in [DFK04].

Proposition 4.2. *The probability density function $P_U(x,N)$ of values of $|\det(A-I_N)|$ for $A \in U(N)$ and the identity I_N in $U(N)$ can be approximated as for $x \to 0$ and N fixed,*

$$P_U(x,N) \sim \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(j)^2}{\Gamma(j-\frac{1}{2})^2}. \tag{4.9}$$

Proof. In (4.8) let $t = \sigma + i\tau$ and $c = 0$. Then, the controlling behaviour of the integrand, for a fixed N, is given by Stirling's formula as $|\tau| \to \infty$ and for $-\pi < \arg t < \pi$:

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}} \left(\frac{t}{e}\right)^t \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{4.10}$$

which gives us some bounds for $|\Gamma(t)|$ as

$$|\Gamma(t)| = |\Gamma(\sigma+i\tau)| = \begin{cases} O\left(|\tau|^{\sigma-1/2} e^{-\pi|\tau|/2}\right) & \text{, if } |\tau| \to \infty \\ O\left(\sigma^{\sigma-1/2} e^{-\sigma}\right) & \text{, if } |\sigma| \to \infty \end{cases}. \tag{4.11}$$

Indeed, let $S_n \in \mathbb{R}$ such that $-(n+1) < S_n < -n$ for $n = 1, 2, \ldots$ and

$$f(t) = x^{-t} \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+t)}{\Gamma^2(j+\frac{t}{2})}.$$

Note that $f(t)$ has poles at $t = -n$ for positive integers n. Now, we consider the rectangular contours C_n with vertices $-S_n + iT, -S_{n+1} + iT, -S_{n+1} - iT,$ and $-S_n - iT$ oriented in the counterclockwise and check the asymptotes of the horizontal lines and the left vertical line of the contour integral

$$\frac{1}{2\pi i x} \oint_{C_n} f(t)dt. \tag{4.12}$$

Some bounds of $|f(t)|$ on the horizontal lines as $T \to \infty$ and the left vertical lines as $n \to \infty$ can be obtained by the Stirling's formula (4.11):

$$\begin{aligned} |f(\sigma+iT)| &= x^{-\sigma} \left|\prod_{j=1}^{N} \Gamma(j)\right| \left|\prod_{j=1}^{N} \frac{\Gamma(j+\sigma+iT)}{\Gamma(j+\sigma/2+iT/2)}\right| \\ &= x^{-\sigma} O\left(\prod_{j=1}^{N} T^{-j+\frac{1}{2}}\right) = x^{-\sigma} O\left(T^{-\frac{N^2}{2}}\right) \text{ as } T \to \infty \end{aligned}$$

and

$$|f(-S_n+i\tau)| = x^{S_n}\left|\prod_{j=1}^{N}\Gamma(j)\right|\left|\prod_{j=1}^{N}\frac{\Gamma(j-S_n+i\tau)}{\Gamma(j-S_n/2+i\tau/2)}\right|$$
$$= x^{S_n}O\left(\prod_{j=1}^{N}S_n^{-j+\frac{1}{2}}\right) = x^{S_n}O\left(S_n^{-\frac{N^2}{2}}\right) \text{ as } n\to\infty.$$

It implies that for $x \to 0$, we have an exponential decay, $x^{-\sigma}$, as $\sigma \to -\infty$ in $f(\sigma+i\tau)$, i.e. the horizontal lines integrals of each (4.12) go to 0 and as $n \to \infty$, the left vertical integral

$$\left|\frac{1}{2\pi ix}\int_{-S_n-iT}^{-S_n+iT} f(t)dt\right| \leq \frac{1}{2\pi x}\int_{-S_n}^{-S_n} |f(\sigma+iT)|\, d\sigma \to 0$$

This allows us to move the vertical lines to left and to pick the poles at $t=-n$ for the contributions of the contour integral (4.8) by applying the Cauchy's theorem. More specifically,

$$P_U(x,N) = \frac{1}{2\pi ix}\int_{-i\infty}^{i\infty} f(t)dt = \frac{1}{x}\sum_{n=-1}^{\infty}\operatorname*{Res}_{t=-n} f(t)$$

where

$$\frac{1}{x}\operatorname*{Res}_{t=-n} f(t) = \frac{1}{x}\operatorname*{Res}_{t=-n}\prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+t)}{\Gamma(j+\frac{t}{2})^2}x^{-t} = x^{n-1}\prod_{j=1}^{N}\frac{\Gamma(j)}{\Gamma(j-\frac{n}{2})^2}\operatorname*{Res}_{t=-n}\prod_{j=1}^{N}\Gamma(j+t).$$

Note that $P_U(x,N)$ can be regarded as a power series of x and for $x \to 0$, the residue of $f(t)$ at $t=-1$ is the main contribution for the series. Therefore, noting $\operatorname*{Res}_{t=0}\Gamma(t) = 1$ we have, as $x \to 0$,

$$P_U(x,N) \sim \frac{1}{x}\operatorname*{Res}_{t=-1}f(t) = \prod_{j=1}^{N}\frac{\Gamma(j)}{\Gamma(j-\frac{1}{2})^2}\prod_{j=2}^{N}\Gamma(j-1) = \frac{1}{\Gamma(N)}\prod_{j=1}^{N}\frac{\Gamma(j)^2}{\Gamma(j-\frac{1}{2})^2} =: R(N). \tag{4.13}$$

□

4.2 Barne's G-function

We define the complex function $G(z)$ called the Barne's G-function as

$$G(1+z) = (2\pi)^{z/2} \exp\left(-\left((\gamma+1)\, z^2 + z\right)/2\right) \prod_{n=1}^{\infty} (1+z/n)^n \exp\left(-z + z^2/(2n)\right),$$

where γ is the Euler constant, i.e. $\gamma = \lim_{n\to\infty} \left(\sum_{m=1}^{n} \frac{1}{m} - \log n \right)$.

The following properties of $G(z)$ are given in [HKO01]. The $G(z)$ is analytic on $\mathbb{C}$, $G(1) = 1$, and has the functional equation $G(1+z) = \Gamma(z)G(z)$ for any $z \in \mathbb{C}$. Furthermore, $\log G(1+z)$ has the following asymptotic formula:

for $|z| \to \infty$ and $-\pi < \arg z < \pi$,

$$\log G(1+z) = \frac{z^2 \log z}{2} - \frac{3z^2}{4} + \frac{z \log 2\pi}{2} - \frac{\log z}{12} + \zeta'(-1) + O(\frac{1}{|z|}), \tag{4.14}$$

where $\zeta(z)$ is the Riemann zeta function. Then, we can represent $R(N)$ in (4.13) by $G(N)$ using the functional equation of $G(z)$ and the asymptotic formula (4.14).

Proposition 4.3. *For $N \to \infty$,*

$$R(N) \;\sim\; N^{\frac{1}{4}} G^2\left(\frac{1}{2}\right).$$

Proof. Using the functional equation of the $G(z)$, we can write

$$\begin{aligned} R(N) &= \frac{1}{\Gamma(N)} \prod_{j=1}^{N} \frac{\Gamma(j)^2}{\Gamma(j-\frac{1}{2})^2} \\ &= \frac{G(N)}{G(1+N)} \prod_{j=1}^{N} \frac{G^2(1+j)}{G^2(j)} \prod_{j=1}^{N} \frac{G^2(j-1/2)}{G^2(j+1/2)} \\ &= \frac{G(N)G(1+N)}{G^2(N+1/2)} G^2(1/2). \end{aligned}$$

So, we need to show

$$\log G(N) + \log G(1+N) - 2\log G(N+1/2) \;\sim\; \frac{\log N}{4}. \tag{4.15}$$

By the above asymptotic formula, we write the LHS of (4.15) as

$$\begin{aligned}
&\log G(N) + \log G(1+N) - 2\log G(N+1/2) \\
&= \frac{(N-1)^2\log(N-1)}{2} - \frac{3(N-1)^2}{4} + \frac{(N-1)\log 2\pi}{2} - \frac{\log(N-1)}{12} \\
&+ \frac{N^2\log N}{2} - \frac{3N^2}{4} + \frac{N\log 2\pi}{2} - \frac{\log N}{12} \\
&- 2\left[\frac{(N-\frac{1}{2})^2\log(N-\frac{1}{2})}{2} - \frac{3(N-\frac{1}{2})^2}{4} + \frac{(N-\frac{1}{2})\log 2\pi}{2} - \frac{\log(N-\frac{1}{2})}{12}\right] + O\left(\frac{1}{N}\right) \\
&= \frac{N^2}{2}\log\left[\frac{(N-1)N}{\left(N-\frac{1}{2}\right)^2}\right] - N\log\left(\frac{N-\frac{1}{2}}{N-1}\right) + \frac{1}{12}\left[5\log(N-1) - \log N - \log\left(N+\frac{1}{2}\right)\right] + O(1).
\end{aligned} \tag{4.16}$$

Note that for $x \in \mathbb{R}$ such that $|x| < 1$, the Taylor series for $\log(1+x)$ is

$$\log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots.$$

Thus, we have for $N \to \infty$

$$\frac{N^2}{2}\log\left[\frac{(N-1)N}{\left(N-\frac{1}{2}\right)^2}\right] = \frac{N^2}{2}\log\left(1 - \frac{1}{4N^2-4N+1}\right) = O(1)$$

and

$$N\log\left(\frac{N-\frac{1}{2}}{N-1}\right) = N\log\left(1 + \frac{1}{2N-2}\right) = O(1).$$

Furthermore, as $N \to \infty$,

$$\frac{1}{12}\left[5\log(N-1) - \log N - \log\left(N+\frac{1}{2}\right)\right] \ \to\ \frac{\log N}{4}.$$

Therefore, we obtained all asymptotic behaviours of all terms in the RHS of (4.16). This completes the proof.

□

4.3 Vanishings of Families of $L_E(1,\chi)$ via Random Matrix Theory

Definition 4.3. *The t-th Moments of $L_E(1,\chi)$ over $\chi \in Y_{E,k}(X)$ is*

$$M_E(t,X) = \frac{1}{|Y_{E,k}(X)|}\sum_{\chi\in Y_{E,k}(X)}\left|L_E(1,\chi)\right|^t. \tag{4.17}$$

Now we study distributions of critical L-values for the family $Y_{E,3}(X)$, in particular $M_E(t,X)$, via some conjectures about the connection of $M_E(t,X)$ and $M_U(t,N)$ proposed by David, Fearnley, and Kisilevsky [DFK04].

Conjecture 3.

$$M_E(t,X) \;\sim\; a_E(t/2)M_U(t,N) \tag{4.18}$$

where $a_E(t/2)$ is a function of t and depends only on E. Furthermore, we have $N \sim 2\log X$.

David, Fearnley, and Kisilevsky [DFK04] also compute the arithmetic factor $a_E(t/2)$ depending on E in the conjecture 3 by using number theory and support it by numerical results. More precisely, they find a main contribution of

$$\frac{1}{|Y_{E,k}(X)|}\sum_{\chi\in Y_{E,k}(X)} \left|L_E(s,\chi)\right|^t \tag{4.19}$$

can be written as $\zeta^{t^2}(s)f(s,t)$ where $f(s,t)$ is analytic at $s=1$ for each t. Then, they conclude $a_E(t/2)=f(1,t)$ and $a_E(t/2)$ converges at $t=-1$.

Moreover, we can evaluate the relation of N and X in the above conjecture by Keating and Snaith's idea that the mean eigenvalue density is asymptotically same as the mean non-trivial zero density of the Riemann zeta function $\zeta(s)$ at a fixed height $T>0$ in the upper half critical strip such that such zeroes all have $0<\mathrm{Im}(s)<T$. We adapt the same idea for $L_E(s,\chi)$. In other words, let

$$N_\chi(T) := \#\{s\in\mathbb{C} \mid 0<\mathrm{Re}(s)<2, 0<\mathrm{Im}(s)<T, \text{ and } L_E(s,\chi)=0\}.$$

Then, the mean non-trivial zero density at a fixed T is given by $N_\chi(T)/T$. Meanwhile, the mean eigenvalue density is the mean density of the conjugacy class representation $\mathrm{diag}\,(\theta_1,\ldots,\theta_N)\in U(N)$ which is given by $N/2\pi$ since each θ_j for $j=1,\ldots,N$ has mean density $1/2\pi$.

Let $f(z)$ be an entire function on $\mathbb{C}$. Then, we say that $f(z)$ has the order $r > 0$ if for $c \geq r$

$$f(z) = O\left(\exp\left(|z|^c\right)\right) \text{ as } |z| \to \infty.$$

Lemma 4.1. $\Lambda_E(s,\chi) = \left(\mathfrak{f}_\chi \sqrt{N_E}/2\pi\right)^s \Gamma(s) L_E(s,\chi)$ *is of order 1.*

Proof. By (3.7) and using $|a_n\chi(n)| = |a_n| \ll n$ from equation 5.7 in [Iwa97],

$$\begin{aligned}\Lambda_E(s,\chi) &= \left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right)^s \Gamma(s)L_E(s,\chi) = \left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right)^s \Gamma(s)L_{f_\chi}(s) \\ &= \left(\mathfrak{f}_\chi\sqrt{N_E}\right)^s \int_0^\infty f_\chi(it)t^s\frac{dt}{t}\end{aligned}$$

is entire on $\mathbb{C}$ where $f(s)$ is an eigenform in $S_2(N_E)$ and $f_\chi(s) = \sum_{n=1}^\infty a_n\chi(n)q^n$ where $q = \exp(2\pi i s)$. Furthermore, equation (3.7) implies that the integral converges for all $s \in \mathbb{C}$ since f_χ is also a cusp form. We check the asymptotic behaviours of the factors of $\Lambda_E(s,\chi)$:

$$\left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right)^s = O\left(\exp\left(|s|\log\left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right)\right)\right) = O\left(\exp\left(c_1|s|\right)\right)$$

and

$$\Gamma(s) = O\left(\exp\left(c_2|s|\log|s|\right)\right)$$

for some constant c_1 and c_2. Corollary 5.2 in [Iwa97] gives us the bound

$$\sum_{n\leq X}|a_n| \ll X^{\frac{3}{2}} \quad \text{for } X \geq 1. \tag{4.20}$$

For $\left|L_{f_\chi}(s)\right|$, we denote $s = \sigma + i\tau$ where σ and $\tau \in \mathbb{R}$. Then, by equation (4.20) and the partial summation formula, we have for $\sigma > 3/2$

$$\begin{aligned}\left|\sum_{n=1}^X \frac{a_n\chi(n)}{n^s}\right| &\leq \sum_{n=1}^X \frac{|a_n|}{n^\sigma} \ll X^{\frac{3}{2}-\sigma} + \sigma\int_1^X u^{\frac{1}{2}-\sigma}du \\ &= X^{\frac{3}{2}-\sigma} + \sigma\left(\sigma - \frac{3}{2}\right)\left(1 - X^{\frac{3}{2}-\sigma}\right) \to \sigma\left(\sigma-\frac{3}{2}\right) \text{ as } X \to \infty.\end{aligned}$$

For $1 \leq \sigma \leq 3/2$, by theorem 5.37 in [IK04] and the functional equation of $L_{f_\chi}(s)$, we have for any $\epsilon > 0$

$$L_{f_\chi}(s) \ll \left(\sqrt{N_E}\left(|s|+2\right)\right)^{1-\sigma+\epsilon}.$$

Therefore, we have

$$|\Lambda_E(s,\chi)| = O\left(\exp\left(c|s|\log|s|\right)\right) \quad \text{for some constant } c.$$

Fix τ. Then, as $\sigma \to \infty$, we have $\Lambda_E(s,\chi) = O\left(\exp\left(c_3\sigma\log\sigma\right)\right)$ for some constant c_3 by equation (4.11) and completes the proof.

□

Therefore, we can apply the Hadamard product for $\Lambda_E(s,\chi)$

$$\Lambda_E(s,\chi) = \left(\mathfrak{f}_\chi\sqrt{N_E}/2\pi\right)^s \Gamma(s)L_E(s,\chi) = e^{a+bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{s/\rho}$$

for some constant a and $b \in \mathbb{C}$ and the product runs over all non-trivial zeroes of $L_E(s,\chi)$. If we consider the logarithmic derivative of $\Lambda_E(s,\chi)$, then

$$\frac{\Lambda_E'(s,\chi)}{\Lambda_E(s,\chi)} = \log\left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right) + \frac{\Gamma'(s)}{\Gamma(s)} + \frac{L_E'(s,\chi)}{L_E(s,\chi)} = b + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right). \tag{4.21}$$

We can easily find the constants a and b as

$$a = \log\Lambda_E(0,\chi) \;\text{ and }\; b = \frac{\Lambda_E'(0,\chi)}{\Lambda_E(0,\chi)} = \frac{\Lambda_E'(2,\overline{\chi})}{\Lambda_E(2,\overline{\chi})}.$$

Denote $s = \sigma + i\tau$ and a non-trivial zero of $L_E(s,\chi)$ by $\rho = \beta + i\gamma$.

Lemma 4.2. *Suppose* $2 \le \sigma \le 7/2$ *and* $|\tau| > 0$. *Then,*

$$\sum_\rho \frac{1}{1+(\tau-\gamma)^2} < c\log|\tau| \text{ for some constant } c$$

where the sum is over all non-trivial zeroes of $L_E(s,\chi)$.

Proof. From equation (4.21), we have

$$-\frac{L_E'(s,\chi)}{L_E(s,\chi)} = \log\left(\frac{\mathfrak{f}_\chi\sqrt{N_E}}{2\pi}\right) - b + \frac{\Gamma'(s)}{\Gamma(s)} - \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Since $\dfrac{\Gamma'(s)}{\Gamma(s)} = O\left(\log|\tau|\right)$ for such s,

$$-\operatorname{Re}\left(\frac{L_E'(s,\chi)}{L_E(s,\chi)}\right) < c_1\log|\tau| - \sum_\rho \operatorname{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Put $s = 7/2 + i\tau$ into the above inequality. Since $\dfrac{L_E'(7/2+i\tau,\chi)}{L_E(7/2+i\tau,\chi)}$ is bounded, we have

$$\sum_\rho \mathrm{Re}\left(\frac{1}{7/2+i\tau-\rho}+\frac{1}{\rho}\right) < c_1 \log|\tau|$$

Furthermore,

$$\mathrm{Re}\left(\frac{1}{7/2+i\tau-\rho}\right) = \frac{7/2-\beta}{(7/2-\beta)^2+(\tau-\gamma)^2} \geq \frac{1}{12+(\tau-\gamma)^2}$$

and by lemma 5.5 in [IK04],

$$\mathrm{Re}\left(\frac{1}{\rho}\right) = \frac{\beta}{|\rho|^2} < \infty.$$

Therefore, for some constant c_2,

$$\sum_\rho \frac{1}{12\left(1+(\tau-\gamma)^2\right)} \leq \sum_\rho \frac{1}{12+(\tau-\gamma)^2} < c_2 \log|\tau|.$$

By letting $c = 12c_2$, we complete the proof. □

Lemma 4.3. *For $-3/2 \leq \sigma \leq 7/2$ and some large $|\tau|$*

$$\frac{L_E'(s,\chi)}{L_E(s,\chi)} = {\sum_\rho}' \frac{1}{s-\rho} + O(\log|\tau|)$$

where the sum runs over ρ such that $|\tau-\gamma| < 1$.

Proof. From equation (4.21),

$$\begin{aligned}\frac{L_E'(s,\chi)}{L_E(s,\chi)} &= \frac{L_E'(\frac{7}{2}+i\tau,\chi)}{L_E(\frac{7}{2}+i\tau,\chi)} + \frac{\Gamma'(\frac{7}{2}+i\tau)}{\Gamma(\frac{7}{2}+i\tau)} - \frac{\Gamma'(s)}{\Gamma(s)} + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{\frac{7}{2}+i\tau-\rho}\right) \\ &= O(\log|\tau|) + \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{\frac{7}{2}+i\tau-\rho}\right).\end{aligned}$$

Thus, if $|\tau-\gamma| \geq 1$, then

$$\begin{aligned}\left|\sum_\rho^* \left(\frac{1}{s-\rho} - \frac{1}{\frac{7}{2}+i\tau-\rho}\right)\right| &\leq \sum_\rho^* \left|\frac{1}{s-\rho} - \frac{1}{\frac{7}{2}+i\tau-\rho}\right| = \sum_\rho^* \frac{\frac{7}{2}-\sigma}{|s-\rho|\left|\frac{7}{2}+i\tau-\rho\right|} \\ &\leq \sum_\rho^* \frac{5}{|\tau-\gamma|^2} \leq \sum_\rho \frac{10}{1+|\tau-\gamma|^2} \leq c_1 \log|\tau|.\end{aligned}$$

by lemma 4.2 and where $\sum_{\rho}^{*}$ sums over ρ such that $|\tau-\gamma| \geq 1$ and c_1 is a constant. If $|\tau-\gamma| < 1$, then

$$\left|\sum_{\rho}{}' \frac{1}{\frac{7}{2}+i\tau-\rho}\right| \leq \sum_{\rho}{}' \frac{1}{|\frac{7}{2}+i\tau-\rho|} \leq \sum_{\rho}{}' 1 \leq \sum_{\rho} 1 \leq c_2 \log|\tau|$$

for some constant c_2. We complete the proof.

□

Proposition 4.4 (Argument Principle)**.** *Let f on $\mathbb{C}$ is a meromorphic function inside and on a closed contour C such that there are neither zeroes nor poles on C, then*

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N - P$$

where N and P are the number of zeroes and poles of f counting multiplicities and orders respectively. In particular, if f is analytic in C and there is no zero on C, then

$$N = \frac{1}{2\pi}\Delta_C \arg f(z)$$

where $\Delta_C \arg f(z)$ is the variation of arguments of $f(z)$ around C.

The second argument of the above proposition follows from the fact that for an analytic function f and a contour C on $\mathbb{C}$ such that there are neither zeroes nor poles on C,

$$\begin{aligned} N &= \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\Delta_C \log f(z) \\ &= \frac{1}{2\pi i}\left(\Delta_C \log|f(z)| + i\Delta_C \arg f(z)\right) \\ &= \frac{1}{2\pi}\Delta_C \arg f(z) \end{aligned}$$

since $\log|f(z)|$ is single valued and, hence, $\Delta_C \log|f(z)|$ around the closed contour C is 0.

The following proposition can be deduced from theorem 5.8 in [IK04].

Proposition 4.5. *For a fixed height $T > 2$, a E of conductor N_E, a Dirichlet character χ of conductor $\mathfrak{f}_\chi$,*

$$N_\chi(T) = \frac{T}{\pi}\log\left(\frac{\mathfrak{f}_\chi\sqrt{N_E}T}{2\pi}\right) - \frac{T}{\pi} + O\left(\log T\right).$$

Proof. First, let

$$M_\chi(T) := \#\{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 2, -T < \mathrm{Im}(s) < T, \text{ and } L_E(s,\chi) = 0\}.$$

Then, by the fact that $L_E(s,\chi) = 0$ if and only if $L_E(\overline{s},\overline{\chi}) = 0$ and by the functional equation (1.1),

$$M_\chi(T) = 2N_\chi(T). \tag{4.22}$$

Let $s = \sigma + i\tau$ and the rectangular contour C be composed of the line segments $l_1 = \{1 - iT \to \frac{7}{2} - iT\}$, $l_2 = \{\frac{7}{2} - iT \to \frac{7}{2} + iT\}$, $l_3 = \{\frac{7}{2} + iT \to 1 + iT\}$, $l_4 = \{1 + iT \to -\frac{3}{2} + iT\}$, $l_5 = \{-\frac{3}{2} + iT \to -\frac{3}{2} - iT\}$, and $l_6 = \{-\frac{3}{2} - iT \to 1 - iT\}$. Then, considering possible trivial zeroes of $L_E(s,\chi)$ at either $s = 0$ or $s = 2$ and either $s = -1$ or $s = 3$ and the variance of arguments through C counter-clockwise, by the argument principle

$$2\pi\left(M_\chi(T) + 2\right) = \Delta_C \arg\Lambda_E\left(s,\chi\right). \tag{4.23}$$

Note that $\Lambda_E\left(s,\chi\right)$ is entire on $\mathbb{C}$. Furthermore, by the functional equation (1.1), we have

$$\arg\Lambda_E\left(\sigma + i\tau,\chi\right) = \arg\Lambda_E\left(2 - \sigma - i\tau,\overline{\chi}\right) + \arg\tau(1,\chi)^2.$$

It implies that $\Delta_{l_i}\arg\Lambda_E\left(s,\chi\right) = \Delta_{l_{i+3}}\arg\Lambda_E\left(s,\chi\right)$ for $i = 1, 2,$ and 3. So, it is enough to compute the variances through $l_1, l_2,$ and l_3, say this right half contour D, and double them. Note that

$$\Delta_D\arg\Lambda_E(s,\chi) = \Delta_D\arg\left(\mathfrak{f}_\chi\sqrt{N_E}/2\pi\right)^s + \Delta_D\arg\Gamma(s) + \Delta_D\arg L_E(s,\chi).$$

Then, we have

$$\Delta_D\arg\left(\mathfrak{f}_\chi\sqrt{N_E}/2\pi\right)^s = \Delta_D\,\mathrm{Im}\left(s\log\left(\mathfrak{f}_\chi\sqrt{N_E}/2\pi\right)\right) = 2T\log\left(\mathfrak{f}_\chi\sqrt{N_E}/2\pi\right), \tag{4.24}$$

and by the Stirling's formula for $\log\Gamma(s)$,

$$\Delta_D \arg\Gamma(s) = \Delta_D \operatorname{Im}\log\left(\Gamma(s)\right) = 2T\log T - 2T + O(1). \tag{4.25}$$

Let non-trivial zero $\rho = \beta + i\gamma$. We compute $\Delta_D \arg L_E(s,\chi) = \Delta_D \operatorname{Im}\log L_E(s,\chi)$. Note that since $\log L_E\left(s,\chi\right)$ is bounded at every $s \in l_2$, $\Delta_{l_2} \arg L_E(s,\chi)$ is also bounded and we only need to investigate the variances of arguments of $L_E(s,\chi)$ on l_1 and l_3. More specifically,

$$\begin{aligned}
\Delta_{l_1} \operatorname{Im}\log L_E(s,\chi) &= \int_{1-iT}^{\frac{7}{2}-iT} \operatorname{Im}\left(\frac{L_E'(s,\chi)}{L_E(s,\chi)}\right) ds \\
&= \int_{1-iT}^{\frac{7}{2}-iT} \operatorname{Im}\left(\frac{1}{s-\rho}\right) ds + O\left(\log T\right) \quad \text{by lemma 4.3} \\
&= \int_{1-iT}^{\frac{7}{2}-iT} \operatorname{Im}\left((\log s-\rho)'\right) ds + O\left(\log T\right) \\
&= \Delta_{l_1}(s-\rho) + O\left(\log|\tau|\right) \ < \ O\left(\log T\right)
\end{aligned}$$

since $\Delta_{l_1}(s-\rho) \leq \pi$. We have the same result for $\Delta_{l_3} \operatorname{Im}\log L_E(s,\chi) = O\left(\log T\right)$. Therefore, we have

$$\Delta_D \arg L_E(s,\chi) = O\left(\log T\right). \tag{4.26}$$

Use $\Delta_C \arg\Lambda_E(s,\chi) = \Delta_D \arg\Lambda_E(s,\chi)$ and by equation (4.22), (4.23), (4.24), (4.25), and (4.26), we complete the proof. □

Note that our family of $L_E(s,\chi)$ is the set of Dirichlet characters χ of order k and conductor $\mathfrak{f}_\chi \leq X$. So, it seems natural to take the maximum conductor among such $\mathfrak{f}_\chi$ call it X. Therefore, we obtain the relation between N and X as

$$N \ \sim \ 2\log X \tag{4.27}$$

since N_E and T are fixed and

$$\frac{N}{2\pi} = \frac{1}{T} N_\chi(T) \ \sim \ \frac{1}{\pi}\log X.$$

Following conjecture 3, we can deduce the probability density function $P_E(x,X)$ for the values of $|L_E(1,\chi)|$ for $\chi \in Y_{E,k}(X)$.

Proposition 4.6. *Assume the conjecture 3. Then, for x small and $N \to \infty$,*

$$P_E(x, X) \;\sim\; a_E(-1/2)N^{1/4}G^2(1/2).$$

Proof. Since $x = |L_E(1,\chi)|$ and $M_E(t, X) = \int_0^\infty x^t P_E(x, X)dx$, by the Mellin transform, we have

$$\begin{aligned} P_E(x, X) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M_E(t, X)x^{-t}\frac{dt}{x} \;\text{ for some } c > -1 \\ &\sim \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} a_E(t/2)M_U(t, N)x^{-t}\frac{dt}{x} \;\text{ by the conjecture 3} \end{aligned} \tag{4.28}$$

We know that $a_E(t/2)$ converges at $t = -1$ from the above argument about $a_E(t/2)$ derived by study of (4.19). Furthermore, a main contribution of $P_E(x, X)$ can be obtained by taking the residue of the integrand of the above line integral in (4.28) at $t = -1$ as shown in the proof of proposition 4.2. So, equation (4.28) becomes

$$\begin{aligned} &\sim a_E(-1/2)P_U(x, N) \\ &\sim a_E(-1/2)N^{1/4}G^2(1/2) \;\text{ by equation (4.13) and proposition 4.3.} \end{aligned}$$

□

For large enough $\mathfrak{f}_\chi$, consider x such that $x < c/\sqrt{\mathfrak{f}_\chi}$ where c is a positive constant. Then, from proposition 4.6 and equation (4.27) we have

$$P_E(x, X) \;\sim\; a_E(-1/2)N^{1/4}G^2(1/2) \;\sim\; C_E \log^{1/4} X \;\sim\; C_E \log^{1/4} \mathfrak{f}_\chi \tag{4.29}$$

where $C_E = 2^{1/4}a_E(-1/2)G^2(1/2)$.

Conrey, Keating, Rubinstein, and Snaith [CKRS02] proposed the conjecture for the number of vanishings for some family of quadratic twists $L_E(s, \chi_d)$ where d is the conductor of a quadratic Dirichlet character. More precisely, let

$$V_{E^+,2} = \#\{L_E(s, \chi_d) \mid \omega\chi_d(-N_E) = 1 \text{ and } d < X \text{ and } L_E(1/2, \chi_d) = 0\}.$$

Then, for some constant c_E depending on E,

$$V_{E^+,2} \;\sim\; c_E X^{\frac{3}{4}} \log^{-\frac{5}{8}} X.$$

David, Fearnley, and Kisilevsky [DFK04] derived and numerically supported the asymptotes of $|V_{E,3}(X)|$ about X for a fixed elliptic curve E using $P_E(x,X)$, theorem 3.3, and conjecture 1 as

$$|V_{E,3}(X)| \sim \begin{cases} b_E X^{1/2}\log^{1/4} X & \text{if } E \text{ has no 3-torsion} \\ \widetilde{b}_E X^{1/2}\log^{9/4} X & \text{if } E \text{ has a 3-torsion} \end{cases}.$$

for some constant b_E and $\widetilde{b}_E$ which depend only on E.

We use the same arguments in [DFK06] and derive the asymptotes of $V_{E,5}(X)$ and $V_{E,7}(X)$ by using the probability density function $P_E(x,X)$ in the rest of this section. Recall equations (3.22) and (3.23) in section 3.2. Consider the probability that $|L_E(1,\chi)| < c/\sqrt{\mathfrak{f}_\chi}$ for some positive constant c and $\mathfrak{f}_\chi$ large enough. Then, by equation (4.29)

$$\begin{aligned} \operatorname{Prob}(|L_E(1,\chi)| < c/\sqrt{\mathfrak{f}_\chi}) &= \int_0^{c/\sqrt{\mathfrak{f}_\chi}} P_E(x,X)dx \\ &\sim \int_0^{c/\sqrt{\mathfrak{f}_\chi}} C_E \log^{1/4}\mathfrak{f}_\chi dx \\ &\sim c\, C_E \frac{\log^{1/4}\mathfrak{f}_\chi}{\sqrt{\mathfrak{f}_\chi}}. \end{aligned} \tag{4.30}$$

Then, we can obtain the asymptotes of $V_{E,5}(X)$ and $V_{E,7}(X)$ by computing the probability that $\operatorname{Prob}(|L_E(1,\chi)| = 0)$ and the partial summation.

From equation (3.20) we know

$$L_E(1,\chi) = 0 \iff n_\chi = 0 \iff \pi(n_\chi) = (n_\chi^{\sigma_1}, n_\chi^{\sigma_2}, \ldots, n_\chi^{\sigma_{(k-1)/2}}) \in R.$$

For $k = 5$, we use equation (3.22) to get

$$\begin{aligned} &\operatorname{Prob}\left(|L_E(1,\chi)| < \frac{A_{E,k}}{\sqrt{\mathfrak{f}_\chi}}\right) \operatorname{Prob}\left(|L_E(1,\chi^\sigma)| < \frac{A_{E,k}}{\sqrt{\mathfrak{f}_\chi}}\right) \\ &\le \operatorname{Prob}\Big(|L_E(1,\chi)| = 0\Big) \\ &\le \operatorname{Prob}\left(|L_E(1,\chi)| < \frac{\sqrt{5}A_{E,k}}{\sqrt{\mathfrak{f}_\chi}}\right) \operatorname{Prob}\left(|L_E(1,\chi^\sigma)| < \frac{\sqrt{5}A_{E,k}}{\sqrt{\mathfrak{f}_\chi}}\right). \end{aligned}$$

Thus, by equation (4.30), Prob $(|L_E(1,\chi)| = 0) \sim C_5 \dfrac{\log^{1/2} \mathfrak{f}_\chi}{\sqrt{\mathfrak{f}_\chi}}$ where C_5 is a constant. Summing it over $\mathfrak{f}_\chi$ up to X we have

$$|V_{E,5}(X)| \sim C_5 \sum_{\mathfrak{f}_\chi \le X} \frac{\log^{1/2} \mathfrak{f}_\chi}{\sqrt{\mathfrak{f}_\chi}}. \tag{4.31}$$

Let $f(x) = \dfrac{\log^{1/2} x}{\sqrt{x}}$. Then, by the partial summation formula and corollary 2.1, we have

$$\begin{aligned}
\sum_{1 \le \mathfrak{f}_\chi \le X} \frac{\log^{1/2} \mathfrak{f}_\chi}{\sqrt{\mathfrak{f}_\chi}} &= \left(\sum_{1 \le \mathfrak{f}_\chi \le X} 1 \right) f(X) - \int_1^X \left(\sum_{1 \le \mathfrak{f}_\chi \le u} 1 \right) f'(u) du \\
&\sim c_5 \log^{1/2} X - c_5 \int_1^X \left(\frac{1}{2u \log^{1/2} u} - \frac{\log^{1/2} u}{u} \right) du \\
&\sim c_5 \int_1^X \frac{\log^{1/2} u}{u} du + O(\log^{1/2} X) \\
&\sim \frac{2c_5}{3} \log^{3/2} X \text{ as } X \to \infty.
\end{aligned}$$

Therefore, equation (4.31) becomes

$$|V_{E,5}(X)| \sim b_{E,5} \log^{3/2} X$$

where $b_{E,5}$ is a constant depending on E.

For $k = 7$, we use equation (3.23) to get

$$\text{Prob}\,(|L_E(1,\chi)| = 0) \le \prod_{1 \le i \le 3} \text{Prob} \left(|L_E(1,\chi^{\sigma_i})| < \frac{A'_{E,7}}{\sqrt{\mathfrak{f}_\chi}} \right).$$

Thus, by equation (4.30), Prob $(|L_E(1,\chi)| = 0) \sim C_7 \dfrac{\log^{3/4} \mathfrak{f}_\chi}{\mathfrak{f}_\chi^{3/2}}$ where C_7 is a constant. Summing it over $\mathfrak{f}_\chi$ up to X we have

$$|V_{E,7}(X)| \sim C_7 \sum_{\mathfrak{f}_\chi \le X} \frac{\log^{3/4} \mathfrak{f}_\chi}{\mathfrak{f}_\chi^{3/2}}. \tag{4.32}$$

Note since $\log^{3/4} \mathfrak{f}_\chi \le \log \mathfrak{f}_\chi \ll (\mathfrak{f}_\chi)^\epsilon$ for any $\epsilon > 0$ the sum in (4.32) converges. Therefore, equation (4.32) becomes

$$|V_{E,7}(X)| \sim O(1).$$

We summarize the above arguments by the following conjecture.

Conjecture 4 (David, Fearnley, and Kisilevsky)**.** *Let $k = 5$ or 7. Fix an elliptic curve E over $\mathbb{Q}$ without a k-torsion point. Let $K = \mathbb{Q}(\zeta_k)$ and $K^+ = \mathbb{Q}(\zeta_k + \zeta_k^{-1})$. Assume that conjecture 3 is true and random variables $|L_E(1, \chi^\sigma)|$ for all $\sigma \in \mathrm{Gal}\,(K/\mathbb{Q})$ restricted to $\mathrm{Gal}\,(K^+/\mathbb{Q})$ are independent identically distributed.*
Then,

$$|V_{E,5}(X)| \sim b_{E,5} \log^{3/2} X$$

for some constant $b_{E,5}$ depending on E and

$$|V_{E,7}(X)| \sim O(1).$$

The above conjecture is also heuristically supported in [DFK06]. When E has a k-torsion point, powers of $\log X$ can be derived by using proposition 2.4 in partial summation.

Conrey, Keating, Rubinstein, and Snaith [CKRS02] introduced and empirically supported the ratio conjecture of the t-th moments of the family

$$F_{E^+}(X) = \{L_E(s, \chi_d) \mid \omega\chi_d(-N_E) = 1 \text{ and } |d| \leq X\} \tag{4.33}$$

where d is the conductor of a quadratic Dirichlet character. More precisely, for a prime p,

$$\begin{aligned} Q_p(t) := \lim_{X \to \infty} & \sum_{\substack{L_E(s,\chi_d) \in F_{E^+}(X) \\ \chi_d(p)=1}} L_E\left(1/2, \chi_d\right)^t \Bigg/ \sum_{\substack{L_E(s,\chi_d) \in F_{E^+}(X) \\ \chi_d(p)=-1}} L_E\left(1/2, \chi_d\right)^t \\ & \sim \frac{(p+1+a_p)^t}{(p+1-a_p)^t} \end{aligned}$$

where $a_p = p + 1 - N_p$ for N_p being the number of points of the reduction $\widetilde{E}$ of E mod p and also the coefficient of term $\chi_d(p)/p^s$ of $L_E(s, \chi_d)$. We propose the ratio conjectures of the t-th moments of some family of critical values of $L_E(s, \chi)$ twisted by Dirichlet characters of order of odd prime k in the next chapter.

Chapter 5

Conjectural Formulae

Choose an elliptic curve E and a prime $p \nmid N_E$, and consider $Y_{E,k}(X)$. We make the following conjectural formula for the asymptotic ratio of t-th moments for $L_E(1,\chi)$ of χ of $\mathfrak{f}_\chi \leq X$ of order k for such p:

Conjecture 5.

$$\begin{aligned} R_p(t,k) &:= \lim_{X\to\infty} \sum_{\substack{\chi\in Y_{E,k}(X)\\ \chi(p)=1}} \left|L_E(1,\chi)\right|^t \Big/ \sum_{\chi\in Y_{E,k}(X)} \left|L_E(1,\chi)\right|^t \\ &\sim\ C_p(t,k) := \left(1+\frac{2F(k,t)}{(p+1-a_p)^{-t}}\right)^{-1} \end{aligned} \tag{5.1}$$

where

$$F(k,t) = \sum_{n=1}^{(k-1)/2} \left(p^2+2p\left(2\cos^2\left(\frac{2\pi n}{k}\right)-a_p\cos\left(\frac{2\pi n}{k}\right)-1\right)+a_p^2-2a_p\cos\left(\frac{2\pi n}{k}\right)+1\right)^{-\frac{t}{2}}.$$

Following the argument for the ratio conjecture in [CKRS02] we observe for each fixed value of $\chi(p)$, say ζ_k^n for some $n \in \{0,1,\ldots,k-1\}$,

$$\sum_{\substack{\chi\in Y_{E,k}(X)\\ \chi(p)=\zeta_k^n}} \left|L_E(1,\chi)\right|^t \sim \left|\left(1-\frac{\alpha_p\chi(p)}{p}\right)\left(1-\frac{\beta_p\chi(p)}{p}\right)\right|^{-t} \times \widetilde{L} \text{ as } X\to\infty$$

where $\alpha_p+\beta_p = a_p, |\alpha_p| = |\beta_p| = p, \alpha_p = \overline{\beta_p}$, and $\widetilde{L}$ is a factor depending on $E,p,t,$ and k but not on $\chi \in Y_{E,k}(X)$. Therefore, $R_p(t,k)$ in conjecture 5 has the cancellation of the common factor $\widetilde{L}$.

Furthermore, for a fixed $\chi(p) = \zeta_k^n$ for some $n \in \{0, 1, \dots, k-1\}$ we have

$$
\begin{aligned}
&\left|\left(1 - \frac{\alpha_p \chi(p)}{p}\right)\left(1 - \frac{\beta_p \chi(p)}{p}\right)\right|^{-t} \\
&= \left|\left(1 - \frac{\alpha_p \zeta_k^n}{p}\right)\left(1 - \frac{\beta_p \zeta_k^n}{p}\right)\left(1 - \frac{\overline{\alpha_p \zeta_k^n}}{p}\right)\left(1 - \frac{\overline{\beta_p \zeta_k^n}}{p}\right)\right|^{-\frac{t}{2}} \\
&= p^{2t}\left|\left(p - \alpha_p \zeta_k^n\right)\left(p - \beta_p \zeta_k^n\right)\left(p - \overline{\alpha_p}\overline{\zeta_k}^n\right)\left(1 - \overline{\beta_p \zeta_k}^n\right)\right|^{-\frac{t}{2}} \\
&= p^t\left(p^2 + 2p\left(2\cos^2\left(\frac{2\pi n}{k}\right) - a_p \cos\left(\frac{2\pi n}{k}\right) - 1\right) + a_p^2 - 2a_p \cos\left(\frac{2\pi n}{k}\right) + 1\right)^{-\frac{t}{2}}.
\end{aligned}
$$

In particular, when $\chi(p) = 1$, we have

$$
\left|\left(1 - \frac{\chi\alpha}{p}\right)\left(1 - \frac{\chi\beta}{p}\right)\right|^{-t} = p^t \left(p + 1 - a_p\right)^{-t}.
$$

For the ratio conjecture of vanishings, Conrey, Keating, Rubinstein, and Snaith take $t = -\frac{1}{2}$ where there is a pole of the t-th moment function, See [CKRS02]. However, by the normalizing issue of L-functions, here we take $t = -1$ where there is a pole of $M_U(t, N)$ in (4.7), See [DFK04]. Therefore, the conjectural formula for the ratio of vanishings of $L_E(1, \chi)$ is

Conjecture 6.

$$
\begin{aligned}
R_p(k) &:= \lim_{X \to \infty} \sum_{\substack{\chi \in V_{E,k}(X) \\ \chi(p)=1}} 1 \Big/ \sum_{\chi \in V_{E,k}(X)} 1 \\
&\sim\ C_p(k) := \left(1 + \frac{2F(k, -1)}{p + 1 - a_p}\right)^{-1}.
\end{aligned}
\tag{5.2}
$$

Appendix A

Numerical Data

In this section, We compute the empirical and conjectural ratios of some t-th moments (5.1) and vanishings (5.2) of critical L-values of twists of order 3, 5, and 7 of prime conductors for various elliptic curves. We used the critical L-values already calculated by Fearnley and Kisilevsky.

The program code for this experiment is based mainly on the script library written by Fearnley and Kisilevsky in PARI/GP [PAR10] and the central L-values of twists of order 3, 5, and 7 they already produced. Furthermore, the elliptic curves used in this experiment are E11A, E14A, E37A, and E37B with Cremona's notations.

Definition A.1.

$$\widetilde{R}_p(t,k) := \lim_{X\to\infty} \sum_{\substack{\chi\in Z_{E,k}(X)\\ \chi(p)=1}} \left|L_E(1,\chi)\right|^t \Big/ \sum_{Z_{E,k}(X)} \left|L_E(1,\chi)\right|^t. \tag{1.1}$$

$$\widetilde{R}_p(k) := \lim_{X\to\infty} \sum_{\substack{\chi\in W_{E,k}(X)\\ \chi(p)=1}} 1 \Big/ \sum_{W_{E,k}(X)} 1. \tag{1.2}$$

In order to save computational time and memory space, we compute $\widetilde{R}_p(t,k)$ and $\widetilde{R}_p(k)$ instead of $R_p(t,k)$ and $R_p(k)$ by using the PARI/GP function to check whether, given an odd prime k and a prime conductor $\mathfrak{f}_\chi$, $\chi(p)=1$ or not. This function is based on the algebraic number theoretic fact: For odd primes $\mathfrak{f}_\chi$ and k such that

$\mathfrak{f}_\chi \equiv 1 \bmod k$, consider the sub-abelian extension field of degree k of the cyclotomic field $\mathbb{Q}(\zeta_{\mathfrak{f}_\chi})$ where $\zeta_{\mathfrak{f}_\chi}$ is a $\mathfrak{f}_\chi$-th root of unity. Then, $\chi(p) = 1$ if and only if p completely splits in this field.

The sample sizes for all experiments conducted are shown in table A.1. Recall

- $Y_{E,k}(X) = \{\chi \mid \chi \text{ is a character of order } k,\, \mathfrak{f}_\chi \leq X, \text{ and } (\mathfrak{f}_\chi, N_E) = 1\}$,
- $Z_{E,k}(X) = \{\chi \in Y_{E,k}(X) \mid \mathfrak{f}_\chi \text{ is a prime}\}$
- $W_{E,k}(X) = \{\chi \in Z_{E,k}(X) \mid L_E(1,\chi) = 0\}$.

Note that the sample sizes of vanishings of twists of order 5 and 7, $|W_{E,5}(X)|$ and $|W_{E,7}(X)|$ respectively for some elliptic curves, are so small, even though their size follow theorem 4, in the our maximum conductors X that there is no valuable statistical information that can be seen for asymptotic conjectural formula $C_p(k)$ in (5.2) for $k = 5$ and 7. Therefore, we show the numerical data for the ratio of vanishings only for cubic twists. Furthermore, In our families, we only consider χ of conductor $\mathfrak{f}_\chi$ such that $\chi(p) \neq 0$, i.e. $p \nmid \mathfrak{f}_\chi$. Hence, for a given prime p, central L-values data with the Dirichlet characters χ of order k and conductor $\mathfrak{f}_\chi$ divisible by p are excluded from the samples.

We compare the accuracy of the data for quadratic twists obtained by Conrey, Keating, Rubinstein, and Snaith [CKRS02] and cubic twists in the following table A.2. They considered the family of $L_E(s,\chi_d)$ twisted by a quadratic character χ_d of conductor $|d|$, $F_{E^+}(X)$ defined in (4.33), such that $\omega_E\chi(-N_E) = 1$ where ω_E is the sign of the functional equation for $L_E(s,\chi_d)$, and conjectured the ratio of vanishings, for a fixed prime p,

$$R_p := \lim_{X\to\infty} \sum_{\substack{L_E(s,\chi_d)\in F_{E^+}(X)\\ L_E(1/2,\chi_d)=0\\ |d|\leq X\\ \chi(p)=1}} 1 \Bigg/ \sum_{\substack{L_E(s,\chi_d)\in F_{E^+}(X)\\ L_E(1/2,\chi_d)=0\\ |d|\leq X\\ \chi(p)=-1}} 1 \sim \sqrt{\frac{p+1-a_p}{p+1+a_p}}. \tag{1.3}$$

k	E	X	$\lvert Y_{E,k}(X)\rvert$	$\lvert Z_{E,k}(X)\rvert$	$\lvert W_{E,k}(X)\rvert$
$k=3$	E11A	999997	317084	78462	420
	E14A	999997	246610	78460	690
	E37A	279211	83982	24308	274
	E37B	364723	109660	31050	306
$k=5$	E11A	377371	72464	31996	8
	E14A	334511	87784	28628	20
	E37A	205775	54140	18392	12
	E37B	205775	54140	18392	0
$k=7$	E11A	199921	39390	17892	12
	E14A	334643	59034	28692	0
	E37A	205759	40446	18408	0
	E37B	205759	40446	18408	6

Table A.1: Sample Sizes

In the table A.2, we compute $\left(1+\frac{1}{R_p}\right)^{-1}$ and $\left(1+\sqrt{\frac{p+1+a_p}{p+1-a_p}}\right)^{-1}$ instead of R_p and $\sqrt{\frac{p+1-a_p}{p+1+a_p}}$ respectively to make the form of the vanishing conjecture for quadratic consistent with that for higher order twists. The average and standard deviation of absolute difference between experimental and conjectural values at p for the quadratic and cubic twists show that the accuracy in the quadratic case is better than that in the cubic case, see the caption in table A.2. This is presumably because the size of the quadradic samples is about 300 times larger than that of cubic samples, i.e. For quadratic twists, $\mathfrak{f}_\chi \leq 333605031$ are experimented while $\mathfrak{f}_\chi \leq 1000000$ are experimented for cubic twists.

Tables A.3 and A.5 show the ratio of vanishings for prime conductors, $\widetilde{R}_p(3)$, for E11A and E14A and for E37A and E37B, and compare them with those for all conductors, $R_p(3)$, for E11A and E14A in table A.3. Furthermore, we also compare the ratio of the 1st moments for prime conductors, $\widetilde{R}_p(1,3)$, with those for all conductors, $R_p(1,3)$, in table A.4. For a summary, figure A.1 allows us to use $\widetilde{R}_p(k)$ and $\widetilde{R}_p(t,k)$ for the support of the ratio conjectures. Note that E37A is the first elliptic curve with rank 1, and E11A, E14A, and E37B have the torsion group of order 5, 6, and 3 respectively. Furthermore, tables A.6 to A.17 are supporting the our ratio conjectures for $\widetilde{R}_p(t, k)$ for $t = 1, 2,$ and 6 and $k = 3, 5,$ and 7.

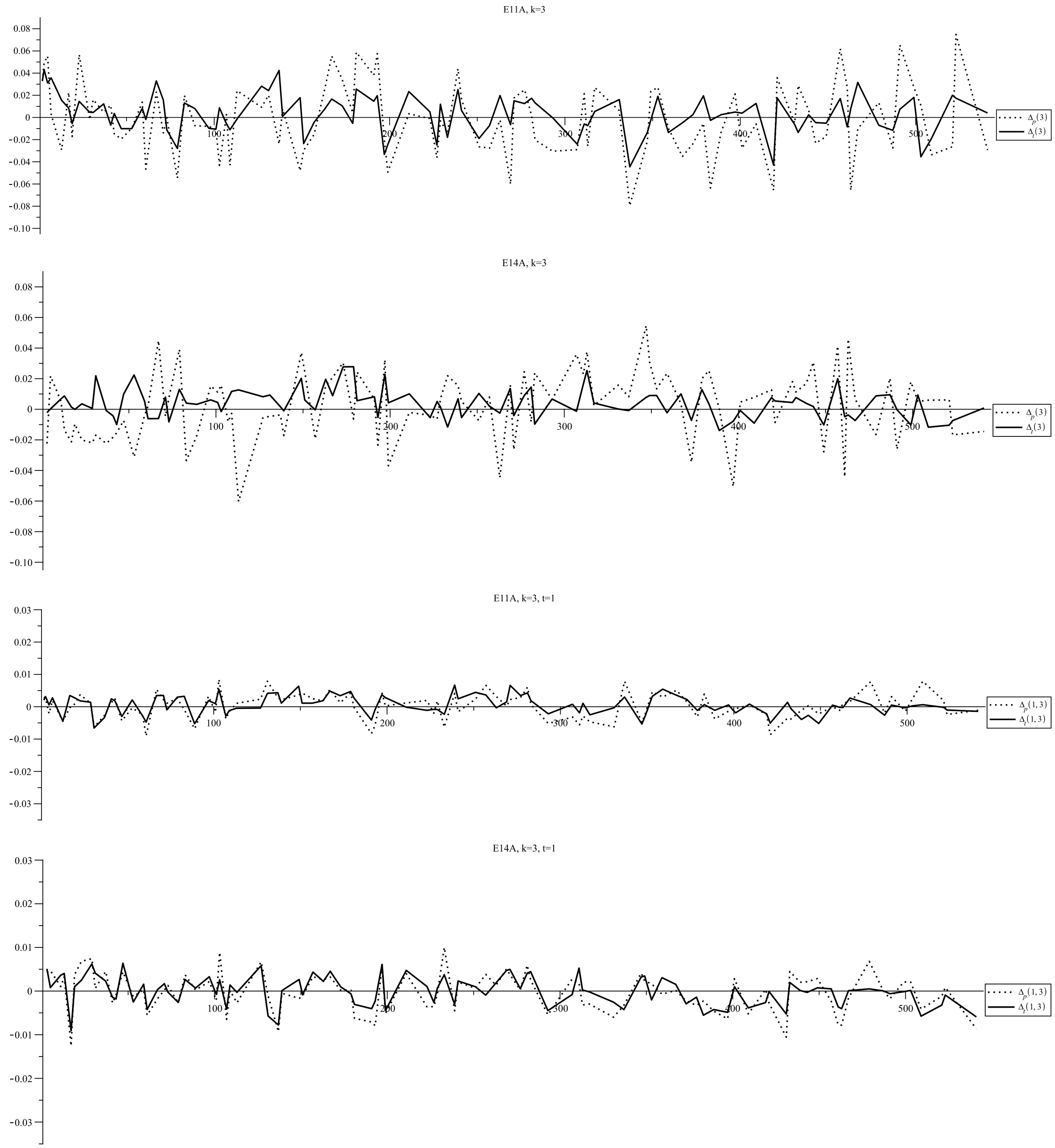

Figure A.1: Line chart for comparison of $\Delta_p(3) := \widetilde{R}_p(3) - C_p(3)$, $\Delta_t(3) := R_p(3) - C_p(3)$, $\Delta_p(1,3) := \widetilde{R}_p(1,3) - C_p(1,3)$, and $\Delta_t(1,3) := R_p(1,3) - C_p(1,3)$ for E11A and E14A. Each horizontal axis denotes primes $p \leq 541$. Note that the errors are bounded by 0.080 for $|\Delta_p(3)|$ and $|\Delta_t(3)|$ and 0.013 for $|\Delta_p(1,3)|$ and $|\Delta_t(1,3)|$ for E11A and E14A.

	Quadratic Twists		Cubic Twists	
p	$\left(1+\frac{1}{R_p}\right)^{-1}$	$\left(1+\sqrt{\frac{p+1+a_p}{p+1-a_p}}\right)^{-1}$	$\widetilde{R}_p(3)$	$C_p(3)$
3	0.5609196	0.5635083	0.6047619	0.5555556
5	0.4592806	0.4580399	0.3761905	0.3208624
7	0.5630551	0.5635083	0.4761905	0.4731370
13	0.4228537	0.4270510	0.2190476	0.2483516
17	0.5301209	0.5278640	0.4142857	0.3917854
19	0.5002159	0.5000000	0.3333333	0.3506279
23	0.5114825	0.5104212	0.4190476	0.3623188
29	0.4994228	0.5000000	0.3428571	0.3447233
31	0.4392511	0.4446421	0.2809524	0.2648378
37	0.4788075	0.4802323	0.3190476	0.3144226
41	0.5535731	0.5480590	0.4142857	0.4032258
43	0.5379498	0.5342509	0.3714286	0.3861801
47	0.4530124	0.4580399	0.2619048	0.2813021
53	0.5312276	0.5278640	0.3666667	0.3764633
59	0.4773404	0.4791304	0.3238095	0.3099607
61	0.4440276	0.4511511	0.2238095	0.2704273
67	0.5286749	0.5258038	0.3952381	0.3723418
71	0.5121421	0.5104212	0.3380952	0.3520408
73	0.4854512	0.4864766	0.3190476	0.3192681
79	0.5362262	0.5313730	0.3238095	0.3784308
83	0.5204884	0.5178800	0.3809524	0.3611040
89	0.4517865	0.4580399	0.2714286	0.2788211
97	0.5210106	0.5178800	0.3523810	0.3604654
101	0.4945570	0.4950976	0.3190476	0.3299606
103	0.5449674	0.5386919	0.3428571	0.3861285
107	0.4511117	0.4580399	0.2809524	0.2783513
109	0.4734995	0.4772256	0.2619048	0.3050658
113	0.4777122	0.4802323	0.3333333	0.3091507
127	0.4820064	0.4843597	0.3238095	0.3145766
131	0.5394984	0.5342509	0.4000000	0.3799609
137	0.5146421	0.5126893	0.3285714	0.3526239
139	0.4778943	0.4821200	0.3190476	0.3112790
149	0.5200115	0.5166852	0.3095238	0.3576034

Table A.2: Accuracy comparison between the ratio of vanishings for quadratic of negative and odd conductors $|\mathfrak{f}_\chi| \leq 333605031$ and cubic twists of prime conductors $\mathfrak{f}_\chi \leq 1000000$. The elliptic curve E11A is used and the data for quadratic twists are from [CKRS02].For the data in this table, the average of errors for the quadratic and cubic cases are 0.003196468 and 0.02217588 respectively, and the corresponding standard deviations are 0.003748201 and 0.02803420.

	E11A				E14A			
p	a_p	$\widetilde{R}_p(3)$	$R_p(3)$	$C_p(3)$	a_p	$\widetilde{R}_p(3)$	$R_p(3)$	$C_p(3)$
2	-2	0.7476190	0.7470665	0.7142857				
3	-1	0.6047619	0.5988024	0.5555556	-2	0.6115942	0.6317808	0.6339746
5	1	0.3761905	0.3520209	0.3208624	0	0.4173913	0.3958724	0.3956439
7	-2	0.4761905	0.5088853	0.4731370				
11					0	0.3681159	0.3696060	0.3628523
13	4	0.2190476	0.2638484	0.2483516	-4	0.4405797	0.4628713	0.4541635
17	-2	0.4142857	0.4002608	0.3917854	6	0.2057971	0.2285178	0.2270957
19	0	0.3333333	0.3451202	0.3506279	2	0.3043478	0.3138226	0.3138916
23	-1	0.4190476	0.3767927	0.3623188	0	0.3275362	0.3512195	0.3476582
29	0	0.3428571	0.3494133	0.3447233	-6	0.3884058	0.4108818	0.4103496
31	7	0.2809524	0.2693878	0.2648378	-4	0.3691860	0.4079665	0.3861155
37	3	0.3190476	0.3266399	0.3144226	2	0.3014493	0.3229508	0.3237999
41	-8	0.4142857	0.3963494	0.4032258	6	0.2724638	0.2863039	0.2906959
43	-6	0.3714286	0.3897849	0.3861801	8	0.2608696	0.2662866	0.2762910
47	8	0.2619048	0.2711864	0.2813021	-12	0.4115942	0.4288931	0.4189635
53	-6	0.3666667	0.3663625	0.3764633	6	0.2695652	0.3230769	0.3006385
59	5	0.3238095	0.3181226	0.3099607	-6	0.3681159	0.3774859	0.3721721
61	12	0.2238095	0.2686170	0.2704273	8	0.2956522	0.2874156	0.2935765
67	-7	0.3952381	0.4053296	0.3723418	-4	0.4028986	0.3519465	0.3580226
71	-3	0.3380952	0.3676662	0.3520408	0	0.3333333	0.3459662	0.3380113
73	4	0.3190476	0.3080000	0.3192681	2	0.3362319	0.3203922	0.3286294
79	-10	0.3238095	0.3504868	0.3784308	8	0.3420290	0.3158513	0.3028586
83	-6	0.3809524	0.3741851	0.3611040	-6	0.3275362	0.3651032	0.3611040
89	15	0.2714286	0.2868318	0.2788211	-6	0.3420290	0.3624765	0.3592557
97	-7	0.3523810	0.3506139	0.3604654	-10	0.3855072	0.3764199	0.3702907
101	2	0.3190476	0.3194263	0.3299606	0	0.3478261	0.3410882	0.3366254
103	-16	0.3428571	0.3948787	0.3861285	-4	0.3652174	0.3479102	0.3494386
107	18	0.2809524	0.2724902	0.2783513	12	0.2898551	0.3050657	0.2980220
109	10	0.2619048	0.2939633	0.3050658	2	0.3275362	0.3418704	0.3302130
113	9	0.3333333	0.3076923	0.3091507	6	0.2579710	0.3309568	0.3182855
127	8	0.3238095	0.3427800	0.3145766	-16	0.3710145	0.3847340	0.3765495
131	-18	0.4000000	0.4041721	0.3799609	18	0.2840580	0.2979362	0.2886557
137	-7	0.3285714	0.3950456	0.3526239	18	0.2869565	0.2923077	0.2906590
139	10	0.3190476	0.3122530	0.3112790	14	0.2840580	0.3002681	0.3013219
149	-10	0.3095238	0.3754889	0.3576034	-18	0.4115942	0.3947467	0.3745406
151	2	0.3014354	0.3074866	0.3310933	8	0.3420290	0.3236641	0.3176053
157	-7	0.3349282	0.3459459	0.3501873	-4	0.3246377	0.3435231	0.3439171

Table A.3: Ratio of vanishings $\widetilde{R}_p(3)$ for cubic twists of the case $\mathfrak{f}_\chi$ is prime and $\leq X$ to the case $\mathfrak{f}_\chi \leq X$ for E11A and E14A. The primes dividing N_E are left as blank. The statistics for the first 100 primes p excluding $p|N_E$: For E11A, the average of $|\widetilde{R}_p(3)-C_p(3)|$ and $|R_p(3)-C_p(3)|$ are 0.02630768 and 0.01448274 with the standard deviations 0.03242929 and 0.01792755 respectively. For E14A, the average of $|\widetilde{R}_p(3)-C_p(3)|$ and $|R_p(3)-C_p(3)|$ are 0.01873202 and 0.007887033 with the standard deviations 0.02290864 and 0.01002376 respectively.

	E11A			E14A		
p	$\widetilde{R}_p(1,3)$	$R_p(1,3)$	$C_p(1,3)$	$\widetilde{R}_p(1,3)$	$R_p(1,3)$	$C_p(1,3)$
2	0.09386460	0.09294287	0.09090909			
3	0.1693487	0.1698208	0.1666667	0.1304314	0.1312307	0.1261320
5	0.3439487	0.3466059	0.3460420	0.2810544	0.2771475	0.2763488
7	0.2196823	0.2205287	0.2177650			
11				0.3061007	0.3086978	0.3050660
13	0.4262546	0.4262275	0.4307305	0.2342846	0.2350774	0.2310429
17	0.2793002	0.2830551	0.2795931	0.4471545	0.4506205	0.4597099
19	0.3166852	0.3194521	0.3164760	0.3571066	0.3543621	0.3533590
23	0.3092655	0.3073771	0.3055556	0.3261298	0.3218110	0.3193098
29	0.3232081	0.3235229	0.3221347	0.2716994	0.2704904	0.2642929
31	0.4033961	0.4030852	0.4096722	0.2850558	0.2885248	0.2844235
37	0.3501137	0.3494443	0.3527961	0.3474976	0.3452887	0.3430051
41	0.2713446	0.2725176	0.2700730	0.3761420	0.3774466	0.3788840
43	0.2869762	0.2862071	0.2843680	0.3948289	0.3938901	0.3957131
47	0.3854371	0.3867518	0.3897692	0.2618438	0.2638169	0.2574500
53	0.2924508	0.2949303	0.2928241	0.3664180	0.3652107	0.3677143
59	0.3550961	0.3547193	0.3575551	0.2969551	0.2982084	0.2966327
61	0.3937564	0.3980339	0.4027935	0.3700689	0.3713823	0.3756115
67	0.3018702	0.2999288	0.2964812	0.3077718	0.3098596	0.3095257
71	0.3173034	0.3186352	0.3151365	0.3289162	0.3303811	0.3286880
73	0.3487922	0.3467376	0.3477017	0.3393684	0.3378474	0.3380707
79	0.2936710	0.2940508	0.2910933	0.3625742	0.3626657	0.3652679
83	0.3057806	0.3099059	0.3066732	0.3103864	0.3093102	0.3066732
89	0.3858922	0.3874449	0.3926999	0.3086220	0.3090684	0.3083812
97	0.3105005	0.3091567	0.3072623	0.3004950	0.3016018	0.2983171
101	0.3350107	0.3376881	0.3367232	0.3279672	0.3292893	0.3300575
103	0.2928706	0.2897919	0.2844123	0.3264863	0.3201832	0.3176079
107	0.3895009	0.3903730	0.3932575	0.3641238	0.3664830	0.3706196
109	0.3613249	0.3616601	0.3628525	0.3365314	0.3378585	0.3364684
113	0.3594699	0.3579544	0.3584262	0.3460388	0.3484346	0.3487286
127	0.3549640	0.3522275	0.3526331	0.2994457	0.2985689	0.2927480
131	0.2976498	0.2939060	0.2897538	0.3801640	0.3755222	0.3812202
137	0.3173663	0.3189231	0.3145852	0.3696290	0.3711556	0.3789261
139	0.3582954	0.3572489	0.3561422	0.3664538	0.3670636	0.3669594
149	0.3135628	0.3162845	0.3099158	0.2928628	0.2971797	0.2945248
151	0.3398065	0.3366882	0.3355810	0.3483512	0.3485709	0.3494414
157	0.3194115	0.3180098	0.3168949	0.3262073	0.3272658	0.3229149

Table A.4: Comparison between $\widetilde{R}_p(1,3)$ and $R_p(1,3)$ for cubic twists of the case $\mathfrak{f}_\chi \leq X$ for E11A and E14A. The primes dividing N_E are left as blank. The statistics for the first 100 primes p excluding $p|N_E$: For E11A, the average of $|\widetilde{R}_p(1,3) - C_p(1,3)|$ and $|R_p(1,3) - C_p(1,3)|$ are 0.003165847 and 0.002452302 with the standard deviations 0.003874501 and 0.002980515 respectively. For E14A, the average of $|\widetilde{R}_p(1,3) - C_p(1,3)|$ and $|R_p(1,3) - C_p(1,3)|$ are 0.003506034 and 0.002705255 with the standard deviations 0.004398328 and 0.003363725 respectively.

	E37A			E37B		
p	a_p	$\widetilde{R}_p(3)$	$C_p(3)$	a_p	$\widetilde{R}_p(3)$	$C_p(3)$
2	-2	0.6350365	0.7142857	0	0.4444444	0.4641016
3	-3	0.6642336	0.6363636	1	0.2875817	0.3021695
5	-2	0.5109489	0.5259316	0	0.4509804	0.3956439
7	-1	0.4306569	0.4285714	-1	0.4144737	0.4285714
11	-5	0.4817518	0.4936752	3	0.2679739	0.2648617
13	-2	0.4452555	0.4095725	-4	0.5065789	0.4541635
17	0	0.3065693	0.3526271	6	0.2549020	0.2270957
19	0	0.2773723	0.3506279	2	0.3398693	0.3138916
23	2	0.2846715	0.3175228	6	0.2222222	0.2556042
29	6	0.2262774	0.2722445	-6	0.4313725	0.4103496
31	-4	0.4160584	0.3861155	-4	0.3137255	0.3861155
41	-9	0.3138686	0.4102277	-9	0.3986928	0.4102277
43	2	0.3014706	0.3251906	8	0.3006536	0.2762910
47	-9	0.3868613	0.4010267	3	0.2549020	0.3185848
53	1	0.3430657	0.3332147	-3	0.3464052	0.3583580
59	8	0.2481752	0.2921869	12	0.2287582	0.2682308
61	-8	0.3455882	0.3811628	8	0.3006536	0.2935765
67	8	0.2919708	0.2972371	-4	0.4313725	0.3580226
71	9	0.2992701	0.2943971	-15	0.3725490	0.4040646
73	-1	0.2627737	0.3424658	11	0.3594771	0.2858718
79	4	0.2627737	0.3203605	-10	0.4444444	0.3784308
83	-15	0.3576642	0.3945050	9	0.3137255	0.3001682
89	4	0.3211679	0.3218478	6	0.3006536	0.3141347
97	4	0.2627737	0.3228130	8	0.3202614	0.3086382
101	3	0.3284672	0.3266059	3	0.3071895	0.3266059
103	18	0.2279412	0.2761587	-4	0.3137255	0.3494386
107	-12	0.3941606	0.3727855	12	0.2941176	0.2980220
109	-16	0.3065693	0.3833607	2	0.3398693	0.3302130
113	-18	0.3722628	0.3870018	-6	0.2941176	0.3538037
127	1	0.3382353	0.3333127	-7	0.3464052	0.3541268
131	-12	0.4233577	0.3657349	-6	0.3856209	0.3510137
137	-6	0.3868613	0.3502453	-6	0.3986928	0.3502453
139	4	0.2408759	0.3260348	-4	0.4379085	0.3452828
149	-5	0.2992701	0.3466924	15	0.3071895	0.3011858
151	16	0.3284672	0.2993191	8	0.3594771	0.3176053
157	23	0.2992701	0.2850738	-13	0.3355263	0.3625038
163	-18	0.3795620	0.3711189	-16	0.3267974	0.3672792
167	-12	0.3430657	0.3588768	18	0.2614379	0.2984841
173	9	0.3138686	0.3176599	9	0.2810458	0.3176599
179	18	0.2481752	0.3008685	18	0.3006536	0.3008685
181	5	0.3138686	0.3258783	-7	0.3750000	0.3479686
191	-4	0.3649635	0.3420380	-24	0.4248366	0.3755112
193	-26	0.3722628	0.3782656	-4	0.3137255	0.3419480

Table A.5: Ratio of vanishings $\widetilde{R}_p(3)$ for E37A and E37B. The error of $C_p(3)$ when $p = 2$ should be interesting.

p	a_p	$\widetilde{R}_p(1,3)$	$C_p(1,3)$	$\widetilde{R}_p(2,3)$	$C_p(2,3)$	$\widetilde{R}_p(6,3)$	$C_p(6,3)$
2	-2	0.09386460	0.09090909	0.02201568	0.01960784	6.624101 E-5	3.199898 E-5
3	-1	0.1693487	0.1666667	0.08148780	0.07407407	0.004426514	0.002043814
5	1	0.3439487	0.3460420	0.3564464	0.3589744	0.4427762	0.4126161
7	-2	0.2196823	0.2177650	0.1424398	0.1341991	0.02145361	0.01467688
13	4	0.4262546	0.4307305	0.5192355	0.5337995	0.7997619	0.8572345
17	-2	0.2793002	0.2795931	0.2359809	0.2315082	0.1682742	0.09857581
19	0	0.3166852	0.3164760	0.3059302	0.3000875	0.3353424	0.2396955
23	-1	0.3092655	0.3055556	0.2845829	0.2791234	0.2153533	0.1884448
29	0	0.3232081	0.3221347	0.3156324	0.3111366	0.3279380	0.2693076
31	7	0.4033961	0.4096722	0.4674089	0.4906275	0.6823854	0.7813946
37	3	0.3501137	0.3527961	0.3734236	0.3727599	0.4715555	0.4563884
41	-8	0.2713446	0.2700730	0.2255424	0.2149474	0.1280076	0.07587376
43	-6	0.2869762	0.2843680	0.2516242	0.2400061	0.1849097	0.1118836
47	8	0.3854371	0.3897692	0.4348643	0.4493203	0.6095199	0.6848264
53	-6	0.2924508	0.2928241	0.2615605	0.2553522	0.1698878	0.1388937
59	5	0.3550961	0.3575551	0.3724268	0.3825270	0.3891642	0.4874493
61	12	0.3937564	0.4027935	0.4475126	0.4763850	0.6026654	0.7507662
67	-7	0.3018702	0.2964812	0.2783081	0.2621015	0.2225088	0.1520092
71	-3	0.3173034	0.3151365	0.3041265	0.2974897	0.2771118	0.2329824
73	4	0.3487922	0.3477017	0.3612615	0.3623528	0.4189200	0.4233085
79	-10	0.2936710	0.2910933	0.2590448	0.2521811	0.1640704	0.1329936
83	-6	0.3057806	0.3066732	0.2856482	0.2812458	0.2602785	0.1933206
89	15	0.3858922	0.3926999	0.4358300	0.4554168	0.5590076	0.7005401
97	-7	0.3105005	0.3072623	0.2935317	0.2823667	0.2780311	0.1959248
101	2	0.3350107	0.3367232	0.3400885	0.3401300	0.3768659	0.3539204
103	-16	0.2928706	0.2844123	0.2618799	0.2400855	0.2199135	0.1120134
107	18	0.3895009	0.3932575	0.4406304	0.4565764	0.5988933	0.7034739
109	10	0.3613249	0.3628525	0.3844576	0.3934431	0.4338133	0.5219130
113	9	0.3594699	0.3584262	0.3863285	0.3843190	0.4394757	0.4931319
127	8	0.3549640	0.3526331	0.3808222	0.3724260	0.5068188	0.4553255
131	-18	0.2976498	0.2897538	0.2694342	0.2497374	0.1653054	0.1285607
137	-7	0.3173663	0.3145852	0.3106325	0.2964227	0.3090944	0.2302537
139	10	0.3582954	0.3561422	0.3813037	0.3796230	0.4009005	0.4782258
149	-10	0.3135628	0.3099158	0.3032984	0.2874341	0.3205116	0.2079474
151	2	0.3398065	0.3355810	0.3459263	0.3378360	0.3997413	0.3469296
157	-7	0.3194115	0.3168949	0.3130163	0.3009013	0.3139450	0.2418187
163	4	0.3413825	0.3396030	0.3470590	0.3459293	0.4186498	0.3717659
167	-12	0.3140971	0.3087325	0.3059590	0.2851706	0.3050954	0.2025270
173	-6	0.3214256	0.3201843	0.3116385	0.3073149	0.2446804	0.2588770
179	-15	0.3096439	0.3053444	0.2945995	0.2787230	0.2260483	0.1875332
181	7	0.3444990	0.3447293	0.3568924	0.3563066	0.3840161	0.4041993
191	17	0.3548997	0.3631844	0.3768748	0.3941282	0.5023090	0.5240610

Table A.6: Ratios of t-th moments $\widetilde{R}_p(t,3)$ and conjectural values $C_p(t,3)$, $t = 1, 2,$ and 6, for E11A.

p	a_p	$\widetilde{R}_p(1,3)$	$C_p(1,3)$	$\widetilde{R}_p(2,3)$	$C_p(2,3)$	$\widetilde{R}_p(6,3)$	$C_p(6,3)$
3	-2	0.1304314	0.1261320	0.04602524	0.04000000	0.0005231723	0.0002892682
5	0	0.2810544	0.2763488	0.2414617	0.2258065	0.09989485	0.09028692
11	0	0.3061007	0.3050660	0.2826947	0.2781955	0.1488009	0.1863359
13	-4	0.2342846	0.2310429	0.1637913	0.1529412	0.02270937	0.02300307
17	6	0.4471545	0.4597099	0.5525699	0.5914894	0.8791886	0.9239082
19	2	0.3571066	0.3533590	0.3768521	0.3739130	0.5748179	0.4600594
23	0	0.3261298	0.3193098	0.3200651	0.3056058	0.2482329	0.2542756
29	-6	0.2716994	0.2642929	0.2251906	0.2051518	0.06779255	0.06434975
31	-4	0.2850558	0.2844235	0.2444513	0.2401055	0.1104665	0.1120462
37	2	0.3474976	0.3430051	0.3601817	0.3528090	0.5840301	0.3932074
41	6	0.3761420	0.3788840	0.4177778	0.4266755	0.6999680	0.6224617
43	8	0.3948289	0.3957131	0.4510925	0.4616822	0.7073018	0.7161773
47	-12	0.2618438	0.2574500	0.2117846	0.1938193	0.07977798	0.05265756
53	6	0.3664180	0.3677143	0.3972248	0.4034951	0.6298010	0.5531795
59	-6	0.2969551	0.2966327	0.2679165	0.2623825	0.1449893	0.1525717
61	8	0.3700689	0.3756115	0.4099510	0.4198747	0.6992534	0.6026288
67	-4	0.3077718	0.3095257	0.2883247	0.2866873	0.1873323	0.2061502
71	0	0.3289162	0.3286880	0.3276231	0.3240759	0.2626655	0.3059728
73	2	0.3393684	0.3380707	0.3520700	0.3428408	0.6047193	0.3622330
79	8	0.3625742	0.3652679	0.3923585	0.3984334	0.6058939	0.5375054
83	-6	0.3103864	0.3066732	0.2920377	0.2812458	0.1833251	0.1933206
89	-6	0.3086220	0.3083812	0.2885726	0.2844998	0.1736096	0.2009362
97	-10	0.3004950	0.2983171	0.2802431	0.2655143	0.4356826	0.1589300
101	0	0.3279672	0.3300575	0.3176546	0.3267980	0.1904977	0.3139289
103	-4	0.3264863	0.3176079	0.3211828	0.3022880	0.2249588	0.2454582
107	12	0.3641238	0.3706196	0.3898235	0.4095147	0.6546922	0.5715990
109	2	0.3365314	0.3364684	0.3430660	0.3396178	0.2696104	0.3523561
113	6	0.3460388	0.3487286	0.3562492	0.3644463	0.2999492	0.4299492
127	-16	0.2994457	0.2927480	0.2725474	0.2552125	0.1413786	0.1386304
131	18	0.3801640	0.3812202	0.4194867	0.4315334	0.3552459	0.6363371
137	18	0.3696290	0.3789261	0.3983496	0.4267630	0.3609211	0.6227138
139	14	0.3664538	0.3669594	0.3978573	0.4019325	0.7142897	0.5483575
149	-18	0.2928628	0.2945248	0.2622022	0.2584828	0.1440634	0.1448827
151	8	0.3483512	0.3494414	0.3613370	0.3659006	0.5848711	0.4345681
157	-4	0.3262073	0.3229149	0.3254101	0.3126694	0.2558357	0.2735446
163	-16	0.3036369	0.3010322	0.2801309	0.2705906	0.1415613	0.1695824
167	-12	0.3119538	0.3087325	0.2985526	0.2851706	0.4821850	0.2025270
173	-12	0.3097688	0.3095413	0.2880726	0.2867172	0.1739063	0.2062220
179	-12	0.3104978	0.3102988	0.2932967	0.2881680	0.1789478	0.2097222
181	20	0.3650846	0.3713207	0.3910248	0.4109686	0.6121355	0.5760078
191	24	0.3702825	0.3773790	0.4075885	0.4235471	0.4371739	0.6133954

Table A.7: Ratios of t-th moments $\widetilde{R}_p(t,3)$ and conjectural values $C_p(t,3)$, $t = 1, 2,$ and 6, for E14A.

p	a_p	$\widetilde{R}_p(1,3)$	$C_p(1,3)$	$\widetilde{R}_p(2,3)$	$C_p(2,3)$	$\widetilde{R}_p(6,3)$	$C_p(6,3)$
2	-2	0.09042620	0.09090909	0.01970293	0.01960784	3.691170 E-5	3.199898 E-5
3	-3	0.1263066	0.1250000	0.04160156	0.03921569	0.0004235209	0.0002719216
5	-2	0.1893701	0.1839046	0.1034252	0.09219858	0.01265852	0.004172958
7	-1	0.2515622	0.2500000	0.1891360	0.1818182	0.06945842	0.04204993
11	-5	0.2140761	0.2040788	0.1385491	0.1162080	0.02910031	0.009011269
13	-2	0.2711888	0.2649181	0.2188025	0.2062016	0.1061912	0.06552016
17	0	0.3160003	0.3145822	0.2967071	0.2964169	0.2518826	0.2302390
19	0	0.3149676	0.3164760	0.2934817	0.3000875	0.2383264	0.2396955
23	2	0.3571909	0.3495279	0.3768999	0.3660773	0.3635321	0.4351293
29	6	0.4032716	0.4005847	0.4698142	0.4718019	0.6830517	0.7403057
31	-4	0.2805610	0.2844235	0.2373505	0.2401055	0.1375861	0.1120462
41	-9	0.2711228	0.2643909	0.2225754	0.2053162	0.1201022	0.06453208
43	2	0.3446860	0.3415768	0.3552376	0.3499171	0.3714129	0.3841627
47	-9	0.2820069	0.2718800	0.2488543	0.2180505	0.1553021	0.07981265
53	1	0.3313198	0.3334519	0.3216609	0.3335706	0.2741414	0.3340453
59	8	0.3772358	0.3771863	0.4148781	0.4231467	0.4833564	0.6122279
61	-8	0.2923600	0.2887057	0.2717426	0.2478318	0.2806855	0.1251723
67	8	0.3623829	0.3714959	0.3938212	0.4113319	0.4890638	0.5771068
71	9	0.3710380	0.3746847	0.4042023	0.4179498	0.4889686	0.5969346
73	-1	0.3342541	0.3243243	0.3429258	0.3154436	0.3834601	0.2812885
79	4	0.3426894	0.3465636	0.3432370	0.3600355	0.2916238	0.4159707
83	-15	0.2770916	0.2773028	0.2270135	0.2274769	0.09464883	0.09266301
89	4	0.3500633	0.3450202	0.3621867	0.3568975	0.3735983	0.4060613
97	4	0.3454859	0.3440224	0.3488767	0.3548717	0.2515407	0.3996834
101	3	0.3396682	0.3401293	0.3485361	0.3469918	0.3879080	0.3750596
103	18	0.3993773	0.3958714	0.4407449	0.4620113	0.5777005	0.7169839
107	-12	0.3059057	0.2960855	0.2925750	0.2613680	0.2790093	0.1505471
109	-16	0.2907674	0.2867980	0.2597577	0.2443778	0.1992951	0.1191836
113	-18	0.2874169	0.2836635	0.2572486	0.2387443	0.2515398	0.1098344
127	1	0.3373957	0.3333540	0.3535695	0.3333747	0.4052645	0.3334573
131	-12	0.3133171	0.3024336	0.3143956	0.2732243	0.3279362	0.1752770
137	-6	0.3236502	0.3168398	0.3215093	0.3007941	0.3148399	0.2415384
139	4	0.3430578	0.3407127	0.3556960	0.3481701	0.3813956	0.3787203
149	-5	0.3169403	0.3202367	0.3055089	0.3074173	0.3529516	0.2591541
151	16	0.3748768	0.3691763	0.4051992	0.4065232	0.4401432	0.5624776
157	23	0.3832892	0.3853590	0.4320064	0.4401438	0.5665574	0.6602837
163	-18	0.3058369	0.2975745	0.2862247	0.2641319	0.2056849	0.1561032
167	-12	0.3074339	0.3087325	0.2898683	0.2851706	0.2102813	0.2025270
173	9	0.3440846	0.3493841	0.3531275	0.3657838	0.3689042	0.4341968
179	18	0.3752621	0.3674601	0.4098103	0.4029688	0.5343709	0.5515569
181	5	0.3435305	0.3408727	0.3540524	0.3484934	0.3902995	0.3797261
191	-4	0.3269976	0.3247409	0.3287364	0.3162649	0.3879614	0.2835993

Table A.8: Ratios of t-th moments $\widetilde{R}_p(t,3)$ and conjectural values $C_p(t,3)$, $t = 1, 2,$ and 6, for E37A.

p	a_p	$\widetilde{R}_p(1,3)$	$C_p(1,3)$	$\widetilde{R}_p(2,3)$	$C_p(2,3)$	$\widetilde{R}_p(6,3)$	$C_p(6,3)$
2	0	0.2292885	0.2240092	0.1535844	0.1428571	0.01913599	0.01818182
3	1	0.3667964	0.3660254	0.4027885	0.4000000	0.4858855	0.5423729
5	0	0.2704294	0.2763488	0.2238061	0.2258065	0.1007520	0.09028692
7	-1	0.2505167	0.2500000	0.1884945	0.1818182	0.05919499	0.04204993
11	3	0.4039068	0.4096424	0.4815684	0.4905660	0.7982570	0.7812684
13	-4	0.2341373	0.2310429	0.1625325	0.1529412	0.03289805	0.02300307
17	6	0.4563635	0.4597099	0.5768667	0.5914894	0.8902237	0.9239082
19	2	0.3547700	0.3533590	0.3766602	0.3739130	0.4603863	0.4600594
23	6	0.4173965	0.4213213	0.4955843	0.5146067	0.7875738	0.8265864
29	-6	0.2662265	0.2642929	0.2173469	0.2051518	0.1143403	0.06434975
31	-4	0.2851673	0.2844235	0.2508179	0.2401055	0.2051618	0.1120462
41	-9	0.2701425	0.2643909	0.2186738	0.2053162	0.06357244	0.06453208
43	8	0.3933797	0.3957131	0.4406108	0.4616822	0.4499224	0.7161773
47	3	0.3539795	0.3484155	0.3721570	0.3638077	0.4133244	0.4279227
53	-3	0.3082147	0.3092140	0.2873109	0.2860910	0.2102183	0.2047213
59	12	0.4083136	0.4054814	0.4877426	0.4819562	0.7952885	0.7630861
61	8	0.3636543	0.3756115	0.3926857	0.4198747	0.3132320	0.6026288
67	-4	0.3078862	0.3095257	0.2881889	0.2866873	0.2984525	0.2061502
71	-15	0.2758590	0.2693866	0.2300192	0.2137738	0.1151865	0.07442077
73	11	0.3776090	0.3844328	0.4135224	0.4382166	0.5662022	0.6549986
79	-10	0.2869151	0.2910933	0.2633960	0.2521811	0.3052795	0.1329936
83	9	0.3672499	0.3682348	0.4015395	0.4045729	0.5347872	0.5564960
89	6	0.3441170	0.3531012	0.3532933	0.3733848	0.3990633	0.4583779
97	8	0.3535775	0.3589784	0.3691945	0.3854558	0.3747469	0.4967329
101	3	0.3343152	0.3401293	0.3388454	0.3469918	0.3486965	0.3750596
103	-4	0.3232624	0.3176079	0.3182594	0.3022880	0.3653410	0.2454582
107	12	0.3691734	0.3706196	0.3964972	0.4095147	0.4663974	0.5715990
109	2	0.3292101	0.3364684	0.3303585	0.3396178	0.2645446	0.3523561
113	-6	0.3190727	0.3134728	0.3062971	0.2942733	0.2608355	0.2248082
127	-7	0.3226832	0.3131688	0.3078243	0.2936868	0.1783739	0.2233343
131	-6	0.3143699	0.3161097	0.2946678	0.2993764	0.1897707	0.2378482
137	-6	0.3134861	0.3168398	0.3056970	0.3007941	0.1881528	0.2415384
139	-4	0.3246224	0.3215943	0.3185769	0.3100764	0.3303446	0.2663946
149	15	0.3646573	0.3671096	0.4106335	0.4022434	0.5285992	0.5493181
151	8	0.3565417	0.3494414	0.3790048	0.3659006	0.4234731	0.4345681
157	-13	0.3051319	0.3053857	0.2873938	0.2788013	0.2237564	0.1877112
163	-16	0.3031936	0.3010322	0.2740339	0.2705906	0.1429760	0.1695824
167	18	0.3602830	0.3701048	0.3820466	0.4084474	0.3733253	0.5683523
173	9	0.3545170	0.3493841	0.3714830	0.3657838	0.5138845	0.4341968
179	18	0.3667798	0.3674601	0.3967357	0.4029688	0.5050972	0.5515569
181	-7	0.3242736	0.3190125	0.3212977	0.3050254	0.4073444	0.2527221
191	-24	0.2994137	0.2936651	0.2811504	0.2568985	0.3715866	0.1418315

Table A.9: Ratios of t-th moments $\widetilde{R}_p(t,3)$ and conjectural values $C_p(t,3)$, $t = 1, 2,$ and 6, for E37B.

p	a_p	$\widetilde{R}_p(1,5)$	$C_p(1,5)$	$\widetilde{R}_p(2,5)$	$C_p(2,5)$	$\widetilde{R}_p(6,5)$	$C_p(6,5)$
2	-2	0.05753610	0.05551541	0.01129648	0.009900990	8.319979 E-6	4.705860 E-6
3	-1	0.1254908	0.1195482	0.07633639	0.06832298	0.01159485	0.005880768
5	1	0.1963133	0.1941767	0.1969122	0.1803279	0.1521302	0.09623310
7	-2	0.1460728	0.1375473	0.1092655	0.09115210	0.02554812	0.01362956
13	4	0.2500794	0.2539902	0.2963060	0.3089380	0.3435367	0.5026080
17	-2	0.1641063	0.1694064	0.1371118	0.1424650	0.04723962	0.06716664
19	0	0.1902358	0.1898683	0.1928153	0.1800001	0.2303949	0.1434057
23	-1	0.1851503	0.1838581	0.1807041	0.1687441	0.1157220	0.1180436
29	0	0.1956090	0.1932758	0.1918654	0.1866667	0.2226235	0.1614531
31	7	0.2442480	0.2448953	0.2869192	0.2926080	0.4317560	0.4857566
37	3	0.2040001	0.2110383	0.2136748	0.2219003	0.2517157	0.2625098
41	-8	0.1645818	0.1624465	0.1435822	0.1295004	0.08910862	0.04437120
43	-6	0.1774004	0.1711683	0.1631838	0.1450990	0.1391303	0.06866693
47	8	0.2285092	0.2333131	0.2595763	0.2683568	0.4545376	0.4141646
53	-6	0.1809806	0.1761028	0.1630362	0.1540835	0.08087317	0.08514223
59	5	0.2206937	0.2141482	0.2347618	0.2284683	0.2132160	0.2860958
61	12	0.2275673	0.2416594	0.2526024	0.2864777	0.3597798	0.4765920
67	-7	0.1709567	0.1781805	0.1533362	0.1578869	0.09530242	0.09250858
71	-3	0.1888658	0.1892437	0.1823561	0.1788964	0.2078968	0.1415826

Table A.10: Ratios of t-th moments $\widetilde{R}_p(t,5)$ and conjectural values $C_p(t,5)$, $t = 1, 2,$ and 6, for E11A.

p	a_p	$\widetilde{R}_p(1,5)$	$C_p(1,5)$	$\widetilde{R}_p(2,5)$	$C_p(2,5)$	$\widetilde{R}_p(6,5)$	$C_p(6,5)$
3	-2	0.09017803	0.08903462	0.03606234	0.03241107	0.0005579618	0.0002616637
5	0	0.1696182	0.1650049	0.1412784	0.1334187	0.06816410	0.04754343
11	0	0.1846127	0.1829541	0.1691642	0.1666687	0.1150391	0.1102131
13	-4	0.1424789	0.1410798	0.1095687	0.09512639	0.1133752	0.01407539
17	6	0.2563366	0.2748782	0.3020466	0.3563713	0.2535983	0.6559468
19	2	0.2105271	0.2103339	0.2240221	0.2197203	0.2803272	0.2464950
23	0	0.1890750	0.1915760	0.1897354	0.1833334	0.2003414	0.1523178
29	-6	0.1675490	0.1594011	0.1449578	0.1244209	0.1564820	0.03861865
31	-4	0.1741439	0.1715109	0.1446353	0.1458808	0.05999775	0.07096018
37	2	0.1960646	0.2053603	0.1969739	0.2104837	0.1341041	0.2283226
41	6	0.2183536	0.2265535	0.2317091	0.2539559	0.2847039	0.3652390
43	8	0.2278089	0.2368454	0.2596761	0.2757970	0.3080570	0.4373802
47	-12	0.1593958	0.1541216	0.1339064	0.1150028	0.05185458	0.02728712
53	6	0.2234441	0.2201176	0.2420396	0.2407261	0.3008623	0.3248111
59	-6	0.1803012	0.1783267	0.1615773	0.1581905	0.1215600	0.09336367
61	8	0.2133355	0.2249568	0.2268477	0.2508975	0.2633101	0.3591673
67	-4	0.1877352	0.1859418	0.1911231	0.1725586	0.2599463	0.1258061
71	0	0.1999966	0.1972124	0.1947948	0.1944444	0.1110958	0.1835734

Table A.11: Ratios of t-th moments $\widetilde{R}_p(t,5)$ and conjectural values $C_p(t,5)$, $t = 1, 2,$ and 6, for E14A.

p	a_p	$\widetilde{R}_p(1,5)$	$C_p(1,5)$	$\widetilde{R}_p(2,5)$	$C_p(2,5)$	$\widetilde{R}_p(6,5)$	$C_p(6,5)$
2	-2	0.05463435	0.05551541	0.01022745	0.009900990	6.268497 E-6	4.705860 E-6
3	-3	0.06557007	0.06373092	0.01459621	0.01356674	1.952732 E-5	1.262545 E-5
5	-2	0.1223415	0.1207519	0.07582518	0.06790855	0.01136526	0.004408021
7	-1	0.1649374	0.1548069	0.1442416	0.1182984	0.08525058	0.03713615
11	-5	0.1156211	0.1226796	0.06300108	0.06818001	0.003336572	0.003621807
13	-2	0.1489676	0.1615774	0.1245767	0.1290056	0.1849199	0.04779165
17	0	0.1887714	0.1887254	0.1727293	0.1777780	0.08749336	0.1375940
19	0	0.1988296	0.1898683	0.2024925	0.1800001	0.2324326	0.1434057
23	2	0.2012919	0.2085689	0.2040170	0.2165020	0.2050416	0.2409843
29	6	0.2463278	0.2391101	0.2849307	0.2800531	0.3047748	0.4438220
31	-4	0.1807964	0.1715109	0.1684810	0.1458808	0.1425174	0.07096018
41	-9	0.1517468	0.1588542	0.1132015	0.1232179	0.04095121	0.03637429
43	2	0.2010408	0.2046186	0.1959875	0.2090634	0.2028240	0.2249126
47	-9	0.1628436	0.1634065	0.1432352	0.1311199	0.09132413	0.04632069
53	1	0.2044057	0.1999624	0.2132508	0.1998524	0.2715027	0.1986937
59	8	0.2195344	0.2258833	0.2301801	0.2528234	0.2569079	0.3652586
61	-8	0.1651462	0.1735396	0.1406141	0.1493079	0.09122098	0.07560577
67	8	0.2232620	0.2225376	0.2499017	0.2458810	0.4638391	0.3432857
71	9	0.2325998	0.2244945	0.2594324	0.2500031	0.2589515	0.3572548

Table A.12: Ratios of t-th moments $\widetilde{R}_p(t,5)$ and conjectural values $C_p(t,5)$, $t = 1, 2,$ and 6, for E37A.

p	a_p	$\widetilde{R}_p(1,5)$	$C_p(1,5)$	$\widetilde{R}_p(2,5)$	$C_p(2,5)$	$\widetilde{R}_p(6,5)$	$C_p(6,5)$
2	0	0.1264809	0.1262577	0.07346294	0.07096774	0.004735500	0.003667525
3	1	0.1823877	0.1819771	0.1480553	0.1463918	0.04396169	0.03267205
5	0	0.1700676	0.1650049	0.1442396	0.1334187	0.1187393	0.04754343
7	-1	0.1550852	0.1548069	0.1187926	0.1182984	0.03958193	0.03713615
11	3	0.2240443	0.2396378	0.2561338	0.2772017	0.4988330	0.3915833
13	-4	0.1310228	0.1410798	0.08660382	0.09512639	0.01392766	0.01407539
17	6	0.2742178	0.2748782	0.3436815	0.3563713	0.5257650	0.6559468
19	2	0.2108410	0.2103339	0.2157985	0.2197203	0.2215943	0.2464950
23	6	0.2416072	0.2513159	0.2829995	0.3057845	0.4087061	0.5191563
29	-6	0.1645495	0.1594011	0.1289777	0.1244209	0.05410465	0.03861865
31	-4	0.1782438	0.1715109	0.1547804	0.1458808	0.08419170	0.07096018
41	-9	0.1569974	0.1588542	0.1220519	0.1232179	0.03410827	0.03637429
43	8	0.2330952	0.2368454	0.2562163	0.2757970	0.2290469	0.4373802
47	3	0.1960803	0.2086500	0.1928195	0.2171938	0.1191786	0.2497482
53	-3	0.1859200	0.1858091	0.1713946	0.1723359	0.1052931	0.1255449
59	12	0.2352567	0.2433126	0.2764578	0.2900271	0.4101452	0.4877777
61	8	0.2226974	0.2249568	0.2549369	0.2508975	0.3648650	0.3591673
67	-4	0.1910745	0.1859418	0.1861348	0.1725586	0.1621258	0.1258061
71	-15	0.1691572	0.1614013	0.1403683	0.1273983	0.08607410	0.04070504

Table A.13: Ratios of t-th moments $\widetilde{R}_p(t,5)$ and conjectural values $C_p(t,5)$, $t = 1, 2,$ and 6, for E37B.

p	a_p	$\widetilde{R}_p(1,7)$	$C_p(1,7)$	$\widetilde{R}_p(2,7)$	$C_p(2,7)$	$\widetilde{R}_p(6,7)$	$C_p(6,7)$
2	-2	0.04913110	0.04609271	0.01200305	0.01043541	1.237683 E-5	1.000782 E-5
3	-1	0.08463728	0.08079968	0.04818549	0.04139219	0.003805425	0.001544920
5	1	0.1484997	0.1406169	0.1667414	0.1342896	0.4187349	0.09130734
7	-2	0.1021764	0.09777003	0.06576331	0.06426646	0.007578723	0.009286417
13	4	0.1784010	0.1813908	0.2114344	0.2203443	0.2670335	0.3493754
17	-2	0.1151408	0.1209330	0.1054121	0.1015777	0.08830989	0.04730918
19	0	0.1316721	0.1356202	0.1220740	0.1285714	0.08140347	0.1024309
23	-1	0.1385726	0.1313072	0.1338864	0.1204731	0.1025456	0.08392626
29	0	0.1486957	0.1380542	0.1542821	0.1333333	0.3621129	0.1153234
31	7	0.1730110	0.1749104	0.1912029	0.2089261	0.2041953	0.3454320
37	3	0.1511551	0.1507542	0.1541723	0.1585402	0.1112527	0.1878774
41	-8	0.1124083	0.1160379	0.08541260	0.09251742	0.01365968	0.03176805
43	-6	0.1219280	0.1222595	0.1093683	0.1036366	0.06801911	0.04908161
47	8	0.1699713	0.1666497	0.1960684	0.1916660	0.4559027	0.2953437
53	-6	0.1329574	0.1257846	0.1299008	0.1100530	0.3647775	0.06081902
59	5	0.1468927	0.1529669	0.1643237	0.1632036	0.3750744	0.2044374
61	12	0.1680037	0.1726089	0.1974791	0.2046085	0.2171306	0.3405553
67	-7	0.1285133	0.1272702	0.1141616	0.1127731	0.2903084	0.06608437
71	-3	0.1461444	0.1351722	0.1422086	0.1277777	0.09974393	0.1010917

Table A.14: Ratios of t-th moments $\widetilde{R}_p(t,7)$ and conjectural values $C_p(t,7)$, $t = 1,2,$ and 6, for E11A.

p	a_p	$\widetilde{R}_p(1,7)$	$C_p(1,7)$	$\widetilde{R}_p(2,7)$	$C_p(2,7)$	$\widetilde{R}_p(6,7)$	$C_p(6,7)$
3	-2	0.06628917	0.06472112	0.02789337	0.02491618	0.0004035197	0.0003050649
5	0	0.1163822	0.1178426	0.09251104	0.09524053	0.02855121	0.03360793
11	0	0.1381275	0.1306811	0.1325231	0.1190476	0.08045956	0.07870424
13	-4	0.09969597	0.1007881	0.06811637	0.06804694	0.01012804	0.01032352
17	6	0.1851287	0.1962096	0.2328267	0.2539743	0.3559521	0.4656809
19	2	0.1463977	0.1503042	0.1525735	0.1571532	0.1337356	0.1779633
23	0	0.1349744	0.1368400	0.1262765	0.1309524	0.04992191	0.1087977
29	-6	0.1181690	0.1138605	0.09481174	0.08889049	0.03984615	0.02771194
31	-4	0.1160061	0.1224931	0.09320674	0.1041664	0.03375662	0.05061976
37	2	0.1424398	0.1466957	0.1450649	0.1503769	0.08411887	0.1633973
41	6	0.1602345	0.1618286	0.1672864	0.1814053	0.1167488	0.2606403
43	8	0.1630341	0.1691703	0.1784936	0.1969689	0.1752733	0.3117558
47	-12	0.1125561	0.1100894	0.08354280	0.08214249	0.01148579	0.01936360
53	6	0.1570047	0.1572308	0.1656038	0.1719576	0.2074344	0.2319959
59	-6	0.1255500	0.1273735	0.1105774	0.1129870	0.04976207	0.06668262
61	8	0.1614936	0.1606841	0.1787722	0.1792113	0.1958162	0.2563910
67	-4	0.1347899	0.1328131	0.1211511	0.1232492	0.07926217	0.08982119
71	0	0.1502588	0.1408660	0.1535452	0.1388889	0.2247749	0.1311239

Table A.15: Ratios of t-th moments $\widetilde{R}_p(t,7)$ and conjectural values $C_p(t,7)$, $t = 1,2,$ and 6, for E14A.

p	a_p	$\widetilde{R}_p$	$C_p(1,7)$	$\widetilde{R}_p(2,7)$	$C_p(2,7)$	$\widetilde{R}_p(6,7)$	$C_p(6,7)$
2	-2	0.05006155	0.04609271	0.01291882	0.01043541	2.641661 E-5	1.000782 E-5
3	-3	0.04846034	0.04636832	0.01156861	0.009656411	1.141031 E-5	5.918942 E-6
5	-2	0.08884075	0.08563361	0.05844347	0.04776894	0.008042595	0.003199329
7	-1	0.1137898	0.1100050	0.08745650	0.08314885	0.02940149	0.02321576
11	-5	0.09077078	0.08780886	0.05315622	0.04900608	0.003439032	0.002597216
13	-2	0.1186768	0.1152739	0.09673843	0.09181756	0.01334054	0.03329789
17	0	0.1276490	0.1348038	0.1157465	0.1269841	0.09469112	0.09827858
19	0	0.1363597	0.1356202	0.1426645	0.1285714	0.09290718	0.1024309
23	2	0.1624662	0.1490162	0.1856160	0.1547673	0.3850514	0.1732868
29	6	0.1792497	0.1707891	0.2087592	0.1999964	0.3915469	0.3155538
31	-4	0.1245198	0.1224931	0.1046868	0.1041664	0.03127698	0.05061976
41	-9	0.1082864	0.1134751	0.08924154	0.08803525	0.1414353	0.02601832
43	2	0.1456202	0.1461624	0.1381384	0.1493512	0.06592424	0.1608531
47	-9	0.1197479	0.1167233	0.1076341	0.09367187	0.03865340	0.03313833
53	1	0.1528981	0.1428321	0.1618207	0.1427574	0.3539702	0.1419800
59	8	0.1457163	0.1613457	0.1578679	0.1805858	0.1810486	0.2607136
61	-8	0.1172275	0.1239563	0.1115960	0.1066492	0.3284859	0.05403718
67	8	0.1729943	0.1589564	0.2192329	0.1756301	0.4179421	0.2451091
71	9	0.1668626	0.1603533	0.1664881	0.1785713	0.1410623	0.2550542

Table A.16: Ratios of t-th moments $\widetilde{R}_p(t,7)$ and conjectural values $C_p(t,7)$, $t = 1, 2,$ and 6, for E37A.

p	a_p	$\widetilde{R}_p(1,7)$	$C_p(1,7)$	$\widetilde{R}_p(2,7)$	$C_p(2,7)$	$\widetilde{R}_p(6,7)$	$C_p(6,7)$
2	0	0.08949890	0.08905240	0.05195172	0.04836895	0.005687344	0.001781755
3	1	0.1364418	0.1376542	0.1215375	0.1233100	0.05504109	0.05256158
5	0	0.1150201	0.1178426	0.1068988	0.09524053	0.1127090	0.03360793
7	-1	0.1178994	0.1100050	0.09284228	0.08314885	0.02406449	0.02321576
11	3	0.1674157	0.1713297	0.1774383	0.1983623	0.1494568	0.2765351
13	-4	0.1072454	0.1007881	0.08439221	0.06804694	0.03861964	0.01032352
17	6	0.1838215	0.1962096	0.2264205	0.2539743	0.2298914	0.4656809
19	2	0.1509049	0.1503042	0.1541333	0.1571532	0.1110045	0.1779633
23	6	0.1750627	0.1794800	0.2059447	0.2182466	0.3727368	0.3676038
29	-6	0.1194192	0.1138605	0.09502093	0.08889049	0.05384385	0.02771194
31	-4	0.1256722	0.1224931	0.1112572	0.1041664	0.05827943	0.05061976
41	-9	0.1162392	0.1134751	0.09705033	0.08803525	0.06322664	0.02601832
43	8	0.1601511	0.1691703	0.1723761	0.1969689	0.1723623	0.3117558
47	3	0.1500661	0.1490422	0.1541436	0.1551592	0.3071541	0.1785911
53	-3	0.1424761	0.1327166	0.1401822	0.1230853	0.2018547	0.08960079
59	12	0.1609705	0.1737894	0.1833572	0.2071432	0.2064015	0.3486604
61	8	0.1570687	0.1606841	0.1686330	0.1792113	0.1986809	0.2563910
67	-4	0.1435695	0.1328131	0.1522614	0.1232492	0.2680308	0.08982119
71	-15	0.1262625	0.1152872	0.1079043	0.09099609	0.08836729	0.02898342

Table A.17: Ratios of t-th moments $\widetilde{R}_p(t,7)$ and conjectural values $C_p(t,7)$, $t = 1, 2,$ and 6, for E37B.

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