STUDENTS' UNDERSTANDING OF REAL, RATIONAL AND IRRATIONAL NUMBERS

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ABSTRACT

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This thesis presents a study of the understanding of real, rational, and irrational numbers by 30 fourth semester college science students in the Montreal region. The written answers to a set of seven questions were analyzed to determine the students' interpretations of mathematical signs according to C. S. Peirce's classifications and to describe their modes of thought according to Vygotsky's theory of concept development. From these interpretations, we are able to reconstruct a facsimile of what the students' concept images are as they pertain to the sets in question. Finding the concept images to be idiosyncratic and rarely in agreement with what conventional mathematics holds to be true, we examine the way the number systems are approached in school and in the field of mathematics and use this, along with the analyses, to make pedagogical recommendations.

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This thesis is dedicated

to the two most important ladies in my life,

my mom, Linda

and my daughter, Vivianne

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Introduction

At the onset of the study, my aim was to assess a group of thirty Canadian college-level science students' understanding of irrational numbers, and compare the results with those presented in the available literature. Students were asked to respond to seven questions that resembled questions asked by other researchers in various studies. This study differs from those in the literature considered in that the participants were science students rather than high school or teacher education students. Presumably, science students have been exposed to more mathematics than high school students or future teachers. One could therefore expect that they would be better able to reason about the real number system; that they would not make the same mistakes. This, sadly, turned out not to be the case.

My initial research goal guided the construction of the research instrument (the questionnaire) and the choice of a theoretical framework for analyzing the responses. The analysis focused on conceptions that could function as obstacles to understanding irrational numbers in a way consistent with present-day mathematical theory of those numbers. However, in the course of analyzing the data, the goal changed and the theoretical framework grew.

Initially, my main concern was whether the students' answers were mathematically correct. Therefore, I sorted out the students' responses according to how they defined irrational numbers and whether their implicit concept images[1] agreed with any of the mathematical characterizations of irrational numbers. In the teaching of

[1] A person's concept image consists of the cognitive structure that is associated with the concept, including all the mental pictures and associated properties and processes. (Tall and Vinner, 1981, p. 152) We will see more on concept images below.

mathematics, until at least a first undergraduate Analysis I course, irrational numbers are typically characterized in one of two ways: as a number that cannot be represented as a ratio of integers (denominator not zero), or a number whose decimal representation is infinite and non-repeating. The participants were asked to define irrational numbers in the first question on the questionnaire. With their definition in mind, I reviewed their remaining answers to see if they applied and understood the definition they provided. Ultimately, I sought to find if any differences or generalizations resulted among students who identify with a particular notion of irrational number.

However, the analysis suggested that some of the misconceptions about irrational numbers in my data set and perhaps across all studies might have something to do with what students understand to be a representation of a number. Each representation of a number tells us something about the number, but not all. For example, √13 represents a non-negative number whose square is equal to 13 for anybody who has such understanding of the square root sign, but it does not tell us anything about its decimal expansion. On the other hand, the output of punching the keys '2nd', '√', '1', '3', ')', and 'ENTER' on a TI-83 Plus calculator, namely, 3.605551275, can also be seen as a representation of the number √13, especially if we know what the input sequence of the keys was. If we don't know the input sequence of keys, then this representation does not directly inform us that the square of the number 3.605551275 is close to 13. Therefore, both number signs, '√13' and '3.605551275,' can be seen as *representing* the same number. But both representations are necessarily limited. For one thing, neither offers any information about the entire sequence of digits which occur after the decimal point, or even about, say, the 100th digit. There is ambiguity in the way we represent numbers and also in the way we ask questions: the representations are open to interpretation. When asked if a number is irrational or rational, the student may be classifying the value of the number or they may be classifying the form of the representation. For some of the students in our research, both $\sqrt{2}$ and $\sqrt{2}/\sqrt{8}$ represented irrational numbers because both expressions contained the square root symbol. Also, a given student may be inconsistent in his or her criteria for classification, sometimes basing their decision on the value and sometimes on the form. This realization led us to examine each student's answer set in the aim of revealing his or her interpretation of the number signs. We used Peirce's classification of signs (1885) into indices, icons and symbols to describe each student's interpretation of the number signs. Paying close attention to what could be considered explicit knowledge possessed by the student and also what knowledge was implicit in the answers set we attempted to reconstruct the student's concept image (Tall & Vinner, 1981) of rational, irrational and real numbers. Based on this understanding of the student's interpretation of the signs and concept image, we assessed the student's mode of thought in terms of Vygotsky's theory of conceptual development (1987).

The structure of the thesis is as follows. Chapter 1 is a review of the literature related to the present research. Some history of the irrational number concept and some obstacles to learning are introduced along with a discussion of previous results found in the literature. Chapter 2 presents the theoretical framework; it is divided into three sections pertaining to sign interpretation, concept images, and modes of thought. In chapter 3 we discuss the meaning and presentation of the sets of rational, irrational, and real numbers in the field of mathematics and in learning environments, both at the university and in secondary school classrooms. Chapter 4 contains a description of methodology and research procedures. Chapter 5 is a presentation of the results of the analysis. In chapter 6, conclusions are drawn and I discuss these results as they compare to results in previous studies. Finally, in chapter 7, I make recommendations for pedagogy and propose some avenues for further research.

Chapter 1. Literature Review

Through history, irrational numbers have provoked debate and caused much struggle. Ancient mathematicians struggled to reconcile the concept of incommensurable lengths with the existing body of geometrical knowledge. Much later, 19^{th} century mathematicians were surprised as they discovered that the cardinality of the set of irrational numbers was larger than that of the set of rational numbers of which there were only as many as there were of natural numbers. Already "the long period in time between Antiquity and the founding of calculus in the seventeenth century can partly be explained by the difficulty in giving up the discrete foundation of numbers…" (Lehtinen *et al.*, 1997, p. 135). We can see that making this cognitive leap was an epistemological obstacle for mathematicians. The fact that there are many more irrational than rational numbers would still be very surprising to today's students who work most of the time with integers or finite decimals and think that irrational numbers are rare. They know only a few examples of them; π and square roots of 2 or 3 are the most commonly quoted by teachers when they discuss irrational numbers in class.

This belief can therefore be considered an epistemological obstacle (Sierpinska, 1994, p. x, 121-8). Fischbein *et al.* (1995) assumed at the onset of their study that epistemological obstacles that mathematicians encountered in developing irrational numbers would reoccur for students who are learning about irrational numbers in classrooms today. Specifically, they assumed that difficulties surrounding the concepts of incommensurability and non-denumerability would appear in student responses in a written questionnaire. However, today's pre-university students are not bothered by the fact that the irrational numbers are non-denumerable, nor are they troubled to find out that not all lengths can be measured by a common unit; they simply do not attend to such questions. Fischbein *et al*. (1995) concluded that a high level of mathematical maturity was required for incommensurability and non-denumerability to appear as obstacles to understanding. These obstacles only manifest themselves at a much higher level of learning when the student becomes well versed in more advanced mathematics, and as such should not to be considered obstacles at the secondary level.

What the Fischbein study found was that students had an insufficient grasp of the definition of irrational number. In the case of defining irrational numbers, for example, periodicity in the decimal representation was often overlooked. Also overlooked was the requirement of a non-zero denominator when irrational numbers are described as numbers which cannot be written as *a/b*. It was also found that students do not possess the algorithmic knowledge of how a periodic infinite decimal may be transformed into a ratio of two whole numbers. This lack of algorithmic knowledge is also discussed by Zazkis and Sirotic (2010) where they conclude that students lack both the formal and algorithmic knowledge that being represented as an infinite, non-repeating decimal is equivalent to not being represented as a fraction. However, it is interesting to note that students who possess the knowledge formally, even on authority alone, were better equipped to call upon one representation or the other as needed despite the lack of deeper understanding of the relationship between the two representations.

If knowledge and experience are lacking, students will naturally fall back on what they are familiar with when given a new definition or concept. In the case of irrational numbers, they rely on their knowledge of the natural numbers. “The discrete nature of numbers is based on innate cognitive mechanisms, powerful experiences of everyday counting, and on formal mathematics instruction." (Merenluoto and Lehtinen, 2004, p. 521) Children are introduced to the discreteness of the counting numbers even before they can speak. Students spend their early education dealing only in discrete quantities and the conceptual change required when they begin to look at fractions and decimals is expected to come quickly, often without explicit attention to what is happening. As a result, when asked if an irrational number is real, students often say it is not. We, as mathematicians and teachers, are using the technical mathematical term "real numbers" while they are using the word "real" in its everyday sense, coupled with a concept image based on their knowledge of the natural numbers, or rather, of "whole numbers" as they are commonly called in school.

What is often the case is that students call upon their own concept image of what an irrational number is instead of relying on a formal definition. Common across all studies considered here were examples of subjects who could give an exact or reasonably good definition of irrational numbers and then went on to misclassify obviously rational examples. Most common was the example of 22/7 which is a rational approximation to π, being classified as irrational because π itself is irrational. Arcavi *et al.* (1987) found that participants (in-service teachers) commonly misclassified 22/7 as an irrational number. The participants did not have a clear distinction between a rational approximation, 22/7, and the irrational number it approximates, π. The researchers wondered if other rational approximations would be regarded as an irrational number, but only after the questionnaire had already been administered.

1.1 Obstacles to understanding related with multiple representations of irrational numbers

Ideas in mathematics are typically represented in many ways across different areas of mathematics. Pre-university students have problems making conceptual relations between the multiple representations of irrational number and keeping them in agreement. They also rely heavily on the decimal representation in particular. This is a natural result of over-reliance on calculators. What can result is that they use rational approximations to irrational numbers without being conscious that they are doing so. Students show a lack of reflection and cohesion in defining and identifying rational, irrational, and real numbers, instead evaluating representations on a case by case basis. When asked to consider an irrational number, students may bring to mind one or more representations which may complement or conflict with their concepts and intuitions regarding numbers generally, and irrational numbers in particular.

Furthermore, mental representations need not be the same across all individuals since what is called to mind is a direct result of individual experience. Thus arises ambiguity and misunderstanding when one's mental representation doesn't match that of another (e.g., teacher and student) and neither is aware of the discrepancy of experience or representation. For example, when the teacher says "irrational number," the student may think of one or more of the following representations which may be, depending on the situation at hand, useful or misleading, adequate or inadequate:

1. a particular example: π
2. a particular class of examples: square roots

3. an infinite decimal
4. a non-periodic, infinite decimal
5. not a fraction

or they may think of something entirely different.

Of course, although there are strengths to every representation, problems can arise alongside any given representation. For instance, thinking in terms of specific examples leads the student to believe that irrational numbers are rare. One student claims that the "[p]robability [of randomly picking a rational number from the unit interval] is 100% because I don't know of many numbers like Pi." (Sirotic and Zazkis, 2007a, p. 51)

Students have far less experience with "ordinary" irrational numbers than they do with special cases like π and *e*. These particular examples are very useful to mathematicians and therefore had names bestowed upon them. This misconception that the student makes above, I believe, is partly due to our only mentioning and using the few named irrational numbers in the classroom, like π or *e* or various square roots, which the student repeatedly sees. The plenitude of irrational numbers which do not fall into special cases is not made explicit.

> *Most reals are inaccessible to us, and will never, ever, be picked out as individuals using* ***any*** *conceivable mathematical tool, because whatever these tools may be they could always be explained in French, and therefore can only "individualize" a countable infinity of reals, a set of reals of measure zero, an infinitesimal subset of the set of all possible reals.*
>
> *Pick a real at random, and the probability is zero that it's accessible – the probability is zero that it* ***will ever be*** *accessible to us as an individual mathematical object. (Chaitin, 2004, p. 6)*

If, instead of specific examples, the student thinks of classes of examples or in incomplete definitions (e.g., an infinite decimal) the student is led astray when they encounter examples which fit their concept image of an irrational number but are in fact rational, e.g., the square root of a square, or an infinite decimal representation with a long period. In fact, periodicity itself is difficult for the student to grasp. The study by Sirotic and Zazkis (2004) showed that even if students recall that there is a non-repeating characteristic of the decimal representations of an irrational number, they are not clear on the details of what constitutes a repeating pattern. Pattern recognition, it turns out, is subjective and students fumble with the following pattern misconceptions:

- What is the common element of the pattern? Some students believed that a pattern must be composed of a single digit repeating as in the case of the decimal expansion of 1/3. Patterns involving a repeated longer sequence of digits often went over-looked.
- Should the pattern start right away, or can it begin further down in the decimal expansion? Students seemed only to recognize patterns that started immediately and which had a short period. Due to the reliance on technology, the pattern is something that must be evident on the limited display of a calculator.
- Does any pattern qualify the number as rational? 0.12122122212… and 0.123456… follow patterns in their own right, but what is important (when talking about irrational numbers) is that it is a *non-repeating pattern*.

Therefore, although many students show a preference for the decimal representation of a number, they confuse irrational decimal expressions with rational ones because they are unable to properly interpret the pattern. Some of these confusions may be explained by students forming a rigid concept image from the examples we, as teachers, give them. Unless the student is using a computer algebra system in class, it is not likely they'll see a rational number in decimal form with a period of 50, say. We use simple, concise examples and then the student relies on these examples instead of formal definitions. If we are to be completely honest with our students, we as teachers need to make clear that a calculator can never be used to decide irrationality. This is unavoidable due to the finite nature of the machine. What calculators can do is produce approximations.

Finally, although students, would define an irrational number as a number that could not be written as a ratio of *a*/*b* (where *b* is non-zero), they could not move freely between fractional representations of numbers and decimal representations. To attend to each representation separately can lead a student, in the most drastic cases, to claim that a number could be both rational and irrational depending solely on how it is written. (Sirotic and Zazkis, 2004, p. 501-502) Zazkis and Sirotic (2010) suggest that bridging the gap in algorithmic knowledge between decimals and fractions can go a long way in enabling students to move flexibly between the different representations of irrational numbers and let them achieve a deeper understanding of the real number system. I very much agree with this pedagogical suggestion as it focuses on the use and understanding of multiple representations which is fundamental for doing mathematics at every level.

1.2 On irrational number as object vs process

Procept is a term that refers to an amalgamation of process and concept. It has been coined by Gray and Tall to describe a device that mathematicians and those successful in mathematics use. According to Gray and Tall, mathematicians exploit ambiguity by "using the same notation to represent both a process and the product of that process"

(Gray and Tall, 1994, p.4) rather than avoiding the dichotomy between object and process. For example 2 + 3 denotes the *process* of adding and the *concept* of sum; *a/b* denotes both the *process* of division and the *concept* of rational number. Allowing the notation to have a dual representation in a proceptual way, the successful student can move from one representation to the other without consciously thinking about the ambiguities resulting from this duality. This proceptual way of thinking is a method of "chunking" information so as to reduce the cognitive load with which our minds must deal. However, some (perhaps most) students will get stuck trying to resolve the ambiguity and will have difficulty developing the flexibility to move in and out of representations.

Irrational numbers, typically in decimal representation, are most often viewed by students as infinite processes and not as mathematical objects in and of themselves. For example, "an 'infinite' decimal representation $\pi = 3.14159\ldots$ is both a process of approximating π by calculating ever more decimal places and the specific numerical limit of that process." (Gray and Tall, 1994, p. 120) Irrational numbers can be written as an infinite sum or a continued fraction. They can be computed on a computer (given infinite running time) or approximated in short periods of time. But despite the notation we use, e.g., π or $\sqrt{2}$, which suggests these numbers exist as objects, they are rarely seen to be a number in the same way as the natural numbers. The sign "π" may be seen as a character or symbol, very much unlike a digit which is a number. The sign "$\sqrt{2}$" might be looked upon as an instruction to carry out an operation.

According to Monaghan (1986), "recurring decimals are perceived as dynamic and qualitatively different from finite decimals." (Pinto and Tall, 1996, p. 141) This is perhaps the root of the problem with identifying and understanding irrational numbers and the most widespread problem across all studies considered herein. Students see a decimal number which actually goes on forever, extending itself digit by digit, continually. As such, irrational numbers are seemingly dependent on time for their very existence. Perhaps this is partly due to a reliance on the calculator display. One can imagine the computer endlessly writing a string of digits of which we only get a glimpse. One student says, "You cannot add two irrational numbers because they both continue forever so you would be adding infinitely." (Sirotic and Zazkis, 2007a, p. 70) Even the language we use to describe irrational numbers, e.g. *goes* on forever, *continues* forever, stresses the procedural nature of irrational numbers. Students look at these numbers, which we as teachers describe with verbs, as active things which occur *in time* and students fail to see them any other way.[2]

To help encapsulate the process into an object, Sirotic and Zazkis (2007b) suggest making good use of the geometric representation of irrational numbers. The geometric approach avoids the error in approximating and shows that to each number, no matter the number of decimal places, there is but one point on the line to which it corresponds. There is an error term to attend to when using a rational approximation to an irrational number in order to estimate its place on the number line. The irrational number can be

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[2] I want to stress that coming to associate irrational numbers with infinite processes isn't a mistake or a misconception in and of itself. It is in the absence of also seeing irrational numbers as a number in their own right that the obstacle presents itself. I also get the impression that the infinite process with which students identify irrational numbers has little or nothing to do with understanding them as limits of sequences of rational numbers. Rather, the infinite process they see is the decimal expansion that goes marching along, without stopping, i.e., they miss the link between decimal place values and rational numbers.

thought of as being included between pairs of ever-more precise rational approximations.

However, relating a constructible irrational length (using the Pythagorean Theorem) to a point on the number line takes an infinite representation, the decimal, and identifies it with a finite representation, the point. This approach provides segue to the history of the development of irrational quantities and also highlights the importance of multiple representations of a number.

That being said, it is also important to explicitly attend to the act of approximating, which becomes even more crucial as the high school students enter the applied sciences. The concept of approximation is especially important when considering irrational numbers which do not lend themselves to geometric consideration, and there is an uncountable infinity of such numbers. The sign 0.4635446836…, which I obtained by mashing the keypad, can be interpreted as an irrational number or as a rational approximation of an irrational number. It has no name to which we can refer. It has no obvious geometric properties that we can exploit. In a sense, the irrational number has become its own representation. "Thus the infinite decimal now is both the approximation and that which is approximated." (Byers, 2007, p. 134) Spurred on by a reliance and misunderstanding of technology, I believe, that misclassification of particular examples may have less to do with understanding the concept definition, or recalling declarative knowledge, and more to do with the student's interpretation of how the number is represented.

Chapter 2. Theoretical Framework

Mathematicians use multiple representations, or ways of understanding, denoting, and describing, to investigate mathematical objects in order to gain greater insight into them. For example, William Thurston mentions seven ways to conceive of the derivative of a function. (Thurston, 1994, p. 39-40) In pre-university classes, the concept of irrational numbers typically relies on one of two representations, i.e., not fractions or infinite, non-repeating decimals. But we can make the following (non-exhaustive) list of how one may interpret irrational numbers using different mathematical ideas:

1. A number which cannot be expressed as a ratio of two integers, *a* and *b*, where $b \neq 0$.
2. An infinite, non-repeating decimal.
3. A Dedekind cut, denoted by (A_1, A_2), is a partition of the rational numbers into two non-empty sets, A_1 and A_2, such that all the members of A_1 are less than (or lie to the left of) all of the members of A_2 and such that A_1 has no greatest element and A_2 has no least element.[3]
4. Equivalence classes of Cauchy sequences. Although Cauchy sequences look like they converge, their terms getting closer and closer together, they do not necessarily converge in the rationals. Cauchy sequence do, however converge in the reals.

[3] It should be noted that Dedekind cuts are used to represent rational numbers as well. When a cut does not correspond to a rational number, we create the irrational number which produces the cut so as to "fill in the gaps" between rational numbers.

It is remarkable that mathematicians may plainly see connections among his or her notations and various definitions that he could not recognize among his thoughts. One may ask if the knowledge of these connections previously existed in the mathematician's mind, perhaps on some unconscious level, or if it is only contained in the symbols themselves. Descartes uses the example of symbolic algebra, saying,

> *By this device not only shall we economize our words, but, which is the chief thing, display the terms of our problem in such a detached and unencumbered way that, even though it is so full as to omit nothing, there will nevertheless be nothing superfluous to be discovered in our symbols, or anything to exercise our mental powers to no purpose, by requiring the mind to grasp a number of things at the same time. (Descartes, 1970, p. 67)*

The truth is that notation lightens the cognitive load on our minds. Any mathematical symbol is open to multiple interpretations, e.g., –3 stands for both the process of subtracting three and the negative number. In effect, the signs do some of the work for us, but only after we can hold the process and object in mind simultaneously without contradiction. The hard part is to come to see –3 as a number, in and of itself, not just the procedure of subtracting three or taking the opposite of the number 3.

The use of multiple representations gives rise to ambiguity, and ambiguity gives rise to new knowledge through acts of incredible creativity. Ambiguity arises when we can understand one thing in two or more possible ways. Typically, when we use the term "ambiguous" we imply a lack of certainty. Each of the modes of understanding may be perfectly acceptable when taken alone, but are mutually incompatible. One may think that mathematics, with all its rigour and concern about contradiction, would be free from ambiguity. The above discussion of multiple representations in mathematics should suggest otherwise. In fact, William Byers (2007) credits ambiguity for some of the creative development of mathematics. Viewing one idea in multiple ways is to be seen as an opportunity for deeper understanding not as a challenge to attain certainty by rejecting one side or the other. In looking for a unique mode of representation for all of mathematics, we lose sight of the richness that the ambiguity of multiple representations has to offer. It is in this regard that mathematicians and philosophers who wish to reduce mathematics to logic, arithmetic, geometry, or any other foundation make their biggest mistake.

The strange thing about ambiguity is that we can come to reconcile, within our own understanding, the many ways of conceiving of a mathematical object or concept. In effect, it becomes no longer ambiguous to us when we perceive the different representations as complementary and not as self-sufficient and excluding each other. One of the dangers in this, for teachers, is that we can forget that students are still struggling with the discomfort that ambiguity can cause. Teachers cover what students ought to learn, while students are "trying to grapple with the more fundamental issues of learning our language and guessing at our mental models." (Thurston, 1994, p. 42) There is a difference in how we write mathematics and how we think and speak about mathematics. Writing mathematics typically employs a formal use of symbols, when we in fact use natural language and metaphor when we are thinking about mathematics. The verbal communication of mathematics also necessitates the use of natural language and metaphor, as well as a very dynamical use of a blackboard. Written mathematics can appear to have lost these human qualities. We will now identify the ways in which written mathematics can be interpreted.

2.1 Interpreting mathematical signs: Peirce's index, icon, and symbol

We will use C. S. Peirce's (1885) classification of signs into indices, icons and symbols to describe the participants' interpretation of number signs in the questionnaire.

Peirce's classification is based on the relationship that the interpreter perceives between the signifier and the signified. A sign is interpreted as an index if there is a perceived contingency between the signifier and the signified. The most literal example of index is the interpretation of a gesture of pointing the index finger at something as referring to this something. A more abstract example of an index is interpreting the proper name of a person, called out or written, as referring to that person. In mathematics, the Greek letter π, or the letter e are often interpreted as proper names of concrete numbers, and in this sense they are interpreted as indices. Another common example of index is interpreting certain signs as symptoms of something physically associated with it; for instance, a low red blood cells count indicates (is a symptom of) anemia in the patient. Some participants in our research appeared to interpret the mark "√" in an expression as a symptom of being irrational; we therefore described their interpretation of such number signs as indices.

A sign is interpreted as an icon if the signifier is perceived as resembling some aspect of the object for which it stands. A common example of icon is a photograph of a person as representing that person. This kind of physical resemblance cannot, of course, be obtained in mathematical representations, but still drawings in geometry, graphs, or even arrow diagrams are customarily interpreted as having an iconic relationship with abstract objects such as geometric figures, functions, or relationships between homomorphisms. They are interpreted as resembling the mathematical objects in quality, in relations to itself, or in relations to something else in a way that can be completely indistinguishable from the object itself. Icons are very much dependent on the person's knowledge and experience. Among the participants in our research, a couple (#9, #22) appeared to interpret expressions written as two number signs separated by a bar as fractions, regardless of their value. The number $\sqrt{2}/\sqrt{8}$ was a "fraction" because it looked like one, and not because its value was equivalent to ½. Interpreting expressions of the form *a/b* pictorially or iconically, could, in principle, lead students to classify, for example, the irrational number $1/\sqrt{2}$ as rational.

Finally, a sign is interpreted as a symbol, if the perceived relationship between the signifier and the signified is based on a cultural convention, not contingency or resemblance. Peirce (1885) actually used the word "token" to describe this relationship: in a symbolic relationship, a sign is interpreted as a token of something else, although it does not resemble the signifier nor is contingent with it. For example, a bouquet of flowers can be offered to someone as a "token of appreciation"; it is by way of a cultural tradition that offering flowers symbolizes appreciation. It would be difficult or impossible to advance in mathematics in any way without perceiving the mathematical signs as symbols of ideas that have little to do with the way the signs representing them look like. The symbolic association is a matter of convention within the mathematical culture. Symbols rely on formal and structural relations and are general in nature. At first glance this may appear to be necessarily a very deep way of interpreting mathematical signs. But we must not assume that the student who interprets a sign symbolically is doing so because s/he recognizes the underlying mathematical structures and relations at play. Take the example of the expression *x is a real number*, used in the context of describing the domain of a function. Mathematicians and teachers could read this as allowing the independent variable to assume any value from the field of real numbers, both rational and irrational, without restrictions, but not values from the field of complex numbers. To the high school student, however, the statement "*x is a real number*" could be interpreted as saying nothing more than "*x is a number*", or even as a conventional phrase to be written after the formula of a function. "Real" could be seen as merely something mathematicians say of numbers. Some students will mimic our habits without knowing why we behave the way we do; this is, in general, how people (children and adults) learn to participate in a culture.

One important aspect of symbols is that they come into being and make sense only in their relation to other symbols; symbols combine according to the conventional rules (of grammar and usage in ordinary language; of mathematical notation, logic and usage in mathematical language) and thus produce other symbols. If a person is processing a mathematical expression according to certain rules – calculating, simplifying an expression, deducing general characteristics of the concept from a given representation – we can consider this as a symptom of his or her interpretation of the sign as a symbol. Again, this can be deep in nature or superficial. A student may rationalize the denominator of a fraction automatically without much thought or reflection. To do so is an algorithmic convention, a form of rule-following which is nevertheless a symbolic interpretation of the sign.

As we see in the example ($x \in \mathbf{R}$) above, a representation of a mathematical object may be symbolic in one respect and iconic in another depending on the context of the problem. If we interpret a sign symbolically, we tend to also be able to see it iconically and, necessarily, some aspects of the representation will function as indices for us if we are to refer to the object or talk about relations among signs. Although symbolic notation may be manipulated in a purely formal way, it requires the addition of a natural language if it is to be truly meaningful, as indices and icons come into play in our natural language. (Grosholz, 2007, p. 127)

Generality cannot be achieved without symbols,

> *[t]he truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (Peirce, 1885, p. 182)*

It is therefore advantageous to avoid creating a dichotomy between formal symbols on the one hand and the metaphor of icons on the other.

2.2 The making of a concept image

An individual's concept image of a mathematical idea or object is something that grows and changes over time as the individual acquires new knowledge, representations, and examples with which to work.

> *"We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes." (Tall and Vinner, 1981, p. 152)*

The pictures, properties, and processes to which Tall and Vinner refer may be conscious or unconscious parts of the concept image. "At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked *simultaneously* need there be any actual sense of conflict or confusion." (ibid.) It is an important consequence that mathematical objects are therefore more than the sum total of their representations. A triangle is more than lines and angles. But these mathematical objects also cannot be distinguished from their representations.

Given that mental representations and concept images of mathematical objects differ from person to person, can we really have objectivity in a science that largely deals with intangible things? This is a question of much debate. On the one hand, we have mathematical objects which cannot be perceived using the senses; only reason can access them and reason is an internal and highly subjective process. On the other hand, there has been an enormous wealth of mathematical knowledge that has been handed down to us through a chain of mathematicians (though not necessarily a continuous or linear chain). We are still doing now the same mathematics that was done by the ancient Greeks! Can we reconcile the inherent subjectivity of individual reasoning with the apparent objectivity of mathematical knowledge? (see, for example, White, 2006; Boyer, 2007, p. 338-343.)

Mathematical objects are born of human thought and activity and humans are fallible creatures; our thoughts are subjective. And it is owing to the different interpretations of the same things by the same or different mathematicians that new theories or links between theories are made. Concept images differ from person to person, but the underlying concepts are invariant across all representations, or at least we hope they are. It is in the sense of there being a unifying, invariant concept underlying the multiple representations we use that mathematical objects remain the same for everyone and, ultimately, achieve a sort of objectivity. Thought content and knowledge becomes something that is inter-subjective, and in fact only objective when we phrase them linguistically or represent them symbolically. Mathematics can therefore be said to have an external existence which is, however, internal with respect to the mathematical society at large.

When we talk about the invariant underlying concept, we are speaking of mature, well-defined mathematical concepts, not the concept images that students construct on their educational path. I do not mean to imply that there is a correct concept that one must strive to obtain. Concepts are such that they are always developing, becoming more nuanced, and are often times inaccessible. Our concepts are only accessible in so far as we can articulate our thought processes and relations among thoughts. This requires a good deal of self-awareness and reflection that not everyone is capable of or willing to engage in. When I say well-defined and mature concepts, I refer to concept which is general and relatively invariant, for which we have multiple representations, notations, and accepted conventions all of which are held together by the concept. In short, it is what mathematicians work with and can work with because of this relative objectivity. Mathematicians and successful students of mathematics must be able to call upon varying representations as the need arises, all the while keeping in mind the underlying concept which holds it all together. However, learners who do not yet possess mathematical maturity are unaware of the structure and relations among mathematical ideas. The concept is unknown for them and their concept image is dynamical and often wildly contradictory in nature. Everyone's concept image changes as new representations are met (unless the representation is rejected outright) so the concept image is always developing, but for students this is not a stable process and they may cling to one representation or another in an effort to avoid the discomfort of dealing with the feelings of ambiguity and contradiction. A concept image may become inextricably linked to a particular representation so that, in effect, the concept image is exactly the representation. This can pose an obstacle to deeper understanding. A certain level of sensitivity to ambiguity and some flexibility is required to overcome this.

2.3 Modes of thought

Concept images are dependent on the individual's experience, perceptions and interpretations. That is to say, concept images are dependent on how the individual thinks and reasons. It is therefore important for us to examine the different modes of thought that students use when reasoning about mathematics. We will borrow ideas of Lev S. Vygotsky's theory of cognitive development wherein he distinguishes between different modes of thought. (Vygotsky, 1987) We will examine Vygotsky's modes of thought with respect to mathematics following Anna Sierpinska's (1994) and Shlomo Vinner's (1997) leads.

Vygotsky used, what he called, the *zone of proximal development* (ZPD) in his analysis of cognitive development. He noted a discrepancy between what the child is capable of doing independently and what he or she can do with the guidance of a mentor. The ZPD is defined as the area of maturing psychological processes; the ZPD separates actual from potential development. Recognizing the existence of maturing processes has great implications for instruction and assessment. The only useful form of instruction, according to Vygotsky, is that which awakens these maturing processes. (Driscoll, 2005, p. 245-261)

One of Vygotsky's techniques involved introducing obstacles that disrupt the problem solving process. In the mathematics classroom we can try to exploit the inherent ambiguity in mathematical ideas. But in Vygotsky's theory of concept development, the most important tool is language in general. Language is a sign-using behaviour. It is a cultural activity that is rooted in the social history of the community. It serves to organize experience and free the individual from their physical environment. It is thanks to language that we can think and reason in the absence of concrete examples. Learning the language of mathematics, not just the signs that are used, but the manner of discourse, can help the learner to think in mathematical concepts instead of examples. Written and verbal activity in the classroom promotes reflection and self-regulation and is essential for the encapsulation of mathematical procepts and conceptual thinking. Language is, therefore, essential if the student is to make the qualitative jump from complexive thinking to conceptual thinking.

To think in complexes is to reason using concrete, factual connections between objects rather than logical ones. *Complexive mode of thought* organizes discrete elements into a group; each element has a common characteristic with another but that commonality continues to change.

> *"No single feature abstracted from others plays a unique role. The significance of the feature that is selected is essentially functional in nature. It is an equal among equals, one feature among others that define the object." (Vygotsky, 1987, p. 140)*

In the worst case scenario the student creates a chain of objects focusing on how the last is similar to the next while ignoring, or not being aware of, contradictions that arise from this behaviour. (Sierpinska, 1994, p. 147) As we noted above, contradiction may only be observed by the individual if he evokes two conflicting aspects of his concept image simultaneously. Using a complexive mode of thought makes it difficult to create a meaningful framework for one's concept image. At some point, the goal is for the student to make the cognitive jump from thinking in these organized complexes to making broader generalizations which lead to conceptual thinking.

The *conceptual mode of thought*, with respect to mathematics, is thinking in abstract, logically coherent ways which are consistent internally, if not with respect to mathematics at large. The conceptual thinker thinks in terms of ideas and relations among ideas. Objects are grouped in classes and membership in the class is governed by a certain set of characteristics which all members must have. Isolated facts about particular elements do not factor in decisions about the class itself. For example, within the class of irrational numbers there are numbers expressed as decimals, numbers expressed as square roots, numbers expressed as arcsines, etc. However, no particular representation is enough to classify a given example as irrational as is the case in complexive thinking. It is also important to note that the conceptual mode of thought is not necessarily linked to the correct concepts. An individual can succeed to think conceptually while using concepts that are in contradiction with the mathematical convention and theory.

In the progression of concept development we have complexes on the one hand and concepts on the other, but we will take care not to force ourselves into a false dichotomy. Not all thinking is either complexive or conceptual. Vygotsky accounts for a middle ground that is referred to as the pseudo-conceptual mode of thinking.

> *"A dominant feature of the pseudo-conceptual thought processes is the uncontrolled associations which fail to become a meaningful framework for further thought." (Vinner, 1997, p. 103)*

On the surface, the pseudo-conceptual mode of thought looks very much like the conceptual mode of thought. Although the resulting concept may be the same, the process by which the student arrives at the concept is different from that of the conceptual thinker's process. Pseudo-conceptual thought is based on classes like conceptual thought, but belonging to a class is based on fact instead of logical ties. Therefore the elements of the class may not be so very consistent with the referential name of the category to which they belong. (Hardy, 2009, p. 38-39) A student may base his or her decisions on one condition, perhaps a superficial similarity and therefore appear to be using a concept instead of a complex which uses multiple, changing criteria.

> *Sometimes they are the natural cognitive reactions to certain cognitive stimuli. The students use them without going through any reflective procedure, control procedure or analysis of any kind. (Vinner, 1997, p. 101)*

Rote learning is therefore a symptom of pseudo-conceptual thought processes.

2.4 Operationalization of the theoretical framework

In analyzing participants' responses, we looked for evidence on how they interpreted the number signs they were asked to classify or reason about, and we attempted to assess the mode of thought they used in responding to the questionnaire. To be able to use Peirce's and Vygotsky's theories consistently in our analyses we needed to operationalize the definitions of the three ways of sign interpretation and the three modes of thought. In this section, we present the criteria we used in the operationalization.

2.4.1 Operationalization of Peirce's classification of sign interpretation

There is a need to operationalize Peirce's classification of sign interpretation for the purpose of data analysis. Below we offer the guideline we followed in determining how the student interprets number signs.

Index: If the student used a number sign declaratively, as names for the value of a number without deducing any facts about that object, they were said to be making an indexical interpretation. Additionally, if they viewed non-digit characters, like "√" as symptoms of irrationality (not as denoting families of expressions (icon), and not as an instruction to perform an operation on a calculator (symbol), etc.), then they were also interpreting "√" as an index.

Icon: Iconic interpretations were found in students who based their decisions on the form of the representation without regard for the value of the number represented. If students based their decisions on visual characteristics of the various representations and visual similarities among examples, they were said to be using an iconic interpretation.

Symbol: Symbolic interpretations were found in the students who made explicit algebraic manipulations or performed (or would perform) operations on a calculator. Also, any student who deduced properties of the object from its representation was said to be making a symbolic interpretation.

We now offer an example that came up in the students' answers and interpret it in the three ways we have identified.

Example: $\sqrt{2} \times \sqrt{2} = 2$

Indexical interpretation: "$\sqrt{2} \times \sqrt{2}$" *is* "2." There is no calculation performed; the product (an object in this case) is simply recognized to be the number 2.

Iconic interpretation: In the course of learning about operations involving radicals, the student may recall hearing or have formed the thought: “when you multiply two identical square roots, you just *get rid of the root sign.*” This is to view the equation as an image or diagram for the rule. The following diagram of the same equation is no doubt seen in high school algebra classes: $\cancel{\sqrt{}}2^{\cancel{2}}$, where the square and the square root are “cancelled out.”

Symbolic interpretation: The student performs a calculation, albeit a simple one, or deduces properties of inverse functions from the equation.

2.4.2 Operationalization of Vygotsky’s modes of thought

There is a need to operationalize Vygotsky’s modes of thinking for the purpose of data analysis. We need to be able to decide which mode of thought the student is using. The following is a guideline for how we went about our decisions.

Complexive thinking: The student will be said to use the complexive mode of thought if he or she bases their decisions about rational, irrational, and real numbers on a variety of criteria. No conscious link is made among like examples. Each new example is considered on its own merits.

Pseudo-Conceptual thinking: The student will be said to use the pseudo-conceptual mode of thought if there is evidence that his or her decisions are based on a specific set of criteria, for example, being irrational is to have infinitely many digits after the decimal point. However, decisions about concrete examples are made for factual reasons which are sometimes, but not always, in keeping with the underlying concept.

Conceptual thinking: The student will be said to be using the conceptual mode of thought if he or she uses a logically consistent method of classification. Decisions about concrete examples are consistently and consciously made by reference to his or her criteria of classification.

Chapter 3. Meaning of Rational, Irrational, and Real Numbers in Mathematics and in Teaching

For the mathematics of every day life rational numbers may suffice. We need to know how to count, estimate, and perform simple arithmetical calculations. Indeed, calculators and other technologies are necessarily limited to rational numbers; we cannot express a number whose representation is infinite (an infinite decimal) with a finite instrument in a finite amount of time. Furthermore, in common dealings with money and time we only work with decimals with at most two digits after the decimal point. Aside from some algorithmic knowledge, there is a need for achieving an understanding of proportions as well. Thinking in proportions is a conceptual mode of thought and much of the school mathematics curriculum is geared at developing the skill of proportional reasoning. Proportional reasoning is the ability to compare two things (e.g., rates, ratios) using multiplicative thinking.

In mathematics, the natural numbers, the integers, and the rational numbers grew out of a need for measuring magnitudes and multitudes. The irrationals numbers did not arise out of some practical problems of measurement. Rather, they are a theoretical by-product of the need for consistency and completeness in mathematics. The irrational numbers are responsible for the intuitive continuity of the real line. The way we think about irrational numbers is conceptually different from the way we think about rational numbers, and our teaching practices should reflect this.

In the sections below we will first look at the way that mathematicians formalize the concept of irrational numbers. There was a need to step away from the usual way of looking at numbers as multitudes and magnitudes and think of irrational numbers in terms of continuity instead. The set of irrational numbers is the part of the real number system which is responsible for the intuitive continuity of the real number line. Next we will examine the approach to irrational number in two different university level classes in real analysis. Finally, we will explore the goals of the pre-university mathematical curriculum.

3.1 In the field of mathematics

In 1858, while teaching differential Calculus, Richard Dedekind felt compelled to find a purely arithmetical definition of continuity. The reliance on geometry and geometric intuitions in dealing with limits and continuity was useful and necessary didactically speaking, but Dedekind considered geometry to be lacking as foundation for the principles of infinitesimal analysis. Usually, the irrational numbers are introduced via measuring. Instead of using magnitude and incommensurability as a vehicle for introducing the irrational numbers, Dedekind endeavoured to develop arithmetic "out of itself" thus defining irrational numbers, and thus the real numbers, by means of rational numbers alone.

Arithmetic can be thought to evolve naturally from the act of counting. Addition can be seen as the combining of counting actions. To my left, I count three people; to my right, four. Combining these separate acts of counting yields one act of addition, which we may express as, $3 + 4 = 7$. Counting to three and then counting another four is the same as counting to seven. Multiplication then arises from counting groups. If I count three groups of people, and then count four people to each group, I can combine these two counting acts into the one operation of multiplication as $3 \times 4 = 12$.

We can always perform these two operations on the natural numbers without restriction. The same cannot be said for their inverses, subtraction and division. With only the counting numbers to work with, subtraction is restricted by the size of the numbers; we can only take a smaller number from a larger one. To lift this restriction requires an act of creativity and what is created in the process are the integers, i.e., the positive and negative numbers together with zero. In the realm of the natural numbers division is restricted to the factors of the dividend if the division is to yield a natural number. Lifting this restriction again requires a creative act and what results are the rational numbers, parts of the whole. (Of course, one restriction remains, that of not dividing by zero.)

As we have said, the lifting of these restrictions is not an intuitive process, but a creative act. Any creative act which causes great change tends to meet with resistance, and the ideas in mathematics are not immune to this. For example, zero found easier acceptance in the East than the West because western philosophy and religion made it very difficult to accept zero as a number. (Siefe, 2000) Negative numbers once had no practical purpose. (Siefe, 2000, p. 70.) The famous philosopher and mathematician René Descartes referred to negative roots of equations as “false roots.” (Siefe, 2000, p. 133.) The Pythagoreans are famously said to have drowned a man for letting slip the secret of $\sqrt{2}$, an irrational number. Even the 19th century mathematician, Leopold Kronecker, is quoted to have said, “God made integers; all else is the work of man.” A creative act was needed to extend the rational numbers to the real numbers and that act was to abandon the discrete foundation of numbers. (Lehtinen *et al*., 1997, p. 135)

Students spend so much of their early mathematical education building a concept image of number which rests on ideas of measuring and counting. "For mathematicians, the hierarchical construction of numbers is logical and coherent because they are already familiar with the structure." (Merenluoto and Lehtinen, 2004, p. 522.) This is not the case for students who see inconsistency when discreteness is abandoned.

Irrational numbers, which can be simultaneously viewed as an infinite process (an infinite sum of rational numbers, infinite decimal expansion) and as a mathematical object (a magnitude, a number), i.e., as a procept (Gray & Tall, 1994), have challenged learners from ancient times through to today. Many ancient cultures (e.g., Indian, Greek, and Chinese) used approximate values for numbers we now know to be irrational. In ancient Greece, the Pythagoreans assumed that nature had an underlying mathematical structure that described and explained the world we perceive using natural numbers and ratios of these. It was the Pythagorean Theorem, which relates the lengths of the legs of a right triangle, *a* and *b*, with the hypotenuse, *c*, by the equation $a^2 + b^2 = c^2$, that served as one of the first links between arithmetic and geometry. This simply stated theorem led Pythagoreans to the discovery of incommensurable lengths. Two (or more) lengths are said to be *incommensurable* if they cannot be measured by a common unit. Therefore, the ratio of incommensurable lengths cannot be expressed as a ratio of whole numbers, i.e., the ratio is an *irrational number*. Irrational numbers are "the numerical representation of incommensurability." (Fischbein *et al.*, 1995, p. 40)

Initially, Greek mathematicians rejected irrational quantities and did not consider them to be numbers at all. This denial was part of the more encompassing rejection of infinity as an object, in and of itself. For the Greeks, potential infinity did not pose any problems and they exploited limiting procedures in their mathematics. Infinity was something procedural or algorithmic in nature: a process only, not an object. That is to say that, for the Greeks, infinity was not a procept and this hindered the development of flexible thinking in mathematics in their time. Actual infinity was problematic for Greek mathematicians and philosophers, as is cleverly illustrated in Zeno's paradoxes of motion.[4] (Tall & Tirosh, 2001) Through time irrational numbers became more used in arithmetic and algebra and began to gain some acceptance among mathematicians. It was only in the 19th century, however, that irrational numbers were finally given a formal mathematical definition, by means of Dedekind cuts, centuries after the Pythagoreans first discovered their existence.

It is seemingly paradoxical that although the rational numbers form an infinite dense set,[5] they do not cover the whole real line. In fact, there are an even greater number of irrational numbers than rational ones. Dedekind himself said, "Of greatest importance, however, is the fact that in the straight line L there are infinitely many points which correspond to no rational number." (Dedekind, 1901, p. 4) Dedekind puts forth the following axiom over the rational numbers:

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[4] Perhaps the most famous of Zeno's paradoxes is that of Achilles and the Tortoise. Achilles and the tortoise are to have a race and the tortoise is given a head start. The paradox is that Achilles will never catch the tortoise despite being the faster runner. The reasoning is that Achilles must first achieve the distance to the tortoise's starting point while the tortoise moves on. Now, Achilles must cover this added distance, but the tortoise is again further ahead. The tortoise continues to be ahead by less each time and Achilles cannot catch up.

[5] Between any two distinct rational numbers, there is another and therefore infinitely many.

> *If all points of the straight line fall into two classes such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions. (Dedekind, 1901, p. 5)*

Dedekind claims that this statement, which is obviously true of the set of rational numbers, is the essence of the continuity of the set of real numbers, **R**. The set of rational numbers, **Q,** is discontinuous. Thus the continuity of the real line depends on the creation of the irrational numbers.

A Dedekind cut on the set of rational numbers, denoted by (A_1, A_2), is a partition of the rational numbers into two non-empty sets, A_1 and A_2, neither equal to **Q**, such that all the members of A_1 are less than (or lie to the left of) all of the members of A_2. He showed that, for some cuts, there is no largest member in A_1 and no smallest member in A_2. For example,

$$A_1 = \{x \in \mathbb{Q}: x \leq 0 \; or \; (x > 0 \; and \; x^2 \leq 2\}$$

$$A_2 = \{x \in \mathbb{Q}: x > 0 \; and \; x^2 > 2\}$$

is a cut of this type and defines the number $\sqrt{2}$. Dedekind (1901) called cuts such as these irrational numbers.

What Dedekind achieved was a strictly formal, arithmetical way of defining irrational numbers which is divorced from the geometric intuition of “continuity of points on the number line.” Understanding this definition, however, requires a level of mathematical culture that is not expected of secondary school students. Indeed, many mathematicians themselves think of irrational numbers in geometrical, rather than arithmetical terms. However, it may be useful for school teachers to at least be aware that such a definition exists.

3.2 In the classroom

In the introduction to his book **R** *is for Real*, David Wheeler (1974) remarks that, in the classroom, numbers are always accompanied by rules for their proper use. These rules are usually not up for discussion and little if any motivation is given for why things are as they are. Often times the rules are an obstacle to obtaining a deeper understanding of the subject matter itself. Mathematics becomes something you do, i.e., a series of algorithms, instead of something to think and reason about.

Real numbers are present in mathematical calculations and other classroom explorations whether acknowledged or not. To not acknowledge them is to allow, even force, students to create their own independent concept image of real number. If history's greatest mathematicians had difficulty doing so, aren't we expecting too much of our students? Therefore there is an advantage to studying the concept of number itself and how it changes alongside numerous important mathematical advancements. At the very least it will illuminate for the student how mathematicians actually work using flexible and conceptual modes of thinking. At best it can help the student to do the same.

The ability to apply definitions and differing representations is useful not only in mathematics but in law, science, and all fields of creative endeavour.

3.2.1 At the university level

Although some familiarity with irrational numbers may be assumed of the student in all mathematics classrooms at the university level, it is in a first Real Analysis course that the student is likely to first encounter a formal introduction to the real number system. However, textbooks vary widely in their treatment of the development of the concept of real numbers themselves, ranging from a cursory review on the one hand to a thorough account on the other. We will look at an example of each.

Ethan Bloch (2011) gives a very thorough account of the construction of the real number system in his real analysis textbook. The reader is directed to one of three starting points in this construction:

1. An axiomatic treatment of the natural numbers, which leads to the construction of the integers, rational and finally real numbers.
2. An axiomatic treatment of the integers, which are shown to contain the natural numbers, and construction of the rational and real numbers.
3. An axiomatic treatment of real numbers and demonstration that the natural numbers, the integers and the rational numbers are contained in the set of real numbers. (Bloch, 2011, p. 1)

The third starting point is the most common in textbooks of a first course in Real Analysis as it is "the most efficient route to the core topics of real analysis, but it gives the least insight into the number systems." (Bloch, 2011, p. 1) To round off his treatment of the number systems, he even includes, in chapter one, a section on the history of the development of numbers. The major players are named and Bloch briefly addresses the motivations behind the development of the number concept and some differing philosophies of thought through the years.

Second, in contrast, we will look at a textbook which takes a cursory approach to conceptualizing real numbers. Russell Gordon (1997) remarks in the preface of his textbook that a book on real analysis should, among other things, keep introductory material, the material which is peripheral to real analysis, to a minimum. (Gordon, 1997,

p. v) From his brief acknowledgment of the nature of real numbers, I can assume that their development is such material. Aside from a couple introductory remarks to the student about the language and terminology we use to describe the real numbers, only the first section of the first chapter, a total of six pages, is devoted to what exactly a real number is. In these few pages he quickly mentions:

1. the natural numbers,
2. the integers,
3. the rationals and their inadequacy for measuring all possible lengths,
4. proof by contradiction that $\sqrt{2}$ is irrational,
5. decimal representations including examples of how to transform a fraction into a decimal number and how to transform a repeating decimal into a fraction,
6. representing numbers as points on a line,
7. Dedekind cuts with a reference for further reading, and finally
8. the author arrives at his preferred way of thinking of real numbers as a set of numbers satisfying certain properties, i.e., the axiomatic definition of **R** as a field.

From this brief introduction to real numbers the author moves on to discuss other properties like completeness, density and (un)countability before moving on, in subsequent chapters, to the featured material of the book.

Both books are adequate for studying real analysis, but only the first gives the student a chance to create a concept image of real numbers which is coherently connected. For the student who goes into the class with a well developed concept image and who is already using a conceptual mode of thinking, the second book gets right to the point and wastes no time. However, this student is the exception, not the rule.

3.2.2 At the pre-university (secondary) level

The Ministère de l'Éducation, du Loisir et du Sport (MELS) published a document which outlines the learning goals of mathematics education at the secondary level. According to this document, the mathematics curriculum in Quebec involves an explicit study of natural numbers, integers and rational numbers. The MELS goals for the first two years of secondary school (cycle 1) are for students to develop a number sense, perform operations on both positive and negative decimal and fractional representations, to achieve proportional reasoning skills, to make graphical representations, and to solve for unknowns in algebra and geometry. In the last three years of secondary school (cycle 2), students are expected to "assimilate the concept of real numbers (rational and irrational), particularly in situations involving exponents, radicals or logarithms." (MELS, 2010, p. 6) Although the tables provided in the MELS document do mention irrational (and real) numbers in the curriculum outline, they are not given the same attention as the other sets of numbers. We can only presume from this the irrational numbers are not taught in the same way as the other number systems. "Real numbers" may be briefly introduced to students as "all the numbers you know" or "all numbers, rational and irrational".

Students spend a lot of time in elementary school working with the counting numbers and fractions. Knowledge of the natural numbers, integers, fractions, and finite decimals is assumed of students entering secondary school and they are expected to reinvest this knowledge by making connections between concepts and processes as they continue their mathematics education.[6] These sets of numbers are treated to an itemized

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[6] See Appendices A and B for the MELS tables which outline the learning goals.

breakdown in the outline for understanding real numbers and operations involving real numbers; goals with respect to reading, representing, approximating, comparing, calculating, simplifying, etc are made explicit.[7] However, this is not the case for irrational numbers.[8] Students in cycle 2 are expected to "distinguish rational numbers from irrational numbers in the set of real numbers" (MELS, 2010, p. 8) but there is no indication of how they are expected to do so. For instance, within the section on fractions[9] there is a secondary school goal to identify the "meanings of fractions: part of a whole, division, ratio, operator, measurement" (MELS, 2010, p. 7) which could tie nicely into the section on decimals and their meanings and characteristics via the algorithmic task of transforming fractions into finite or repeating decimals and vice versa.

However, attaching meaning (other than as approximations) to decimal representations is not a goal of understanding decimals. The goal with respect to decimals and fractions is to be able to switch, as needed, between the two representations. (MELS, 2010, p. 11) But this is an operational goal, not a conceptual goal of understanding the representations, i.e., it is a rule to which meaning may go unattached. To be very clear, in the curriculum as it is outlined in the MELS document about the progression of learning mathematics there is no explicit mention of the relationship between fractions and finite decimals, fractions and infinite repeating decimals, and fractions and infinite decimals. This is a "missing link" in the curriculum which has been identified in the literature among students' understanding and processing of irrational numbers. (Sirotic and Zazkis, 2007a)

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[7] See items 1, 2, 3, and 4 in Appendix A and also items 1, 2, and 3 in Appendix B.
[8] See item 8 in Appendix A.
[9] See item 2b in Appendix A.

All of this is not to say that individual teachers do not address irrational numbers in their lesson plans. I'm sure there are some that do, and do so well. The point is that failing to make explicit goals regarding the understanding and use of irrational numbers is to de-emphasize their importance in mathematics and as a tool to learning different conceptual modes of thinking. While rational numbers are adequate for life's daily tasks, and rational approximations will suffice in applications, students who pursue sciences in college and university will be expected to understand and work with real numbers in general. It is therefore important at some point in the secondary curriculum to introduce the irrational number concept. This should not be left until a first course in real analysis, as there are many students who will not take this course, yet will continue on into engineering, sciences and computer programs which rely, if not on irrational numbers, at least on a distinction between them and their rational approximations.

Considering how heavily students rely on technology, it is important to understand how the technology works and what its limitations are. In secondary school, "emphasis is placed on technological tools, as these not only foster the emergence and understanding of mathematical concepts and processes, but also enable students to deal more effectively with various situations." (MELS, 2010, p. 5) I would argue that there is no point in emphasizing technological tools if we do not at least acknowledge, and preferably discuss, both the advantages and disadvantages to having the tool and using it. I would also argue that calculators make students more efficient in various situations, but not necessarily more effective. In the case of irrational numbers, relying on a calculator is always inaccurate and inadequate. The act of approximating has literally become a black box phenomenon; it often goes unnoticed. Approximations are necessary, but it is a mistake to make the act of approximating automatic without conscious awareness.

Chapter 4. Methodology

The participants in this study were 30 fourth semester science students at a college[10] in the Montreal region who were currently enrolled in a class which taught computer applications in mathematics. They were given a questionnaire consisting of seven questions aimed at testing students' knowledge of irrational numbers and the real number system, and the limitations of technology in representing irrational numbers.

The questionnaire was administered during class time and no student took more than 30 minutes to complete the task.[11] Previous studies concerning the understanding of the irrational number concept typically involved high school students and pre-service or in-service teachers. Assuming that science students take more math classes and at a more advanced level than the average student teacher, one might feel safe in assuming that they would fare better on questions about the real number system. For this reason, I developed a questionnaire that borrowed questions from previous studies, so as to be sufficiently similar in task, but also incorporate some original questions geared at the participants' specific skill set.

4.1 The tasks

The first three questions were aimed at establishing the participants' basic knowledge or irrational numbers and the real number system. They were asked first to define what an irrational number is. Next, they were asked to classify a sample of eight numbers as

[10] In Quebec, students attend college after high school (école secondaire) and before going to university.
[11] You may view the complete questionnaire in Appendix C.

rational, irrational and/or real. In previous studies (e.g., Arcavi *et al.*, 1987; Pinto and Tall, 1996; Zazkis and Sirotic, 2004), it was often observed that the definition provided and the manner of classifying concrete examples did not always agree. Thirdly, the students were explicitly asked if the integers and the irrational numbers were real numbers. This was to determine if the mathematical term “real” had any meaning for them, mathematical or otherwise.

Question 4 was inspired by the work of Arcavi *et al*. (1987) who found that participants (in-service teachers) frequently misclassified 22/7, a common approximation of π, as an irrational number. The participants did not have a clear distinction between a rational approximation, 22/7, with the irrational number it approximates, π. Arcavi *et al.* wondered if other rational approximations would be regarded as an irrational number, but it was too late to incorporate this type of question in the above mentioned paper. To see if it was something special about π, I asked the students to consider a rational approximation of $\sqrt{2}$.

Question 5 asked if the intersection of the rational numbers and the irrational numbers is non-empty. Essentially, they were being asked if a number can be both rational and irrational, but the question was asked in such a way (using set theoretic language) that would also give a sense of their mathematical maturity and their understanding of the definition of irrational numbers and of the real number system.

Given that the participants were science students who were taking a computers course, question 6 was aimed at that specific skill set. According to the professor, the first topic covered in the class was concerning the limitations of technology in representing real numbers. They were therefore explicitly asked whether they could represent an irrational number with a calculator.

Finally, question 7 asked if an irrational number would be obtained by adding or multiplying irrational numbers. The question was formulated as in the study by Sirotic and Zazkis (2007a). This question was asked to determine the level of comfort and understanding the students had for irrational quantities as numbers in and of themselves. If the students understand irrational numbers in a formal or algorithmic way, they should be able to reason about operations on them. (Sirotic and Zazkis, 2007a) Question 7 could be answered by means of counterexample or by general reasoning about the nature of the numbers. Students, therefore, had two tools at their disposal: trial and error and reasoning, with which they might answer this question.

4.2 Interpretive analysis of the data

Interpretive research, also referred to as qualitative research, rejects the notion that the researcher and the research can be separated from that which is to be studied.

Interpretive analysis recognizes and even highlights the inter-subjectivity of the activity between researcher and subjects.

> *An interpretive researcher seeks to learn through systematic activity focused on efforts to understand the interactions between participants in social settings [the classroom] in terms of the perspectives of the participants. (Tobin, 2000, p. 488)*

There are many factors that carry influence: individual goals, context, beliefs, and behaviours to name a few. Where quantitative analysis may focus on generality and predictability, interpretive analysis shifts the focus and recognizes that understanding can be gained, not only from central tendencies, but from "data sources on the periphery of a community." (Tobin, 2000, p. 489) Outliers are not discounted. Instead, the balance between the mainstream tendencies and the outlying cases is explored.

The interpretive analysis for the thesis was performed by me and my supervisor. We analyzed the data set individually and in company, coming to an agreement through discussion of the various points of interest: explicit and implicit knowledge possessed by the student, student's interpretation of mathematical signs, reconstructions of students' concept images, and modes of thought used by the student in coming to a decision.

4.2.1 Explicit and Implicit knowledge and reconstructed concept images

Whenever possible there was an attempt made to reconstruct the student's concept image of rational number, irrational number, and real number. Some answer sets were so inconsistent and incoherent that no concept image could reliably be reconstructed. When we speak of consistency with respect to a student's concept image we are referring to a self-consistency which results from having a concept image that is free from variation. This does not imply that it is also consistent with conventional mathematics. Similarly, a student's concept image is coherent if it is clear, cohesive and conceptually connected. Again, it need not agree with mathematical convention.

The reconstructions were based on the explicit knowledge extracted from the answers to the questionnaire and the implicit knowledge that was gleaned from how the questions were explicitly answered. So for example, a student may answer explicitly that irrational numbers are real in question 3b, but fails to classify irrational examples in question 2 as real. This implies that the term "real number" is nothing more than an expression of the mathematical jargon. Another example: if a student explicity defines irrational numbers as not being expressible as a ratio of integers, but also classifies as expected the examples written in decimal form, then part of that student's implicit knowledge is the knowledge that, say, finite decimals are rational and infinite decimals are irrational. Note that we do not mention periodicity because there is no implication in the definition given or classifications made that the student understands the concept. Without further evidence, understanding of periodicity cannot be said to be a part of this student's implicit knowledge set.

4.2.2 Interpretation of signs

The operationalization, outlined in chapter 2, was used to perform the semiotic analysis. Recall briefly that indices are names or symptoms, icons are images or resemblances, and symbols are general and rely on formal or structural relations. Indexical interpretations were common. Most students used number signs as names for numbers and viewed non-digit characters, like "√" as symptoms of irrationality. That is to say that the presence of a square root sign overwhelmed the representation, so that the number was deemed irrational even when it was not, e.g., $\sqrt{2}/\sqrt{8}$. Iconic interpretations were seen among students who based their decisions solely on the form of the represention without conducting further investigation. Symbolic interpretations were found in the students' explicitily performing some calculation, e.g., simplifying the expression $\sqrt{2}/\sqrt{8}$, or in their deduction of properties of the object, the number, from the respresentation on logical or conventional grounds as opposed to visual ones.

Student's answers were examined for any evidence that they were interpreting mathematical signs as indices, as icons, and as symbols. We required there to be explicit interpretation of at least one sign as a symbol for us to say that the student uses that form of interpretation. There were, however, cases where an interpretation was implied, particularly when considering symbolic interpretations. In some cases the use of a symbol was isolated to one example, namely, $\sqrt{2} \times \sqrt{2} = 2$ (see the example in section 2.4.1 above) or $\sqrt{4} = 2$, which because of their simplicity can be explained by an iconic or indexical interpretation as well. In this case, we required more evidence of symbolic interpretation in order to make a definitive decision on the student's use of symbols.

4.2.3 Modes of thinking

Again, the operationalization, outlined in chapter 2, was used to perform analysis of students' modes of thought. Recall that the modes of thought we are interested in are the complexive, the pseudo-conceptual, and the conceptual mode of thought. Students' modes of thought were evaluated based on the decisions they made in classifying the particular examples in question #2 and from the types of reasoning they used to answer questions #4, 6, and 7.

We required there to be explicit evidence that a mode of thought was in use. If the answers were deemed insufficient for deciding on a mode of thought, we assigned that student to the lower mode of thinking. So, for example, a student might offer a defintion of irrational numbers and seem to use that concept and related concepts (for example, having infinitely many digits after the decimal point and the concept of being exact) in making his decisions. However, if there was confusion in the way the classifications were made and the student failed to give justifications for his decision the student was then said to be, for example, a complexive thinker instead of a pseudo-conceptual one.

Chapter 5. Results

This chapter will present our analysis of the participants' responses to the questionnaire, using the frameworks outlined in chapter 2. From Peirce's semiotic perspective, we will look at the participants' interpretations of number representations. Vygotsky's theory of modes of thinking will inform our assessment of the level participants' thinking about numbers. The results will be presented in a synthetic fashion, providing qualitative and quantitative information about the whole group. The results will be illustrated with examples of particular participants' behaviours and the way they were interpreted and analyzed. In our examples, we will use the generic "he" to refer to the participants, regardless of their actual gender. Gender was not a factor in our study. Individual participants' responses and their interpretation are available in the Appendices.

The chapter starts with summaries of participants' responses to each question (section 5.1). Section 5.2 contains the results of our semiotic analysis and section 5.3 – our analysis of participants' modes of thought. Section 5.4 presents the relationship between the semiotic profile and the modes of thought.

Before, however, we turn our attention to how students responded to the particular questions, I wish to examine some of the more interesting responses which pertain to students' intuitions regarding infinity, as they appear to have an impact on their definitions of irrational number and classifications. These observations are readily made among students who identify irrational numbers with their decimal representation. But the obstacle that infinity presents can be seen in the majority of students (regardless of the definition they use) when they are asked to operate on irrational numbers in question

7. Students' beliefs about infinity may function as obstacles to the understanding of irrational numbers and can compound the problems that students have with irrational numbers. Beliefs or conceptions about infinity may have an impact on the student's constructed concept image of irrational number if the student thinks of irrationals in terms of their decimal expansions. If one representation is confused, it should follow that the student will have difficulty moving between representations. Finally, confusion about the nature of infinity will hinder the development of a proceptual (Gray & Tall, 1994) understanding of irrational numbers.

The following are a sample of answers which revealed students' associations of irrationals with their beliefs about infinity as something that never ends, has no specific value, is indeterminate, or is unknowable. All emphasis is mine.

- Numbers that have *no definite solution*. Essentially numbers with decimals that *seem to never end*. (Participant #20, Question 1)
- A number that *you don't really know when it ends* such as π. (Participant #24, Question 1)
- Numbers that *do not have a fixed value*. (Participant #28, Question 1)
- No because a rational number is *definable* but an irrational number is *not*. (Participant #15, Question 5)

Implicit in the first two quotes from participants #20 and #24 is the belief that the sequence of decimal digits in irrational numbers never ends. Participant #20 adds that irrational numbers "have no definite solution", which suggests that both may think of the sequence of digits as a process that will never produce a number, an object, as the end result. Therefore irrational numbers are not numbers because we cannot obtain that end result; they are not objects in their own right. This interferes in their ability to define the concept (which is what the question #1 is asking). The next two participants, #28 and #15, do not explicitly mention the infinite process but they also do not see the irrational number as a fixed, well-defined object.

5.1 Summary of students' answers

Before getting into the analysis that was performed on individual students, I wish to summarize how each question was answered by the class.[12]

5.1.1 Question 1

"Provide a definition of irrational numbers."

In defining irrational numbers, one of two answers was considered acceptable, namely,

a) A real number that cannot be written in the form *a/b* where *a*, *b* are integers and *b* is non-zero.
b) A real number with an infinite, non-periodic (non-repeating) decimal representation.

Ten of the thirty students used the definition a) or something approximately close to it, for example, omitting the condition that *b* should be non-zero and/or that *a* and *b* should be integers. This includes students who defined irrational numbers as numbers that cannot be written as a "fraction", or a "ratio". I recognize that further questioning would be required to confirm that these particular students understood fractions or ratios as ratios of integers with non-zero denominators, but for the purposes of this study they

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[12] See all of the transcribed answers and interpretation in Appendix D.

were given the benefit of the doubt. Two students cited exactly definition a). Five of the remaining students defined irrational numbers in terms of definition b). That is to say that only half of the class provided an acceptable definition of irrational number.

The remaining half of the students provided definitions that were either entirely unacceptable as a definition of irrational number or were somewhat confused. For example, two students confused irrational numbers and prime numbers. This can be attributable to the way in which both irrational numbers and prime numbers are defined in terms of what they are not, namely, a product of distinct factors and a ratio of integers, respectively. Other students showed an inadequate understanding of what it meant to be infinite, e.g., “A number that you don’t really know *when* it ends,” (Participant #24, emphasis mine). There were also several students who defined irrational numbers as infinite decimals but neglected to mention anything about periodicity. This omission is not acceptable as it is crucial to the definition of irrational number.

5.1.2 Question 2

“Classify the following as rational, irrational, and/or real numbers. Justify all that apply.

a) 0.123456…

b) 0.777778

c) π

d) $\sqrt{2} / \sqrt{8}$

e) 22/7

f) $1 + 2\sqrt{4}$

g) 30.450111…

h) 3.14"

Although the participants were asked to justify their classifications of these numbers, only eleven did so. Widespread was an inability to recognize that all these numbers are real numbers. The number that most students (50%) classified as real was $1 + 2\sqrt{4}$. Participant #2, for example, classified it as real but not as rational or irrational. Fifty percent, however, is not much, considering that the participants were science students. This suggests that the students, by and large, are not aware of the mathematical definition of real numbers. As we have noted in section 2, students are first introduced to the formal definition of real numbers in a university level analysis class. The participants in this study are perhaps using the term "real" in a colloquial sense or as meaningless mathematical jargon, i.e., they are mimicking the way the term is used by teachers and mathematicians without attaching any meaning to the term. In the absence of an adequate mathematical definition, whether formal or informal, students fall back on their concept image of the familiar counting numbers to interpret what is real and what is not.

It should be noted that 30.450111… is purposely ambiguous so that students would be forced to justify their response in terms of what they consider adequate information for establishing a pattern. I would have considered either irrational or rational as an acceptable answer provided that the student indicated that either there was not enough information to establish a pattern or that he or she assumed the 1's repeated forever, respectively. This particular example would perhaps best be asked in a face-to-face interview where one could engage the participants into explicitly considering what constitutes a pattern.

5.1.2.1 Some general observations

We will make some general remarks about how the students classified rational, irrational, and real numbers. First, consider the breakdown of answers in Table 1 below which shows how many students (out of 30) chose to classify each number as rational, irrational or real.

	Rational	Irrational	Real
0.123456…	1	27	6
0.777778	27	3	9
π	2	26	8
$\sqrt{2}/\sqrt{8}$	9	18	10
22/7	20	9	9
$1 + 2\sqrt{4}$	18	6	17
30.450111…	3	25	9
3.14	23	3	13

Table 1.Frequency of classifications (out of 30 participants)

From Table 1 we can see that the class was fairly successful on the first three examples. The examples which required some simplification proved to be a little more difficult to recognize as rational numbers.

Consistent with previous studies, the rational approximation to π, 22/7, was misclassified as an irrational number by almost one third of the participants. However, 3.14, which can also be considered an approximation to π did not fool as many people. Only 10% of the class (or three participants) said that 3.14 is an irrational number.

Students consistently answered better on the examples written in decimal form. The number π was also frequently correctly classified. Since teachers usually stress that π has an infinite decimal expansion, this suggests an overall comfort level with viewing irrational numbers as infinite decimals.

In comparing the classifications it is interesting to note that the students who identify irrational numbers in terms of their decimal representations were more likely to fail to recognize that 22/7 as a rational number. Of the ten students who defined irrational numbers as not a ratio of integers in an acceptable way, there was only one participant who misclassified 22/7 as an irrational number. Of the twenty remaining students, who used either an unacceptable definition of irrational number or one that was based on interpreting the decimal representation, eight misclassified 22/7. We should also explicitly mention that almost 30% (i.e. nine out of thirty students) said that 22/7 was irrational, which is consistent with previous results by other researchers. (e.g. Arcavi *et al.*, 1987)

Also consistent with previous studies is the observation that being able to properly formulate a definition of irrational numbers does not give an advantage when it comes to classifying concrete examples. (e.g., Arcavi *et al.*, 1987; Pinto and Tall, 1996; Zazkis and Sirotic, 2004) It is evident that students do not reflect on the definitions they provide; defining is a rote skill rather than internalized knowledge.

5.1.3 Question 3

"Are integers real numbers? Are irrationals real numbers?"

All but one participant recognized that integers are real numbers. This one student admitted to being unsure about the definition of a real number. Exactly half of the participants claimed that the irrational numbers were also real numbers. This result is completely inconsistent with the answers given in question #2. Consider the examples in the list which are irrational.

- 0.123456… was recognized to be irrational by 90% of participants, but recognized to be a real number by only 17%.
- π was recognized to be irrational by 87% of participants, but recognized to be a real number by only 23%.
- 30.450111… was recognized to be irrational by 83% of participants, but recognized to be a real number by only 27%.

This is a large gap that suggests that the students possess some propositional knowledge about rational, irrational, and real numbers, perhaps having heard the expected answers before, but have not internalized this knowledge into their concept image of number.

5.1.4 Question 4

"99/70 provides us with a good approximation of $\sqrt{2}$. Given that $\sqrt{2}$ is irrational, is 99/70 also irrational? Explain."

Question 4 was prompted by the misclassification of 22/7 as irrational in previous studies. This question was asked in an effort to determine if the misclassification occurs because of some confusion in general between rational approximations and the irrational number they approximate or if there is something special in the case of π. Of the nine participants in this study who classified 22/7 as irrational, six also classified 99/70 as irrational. Students were learning in class about fixed point methods for approximating real numbers so the subject matter of this question should have been somewhat familiar to them.

Wording this question in a leading way was inspired by a result obtained by Arcavi *et al*. (1987) who found that some in-service and pre-service teachers continued to classify 22/7 as an irrational number even after doing a series of worksheets aimed at giving these teachers a better understanding of irrational numbers. At the end of such a workshop there should have been no confusion about the rationality of 22/7. Because the confusion remained, I chose to word this question in a leading way to mimic the effect of Arcavi *et al.*

It was found that those who identify irrational numbers as “not fractions” performed well on the example of 22/7 above (eight out of ten classified 22/7 as rational), and were able to answer Question 4 correctly using correct thinking in nine out of ten cases. It was evident from those responses that approximations were distinguished from the numbers they approximate. The eight students explicitly noted that the approximation in question is expressed as a fraction which is at odds with their definition of an irrational number showing that they were in fact reflecting on the definition that they had provided in Question 1. Of the remaining two students who defined irrational numbers as “not fractions,” one (#2) classified 22/7 and 99/70 as real (but not as rational and not as irrational), and the other student (#10) classified 99/70 as irrational, writing, “Yes, as 99/70 = √2. Yes, technically, as √2 can be expressed as a fraction, 99/70, it can be said to be rational.”

Four out of five students who defined irrational numbers in terms of an infinite, non-repeating decimal recognized that the approximation was not irrational, but their explanations weren’t as clear or convincing as those of their peers who used the “not a fraction” type of definition.

Five of the remaining fifteen students who did not provide an acceptable definition failed to recognize the rationality of 99/70. Several of the remaining ten who managed to answer correctly did so for wrong or confused reasons. For example, participant #18 says, “No, 99/70 does not have an infinite amount of decimals.” This was a common type of reasoning in response to Question 4.

There is no doubt some confusion between what a rational approximation is and how it differs from the irrational number it approximates. However, I’m not sure that √2 escapes the problem that π (and its rational approximation 22/7) faces, because it is also a commonly used example of irrational number. In retrospect, it may have been better to use a less familiar irrational number to approximate.

5.1.5 Question 5

“The intersection of two sets consists of the elements common to both sets. Is the intersection of the rational numbers and irrational numbers non-empty? Justify your answer.”

Simply stated, this question is asking if a number can be both rational and irrational. In retrospect, I wish I had worded this question in a simpler way because I think some participants were more confused by the words “intersection” and “non-empty” than by what was actually being asked. (20% of participants did not provide an answer to the question.) This being said, half of all the participants could reason properly about this question.

It was noticed that possessing a good definition of irrational numbers did not help the students to perform better on this question. It is also important to note that no participant classified any example in Question 2 as both rational and irrational, yet half of all students were unable to correctly reason towards the right answer concerning the intersection of the rational and irrational numbers.

5.1.6 Question 6

"Can you represent an irrational number using a calculator? If yes, how? If no, why not?"

Given that the participants were students in a Computer Applications in Mathematics course, I expected them to answer Question 6 with a resounding "no!" especially since the limitations of technology with respect to real number were specifically addressed in class and it was, in fact, one of the first topics. This was not the case; while there were many "no" answers there were also many which took the form of "yes, but…." Participant #1 answered, "Yes, but the calculator will not show all of the irrational number's decimals since irrational numbers have an infinite number of decimals" which is mostly correct but reveals how ambiguous the term "representation of a number" actually is. In saying, 'yes", the student considers an approximation of a number to be its representation; in 'but the calculator will not show all of the irrational number's decimals" he or she points out that this representation does not provide complete information about the represented number. Upon reflection, we came to realize that this is a legitimate response, and not one that we should readily dismiss as mathematically incorrect. Participant #2 said, "Aside from specific buttons dedicated to the numbers π and e, we cannot express irrational numbers since there is a limited number of decimal places on the calculator and the irrational decimals go on to infinity." This response reveals that participant #2 can reason well enough about the limitations of technology and what it means to be a representation but he also reveals an unexpected view of what is sufficient to represent a number, namely that a label on a calculator key will suffice. The fact that over 15% of the class specifically mentioned the π, e, or square root keys as representing irrational numbers with a calculator further supports the claim that there is ambiguity inherent in what it means to represent a number. As mentioned in the Introduction, this realization led us to going beyond the correct-incorrect assessment of students' responses and conducting a semiotic analysis of these responses.

5.1.7 Question 7

"a) If you add two positive irrational numbers, is the result irrational? Explain.

b) If you multiply two different irrational numbers, is the result irrational? Explain."

The overwhelming majority of students (24/30) answered that it was in fact true that the sum of two irrational numbers was again irrational. Of the students who answered correctly, they either did so for the wrong reasons, e.g. participant #29 reasoned that $\sqrt{2} + \sqrt{2} = \sqrt{4} = 2$, or they answered that it depends on the numbers which although true, does not give us any additional insight into the specifics of the students' reasoning about operations on irrational numbers. Many students reasoned that the sum would be irrational because there would still be an infinite number of digits following the decimal place and did not seem to consider that two non-periodic decimal expansions could sum to a finite or a periodic one.

The frequency of "yes" answers in the case of multiplication was similar: the majority of students (23/30) answered that the product would be irrational. Again their reasoning had to do with there being an infinite number of digits in the decimal representation of the product. Periodicity was not mentioned. Only two students, participants #1 and #6, were able to create a proper counterexample. Both gave $\sqrt{2} \cdot \sqrt{8} = 4$ as a counterexample. The remaining five who answered that the product was not necessarily irrational did so by ignoring the requirement in the question that the factors be different, and offered the product of identical square roots as a counterexample, e.g., $\sqrt{2} \cdot \sqrt{2} = 2$.

Very few performed much trial and error and those who did were not making reasoned guesses. In attempting to add square roots, they should have quickly realized it was futile and tried some decimal examples. No one attempted the question by considering a known rational number with infinitely many decimal places and what might add together to give us this rational. For example, 0.333333… can be viewed as the sum of 0.121121112… and 0.212212221…. Students do have a preference for decimal representations but all they seem to see in this representation is a sequence of digits which is "endless", "infinite" or "unknowable"; they ignore the values of the digits in function of their place in the sequence and that this sequence represents an infinite sum of fractions that can be added and multiplied. That is, they focus on the form of the representation rather than on its meaning as a value (or measure) of a certain quantity.

5.2 A semiotic analysis of the way participants interpreted number signs

Students' answers were examined for any evidence that they were interpreting mathematical signs as indices, as icons, and as symbols. We will quickly recall that indices are names or symptoms, icons resemble an aspect of the object, and symbols are generalities which depend on the formal or structural relations with the object they represent.

5.2.1 Examples of indexical interpretation of number signs

In all students, except perhaps for participant #22, we found symptoms of using indexical interpretations in classifying number signs. For example, participant #2 makes reference to "specific buttons dedicated to the numbers π and e" as the only irrational numbers which can be represented on a calculator; the labels on the calculator are indices for the corresponding irrational numbers. Several students interpreted $\sqrt{2}$ as an index for irrationality; the square root sign was interpreted as a symptom of being an irrational number. This indexical interpretation of $\sqrt{2}$ led some students (#4, #5, #23) to conclude that $\sqrt{2}/\sqrt{8}$ was also irrational. Participant #22's interpretations were based more on the perception of number signs as diagrams (e.g., $\sqrt{2}/\sqrt{8}$ was classified as an irrational number because it had "the form of a fraction that cannot be simplif[ied]") and on symbolic manipulation, such as, in response to Question 7:

a) Not necessarily, ex: 0.314159… = a and 2.314159… = b

~~*a/b =*~~

~~*c = b – a*~~

c = 1 – a

I think that a + c = 1 and 1 is not irrational.

b) a · b-1 = a/b = serie/ 2+ serie

a-1 · b = b/a = 2/serie +serie/serie

~~*No*~~ *Yes.*

5.2.2 Examples of iconic interpretations of number signs

60% of the students seemed to be making iconic interpretations of number signs. Frequently, "…" at the end of a number was interpreted as a diagram for infinity. Participant #9 was particularly clear is his use of iconic interpretations: he used the phrase "expressed as it is" in question #4 which explicitly draws attention to the iconic interpretation of 99/70. Therefore the category of a number may depend on the way it is expressed, i.e. the way the sign looks determines what the number is. The number 99/70, represented otherwise, e.g., 1.41428571…, would be classified as irrational since it has "many decimal places" which is the resemblance that this same student used in question 7 to identify irrational numbers. The student's classification of $\sqrt{2}/\sqrt{8}$ as rational could be based on the fact that it appears as a fraction, i.e. as two quantities separated by a bar. Also, the fact that he stated in question 3b that integers are real, but classified $1 + 2\sqrt{4}$ as rational, likely viewing "$\sqrt{4}$" as another name (index) for "2" and not real, supports our claim that this student is using a predominantly iconic interpretation of signs. That is, $1 + 2\sqrt{4}$, *expressed as it is*, is not an integer and therefore is not real. Additionally there are no explicit traces of interpreting signs as symbols.

5.2.3 Examples of symbolic interpretations of number signs

Fourteen of the thirty students appeared to be using symbolic interpretations. As mentioned in chapter 4, this number may be slightly larger, but it was difficult to tell based on the available evidence. Students who were processing number signs according to the rules of algebraic notation were, in principle, considered as interpreting these signs symbolically, although students who only wrote simple equalities such as $\sqrt{2} \cdot \sqrt{2} = 2$

and/or $1 + 2\sqrt{4} = 5$, and did not display any other processing behaviors, were not counted. The expression $\sqrt{2} \cdot \sqrt{2} = 2$ could be seen by such students as a diagram in which case it would be interpreted iconically. In the expression $1 + 2\sqrt{4}$, the sign $\sqrt{4}$ could be seen as another name for 2, and thus interpreted indexically. If the iconic or indexical interpretation better explained these students' overall behaviour, and there was no other evidence supporting the use of symbols, the students were not said to be using the symbolic interpretation.

Certain students who relied heavily on the decimal representation for making classification decisions interpreted the bar in the representation of a fraction symbolically as an instruction to perform a division operation on their calculator. Other students showed comfort in the algebraic manipulation of signs and were deemed to be interpreting those signs symbolically. The students (#1 and #6) who offered the counterexample, $\sqrt{2} \cdot \sqrt{8} = 4$, in question 7b were two such students.

5.3 Analysis of participants' modes of thought

Five of the thirty participants were using the conceptual mode of thought, nine were using the pseudo-conceptual mode of thought, and the remaining sixteen were complexive thinkers.

5.3.1 Complexive thinkers

Sixteen participants out of thirty (53%) were found to be using the complexive mode of thought. All students used multiple facts on which they based their decisions. Although some of the facts could be linked conceptually, they either were not linked by the student or the student did not provide enough information for us to decide that such a conceptual link existed in the mind of the student. It is difficult to form a concise summary of the complexive thinkers because of the very nature of their mode of thinking. Instead we will look at a couple cases which illustrate the complexive mode of thought well.

5.3.1.1 Participant #2

This participant's thinking was classified as complexive because he appeared to apply different kinds of criteria to decide about each number where it belongs and there did not seem to be any conceptual relations among them. He provided an approximately acceptable definition for irrational numbers – "A number that can't be written as *a/b* with integers as *a* and *b*." – and then proceeded to apply different criteria to decide about each number in question 2. His definition seemed to be nothing more than memorized words because he never used it in making decisions. He also justified his answers to questions 6 and 7 saying that "irrational numbers go on to infinity," but this criterion was not consistently applied. For instance 0.123456…, was classified as rational but 30.450111… as real, not rational or irrational. The number, 0.777778 was said to be rational, but 3.14 was real and not rational or irrational. Perhaps knowing that $1 + 2\sqrt{4}$ is equal to 5, the student classified this number as real, not rational or irrational. After much discussion we conjectured that this student's concept image must be that numbers less than 1 (in absolute value) are rational, numbers greater than 1 (in absolute value) are real and irrational numbers are non-digit characters like π, e, or $\sqrt{2}$ (which are the only numbers explicitly classified by this student as irrational). This conjecture was quashed when, at the end of the questionnaire, we saw that the student said that 2 is rational rather than real.

5.3.1.2 Participant #5

Participant #5's thinking was classified as complexive because he appears to classify individual numbers on a case by case basis, applying a different criterion each time. In question 2, the classification of π as irrational was justified by "it goes to infinity decimal." Although, 0.123456… and 30.450111… are also the 'go to infinity decimals', they were classified as irrational because "[0.123456…] can't be written as *a/b*" and, in 30.450111… , there is "no pattern present". Although 3.14 is a finite decimal, it was nevertheless classified as irrational because "it can't be written as *a/b*." Since 3.14 immediately followed 30.450111… which was classified as irrational because of "no pattern," 3.14 could have been classified as irrational because it also displays no pattern. Such behavior is quite typical of a complexive thinker.

5.3.1.3 Participant #10

Participant #10 used several deciding factors which led him to contradictory claims sometimes. He appeared to believe that a number can be rational or irrational depending on how it is represented. For example, in question 4, he said that, represented as 99/70, √2 is rational. In question 7b, he contradicted himself by using √2 as an example of an irrational number. He used it to show that the set of irrational numbers is not closed under multiplication (he multiplied it by itself: "$\sqrt{2} \cdot \sqrt{2} = 2$"). In question 2, he classified √2/√8 as irrational because "of infinite different numbers after the decimal". He obviously did not check if what he was saying was true. He may have decided that √2/√8 is irrational just because √2 in a number sign could be an indication of irrationality for him. Thus, his concept image appears to be made up of several independent ideas about the same type of objects.

5.3.2 Pseudo-conceptual thinkers

Nine of the thirty students (30%) were found to be using a pseudo-conceptual mode of thinking – participants #6, 7, 8, 9, 12, 13, 18, 22, and 25. We should also note that four of the five conceptual thinkers used symbolic interpretations explicitly in their answers, the fifth doing so only implicitly. This last is the only one of the five who did not have an internally consistent and coherent concept image.

5.3.2.1 Pseudo-conceptual thinkers who based their decisions on a single criterion

One third of the pseudo-conceptual thinkers (participants #13, 18, and 25) based their decisions on one conception of irrational number which can be summarized as a never-ending decimal representation. They were classified as pseudo-conceptual thinkers because they treated similar examples differently suggesting a factual basis for deciding if a given number "ends." They made decisions on what Vinner (1997) termed as an uncontrolled association or superficial similarity.

Participant #13, for example, interpreted 99/70 as rational while 22/7 is irrational. In fact, neither decimal representation ends, so, by this student's definition, both should have been classified as irrational if the student were thinking conceptually. Classification of 99/70 as not irrational in question 4 could be triggered by the explicit mention that 99/70 is an "approximation" of $\sqrt{2}$; it could have provoked an association with classroom contexts where irrational numbers were discussed by the teacher. Making decisions on the basis of momentary, factual associations is a symptom of pseudo-conceptual behaviour.

5.3.2.2 Pseudo-conceptual thinkers who based their decision on two criteria

Four of the nine pseudo-conceptual thinkers – participants #6, 7, 8, and 9 – based their classification of a number as irrational on two criteria: being not expressible as a fraction and having an infinite decimal representation. Negations of these criteria were used to classify a number as rational. There was no evidence that these students were formally or algorithmically familiar with the conceptual link between these two conditions

Participant #6's thinking was deemed pseudo-conceptual because belonging to the same category was decided sometimes on the basis of a common property, and sometimes on the basis of factual reasons. An example of classification based on a common property is: 99/70 is not irrational because it is "written under a fraction form and does not apply the irrationality theorem", meaning his or her definition of irrational numbers as numbers that cannot be expressed as "a simple fraction". Otherwise "irrational number" was treated as a family name for numbers that are not fractions, have decimal expansions with three dots, are standard examples of irrational numbers such as $\sqrt{2}$, $\sqrt{3}$ and π, and 3.14 which stands for π in calculations.

Participant #7's likewise used the properties "cannot be written in fractions" and "is infinite" as common to all irrational numbers and appeared to see the relationship between them as factual and not conceptual. The student (wrongly) remembered the two as equivalent but did not even check this equivalence in particular cases (e.g., did not check that 99/70 has, in fact, an infinite decimal expansion so it cannot be dismissed as not irrational just on this basis as he did in Question 4). There were other symptoms of non-conceptual thinking. For example, $\sqrt{2}/\sqrt{8}$ was classified as real but neither rational nor irrational and no reason was given; π was classified as irrational but not real and it is the only non-real number in the student's classification. Yet, in Question 3b, the student claimed that all irrationals are real. Therefore the student couldn't have had conceptual reasons for denying the status of a real number.

5.3.2.3 Examples of other pseudo-conceptual thinkers

Participant #12 was one of the few students who mentioned the condition of a pattern in the decimal expansion, and used this condition in making decisions about particular numbers. He defined an irrational number as “a number that contains a variety of different numbers following the decimal point and doesn’t follow any particular pattern”. Rational numbers were also required to have “many” decimal digits but these digits had to “follow a particular pattern”. He classified 0.777778 and 30.450111… as rational, 3.14 as real but neither irrational or rational. Presumably, 0.777778 is being interpreted as an index (name) for the number 0.77777… and the pattern that is perceived is that of a repeating single digit as it is in the example 30.450111…. This is a factual basis instead of a conceptual one. He classified both 22/7 and 99/70 as irrational claiming that they “have a variety of different # following the decimal that [do] not follow any particular pattern”. This decision, again, could not have been based on a conceptual basis, but perhaps on the fact that when these fractions are entered into a simple calculator the period is not apparent.

The last student in the pseudo-conceptual group is participant #22. This participant's concept image was particularly difficult to describe as he preferred to express his uncertainty or admit lack of knowledge rather than give straightforward answers to the questions. In question 3, for example, he said, "I am not sure about the real numbers definition". He answered question 4 by saying that "this depends on what is understood by good approximation. Is 2.12798 a good approximation for 2.12798010….?" He offered opposite possible answers in question 5: "I guess one could either say that all rational numbers are elements of irrational # or none are element of irrational #". This student is obviously very reflective and therefore already open to conceptual thinking but unable to achieve it because of lack of knowledge.

5.3.3 Conceptual thinkers

It was expected that only a minority of students would be using the conceptual mode of thinking. In fact, five of the thirty students (17%) were found to be using a conceptual mode of thought – participants #1, 14, 21, 23, and 28. As we will see in the examples below, conceptual thinking was not always associated with "mathematically correct" answers, relative to conventional mathematical knowledge.

5.3.3.1 Participant #1

On one end of the spectrum is participant #1 who had a fairly complete concept image of rational, irrational, and real numbers which agreed with mathematical convention. This student reasoned using both fractional representations and decimal ones. The only thing missing from his answers was some attention to periodicity. He did not offer any justification for his classifications in question #2, so it is unclear if he classified 30.450111… as irrational because of questions involving periodicity or not.

Participant #1's thinking was classified as conceptual because he consistently justified his decisions based on two criteria for irrationality: cannot be expressed as a ratio *a/b* with *a, b* integers and *b* not 0, which he gave as his definition in question 1, and has "an infinite amount of decimals", which he used only in questions 6 and 7a. There are no inconsistencies in his responses to question 2, whether within the question or relative to his definition.

In question 4, the student wrote:

> *99/70 is not irrational since it only provides an approximation of √2, not the exact value of √2. Also, 99/70 expresses a ratio where 99 and 70 are integers and 70 is obviously not equal to 0. Therefore, 99/70 is a rational number.*

which reads as an acceptable, definition based proof of the rationality of 99/70.

His response to question 6 was:

> *Yes, [we can represent irrational numbers on a calculator] but the calculator will not show all of the irrational number's decimals since irrational numbers have an infinite number of decimals.*

His decision that the set of irrational numbers is closed under addition in Question 7a was based on the characterization of irrational numbers as having infinite decimal expansions: "the result of adding two positive irrational numbers is indeed irrational since it will still have an infinite amount of decimals".

We cannot claim – for lack of evidence – that the student entertained a conceptual (rather than factual) relationship between the two criteria he used. We will only grant him the benefit of the doubt because we have no evidence to the contrary in his responses.

5.3.3.2 Participant #21

On the other end of the spectrum is participant #21 who was also a conceptual thinker, but his concept image was an overly simplified one and far from conventional mathematical knowledge. For this student, irrational numbers were defined as all numbers that are not "whole numbers" or integers; integers are rational and all the other numbers are irrational. The only real number was π which was also irrational. However, his reasoning was quite consistent within this notion of irrational numbers. As he believed that supporting a general statement with an example is enough (in Question 7), we cannot say that he was contradicting himself when saying, in question 3, that irrational numbers are real, yet classifying only π as real in question 2. In question 3, he was probably just saying that *some* irrationals are real. This student's concept image was so simplified and the student so consistently referred to it in making decisions that the student could hardly help but be a conceptual thinker, even if the concepts he used were erroneous ones.

5.4 Relationship between the semiotic profiles and modes of thought in the participants

As we mentioned in chapter 2, mathematical signs are sufficiently complex semiotic entities that they permit interpretations that are partly indexical, partly iconic, and partly symbolic all at once. Some of the participants, however, interpreted mathematical signs one-sidedly, basing their interpretations on perhaps just one aspect of the sign. The interpretation of signs is similar to the process-object interpretation of mathematical due to Gray and Tall (1994). "Proceptual understanding" refers to the flexibility to see mathematical symbols in multiple ways. Peirce's classification of signs and Gray and Tall's procept theory complement each other nicely. Procepts can be seen as a further analysis of the interpretation of a sign as a symbol.

As we can see in Table 2, all five of the conceptual thinkers in this research used symbolic interpretations of mathematical signs. Additionally, three of the five (60%) used all three interpretations – index, icon, and symbol.

Participant #	Index	Icon	Symbol
1	1	1	1
14	1	1	1
21	1	0	1
23	1	1	1
28	1	0	1

Table 2. Semiotic analysis for conceptual thinkers

Let us now compare the conceptual thinkers' use of symbolic interpretations with their classmates in the pseudo-conceptual (Table 3) and complexive (Table 4) groups.

Participant #	Index	Icon	Symbol
6	1	1	1
7	1	1	0
8	1	1	0
9	1	1	0
12	1	1	1
13	1	1	1
18	1	0	0
22	0	1	0
25	1	0	0

Table 3. Semiotic analysis for pseudo-conceptual thinkers

The percentage of students who used the pseudo-conceptual mode of thought and used symbolic interpretations of mathematical signs was 33%, compared to 100% of conceptual thinkers. Furthermore, the same 33% are found to be using all three interpretations of mathematical signs, compared with 60% of the conceptual thinkers.

Participant #	Index	Icon	Symbol
2	1	0	1
3	1	1	1
4	1	0	0
5	1	0	0
10	1	1	0
11	1	1	1
15	1	1	0
16	1	0	0
17	1	1	0
19	1	0	0
20	1	1	0
24	1	0	1
26	1	1	1
27	1	0	0
29	1	1	1
30	1	0	0

Table 4. Semiotic analysis for complexive thinkers

The percentage of students who used the complexive mode of thought and use symbolic interpretations of mathematical signs was 38%, compared to 100% of conceptual thinkers and 33% of pseudo-conceptual thinkers. Furthermore, only 25% are found to be using all three interpretations of mathematical signs, compared with 60% of the conceptual thinkers and 33% of pseudo-conceptual thinkers.

We can say that the students who were able to think conceptually and interpret mathematical signs in multiple ways were the students who were thinking most like mathematicians. Note, however, that this is not to say that these students were thinking about the number systems in a way that reflects the accepted mathematical knowledge. Just the opposite is true. Participant #14 came reasonably close with a concept image that was dependent on how many digits follow the decimal point and whether those digits repeat. Participant #1 was closer still with a concept image that combined concepts about fractional representations and decimal representations but made no explicit mention of periodicity (though it may be understood). But participant #28 was far from the conventional wisdom with an incoherent and inconsistent concept image built upon the student's own interpretation of what it means to have a "fixed value."

Chapter 6. Conclusions

Science students encounter the same obstacles as teacher education students and high school students: they fail to see links among multiple representations and do not succeed in forming proceptual or symbolic interpretations of numbers. Students who think conceptually and used symbolic interpretations were a minority, and there were fewer still who did so while being consistent with mathematical convention.

Among these 30 science students, half could offer an acceptable definition of irrational numbers. Those who did not offer an acceptable definition did not necessarily lack knowledge of the concept. However, as expected, the students who were unable to formulate an adequate definition performed worse on questions that called upon their reasoning skills or asked them to operate on irrational numbers. Regardless of the quality of the definition, there was a widespread inability to apply the definition consistently to get expected results.

In analyzing the answers of 30 college science students, we expected to find a dependence on decimal representation of irrational numbers (and numbers in general) in keeping with what had been reported in previous studies of high school and students in education programs. We found this, and more to the point, we perceived an overall reliance on *how* the number is represented in general. That is to say, that students often seemed to be interpreting the representation of the number (be that a fraction, decimal, square root, or some other sign or character) *solely* based on its form without regard to the value of the number and without reflecting on the mathematical links among representations. That half of the class think about irrational numbers using a complexive mode of thought supports this claim.

We additionally found students' concept images of rational number, irrational number, real number, decimal number and even number in general varied dramatically and more often than not was deficient in some way and therefore did not agree with conventional mathematics. Many students, relying heavily on the decimal representation, have overly simplified concept images which depend on classifying concrete examples in terms of the following criteria:

- finite vs infinite decimal expansion
- letter characters and special characters such as the square root sign vs digits
- known vs unknown (and variations of this theme: defined vs undefined, fixed vs not fixed, object vs process).

If the students were given the chance to explore the relationship between decimal and fractional representations, their conceptual framework for interpreting number signs could become more meaningful and they could graduate to become conceptual thinkers.

Students at the college and university level are expected to have some knowledge of irrational numbers and the real number system in general. However, these topics are not adequately covered in the high school curriculum. At most, it seems that the typical high school student will be exposed to the definition of irrational numbers as numbers which cannot be expressed as ratios of integers a and b, where $b \neq 0$ which only tells them what irrational numbers are not. They will also likely receive a description of the decimal representation as that of being infinite and non-periodic, while rational numbers are described having finite decimal representations or infinite periodic ones. Our research here and the results in the literature show that the concept of periodicity is lost on most students. Several students in this research based their concept image on the idea of finite vs infinite decimal representations only; very few mentioned, let alone used, periodicity. Finally, high school students experience irrational numbers (or rather rational approximations to them) in the context of using certain operations, like square roots, or in studying certain functions, like the exponential, logarithmic and trigonometric functions.

The lack of explicit attention to the nature of irrational numbers in the high school curriculum (and even at the college and university levels) leaves students to construct concept images *independently* from what they know about natural number, integers and rational numbers. While engaging students to be active participants in their own learning, allowing them to construct concept images in the absence of guiding mathematical principles and definitions is ill-advised at best.

Chapter 7. Recommendations

In the following, recommendations are made for further research in this area and pedagogical considerations.

7.1. Further study

I recognize that this interpretive research would have ideally been conducted in an interview-style setting. For one thing, this would have enabled us to probe further into the classifications that each student made and why decisions were made the way they were. In particular, improvements can be made to the question set which was used in this research. To name a few:

1. Periodicity could have been more explicitly addressed in the questions.
2. It would be interesting to ask the contrapositive of question 4 and see if students make the same mistake. That is, ask "99/70 provides us with a good approximation of $\sqrt{2}$. Given that 99/70 is rational, is $\sqrt{2}$ also rational? Explain." Also, as mentioned above, it may be wise to use an example that is less familiar to the student. The square root of a larger number would suffice.
3. The obstacle of formal language should be removed from question 5. The question should be worded in a plain way that is easy for the student to understand. For example, "Can a number be both rational and irrational?" is a better formulation of question 5.

This research suggests that students' reasoning is highly influenced by the form of the number representations used. It would, therefore, be informative to perform a review of previous studies which require students to classify numbers to see if student answers could be re-analyzed in terms of their interpretations of the way the number is represented. For example, Fischbein *et al.* (1995) state that 97% of pre-service teachers interpret $\sqrt{16}$ as a number, 76% interpret it as a whole number, 69% interpret it as a rational number, 3% interpret it as an irrational number, and 90% interpret it as a real number. The 30% who do not interpret $\sqrt{16}$ as a rational number may be doing so because they are interpreting the square root sign as an index for "not rational." We will take another example from the research done by Zazkis and Sirotic (2010). One of their participants, Anna, has an iconic view of rational numbers: "Because if you divide something by something else, that means you can put it in a fraction, because of what a fraction is, something divided by something." (Zazkis and Sirotic, 2010, p. 17) Anna classifies $\sqrt{5}/\sqrt{2}$ as a rational number based on her iconic interpretation, and in fact, under this interpretation no quotient will ever be irrational. The researchers conclude that "Anna's concept of rational number is still very much tied to its operational origins." (*ibid.*) I think we can equally say that Anna is using iconic interpretations and pseudo-conceptual mode of thought. This particular student is led to see the errors in her concept image when she is asked to consider that π, which she knows to be irrational, is defined as a ratio of circumference and diameter.

7.2. Concerning pedagogy

Student, educators, and parents may rightfully ask, "Why should we teach irrational numbers to our high school students?" After all, what we use in scientific applications and everyday calculations are rational numbers. Students who pursue higher mathematics will learn about irrational numbers and the construction of the real number system when they take their first real analysis.

To this question my answer is: There is more to learning about irrational numbers than just recognizing one. The move from understanding rational numbers to understanding irrational numbers is a conceptual leap that is difficult to make, but in doing so the student stands to gain a deeper interpretation of number signs and a more conceptual mode of thought which will positively impact other (non-mathematical) areas of thought and reasoning. Your student may not become a mathematician, but he may become a lawyer who must know how to apply definitions and existing case logic to novel situations. Furthermore, while the lack of knowledge of concept definitions among science students was disturbing, the lack of understanding concerning approximations was even more so. Approximating is something that students begin to do very early in the course of their mathematical education. The distinction between an approximation and the number it approximates should be clear by the time the student reaches college. Somewhere along the line the boundaries of approximating are being blurred be it by convention, notation ($\approx$ or =), or language used in the classroom. It is important that all teachers at all levels explicitly note the instances when they are using an approximation, why they are using an approximation, and what they are approximating. For science students, in particular, this is of great concern because it will likely be a part of their daily work.

One strategy proposed in the literature to improve pedagogy is to make explicit the historical time-line of the development and acceptance of irrational numbers. (Arcavi *et al.*, 1987)

1. The preliminary stage in which a concept is motivated by some need and may be vague or met with resistance from some practitioners.
2. The familiarization stage wherein the concept is more widely used and understood, and therefore more widely accepted.
3. The axiomatization stage wherein the fully developed concept is made mathematically formal and axiomatized where applicable.

Arcavi *et al.* highlight the above three stages in the development of the irrational number concept with the pre-service and in-service teachers in their study and it is well received.

Fischbein *et al*. recommend that

1. "the idea of mathematics as a coherent, structurally organized body of knowledge [be] systematically conveyed to the student," (Fischbein *et al*., 1995, p. 29)
2. incommensurability should be addressed in a hands-on fashion and not totally disregarded,

and 3. students be made aware of the concept of infinity in a meaningful way.

Troubles with infinity, incommensurability and mathematical structure may be overcome, first, by focusing on teacher knowledge and awareness. Although the concepts of infinity and cardinality are not discussed in secondary school curriculum, they should be addressed in teacher training. It may only take a simple example, like that of Hilbert's Hotel,[13] to give teachers a better feel of the very complex concept of infinity.

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[13] In Hilbert's Hotel there is a countably infinite number of rooms. The hotel is full, but when a new guest arrives, all guests are asked to move from their room, n, to room $n + 1$ and the new guest moves into room #1. Then countably infinitely many new guests arrive and everyone is asked to move from their room, n, to

I like the Hilbert Hotel metaphor because it gets at the almost paradoxical nature of infinity in an accessible, almost visual, way. It is important to note that although this is a great metaphor for countable infinity, it does nothing to instruct the learner about uncountable infinity. But once the teacher has a greater comfort with "regular" infinity she can tackle the notion of uncountably infinite sets. Specifically, they should see the proof that the set of irrational numbers is uncountably infinite so they may dispel the feeling that students frequently have that rational numbers are more common, and therefore more plentiful. The knowledge the teacher possesses with respect to these concepts can go a long way in helping students to understand the irrational number concept.

Sirotic and Zazkis note that students possess underdeveloped intuitions regarding irrational numbers due to a lack of formal knowledge and algorithmic experience. Emphasis in the high school classroom should be placed on the relationship between fraction and decimal representations to lessen student dependence on their calculator and the decimal representation of numbers. As we mention above, awareness of the link between fractions and decimals and algorithmic experience with the transformations of one representation to another can help the student to create more meaningful interpretation of signs and conceptual thought processes.

From the current research we can recommend that teaching should aim at fostering the use of symbolic interpretations alongside iconic and indexical ones. Also, multiple representations of irrational numbers and the connections between these

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room $2n$ and the new guests move into all the odd numbered rooms which are left vacant. And Hilbert's Hotel can further accommodate countably infinitely many buses carrying infinitely many guests.

representations must be emphasized in the classroom. In particular, it must be made explicit that the definition of an irrational number as not a number a/b, where b is non-zero and a, b are integers is **equivalent** to the definition of an irrational number as an infinite non-repeating decimal. Students should be encouraged to examine many representations (e.g., decimal, fraction, geometric, symbolic, continued fraction) and the conclusions we can draw from each. This will go a long way in helping the student to achieve a conceptual mode of thought with respect to rational, irrational and real numbers.

The end benefit to the student is a greater understanding of the number system, a better understanding of approximations in general, a wider conception of what a real number is in the mathematical sense of the word, and a clearer notion of the limitations of technology. This will translate into more well-informed scientists and teachers, post-university.

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Appendix A: MELS table for the understanding of real numbers

Understanding real numbers

	Elementary	Secondary				
→ Student constructs knowledge with teacher guidance. ★ Student applies knowledge by the end of the school year. Student reinvests knowledge.[1]		Cycle One		Cycle Two		
	6	1	2	3	4	5
1. Natural numbers less than 1 000 000						
a. Reads and writes any natural number	★					
b. Represents natural numbers in different ways	★					
c. Composes and decomposes a natural number in a variety of ways and identifies equivalent expressions	★					
d. Approximates a natural number	★					
e. Compares natural numbers or arranges natural numbers in increasing or decreasing order	★					
f. Classifies natural numbers in various ways, based on their properties (e.g. even numbers, composite numbers)	★					
2. Fractions						
a. Represents a fraction in a variety of ways (using objects or drawings)	★					
b. Identifies the different meanings of fractions: part of a whole, division, ratio, operator, measurement	→	→	★			
c. Verifies whether two fractions are equivalent	★					
d. Compares a fraction to 0, $\frac{1}{2}$ or 1						
e. Orders fractions with the same denominator or where one denominator is a multiple of the other or with the same numerator						
3. Decimals up to thousandths						
a. Represents decimals in a variety of ways (using objects or drawings) and identifies equivalent representations	★					
b. Reads and writes numbers written in decimal notation	★					
c. Approximates a number written in decimal notation	★					
d. Composes and decomposes a number written in decimal notation and recognizes equivalent expressions	★					
e. Compares numbers written in decimal notation or arranges them in increasing or decreasing order	★					
4. Integers						
a. Represents integers in a variety of ways (using objects or drawings)	★					
b. Reads and writes integers	★					
c. Compares integers or arranges integers in increasing or decreasing order	★					

5. Expresses numbers in a variety of ways (fractional, decimal percentage notation)	★						
6. Represents, reads and writes numbers written in fractional or decimal notation		★					
7. Approximates, in various contexts, the numbers under study (e.g. estimates, rounds off, truncates)		★					
8. Distinguishes rational numbers from irrational numbers in the set of real numbers Note : Although students do not systematically study sets of numbers in Secondary Cycle One, they should still be encouraged to use the proper terms learned in elementary school (natural numbers, integers, decimals).				★			
9. Represents, in different types of notation, various subsets of real numbers (discrete or continuous): interval, list/roster, on a number line Note : In TS and S, set builder notation may be introduced as needed.				★			
10. Defines the concept absolute value in context (e.g. difference between two numbers, distance between two points) Note : In Cycle One and Secondary III, the concept of *absolute value* is introduced informally, using examples.		→	→	→	★		
11. Represents and writes							
a. the power of a natural number	★						
b. squares and square roots		→	★				
c. numbers in exponential notation (integral exponent)		→	★				
d. numbers in scientific notation				★			
e. cubes and cube roots				★			
f. numbers in exponential notation (fractional exponents)				★			
g. numbers using radicals or rational exponents							CST
					★		TS
						★	S
h. numbers in logarithmic notation using the equivalence $\log_a x = n \Leftrightarrow a^n = x$, if necessary							CST
					→	★	TS
						★	S
12. Estimates the value of the power of an exponential expression with respect to its components: base (between 0 and 1, greater than 1), exponent (positive or negative, integral or fractional) Note : The same applies for a logarithmic expression in TS and S.					★		CST
				→	★		TS
						★	S
13. Estimates the order of magnitude of a real number in different contexts	→	→	→	★			
14. Estimates the order of magnitude of a real number using scientific notation				★			
15. Compares and arranges in order							
a. numbers written in fractional or decimal notation		★					
b. numbers expressed in different ways (fractional, decimal, exponential [integral exponent], percentage, square root, scientific notation) Note : Scientific notation is introduced in Secondary III.		→	★				

1. Mathematical knowledge is constructed using prerequisites or by making connections between concepts and processes. The elements described in the tables will be reinvested and further developed as students progress through secondary school. When actions are included as part of other actions carried out in subsequent years, the shading in the table is not extended to cover all five years of secondary school.

Appendix B: MELS table for the understanding of operations involving real numbers

Understanding operations involving real numbers						
→ Student constructs knowledge with teacher guidance. ★ Student applies knowledge by the end of the school year. Student reinvests knowledge.	Elementary	Secondary				
		Cycle One		Cycle Two		
	6	1	2	3	4	5
1. Natural numbers less than 1 000 000						
a. Determines the operation(s) to perform in a given situation	★					
b. Uses objects, diagrams or equations to represent a situation and, conversely, describes a situation represented by objects, diagrams or equations (use of different meanings of the four operations)	★					
c. Establishes equality relations between numerical expressions (e.g. 3 + 2 = 6 – 1)	★					
d. Determines numerical equivalencies using relationships between operations, the commutative and associative properties of addition and multiplication, the distributive property of multiplication over addition or subtraction	★					
e. Translates a situation using a sequence of operations in accordance with the order of operations	★					
2. Fractions						
a. Uses objects, diagrams or an operation to represent a situation and, conversely, describes a situation represented by objects, diagrams or an operation (use of different meanings of addition, subtraction and multiplication by a natural number)	★					
b. Uses an operation to represent a situation (use of different meanings of operations)	→	★				
3. Decimals						
a. Uses objects, diagrams or equations to represent a situation and, conversely, describes a situation represented by objects, diagrams or equations (use of different meanings of the four operations)	★					
b. Determines numerical equivalencies using relationships between operations (inverse operations), the commutative and associative properties of addition and multiplication, the distributive property of multiplication over addition or subtraction	★					
c. Translates a situation using a sequence of operations in accordance with the order of operations	★					
4. Chooses an appropriate way of writing numbers for a given context **Note** : Over the years, new notation systems such as scientific notation are added to the students' repertoire.	★					
5. Looks for equivalent expressions: decomposing (additive, multiplicative, etc.), equivalent fractions, simplifying and reducing, factoring, etc.		★				
6. Translates (mathematizes) a situation using a sequence of operations (no more than two levels of parentheses)		★				
7. Anticipates the results of operations		★				
8. Interprets the results of operations in light of the context		★				

Appendix C: Questionnaire

1. Provide a definition of irrational numbers.

2. Classify the following as rational, irrational, and/or real numbers. Justify all that apply.

   a) 0.123456…

   b) 0.777778

   c) π

   d) $\sqrt{2} / \sqrt{8}$

   e) 22/7

   f) $1 + 2\sqrt{4}$

   g) 30.450111…

   h) 3.14

3. a) Are integers real numbers?

b) Are irrationals real numbers?

4. 99/70 provides us with a good approximation of $\sqrt{2}$. Given that $\sqrt{2}$ is irrational, is 99/70 also irrational? Explain.

5. The intersection of two sets consists of the elements common to both sets. Is the intersection of the rational numbers and irrational numbers non-empty? Justify your answer.

6. Can you represent an irrational number using a calculator? If yes, how? If no, why not?

7. a) If you add two positive irrational numbers, is the result irrational? Explain.

b) If you multiply two different irrational numbers, is the result irrational? Explain.

Appendix D: Answers provided by students and analytical assessments of the data

Please keep in mind the following as you read the answer sets and the analyses:

- Answers are presented exactly as the student wrote them, and therefore contain grammatical and spelling errors.
- The answers to question #2 are presented in tabular form. The highlighting refers to the expected answers. Recall that they students were asked to write their answer and provide justification; they were not asked to check boxes.
- The generic "he" will be used throughout regardless of the actual gender of the participant. Gender was not a factor in this study.
- Please remember that when we speak of consistent and/or coherent concept images were are referring to the self-consistency and self-coherence that we mention in the body of the thesis and not necessarily consistency and coherence with accepted mathematical practice.

The table following each answer set is our analysis of the student's answers. The row headings are as follows:

Q: Rational numbers

IQ: Irrational numbers

R: Real numbers

Rep: Interpretations of representations

C.I.: Reconstruction of concept images

M.o.T.: Analysis of modes of thought

G.R.: General remarks

Participant #1

1. An irrational number is any real number that cannot be expressed as a ratio *a/b*, where *a* and *b* are integers, with *b* non-zero, and is therefore not a rational number.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*	*	*		*	*		*	*		*		*	*	*		*

3. a) Integers: Yes
   b) Irrationals: Yes

4. 99/70 is not irrational since it only provides an approximation of √2, not the exact value of √2. Also, 99/70 expresses a ratio where 99 and 70 are integers and 70 is obviously not equal to 0. Therefore, 99/70 is a rational number.

5. No idea.

6. Yes, but the calculator will not show all of the irrational number's decimals since irrational numbers have an infinite number of decimals.

7. a) The result of adding two positive irrational numbers is indeed irrational since it will still have an infinite amount of decimals.

   b)The result of multiplying two different irrational numbers is not irrational. For example, √2 · √8 = 4 which is not an irrational number.

#1	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - A rational number is a real number that can be expressed as a ratio a/b, where a and b are integers, with b non-zero (Q4). - 0.777778, 3.14 are rational. (Q2) - $\sqrt{2} / \sqrt{8}$, $1 + 2\sqrt{4}$ are rational. (Q2) - 22/7 (Q2), 99/70 are rational. (Q4) Implicit knowledge - Finite decimals are rational (Q2) - Fractions are rational (Q2e, Q4) - Integers are rational (Q2f) - - If an expression can be converted into a fraction or an integer then it represents a rational number. (Q2f, Q7b)
IQ	Explicit knowledge: - Irrationals are numbers that cannot be expressed as a ratio a/b, where a and b are integers with b non-zero. (Q1) - 0.123456…, 30.450111… are irrational. (Q2) - π is irrational. (Q2) Implicit knowledge: - Infinite decimals (no matter if they are repeating or not) are irrationals. (Q2a, Q6, Q7a) - Unclear if the student is interpreting 30.450111… as repeating or not. (Q2g) - Irrational numbers can be approximated by fractions or finite decimals but are not identical to their approximations. (Q4, Q6)
R	- "Real number" refers to all the numbers. (Q1, Q2, Q3)
Rep'n	- Appears to interpret signs for numbers as symbols: In classifying numbers, makes decisions based on the value of the number, not on the form of its representation. (Q2) - The representation of a number and the number itself is that these are two different things based on convention, not on resemblance alone. (Q6) - Uses indexes, icons and symbols in calculating (Q7) and reasoning. - 99/70 is interpreted symbolically. The student deduces certain properties of the object from the representation. (Q4) - "/" is an index for ratio.
C.I.	- **R** = all numbers. - **Q** = integers, ratios of integers, and numbers with a finite number of digits after the decimal point. - **IQ** = numbers that cannot be expressed as ratios of integers, numbers with infinite digits after the decimal point.

	- No obvious contradictions in the student's concept image, assuming that the student does not perceive 30.450111… as a repeating decimal or doesn't know how to convert repeating decimals into fractions.
M.o.T.	- CONCEPTUAL: consistently uses two criteria for irrationality: cannot be expressed as a ratio a/b with a, b integers and b not 0, and has "an infinite amount of decimals".
G.R.	- Not only remembers accurately some declarative knowledge about numbers (Q1 - recalled exactly the definition of irrational number) but also applies this knowledge in deciding about numbers. - Has some mathematical culture: justifies his classifications of numbers by reference to definitions most of the time; disproves a general statement by an example; justifies a general statement with a general argument; possesses some algebraic skills. - Sensitive to contradiction. - Reflective.

Participant #2

1. Any number that cannot be expressed as a fraction a/b, with $b \neq 0$.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
*			*				*		*					*			*			*			*

3. a) Integers: Yes
   b) Irrationals: ~~No~~ Yes

4. 99/70 is not irrational because only the exact number $\sqrt{2}$ is the irrational number. 99/70 is a real number that happens to be close to $\sqrt{2}$.

5. The intersection is empty because it is impossible for a number to be rational and irrational.

6. Aside from specific buttons dedicated to the numbers π and e, we cannot express irrational numbers since there is a limited number of decimal places on the calculator and the irrational decimals go on to infinity.

7. a) Yes, $\pi + \sqrt{2}$ will give a number that has an infinite amount of decimal numbers as well.

   b) No. $\sqrt{2} \cdot \sqrt{2} = 2$, which is a rational number.

#2	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.123456… is rational (Q2) - 0.777778 is rational (Q2) - √2 / √8 is rational (Q2) - 2 is rational. (Q7b) Implicit knowledge: - Rational numbers are less than 1. (Q2a, b, d are rational, but Q2e, g, h are not; 99/70 is also real but not rational.) - "fraction" could mean smaller than 1. (√2 / √8 is rational because it is ½ which is less than 1)
IQ	Explicit knowledge - Irrationals are numbers that cannot be expressed as fractions a/b, with b not equal to 0. (a, b being integers is not mentioned) (Q1) - π is irrational. (Q2, Q6) - √2 is irrational. (Q4) - *e* is irraitional. (Q6) Implicit knowledge - Irrational implies being expressed as an infinite decimal (Q6), but this is not a sufficient condition for deciding irrationality. (Q2a, b, g are not classified as irrational.) - Irrationals can be approximated by fractions or finite decimals but are not identical with their approximations. (Q4, Q6) - Irrational numbers must contain signs that are not digits, e.g. letters like π or *e*, root signs, which play the role of indices.
R	Explicit knowledge - The intersection of **Q** and **IQ** is empty. (Q5) - 99/70 and 22/7 are real numbers (Q2, Q4) - Integers are real numbers - Irrational numbers are real Implicit knowledge - There are real numbers that are neither in **Q** nor **IQ**: 30.450111…, 3.14, 1 + 2√4. (Q2)
Rep	- The value of the number is very important in deciding rationality. (Q2) - Appears to treat the signs of numbers as symbols. - Labels on calculator buttons (indices) indicate which numbers are represented in the calculator and which are not. (Q6) - Participant switches, unknowingly, between the name of the number (index) and its value (symbol) in Q6 - No use of icons.
C.I.	- It first appears that rational numbers are all numbers less than 1 (in absolute value), reals are greater than 1 (in absolute value), and irrationals are numbers that contain non-digit characters, like π. But in Q7b he says that 2 is rational contradicting this original reconstruction.

	- Unable to construct a coherent concept image.
M.o.T.	- COMPLEXIVE: appears to apply different criteria to decide about each number. States a more or less accurate and correct definition of irrational numbers but this seems to be just memorized words for him because he never uses it in making decisions. Justifies answers to Q6 and Q7a by saying that "irrational numbers go on to infinity" but this criterion is not applied consistently, because in Q2, one infinite decimal - 0.123456... - is classified as rational and another one - 30.450111... as real but not irrational or rational. It is difficult to identify one consistent criterion applied to decide that a number is rational. In Q2, one finite decimal - 0.777778 - is classified as rational, and another one - 3.14 - real but neither rational or irrational. The student probably knows that $1 + 2\sqrt{4} = 5$, classified as real but not rational or irrational. Claims that another integer, 2, is rational in Q7b.
G.R.	- Responds hesitantly (Q3) - The student's definition of irrational numbers (Q1) appears to be part of his memorized declarative knowledge, but it hasn't been internalized as practical knowledge to be used in making decisions or reasoning.

Participant #3

1. A number that can’t be written as *a/b* with integers as *a* and *b*.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*			*				*		*		*				*			*		*		

Participant gives the following reasons for the above classifications:

a) because it can’t be written as a/b with integers
b) because it can’t be written as a/b with integers
c) because ~~it~~ there is a never ending number of decimals.
d) because it is not integers (a/b).
e) because it is integers and can be written as a/b
f) because can’t be written as a/b
g) can’t be written as a/b
h) because I think it is. It follows the definition.

3. a) Integers: Yes
   b) Irrationals: Yes

4. No because it ~~follows a~~ falls in the rational category.

5. No, it can’t because they ~~don’t~~ can be only one of them. Two different type of numbers.

6. Yes, irrational number can be represented by a number with decimal that can’t be ~~written~~ changed into fractions.

7. a) ~~Yes~~ Maybe, I think that can happen but not to all irrational numbers.

   b) ~~Yes~~ Maybe, $\sqrt{2} \cdot \sqrt{2} = 2$ this falls into the category of rational numbers. Some of them might work out.

#3	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - Numbers that can be written as a/b with a, b integers (Q2e explanation) - 22/7 (Q2), 99/70 (Q4) - 3.14 (Q2) - 2 (Q7b) Implicit knowledge: - When a number is written in the form of a fraction, then it is certainly rational. (Q2e: 22/7; Q4: 99/70) - Rational numbers are written as fractions, but this is not a sufficient condition perhaps because the person does not know how to transform one to the other. (Q2b: 0.777778 can be written as a fraction but it is classified as irrational; Q2h: 3.14 is classified as rational, but the reason given is not that it can be represented as a fraction.)
IQ	Explicit knowledge: - Irrationals are numbers that cannot be expressed as a/b with a, b integers. (b not equal to 0 not mentioned) (Q1) - 0.123456…; 0.777778; 30.450111… (Q2) - $\sqrt{2} / \sqrt{8}$; $1 + 2\sqrt{4}$ - $\sqrt{2}$ (Q7b) - A number cannot be irrational and rational at the same time (Q5) - Implicit knowledge: - Decimals, even with a finite number of decimals, can be irrational numbers (Q2b); but not all decimals are irrational (e.g. Q2h: 3.14 is not irrational). - Irrational numbers can be represented on a calculator because some finite decimals cannot be changed into fractions (Q6) - If a number is written as a/b and neither a nor b are integers, then it is irrational. (Q2d) - "Never-ending" decimals apply only to π (Q2c explanation) or perhaps other cases which are explicitly said to be infinite. Infinite decimals have nothing to do with irrationality. - Irrational numbers have lots, i.e., more than 2, digits after the decimals or with square root signs which function as indices. (Q2)

R	Explicit knowledge - π (Q2c) - Irrational numbers are real. (Q3b) - Integers are real. (Q3a) Implicit knowledge - **R** is disjoint with **IQ** (no number is classified as both rational and real, or both irrational and real in Q2); - A number with a never ending number of decimals is real (Q2c explanation)
Rep	- Sometimes makes decisions mainly based on the form of representation (icon), not the value of the number. (Q2d, e explanations, Q4) - Sometimes argues as if he was making decisions based on the value (symbol) of the number and not on how it is written (Q2a, b, f, g explanations – says "can't be written" as opposed to "is not written") - "…" does not mean infinite, but only that we do not know all the digits. It could be an approximation or simply a number that is too long to write out. (Q2a, g) "…" is an icon. - √ is an index of an irrational number. (Q2d, f)
C.I.	- Inconsistencies in the concept image of real, rational and rational numbers: e.g., claims that irrational numbers are real in Q3b, but, in Q2, numbers classified as irrational are not also classified as real; a finite decimal is sometimes classified as rational (3.14) but sometimes as irrational (0.777778). - Unable to reconstruct a concept image for this student.
M.o.T.	- COMPLEXIVE: appears to classify individual numbers on a case by case basis, applying a different criterion each time. All our efforts at identifying some possible implicit general principles governing his decisions have failed because there were just too many contradictions. The student does not try to apply some general principles himself and declares so in response to Q7b: "some of them might work out".
G.R.	- Poor computational and algebraic skills. - Appears to know that a method exists for changing decimals into fractions but lacks the algorithmic knowledge necessary to do so. - Poor intuition regarding what it means to decide upon the truth of a general statement: a statement in mathematics is not either true or false; it could be "maybe true." (Q7) - Lacking mathematical culture.

Participant #4

1. In math, a irrational # is a real number that cannot be expressed as a ratio of *a/b*. For example, π.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*	*		*	*	*		*	*		*		*	*	*		*

Participant gives the following reasons for the above classifications:

a) cannot be expressed in totality as a ratio of a/b
b) has a limited number of digits, hence can be expressed as a/b
c) π known to be irrationals
d) since √2 is irrational that makes the whole fraction irrational.
e) rational since it is expressed as a ratio.
f) 1 + 2·2 = 5 = rational and real
g) (*left blank*)
h) can be expressed as a/b.

3. a) Integers: Yes
   b) Irrationals: Yes

4. ~~Yes, since~~ No, since a irrational number cannot be expressed as a ratio of *a/b*.

5. Yes it is but I am not sure how though.

6. No, since the calculator does not give all digits!

7. a) Irrational $\pi + \pi = 2\pi$ still irrational since it cannot be expressed as a ratio of *a/b*.

   b) $\pi \cdot \pi = \pi^2$ still the same explanation as above.

#4	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.7777778, 3.14 are rational. (Q2) - 22/7 (Q2), 99/70 (Q4) are rational. - $1 + 2\sqrt{4}$ is rational. (Q2) Implicit knowledge - A rational number is a ratio a/b (the condition that a and b be integers is not mentioned) (Q2e) - Decimals with a limited number of digits are rational. (Q2 explanations) - Fractions are rational. (Q2e, Q4) - Integers are rational. (Q2f explanation)
IQ	Explicit knowledge: - Irrationals are numbers that cannot be expressed as a ratio a/b. (a, b integers and b non-0 is not mentioned.) (Q1) - π is "known to be" irrational (Q1, Q2c explanation) - π^2 and 2π are irrational. (Q7) - $\sqrt{2}$ is irrational. (Q2d explanation) - If $\sqrt{2}$ appears in a fraction, the fraction is irrational Implicit knowledge: - Irrationals can be approximated by fractions but are not identical with their approximations. (Hesitant) (Q4) - Infinite decimals, regardless of periodicity, are irrational. (Q2) - Hesitant about what should be considered a repeating pattern. (Q2g explanation left blank)
R	Explicit knowledge: - "Real number" refers to all numbers. (Q1, Q2, Q3) - The intersection of **Q** and **IQ** is empty. (Q5)
Rep	- Representations must accurately express the numerical value of the number. (Q6) - In classifying numbers, makes decisions based on the form of its representation and not on the value of a number. (Q2d, Q7) - $\sqrt{2}$ is an index. (Symptom of irrationality.) (Q2) - π is an index. - Although he processes $1 + 2\sqrt{4} = 5$, this is not counted as a symbolic interpretation because it is the only possible occurance of symbol use and this can be better explained by his interpretation of $\sqrt{4}$ as an index

	for 2. (Q2f) - In Q7 he relies on an example that he may not fully understand ("π is known to be irrational") and therefore, like √2, any expression involving π is also going to be irrational.
C.I.	- **R** = all numbers. - **Q** = integers, ratios of integers, and numbers with a finite number of digits after the decimal point. - **IQ** = numbers that have an infinite number of digits after the decimal point, or contains characters that are not digits, e.g., √, π. - No obvious contradictions in the concept image, assuming that what appears to be a contradiction in Q2b is a consequence of the student's lack of algebraic skills, i.e., he doesn't know that √2/√8 can be simplified to ½.
M.o.T.	- Displays symptoms of COMPLEXIVE thinking because he gives a variety of factual (and not conceptual) reasons for classifying a number as rational or irrational or real. For example, 22/7 is classified as rational because it is "expressed as a ratio" (and he had no conditions on the terms of the ratio); √2/√8 is also expressed as a ratio, but the student classified it as an irrational number because "since √2 is irrational that makes the whole fraction irrational". So the connections are "factual" in the sense that they refer to the visible aspects of the expression or to associations with statements mentioned by the teacher as facts.
G.R.	- Has some limited mathematical culture: does justify her or her classifications by reference to his definition sometimes (Justifications in Q2; Q4); is concerned about why what he or she claims holds (see Q5); but believes that one example is sufficient to prove a general statement (Q7); responses to Q7 beg the question. - Aware that a decimal can be transformed into a fraction but likely lacks the algorithmic knowledge to do so.

Participant #5

1. A number that can't be written as (*a/b*) where is *a* and b is a ~~"nombre entier"~~ integers.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*	*		*	*	*		*	*		*		*	*		*	*

Participant gives the following reasons for the above classifications:

a) because it can't be written as (a/b)
b) because there is a patern in the decimal it can surely be written as (a/b)
c) It goes to infinity decimal.
d) √2 is a irrational number, therefore √2 / √8 is irrational.
e) It is written as (a/b)
f) √4 is 2 then 1 + 2(2) = 5 that can be written as (a/b)
g) no pattern present
h) cannot be written as (a/b)

3. a) Integers: Yes
   b) Irrationals: Yes

4. No because it is not the same number, only an approximation therefore 99/70 is a rational number but √2 is irrational.

5. It is empty because a number cannot be irrational and rational at the same time. I can only be one.

6. ~~No because~~ Yes we can represent an irrational number, but maybe not the entire number because sometime the number of decimal number is too big.

7. a) ~~Yes~~ Yes the numbers will stay irrational because both number were irrational at the beginning so by adding them it only make a bigger irrational number.

   b) ~~Yes because~~ No it will give us a rational number, exemple $\sqrt{2} \cdot \sqrt{2} = 2$

   (*participant circled the 2)*

#5	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 is rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - $1 + 2\sqrt{4}$ (Q2), 2 (Q7b) are rational. - If a number is written as a/b then it is rational (Q2e explanation) - If there is a pattern in a decimal then it can surely be written as a/b and therefore it represents a rational (Q2b explanation) - Products of irrational numbers are rational. (Q7b) Implicit knowledge - Decimals with a pattern are rational. However, the pattern must be a single digit repeating immediately following the decimal point. (Q2b explanation) - Integers are rational. (Q2f explanation, Q7b)
IQ	Explicit knowledge - Irrationals are numbers that cannot be expressed as a/b where a, b are integers (b non-zero not mentioned). (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - 3.14 is irrational. (Q2) - $\sqrt{2}$ is irrational. (Q2d explanation, Q4, Q7b) - $\sqrt{2} / \sqrt{8}$ is irrational because $\sqrt{2}$ is irrational. (Q2) - π is irrational. (Q2) Implicit knowledge - Decimals without an immediate, repeating pattern are irrational. (Q2g) - Infinite decimals are irrational. (Q2c explanation, Q7a) - Sums of irrational numbers are always irrational. (The "Addition makes bigger" obstacle.) (Q7a) - Irrationals can be approximated by fractions but are not identical with their approximations. (Q4)

R	- "Real number" refers to all numbers. (Q2, Q3)
Rep	- In classifying numbers, makes decisions based on the form of the representation: presence of a pattern, or √2 in the expression. (Q2) - Irrationals can be represented by finite decimals on a calculator. (Q6) - Finite decimals are indices for infinite decimals, e.g. 0.777778 is an index for 0.777777… (Q2b explanation) - √4 is an index for 2. (Q2f) - Representations of numbers may have a different value than the numbers they represent. (Q6) Hesitates.
C.I.	- **R** = all numbers. - **Q** = integers and numbers with a single digit repeating after the decimal point, or can be written as a/b. - **IQ** = numbers with an infinite number of digits after the decimal point which do not follow the single-digit-repeating pattern, expressions containing √2, or which cannot be written as a/b. - Some inconsistencies in the concept image: Student claims that **Q** and **IQ** are disjoint, but fails to recognize overlaps in his concept image of these. (Q5, Q2) - Concept image of rational numbers is restricted by his interpretation of pattern.
M.o.T.	- COMPLEXIVE: appears to classify individual numbers on a case by case basis, applying a different criterion each time. Pi is irrational because "it oes to infinity decimal"; 0.123456... and 30.450111... also "go to infinity" but 0.123456... is irrational because "it can't be written as a/b" and 30.4450111... is irrational because of "no pattern present"; 3.14 doesn't go to infinity but is nevertheless classified as irrational because "it cannot be written as a/b". Since 3.14 immediately followed 30.450111... which was classified as irrational because of "no pattern", it could have been implicitly classified as irrational because it also displays no pattern. This would be typical of complexive thinking (a chain complex).
G.R.	- A conceptual link is missing between the decimal and fractional representations of rational numbers; student's notion of rational number is based on a memorized declarative knowledge (a definition of rational number as a ratio of integers) that has not been internalized and connected with other elements of his/her concept image. Similarly, the definition of irrational number is only part of memorized knowledge and only applies to obvious cases. - Refers to definition in Q2e

Participant #6

1. An irrational number is any real number that cannot be expressed as the ratio *a/b*. This means that an irrational number cannot be represented as a simple fraction.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*	*	*		*	*		*	*		*		*	*		*	*

3. a) Integers: Yes
   b) Irrationals: Yes

4. No, because 99/70 is written under fraction form and does not apply the irrationality theorem.

5. Yes, the intersection is non-empty because a number cannot at the same time rational and irrational.

6. Yes, but one must be careful because the calculator might not express to many digits.

7. a) Yes, because in any case the result will be irrational.

   $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$

   $\sqrt{2} + \sqrt{3} = \sqrt{2} + \sqrt{3}$

   etc...

b) No, because $\sqrt{2} \cdot \sqrt{8} = \sqrt{16}$ which is 4, a rational number.

#6	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 is rational (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are rational (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - If a number is not irrational then it is rational (Q2, Q5) Implicit knowledge - A number that can be expressed/written as a ratio/simple fraction/fraction form a/b is rational (the condition that a, b be integers is not necessary, see Q2d; b non-zero also not mentioned) - Integers are rational (Q2f, Q7b)
IQ	Explicit knowledge - An irrational number is any real number that cannot be expressed as the ratio *a/b*. This means that an irrational number cannot be represented as a simple fraction. (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - 3.14 is irrational. (Q2) - $\sqrt{2}$, $\sqrt{3}$, $\sqrt{8}$ are irrational. (Q7) - π is irrational. (Q2) Implicit knowledge - Decimal representations of irrational numbers may have many digits (Q6), but do not necessarily have many digits, e.g. 3.14 is irrational (Q2h) - No mention of a repeating pattern on the decimals.
R	- "Real number" refers to all numbers. (Q1, Q2, Q3) - **Q** is disjoint with **IQ**. (Q5)
Rep	- In classifying numbers, the form of the representation sometimes prevails over its value (Q2d), although not always (e.g. Q2f) - Seems to sometimes perceive the signs used to represent numbers not as symbols, but as indices or icons. In particular, he seems to interpret "3.14" as a nickname and therefore as an index for the number π (Q2h). It is the conventional value of π in applications. - The interpretation of the sign $\sqrt{2}/\sqrt{8}$ as an icon: $\sqrt{2}/\sqrt{8}$ looks like a fraction and therefore it is classified as a rational number. - $\sqrt{}$ is interpreted as a symbol. We can see that he is not treating it as only an index because of his/her level of comfort in performing

	algebraic operations. (Q7) - "…" is an icon (diagram) which is a place holder for more, though not necessarily infinite, digits. (Q6)
C.I.	- **R** = all numbers - **Q** = ratio of two quantities, integers - **IQ** = many digits after the decimal point, in general - There are no very obvious contradictions in the student's concept image but it is difficult to find the logic behind his decisions: on the one hand, one would want to infer his belief that irrational numbers have many digits in their decimal expansions (confirmed by Q2a, g), but the classification of 3.14 as irrational undermines this conjecture. One could explain this by the hypothesis of "3.14" being treated as an index, or by a belief that mathematical laws may have exceptions. This second conjecture is, however, undermined by the student's response to Q7, where, in Q7b, he considers one example to be sufficient to refute a general statement, and, in Q7a, he claims that the sum of two irrational numbers will be, "in any case", irrational.
M.o.T.	- PSEUDO-CONCEPT because belonging to the same category is decided sometimes on the basis of a common property, and sometimes on the basis of factual reasons. An example of classification based on a common property is: 99/70 is not irrational because it is "written under a fraction form and does not apply the irrationality theorem", meaning his or her definition of irrational numbers as numbers that cannot be expressed as "a simple fraction". But otherwise "irrational number" is treated as a family name for numbers that are not fractions, have decimal expansions with three dots, $\sqrt{2}$, $\sqrt{3}$ and π which are known as standard examples of irrational numbers, and 3.14 which stands for π in calculations.
G.R.	- Student is not knowledgeable about decimal representations. (Q2, Q6) - Lacking formal and algorithmic knowledge about the relationship between decimals and fractions. - Avoids mentioning infinity. Uses "many" instead. (Q6)

Participant #7

1. Numbers that cannot be written in fractions such as π.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*				*	*		*	*		*		*	*	*		*

3. a) Integers: Yes
   b) Irrationals: Yes

4. No, because √2 cannot be express as a fraction, 99/70 is a fraction that gives a result close to √2 but it remains a rational number that is not infinite.

5. Intersection of rational and irrational number is Ø. They do not overlap they are in independent subsets ~~even~~ but they are both included in the family of real numbers.

6. Yes, for example √2. We can do operations such as square, cubic roots on the calculator, but it may not shows us all the decimals it really have.

7. a) No, the adition of 2 irrationals in always going to be irrational because a number with infinit decimals plus another number with ∞ decimals is always going to give a result as such (∞ decimals).

   b) I think that they re going to remain irrational always, even if we do a multiplication. At least there is a theorem that proof the opposite.

#7	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - $1 + 2\sqrt{4}$ is rational. (Q2) Implicit knowledge - Finite decimals are rational. (Q2) - Ratios of integers are rational. (Q2, Q4) - Rational implies not infinite. (Q4)
IQ	Explicit knowledge - Irrationals are numbers that cannot be written in fractions such as π. (condition that numerator and denominator are integers and denominator not 0 are not mentioned). (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2}$ is irrational. (Q6) - Product and sum of irrational numbers is an irrational number. (Q7) Implicit knowledge - Infinite decimals implies irrational. (Non-periodicity is not required.) (Q6) - Irrationals can be approximated by fractions but are not identical with their approximations. (Q4)
R	Explicit knowledge - **Q** and **IQ** are subsets of **R**. (Q5) - π is irrational but not real. (Q2c) Implicit knowledge - Numbers written with digits and operations are real. (Q2) - Numbers written with letter signs are not real. (Q2c)
Rep	- "Representation" in Q6 is taken as referring to names of irrational numbers that can be produced using the buttons on a calculator (indices) and this prompts the answer that "Yes, irrational numbers can be represented in a calculator" in Q6. These representations, it is suggested, are accurate. In the second part of the answer, after "but", "representation" refers to the display on the calculator of an irrational number. This display also seems to be perceived as an icon/picture of a number. But this may not be the whole name: "it may not show us all the decimals it really has". Thus the representation on the display may not be accurate. Thus, it seems that "representation" in Q6 is not interpreted as referring to the value of a number, but only to its name or picture. - In keeping with other answers in Q2, this student would surely say that 3.14159… is real. However, π is not real. Therefore, the sign "π" is an

	index for 3.14159… (Q2) - Predominantly indexical interpretations with iconic interpretations of the calculator display.
C.I.	- **R** = anything involving digits and operation, but not letters (π) - **Q** = not infinite: finite decimals and ratios of integers - **IQ** = not a fraction and infinite decimals. - Concept image is consistent but incoherent because "π index for the number whose digital representation is 3.14159…
M.o.T.	- PSEUDO-CONCEPTUAL: uses two properties as common to all irrational numbers: cannot be written "in fractions" and "is infinite"; however, the relationship between these two properties is factual and not conceptual: the student remembers the two as equivalent (wrongly) but does not even check this equivalence in particular cases (e.g. does not check that 99/70 has, in fact, an infinite decimal expansion so cannot be dismissed as not irrational just on this basis). - There are other symptoms of non-conceptual thinking. For example, √2)/√8 is classified as real but neither rational nor irrational and no reason is given; π is irrational but not real and it is the only non-real number in the student's classification. Yet, in Q3b, the student claims that all irrationals are real. Therefore the student couldn't have had conceptual reasons for denying π the status of a real number.
G.R.	- Traces of mathematical culture: refers to definition in classifying numbers in Q4; Justifies general statements with general arguments, not examples in Q7; mentions theorem and proof. (Q7b) - No mention of periodicity.

Participant #8

1. Any number that cannot be expressed as a fraction.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
			*				*		*			*			*				*		*		

3. a) Integers: Yes

   b) Irrationals: Yes

4. No, because 99/70 is a fraction and rational number can be expressed as fractions.

5. No I believe the intersection is empty because you cannot have a number that is both rational and irrational.

6. Yes you can, for example you could use π, which is irrational.

7. a) Yes because the irrational numbers have an infinite number of decimal places, meaning their sum will have an infinite number of decimal places.

   b) In the case of multiplication you would still get an irrational product because of the infinite number of decimal places.

#8	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are rational. (Q2) - "rational number can be expressed as fractions" (Q4) Implicit knowledge - Fractions, even if represented with square roots, are rational. (Q2) - Finite decimals are rational. (Q2) - Integers, even if represented with square roots, are rational. (Q2)
IQ	Explicit knowledge: - Irrationals are numbers that cannot be expressed as fractions. (Q1) - π is irrational. - 30.450111… is irrational (Q2g) - Irrational implies infinite decimals (Q7) - intersection of rational and irrational numbers is empty (Q5) Implicit knowledge - Infinite decimals does not necessarily imply irrational: 0.123456… (Q2a) is not classified as irrational (it is not classified at all) - Sums and products of irrational numbers are irrational. (Q7) - Irrationals can be approximated by fractions but are not identical with their approximations. (Q4)
R	Explicit knowledge - Integers are real numbers (Q3a) - Irrational numbers are real (Q3b) Implicit knowledge - no number listed in Q2 is real
Rep	- Representation is interpreted as referring to names rather than values of numbers (i.e. "representation" are indices for numbers). (Q6) - π is an index. (Q6) - Fractions are interpreted as icons. (Q2) - "Real" is interpreted as mathematical jargon. It is something that is conventionally said of numbers but has no meaning as it pertains to concrete examples. (Q2, Q3, Q5) - No symbolic interpretation: It might appear that he interprets $\sqrt{2}/\sqrt{8}$ as 1/2 and this is why he classifies it as rational. This would imply that he processes the expression and therefore interprets it as a symbol. But there are no traces of processing of number signs in his answers. There is only an appearance of formal logical processing of verbal propositions. Moreover, he claims that π can be represented in a calculator which suggests that he interprets it as an index (a name). Therefore $\sqrt{2}/\sqrt{8}$ might have been classified as rational because it looks like a fraction (icon). He classifies $1+2\sqrt{4}$ as rational maybe because it is equal to 5 which he finds not by processing the $\sqrt{4}$ to obtain 2 but by

	remembering that √4 is another name of 2 (index).
C.I.	- **R** = mathematical jargon: in Q3, student claims that all irrationals are real, but in Q2, no irrational is classified as real - **Q** = numbers with a finite number of digits after the decimal point - **IQ** = not a ratio and numbers with an infinite number of digits after the decimal point.
M.o.T.	- PSEUDO-CONCEPTUAL THINKING: reasoning based on logical processing of propositions such as "if a number is rational then it can be expressed as a fraction"; "irrational numbers have an infinite number of decimal places:; "π is irrational"; "99/70 is a fraction". Number signs are not processed; they are treated as any other word. This implies that he uses a conceptually homogeneous system of properties to justify his/her classifications, but number signs are still interpreted as family names or icons, like in a complex.
G.R.	- Traces of mathematical culture: refers to definition in classifying numbers in Q4; Justifies general statements with general arguments, not examples in Q7. - Periodicity not mentioned.

Participant #9

1. Any number that cannot be expressed as a fraction.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*		*			*			*				*		*		

3. a) Integers: Yes
   b) Irrationals: No

4. 99/70 is not irrational since it gives an approximation, but not a value for $\sqrt{2}$. 99/70 expressed as it is is rational.

5. No the set is empty since you cannot have a number that is both rational and irrational.

6. Yes you may, an example of this is π.

7. a) If you were to add two irrational numbers you would obtain a irrational number with many decimal places.

   b) If you were to multiply 2 irrational number you would still obtain a irrational number with again many decimal places.

#9	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are rational. (Q2) Implicit knowledge - Fractions and integers are rational, even if there is a square root in their representation (Q2, Q4) - Finite decimals are rational. (Q2)
IQ	Explicit knowledge: - Irrational is "any number that cannot be expressed as a fraction." (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - π is irrational. (Q2) Implicit knowledge - Irrationals can be approximated by fractions but are not identical with their approximations. (Q4) - Irrational numbers have many decimal places (Q7) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - Integers are real (Q3a) but $1 + 2\sqrt{4}$ is not real (Q2f). - No number in Q2 is classified as real - Irrational numbers are not real. (Q3b) Implicit knowledge: - **Q**, **IQ**, and **R** are all disjoint. (Q2, Q5)
Rep	- Representations do not correspond to the numerical value of the number. (Q6) Classifications based on form. - Two answers suggest that he or she makes decisions based on the form rather than the value of a number. In particular, in Q4 he or she says: "99/70 expressed as it is is rational". This suggests that, if represented otherwise, e.g., as 1.41428571..., it would be classified as "irrational" since it has "many decimal places" (Q7) - There are no explicit traces of processing signs symbolically. - Representations are indices, i.e., names like π. He says that yes, one can represent irrational numbers in a calculator, and "an example of this is π". This probably refers to the dedicated key, labelled π, and therefore " π " is interpreted here as an index. (Q6) - $\sqrt{}$ is interpreted as an icon. (Q2) - "Real" is interpreted as an icon, i.e., real number signs contain digits and no other characters. (Q2, Q3) - Integers are considered to be real numbers (Q3a), yet $1 + 2\sqrt{4}$ is not classified as real, therefore no link exists between multiple representations. Representations are distinct entities. - The student's classification of $\sqrt{2}/\sqrt{8}$ in Q2d as rational could be based on the fact that is is written as a fraction, so "written as it is, it is

	rational". - In Q2f, √4 could have been interpreted as another name of 2, without any processing and thus as an index.
C.I.	- **R** = integers or number without any digits following the decimal point - **Q** = numbers with finitely many digits following the decimal point - **IQ** = numbers with infinitely many digits following the decimal point - Consistent concept image which results in serious inconsistencies with conventional mathematics. (e.g., interpretation of real numbers.)
M.o.T.	- PSEUDO-CONCEPTUAL: He consistently uses two reasons to justify that a number is irrational: is not expressed as a fraction, and has many decimal places
G.R.	- Poor concept image for infinity. Infinite is understood as "many." - Small trace of mathematical culture in Q7: Justifies general statements with general arguments, not examples in Q7.

Participant #10

1. Any numbers that cannot be expressed into fractions.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*		*		*			*			*		*		*	*		*	*		*

Participant gives the following reasons for the above classifications:

a) because of infinite different numbers after the decimal.
b) because it is a fraction.
c) because of infinite different numbers after the decimal.
d) same as a)
e) same as a)
f) gives a number (= 5)
g) (periodic number)
h) (fraction)

3. a) Integers: Yes
   b) Irrationals: No

4. Yes, as 99/70 = √2. Yes, technically, as √2 can be expressed as a fraction, 99/70, it can be said to be rational.

5. Yes, it is non-empty, as it composed of common elements of both sets. However, the intersection can only be of rational numbers or irrational numbers.

6. No, because the calculator is not a precise tool to represent an irrational number in its entirety. (i.e., the <u>complete</u> number)

7. a) Yes, because only different digits will appear after the decimal and it will go on as an irrational number.

   b) No, it is rational. $\sqrt{2} \cdot \sqrt{2} = 2$.

#10	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 and 3.14 are fractions, therefore rational. (Q2) - 30.450111… is periodic, therefore rational. (Q2) - 2 and $1 + 2\sqrt{4}$ is a number equal to 5, therefore rational. (Q2) - 99/70 is a fraction, therefore rational. (Q2) - $\sqrt{2}$ is technically rational, as $\sqrt{2} = 99/70$. (Q4) Implicit knowledge - Finite decimals are identified with fractions and considered rational. (Q2) - Periodic decimals represent rationals (Q2g, explanation) - Integers are rational. (Q2, Q7b)
IQ	Explicit knowledge - Irrationals are "any numbers that cannot be expressed into fractions" (Q1) - 0.123456…, π, $\sqrt{2} / \sqrt{8}$, and 22/7 are irrational "because of infinite different numbers after decimal." (Q2 explanations) Implicit knowledge - $\sqrt{2}$ is irrational (Q7b) - Irrationals can be approximated by fractions and can be "technically" identified with their approximations. (Q4) - A truncated decimal does not represent an irrational number in its entirety (Q6) - Some numbers are both rational and irrational: e.g. $\sqrt{2}$ is irrational (Q7b) but it is also "technically" rational since it can be expressed as a fraction.
R	Explicit knowledge - 0.777778 and 3.14 are real. (Q2) - 30.450111… is real. (Q2) - $1 + 2\sqrt{4}$ is real. (Q2) - Integers are real (Q3a) - Irrational numbers are not real (Q3b) Implicit knowledge - Rationals are real (Q2)

Rep	- Rationality and irrationality are entirely dependent on the representation: depending how one represents a number, it can be classified as rational or irrational. (Q4, Q7b) Therefore signs are icons or indices. - Representations are names (indices): calculators don't show the "complete number" can be interpreted as "calculators don't display the complete name of a number". - Multiple representations are identified with each other, i.e., treated as indices. (Q2 explanations, Q4) - Interpretation involving $\sqrt{2}$ in Q7b is not symbolic. $\sqrt{2} \cdot \sqrt{2} = 2$ can be viewed as an icon for the trick of "getting rid of the square root sign" when you multiple identical square roots. This interpretation is more consistent with the student's other interpretations.
C.I.	- Incoherent and inconsistent concept image: conceptions are not linked; the concept image is made of several independent ideas about the same type of objects; sometimes one is used, sometimes another. - Unable to reconstruct a concept image for this student
M.o.T.	- COMPLEXIVE: There are several deciding factors which heavily depend on the form of the representation. E.g., Irrationals "cannot be expressed as a fraction," 0.777778 "*is* a fraction" (emphasis mine), π has "infinite different numbers," $1+2\sqrt{4}$ "gives a number," 30.450111... is "periodic," $\sqrt{2}$ "can be expressed as a fraction"
G.R.	- Student makes contextual interpretations of representations (icons). - Lack of mathematical culture: use of = freely to mean equal or approximately equal; does not think in systems, not reflective.

Participant #11

1. Numbers that do not follow a constant pattern on the decimal numbers.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
			*				*			*			*		*				*		*		

3. a) Integers: Yes
   b) Irrationals: ~~No~~ Yes

4. No because with the fraction you can multiply irrational # and get a rational. The fraction of a irrational # is rational.

5. I don't know!

6. No, because irrational # do not have a serie of decimal places (constant repetition).

7. a) ~~No~~ Yes, because adding two numbers with no pattern in the decimal places, you would get a number with also no pattern on decimal places.

   b) Yes, ex: $\sqrt{2} \times \pi = 4.0391905...$

#11	Analysis of individual's knowledge and concept image

Q	Explicit knowledge - 0.777778 and 3.14 are rational. (Q2) - $1 + 2\sqrt{4}$ is rational. (Q2) - 99/70 is rational. (Q4) - “the fraction of an irrational” is rational. (Q4) Implicit knowledge - Finite decimals are rational. (Q2) - Integers (even under the disguise of square roots) are rational. (Q2)
IQ	Explicit knowledge: - Irrationals are decimals with no constant pattern. (Q2) (“constant repetition”, Q6) - π is irrational. (Q2) - 22/7 is irrational. (Q2) - 30.450111… is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ is irrational (Q2) - $\sqrt{2}$ is irrational. (Q7b) - Sums and products of irrational numbers are irrational. (Q7) Implicit knowledge: - Irrationals can be approximated by fractions but are not identical with their approximations (based on “No” in Q4. But this is not clear from the explanation given in the full answer). - If no pattern of “constant repetition” appears in the decimal, then the number can be classified as an irrational. (Q1, Q6, Q7) - 30.450111… is irrational, because the pattern is not “constant.” (Q2, Q6) - The product of a rational number and an irrational number is a rational number; (Q4. Not clear what the student means by a “rational”, or “fraction of a rational” however) - The student is not good at simplifying number representations algebraically; he “calculates” numbers by punching buttons on a calculator. For example, the student might have “calculated” the number “$\sqrt{2} / \sqrt{8}$” by punching √, 2, /, √, 8, =. This sequence of keys, on an iPod calculator, for example, produces 0.176776695296637, which does not exhibit any clear visible repeating sequence and so, by the student’s definition, qualifies as an irrational number. The proper sequence of buttons would have to be 2, √, /, (8, √). That the student is not using his calculator properly shows also in Q7b, where $\sqrt{2} \times \pi$ gives him 4.039…, while the correct sequence of calculator buttons should produce 4.442… (“Reading left to right” obstacle to performing algebraic operations.)
R	Explicit knowledge: - Intergers are real. (Q3a) - Irrationals are real. (Q3b) - No number in Q2 is real. Implicit knowledge:

	- Difficult to say if any given concrete example of a number is real. (Q3b hesitation, Q2) - "integers are real", "irrationals are real" are part of the student's memorized declarative knowledge, but these statements have not been internalized as the student's practical knowledge to use in classification tasks.
Rep	- Representations are sometimes just names, i.e., indices. In particular, the sign "22/7" could be interpreted by the student as an index in the sense of another name of the number π. (Q2) - Alternatively, dividing 22 by 7 on a calculator leads to the assumption that there is no "constant repetition." The division sign is a symbol which instructs the student to perform an operation on the calculator. (Q2) - "Real" is a mathematical convention, mathematical jargon. (Q2, Q3) - The primary representation of a number is its decimal representation, probably as it appears in the calculator. There is no mention of the decimal expansion ever being infinite. - "…" is an icon for more.
C.I.	- Student's concept image is not a consistent entity: claims that integers and irrational numbers are real in Q3 but fails to acknowledge this in Q2. In general, his or her concept images of Q, IQ and R are not very robust. - Unable to reconstruct a concept image for this student.
M.o.T.	- COMPLEXIVE thinking: Decisions are based on a number of criteria having to do with the student's idea of "constant repretition." This may be pseudo-conceptual in nature, but we cannot conclude this based on the answers given. Maybe if he had provided justifications for his classifications we could better understand the criteria he used.
G.R.	- Over-reliance on the calculator in particular and finite decimal representations in general. - Remembers little of the formal declarative knowledge about numbers. "Real number" has little practical meaning to the student. - Poor algebraic skills. (Q2, unable to simplify $\sqrt{2} / \sqrt{8}$) - Some mathematical culture appears in the student's answers: refers to his/her "definition" of irrational number but uses an example to prove a general statement. (Q7b)

Participant #12

1. ~~An irrational number is a # that has many number~~ is a number than contains ~~many~~ a variety of different numbers following the decimal point and doesn't follow any particular pattern.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*				*		*				*	*					*

Participant gives the following reasons for the above classifications:

d) = 0.5

3. a) Integers: Yes
   b) Irrationals: No

4. Yes, because they both have a variety of different # following the decimal that doesn't follow any particular pattern.

5. ?

6. No because a calculator is incapable of showing us exactly how many numbers follow the decimal point.

7. a) If you would add two irrational numbers together then the result would be irrational - ~~because~~ unless the two numbers where to add up perfectly to give one

an integer or a rational # - but because we unaware of exactly how many #'s there are after the decimal it would be close to impossible.

b) The result would still be irrational.

#12	Analysis of individual's knowledge and concept image
Q	Explicit knowledge - 0.777778 is rational. (Q2) - 30.450111… is rational. (Q2) Implicit knowledge - Being expressed in fraction form is not a sufficient criterion for rationality. (22/7 and 99/70 are classified as irrational, Q2, Q4) - Rationality depends on whether a pattern can be found on the decimal representation. - A rational number has many digits (Q2) (3.14 is classified as real but not rational, maybe because it has few digits) - "Pattern" refers to periodic pattern, not just any pattern (30.450111… is periodic; 0.777778 is an approximation of a periodic decimal and so they are classified as rational; 0.123456… has a definite pattern, but it is not periodic, so it was classified as irrational in Q2)
IQ	Explicit knowledge: - Irrational number is "a number that contains a variety of different numbers following the decimal and doesn't follow any particular pattern" (Q1). - π is irrational. (Q2) - 0.123456… is irrational. (Q2) - 22/7 is irrational. (Q2) - 99/70 and √2 "because they both have a variety of different # following the decimal that doesn't follow any particular pattern." (Q4) Implicit knowledge: - Numbers are identified with their decimal representations. (Q4, Q6, Q2) - Irrational numbers are unknowable, because we cannot view all their digits on a calculator and we cannot guess what they could be because there is no pattern. (Q6, Q7) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - Integers are real. (Q3a) - 0.5 is real. (Q2d explanation) - √2 / √8 (Q2, because it is equal to 0.5) - 1 + 2√4 is real. (Q2) - 3.14 is real. (Q2)

	Implicit knowledge: - √2 / √8 is classified as real, but not rational or irrational (Q2); student claims it is equal to 0.5 – not ½ - and this is supposed to justify why the number was classified as real (Q2d, explanation); perhaps because '0.5' is a short sequence of digits) - Rational numbers are not real. (Q2) - Irrational numbers are not real. (Q3b) - Real numbers have very few decimals places or none. (Q2)
Rep	- Calculators are insufficient tools for expressing real numbers. (Q6) - Rationality or irrationality depends on the representation, in particular, the decimal representation and the existence of a pattern therein. (Q4, Q2e, g) - 22/7 and 99/70 are irrational because they have a "variety of different numbers following the decimal point" suggests that "/" is a SYMBOL which instructs the student to perform an operation on the calculator. (Q2, Q4) - "Real" is an index for integers or numbers with very few decimals places. (Q2, Q3) - A number is essentially a sequence of digits; thus numbers 'resemble' their decimal representations and therefore we could say that decimal representations are interpreted as icons by the student. - A number can be given by a sign other than a sequence of digits, and then the sign is interpreted as a symbol. If the symbol can be converted into a small sequence of digits, then it represents a real number. If the symbol can be converted into a larger number of digits then it is rational or irrational. If there is a periodic pattern in the sequence, the number is rational; no periodic pattern indicates an irrational number. - Pattern in the sequence of digits is interpreted as an index: it indicates whether the number is rational or irrational.
C.I.	- **R** = number that have few or no digits after the decimal point, or representation which can be converted into this form - **Q** = numbers which have a "variety of different" digits following the decimal point which follow a pattern. - **IQ** = number which have a "variety of different numbers following the decimal point and doesn't follow any particular pattern." (Q1) - Consistent concept image which depends solely on the decimal representation and the use of a calculator to find this representation when need. Can result in some serious inconsistencies which go unnoticed. - Contradicts accepted mathematics
M.o.T.	- PSEUDO-CONCEPTUAL: Irrationality is decided based on an association with the concept of infinite, non-periodic decimal representations, therefore all representations must be converted into decimal form for consideration. Numbers in decimal form which do not have an evident pattern are either rational or real, e.g., 0.777778 and

	30.450111... are rational, but 3.14 is real. Presumably 0.777778 in an index for 0.77777... and a pattern is perceived as is the repeating 1s in the second example. However, 3.14 doesn't fit the student's own irrational number defnition but is also not rational; it's real, as are $\sqrt{2}/\sqrt{8}$ and $1+2\sqrt{4}$ - Thinking is not conceptual because the student bases his decisions on patterns in a factual way.
G.R.	- Over-reliance on the decimal representation: numbers appear to be identified with the decimal representation. - Avoids infinity: Uses phrases like "variety of numbers" or "exactly how many" when talking about numbers with infinitely many digits after the decimal point. - Missing the link between decimals and fractions. - Some mathematical culture: consistently refers to his definition of irrational numbers in justifying responses; reasons in generality about general statements. (Q7)

Participant #13

1. Irrational numbers are not ~~whole numbers~~ ~~real~~ integers (nombres entiers). It contains many non-sense, random decimals.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.1450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*			*			*				*		*				*		*		

Participant gives the following reasons for the above classifications:

d) √2 / √8 = √2 / √4·2 = ¼ = 0.25

3. a) Integers: Yes
   b) Irrationals: No

4. By the definition, even if it is a good approximation, it doesn't mean it is irrational as well.

5. ?

6. By the calculator, you can get a good approximation of it but it will not show all the decimals of it. Therefore it would not be super precise.

7. a) Yes & no, it depends on the irrational numbers you add. If the sum still has many random decimals, it is still an irrational number.

b) Yes the multiplication answer will still be irrational since the decimals of the two numbers are random & there're many.

#13	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - √2 / √8 and 1 + 2√4 are rational. (Q2) - π is rational. (Q2) Implicit knowledge: - Finite decimals are rational. (Q2) - Integers are rational. (Q2f) - Fraction form is not sufficient for deciding rationality. (Q2e) - Sums or irrational numbers might be rational. (Q7a)
IQ	Explicit knowledge: - Irrational numbers "contain many non-sense, random decimals." (Q1) - 0.123456… and 30.1450111… are irrational. (Q2) - 22/7 is irrational. (Q2, perhaps because s/he is "calculating" the number.) - 99/70 is not necessarily irrational. (Q4) Implicit knowledge: - Irrationals can be approximated by fractions or finite decimals but are not identical with their approximations. (Q4, Q6) - Pattern should begin immediately following the decimal. (Q2g) - Products of irrational numbers are irrational. (Q7b) - Sums or irrational numbers might be irrational. (Q7a)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are not real. (Q3b) - No number in Q2 is real. Implicit knowledge: - Inconsistent concept image of real numbers. (Q2, Q3) - "integers are real", "irrationals are real" are part of the student's memorized declarative knowledge, but these statements have not been internalized as the student's practical knowledge to use in classification tasks. (Q2, Q3)
Rep	- Representations correspond to numerical values, specifically decimal numbers. (Q4, Q6) - The division sign and the root sign are symbols which instruct the student to perform an operation on the calculator. (Q2) - Student explicitly processed the expression in Q2d. - "…" is an icon for more because infinity cannot be achieved. (Q6)

	- “Real” is mathematical jargon. (Q2, Q3) - π is not a symbol. (Q2c) It is an icon: resembles a rational number because, as written, it has no decimals.
C.I.	- Inconsistent concept image which relies on decimal representations: π is rational (Q2c), no clear concept of real numbers, 22/7 is irrational (Q2) but 99/70 is rational (Q4). - Unable to reconstruct a concept image for this student.
M.o.T.	- PSEUDO-CONCEPTUAL: All reasoning depends on one concept, namely that of having many, random decimals. - This is pseudo-conceptual in the sense that Vinner uses the word as an "uncontrolled association" or a "superficial similarity." It is not conceptual 99/70 and 22/7 are interpreted differently (factually). It may just be that the word "approximation" triggers yet another association, i.e., a pseudo-conceptual behaviour. - If the student were made aware of the link between repeating decimals and fractions, he could possibly achieve a conceptual mode of thinking.
G.R.	- Avoids infinity, using “many” instead. - Missing link between decimals and fractions. Lacks algorithmic knowledge to go from decimal to fraction. - Some mathematical culture: refers to definition (Q1, Q4); answer general questions with general statements (Q7b); but truth values of a mathematical statement are True, False, and It Depends. (Q7a)

Participant #14

1. Can only be divided with no remainders by itself and never ends and doesn't repeat itself.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.1450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*		*		*		*		*	*			*		*		*		*		*

Participant gives the following reasons for the above classifications and made the following hesitations:

d) $\sqrt{\frac{1}{4}}$

~~irrational~~

e) ~~irrational~~

3. a) Integers: Yes
   b) Irrationals: No

4. No, because 99/70 is a repeating ~~number~~ decimal number while $\sqrt{2}$ doesn't repeat itself.

5. Yes the intersection is empty. Numbers are either rational or irrational not both.

6. No, because calculators can only approximate irrational numbers. They quickly become inaccurate.

7. a) Yes. Adding two number that don't repeat themselves and never end will give another number that doesn't repeat itself and never ends.

   b) Yes, π^2 is irrational just as π is. Not sure why.

#14	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. Implicit knowledge: - Finite decimals are rational. (Q2) - Fractions are rational. (Q2, Q4)
IQ	Explicit knowledge: - Irrational numbers never end and "doesn't repeat itself." (Q1) - Irrational numbers "can only be divided with no remainders by itself." (Q1) - 0.123456… and 30.1450111… are irrational. (Q2) - π is irrational. (Q2) Implicit knowledge: - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Pattern should be evident and/or begin immediately following the decimal. (Q2g)
R	Explicit knowledge: - 0.777778 and 3.14 are real. (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are real. (Q2) - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b) - **Q** and **IQ** are disjoint. (Q5) Implicit knowledge: - Finite decimals are real. (Q2) - Fractions are not real. (Q2e)
Rep	- Representations correspond to exact numerical values: Finite decimal approximations are seen to be inaccurate. (Q6) - Calculators are insufficient tools for representing irrational numbers. (Q6) - Identifies fractions with repeating decimals. Fractions are indices. (Q4) - $\sqrt{2}$ is an icon for it's decimal representation (Q4, student refers to non-

	repeating decimal.), or as a symbol (Q2, instruction to perform a calculation), and likely as an index as well (Q2d, hesitation in explanation) - “Real” is not convention, it is a name (index) which applies to finite decimals and integers.
C.I.	- **R = Q** - **Q** = integers, ratios of integers, numbers with finitely many digits after the decimal point. - **IQ** = number with infinitely many non-repeating digits after the decimal point. - Consistent concept image which is fairly close to conventional mathematics. - Only exception to an otherwise consistent concept image is that 22/7 is not classified as real, but $\sqrt{2}/\sqrt{8} = 1/2$ is real.
M.o.T.	- CONCEPTUAL: Bases decisions on "never ending" and "not repeating." - Views 99/70 as a repeating decimal which suggests that he is at least informally aware of the link between fractions and repeating decimals. - Confusing the definition of irrational number with prime numbers is a pseudo-conceptual behaviour based on the way the two concepts are similarly defined (i.e., defined in terms of what they are not) and not a part of his understanding of irrational numbers. He nowhere uses the properties of prime numbers to make decisions about irrational numbers. There is a small inconsisitency with the case of how 22/7 is classified. The fact that there are some hesitations (Q2) and some self-questioning (Q7b) suggests that this student is using a conceptual modes of thinking.
G.R.	- Student is hesitant about fractions. - Greater reliance on and comfort with decimals: views fractions as decimals. - Some mathematical culture: Multiple representations of $\sqrt{2}$ co-exist simultaneously (Q4), uses general statements to answer general questions (Q7), is concerned about why a claim holds (Q7b), but student confuses prime number characteristics as those of irrational numbers (Q1). - Able to do calculations/simplifications. (Q2)

Participant #15

1. Irrational number cannot be defined. They "keep going," meaning they don not repeat a sequence after the decimal, for ex. π.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*		*	*		*		*		*		

Participant gives the following reasons for the above classifications:

a) cannot be defined and numbers after decimal do not repeat.
b)
c) (does not repeat a sequence of numbers)
d) (same explanation as above)
e) It can be defined.
f) *(left blank)*
g) cannot be defined
h) It is defined.

3. a) Integers: Yes
   b) Irrationals: No

4. No because it is a fraction. Fractions are rational.

5. No because a rational number is definable but an irrational number is not.

6. No because the sequence of numbers after the decimal does not repeat, so thus showing only 8 or 9 digits does not suffice to represent the number entirely.

7. a) No because there will still be no sequence that repeats and so it cannot be a rational number. 3.1415… + 3.1415… = 6.2830….

   b) No because again, there will be no repeating sequence. This is just a guess. I don't feel comfortable with irrational and rational numbers because I do not have a good understanding of them. I believe they should be introduced earlier in education.

#15	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - $1 + 2\sqrt{4}$ is rational. (Q2) - Fractions are rational. (Q4) - Rational numbers are defined. (Q4) Implicit knowledge: - Finite decimals are rational. (Q2) - Decimals with a repeating sequence are rational. (Q2) - Fractions are rational.
IQ	Explicit knowledge: - Irrationals are infinite decimals with no constant pattern: "They 'keep going," meaning they do not repeat a sequence after the decimal. (Q1) - Irrational numbers "cannot be defined." (Q1, Q5) - π is irrational. (Q1, Q2) - 0.123456… and 30.450111… are irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ is irrational. (Q2) Implicit knowledge: - Irrationals can be approximated by fractions or finite decimals but are not identical with their approximations. (Q4, Q6) - Pattern must begin immediately following the decimal and/or be short enough to show on a calculator display. (Q6) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - 22/7 is real. (Q2) - $1 + 2\sqrt{4}$ is real. (Q2) - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b) Implicit knowledge:

	- No clear concept image of real number.
Rep	- Concept image of infinity includes a notion of patterning. (Q1) - π is an icon: it resembles a number which can be defined (Q2 explanations) and also an index: the name of a particular number that "keeps going" without repeating. Student chooses to explain π in terms of periodicity instead of definablity. (This also point to complexive m.o.t.) - $\sqrt{2}$ is an index (symptom) of irrationality. (Q2d explanation) - $\sqrt{4}$ is an index (name) for 2. - Real numbers are judged based on their representation: fractions and integers are real (Q2, Q3a) but decimals are not (Q2). So "Real" is an index, a symptom of the representation. - Calculators are insufficient tools for representing irrational numbers. (Q6) - You can not produce a pattern from things which have no pattern. (Q7)
C.I.	- **R** = meaningless jargon. - **Q** = ratios of integers, definable numbers, repeating decimals. - **IQ** = undefined numbers, number with infinitely many digits following the decimal point with no repetition, - Inconsistent concept image: classifications are based on representation and not value; no clear concept image of real numbers.
M.o.T.	- Uses ambiguous notion of being definable. - Interprets periodicity incoherently. - COMPLEXIVE: Decisions are based on whether a number cannot be defined, can be defined, or is defined. (Q2) Decisions are also based on if the number "keeps going" (Q1) or is a fraction (Q4), and if it does not have a repeating sequence (Q1, Q2a, Q6, Q7). These are factual interpretations.
G.R.	- Infinity is a process, not an object. (Q1: "keep going", Q2, Q5) - Student reflects on his/her knowledge and abilities and readily admits discomfort with and lack of understanding of rational and irrational numbers. (Q7b) - Missing link between decimals and fractions. (Q2) - Some mathematical culture: usually responds in generality to general questions; refers back to his definition.

Participant #16

1. A number with infinite decimals.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*					*		*		*		

Participant gives the following reasons for the above classifications:

a) *(left blank)*
b) (Decimals without ∞)
c) (Infinite decimal)
d) (Infinite decimal)
e) (Decimals without ∞)
f) real numbers
g) (Infinite decimals)
h) (Decimals without being infinite)

3. a) Integers: Yes
   b) Irrationals: No

4. No, because 99/70 is only an approximation, there it does not have an infinite number of decimals like √2.

5. No, they don't intersept.

6. No, since a calculator can not show more than a certain number of digits.

7. a) Yes, since the infinite # of decimals of the first irrational number will add to the infinite # of decimals of the second irrational #. Therefore giving an infinte # of decimals.

   b) ~~Depends, some might give rational numbers.~~ Yes, since the decimals are infinite, a multiplication has to give infinite # of decimals also.

#16	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. Implicit knowledge: - Finite decimals are rational. (Q2 explanations) - Rational number implies finite decimal. (Q4) - Integers are not rational. (Q2f)
IQ	Explicit knowledge: - Infinite decimals are irrational. (Periodicity is not mentioned.) (Q1) - 0.123456… and 30.1450111… are irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ is irrational. (Q2) Implicit knowledge: - Irrationals can be approximated by fractions or finite decimals but are not identical with their approximations. (Q4, Q6) - Sums of irrational numbers are irrational. (Q7a) - Products of irrational numbers are irrational. (Q7b, but hesitantly)
R	Explicit knowledge: - $1 + 2\sqrt{4}$ is real. (Q2) - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b) Implicit knowledge: - Rational numbers are not real. (Q2) - "Real number" refers to integers only. (Q2, Q3a)
Rep	- Calculators are insufficient tools for representing irrational numbers. (Q6) - $\sqrt{2}$ is an index for an infinite decimal number. (Q2d) - π is an index for an infinite decimal number. (Q2c) - Fractions are indices for finite decimals because they are rational and rational numbers have finite decimals. - No explicit interpretations of signs as symbols, no evidence of calculation or algebraic manipulations.

C.I.	- **R**, **Q**, and **IQ** are all disjoint. (Q5, Q2) - **R =** integers, i.e., no decimals. - **Q** = numbers with finitely many digits after the decimal point. - **IQ** = numbers with infinitely many digits after the decimal point. - Concept image is self-consistent but inconsistent with conventional mathematics.
M.o.T.	- COMPLEXIVE: Uses many different criteria to make decisions. - Looks at odd characters (index) like letters, √; if they appear, the number is a candidate for irrational number except perhaps for √4 which is an index for 2. - If it's written as a fraction, it implies a finite number of decimals (so 22/7 is finite, 99/70 is finite); then decides upon the number of decimals after the decimal point: if infinite (... = infinite) then irrational, if finite then rational. - No decimals- it's real.
G.R.	- Student lacks formal knowledge of the number system. - Consistent concept image. - Relies heavily on decimal representation. - Some mathematical culture: uses general statements to answer general questions (Q7).

Participant #17

1. A number with infinite decimal places according to modern day technology.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*				*			*		*		

3. a) Integers: Yes
   b) Irrationals: No

4. No because it converges to √2.

5. I'm not sure what non-empty is but when they intersect the real number won't be known and we can consider it "empty" so no.

6. You can approximate it but cannot represent a number with infinite decimals.

7. a) Yes because no computer or calculator can take into account all the decimals and therefore its not a rational # → not a real #.

   b) Same as a, but it is possible that it is rational for both a and b if we can take

   into account all decimals but they aren't all known.

#17	Analysis of individual's knowledge and concept image

Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. Implicit knowledge: - Finite decimals and fractions are rational.
IQ	Explicit knowledge: - Infinite decimals “according to modern day technology” are irrational. (No mention of periodicity.) (Q1) - 0.123456… and 30.4510111… are irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are irrational. (Q2) Implicit knowledge: - Numbers expressed using the square root sign are irrational. - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. - Sums and products of irrational numbers are irrational. (Q7) - Irrational numbers are those that cannot be stored in a calculator because of infinitely many decimal digits. (Q7a)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b, Q7) - Intersection of **Q** and **IQ** is empty. (but her notion of intersection is very strange, so I don’t know if we can say that) (Q5) Implicit knowledge: - No number in Q2 is real. - Decimals are not real. (Q2) - Fractions are not real. (Q2) - If a number is not rational then it is not real (Q7); by contrapositive, this implies that if a number is real then it is rational. Perhaps real means the same as rational for her? Therefore he doesn’t classify the numbers in Q2 as real because it would perhaps be redundant.
Rep	- Representations should correspond to the value of the number. (Q6) - Signs are treated as indices. (Q2); e.g. $\sqrt{}$ is an index of irrational number. - “…” could be interpreted as a diagram (icon) for writing many more digits, and/or as an index: indicating that we don’t know theses digits that are unknown, or just indicating that the number has infinitely many digits. - Relies on decimal representation. - “to add” also seems to refer to calculating the sum on a calculator; what he is saying in response to Q7a is that one cannot add two irrational numbers on a calculator. He may not be thinking of “sum” as the result of an operation but as a process of adding; there is evidence of him thinking in terms of process rather than object in answer to Q4: 99/70 “converges” to $\sqrt{2}$. He doesn’t think of 99/7 as an actual, completed

	object.
C.I.	- Consistent but incoherent concept image. - **R** = **Z** - **Q** = explicit ratios of integers, numbers with finitely many digits after the decimal point. - **IQ** = numbers with infinitely many digits after the decimal point, any expression involving characters (√, π) which are not digits.
M.o.T.	- COMPLEXIVE: Most justifications/reasonings are in terms of infinite decimal places. 99/70 is not irrational because it converges to √2. - Finite decimals are correctly identified as rational, but it is unclear that this is because the two sets are disjoint. - Unable to tell from the answers given how the student is reasoning about √2/√8 and 1+2√4 being irrational; likely the presence of the square root sign qualifies the expression as that of an irrational number.
G.R.	- Poor at calculations. (Q2d, f) - Infinity is a process. (Q7) - Some mathematical culture: uses general statements to answer general questions (Q7), is concerned why a claim holds (Q5), but mathematical statement can have the truth value of True, False, or Possible. (Q7b)

Participant #18

1. A number that no matter how many decimals you go down it never stops.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*		*		*			*		*		*	*		*		*		*		

Participant gives the following reasons for the above classifications:

a) inf decimals
b) it's a number that we know exactly what it is
c) (*left blank*)
d) inf decimals
e) we know the exact value of this number
f) it equals 5 obviously rational and a real number
g) inf decimals don't know the actual value of this number
h) we know the exact value of this number

3. a) Integers: Yes
   b) Irrationals: No

4. No, 99/70 does not have an infinite amount of decimals.

5. No a ~~ratio number cannot be~~ set cannot have irrational and rational aspects.

6. No, a calculator will give you a very general approximation of an irrational number but not the number exact value.

7. a) ~~Not in all cases, lets say we add 1.1111 and 1.9999~~
Yes adding two irrationals gives an irrational because of the nature of having inf decimals.

b) Yes the variance in numbers and the amount of numbers makes it impossible for the result to be rational.

#18	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - 1 + 2√4 is rational. (Q2) - Exact value is known. (Q2b) Implicit knowledge: - Rational numbers are precise: they are expressed finitely as a finite decimal or a fraction (a finite diagram). (Q2 explanations) However, π is also a finite diagram but is classified as irrational without any explanation (Q2c) suggesting that knowledge of π's irrationality is a matter of convention (a symbol), relying on authority. - Rational numbers do not have an infinite number of decimals. (Q4)
IQ	Explicit knowledge: - Infinite decimals which "never stop" are irrational. (Periodicity is not mentioned.) (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - π is irrational. (Q2) - √2 / √8 is irrational. Implicit knowledge: - Irrational numbers are imprecise. (Q2 explanations) - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Sums (hesitantly) and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - 0.777778 is real (but 3.14 is not) (Q2) - 22/7 is real. (Q2) - 1 + 2√4 is real because it is equal to 5. (Q2f explanation) - 5 is real. (Q2f explanation) - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b)

	- **Q** and **IQ** are disjoint. (Q5) Implicit knowledge: - Fractions with integer numerators and denominators are real.
Rep	- Representations correspond to exact numerical values. (Q6) - Relies on the decimal representation. - π is an index. - 3.14 is perhaps being interpreted as a nickname for π. It is an approximation, not a number in its own right. - Iconic and symbolic interpretations are questionable. (See M.o.T. section in this table for more details.)
C.I.	- **R** = known values (except 3.14) - **Q** = known values - **IQ** = unknown values - Consistent and coherent concept image which nevertheless contains inconsistencies with conventional mathematics.
M.o.T.	- PSEUDO-CONCEPTUAL: Classifications are not completely chaotic as in complexive thinkers. Decisions are made based on the student's idea of "exact value" as it relates to number having infinitely many digits after the decimal point. This has less to do with exactness and more to do with the appearance of the representation. There are only factual links between elements: classifies $\sqrt{2}/\sqrt{8}$ as irrational because "inf decimals", which suggests that he has not attempted to process the expression even on a calculator. (non-symbolic interpretation, indexical interpretation of $\sqrt{}$) He also obviously does not process 99/70 since he states that "99/70 doesn't have an infinite number of decimals" (non-symbolic interpretation) - Knowing the exact value of 22/7 (Q1f) may mean that there are no dots or square roots in the expression. The categories are subjective, and already this fact would be enough to declare his thinking as pseudoconceptual rather than conceptual. - There is an uncontrolled association of irrational numbers with infinite decimals and rational numbers with finite decimals. Infinite decimals = we do not know the value. So 99/70 has finitely many digits after the decimal because we know its value (just as we do 22/7). It is unclear if 99/70 is known because it if finitely expressed (iconically, i.e., the representation does not have a decimal and some digits followed by a ...) or if it is because the student recalls that fractions are rational and can be expressed as a decimal (symbolic interpretation) and that decimal necessarily has finitely many digits according to this student's concept image of rational number. These categories are subjective. - Decisions are not justified on the basis of these two categories alone, but also on a kind of "proposition" saying that if a number has infinite decimals then its exact value is not known (Q2g). The proposition is mathematically false, but he applies it as if it was a known "fact" and does not question or verify the conclusions he draws from it. (pseudo-

	conceptual behaviour) In Q4 he appears to apply the contrapositive of this proposition: since 99/70 is a number whose "exact value" he knows, then it cannot have infinite decimals. The conclusion is false which suggests that knowing the exact value does not necessarily mean that he would be able to indicate where this number is situated relative to other numbers, for example, to numbers 7/5 and 36/25. In one case, $\sqrt{2}/\sqrt{8}$, he uses the converse of this proposition: he doesn't know the exact value of this number so he claims that it has an infinite number of decimals, without even checking. - In classifying π (Q2c) as irrational, he gives no justification although he could have said that "we don't know the exact value of this number" or that the number has an infinite number of decimals. He probably heard the teacher saying that Pi is irrational so his classification is based on this received "factual" knowledge, which again, points to the pseudo-conceptual nature of his thinking.
G.R.	- Infinity is a process. (Q1) - Some mathematical culture: uses general statements to answer general questions (Q7), refers to definition (Q2 explanations).

Participant #19

1. It is a number that cannot be simplified in a fraction form.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*			*			*			*			*			*			*		*		

3. a) Integers: Yes
   b) Irrationals: No

4. Yes because 99/70 cannot be simplified.

5. I think that the intersection would be empty since both are different. (Rational and irrational are opposites.).

6. Yes, since you can write fractions that cannot be simplified.

7. a) I think it could be rational since adding two fractions could give a fraction that can be simplified.

   b) I think it could be rational also for the same reason as in part a).

#19	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - Rationals are the opposite of irrationals. (Q5)

	- 3.14 is rational. (Unclear how this fits the definition.) (Q2) Implicit knowledge: - Rational numbers can be "simplified in a fraction form." (Q1, Q4) i.e., if the numerator and the denominator have a common factor, the number is rational. - Sums and products of irrational numbers "could be" rational. (Q7)
IQ	Explicit knowledge: - Numbers "that cannot be simplified in a fraction form." (Q1) - 0.123456, 0.777778, and 30.450111… are irrational. (Q2) - π is irrational. (Q2) - √2 / √8 and 1 + 2√4 are irrational. (q2) - 22/7 (Q2) and 99/70 (Q4) are irrational. Implicit knowledge: - Fractions, written in lowest terms, are irrational. (Q1, Q4, Q6) - Infinite decimals are irrational. (Q2) - Irrationals can be approximated by fractions and finite decimals and are identical with their approximations. (Q4, Q6) - The use of a sign that is not a number makes the number irrational. (Q2c, d, f).
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are not real. (Q3b) - No number in Q2 is real. - **Q**, **IQ**, and **R** are disjoint. (Q5, Q2, Q3) - "Rationals and irrationals are opposites." (Q5) Implicit knowledge: - The only real numbers are integers.
Rep	- Representations (signs) are indices. - Relies on representing numbers as fractions. - All numbers can be represented on the calculator. (Q6)
C.I.	- **Q** = reducible fractions. - **IQ** = irreducible fractions. - Concept image contradicts accepted mathematical conventions as it is based on the idea of "simplified fractions." Student lacks the informal, formal, and algorithmic experience with number systems or is avoiding decimals all together. In any case, the student has constructed a concept image which is meaningless for dealing with real numbers.
M.o.T.	- COMPLEXIVE: The student is aware that rationals and irrationals are "opposites" and for whatever reason is using an overly simplified concept image to deal with them. The student makes the following associations: Q = fractions which can be simplified, e.g., 4/6. (my example) IQ = fractions which cannot be simplified, e.g. 22/7. However, answers show that anything that is not obviously an irreducible fraction is said to be irrational, so all infinite decimals and anything involving characters which are not digits (π, √).

	- 3.14 is said to be rational but 0.777778 is not. Unclear if he is actually interpreting 3.14 as 314/100 because it is equally easy to view 0.777778 as 777778/1000000, yet he doe not classify these examples in the same way. Obviously, there are different criteria for at least the finite decimals. - Student is not reflecting on his claims and therefore is missing some contradictions: an integer is a fraction with denominator 1. Because this can be simplified, integers should be considered rational. Alternatively, an integer does not appear as/is not written as a fraction which can be simplified, so he could interpret it as an irrational number. Implicit is that integers are neither rational, nor irrational; they are real.
G.R.	- Student does not explicitly mention decimals anywhere, so it is difficult to say if he knows (has heard) that some decimals can be represented as fractions. In any case, student lacks formal and algorithmic knowledge about the link between fractions and decimals since the concept image of rationals and irrationals is constructed solely based on his concept image of fractions. - Lacking mathematical culture: answers are very one-dimensional and there is little or no justification, but student does use general statements to answer general questions. (Q7) - Student lacks formal knowledge of irrational numbers.

Participant #20

1. Numbers that have no definite solution. Essentially numbers with decimals that seem to never end.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*		*			*			*				*		*		

Participant gives the following reasons for the above classifications:

b) ~~real~~
f) ~~real~~
h) ~~real~~

3. a) Integers: Yes
   b) Irrationals: Yes

4. No. Since 99/70 is an approximation it actually has a finite number of decimals while $\sqrt{2}$ has an infinite number of decimals.

5. No. It's empty. By definition a rational number is not irrational and vice versa.

6. Not really. It's always rounded at some point.

7. a) Yes.

b) Yes. Since you have an infinite number of decimals adding or multiplying two values will always give you an irrational. It's similar to adding ∞ to ∞, you get ∞, similarly ∞ × ∞ give ∞.

#20	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - √2 / √8 and 1 + 2√4 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. Implicit knowledge: - Having finitely many decimals is a necessary and sufficient condition for rationality. (Q4) - Integers, are rational. (Q2) - Fractions are rational. (Q2, Q4)
IQ	Explicit knowledge: - Irrational numbers have decimals representations "that seem to never end." (Periodicity is not mentioned.) (Q1) - Irrational numbers "have no definite solution." (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - π is irrational. (Q2) Implicit knowledge: - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are real. (Q3b) - No number in Q2 is real, but 0.777778, 1 + 2√4, and 3.14 were briefly considered to be real. (Q2: hesitation in the explanations) - **Q** and **IQ** are disjoint. (Q5) Implicit knowledge: - Inconsistent concept image for "real number".
Rep	- Representations do not necessarily represent the exact value of the number. (Q6) - Relies on decimal representation. - Fractions are indices for finite decimals. (Q4) - "…" is an icon for more or infinite. - "Real" is mathematical jargon. (Q2, Q3) - No evidence of symbolic interpretation.

C.I.	- **R** = mathematical jargon which applies to general cases. - **Q** = numbers with finitely many digits after the decimal point. - **IQ** = numbers with infinitely many digits after the decimal point having no "definite solution." - Concept image is consistent but incoherent because it is unclear how judgements are being made about √2/√8. Is it being treated as an icon (diagram resembling a ratio) or symbolically (having been recognized to be equal to 1/2 but then why is it not real)?
M.o.T.	- COMPLEXIVE: There is not enough information to definitely say that this student is using a pseudo-conceptual mode of thinking. Thinking is certainly complexive wrt real numbers but may be pseudo-conceptual wrt irrational numbers. There is some evidence (the wording in his response to Q4) that the student is operating on the distinction between infinite and finite decimal representations. But it is unclear how √2/√8 is being evaluated. For these reasons, this student is said to be using the COMPLEXIVE mode of thinking.
G.R.	- Infinity is a process which may or may not end. (Q1, Q6: Uses the phrase “Not really” when discussing whether a finite decimal can represent an infinite one.) - No mention of periodicity anywhere. - Some mathematical culture: uses general statements to answer general questions. (Q7)

Participant #21

1. Irrational numbers can't be (*hand-writing unclear*: -i-plied) into whole numbers, they are fractions.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*			*			*	*		*			*		*		*		*			*	

Participant gives the following reasons for the above classifications:

d) = √2 / √2·√4 = 1/√4 = ½
f) = 1 + 2(2) = 5

3. a) Integers: Yes
   b) Irrationals: Yes

4. Yes, given that √2 is irrational, and 99/70 is an approximation then 99/70 is irrational as well.

5. Yes because rational numbers can't be irrational & vice versa.

6. Yes but the accuracy of the number depends on the calculator.

7. a) Yes, for example:
   2/3 + 2/4 = 8/12 + 6/12 = 14/12

b) Yes, for example:

$2/3 \cdot 2/4 = 4/12$

#21	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - $1 + 2\sqrt{4}$ is rational. (Q2f) Implicit knowledge: - Integers are rational. (Q2f explanation)
IQ	Explicit knowledge: - Fractions are irrational numbers. (Q1) - 0.123456…, 0.777778, 30.450111…, and 3.14 are irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ is irrational. (Q2 explanation: it equals ½) - 22/7 (Q2) and 99/70 (Q4) are irrational. Implicit knowledge: - All decimals are irrational. (Q2) He may think that "irrational" refers to the union of the set of fractions that are not integers and decimals of whichever length and pattern. (Q1, Q2) - Irrationals can be approximated by fractions and finite decimals with varying degree of accuracy. The approximations are identified with the number they approximate. (Q4, Q6) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - π is a real. (Q2c) - Integers are real. (Q3a) - Irrationals are real. (Q3b) - **Q** and **IQ** are disjoint. (Q5)
Rep	- Representations depend more on how the number is expressed than the numerical value. (Q2, Q4, Q6) - Accuracy of the representation is sensitive to technology: "accuracy depends on the calculator" so infinity is a process which is executable by ever-more powerful calculators. - Decimal point is interpreted as an index: it indicates an irrational number. A fractional form is also an index of an irrational number. - "Real" is mathematical jargon. (Q2, Q3) - Signs are interpreted as symbols. (processing expressions in Q2d,f and Q7a,b)
C.I.	- **R** = mathematical jargon. - **Q = Z** - **IQ** = everything else, i.e., all decimal, fractions which are not integers, and any other type of non-digit character.

	- Concept image is consistent and coherent but it is deficient and not in agreement with conventional mathematics.
M.o.T.	- CONCEPTUAL but using erroneous concepts. - Student has an inadequate understanding of the topic. His concept image of numbers seems to be broken down into integers (or perhaps natural numbers) and everything else. I don't think this student uses any term (rational, real, irrational, integer) reliably. He is very consistently uses his definition that irrational numbers as everything that is not a whole number and for this reason he is classified as a conceptual thinker.
G.R.	- Student lacks formal knowledge of irrational numbers. - Student is mistaken about the link between fractions and decimals - Lacks mathematical culture: uses examples to prove general questions (Q7).

Participant #22

1. A number that is written either in the form of a fraction that can’t be simplify or a number with endless digits. (ex. 22/7, 1.56712…)

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*			*			*			*		*		

Participant gives the following reasons for the above classifications:

a) (*participant circled “…”*)
b) (*left blank*)
c) has infinite dig. = 2
d) fraction that can’t be simplify = 1
e) 1
f) 1
g) 2
h) (*left blank*)

3. a) Integers: ~~Yes~~ I am unsure about real numbers’ definition
   b) Irrationals: *(blank)*

4. This depends on what is understood by good approximation. Is 2.13798 a good approximation for 2.13798010…?

5. ~~Yes.~~ ~~No, It is empty.~~ Rational numbers have finite number of digits whereas irrational have infinite. I guess one could either say that all rational numbers are elements of irrational # or none are element of irrational #.

6. Yes, with the division symbol ex: 22/7, 22 (division sign) 7.

7. a) Not necessarily, ex: 0.314159… = a and 2.314159… = b
   ~~a/b =~~

   ~~c = b – a~~

   c = 1 – a

   I think that a + c = 1 and 1 is not irrational.

   b) $a \cdot b^{-1}$ = a/b = serie/ 2+ serie

   $a^{-1} \cdot b$ = b/a = 2/serie +serie/serie

   ~~No~~ Yes.

#22	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - Finite decimals are rational. (Q5)
IQ	Explicit knowledge: - Irrational numbers are "is written either in the form of a fraction that can't be simplify or a number with endless digits." (Periodicity is not mentioned.) (Q1) - 0.123456…, 30.450111… (Q2), 1.56712… (Q1), 0.314159…, and 2.314159… (Q7) are irrational. - π is irrational. (Q2) - √2 / √8 and 1 + 2√4 are irrational. (Q2) - 22/7 is irrational. (Q1, Q2)
R	Explicit knowledge: - Student admits to being unsure about the definition of real number. (Q3a) - Hesitates to call integers real. (Q3a)
Rep	- Student is hesitant about approximations. (Q4) - "…" is an icon for infinite digits. (Q2a explanation) - Calculators are sufficient tools for representing irrational number: representation is a sequence of buttons (Q6) - Signs are icons: resemblances which are subjective. This has

	consequences for the student's understanding of number. The sequence of buttons 2, 2, /, 7 "represents" the number 22/7 (Q6). Therefore the student interprets this sequence as "resembling" the number, which, for this student, is a graphical form itself. The student doesn't seem to take into account the value that the digits in the decimal expansion or fraction represent, which he would be, were he interpreting 22/7 or 0.314159… as symbols. In Q7a,to show that the sum of two irrationals can be not an irrational, he takes two sequences of digits with three dots, labels them with a and b, and then claims that their sum is 1, which it is not, if one takes into account the value of these numbers.
C.I.	- Unable to reconstruct a concept image for this student. - This student's inconsistency may appear in the inconsistent interpretation of the sqrt sign in Q2d and in Q2f; in the former he seems to interpret it as an operator of one number; it the latter as an operator on two numbers, namely the division sign. Otherwise, the contradiction appears in the context of an expression of uncertainty. He does not utter contradictions but rather states opposite possibilities.
M.o.T.	- PSEUDO-CONCEPTUAL: Asking questions and searching for the right concepts is a pseudo-conceptual behaviour that shows he is reflecting on his own thoughts. - The choice between the possibilities is conditional upon knowledge of definitions or theorems that he admits not knowing: Q3: I am unsure about real numbers' definition"; Q4 "this depends on what is understood by good approximation"; Q5. "Rational numbers have finite number of digits whereas irrational have infinite. I guess one could either say that all elements of irrational numbers or none are elements of irrational numbers". Open to learning something new, changing his understanding. - Conceptual thinking is possible without a symbolic understanding of number signs (see #8), but this is only a type of "nominal" thinking about numbers where attention is focused on the form of their representation and not on their values and relations between the values of different representations of the same number.
G.R.	- Unique concept image of infinity: all rationals are irrationals because the finite is contained in the infinite, and then he reverses this claim. (Q5) This student makes his answer conditional upon the assumption about the inclusion of the finite in the finite, i.e. whether one can say that a finite sequence of digits is a special case of an infinite sequence of digits. He doesn't know what is actually being assumed in mathematics so he considers both cases. - This student makes answer to Q4 also conditional upon a definition of "good approximation", which he admits he doesn't know. This awareness of the fact that the truth of claims in mathematics depends on initial assumptions and admitted definitions is a sign of some rudimentary mathematical culture. It's just that this student is very

	ignorant of the very basic assumptions about decimal representation of numbers, and definitions of rational and irrational numbers, and therefore this intuitive mathematical culture cannot be put to any use in answering the questions in the questionnaire. - Student is lacking formal and algorithmic knowledge of the link between decimals and fractions. - Students perceives subjectivity in mathematics: “good approximation” is subjective or contextual (Q4), truth values are Yes, No, “Not necessarily” (Q7) - Lacking mathematical culture: Answers general questions with examples (Q7), interpretation of infinity (Q5).

Participant #23

1. Irrational number are those numbers that can't be count.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*			*		*				*		*		

Participant gives the following reasons for the above classifications:

a) because the number is not defined (it doesn't seem to end)
b) can be counted
c) can't be counted
d) because $\sqrt{2}$ is irrational
e) can't be counted
f) can be counted
g) can't be counted
h) can be counted

3. a) Integers: Yes
   b) Irrationals: No

4. No, if 99/70 can't be counted then, yes it is an irrational number. If 99/70 can be counted then no.

5. A number can't be rational and irrational at the same time. They are opposites one is a set of countable numbers and the other is not.

6. It can be represented as a fraction, the π key or the square root of a number.

7. a) Yes, because they never stop adding.

   b) Yes, because the multiplication goes to infinity. Meaning the answer can't be count.

#23	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 1 + 2√4 is rational. (Q2) - 99/70 might be rational depending on whether or not it is countable. (Q4) (the student either didn't use a calculator or did, but could not tell if the digits appearing in the window are all digits or only the first few) - Rational numbers have decimal representations which have a countable number of digits after the decimal point. Implicit knowledge: - Finite decimals and integers are rational. ("countable" appears to refer to a finite number of digits either in the way the number is given, or it can be represented with a finite number of digits after some conversion or calculation: 1 + 2√4 is classified as rational perhaps because the student calculated it as equal to 5)
IQ	Explicit knowledge: - Irrational numbers can't be counted. (Periodicity is not mentioned.) (Q1) - 0.123456… and 30.450111… are rational, because they don't end, can't be counted, are not defined. (Q2) - π is irrational, because it can't be counted. (Q2) - √2 / √8 is irrational, because sqrt(2) is irrational. (Q2) - 22/7 is irrational, because it can't be counted. (Q2) - 99/70 is irrational, if it can't be counted. (Q4) - √2 is an irrational. (Q2d explanation) Implicit knowledge: - If any number contains √2, then it is irrational. (Q2a explanation) - Irrational numbers are not defined. (Q2a) - Sums and products of irrational numbers are irrational. (Q7) - Irrational numbers are made of infinite number of decimal digits ("can't be counted" refers to "infinite": Q2a, c, g)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are not real. (Q3b) - No number in Q2 is real.

	- **Q** and **IQ** are disjoint, because a number cannot be countable and not countable at the same time. (Q5) Implicit knowledge: - **Q**, **IQ**, and **R** are disjoint. (Q5, Q2, Q3)
Rep	- Over-reliance on the decimal representation: decimal representation is required to decide irrationality/rationality. (Q4) - Representations are symbols or operations that are either written down or appear as keys on a calculator, i.e, indices. - "…" is an symbol for infinite or unknown end. (Q2a explanation) - "Real" is mathematical jargon. (Q2, Q3) - Fractions, π and √ signs are indices, icons or symbols; they name a number or are the diagram of a number and a calculation must be performed to see the number itself, i.e., the decimal. (Q6)
C.I.	- **R** = mathematical jargon - **Q** = numbers which have a countable number of digits after the decimal point. - **IQ** = numbers which have a uncountable number of digits after the decimal point. - Consistent and coherent concept image except for one case: integers are said to be real but $1 + 2\sqrt{4}$ is said to be rational. - Concept image relies on representing the number in decimal form.
M.o.T.	- CONCEPTUAL: Student consistently uses the same criteria (ctbl/unctbl) to reason about Q and IQ whether deciding about classifications or reasoning about approximations, and operations on irrational numbers. The only exception is that $\sqrt{2}/\sqrt{8}$ is said to be irrational because $\sqrt{2}$ is irrational (index). Student does not say that $\sqrt{2}$ can or cannot be counted. - Ultimately this conceptual mode of thinking can be made more meaningful if the student were to incorporate the concept of periodicity and were aware of the link between decimals and fractions.
G.R.	- Being infinite means being undefined or imprecise. (Q2a) - Some mathematical culture: uses general statements to answer general questions (Q7), evidence of some formal knowledge of infinity: countable and uncountable (Q5). - Infinity is a process. (Q7a) - Missing link between decimals and fractions. (Q4)

Participant #24

1. A number that you don't really know when it ends such as π.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*			*				*		*		

3. a) Integers: Yes
   b) Irrationals: Yes

4. No, because its only an approximation and doesn't give the exact value.

5. I am not familiar with what is being asked in this question. ☹.

6. No, because there is not known ends to irrational numbers, calculators can only use good approximations.

7. a) I suppose so because if ~~you do~~ one is never ending no matter what you add to it it will still be like that.

   b) Not necessarily, if you multiply $\sqrt{2}$ by $\sqrt{2}$ you get 2 which isn't irrational…

   nvm I just re-read the question and saw it was two DIFFERENT irrational

   numbers, now I think the result would be irrational pretty much for the same

   reason as A.

#24	Analysis of individual’s knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational. - 1 + 2√4 (Q2) and 2 (Q7b) are rational. Implicit knowledge: - Finite decimals are rational. (Q2) - Fractions are rational. (Q2) - Integers are rational. (Q2, Q7b)
IQ	Explicit knowledge: - You do not know when an irrational numbers ends. (Periodicity is not mentioned.) (Q1) - Never ending (Q7a) - π is irrational. (Q1, Q2) - 0.123456… and 30.450111… are irrational. (Q2) - √2 / √8 is irrational. (Q2) - √2 is irrational. (Q7b) Implicit knowledge: - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Sums of irrational numbers are irrational. (Q7a) - Products of irrational numbers might be irrational. (Q7b)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are real. (Q3b) - No number in Q2 is real.
Rep	- Representations correspond to exact numerical values. (Q4, Q6) For this student, representations of numbers are taken as symbols of their value. Although 99/70 and 22/7 have never ending decimal expansions, they are compared to √2 and π based on their value, not the length of their decimal expansions. Therefore, the student is deducing the relationship between the representation and the value of the number from the representation and that’s why we can say that the sign is interpreted as a symbol - π is an index. (Q1) The letter “π” is a proper name of a number, so in that sense it is an index. However, the number it names has infinite and unknown decimal expansion which is probably not interpreted as just a diagram, because in other questions (Q4, for example) the student explicitly compares values of numbers, not just the diagrams they are represented with. - “Real” is mathematical jargon. (Q2, Q3) - √2 is an index: symptom of irrationality. (Q2d)

	- √4 is an index (name) of 2. (Q2f)
C.I.	- **R** = mathematical jargon. - **Q** = numbers which have decimal representations that end. - **IQ** = numbers which have decimal representations that do not end.
M.o.T.	- COMPLEXIVE: Student uses two criteria to make decisions: decimal expansion is infinite, and that of having an exact value. - In Q7b the student finds an example of a product of two irrationals which yield a rational ($\sqrt{2} \cdot \sqrt{2} = 2$) which does not fit the question parameters. When he notices this he abandons the possibility that it may also work for two different irrational numbers; instead of searching for a common thread he treats the cases separately which is a complexive behaviour. - Although the students two criteria are conceptually linked in mathematics, the lack of reasons in Q2 make it impossible to conclude that the student sees this link and we therefore cannot conclude that his thinking could be pseudo-conceptual.
G.R.	- Lacking formal and algorithmic knowledge of the link between repeating decimals and fractions. - Infinity is a process. (Q1) - Knowledge of real numbers is rote and has not been internalized.

Participant #25

1. A number infinitely long.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*			*				*		*		

Participant gives the following reasons for the above classifications:

a) it doesn't have an end
b) it has an end
c) no end
d) no end
e) has an end
f) has an end
g) no end
h) has an end

3. a) Integers: Yes
   b) Irrationals: ~~No~~ Yes

4. No. 3.14 is an approximation of π. 3.14 is rational and π is irrational.

5. No, a number cannot be rational and irrational at the same time. Those are complete different definitions.

6. No, because it is infinitely long. So, the calculator can only show an approximation.

7. a) Yes, ~~the answer won't because~~ if you add infinitely long numbers together, the answer cannot be exactly precise.

   b) ~~√2 × √2= 2~~

   ~~irrational rational~~

   ~~I think that~~

   I am not able to find a result that could be rational. So, I guess it is yes.

<table>
<tr><td>#25</td><td>Analysis of individual's knowledge and concept image</td></tr>
<tr><td>Q</td><td>Explicit knowledge:
- 0.777778 and 3.14 are rational. (Q2)
- 22/7 is rational. (Q2)
- 1 + 2√4 is rational. (Q2)
Implicit knowledge:
- 99/70 is rational. (Q4)
- Finite number of decimals is a necessary and sufficient condition of rational numbers. (Q2 explanations)</td></tr>
<tr><td>IQ</td><td>Explicit knowledge:
- Irrational numbers are "infinitely long." (Periodicity is not mentioned. Decimal is not mentioned, but I don't think the student has considered that a number can be infinitely long to the left of the decimal.) (Q1)
- 0.123456… and 30.450111… are irrational. (Q2)
- π is irrational. (Q2)
- √2 / √8 (Q2) and √2 (Q7b) are irrational.
Implicit knowledge:
- Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6)
- Sums and products of irrational numbers are irrational. (Q7)</td></tr>
<tr><td>R</td><td>Explicit knowledge:
- Integers are real. (Q3a)
- Irrationals are real. (Q3b, but hesitates)
- Q and IQ are disjoint. (Q5)
- No number in Q2 is real.</td></tr>
<tr><td>Rep</td><td>- Representations should correspond to exact numerical values. (Q6)
- "Real" is mathematical jargon. (Q2, Q3)
- Fractions are indices for finite decimals. (Q2e)
- √2 is an index of irrationality. (Q2d)
- √4 is an index for 2. (Q2f)</td></tr>
</table>

	- Missing link between decimals and fractions: does not explicitly use fractions in reasoning; relies on decimal representation. - No use of icons: resemblances are not recognized, no sets of numbers.
C.I.	- Concept image is inconsistent (22/7 has an infinitely long decimal representation yet it is not classified as irrational) and incoherent (case by case decisions on individual numbers not sets). - Unable to reconstruct a concept image for this student.
M.o.T.	- PSEUDO-CONCEPTUAL: All reasoning is in terms of whether the number (in its decimal representation) ends, but the decisions about putting the number signs into each class are conceptually unrelated to those criteria and decisions are made on the basis of the form of the sign or indices in the sign. - There is an uncontrolled association of irrational numbers with infinite decimals and rational numbers with finite decimals. It is unclear if 22/7 is said to be rational because it is finitely expressed (iconically, i.e., the representation does not have a decimal and some digits followed by a ...) or if it is because the student recalls that fractions are rational and can be expressed as a decimal (symbolic interpretation) and that decimal necessarily has finitely many digits according to this student's concept image of rational number. Alternatively, fractions may be indices for finite decimals which is the interpretation which we settled on as it agrees with his interpretations in general.
G.R.	- Infinity is a process and is associated with lack of precision. (Q2 expanations, Q7a) - Knowledge of real numbers is rote and has not been internalized. - Lacking mathematical culture: uses general statements to answer general questions (Q7a), but uses trial and error in Q7b and contradicts himself. - He seems to be conscious of the fact that he should be consistent. Resistant to contradiction (Q7b)

Participant #26

1. Irrational numbers are all unreal numbers and/or numbers with infinite decimals.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*			*				*		*					*		*		*		

Participant gives the following reasons for the above classifications:

a) because it has infintly many decimals.
e) because it is the division of two real numbers
g) it has many decimals

3. a) Integers: Yes
   b) Irrationals: No, they are not.

4. No, because 99 & 70 are real numbers and the division of both real numbers should have a real solution.

5. No because in between rational numbers, there exist some irrational numbers.

6. No, you cannot because calculator cannot have infinitely many decimals on its screen, you can only have rational and real numbers.

7. a) Yes, because both numbers will have infinitely many decimals & both are irrational.

b) No, because in some cases, it might happen that the multiplication gives a rational or real number.

#26	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - π is rational. (Q2c) - 22/7 is rational. (Q2e) Implicit knowledge: - Fractions are rational: "division of real numbers" (Q2e explanation, Q4) - Finite decimals are rational. (Q2) - Products of irrational numbers might be rational. (Q7b)
IQ	Explicit knowledge: - Irrational numbers are "unreal numbers and/or numbers with infinite decimals." (Periodicity is not mentioned.) (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - √2 / √8 is irrational. (Q2) - Implicit knowledge: - √2 is an irrational. (Q2d) - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Having infinite decimals is a necessary and sufficient condition of irrationality. - Sums of irrational numbers are irrational. (Q7a)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrational numbers are not real. (Q3b) - 1 + 2√4 is real. (Q2f) - 22, 7, 99, and 70 are real. - **Q** and **IQ** are not disjoint. (Q5) Implicit knowledge: - **Q**, **IQ**, and **R** are all disjoint. - Products of irrational numbers might be real. (Q7b)
Rep	- Representations correspond to exact numerical values. (Q6) - The division sign is a symbol which instructs the student to perform an operation on the calculator. 99/70 "should have a real solution" indicates that 99/70 is interpreted as a process to execute. (Q4) - The fraction itself is an index, a name, of an executed value. - Concept image of real numbers is dominated by the image of a number line. (icon) When asked if the intersection of **Q** and **IQ** is empty the student seems to be considering an interval of the line instead of the

	intersection of two sets. (Q5) - Calculators are insufficient tools for representing irrational numbers; they can accurately represent reals and rationals. (Q6) - π is an icon: π is rational because it resembles rational numbers, i.e., it does not have "…" and is therefore not infinite. (Q2c) - "…" is an index for infinite. It is taken to denote infinite, not many.
C.I.	- Unable to reconstruct a concept image for this student. - Inconsisitent and incoherent.
M.o.T.	- COMPLEXIVE: Student uses multiple criteria based on fact to make decisions. - Case by case assessments: 99/70 "should be real" but 22/7 is rational. π is rational (reason unknown). Many or infnite decimals implies irrational. - Student answers "no, because in some cases..." (Q7b) which is an explicit acknowledgment that concepts are not logically linked together.
G.R.	- Lacks knowledge of the nature of repeating infinite decimals; missing the link between decimals and fractions. - Some mathematical culture: uses general statements to answer general questions (Q7), however truth values of a statement include True, False and Maybe (Q7).

Participant #27

1. Any number that cannot be divided into a round number by more than two integers other than 0.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*	*	*		*		*	*		*	*	*		*		*	*		*	*	*		*

3. a) Integers: Yes
   b) Irrationals: Yes

4. Yes, since this number has an infinite amount of decimal numbers.

5. No since a number cannot have the possibility to ~~be divided by both~~ be both irrational and rational.

6. No idea.

7. a) No since adding two numbers together results in a number capable of being divided by 2 (pair).

   b) No since the resulting number can be divided by 1, itself and also either one of the numbers multiplied to make it.

#27	Analysis of individual’s knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational. (Q2) - 22/7 is rational. (Q2) - Sums and products of irrational numbers are rational. (Q7) Implicit knowledge: - Finite decimals are rational. (Q2)
IQ	Explicit knowledge: - Irrational number “cannot be divided into a round number by more than two integers other than 0.” (Q1) Fractions, decimals, and periodicity are not mentioned in the definition, although infinite decimals are mentioned later in Q4. - 0.123456… and 30.450111… are irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are irrational. (Q2) - Infinite decimals are irrational. (Q4, also implicit in his/her classifications in Q2) Implicit knowledge: - The square root in an expression makes a number irrational.
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are real. (Q3b) - All examples in Q2 are real. - **Q** and **IQ** are disjoint. (Q5) Implicit knowledge: - “Real number” refers to all numbers and characters (π) that mathematicians work with.
Rep	- Over-reliance on the decimal representation. (Q4) - $\sqrt{}$ signs are indices for irrationality (Q2d, f) - π is an index. - Does not understand approximations and how they differ from that which they approximate. (Q4, Q6) Approximations are indices. - Fractions are not icons: different treatment of 22/7 and 99/70; they are indices. 99/70 is another name for $\sqrt{2}$. - Purely indexical interpretations.
C.I.	- Inconsistent concept image: The sum of two numbers is divisible by 2 (Q7a), 22/7 is rational but 99/70 is irrational. - Unable to reconstruct a concept image for this student.
M.o.T.	- COMPLEXIVE: Purely indexical interpretations on a case by case basis. Each decision is based on facts and stands alone. - Inconsistency appears in classifying 22/7 as rational and 99/70 as irrational. The latter's classification is justified by "infinite amount of decimal numbers" (note the lack of distinction between digits and numbers). Unclear why 22/7 is classified as rational. If the student

	interpreted the sign "/" in 99/70 as the instruction to perform an operation of division (an hence as a symbol) and were consistent with her notion of irrationals as having infinite decimal expansions, then she would have classified 22/7 also as irrational
G.R.	- Confuses irrational and prime number definitions. - Poor with calculations (Q2d, f, Q7) - Has not internalized rote (and poorly recalled) knowledge of irrational or prime numbers. - Poor mathematical culture: although s/he uses general reasoning where it is required, s/he does not refer to definitions and has muddled many mathematical ideas into one confused concept image.

Participant #28

1. Numbers that do not have a fixed value.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*		*		*	*					*	*			*		*

3. a) Integers: Yes
   b) Irrationals: No

4. No because √2 is the result of an algorithm that converges to a fixed point but the value of √2 continues to n amount of values after the decimal.

5. Yes because in the real numbers there are rational and irrational numbers that are part of the same set.

6. No because a calculator rounds up to a certain decimal such that not all irrational number can be displayed to their full extent (∞).

7. a) Yes, because in order to obtain a rational number all numbers in each value must add up giving a rational number and since irrational numbers do not end it is very likely that they will not add up to a rational number..

   b) Yes because two irrational numbers will give the result of a multiplication of all infinite values after the decimal such that there is no fixed value thus being irrational.

#28	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 and 3.14 are rational (Q2) - √2 / √8 is rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) is rational. - 30.450111… is rational. (Q2) Implicit knowledge: - Finite decimals are rational. (Q2) - Infinite decimals with a repeating digit pattern are rational. (Q2g) - Fractions are rational. (Q2e, Q4)
IQ	Explicit knowledge: - Numbers that do not have fixed value are irrational. (Q1) - 0.123456… is irrational. (Q2) - π is irrational. (Q2) - √2 is irrational. (Q4) Implicit knowledge: - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Non-periodic decimals are irrational. (Q2) - Sums and products of irrational numbers are irrational. (Q7)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrationals are not real. (Q3b) - √2 / √8 is real. (Q2d) - 1 + 2√4 is real. (Q2f) - 3.14 is real. (Q2h) - **Q** and **IQ** are not disjoint. (Q5) Implicit knowledge: - There are elements in **R** which are in neither **Q** nor **IQ**. (Q2, Q5) - A calculator cannot represent infinite decimals whether periodic or not. (Q6) - Real numbers are fractions which are less than 1, integers, and short decimal number (e.g. two places after the decimal) (Q2)
Rep	- Representations correspond to exact numerical values.: "not all rational number can be display to their full extent" (Q6) - "Real" is mathematical jargon. (Q2, Q3) - Fractions are indices: they are not mentioned explicitly and do not seem to be used in calculations or reasoning. - It does not matter the way a number is written (this is an index), ir/rationality is decided based on value (symbolic interpretation). (Q2d, f) - Classifies √2/√8 as rational which, together with his consistent notion

	of irrational numbers as not having a fixed value (in the sense that they are sequences in which every next term is different than the preceding one), and the fact that he classified 30.450111... as rational (noticing probably that the sequence of digits stabilizes) suggests that he must have discovered that this expression represents a "fixed value", or 0.5. Therefore he may have processed this sign. (symbolic interpretation.)
C.I.	- **R** = integers and short decimals, i.e., one or two digits after the decimal point. - **Q** = numbers with a fixed value, but not integers. - **IQ** = number with no fixed value, sequences of digits. - Inconsistent concept image: Concept image relies on the notion of knowable values, but if there are no decimal places then the number is not eligible for consideration as fixed or not. - Incoherent concept image: Integers, which would most certainly be said to have a fixed value, are written with no digits/no decimal point so do not need to be evalated as rational/irrational and results in incoherence. Student is not aware of the contradiction.
M.o.T.	- CONCEPTUAL: Decisions are based on the concept of a fixed value and student uses logical argumentation in Q7a: "must add up giving a rational number." "Add up" likely refers to resulting in a finite number of digits, i.e., pairs of digits sum to 10 (or 0) resulting in a 0 place-holder. Less likely is that it must result in some predictable pattern or repeating digit. This shows that the student is reflecting on and reasoning from his concept images of rational and irrational numbers. Irrational numbers are sequences and not the limit ie the fixed point.
G.R.	- Infinity is imprecise and procedural. (Q1, Q7b) - Student seems to have a working knowledge of rational and irrational numbers which may be greater than his ability to articulate it. This was one of the few students who seems to consider periodicity when deciding about irrationality. - Some mathematical culture: uses general statements to answer general questions (Q7).

Participant #29

1. It is ~~not~~ a fictional number / not regular number.

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			$\sqrt{2} / \sqrt{8}$			22/7			$1 + 2\sqrt{4}$			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*					*			*		*				*		*				*

Participant gives the following reasons for the above classifications:

a) the numbers after the decimal go toward ∞ amount of other number
b) fraction form: makes me thought of rational function
c)
d) $\cancel{\sqrt{2}} / 2\cancel{\sqrt{2}} = \frac{1}{2} = 0.5$ ~~rational~~ real number
e) fraction form: makes me thought of rational function
f) $1 + 2\sqrt{4} \rightarrow 1 + 2\cdot 2 = 5$ real number = integer
g) the numbers after the decimal go toward ∞ amount of other number
h) fix number

3. a) Integers: Yes
   b) Irrationals: No

4. No, it can be written in a fraction form therefore it is a rational number.

5. No, as proven in #4, a two number can be extremely close, but there is always a difference. $\sqrt{2}$, irrational, cannot be written in fractional form. 99/70, rational, can be written in a fraction form.

6. No, because the irrational number never ends… therefore the ~~compu~~ calculator is going to give an answer by approximating the answer however, it can be given in

this for ex: √2 but as ~~it~~ the calculator expands the number it is nearly an approximation.

7. a) Not always, but it is possible, for example. √2 + √2= √4 = 2 (real number) however we could have. √7 + √6 = √13 it stays irrational.

   b) Yes, because it take of the square root ex: $\sqrt{3} \cdot \sqrt{3} = \sqrt{3^2} = 3$.

#29	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 is rational. (Q2) - Rational numbers can be written in fraction form. (Q4) - 99/70 is rational. (Q4) Implicit knowledge: - Decimals with a singles repeating digit are rational: 0.777778 is possibly being interpreted as 0.77777… which would imply that rational numbers can be always be represented by finite decimals.
IQ	Explicit knowledge: - Irrationals are "fictional numbers." (Q1) - Irrationals are "not regular numbers." (Q1) - 0.123456… and 30.450111… are irrational. (Q2) - 22/7 is irrational. (Q2) - Infinite decimals are irrationals. (Q7a) Implicit knowledge: - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6)
R	Explicit knowledge: - π is real. (Q2) - √2 / √8 is real because it is equal to 0.5. (Q2d explanation) - 0.5 and 3.14 are real. (Q2: fixed numbers) - 5 is a real number. (Q2f) - Integers are real. (Q3a) - Irrationals are not real. (Q3b) - **Q** and **IQ** are disjoint. (Q5) Implicit knowledge: - There are elements in **R** which are in neither **Q** nor **IQ**. (Q2, Q5)
Rep	- Depends on the way the number is written: iconic representation (Q2e, Q4). - √2 is interpreted symbolically (Q2) and indexically. (Q6)

C.I.	- Unable to reconstruct a concept image for this student. - Inconsistent concept image: 22/7 is classified differently from 99/70, uses one statement to justify two different things (Q2b, e explanations).
M.o.T.	- COMPLEXIVE: Student uses multiple criteria to make decisions: infinite decimals, not "regular," fraction form, rational function, fixed number. - 22/7 is classified differently than 99/70. - Student uses concept image of rational function to support contradictory statements, namely that 0.777778 is rational and 22/7 is irrational.
G.R.	- Lack of mathematical culture: literal/colloquial use and interpretation of terminology (fictional number, regular number), uses examples to answer general questions (Q7), misunderstands “proof” (Q5), uses one statement to justify two different things (Q2b, e explanations). - Infinity is a process: going towards infinity means you are approaching another number (Q2a, g explanation); “never ends” (Q7a)

Participant #30

1. *(left blank)*

2.

a			b			c			d			e			f			g			h		
0.123456…			0.777778			π			√2 / √8			22/7			1 + 2√4			30.450111…			3.14		
Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R	Q	I	R
	*		*				*			*		*				*				*			*

3. a) Integers: Yes
   b) Irrationals: Yes

4. No because it is a fraction.

5. *(left blank)*

6. No, you can only approximate, the calculator will only compute so far.

7. a) The addition of two irrational numbers is a bigger irrational number because addition simply adds them.

   b) Multiplying two will also give an irrational number.

#30	Analysis of individual's knowledge and concept image
Q	Explicit knowledge: - 0.777778 is rational. (Q2) - 22/7 (Q2) and 99/70 (Q4) are rational.

	- Fractions are rational. (Q4) Implicit knowledge: - Decimals with a singles repeating digit are rational: 0.777778 is possibly being interpreted as 0.77777…, however not all finite decimals are rational. (Q2h) - Rational numbers are less than 1. (Q2)
IQ	Explicit knowledge: - 0.123456… is irrational. (Q2) - π is irrational. (Q2) - $\sqrt{2} / \sqrt{8}$ and $1 + 2\sqrt{4}$ are irrational. (Q2) Implicit knowledge: - $\sqrt{}$ make an expression irrational. (Q2) - Irrationals can be approximated by fractions and finite decimals but are not identical with their approximations. (Q4, Q6) - Having infinite decimals is not necessary and sufficient condition of irrationality. - Sums and products of irrational numbers are irrational. (Q7) - Irrational numbers are less than 1. (Q2a compared to Q2g)
R	Explicit knowledge: - Integers are real. (Q3a) - Irrational numbers are real. (Q3b) - 30.450111… is real. (Q2g) - 3.14 is real. (Q2h) Implicit knowledge: - **R**, **Q**, and **IQ** are all disjoint. (Q2) - Real numbers are greater than 1. (Q2)
Rep	- $\sqrt{}$ is an index: symptom of irrationality. (Q2) - "Real" is mathematical jargon. (Q2, Q3) - No evidence of symbolic or iconic interpretations.
C.I.	- Inconsistent concept image: real number is not a meaningful expression, classifications in Q2 are contradictory. - Lacking any definition of irrational numbers and contradictory classifications in Q2 make it difficult to reconstruct a concept image for this participant.
M.o.T.	- COMPLEXIVE: Since the student does not provide definitions or justifications for his answers, we can only assume that he is thinking in clusters of facts, not concepts.
G.R.	- Lacks mathematical culture: although s/he uses general statements to answer general questions (Q7), s/he is unable to offer a definition of irrational numbers (Q1) and does not understand set-theoretic terminology. (Q5) - "Addition makes bigger" obstacle. (Q7a) - Poor at calculations. (Q2d, f)