A Unified Dissipativity Approach for Stability Analysis of Piecewise Smooth Systems ⋆

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Abstract

The main objective of this paper is to present a unified dissipativity approach for stability analysis of piecewise smooth (PWS) systems with continuous and discontinuous vector fields. The Filippov definition is considered for the solution of these systems. Using the concept of generalized gradients for nonsmooth functions, sufficient conditions for the stability of a PWS system are formulated based on Lyapunov theory. The importance of the proposed approach is that it does not need any a-priori information about attractive sliding modes on switching surfaces, which is in general difficult to obtain. A section on application of the main results to piecewise affine (PWA) systems followed by a section with extensive examples clearly show the usefulness of the proposed unified methodology. In particular, we present an example with a stable sliding mode where the proposed method works and previously suggested methods fail.

Key words: Piecewise smooth systems; Piecewise polynomial systems; Piecewise affine systems; Lyapunov stability; Dissipativity.

1 Introduction

There have been different approaches to construct a Lyapunov function to provide sufficient conditions for the stability of PWS systems. Surveys of stability analysis for hybrid and switched linear systems can be found in Decarlo et al. (2000); Liberzon (2003) and a general framework for analyzing stability of nonlinear switched systems using multiple Lyapunov functions can be found in Chatterjee and Liberzon (2006). A widely used approach in the literature has been to search for quadratic Lyapunov functions. An advantage of searching for such quadratic functions is that sufficient conditions for stability of a class of PWS systems called piecewise affine (PWA) systems can be formulated as convex optimization problems subject to linear matrix inequality (LMI) constraints as shown in Hassibi and Boyd (1998); Rodrigues and Boyd (2005). Furthermore, a common quadratic Lyapunov function has been used to analyze the stability of switched linear systems under arbitrary switching in Liberzon (2003). Reference Pavlov et al. (2005a) shows that the existence of a common quadratic Lyapunov function for the linear parts of a PWA system in every mode is sufficient for exponential convergence of the system if the vector field of the PWA system is continuous. The case of discontinuous vector fields is studied in Pavlov et al. (2005b) and it is shown that the existence of a common quadratic Lyapunov function for linear parts of the system is not a sufficient condition for convergence. Necessary and sufficient conditions for quadratic convergence of the special case of bimodal PWA systems are then derived.

Despite the attractive features of quadratic Lyapunov functions, there are stable PWA systems for which a quadratic Lyapunov function does not exist. Examples of such systems are shown in (Johansson, 2003, p. 47). Conservativeness of a quadratic form has been the motivation for studying nonquadratic Lyapunov functions. One of the first approaches in this direction was to search for continuous piecewise quadratic (PWQ) Lyapunov functions in Branicky (1998); Pettersson (1999); Johansson (2003); Rodrigues et al. (2000). However, it is a common misunderstanding in the literature to believe that

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if there is a continuous PWQ or piecewise polynomial (PWP) function that is positive definite and decreasing with time along each vector field of a switched affine system then the system is stable. A counter-example will be provided in this paper, in section 4.3. Sum of squares (SOS) polynomials were also proposed as candidate Lyapunov functions. In fact, quadratic Lyapunov functions are a special class of SOS Lyapunov functions as shown in Prajna et al. (2005). In addition, by using the SOS approach, it is possible to analyze the stability of systems with nonlinear polynomial vector fields. Stability analysis tools based on the SOS decomposition for classes of nonlinear systems, hybrid systems, switched systems, and time-delay systems are presented in Papachristodoulou and Prajna (2005). In the same reference it is proposed to use PWP Lyapunov functions for hybrid systems, which is a generalization of PWQ Lyapunov functions. However, systems with infinitely fast switching or sliding modes are excluded from the discussion in Papachristodoulou and Prajna (2005), as well as from the discussion in most of the available literature.

Although there is a vast amount of work on stability of switched linear and PWA systems, sliding modes or infinitely fast switching are not usually considered in the literature. Important exceptions are found in Branicky (1998); Johansson (2003); Rodrigues and How (2003); Pavlov et al. (2005b). It is proposed in Branicky (1998) to add the sliding modes and their associated sliding dynamics to the modes of the system before doing the stability analysis. However, this needs a-priori information about the sliding modes of the system, which is typically hard to get. In another approach, an extra condition is introduced in (Johansson, 2003, p.64) to extend the analysis of PWA systems to systems with attractive sliding modes. However, one needs to identify potential sets in which sliding modes can occur and then the corresponding condition can be formed and added to the analysis problem. This might again be hard and make the problem complex if there is no previous information about sliding modes. In Rodrigues and How (2003), a synthesis method based on bilinear matrix inequalities was proposed for state and output feedback stabilization of PWA systems. The synthesis method includes linear constraints on controller gains to guarantee that sliding modes are not generated at the switching. Finally, the work in Pavlov et al. (2005b) has addressed sliding modes but has concentrated on the specific case of common quadratic Lyapunov functions for bimodal PWA systems. A question that still remains to be answered in the literature is when can one remove the necessity to check the existence of unstable sliding modes or add additional constraints for a general PWA system. This question will be answered in this paper using a unified dissipative approach where the supply rate is proportional to the candidate Lyapunov function, the proportionality constant being related to the rate of decay of the Lyapunov function. The paper is structured as follows. We start by some mathematical preliminaries in section 2. Then we present the main results of the paper followed by application to PWA systems and examples.

2 Mathematical Preliminaries

This section presents the background mathematical notions used in the rest of the paper.

2.1 PWS Systems

The dynamics of a PWS system can be written as

$$\dot{x} = f_i(x), \; x \in \mathcal{R}_i \tag{1}$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector and the initial state is $x(0) = x_0$. A subset of the state space $\mathcal{X}$ is partitioned into M regions, $\mathcal{R}_i$, $i = 1, \ldots, M$, such that

$$\cup_{i=1}^{M} \overline{\mathcal{R}}_i = \mathcal{X}, \; \mathcal{R}_i \cap \mathcal{R}_j = \emptyset, \; i \neq j$$

where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. The function $f_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^n$ is continuous and locally bounded. The Filippov definition of trajectories is considered for the solution of (1) (see Filippov (1960) and Acary and Brogliato (2008)).

Definition 1 *(Filippov solution) A continuous function $x(t)$ is regarded to be a Filippov solution to (1) if it is a solution of the differential inclusion*

$$\dot{x}(t) \in \mathcal{F}(x) \tag{2}$$

for almost all $t \geq 0$ where

$$\mathcal{F}(x) \triangleq \boldsymbol{conv}\{f_i(x) | i \in \mathcal{I}(x)\}, \; \mathcal{I}(x) = \{i | x \in \overline{\mathcal{R}}_i\},$$

*and **conv** stands for the convex hull of a set. Note that if $x \in \mathcal{R}_i$, then $\mathcal{F}(x) = \{f_i(x)\}$.*

2.2 PWP and PWA Systems

The dynamics of a PWP system can be written in the form (1) where $f_i(x) \in \mathbb{R}^n$ are polynomial functions of x. Each region in the partition is described by

$$\mathcal{R}_i = \{x | E_i(x) \succ 0\} \tag{3}$$

where $E_i(x) \in \mathbb{R}^{p_i}$ is a vector polynomial function of x and $\succ$ represents an elementwise inequality. A PWA system is a PWP system for which $f_i(x) = A_i x + a_i$ and

$$\dot{x} = A_i x + a_i, \quad \text{for } x \in \mathcal{R}_i \tag{4}$$

where $A_i \in \mathbb{R}^{n \times n}$, $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, M$. It is assumed that $a_i = 0$ for $i \in \mathcal{I}(0)$. Therefore, the origin is an equilibrium point of the system. Each region in the partition is described by (3) with $E_i(x) = E_i x + e_i$ where $E_i \in \mathbb{R}^{p_i \times n}$, $e_i \in \mathbb{R}^{p_i}$ and $\succ$ represents an elementwise inequality. For system (4) we define

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \quad \bar{E}_i = \begin{bmatrix} E_i & e_i \\ 0 & 1 \end{bmatrix}$$

Equation (4) can then be rewritten as

$$\dot{\bar{x}}(t) = \bar{A}_i \bar{x}(t), \; x(t) \in \mathcal{R}_i$$

where $\bar{x} = [x \;\; 1]^T$. Each polytopic region $\mathcal{R}_i$ can be outer approximated by a quadratic curve

$$\mathcal{R}_i \subseteq \varepsilon_i = \{x | \bar{x}^T \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \bar{x} > 0\},$$

where $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ is a matrix with nonnegative entries. A parametric description of the boundaries between two regions $\mathcal{R}_i$ and $\mathcal{R}_j$ where $\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}}_j \neq \emptyset$ can also be obtained as (see Hassibi and Boyd (1998) and Rodrigues and How (2003) for more details)

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}}_j \subseteq \{x | x = F_{ij}s + f_{ij}, \; s \in \mathbb{R}^{n-1}\} \qquad (5)$$

A particular class of interest in applications is the class of PWA slab systems (see Rodrigues and Boyd (2005)). For such systems, the slab regions $\mathcal{R}_i$, $i = 1, \ldots, M$ partitioning a slab subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ are defined as

$$\mathcal{R}_i = \{x \mid \sigma_i < C_{\mathcal{R}} x < \sigma_{i+1}\},$$

where $C_{\mathcal{R}} \in \mathbb{R}^{1\times n}$ and σ_i for $i = 1, \ldots, M+1$ are scalars such that

$$\sigma_1 < \sigma_2 < \ldots < \sigma_{M+1}$$

Each slab region can alternatively be described by the following degenerate ellipsoid

$$\mathcal{R}_i = \{x \mid \|L_i x + l_i\| < 1\} \qquad (6)$$

where $L_i = 2C_{\mathcal{R}}/(\sigma_{i+1} - \sigma_i)$ and $l_i = -(\sigma_{i+1} + \sigma_i)/(\sigma_{i+1} - \sigma_i)$.

2.3 Dissipativity

We now consider PWS systems with inputs and outputs described by

$$\begin{aligned} \dot{x} &= f_i(x) + g_i(x)w, \; x \in \mathcal{R}_i \\ y &= h(x, w) \end{aligned} \qquad (7)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ denotes the state, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous input and $y(t) \in \mathbb{R}^{n_y}$ is the output. The functions $f_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^n$, $g_i(x) : \overline{\mathcal{R}}_i \to \mathbb{R}^{n \times n_w}$ and $h(x, w) : \mathcal{X} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_y}$ are continuous and locally bounded. Roughly speaking, a system is considered dissipative if the amount of energy that the system can provide to its environment is less than what it receives from external sources according to the following definition (Willems and Takaba, 2007).

Definition 2 *The system (7) is* dissipative *with* supply rate $W(y, w)$ *and* storage function $V(x)$, *if $V(x)$ is nonnegative and if $S(x, t) = V(x) - \int_0^t W(y(\tau), w(\tau))d\tau$ is nonincreasing along the trajectories of (7), i.e, if*

$$t_1 \le t_2 \Rightarrow V(x(t_1)) + \int_{t_1}^{t_2} W(y(\tau), w(\tau))d\tau \ge V(x(t_2)) \qquad (8)$$

The link of the stability results with dissipativity is given in Theorem 3.

Theorem 3 *For the PWS system (1), if there exists a continuous function $V(x)$ defined in a forward invariant set $\mathcal{X}$ such that*

$$\begin{aligned} &V(x^\star) = 0, \\ &V(x) > 0 \textit{ for all } x \neq x^\star \textit{ in } \mathcal{X}, \\ &t_1 \le t_2 \Rightarrow V(x(t_1)) \ge V(x(t_2)), \end{aligned}$$

then $x = x^\star$ is a stable equilibrium point. Moreover if there exists a continuous function $Q(x)$ such that

$$\begin{aligned} &Q(x^\star) = 0, \\ &Q(x) > 0 \textit{ for all } x \neq x^\star \textit{ in } \mathcal{X}, \end{aligned}$$

and if the system (7) with $h(x, w) = x$ is dissipative *with* supply rate $-Q(x)$ *and* storage function $V(x)$, *i.e,*

$$t_1 \le t_2 \Rightarrow V(x(t_1)) \ge V(x(t_2)) + \int_{t_1}^{t_2} Q(x(\tau))d\tau$$

then all trajectories in $\mathcal{X}$ asymptotically converge to $x = x^\star$ provided V is radially unbounded, i.e,

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \qquad (9)$$

PROOF. See Samadi and Rodrigues (2008).

Note that more sophisticated versions of this result exist in the literature of nonsmooth Lyapunov functions (see for example (Clarke et al., 1998; Ryan, 1998; Byrnes and Martin, 1995; Teel et al., 2002)). The problem with the monotonicity conditions in Theorem 3 is that they can be hard to check. Therefore, the next section presents alternative conditions.

2.4 Monotonicity of Nonsmooth Functions

Necessary and sufficient conditions for monotonicity of nonsmooth functions are described in this subsection.

Definition 4 *(Clarke et al., 1998) For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the generalized gradient is defined as*

$$\partial_C V(x) = \cap_{\epsilon>0}\mathrm{conv}\{\nabla V(y)|y \in \mathcal{B}_\epsilon(x), y \notin N\} \quad (10)$$

where N is the set of measure zero where the gradient of V does not exist and $\mathcal{B}_\epsilon(x)$ is a ball of radius ϵ centered at x.

Proposition 5 *(Ceragioli (1999)) Let $\mathcal{F} : \mathbb{R}^n \to 2^{\mathbb{R}^n} \backslash \emptyset$ be continuous and let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous. V is nonincreasing along all solutions of*

$$\dot{x} \in \mathcal{F}(x)$$

if and only if

$$\forall x \in \mathbb{R}^n, \forall f \in \mathcal{F}(x), \max\{p^T f | p \in \partial_C V(x)\} \leq 0 \quad (11)$$

Proposition 5 will be used in the next section to prove sufficient conditions for the dissipativity of PWS systems.

3 Main Results

The importance and contribution of the main results lie in the fact that to check the dissipativity of the system, it suffices to verify a condition on the storage function, the supply rate and the vector field of the subsystem in each region separately. There is therefore no need to examine the storage function in one region with the vector field of another region, which would make the problem much more complex.

3.1 PWS Systems With Discontinuous Vector Fields

Proposition 6 *(Smooth storage functions) The piecewise smooth system (7) is dissipative with a storage function $V(x)$ and a supply rate $W(y, w)$ if $V(x)$ is a nonnegative $\mathcal{C}^1$ function, $W(y, w)$ is a continuous function and for all $x \in \overline{\mathcal{R}}_i$, $i = 1, \ldots, M$ and any $w \in \mathbb{R}^{n_w}$*

$$\nabla V(x)^T (f_i(x) + g_i(x)w) \leq W(y, w) \quad (12)$$

PROOF. The inequality (12) can be rewritten as

$$\begin{bmatrix} \nabla V(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix} \leq 0 \quad (13)$$

By appending time (t) to the state vector of the system (7), we have the following differential equation

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f_i(x) + g_i(x)w \\ 1 \end{bmatrix}, \ x \in \mathcal{R}_i \quad (14)$$

The fact that $V(x)$ is a $\mathcal{C}^1$ function implies that

$$\partial_c S(x,t) = \mathrm{conv}\left\{ \begin{bmatrix} \nabla V(x(t)) \\ -W(y(t), w(\tau)) \end{bmatrix} \middle| \tau \to t \right\} \quad (15)$$

where $S(x, t)$ is given in definition 2. Let $x(t)$ be a Filippov solution of (7). Therefore, $x(t)$ is a solution of the following differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \mathrm{conv}\left\{ \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(\tau) \\ 1 \end{bmatrix} \middle| \begin{array}{c} i \in \mathcal{I}(x), \\ \tau \to t \end{array} \right\} \quad (16)$$

Since (13) is satisfied for any x and w in the domain, it follows from (15) that (11) is satisfied for the differential inclusion (16). Therefore by Proposition 5, $S(x, t)$ is nonincreasing along the trajectories of (7) in $\mathcal{X}$. Therefore (8) is satisfied and the system (7) is dissipative with storage function $V(x)$ and supply rate $W(y, w)$.

3.2 PWS Systems With Continuous Vector Fields

Proposition 7 *(Piecewise smooth storage functions) The piecewise smooth system (7) is dissipative with a storage function $V(x)$ and a supply rate $W(y, w)$ if*

- *$V(x)$ is a nonnegative continuous function where*

$$V(x) = V_i(x), x \in \overline{\mathcal{R}}_i$$

  and $V_i : \overline{\mathcal{R}}_i \to \mathbb{R}$ is a $\mathcal{C}^1$ function,
- *$W(y, w)$ is a continuous function,*
- *the vector field of the system (7) is continuous in x, i.e., $w(t)$ is continuous and the following conditions hold for any $i, j \in \{1, \ldots, M\}$ such that $\overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$:*

$$\begin{cases} f_i(x) = f_j(x) \\ g_i(x) = g_j(x) \end{cases}, \ \forall x \in \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \quad (17)$$

- *for all $x \in \overline{\mathcal{R}}_i$, $i = 1, \ldots, M$ and any continuous function $w \in \mathbb{R}^{n_w}$*

$$\nabla V_i(x)^T (f_i(x) + g_i(x)w) \leq W(y, w) \quad (18)$$

PROOF. By appending time (t) to the state vector of the system (7), we obtain the differential equation

(14). In the following, using Proposition 5 the function $S(x,t) = V_i(x) - \int_0^t W(y,w)d\tau, x \in \overline{\mathcal{R}}_i$ is shown to be non-increasing along the trajectories of (14).

Let $x(t)$ be a Filippov solution of (7). Therefore, $x(t)$ is a solution of the following differential inclusion

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} \in \operatorname{conv}\left\{ \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(t) \\ 1 \end{bmatrix} \middle| \; i \in \mathcal{I}(x) \right\} \tag{19}$$

Consider the following two cases

- If $x(t) \in \mathcal{R}_i$, we have

$$\partial_c S(x,t) = \begin{bmatrix} \nabla V_i(x(t)) \\ -W(y(t),w(t)) \end{bmatrix} \tag{20}$$

  and

$$\begin{bmatrix} \dot{x} \\ \dot{t} \end{bmatrix} = \begin{bmatrix} f_i(x(t)) + g_i(x(t))w(t) \\ 1 \end{bmatrix} \tag{21}$$

  Since (18) is satisfied, it follows from (20) that (11) is satisfied for the differential equation (21).
- If $x(t)$ is on the boundary of two or more regions i.e. $x(t) \in \bigcap_{i\in\mathcal{I}(x)} \overline{\mathcal{R}}_i$,

$$\partial_c S(x,t) = \operatorname{conv}\left\{ \begin{bmatrix} \nabla V_i(x(t)) \\ -W(y(t),w(t)) \end{bmatrix} \middle| \; i \in \mathcal{I}(x) \right\} \tag{22}$$

  and it follows from (18) that for any j and k in $\mathcal{I}(x)$,

$$\begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} \le 0 \tag{23}$$

$$\begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} \le 0 \tag{24}$$

  In addition, the continuity condition (17) implies that

$$\begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla V_j(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} \le 0 \tag{25}$$

  and

$$\begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_j(x) + g_j(x)w \\ 1 \end{bmatrix} = \begin{bmatrix} \nabla V_k(x) \\ -W(y,w) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} f_k(x) + g_k(x)w \\ 1 \end{bmatrix} \le 0 \tag{26}$$

  From (23-26), it follows that (11) is satisfied for the differential inclusion (19).

In conclusion, by Proposition 5, $S(x,t)$ is nonincreasing along the trajectories of (7) in $\mathcal{X}$. Therefore (8) is satisfied and the system (7) is dissipative with storage function $V(x)$ and supply rate $W(y,w)$.

4 Application to PWA Systems

In the following, three types of candidate Lyapunov functions (quadratic, piecewise quadratic and polynomial) are used for stability analysis of PWA systems.

4.1 Quadratic Lyapunov Functions For PWA Systems

Perhaps the simplest candidate for a $\mathcal{C}^1$ Lyapunov function is the quadratic form

$$V(x) = x^{\mathrm{T}} P x$$

where $P = P^{\mathrm{T}} > 0$. Proposition 8 describes sufficient conditions for the stability of the PWA system (4) using a quadratic Lyapunov function.

Proposition 8 *If for a given decay rate $\alpha > 0$, there exists $P = P^T > 0$ satisfying*

$$\begin{cases} PA_i + A_i^T P \le -\alpha P, & \text{if } a_i = 0 \text{ and } e_i \neq 0 \\ PA_i + A_i^T P + E_i^T \Lambda_i E_i \le -\alpha P, & \text{if } a_i = 0 \text{ and } e_i = 0 \\ \bar{P}\bar{A}_i + \bar{A}_i^T \bar{P} + \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \le -\alpha \bar{P}, & \text{otherwise} \end{cases} \tag{27}$$

for $i = 1, \ldots, M$ where

$$\bar{P} = \begin{bmatrix} P & 0_{n\times 1} \\ 0_{1\times n} & 0 \end{bmatrix},$$

$\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ and $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ have nonnegative entries, $x = 0$ is asymptotically stable for the PWA system (4).

PROOF. Consider $V(x) = x^{\mathrm{T}}Px$ as the candidate Lyapunov function. For this function, $\nabla V(x) = 2Px$. In the following, the regions $\mathcal{R}_i$ will be divided into three groups:

(1) If $a_i = 0$ and $e_i \neq 0$, we conclude from (27) that for all $x \in \mathbb{R}^n$

$$\begin{aligned}\nabla V(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P A_i x = x^{\mathrm{T}}(PA_i + A_i^{\mathrm{T}} P)x \\ &\leq -\alpha x^{\mathrm{T}} P x = -\alpha V(x)\end{aligned}$$

(2) If $a_i = 0$ and $e_i = 0$, we have $\overline{\mathcal{R}}_i = \{x | E_i x \geq 0\}$ and for any $\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ with nonnegative entries and for all $x \in \overline{\mathcal{R}}_i$, $x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \geq 0$. In this case, (27) leads to the following inequality for all $x \in \overline{\mathcal{R}}_i$.

$$\begin{aligned}\nabla V(x)^{\mathrm{T}} A_i x &\leq -\alpha x^{\mathrm{T}} P x - x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \\ &\leq -\alpha x^{\mathrm{T}} P x = -\alpha V(x)\end{aligned}$$

(3) If $a_i \neq 0$, we have $\overline{\mathcal{R}}_i = \{x | \bar{E}_i \bar{x} \geq 0\}$. Condition (27) implies that for all $x \in \overline{\mathcal{R}}_i$

$$\begin{aligned}\nabla V(x)^{\mathrm{T}}(A_i x + a_i) &= \bar{x}^{\mathrm{T}}(\bar{P}\bar{A}_i + \bar{A}_i^{\mathrm{T}}\bar{P})\bar{x} \\ &\leq -\alpha \bar{x}^{\mathrm{T}} \bar{P} \bar{x} - \bar{x}^{\mathrm{T}} \bar{E}_i^{\mathrm{T}} \bar{\Lambda}_i \bar{E}_i \bar{x} \\ &\leq -\alpha \bar{x}^{\mathrm{T}} \bar{P} \bar{x} = -\alpha x^{\mathrm{T}} P x \\ &= -\alpha V(x)\end{aligned}$$

In summary, for all $x \in \overline{\mathcal{R}}_i, i = 1, \ldots, M$,

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \leq -\alpha V(x)$$

Therefore using Proposition 6, the system (4) is dissipative with the storage function $V(x)$ and the supply rate $-\alpha V(x)$. Invoking Theorem 3 finishes the proof.

Remark 9 *In Proposition 8, the origin is not required to be the equilibrium point of all the subsystems of the PWA system (4). This makes Proposition 8 different from the common Lyapunov function approach in Lin and Antsaklis (2005) which requires the origin to be the equilibrium point for all vector fields of the system.*

Proposition 8 provides sufficient conditions for quadratic stability of PWA systems as a set of linear matrix inequalities (LMIs). LMIs can be solved efficiently using interior point algorithms implemented in software packages such as Yalmip in Löfberg (2004) and SeDuMi in Strum (2001).

4.2 Quadratic Lyapunov Functions For PWA Slab Systems

Proposition 10 provides sufficient conditions for the stability of system (4) with slab regions.

Proposition 10 *All trajectories of the PWA slab system (4) in $\mathcal{X}$ asymptotically converge to $x = 0$ if for a given decay rate $\alpha > 0$, there exist $P \in \mathbb{R}^{n \times n}$ and $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, M$ such that*

$$P > 0,$$

$$A_i^T P + PA_i + \alpha P \leq 0, \quad \textit{for } i \in \mathcal{I}(0), \tag{28}$$

$$\begin{cases} \lambda_i < 0, \\ \begin{bmatrix} A_i^T P + PA_i + \alpha P + \lambda_i L_i^T L_i & Pa_i + \lambda_i l_i L_i^T \\ a_i^T P + \lambda_i l_i L_i & \lambda_i(l_i^2 - 1) \end{bmatrix} \leq 0, \end{cases} \tag{29}$$

for $i \notin \mathcal{I}(0)$.

PROOF. Consider the candidate Lyapunov function $V(x) = x^{\mathrm{T}}Px$ for the PWA slab system (4) where $P > 0$. One of the following situations is true:

(1) For $x \in \overline{\mathcal{R}}_i$ where $i \in \mathcal{I}(0)$, multiplying the inequality (28) by x^{T} and x from left and right, respectively, implies

$$\nabla V(x)^{\mathrm{T}} A_i x + \alpha V(x) \leq 0, \text{ for } x \in \overline{\mathcal{R}}_i, i \in \mathcal{I}(0) \tag{30}$$

(2) For $x \in \overline{\mathcal{R}}_i$ where $i \notin \mathcal{I}(0)$, it follows from the constraint (29) that

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) + \alpha V(x) + \lambda_i(\|L_i x - l_i\|^2 - 1) \leq 0, \tag{31}$$

Since $\lambda_i < 0$, conditions (31) and (6) imply

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) + \alpha V(x) \leq 0, \text{ for } x \in \overline{\mathcal{R}}_i, i \notin \mathcal{I}(0) \tag{32}$$

Now, it follows from (30), (32) and Proposition 6 that the system (4) is dissipative with the storage function $V(x)$ and the supply rate $-\alpha V(x)$. Invoking Theorem 3 finishes the proof.

4.3 Piecewise Quadratic Lyapunov Functions For PWA Systems

For stability analysis of PWA systems, PWQ functions are less conservative than quadratic Lyapunov functions as shown in Johansson (2003). However, PWA systems with sliding modes are not usually considered. The reason is that the existence of a continuous positive definite PWQ function that decreases with time inside the regions is not a sufficient condition for stability of a PWA system. This is shown by the following counter-example.

Example 11 *Consider the PWA system*

$$\dot{x} = \begin{cases} A_1 x, \ x_2 > 0 \\ \\ A_2 x, \ x_2 < 0 \end{cases} \tag{33}$$

where

$$A_1 = \begin{bmatrix} 1 & -2 \\ 2 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 2 \\ -2 & -2 \end{bmatrix} \tag{34}$$

For this system, we have $E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}$. *Consider the following PWQ Lyapunov function candidate*

$$V(x) = \begin{cases} x^T P_1 x, \ \ x_2 \geq 0 \\ \\ x^T P_2 x, \ \ x_2 \leq 0 \end{cases} \tag{35}$$

The following set of constraints is a sufficient condition for (35) to be continuous, positive definite and decreasing with time inside the regions.

$$\begin{cases} (P_1 - P_2)_{11} = 0, \ P_1 > 0, \ P_2 > 0 \\ A_1^T P_1 + P_1 A_1 + \lambda_1 E_1^T E_1 < -\alpha P_1 \\ A_2^T P_2 + P_2 A_2 + \lambda_2 E_2^T E_2 < -\alpha P_2 \\ \lambda_1 > 0, \ \lambda_2 > 0, \ \alpha = 0.1 \end{cases}$$

where $(P_1 - P_2)_{11}$ *means the element in row one and column one. One solution of the above problem is* $\lambda_1 = 0.5755, \lambda_2 = 0.5755$ *and*

$$P_1 = \begin{bmatrix} 1.8073 & -1.0745 \\ -1.0745 & 1.4261 \end{bmatrix}, \ P_2 = \begin{bmatrix} 1.8073 & 1.0745 \\ 1.0745 & 1.4261 \end{bmatrix}$$

$V(x)$ in (35) is a continuous PWQ positive definite function that decreases with time inside the regions of system (33). However, system (33) is unstable. The problem here is that a PWQ candidate Lyapunov function has been used for a PWA system with a discontinuous vector field. In this case, it is possible to have unstable sliding modes at the boundaries of the regions.

Remark 12 *In (Johansson, 2003, p.64), an extra condition is introduced for PWA systems which have sliding modes. However, the limitation of this method is that it requires previous knowledge of geometrical properties of the sliding modes. One way to solve this problem is to use $\mathcal{C}^1$ PWQ Lyapunov functions, which is also proposed in (Johansson, 2003, p. 84) but this is more conservative because there are cases for which $\mathcal{C}^1$ Lyapunov PWQ functions cannot be found as shown in example 18*

Consider the piecewise quadratic candidate Lyapunov function continuous at the boundaries and defined in $\mathcal{X}$ by the expression

$$V(x) = \bar{x}^T \bar{P}_i \bar{x}, \text{ for } x \in \overline{\mathcal{R}}_i$$

where $\bar{P}_i = \bar{P}_i^T \in \mathbb{R}^{(n+1)\times(n+1)}$. Define

$$\bar{P}_i = \begin{bmatrix} P_i & q_i \\ q_i^T & r_i \end{bmatrix}, \quad \bar{F}_{ij} = \begin{bmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{bmatrix} \tag{36}$$

where $P_i \in \mathbb{R}^{n\times n}, q_i \in \mathbb{R}^n, r_i \in \mathbb{R}$ and F_{ij}, l_{ij} are defined in (5). Proposition 13 describes sufficient conditions for the stability of the PWA system (4) based on a PWQ Lyapunov function.

Proposition 13 *Let there exist matrices* $\bar{P}_i = \bar{P}_i^T$ *defined in (36),* Z_i, $\bar{Z}_i$, Λ_i *and* $\bar{\Lambda}_i$ *that verify the following conditions for all* $i = 1, \ldots, M$ *and a given decay rate* $\alpha > 0$

- *Conditions on the vector field:*

$$a_i = 0, \ \textit{if}\ 0 \in \overline{\mathcal{R}}_i$$
$$(\bar{A}_i - \bar{A}_j)\bar{F}_{ij} = 0, \ \textit{if} \ \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset$$

- *Continuity of the Lyapunov function:*

$$\bar{F}_{ij}^T (\bar{P}_i - \bar{P}_j)\bar{F}_{ij} = 0, \ \textit{if} \ \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j \neq \emptyset \tag{37}$$

- *Positive definiteness of the Lyapunov function:*

$$q_i = 0, \ r_i = 0, \ \textit{if}\ a_i = 0 \tag{38}$$
$$P_i \geq \epsilon I, \ \textit{if}\ a_i = 0 \textit{ and } e_i \neq 0 \tag{39}$$
$$\begin{cases} Z_i \succeq 0 \\ P_i - E_i^T Z_i E_i \geq \epsilon I \end{cases}, \ \textit{if}\ a_i = 0 \textit{ and } e_i = 0 \tag{40}$$
$$\begin{cases} \bar{Z}_i \succeq 0 \\ \bar{P}_i - \bar{E}_i^T Z_i \bar{E}_i \geq \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{cases}, \ \textit{if}\ a_i \neq 0 \tag{41}$$

- *Monotonicity of the Lyapunov function:*

if $a_i = 0$ *and* $e_i \neq 0$,

$$P_i A_i + A_i^T P_i \leq -\alpha P_i \tag{42}$$

if $a_i = 0$ *and* $e_i = 0$,

$$\begin{cases} \Lambda_i \succeq 0 \\ P_i A_i + A_i^T P_i + E_i^T \Lambda_i E_i \leq -\alpha P_i \end{cases} \tag{43}$$

if $a_i \neq 0$,

$$\begin{cases} \bar{\Lambda}_i \succeq 0 \\ \bar{P}_i \bar{A}_i + \bar{A}_i^T \bar{P}_i + \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \leq -\alpha \bar{P}_i \end{cases} \tag{44}$$

where $\succeq$ *denotes an elementwise inequality. Then all the trajectories of (4) in* $\mathcal{X}$ *asymptotically converge to* $x = 0$.

PROOF. Consider $V(x) = \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x}$ for $x \in \overline{\mathcal{R}}_i$ as the candidate Lyapunov function. It follows from (5) and (37) that for any $x \in \overline{\mathcal{R}}_i \bigcap \overline{\mathcal{R}}_j$, $V_i(x) = V_j(x)$. Therefore $V(x)$ is continuous over $\mathcal{X}$. In addition, constraint (38) implies that $V(0) = 0$. The rest of the proof is divided into three parts:

(1) If $a_i = 0$ and $e_i \neq 0$, we conclude from (39) that for all $x \neq 0$ in $\mathcal{R}_i$, $V(x) = x^{\mathrm{T}} P_i x \geq \epsilon \|x\|^2 > 0$, and from (42) that for all $x \neq 0$

$$\begin{aligned} \nabla V_i(x)^{\mathrm{T}} A_i x &= 2x^{\mathrm{T}} P_i A_i x = x^{\mathrm{T}} (P_i A_i + A_i^{\mathrm{T}} P_i) x \\ &\leq -\alpha x^{\mathrm{T}} P_i x = -\alpha V(x) \end{aligned}$$

(2) If $a_i = 0$ and $e_i = 0$, we have $\overline{\mathcal{R}}_i = \{x | E_i x \geq 0\}$ and for any $Z_i \in \mathbb{R}^{p_i \times p_i}$ and $\Lambda_i \in \mathbb{R}^{p_i \times p_i}$ with nonnegative entries and for all $x \in \overline{\mathcal{R}}_i$, $x^{\mathrm{T}} E_i^{\mathrm{T}} Z_i E_i x \geq 0$, $x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \geq 0$. In this case, (40) yields

$$V_i(x) = x^{\mathrm{T}} P_i x > x^{\mathrm{T}} E_i^{\mathrm{T}} Z_i E_i x + \epsilon \|x\|^2 \geq \epsilon \|x\|^2 > 0$$

for all $x \neq 0$ in $\overline{\mathcal{R}}_i$, and (43) yields

$$\begin{aligned} \nabla V_i(x)^{\mathrm{T}} A_i x &\leq -\alpha x^{\mathrm{T}} P_i x - x^{\mathrm{T}} E_i^{\mathrm{T}} \Lambda_i E_i x \\ &\leq -\alpha x^{\mathrm{T}} P_i x = -\alpha V(x) \end{aligned}$$

for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

(3) If $a_i \neq 0$, we have $\overline{\mathcal{R}}_i = \{x | \bar{E}_i \bar{x} \geq 0\}$ and similarly to the previous case, condition (41) implies that for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$V_i(x) = \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x} > \bar{x}^{\mathrm{T}} \bar{E}_i^{\mathrm{T}} \bar{Z}_i \bar{E}_i \bar{x} + \epsilon \|x\|^2 \geq \epsilon \|x\|^2 > 0$$

and condition (44) implies that for all $x \neq 0$ in $\overline{\mathcal{R}}_i$

$$\begin{aligned} \nabla V_i(x)^{\mathrm{T}} (A_i x + a_i) &\leq -\alpha \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x} - \bar{x}^{\mathrm{T}} \bar{E}_i^{\mathrm{T}} \bar{\Lambda}_i \bar{E}_i \bar{x} \\ &\leq -\alpha \bar{x}^{\mathrm{T}} \bar{P}_i \bar{x} = -\alpha V(x) \end{aligned}$$

In summary, for all $x \in \overline{\mathcal{R}}_i$ and for $i = 1, \ldots, M$,

$$V_i(x) \geq \epsilon \|x\|^2$$

$$\nabla V_i(x)^{\mathrm{T}} (A_i x + a_i) \leq -\alpha V_i(x)$$

Therefore using Proposition 7, the system (4) is dissipative with the storage function $V(x)$ and the supply rate $-\alpha V(x)$. Invoking Theorem 3 finishes the proof.

PWQ Lyapunov functions are less conservative than quadratic Lyapunov functions. However, it is stated in Johansson (2003) that for the PWA system (33), it is not possible to find a $\mathcal{C}^1$ PWQ Lyapunov function although the system is stable when we have

$$A_1 = \begin{bmatrix} -2 & -2 \\ 4 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & 2 \\ -4 & 1 \end{bmatrix} \tag{45}$$

In the next section, it is shown that a polynomial Lyapunov function exists for this system.

4.4 Polynomial Lyapunov Function for PWA Systems

In this section, it is proposed to consider a sum of squares (SOS) polynomial as a candidate Lyapunov function. For a tutorial about system analysis techniques based on the SOS decomposition see Papachristodoulou and Prajna (2005).

Definition 14 *Prajna et al. (2005) A multivariate polynomial*

$$p(x_1, \ldots, x_n) \triangleq p(x)$$

is SOS if there exist polynomials $p_1(x), \ldots, p_m(x)$ *such that*

$$p(x) = \sum_{i=1}^{m} p_i^2(x).$$

SOS polynomials $p(x)$ are globally nonnegative. Although verifying nonnegativity of a polynomial is an NP-hard problem (see Murty and Kabadi (1987)), the SOS condition can be formulated as a convex problem in polynomial coefficients (see Prajna et al. (2005)). However, note that not all nonnegative polynomials are SOS.

Proposition 15 *If for the PWA system (4), there exists a polynomial* $V(x)$ *satisfying the following conditions for* $i = 1, \ldots, M$

$$V(x) - \lambda(\|x\|^2) \text{ is SOS.} \tag{46}$$

$$\begin{aligned} &- \nabla V(x)^T (A_i x + a_i) - \Gamma_i(x)^T (E_i x + e_i) \\ &- \alpha V(x) \text{ is SOS for all } i. \end{aligned} \tag{47}$$

where $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ *is a strictly increasing polynomial function,* $\lambda(0) = 0$, $\alpha > 0$ *and* $\Gamma_i : \mathbb{R}^n \to \mathbb{R}^{p_i \times 1}$ *is a vector of SOS polynomials,* $x = 0$ *is asymptotically stable.*

PROOF. Conditions (46) imply

$$V(x) \geq \lambda(\|x\|^2)$$

Note that λ is radially unbounded since it is a strictly increasing polynomial function. Therefore, V is radially

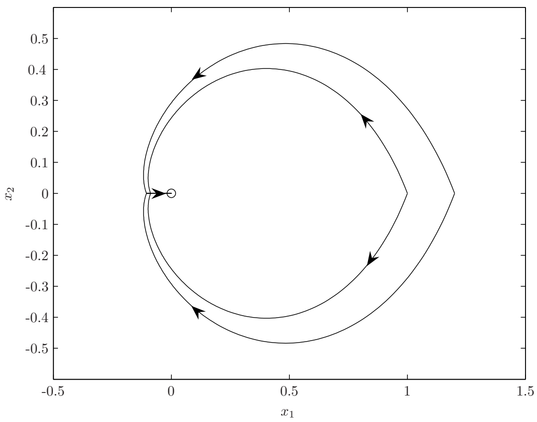

Fig. 1. Trajectories of a stable PWA system with a sliding mode

unbounded. Condition (47) leads to the following inequality

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \le -\Gamma_i(x)^{\mathrm{T}}(E_i x + e_i) - \alpha V(x) \quad (48)$$

Since $\Gamma_i(x)$ is a vector of SOS polynomials and $E_i x + e_i \ge 0$ for all x in $\overline{\mathcal{R}}_i$, we have

$$\Gamma_i(x)^{\mathrm{T}}(E_i x + e_i) \ge 0, \ \forall x \in \overline{\mathcal{R}}_i \quad (49)$$

Therefore (48) and (49) imply

$$\nabla V(x)^{\mathrm{T}}(A_i x + a_i) \le -\alpha V(x) < 0, \ \forall x \in \overline{\mathcal{R}}_i, x \ne 0$$

Thus, using Proposition 6, the system (4) is dissipative with the storage function $V(x)$ and the supply rate $-\alpha V(x)$. Invoking Theorem 3 finishes the proof.

Example 16 *Consider the PWA system (33) where*

$$A_1 = \begin{bmatrix} -1 & -2 \\ 2 & -2 \end{bmatrix}, \ A_2 = \begin{bmatrix} -1 & 2 \\ -2 & -2 \end{bmatrix}$$

For this system, we have $E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & -1 \end{bmatrix}$. *Solving the following LMIs based on Proposition 8*

$$\begin{cases} A_1^T P + P A_1 + \lambda_1 E_1^T E_1 < -\alpha P \\ A_2^T P + P A_2 + \lambda_2 E_2^T E_2 < -\alpha P \\ P > 0, \ \lambda_1 > 0, \ \lambda_2 > 0 \end{cases} \quad (50)$$

where $\alpha = 0.1$, *yields*

$$P = \begin{bmatrix} 0.6002 & 0 \\ 0 & 0.5817 \end{bmatrix}, \ \lambda_1 = 1.1329, \ \lambda_2 = 1.1329.$$

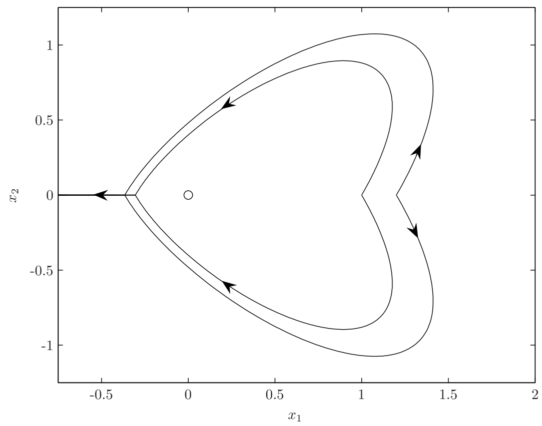

Fig. 2. Trajectories of an unstable PWA system with a sliding mode

Therefore $x = 0$ *is asymptotically stable since* $\mathcal{X} = \mathbb{R}^2$. *It is interesting to note that this system has an attractive sliding mode on the negative side of the* x_1 *axis (Fig. 1). However, no separate condition was considered to check the existence or stability of the sliding mode.*

Example 17 *Consider system (33) in* $\mathbb{R}^2$ *with matrices (34). For this system,* E_1 *and* E_2 *are defined as in Example 16. The LMI set (50) is infeasible in this case. In fact, although* A_1 *and* A_2 *are Hurwitz, there exists an unstable sliding mode and system (33) is unstable (Fig. 2).*

Example 18 *Consider the PWA system (33) in* $\mathbb{R}^2$ *with matrices (45). There is no quadratic or* $\mathcal{C}^1$ *PWQ Lyapunov function for this system (Johansson, 2003, p.84). However, by solving the following SOS program we can find a sixth order polynomial Lyapunov function.*

$V(x) - 0.001\|x\|^2$ *is SOS.*
$-\nabla V.(A_1 x) - \Gamma_1(x)(x_2) - 0.01V(x)$ *is SOS.*
$-\nabla V.(A_2 x) - \Gamma_2(x)(-x_2) - 0.01V(x)$ *is SOS.*

where $\Gamma_1(.)$ *and* $\Gamma_2(.)$ *are fourth order SOS polynomials. This is a convex problem and can be solved by SOSTOOLS in Prajna et al. (2004) or Yalmip in Löfberg (2004). Trajectories of the system (45) and contours of the obtained SOS Lyapunov function are shown in Fig. 3. Notice that there is a stable sliding mode.*

5 Conclusions

Sufficient conditions for stability of piecewise smooth systems were formulated as convex problems. Departing from the work in the literature, our results show that sufficient conditions for the stability of PWA and PWP systems can be formed without any need for a-priori information about attractive sliding modes on switching surfaces. Example 18 shows that the proposed method works where previously suggested methods fail.

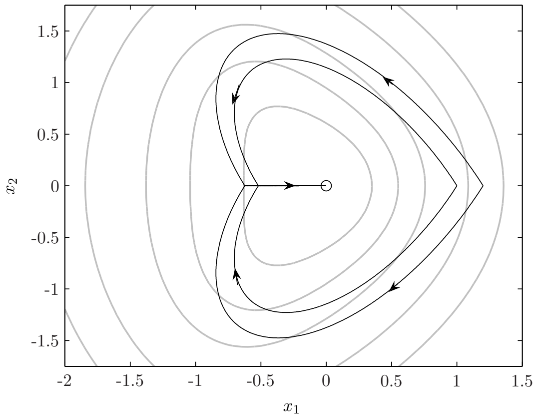

Fig. 3. Trajectories of a stable PWA system (black) and contours of the Lyapunov function (gray)

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