Multiple solutions of a generalized one-dimensional Bratu problem

Taoufik Bakri, Yuri A. Kuznetsov, Ferdinand Verhulst

Department of Mathematics, Utrecht University
Budapestlaan 6, 3508 TA Utrecht
The Netherlands

Eusebius Doedel

Department of Computer Science, Concordia University
1455 Boulevard de Maisonneuve O., Montreal
Quebec, H3G 1M8
Canada

Abstract

Nonlinear two-point boundary value problems (BVPs) may have none or more than one solution. For the singularly perturbed two-point BVP

$$\varepsilon u'' + 2u' + f(u) = 0, \quad 0 < x < 1, \quad u(0) = 0,\ u(1) = 0,$$

a condition is given to have one and only one solution; also cases of more solutions have been analyzed. After attention to the form and validity of the corresponding asymptotic expansions, partially based on slow manifold theory, we reconsider the BVP within the framework of small and large values of the parameter. In the case of a special nonlinearity, numerical bifurcation patterns are studied that improve our understanding of the multi-valuedness of the solutions.

Keywords: boundary value problems, branch points, asymptotic expansions, numerical continuation

1. Introduction

It is well known that existence of solutions of nonlinear boundary value problems does not necessarily imply uniqueness. An example is the following strongly damped equation with a small parameter ε subject to the Dirchlet boundary conditions

$$\varepsilon u'' + 2u' + f(u) = 0, \quad 0 < x < 1, \quad u(0) = 0,\ u(1) = 0, \quad (1)$$

where for certain nonlinearities $f(u)$ more than one solution exists. Interestingly, classical matched asymptotic expansions immediately produces an approximation of one of the solutions; we will call this the "small" solution. A proof of asymptotic validity of the expansions can be given with various methods; we choose here a shooting method employing slow manifold theory. It turns out that the boundedness of the nonlinearity $f(u)$ will guarantee the uniqueness of the solution. Insight in the existence and the approximate character of a possible second solution is obtained by considering a neighbouring conservative problem and using mixed analytic-numerical methods; the asymptotics is definitely non-standard. In nonlinear two-point boundary value problems with a small parameter ε one can distinguish between cases where 'routine' matching or multiple scale methods apply, and cases showing unexpected behaviour. For the latter it helps if we can identify a slow manifold which is stable. The analysis is supplemented by numerical continuation that explores the behaviour of the solutions for both small and large values of the parameter ε, producing conditions for the existence and bifurcations of "small" and "large" solutions of the boundary value problem (1) with $f(u) = e^u$ (a dissipative modification of the classical one-dimensional Liouville-Bratu-Gelfand problem). In particular, we show that the corresponding solutions exhibit a generic branch point, thus making the bifurcation diagrams of (1) qualitatively different from that of the undumped problem.

2. The General case

Consider the boundary value problem (1). We assume $f(u) \geq a_0 > 0$, $f \in C^2(\mathbb{R})$, and without loss of generality we take $f(0) = 1$. Existence of solutions of this problem has been demonstrated in [3] and [7]. These existence proofs also imply local uniqueness, which means that in a neighbourhood of the solution there exists no other solution of the Dirichlet boundary value problem. Conditions for the existence of one or two solutions of (1) are presented in [12].

2.1. Introductory lemmas

We start with a number of general observations.

Lemma 1. *A solution of Eqn.* (1) *has one and only one interior maximum (no interior minimum).*

Proof: When $u(x)$ is identically zero, it does not satisfy the equation. Thus $u(x)$ has interior extreme values. At a stationary point p we have

$$\varepsilon u''(p) = -f(u(p)) ,$$

so the curvature is negative, and we have a maximum with respect to p. The presence of more than one interior maximum would imply the existence of a minimum. Contradiction.

□

A simple corollary of Lemma 1 is that $u(x) \geq 0$ in $[0, 1]$. We will reformulate Eqn. (1) as an initial value problem with

$$u(0) = 0 \quad , \quad u'(0) = \alpha > 0 ,$$

where α is to be determined later by imposing a second boundary condition.

Lemma 2. *For a solution $u(x)$ of Eqn.* (1), *as $\varepsilon \to 0$, we have*

$$u'(1) = -\frac{1}{2} + O(\varepsilon) .$$

Proof: Putting $\varepsilon = 0$ in the equation of (1) and replacing u by u_0 we have

$$2u_0' = -f(u_0) .$$

We put $u_0(1) = 0$, and we note that the equation has a stable slow manifold M, with a $O(\varepsilon)$ approximation by the manifold M_0 described by $u_0(x)$. This obervation follows from Fenichel's geometric singular perturbation theory; for an introduction see [13]. Explicitly this implies that for x away from the boundary layer near $x = 0$ we have

$$u(x) = u_0(x) + O(\varepsilon) .$$

The lemma follows from this estimate.

□

2.2. Construction by matched asymptotic expansions

For the outer (regular) solution we expand

$$u(x) = u_0(x) + \varepsilon u_1(x) + \varepsilon^2 \cdots ,$$

which, as seen above, produces

$$2u_0' = -f(u_0) ,$$

for the first term. This leads to

$$F(u_0) = \int_0^{u_0} \frac{ds}{f(s)} = -\frac{1}{2}x + c .$$

Since $x(u_0)$ is monotonic, we apply the condition $u_0(1) = 0$ to the inverse, so that $c = \frac{1}{2}$. The reasoning of classical singular perturbation theory is that the function $u_0(x)$ will generally not satisfy the boundary condition, *i.e.*, $u_0(0) \neq 0$. Thus we expect a boundary layer near $x = 0$, which agrees with Fenichel's theory. Rescaling

$$\frac{x}{\lambda(\varepsilon)} = \xi, \;\; \lambda(\varepsilon) = o(1) ,$$

and transforming $u(x) \to w(\xi)$ yields

$$\frac{\varepsilon}{\lambda^2(\varepsilon)}\frac{d^2w}{d\xi^2} + \frac{2}{\lambda(\varepsilon)}\frac{dw}{d\xi} + f(w) = 0 .$$

If we assume that $f(w)$ is bounded by a constant independent of ε then we have a significant degeneration for $\lambda(\varepsilon) = \varepsilon$. We find

$$\frac{d^2w}{d\xi^2} + 2\frac{dw}{d\xi} + \varepsilon f(w) = 0 .$$

Upon expanding

$$w(\xi) = w_0(\xi) + \varepsilon w_1(\xi) + \varepsilon^2 + \cdots$$

we find

$$\frac{d^2w_0}{d\xi^2} + 2\frac{dw_0}{d\xi} = 0 .$$

With the boundary condition $w(0) = 0$, and introducing a constant A, we have

$$w_0(\xi) = Ae^{-2\xi} - A .$$

The matching rule

$$\lim_{\xi\to\infty} w_0(\xi) = \lim_{x\to 0} u_0(x)$$

yields

$$-A = u_0(0) .$$

The composite expansion $\tilde{u}(x)$ is a first order formal asymptotic approximation of the form

$$\tilde{u}(x) = -u_0(0)e^{-2\frac{x}{\varepsilon}} + u_0(x) . \tag{2}$$

Note that the maximum is located in the $O(\varepsilon)$ boundary layer near $x = 0$. We still have to give a proof of the asymptotic validity. We can obtain this by a maximum principle, see for instance [3], [5] or [7], but we will explore an alternative route that produces additional information.

2.3. Construction and proof by shooting

Putting $u(0) = 0, \; u'(0) = \alpha > 0$ for the equation of Eqn. (1), we transform $u, u' \to A, B$ by

$$\begin{aligned} u(x) &= A(x) + B(x)e^{-\frac{2}{\varepsilon}x}, \\ u'(x) &= -\frac{2}{\varepsilon}B(x)e^{-\frac{2}{\varepsilon}x}. \end{aligned}$$

We find $A(0) = \frac{1}{2}\varepsilon\alpha$, $B(0) = -\frac{1}{2}\varepsilon\alpha$, and by variation of constants the integral equations

$$u(x) = \frac{1}{2}\varepsilon\alpha - \frac{1}{2}\varepsilon\alpha e^{-\frac{2}{\varepsilon}x} - \frac{1}{2}\int_0^x f(u(s))ds + \frac{1}{2}e^{-\frac{2}{\varepsilon}x}\int_0^x e^{\frac{2}{\varepsilon}s} f(u(s))\,ds , \tag{3}$$

and

$$u'(x) = \alpha e^{-\frac{2}{\varepsilon}x} - \frac{1}{\varepsilon}e^{-\frac{2}{\varepsilon}x}\int_0^x e^{\frac{2}{\varepsilon}s} f(u(s))\,ds . \tag{4}$$

Lemma 3. *The interior maximum of $u(x)$ is assumed for $x = m$, $0 < m < 1$ with*

$$0 < u(m) < \frac{1}{2}\varepsilon\alpha .$$

Proof: From the requirement $u'(m) = 0$ and (3) we find for $0 < x < 1$

$$u(m) = \frac{1}{2}\varepsilon\alpha - \frac{1}{2}\int_0^m f(u(s))ds < \frac{1}{2}\varepsilon\alpha \ .$$

□

We derive an equation for α by the following lemma:

Lemma 4. *With $u(1) = 0$, the solution $u(x)$ of the initial value problem and the initial value α have to satisfy the equation*

$$\varepsilon\alpha - \varepsilon u'(1) = \int_0^1 f(u(s))\, ds \ .$$

Proof: Put $x = 1$ in Eqs. (3-4) and eliminate one of the integrals by using (4).

□

Lemmas 3 and 4 have as an interesting consequence that α has to depend on ε and that it has to become unbounded as $\varepsilon \to 0$. For suppose we have $0 < \alpha < c$, where c is a constant independent of ε. From Lemma 2 we have

$$0 < u(m) < \varepsilon c \ ,$$

so that

$$f(u(s)) = 1 + O(\varepsilon),\ \int_0^1 f(u(s))ds = 1 + O(\varepsilon),\ 0 < x < 1 \ .$$

It follows from Lemma 4 that the equation linking α and $u(x)$ can not be satisfied.

We will now apply an O'Malley-Vasil'eva expansion to obtain an asymptotic approximation of $u(x)$; for references see [11] or [13]. We find

$$\tilde{u}(x) = \frac{1}{2}\varepsilon\alpha - u_0(0) - \frac{1}{2}\varepsilon\alpha e^{-\frac{2}{\varepsilon}x} + u_0(x) \ , \qquad (5)$$

with the estimate $u(x) - \tilde{u}(x) = O(\varepsilon)$ on $[0, 1]$. Requiring that $\tilde{u}(1) = 0$ results in the condition

$$\frac{1}{2}\varepsilon\alpha = u_0(0) \ ,$$

which is the same expression obtained in the preceding subsection by matched asymptotic expansions. Note that the actual construction of the asymptotic approximation by shooting is more complicated, but that it also provides a proof of asymptotic validity. Another point of interest is that for the initial value problem the solutions approach the slow manifold M_0 described by the outer solution.

2.4. A condition for uniqueness

In Section 2.2 we found that there is only one significant degeneration if $f(u)$ is bounded for $x \in [0, 1]$ and $\varepsilon \to 0$. For a case like $f(u) = e^u$ we have found two solutions numerically; see also [12]. On the other hand in [3] and [7] theorems are given guaranteeing existence and uniqueness of boundary value problem (1), *e.g.*, in [7], Chapter 11, Theorem 1. The apparent contradiction is solved when one considers the proof of this Theorem 1. It is constructed by translating the boundary value problem into a formulation as a nonlinear map $p \mapsto F(p)$ of a normed space N into a Banach space and then applying fixed point theory. For the linear space we have

$$N := \{p | p \in C^2[0, 1], p(0) = p(1) = 0\}$$

with

$$||p|| = \max_{[0,1]} |p(x)| + \varepsilon \max_{[0,1]} |\frac{dp(x)}{dx}| + \varepsilon^2 \max_{[0,1]} |\frac{d^2p(x)}{dx^2}| \ .$$

Note that there is no guarantee that this norm is bounded for $\varepsilon \to 0$ in Eqn. (1). However, as the matching process suggests, in the case of bounded $f(u)$ we have uniqueness of the solution of Eqn. (1).

Theorem 1. *For the nonlinear term in Eqn.* (1), *assume that*

$$0 < a_0 \leq f(u(x)) \leq a_1 \ ,$$

for constants a_0, a_1 that are independent of ε. Then the solution of Eqn. (1) *is unique and has an $O(1)$ bound.*

Proof: The solution constructed in Section 2.2 is bounded to $O(1)$; see also Lemma 2. Suppose we have two solutions of Eqn. (1) with determining conditions $u(0) = 0, u'(0) = \alpha$, where either $\alpha = \alpha_1 = 2u_0(0)/\varepsilon + O(1)$ or $\alpha = \alpha_2$. The value of α_1 follows from Sections 2.2 and 2.3. Substituting α_1 and α_2 into the equation of Lemma 3 and subtracting we have

$$2u_0(0) - \varepsilon\alpha_2 + O(\varepsilon) = \int_0^1 (f(u_{\alpha_1}(x) - f(u_{\alpha_2}(x))\, dx \ ,$$

so that from the estimates for the nonlinearity f we have $\alpha_2 = C/\varepsilon + O(1)$, where C is a constant independent of ε. Lemma 2 produces the bound $C/2 + O(\varepsilon)$ for the solution. Note that from the approximation $\tilde{u}(x)$ in (5) we have for the two solutions

$$u_{\alpha_1}(x) - u_{\alpha_2}(x) = O(\varepsilon) \ .$$

From (4) it is now easy to conclude that $\varepsilon du/dx$ and $\varepsilon^2 d^2u/dx^2$ are also bounded by an $O(1)$ constant. We conclude from the uniqueness Theorem 1 of Chapter 11 of [7] that $u_{\alpha_1}(x) = u_{\alpha_2}(x)$.

□

3. The Case $f(u) = \exp(u)$

Consider the equation

$$\varepsilon u'' + 2u' + \exp(u) = 0,\ 0 < x < 1 \ , \qquad (6)$$

with boundary conditions $u(0) = 0$, $u(1) = 0$.

3.1. Asymptotic expansions

Following the procedure outlined in Section 2 we find

$$u_0(x) = \ln\left(\frac{2}{x+1}\right),$$

and for the asymptotic approximation

$$\tilde{u}(x) = \ln\left(\frac{2}{x+1}\right) - \ln 2e^{-\frac{2}{\varepsilon}x}.$$

In the construction of this example there is the explicit assumption that e^u is bounded by a constant independent of ε. The maximum of $u(x)$ is found in the boundary layer near $x = 0$. If it becomes large we have a different significant degeneration; unfortunately this is not a 'degeneration', it is described by the full equation.

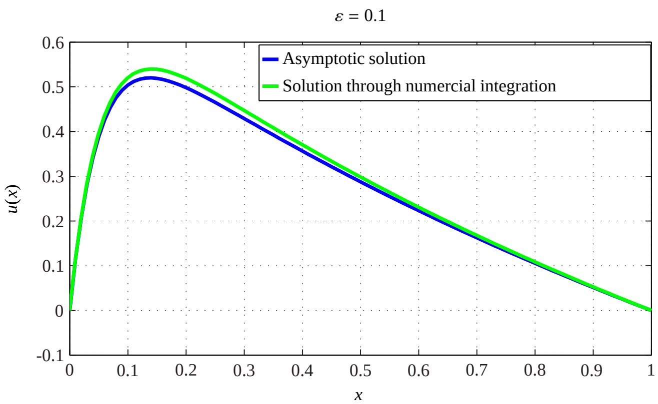


Figure 1: Numerical and first-order asymptotic approximations of the two-point boundary problem with $f(u) = \exp(u)$ in Eqn. (1).

Thus we have constructed a solution through first order asymptotic approximation for small ε. In [12] it has been shown that a second solution exists, which becomes unbounded as ε tends to zero. In an attempt to understand the origin of the second solution, we focus in this section on the asymptotic behaviour of the solution of this specific BVP for large ε. We will show that this BVP has two solutions u_1 and u_2 such that

$$\alpha_1 := u_1'(0) = 2\ln(\varepsilon) + O(\ln(|\ln(\varepsilon)|)),$$

and

$$\alpha_2 := u_2'(0) = 1/(2\varepsilon) + O(\varepsilon^{-3/2}),$$

as $\varepsilon \to \infty$. A numerical bifurcation study producing several bifurcation diagrams clarifies the origin of the second solution.

Introducing the perturbation parameter $0 < \lambda = 1/\varepsilon \ll 1$, the BVP (6), which from now on will be referred to as the nonconservative BVP, can be rewritten as follows:

$$\begin{aligned} u'' + 2u' + \lambda\exp(u) &= 0, \\ u(0) = u(1) &= 0. \end{aligned} \tag{7}$$

3.2. The conservative case

The conservative part of (7) is

$$\begin{aligned} u'' + \lambda\exp(u) &= 0, \\ u(0) = u(1) &= 0, \end{aligned} \tag{8}$$

which is a special case of the well-known 1D *Liuoville-Bratu-Gelfand BVP* [10, 6, 1], used as a test-example in BVP-continuation packages, *e.g.* AUTO [4].

Remark: A problem equivalent to (8) appears for $N = 1$ from the classical 1D Liuoville-Bratu-Gelfand problem

$$\begin{aligned} u'' + \frac{N-1}{x}u' + \lambda\exp(u) &= 0, \\ u'(0) = u(1) &= 0, \end{aligned} \tag{9}$$

which describes radially symmetric solutions of $\Delta u + \lambda\exp(u) = 0$ inside the unit sphere subject to the Dirichlet boundary conditions. It is remarkable that the solution behaviour in (9) with respect to parameter λ depends strongly on N (see [8]) and is very different for $N = 1$ [10], $N = 2$ [10], and $N = 3$ [6]. We will not further discuss these phenomena here and concentrate on case $N = 1$ when the u'-term vanishes. $\diamond$

We shall prove that solutions of the conservative BVP (8) remain close to those of the nonconsevative BVP on the time scale $O(1)$. Thus, we now focus on the asymptotics of the conservative BVP. The differential equation in this case can be solved implicitly through integration. We get the following expression from the integral:

$$u' = \pm\sqrt{c - 2\lambda\exp(u)},$$

with

$$c = \alpha^2 + 2\lambda, \tag{10}$$

and α as defined in Section 2.3. The orbits in this case are given in Fig. 2 for $\lambda = 0.01$.

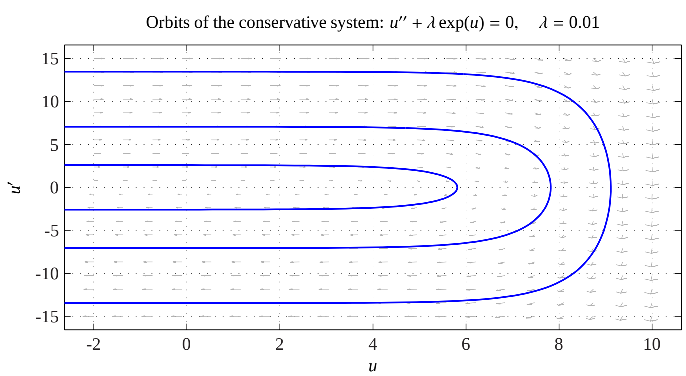


Figure 2: Orbits corresponding to the conservative case $u'' + \lambda\exp(u) = 0$

It is not difficult to see that the conservative BVP has a solution y if and only if

$$\int_0^{\ln(\frac{c}{2\lambda})} \frac{du}{\sqrt{c - 2\lambda\exp(u)}} = \frac{1}{2}. \tag{11}$$

This can be deduced from Fig. 2, in the sense that using the symmetry, one can see that the time τ needed for a particle to travel in the phase space from point $(0, \alpha)$ to $(0, y'(\tau))$ is twice the time needed to travel from $(0, \alpha)$ to $(\ln(\frac{c}{2\lambda}), 0)$. Requiring τ to be equal to 1 yields equation (11). Integrating the left hand side of (11) gives

$$\frac{\ln\left[\sqrt{\alpha^2+2\lambda}+\alpha\right]-\ln\left[\sqrt{\alpha^2+2\lambda}-\alpha\right]}{\sqrt{\alpha^2+2\lambda}}=\frac{1}{2}\,. \tag{12}$$

Note that (12) gives a necessary and sufficient condition for the conservative BVP to have solutions. In other words the number of real positive roots α of (12) is equal to the number of solutions of the conservative BVP.

Lemma 5. *Eqn.* (12) *has exactly two solutions* $\alpha_1 > \alpha_2 > 0$ *for* $\lambda > 0$ *and small enough, such that* $\alpha_1 \to +\infty$ *en* $\alpha_2 \to 0$ *as* $\lambda \to 0$.

Proof: Eqn. (12) can be rewritten as

$$\frac{2(Q^2-1)}{Q^2}\ln^2\!\left(\frac{Q+1}{Q-1}\right)=\lambda\,, \tag{13}$$

where $Q = \sqrt{1+2\lambda/\alpha^2} \geq 1$. We see from Fig. 3 that

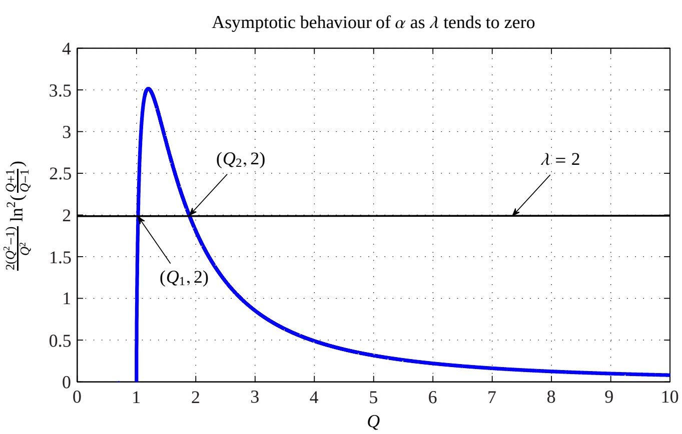


Figure 3: Asymptotic behaviour of the roots of (13) as $\lambda \to 0$.

(12) has exactly 2 solutions $\alpha_1 = \sqrt{2\lambda/(Q_1^2-1)}$ and $\alpha_2 = \sqrt{2\lambda/(Q_2^2-1)}$, provided λ is small enough. It also follows from Fig. 3 that $Q_1 \to 1$ and $Q_2 \to +\infty$ as $\lambda \to 0$. This is equivalent to $\alpha_1 \to +\infty$ and $\alpha_2 \to 0$ as $\lambda \to 0$, which concludes the proof. □

Corollary 1. *The conservative BVP* (8) *has 0, 1 or 2 solutions, depending on whether* $\lambda > \lambda_c$, $\lambda = \lambda_c$, *or* $0 < \lambda < \lambda_c$ *respectively, with* $\lambda_c \approx 3.513830719125$.

Proof: This follows immediately from Lemma 5. For the computation of λ_c we look for the maximum of the curve in Fig. 3. □

Remark: Lemma 1 is well-known and is given here for completeness. Moreover, there exists an explicit formula originally due to Liouville [10] for the solutions $u(x)$ of the conservative BVP (8), where roots of a transcendental equation equivalent to (13) are involved (see, e.g. [2]). The critical value λ_c corresponds to a fold bifurcation, at which these two solutions coalesce and disappear. We note that the value of λ_c can be computed to any desired accuracy without the use of continuation software. ⋄

Corollary 2. *The following asymptotic expressions hold for* α_1 *and* α_2*, as* λ *tends to* 0 *:*

$$\alpha_1 = -2\ln\lambda + O(\ln(|\ln\lambda|)) \quad , \quad \alpha_2 = \lambda/2 + O(\lambda^{3/2})\,.$$

Proof: One easily derives from (13) that:

$$\alpha_1 - 4\ln\alpha_1 = -\ln\lambda^2 + \ln 4 + o(\lambda) \tag{14}$$

$$\alpha_2^2 = \lambda^2/4 + O(\lambda^{5/2}) \tag{15}$$

from which it is straightforward to derive the asymptotic estimates for α_1 and α_2.

□

We now prove that the two solutions of the conservative BVP yield two neighboring solutions of the nonconservative BVP.

Theorem 2. *The nonconservative BVP has two solutions* $u_{1,2}$ *such that* $\sup_{t\in[0,1],\, y\in D}|u_1 - w_1| = O(\lambda|\ln\lambda|)$,
and $\sup_{t\in[0,1],\, y\in D}|u_2 - w_2| = O(\lambda^2)$ *as* $\lambda \to 0$, *with validity on the timescale* $t = O(1)$. *Here* $w_{1,2}$ *are the solutions of the conservative BVP as described above.*

Proof: It is easy to see that $w'_{1,2}(t) \leq \alpha_{1,2}$ for $0 \leq t \leq 1$. Because of the damping, we have the following relationship: $|u'_{1,2}(t)| \leq |w'_{1,2}(t)|$ for $0 \leq t \leq 1$. Combining these results with Corollary 2 yields the following estimate:

$$u'_1(t) = O\ln(\lambda)\,, \tag{16}$$

$$u'_2(t) = O(\lambda)\,. \tag{17}$$

$$\tag{18}$$

The BVP for the nonconservative case can be rewritten in vector form as follows:

$$\Phi'_1 = F_0(t,u_1) + \lambda\ln(\lambda)F_1(t,u_1) \tag{19}$$

$$\Phi'_2 = F_0(t,u_2) + \lambda^2 G_1(t,u_1) \tag{20}$$

with

$$F_0(t,x,x') = (x', -\lambda\exp(x))^T \tag{21}$$

$$F_1(t,x,x') = (0, -2\lambda x')^T = (0, -2\lambda\ln(\lambda)f_1(x'))^T \tag{22}$$

$$G_1(t,x,x') = (0, -2\lambda x')^T = (0, -2\lambda^2 g_1(x'))^T \tag{23}$$

Here the term F_0 represents the conservative part of the BVP. In this setting the nonconservative BVP can be seen as a perturbation of the conservative BVP. Theorems on formal expansion, (see, it e.g., [14], Chapter 9), guarantee the solutions will remain close; in the one case $O(\lambda\ln(\lambda)$ close and in the other case $O(\lambda^2)$, on the timescale $O(1)$. This concludes the proof.

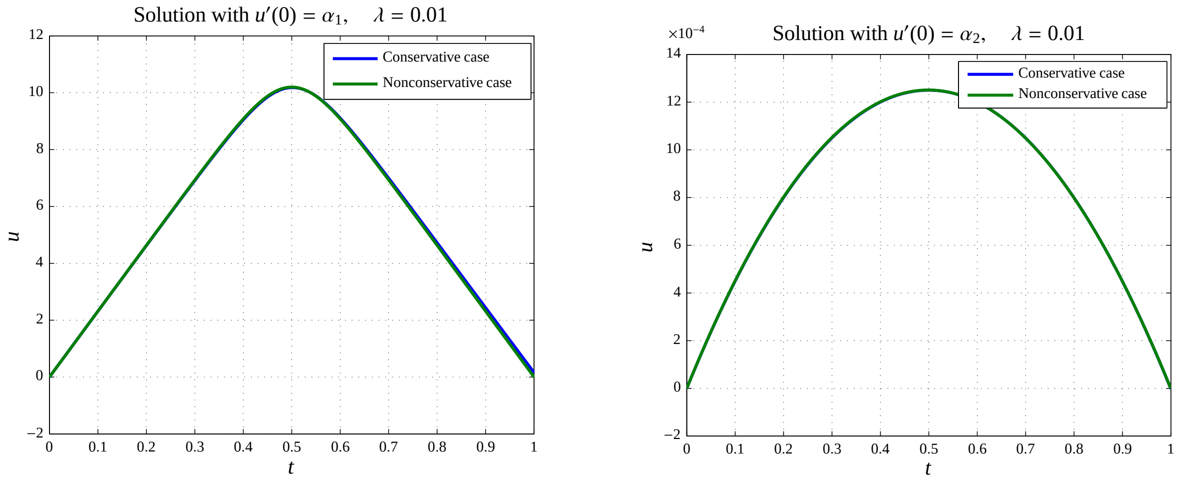


Figure 4: Plot of the solutions of the conservative and nonconservative BVP with initial condition $u_{1,2}(0) = w_{1,2}(0) = 0$, $u_1'(0) \simeq w_1'(0) = \alpha_1$ (left) and $u_2'(0) \simeq w_2'(0) = \alpha_2$ (right).

□

Fig. 4 shows that the solutions, here obtained by shooting, of both the conservative BVP and nonconservative BVP are close. Note that the values of α_1 and α_2 can be derived from Eqs. 14 and 15. We find in the case $\lambda = 0.01$

$$\alpha_1 = 23.16\,, \tag{24}$$
$$\alpha_2 = 0.005\,, \tag{25}$$

which is in agreement with the slopes witnessed in Fig. 4.

3.3. *Numerical bifurcation analysis*

One might try to obtain the second solution by numerical continuation of the "small" solution of the nonconservative BVP with respect the parameter λ, *e.g.*, using the sofware package AUTO [4]. Unfortunately no folds are encountered. This suggests that the two solutions probably never merge. However, unfolding the fold bifurcation encountered for large ε (i.e. small λ) explains what happens.

Consider a linear homotopy between (7) and (8), that we write as

$$\begin{aligned} u'' + \lambda\mu u' + \lambda \exp(u) &= 0\,, \\ u(0) = u(1) &= 0\,, \end{aligned} \tag{26}$$

where the homotopy parameter takes values in $[0, 2]$.

When $\mu = 0$, this BVP has the trivial solution $u \equiv 0$ at $\lambda = 0$ that can be continued w.r.t. the parameter λ, and which exhibits a fold (or "limit point" LP) at $\lambda_{\mathsf{LP}} \approx 3.5138$ (see Fig. 5). This agrees with the results of Section 3.2 and implies that for $\lambda > \lambda_{\mathsf{LP}}$ the Eqn. (26) has no solutions, while for $0 < \lambda < \lambda_{\mathsf{LP}}$ two solutions coexist, one "small" and one "large".

Our aim is to study the evolution of the bifurcation diagram when μ changes between 0 and 2. For $\mu = 2$, at least two different solutions still exist for small λ. It turns out that these solutions belong to two different families (or "branches"), one of which does not pass through the trivial solution at $\lambda = 0$.

The numerical results are presented in Fig. 6 indicate that for all $0 < \mu < 2$ the BVP (26) has two different solutions with

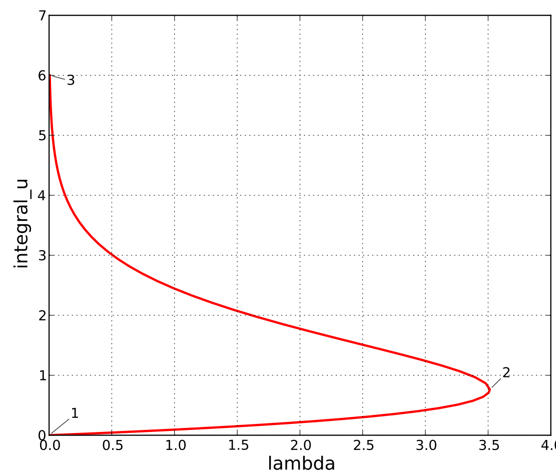


Figure 5: The bifurcation diagram for $\mu = 0$; label 2 indicates a fold.

small λ. However, these solutions either belong to the same branch or to two disconnected branches. In particular, this implies that for $\mu = 2$ the upper solution cannot be obtained by continuation of the trivial solution, since no folds occur. It is also clear that Eqn. (26) with $\mu = 2$ also has two solutions for sufficiently big values of λ. Notice that such values correspond to *small* values of the original parameter ε, when the BVP becomes singularly perturbed.

At $\mu \approx 1.076$ a *branch point* occurs where the solution branches cross. The figures clearly illustrate that this branching is generic [9] and disappears under small variations of parameter μ. Since this branch point can be viewed as a collision of two folds on different branches, we can locate it accurately by continuing a curve of folds in the two parameters (λ, μ), and by locating a (maximum) on this curve with respect to the parameter μ. The results are shown in Fig. 7. The maximum value $\mu_{\mathsf{BP}} \approx 1.07532$ corresponds to the branch point.

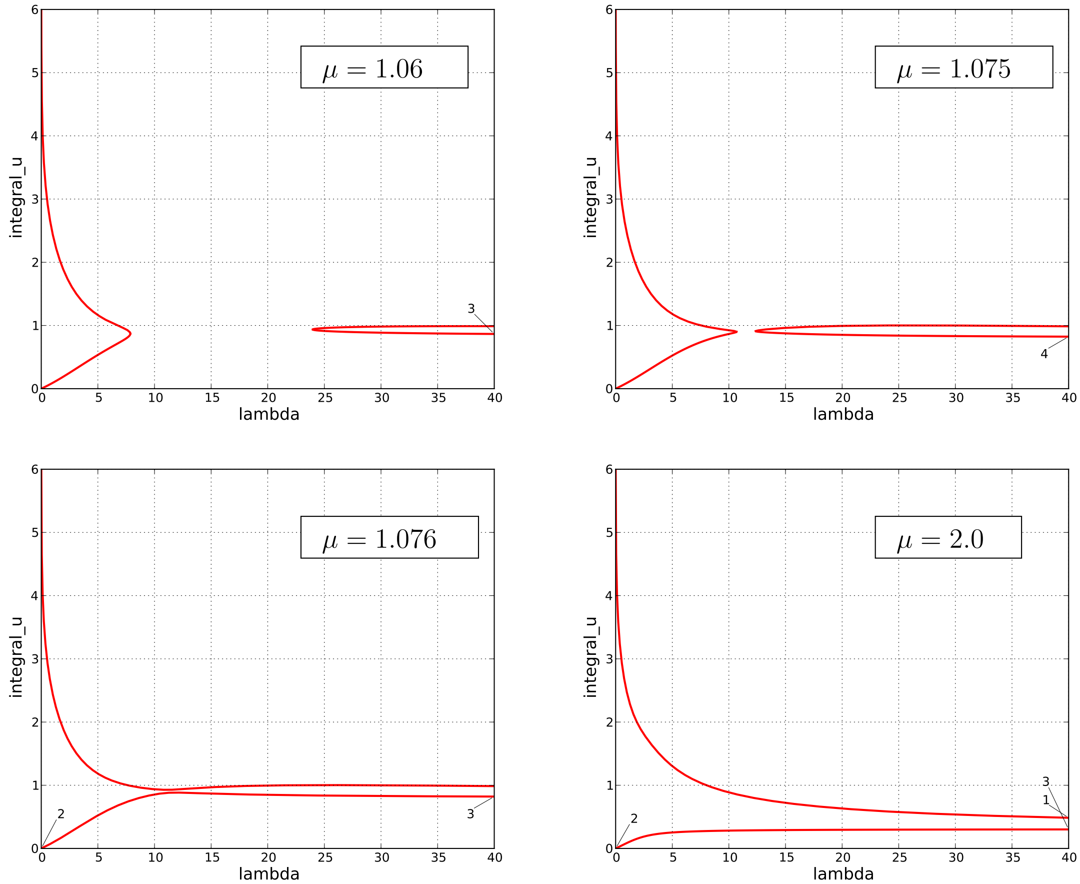
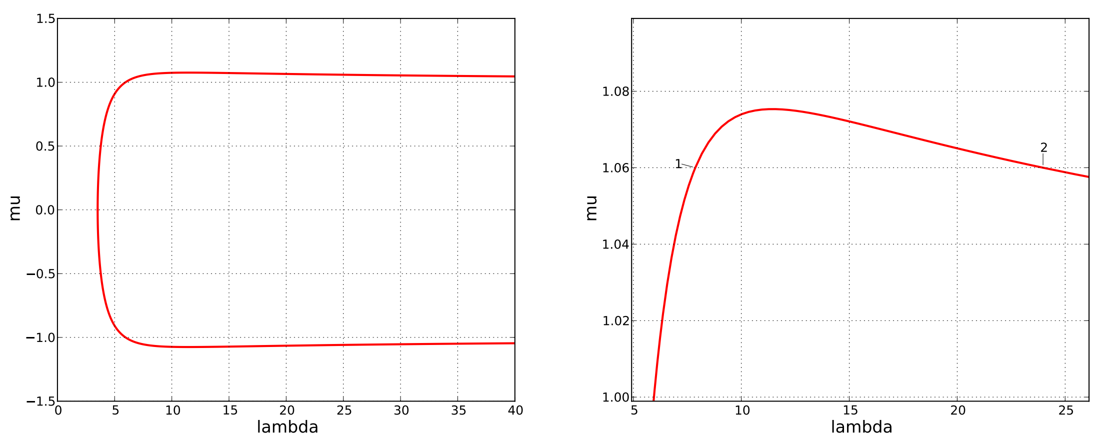


Figure 6: One-parameter bifurcation diagrams for different values of μ.



(a) (b)

Figure 7: (a) Fold curve; (b) Detail of the fold curve: The maximum value $\mu_{\mathrm{BP}} \approx 1.07532$ corresponds to the branch point.

4. Conclusions

We studied a nonlinear boundary value problem by traditional methods. The identification of a slow manifold in the shooting approach provides some information. However, approximation by local expansions and matching is more efficient. For the existence of two solutions, the behaviour of the "large" solution is quite surprising, but can be fully understood by a combination of analytical and numerical methods, which reveal a generic branch point where two folds collide, thereby disconnecting the "large" solution from the "small" one.

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