Stability of Sampled-Data Piecewise Affine Systems: A Time-Delay Approach ⋆

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Abstract

This paper addresses stability analysis of sampled-data piecewise-affine (PWA) systems consisting of a continuous-time plant in feedback connection with a discrete-time emulation of a continuous-time state feedback controller. The sampled-data system is considered as a continuous-time system with a variable time delay. Conditions under which the trajectories of the sampled-data closed-loop system will converge to an attracting invariant set are then presented. It is also shown that when the sampling period converges to zero, these conditions coincide with sufficient conditions for non-fragility of the stabilizing continuous-time PWA state feedback controller.

Key words: Piecewise affine systems; Sampled-data systems; Lyapunov stability.

1 Introduction

State feedback control of PWA systems has received increasing interest over the last few years. Hassibi and Boyd (1998); Johansson (2003); Rantzer and Johansson (2000); Rodrigues and Boyd (2005). These references consider continuous-time processes controlled by continuous-time controllers. However, the implementation in a microprocessor requires emulation of a continuous-time controller as a discrete-time controller. Although linear sampled-data control is a well-studied topic Chen and Francis (1995), controller emulation for systems with possible discontinuities at the switching, such as sampled-data PWA systems, has not had many research contributions. In fact, only recently these systems have started to be addressed in the literature in references such as Imura (2003a,b); Azuma and Imura (2004); Sun and Ge (2002); Sun (2004); Zhai et al. (2004); Rodrigues (2007). See Rodrigues (2007) for a more detailed description of previous work on sampled-data switched systems.

The approach by Imura (2003a,b); Azuma and Imura (2004) was probably the first where the term "sampled-data PWA systems" is used, although the systems described in that work do not posses the typical structure of a continuous-time plant being controlled by a discrete-time controller. The problem addressed in Imura (2003a,b); Azuma and Imura (2004) is one where the controller is continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, in these systems it is assumed that the designer has command over the switching times of the system, which is not always possible. For this class of systems, Azuma and Imura (2004) present a probabilistic analysis of controllability. By contrast, the work in Rodrigues (2007) addresses the classical structure of a sampled-data system whereby the system is continuous-time and the controller is being implemented (emulated) in discrete-time inside a computer. However, the sampling time was considered to be constant in that work. For a general and unified framework for the design of nonlinear controllers using the emulation method, the reader is referred to Laila et al. (2002).

Departing from previous research, this paper addresses stability analysis of sampled-data PWA systems using a time delay approach, whereby the discrete-time PWA controller is seen as a continuous-time PWA controller

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with a delay that varies with time. To the best of our knowledge, the first use of time varying delays to model sampled-data nonlinear systems was given in Teel et al. (1998). The proposed method in Teel et al. (1998) exploited a Razumikhin type theorem for input-to-state stability of functional differential equations that was proved in Teel (1998) to analyze sampled-data systems. In Razumikhin type theorems, a Lyapunov function is used for stability analysis of time delay systems. The main problem of this approach is to obtain an upper bound of the time derivative of the Lyapunov function which does not depend on past states (Kharitonov, 1999). Avoiding this problem by using a Lyapunov-Krasovskii functional usually leads to less conservative results (Jianga and Han, 2008). In this paper, using a Lyapunov-Krasovskii functional, linear matrix inequalities (LMIs) are derived to describe sufficient conditions for convergence of the sampled-data PWA system trajectories to an attracting invariant set. One of the advantages of the proposed method is that it can be applied to sampled-data PWA systems with variable sampling time as opposed to Rodrigues (2007) that deals with a constant sampling time. Furthermore, a very important property of the conditions derived in this paper is that when the sampling time converges to zero, they reduce to LMIs for non-fragility of the continuous-time PWA controller. Therefore, for a correct implementation in discrete-time, the result derived in this paper requires that the controller be robust to variations in its parameters. This is in itself an interesting result.

The paper starts by analyzing the stability of a sampled-data system when a PWA continuous-time controller is emulated in discrete-time. A numerical example is then included to show the performance of the proposed method. Finally, the paper closes by stating the conclusions.

2 Stability of Sampled-Data PWA Systems

Consider the following continuous time PWA system

$$\dot{x} = A_i x + a_i + Bu, \quad \text{for } x \in \mathcal{R}_i \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $A_i \in \mathbb{R}^{n\times n}$, $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, M$ and $B \in \mathbb{R}^{n\times m}$. The region $\mathcal{R}_i$ defined as

$$\mathcal{R}_i = \{x | E_i x + e_i \succ 0\}, \tag{2}$$

where $E_i \in \mathbb{R}^{p_i \times n}$ and $e_i \in \mathbb{R}^{p_i}$ and $\succ$ represents an elementwise inequality. Each polytopic region $\mathcal{R}_i$ can be outer approximated by a (possibly degenerate) quadratic curve as

$$\mathcal{R}_i \subseteq \varepsilon_i = \{x | \bar{x}^T \bar{E}_i^T \bar{\Lambda}_i \bar{E}_i \bar{x} > 0\} \tag{3}$$

where $\bar{\Lambda}_i \in \mathbb{R}^{(p_i+1)\times(p_i+1)}$ is a matrix with nonnegative entries and

$$\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \ \bar{E}_i = \begin{bmatrix} E_i & e_i \\ 0 & 1 \end{bmatrix} \tag{4}$$

Considering a stabilizing PWA controller of the form

$$u(t) = K_i x(t) + k_i, \ x(t) \in \mathcal{R}_i \tag{5}$$

where $K_i \in \mathbb{R}^{m\times n}$ and $k_i \in \mathbb{R}^m$, the closed-loop system is assumed to be asymptotically stable. It is also assumed that the vector field of the open-loop PWA system (1) with $u(t) = 0$ is continuous across the boundaries of two or more regions and $a_i = 0$ for i such that $0 \in \mathcal{R}_i$.

If the PWA controller (5) is implemented as a digital controller and is connected to the PWA system (1) through a sample-and-hold, there is no guarantee that $x(t)$ and its sample at t_k would be in the same region. Therefore, assuming $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$, the closed-loop system can be described by

$$\dot{x}(t) = A_i x(t) + a_i + B(K_j x(t_k) + k_j), \tag{6}$$

where t_k for $k \in \mathbb{N}$ is the sampling time and $t_k \le t < t_{k+1}$. The closed-loop system (6) can be rewritten as

$$\dot{x}(t) = A_i x(t) + a_i + B(K_i x(t_k) + k_i) + Bw, \tag{7}$$

for $x(t) \in \mathcal{R}_i$ and $x(t_k) \in \mathcal{R}_j$ where

$$w(t) = (K_j - K_i)x(t_k) + (k_j - k_i), \ x(t) \in \mathcal{R}_i, \ x(t_k) \in \mathcal{R}_j \tag{8}$$

The input $w(t)$ is a result of the fact that $x(t)$ and $x(t_k)$ are not necessarily in the same region.

Following Naghshtabrizi et al. (2006), the time elapsed since the last sampling time will be denoted by

$$\rho(t) := t - t_k, \ t_k \le t < t_{k+1} \tag{9}$$

and τ_{max} is defined as the maximum interval between sampling times.

$$t_{k+1} - t_k \le \tau_{max}, \forall k \in \mathbb{N} \tag{10}$$

Consider a Lyapunov-Krasovskii functional of the form

$$V(x_s, \rho) := V_1(x) + V_2(x_s) + V_3(x_s, \rho) \tag{11}$$

where

$$x_s(t) := \begin{bmatrix} x(t) \\ x(t_k) \end{bmatrix}, \ t_k \le t < t_{k+1}$$

$$V_1(x) := x^{\mathrm{T}} P x$$
$$V_2(x_s) := \int_{-\tau_{max}}^{0} \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds dr$$
$$V_3(x_s, \rho) := (\tau_{max} - \rho)(x(t) - x(t_k))^{\mathrm{T}} X (x(t) - x(t_k))$$

and P, R and X are positive definite matrices. It is shown in Samadi (2008) that $V(x_s, \rho)$ satisfies

$$\lambda_{\min}(P)\|x\|^2 \leq V(x_s, \rho) \leq \sigma_a \|x_s\|^2 + \sigma_b \tag{12}$$

where

$$\sigma_a = \lambda_{\max}(P) + 2(\tau_{max} - \rho)\lambda_{\max}(X) + \frac{\tau_{max}^2}{2}\lambda_{\max}(\bar{R}),$$
$$\sigma_b = \frac{\tau_{max}^2}{2}\lambda_{\max}(\bar{R}),$$

$\lambda_{\min}(.)$ and $\lambda_{\max}(.)$ mean the minimum and maximum eigenvalues of a matrix, respectively, and

$$\bar{R} = \arg\max_{i,j} \lambda_{\max}(\bar{R}_{ij}) \tag{13}$$

$$\bar{R}_{ij} = \begin{bmatrix} A_i^{\mathrm{T}} \\ K_j^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_j^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} R \begin{bmatrix} A_i & BK_j & a_i + Bk_j \end{bmatrix} \tag{14}$$

The main result of this paper is now presented.

Theorem 1 *For the sampled-data PWA system (7), assume there exist symmetric positive definite matrices P, R, X and matrices N_i, $i = 1, \ldots, M$ such that*

- *for all $i \in \mathcal{I}(0)$,*

$$\Omega_i + \tau_{max} M_{1i} + \tau_{max} M_{2i} < 0 \tag{15}$$

$$\begin{bmatrix} \Omega_i + \tau_{max} M_{1i} & \tau_{max} \begin{bmatrix} N_i \\ 0 \end{bmatrix} \\ \tau_{max} \begin{bmatrix} N_i^T & 0 \end{bmatrix} & -\tau_{max} R \end{bmatrix} < 0 \tag{16}$$

- *for all $i \notin \mathcal{I}(0)$, $\bar{\Lambda}_i \succ 0$,*

$$\overline{\Omega}_i + \tau_{max} \overline{M}_{1i} + \tau_{max} \overline{M}_{2i} < 0 \tag{17}$$

$$\begin{bmatrix} \overline{\Omega}_i + \tau_{max} \overline{M}_{1i} & \tau_{max} \begin{bmatrix} N_i \\ 0 \\ 0 \end{bmatrix} \\ \tau_{max} \begin{bmatrix} N_i^T & 0 & 0 \end{bmatrix} & -\tau_{max} R \end{bmatrix} < 0 \tag{18}$$

where

$$\Omega_i = \begin{bmatrix} \Psi_i & \begin{bmatrix} P \\ 0 \end{bmatrix} B \\ B^T \begin{bmatrix} P & 0 \end{bmatrix} & -\gamma I \end{bmatrix},$$

$$\Psi_i = \begin{bmatrix} P \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \end{bmatrix} \begin{bmatrix} P & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} X \begin{bmatrix} I & -I \end{bmatrix} - N_i \begin{bmatrix} I - I \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} N_i^T + \eta I_{2n \times 2n},$$

$$M_{1i} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \end{bmatrix} R \begin{bmatrix} A_i & BK_i & B \end{bmatrix},$$

$$M_{2i} = \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & B \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ B^T \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix},$$

$$\overline{\Omega}_i = \begin{bmatrix} \overline{\Psi}_i & \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} B \\ B^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} & -\gamma I \end{bmatrix},$$

$$\overline{\Psi}_i = \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} A_i & BK_i & Bk_i + a_i \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \end{bmatrix} \begin{bmatrix} P & 0 & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix}$$
$$- \begin{bmatrix} N_i \\ 0 \end{bmatrix} \begin{bmatrix} I - I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} N_i^T & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \eta I_{2n \times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} E_i^T & 0 \\ 0 & 0 \\ e_i^T & 1 \end{bmatrix} \bar{\Lambda}_i \begin{bmatrix} E_i & 0 & e_i \\ 0 & 0 & 1 \end{bmatrix},$$

$$\overline{M_{1i}} = \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \\ B^T \end{bmatrix} R \begin{bmatrix} A_i & BK_i & Bk_i + a_i & B \end{bmatrix},$$

$$\overline{M_{2i}} = \begin{bmatrix} I \\ -I \\ 0 \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & Bk_i + a_i & B \end{bmatrix} + \begin{bmatrix} A_i^T \\ K_i^T B^T \\ k_i^T B^T + a_i^T \\ B^T \end{bmatrix} X \begin{bmatrix} I & -I & 0 & 0 \end{bmatrix}$$

Let there be nonnegative constants Δ_K and Δ_k such that

$$\|w\| \leq \Delta_K \|x(t_k)\| + \Delta_k \tag{19}$$

Define

$$\mu_\theta = \frac{\sqrt{\gamma}\Delta_k}{\sqrt{\theta\eta} - \sqrt{\gamma}\Delta_K} \tag{20}$$

and the region

$$\Phi_\theta = \{x_s | \; \|x_s\| \leq \mu_\theta\} \tag{21}$$

for some positive constant η and $0 < \theta < 1$ that verify

$$\Delta_K < \sqrt{\frac{\theta\eta}{\gamma}} \tag{22}$$

Then, all the trajectories of the system (7) in $\mathcal{X}$ converge to the following invariant set

$$\Omega = \{x_s | \; V(x_s, \rho) \leq \sigma_a \mu_\theta^2 + \sigma_b\} \tag{23}$$

□

PROOF. Note that the Lyapunov-Krasovskii functional $V(x_s, \rho)$ is positive definite and $\dot{x}$ in (6) is continuous for $t_k < t < t_{k+1}$. The proof is now divided into two parts.

(1) First, it is shown that the inequalities (15), (16), (17) and (18) are sufficient conditions for the following inequality to hold

$$\dot{V}(x_s, \rho) \leq -\eta x_s^{\mathrm{T}} x_s + \gamma w^{\mathrm{T}} w \tag{24}$$

for $t_k < t < t_{k+1}$.

Since $V_1(x) = x^{\mathrm{T}} P x$, one has

$$\dot{V}_1(x) = \dot{x}^{\mathrm{T}} P x + x^{\mathrm{T}} P \dot{x} \tag{25}$$

$V_2(x_s)$ can be written in the following form

$$V_2(x_s) = \int_{-\tau_{max}}^{0} g(t, r) dr \tag{26}$$

where

$$g(t, r) = \int_{t+r}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{27}$$

Thus,

$$\dot{V}_2(x_s) = \int_{-\tau_{max}}^{0} \frac{\partial}{\partial t} g(t, r) dr \tag{28}$$

The expression

$$\frac{\partial}{\partial t} g(t, r) = \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \dot{x}^{\mathrm{T}}(t+r) R \dot{x}(t+r) \tag{29}$$

then yields

$$\dot{V}_2(x_s) = \tau_{max} \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \int_{t-\tau_{max}}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{30}$$

From (10) one has $\rho \leq \tau_{max}$ and considering the fact that R is positive definite, this leads to

$$\dot{V}_2(x_s) \leq \tau_{max} \dot{x}^{\mathrm{T}}(t) R \dot{x}(t) - \int_{t-\rho}^{t} \dot{x}^{\mathrm{T}}(s) R \dot{x}(s) ds \tag{31}$$

Since R is positive definite, for any matrix $N_i \in \mathbb{R}^{n \times 2n}$ one has

$$\begin{bmatrix} \dot{x}^{\mathrm{T}}(s) & x_s^{\mathrm{T}}(t) N_i \end{bmatrix} \begin{bmatrix} R & -I \\ -I & R^{-1} \end{bmatrix} \begin{bmatrix} \dot{x}(s) \\ N_i^{\mathrm{T}} x_s(t) \end{bmatrix} \geq 0 \tag{32}$$

and therefore

$$\begin{aligned} -\dot{x}(s)^{\mathrm{T}} R \dot{x}(s) \leq & x_s^{\mathrm{T}}(t) N_i R^{-1} N_i^{\mathrm{T}} x_s(t) \\ & - 2 x_s^{\mathrm{T}}(t) N_i \dot{x}(s) \end{aligned} \tag{33}$$

Integrating both sides from $t - \rho$ to t and using (9) yields,

$$\begin{aligned} -\int_{t-\rho}^{t} \dot{x}(s)^{\mathrm{T}} R \dot{x}(s) ds \leq & \rho x_s^{\mathrm{T}}(t) N_i R^{-1} N_i^{\mathrm{T}} x_s(t) \\ & - 2 x_s^{\mathrm{T}}(t) N_i \begin{bmatrix} I & -I \end{bmatrix} x_s(t) \end{aligned} \tag{34}$$

It follows from (31) and (34) that

$$\dot{V}_2(x_s) \le \tau_{max}\dot{x}^{\mathrm{T}}R\dot{x} + \rho x_s^{\mathrm{T}}N_iR^{-1}N_i^{\mathrm{T}}x_s - 2x_s^{\mathrm{T}}N_i\begin{bmatrix} I & -I \end{bmatrix}x_s \tag{35}$$

For $V_3(x_s,\rho)$, since $\dot{\rho}=1$ for $t_k<t<t_{k+1}$, one can write

$$\dot{V}_3(x_s,\rho) = -\left(x(t)-x(t_k)\right)^{\mathrm{T}}X(x(t)-x(t_k)) + 2(\tau_{max}-\rho)(x(t)-x(t_k))^{\mathrm{T}}X\dot{x}(t) \tag{36}$$

From (25), (35) and (36), it follows that a sufficient condition for (24) is the following inequality

$$\begin{aligned}&\dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} + \tau_{max}\dot{x}^{\mathrm{T}}R\dot{x} + \rho x_s^{\mathrm{T}}N_iR^{-1}N_i^{\mathrm{T}}x_s\\ &-2x_s^{\mathrm{T}}N_i\begin{bmatrix} I & -I \end{bmatrix}x_s\\ &-x_s^{\mathrm{T}}\begin{bmatrix} I\\ -I \end{bmatrix}X\begin{bmatrix} I & -I \end{bmatrix}x_s\\ &+2(\tau_{max}-\rho)x_s^{\mathrm{T}}\begin{bmatrix} I\\ -I \end{bmatrix}X\dot{x} + \eta x_s^{\mathrm{T}}x_s - \gamma w^{\mathrm{T}}w \le 0\end{aligned} \tag{37}$$

For $i \in \mathcal{I}(0)$, one has

$$\dot{x} = \begin{bmatrix} A_i & BK_i \end{bmatrix}x_s + Bw, \tag{38}$$

for $x(t)\in\mathcal{R}_i$ and $x(t_k)\in\mathcal{R}_j$. Replacing $\dot{x}$ from (38) into (37) yields

$$\begin{aligned}&x_s^{\mathrm{T}}\Bigg(\begin{bmatrix} P\\ 0 \end{bmatrix}\begin{bmatrix} A_i & BK_i \end{bmatrix} + \begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} P & 0 \end{bmatrix}\\ &+\tau_{max}\begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}R\begin{bmatrix} A_i & BK_i \end{bmatrix} + \rho N_iR^{-1}N_i^{\mathrm{T}}\\ &-N_i\begin{bmatrix} I & -I \end{bmatrix} - \begin{bmatrix} I\\ -I \end{bmatrix}N_i^{\mathrm{T}}\\ &-\begin{bmatrix} I\\ -I \end{bmatrix}X\begin{bmatrix} I & -I \end{bmatrix}\\ &+(\tau_{max}-\rho)\begin{bmatrix} I\\ -I \end{bmatrix}X\begin{bmatrix} A_i & BK_i \end{bmatrix}\\ &+(\tau_{max}-\rho)\begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}X\begin{bmatrix} I & -I \end{bmatrix} + \eta I\Bigg)x_s\\ &+x_s^{\mathrm{T}}\begin{bmatrix} P\\ 0 \end{bmatrix}Bw + w^{\mathrm{T}}B^{\mathrm{T}}\begin{bmatrix} P & 0 \end{bmatrix}x_s\\ &+\tau_{max}x_s^{\mathrm{T}}\begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}RBw\\ &+\tau_{max}w^{\mathrm{T}}B^{\mathrm{T}}R\begin{bmatrix} A_i & BK_i \end{bmatrix}x_s\\ &+(\tau_{max}-\rho)x_s^{\mathrm{T}}\begin{bmatrix} I\\ -I \end{bmatrix}XBw\\ &+(\tau_{max}-\rho)w^{\mathrm{T}}B^{\mathrm{T}}X\begin{bmatrix} I & -I \end{bmatrix}x_s\\ &+\tau_{max}w^{\mathrm{T}}B^{\mathrm{T}}RBw - \gamma w^{\mathrm{T}}w < 0\end{aligned} \tag{39}$$

Since (39) is affine in ρ, if it holds for $\rho=0$ and $\rho=\tau_{max}$, then it is satisfied for any $\rho\in[0,\tau_{max}]$. For $\rho=0$, the inequality (39) can be written as (15). Using Schur complement for $\rho=\tau_{max}$, the inequality (39) can be converted to (16).

For $i \notin \mathcal{I}(0)$, one has

$$\dot{x} = \begin{bmatrix} A_i & BK_i & a_i+Bk_i \end{bmatrix}\bar{x}_s + Bw,\ x\in\mathcal{R}_i \tag{40}$$

where

$$\bar{x}_s = \begin{bmatrix} x_s\\ 1 \end{bmatrix} \tag{41}$$

It follows from (3) that

$$\begin{bmatrix} x^{\mathrm{T}} & 1 \end{bmatrix}\begin{bmatrix} E_i^{\mathrm{T}} & 0\\ e_i^{\mathrm{T}} & 1 \end{bmatrix}\bar{\Lambda}_i\begin{bmatrix} E_i & e_i\\ 0 & 1 \end{bmatrix}\begin{bmatrix} x\\ 1 \end{bmatrix} > 0,\ x\in\mathcal{R}_i \tag{42}$$

where $\bar{\Lambda}_i \succ 0$. Using (40) and (42), a sufficient condition for (39) when $x\in\mathcal{R}_i$ with $i\notin\mathcal{I}(0)$ can be written as

$$\begin{aligned}&\bar{x}_s^{\mathrm{T}}\Bigg(\begin{bmatrix} P\\ 0\\ 0 \end{bmatrix}\begin{bmatrix} A_i & BK_i & a_i+Bk_i \end{bmatrix}\\ &+\begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}}\\ a_i^{\mathrm{T}}+k_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} P & 0 & 0 \end{bmatrix}\\ &+\tau_{max}\begin{bmatrix} A_i^{\mathrm{T}}\\ K_i^{\mathrm{T}}B^{\mathrm{T}}\\ a_i^{\mathrm{T}}+k_i^{\mathrm{T}}B^{\mathrm{T}} \end{bmatrix}R\begin{bmatrix} A_i & BK_i & a_i+Bk_i \end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&+\rho \begin{bmatrix} N_i \\ 0 \end{bmatrix} R^{-1} \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} \\
&- \begin{bmatrix} N_i \\ 0 \end{bmatrix} \begin{bmatrix} I & -I & 0 \end{bmatrix} - \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} \begin{bmatrix} N_i^{\mathrm{T}} & 0 \end{bmatrix} \\
&- \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \\
&+ (\tau_{max} - \rho) \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} X \begin{bmatrix} A_i & BK_i & a_i + Bk_i \end{bmatrix} \\
&+ (\tau_{max} - \rho) \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ a_i^{\mathrm{T}} + k_i^{\mathrm{T}} B^{\mathrm{T}} \end{bmatrix} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \\
&\left. +\eta \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \bar{x}_s + \bar{x}_s^{\mathrm{T}} \begin{bmatrix} P \\ 0 \\ 0 \end{bmatrix} Bw \\
&+ w^{\mathrm{T}} B^{\mathrm{T}} \begin{bmatrix} P & 0 & 0 \end{bmatrix} \bar{x}_s + \tau_{max} \bar{x}_s^{\mathrm{T}} \begin{bmatrix} A_i^{\mathrm{T}} \\ K_i^{\mathrm{T}} B^{\mathrm{T}} \\ k_i^{\mathrm{T}} B^{\mathrm{T}} + a_i^{\mathrm{T}} \end{bmatrix} RBw \\
&+ \tau_{max} w^{\mathrm{T}} B^{\mathrm{T}} R \begin{bmatrix} A_i & BK_i & Bk_i + a_i \end{bmatrix} \bar{x}_s \\
&+ (\tau_{max} - \rho) \bar{x}_s^{\mathrm{T}} \begin{bmatrix} I \\ -I \\ 0 \end{bmatrix} XBw \\
&+ (\tau_{max} - \rho) w^{\mathrm{T}} B^{\mathrm{T}} X \begin{bmatrix} I & -I & 0 \end{bmatrix} \bar{x}_s \\
&+ \tau_{max} w^{\mathrm{T}} B^{\mathrm{T}} RBw - \gamma w^{\mathrm{T}} w \\
&+ \bar{x}_s^{\mathrm{T}} \begin{bmatrix} E_i^{\mathrm{T}} & 0 \\ 0 & 0 \\ e_i^{\mathrm{T}} & 1 \end{bmatrix} \bar{\Lambda}_i \begin{bmatrix} E_i & 0 & e_i \\ 0 & 0 & 1 \end{bmatrix} \bar{x}_s < 0
\end{aligned}
\tag{43}
$$

Inequality (17) is equivalent to (43) for $\rho = 0$ and using Schur complement, inequality (18) is equivalent to (43) for $\rho = \tau_{max}$. Since (43) is affine in ρ, inequalities (17) and (18) imply that (43) is satisfied for any $\rho \in [0, \tau_{max}]$.

In conclusion, (24) is satisfied for $t_k < t < t_{k+1}$, $k = 0, 1, 2, \ldots$ and any $x \in \mathcal{R}_i$, $i = 1, 2, \ldots, M$.

(2) In the second part of the proof, it will be shown that for $0 < \theta < 1$, Ω is an attracting invariant set. For any $x_s \notin \Omega$, one has

$$V(x_s, \rho) > \sigma_a \mu_\theta^2 + \sigma_b \tag{44}$$

It follows from (12) that $\|x_s\| > \mu_\theta$ and therefore (20) and (22) lead to

$$\sqrt{\theta\eta}\|x_s\| > \sqrt{\gamma}(\Delta_K \|x_s\| + \Delta_k) \tag{45}$$

It now follows from (19) and (45) that

$$\theta\eta x_s^{\mathrm{T}} x_s > \gamma w^{\mathrm{T}} w \tag{46}$$

Since the inequality (24) can be written as

$$\dot{V}(x_s, \rho) \le -(1-\theta)\eta x_s^{\mathrm{T}} x_s - \theta\eta x_s^{\mathrm{T}} x_s + \gamma w^{\mathrm{T}} w \tag{47}$$

for $0 < \theta < 1$, it follows from (46) that

$$\dot{V}(x_s, \rho) < -(1-\theta)\eta x_s^{\mathrm{T}} x_s \tag{48}$$

and from $\|x_s\| > \mu_\theta$, one has

$$\dot{V}(x_s, \rho) < -(1-\theta)\eta\mu_\theta^2, \text{ for } t_k < t < t_{k+1} \tag{49}$$

Therefore $V(x_s, \rho)$ decreases between the sampling times for $\|x_s\| > \mu_\theta$. At the sampling times, $V(x_s, \rho)$ does not increase because $V_1(x_s)$, $V_2(x_s)$ are continuous and $V_3(x_s, \rho)$ is non-negative right before each sampling time and it becomes zero right after the sampling time. Note that no fast switching can occur because for $t_k < t < t_{k+1}$, the control input is constant and $\dot{x}$ is continuous.

Thus, there is a finite time t^θ such that $x_s(t^\theta) \in \Phi_\theta$ and therefore from (20), (21) and (22), one has $V(x_s(t^\theta), \rho) \le \sigma_a \mu_\theta^2 + \sigma_b$, which means $x_s(t^\theta) \in \Omega$. Therefore, Ω is an attracting invariant set.

Remark 1 *The upper bound for $\|w\|$ defined in (19) can be obtained as*

$$
\begin{aligned}
\Delta_K &= \max_{i,j=1,\ldots,M} \|K_i - K_j\| \\
\Delta_k &= \max_{i,j=1,\ldots,M} \|k_i - k_j\|
\end{aligned}
\tag{50}
$$

Note that for the case where $K_i = K_j$, $\Delta_K = 0$ and (22) is automatically satisfied. If, furthermore, $k_i = k_j$, then $\Delta_k = 0$,which implies that $w = 0$ and $\mu_\theta = 0$.

Remark 2 *For $\tau_{max} \to 0$ and*

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \; N_i = \begin{bmatrix} -PBK_i + I \\ -I \end{bmatrix}, \; X = (\beta - 2)I \tag{51}$$

where $\beta > \max(\eta, 2)$ and

$$\eta_c = \eta + \frac{\eta\beta}{\beta - \eta}, \tag{52}$$

the conditions (15), (16) are reduced to the inequality (53) for all $i \in \mathcal{I}(0)$

$$\begin{bmatrix} (A_i + BK_i)^T P + P(A_i + BK_i) + \eta_c I & PB \\ B^T P & -\gamma I \end{bmatrix} < 0 \tag{53}$$

and the conditions (17) and (18) are reduced to the inequality (54) for $i \notin \mathcal{I}(0)$

$$\begin{bmatrix} \begin{pmatrix} (\bar{A}_i + \bar{B}\bar{K}_i)^T \bar{P} + \bar{P}(\bar{A}_i + \bar{B}\bar{K}_i) \\ +\bar{E}_i^T \bar{\Lambda}_i \bar{E}_i + \eta_c \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \end{pmatrix} & \bar{P}\bar{B} \\ \bar{B}^T \bar{P} & -\gamma I \end{bmatrix} < 0 \tag{54}$$

Inequalities (53) and (54) are sufficient conditions for input to state stability of the continuous-time PWA system (1). More specifically, for $V(x) = x^T P x$ one has

$$\dot{V}(x) < -\eta_c x^T x + \gamma w^T w \tag{55}$$

This result establishes that the continuous-time PWA controller should satisfy a very important property: non-fragility. In other words, if there exists an error w in the implementation of the continuous-time PWA controller (5) as shown in the following

$$u(t) = K_i x(t) + k_i + w(t) \tag{56}$$

and the norm of w is bounded, the norm of the state vector $x(t)$ remains bounded.

3 Numerical Example

Example 1 *A state space model was built for an experimental setup of a two degrees of freedom helicopter in Endreß (2008). In this example, a simplified version of the pitch model of the helicopter (Fig. 1) is considered. This model is described by the following equations*

$$\begin{aligned} \dot{x}_1 =& x_2 \\ \dot{x}_2 =& \frac{1}{I_{yy}}(-m_{heli} l_{cgx} g \cos(x_1) - m_{heli} l_{cgz} g \sin(x_1) \\ & - F_{kM}\,\mathrm{sgn}(x_2) - F_{vM} x_2 + u) \end{aligned} \tag{57}$$

where x_1 and x_2 represent pitch angle and pitch rate, respectively. The values of the parameters are shown in Table 1.

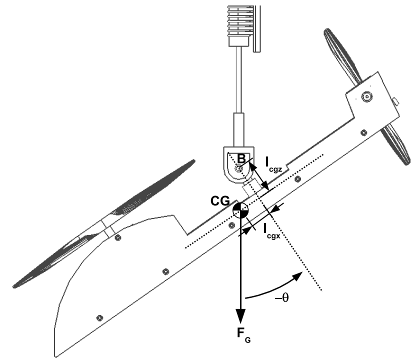

Fig. 1. Pitch model of the helicopter from Endreß (2008)

Table 1
Parameters of the helicopter model

Parameter	Value	Unit
I_{yy}	0.0283	kgm^2
m_{heli}	0.9941	kg
l_{cgx}	0.0134	m
l_{cgz}	0.0289	m
F_{kM}	0.0003	Nm
F_{vM}	0.0041	Nm/rad/s
g	9.81	m/s^2

First, the PWA approximation $\hat{f}(x_1)$ of

$$f(x_1) = -m_{heli} l_{cgx} g \cos(x_1) - m_{heli} l_{cgz} g \sin(x_1) \tag{58}$$

is computed based on a uniform grid for x_1. The resulting approximation is shown in Fig. 2. A PWA model is obtained by replacing $f(x_1)$ by $\hat{f}(x_1)$ in (57). The PWA model is described by the following equations

$$\begin{aligned}
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 5.3058 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 22.2968 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_1 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -8.1786 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_2 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -10.5751 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_3 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 1.9210 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -12.4780 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_4 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 10.7980 & -0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -29.2108 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_5 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ 5.3058 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ 22.2968 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_6 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -8.1786 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_7 \\
\dot{x} &= \begin{bmatrix} 0 & 1 \\ -10.5751 & +0.1447 \end{bmatrix} x + \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix} + \begin{bmatrix} 0 \\ 35.3012 \end{bmatrix} u \\ & \textit{for } x \in \mathcal{R}_8
\end{aligned}$$

$$\dot{x} = \left[\begin{smallmatrix} 0 & 1 \\ 1.9210 & +0.1447 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0 \\ -12.4780 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 35.3012 \end{smallmatrix}\right] u$$
for $x \in \mathcal{R}_9$

$$\dot{x} = \left[\begin{smallmatrix} 0 & 1 \\ 10.7980 & +0.1447 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0 \\ -29.2108 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 \\ 35.3012 \end{smallmatrix}\right] u$$
for $x \in \mathcal{R}_{10}$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ *and*

$$\begin{aligned}
\mathcal{R}_1 =& \{x| -\pi < x_1 < -\frac{3\pi}{5}, x_2 > 0\} \\
\mathcal{R}_2 =& \{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}, x_2 > 0\} \\
\mathcal{R}_3 =& \{x| -\frac{\pi}{5} < x_1 < \frac{\pi}{5}, x_2 > 0\} \\
\mathcal{R}_4 =& \{x| \frac{\pi}{5} < x_1 < \frac{3\pi}{5}, x_2 > 0\} \\
\mathcal{R}_5 =& \{x| \frac{3\pi}{5} < x_1 < \pi, x_2 > 0\} \\
\mathcal{R}_6 =& \{x| -\pi < x_1 < -\frac{3\pi}{5}, x_2 < 0\} \\
\mathcal{R}_7 =& \{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}, x_2 < 0\} \\
\mathcal{R}_8 =& \{x| -\frac{\pi}{5} < x_1 < \frac{\pi}{5}, x_2 < 0\} \\
\mathcal{R}_9 =& \{x| \frac{\pi}{5} < x_1 < \frac{3\pi}{5}, x_2 < 0\} \\
\mathcal{R}_{10} =& \{x| \frac{3\pi}{5} < x_1 < \pi, x_2 < 0\}
\end{aligned} \tag{59}$$

The following PWA controller is then designed to stabilize the origin ($x_1 = x_2 = 0$) for the PWA system (59) using the backstepping method in Samadi (2008).

$$\begin{aligned}
u =& -0.2919x_1 - 0.1092x_2 - 0.6313, \quad \text{for } x \in \mathcal{R}_1 \\
u =& 0.0900x_1 - 0.1092x_2 + 0.0887, \quad \text{for } x \in \mathcal{R}_2 \\
u =& 0.1579x_1 - 0.1092x_2 + 0.1314, \quad \text{for } x \in \mathcal{R}_3 \\
u =& -0.1961x_1 - 0.1092x_2 + 0.3538, \quad \text{for } x \in \mathcal{R}_4 \\
u =& -0.4475x_1 - 0.1092x_2 + 0.8278, \quad \text{for } x \in \mathcal{R}_5 \\
u =& -0.2919x_1 - 0.1092x_2 - 0.6319, \quad \text{for } x \in \mathcal{R}_6 \\
u =& 0.0900x_1 - 0.1092x_2 + 0.0881, \quad \text{for } x \in \mathcal{R}_7 \\
u =& 0.1579x_1 - 0.1092x_2 + 0.1308, \quad \text{for } x \in \mathcal{R}_8 \\
u =& -0.1961x_1 - 0.1092x_2 + 0.3532, \quad \text{for } x \in \mathcal{R}_9 \\
u =& -0.4475x_1 - 0.1092x_2 + 0.8272, \quad \text{for } x \in \mathcal{R}_{10}
\end{aligned}$$

Using Theorem 1, a sampling time for discrete-time implementation of the proposed PWA controller can be computed so that the closed-loop sampled-data system converges to a bounded invariant set. In this example, we consider η and γ as optimization parameters. However, to provide a larger upper bound on Δ_K, we require that $\eta > \gamma$ and $\gamma > 1$. Solving an optimization problem to maximize τ_{max} subject to the constraints of Theorem 1 and $\eta > \gamma > 1$ yields

$$\tau^{\star}_{max} = 0.1465, \ \eta = 4.2403, \ \gamma = 4.2403 \tag{60}$$

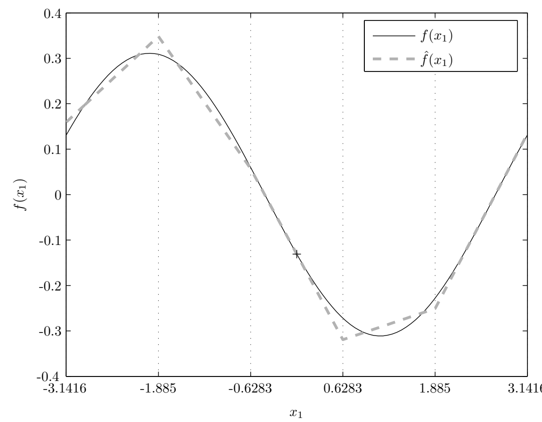

Fig. 2. PWA approximation for helicopter model

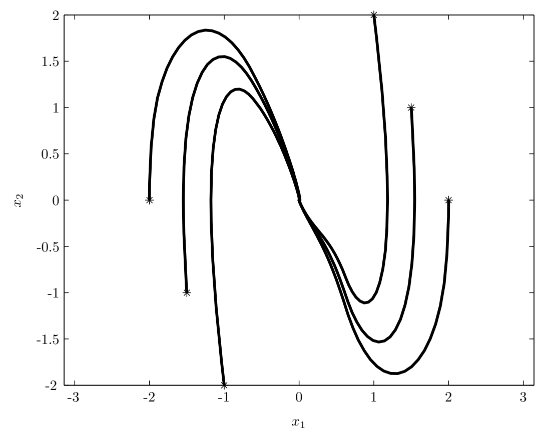

Fig. 3. Trajectories of the nonlinear Helicopter model - continuous-time PWA controller

$$P = \begin{bmatrix} 30.4829 & 2.4706 \\ 2.4706 & 4.4771 \end{bmatrix}, \ R = \begin{bmatrix} 44.9622 & 9.0745 \\ 9.0745 & 3.1994 \end{bmatrix}, \tag{61}$$

$$X = \begin{bmatrix} 499.9799 & 11.6429 \\ 11.6429 & 24.1825 \end{bmatrix} \tag{62}$$

Fig. 3 shows the trajectories of the nonlinear model (57) in feedback connection with the continuous-time PWA controller. The trajectories of a sampled-data PWA controller with a sampling time of 0.1465 *s are shown in Fig. 4, where it is clear that the trajectories converge.*

4 Conclusions

This paper presented stability results for closed-loop sampled-data PWA systems under state feedback. PWA sampled-data systems were considered as delay systems with a variable time delay. It was also shown that the stability result for PWA systems is equivalent to the non-fragility of the continuous-time PWA controller when the sampling time converges to zero.

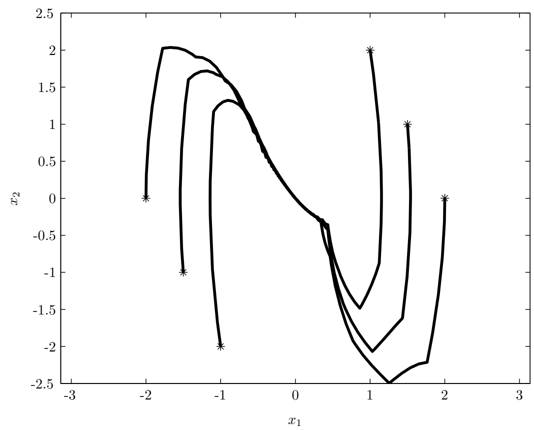

Fig. 4. Trajectories of the nonlinear helicopter model - sampled-data PWA controller

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