Stability of Sampled-Data Piecewise-Affine Systems Under State Feedback ⋆

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Abstract

This paper addresses stability of sampled-data piecewise-affine (PWA) systems consisting of a continuous-time plant and a discrete-time emulation of a continuous-time state feedback controller. The paper presents conditions under which the trajectories of the sampled-data closed-loop system will exponentially converge to a neighborhood of the origin. Moreover, the size of this neighborhood will be related to bounds on perturbation parameters related to the sampling procedure, in particular, related to the sampling period. Finally, it will be shown that when the sampling period converges to zero the performance of the stabilizing continuous-time PWA state feedback controller can be recovered by the emulated controller.

Key words: Piecewise-Affine Systems, Switched Systems, Sampled-data Systems, Stability Analysis, Linear Matrix Inequalities.

1 Introduction

PWA systems are multi-model systems that offer a good modeling framework for complex dynamical systems involving nonlinear phenomena. State and output feedback control of continuous-time PWA systems have received increasing interest over the last years. The research work has concentrated on Lyapunov-based controller synthesis methods for continuous-time PWA systems Hassibi & Boyd (1998); Johansson (2003); Johansson & Rantzer (2000); Rodrigues & How (2003); Rodrigues & Boyd (2005). However, none of these approaches would be applicable directly to controller synthesis for computer-controlled or sampled-data PWA systems. This is the scenario mostly encountered in applications given the flexibility of control implementation in a microprocessor. References Hassibi & Boyd (1998); Johansson (2003); Johansson & Rantzer (2000); Rodrigues & How (2003); Rodrigues & Boyd (2005) consider continuous-time processes controlled by continuous-time controllers while the implementation in a microprocessor requires emulation of a continuous-time controller as a discrete-time controller. Although linear sampled-data systems are a well-studied matter Chen & Francis (1995), controller emulation for systems with possible discontinuities at the switching, such as sampled-data PWA systems, has not had many research contributions. In fact, only recently these systems have started to be addressed in the literature in references such as Imura (2003a,b); Azuma & Imura (2004); Sun & Ge (2002); Sun (2004); G. Zhai & Yasuda (2004). The approach by Sun & Ge (2002) established that, under certain conditions, the controllable subspaces of a continuous-time switched linear system and its discrete-time counterpart are the same. Canonical forms of switched linear systems based on controllability are presented in the more recent work of Sun (2004). The approach by G. Zhai & Yasuda (2004) considers stability analysis of switched systems that can switch between a set of continuous-time plants and a set of discrete-time plants but does not handle sampled-data systems involving a cascade of a discrete-time system between a sample-and-hold and a continuous-time system. Furthermore, it does not address controller design. The approach by Imura (2003a,b); Azuma & Imura (2004) was probably the first where the term "sampled-data PWA systems" is used, although the systems described in this work do not posses the typical structure of a continuous-time plant being controlled by a discrete-time controller. The problem addressed in Imura (2003a,b); Azuma &

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Imura (2004) is one where the controller is continuous-time and the switching events are the ones controlled by the system logic inside a computer. In other words, in these systems it is assumed that the designer has command over the switching times of the system. The preliminary study of Imura (2003a,b) is interesting as it highlights important limitations of current discrete-time PWA control methodologies when applied to the control of a physical continuous-time system. As mentioned in Imura (2003a) unexpected phenomena such as chattering can occur, depending on the switching times. This increases the interest in studying computer implementations of controllers designed in continuous-time.

This paper addresses the classical structure of a sampled-data system whereby the system is continuous-time and the controller is being implemented (emulated) in discrete-time inside a computer. Previous approaches to this classical structure can be classified into two categories: i) discrete-time controller design to a discrete-time approximation of the continuous-time plant and ii) continuous-time controller design to a continuous-time plant followed by discrete-time emulation of the controller. To the best of the author's knowledge the only previous work in sampled-data PWA systems is the work of Imura *et. al.* Imura (2003a,b); Azuma & Imura (2004) which, as already stated, does not address the classical structure of interest in this paper. For papers in sampled-data control for nonlinear systems that fall into category i) we refer the reader to Nesic & Laila (2002) and references therein. For papers in sampled-data control for nonlinear systems that fall under category ii) we refer the reader to Khalil (2004) and references therein. Note that these papers always assume the plant dynamics to be locally Lipschitz. Therefore they do not include the possibility of having PWA dynamics that are switched with possible discontinuities in the plant dynamics at the switching. The interesting paper by Nesic & Teel (2004) also falls under category i) described above but offers the advantage of treating the plant model as a differential inclusion, thus possibly enabling discontinuous vector fields. In fact, one of the examples described in Nesic & Teel (2004) deals with a hysteresis switched controller. Although potentially applicable to PWA systems, Nesic & Teel (2004) does not address the problem of interest here, namely stability and performance recovery by emulation of a continuous-time PWA controller. Furthermore, in the framework of Nesic & Teel (2004), the plant dynamics must be embedded in a differential inclusion, which can potentially lead to conservative results instead of handling the PWA dynamics directly.

The paper starts by stating the problem assumptions. Then, the stability of the sampled-data system when a continuous-time controller is emulated in discrete-time is analyzed. A numerical example is included to show an application of the main stability result. Finally, the paper closes by stating the conclusions.

2 Problem Assumptions

It is assumed that a PWA system and a corresponding partition of the state space with polytopic cells $\mathcal{R}_i$, $i \in \mathcal{I} = \{1, \ldots, M\}$ are given (see Rodrigues & How (2001) for generating such a partition). Following Johansson (2003); Hassibi & Boyd (1998), each cell is constructed as the intersection of a finite number (p_i) of half spaces

$$\mathcal{R}_i = \{z \in I\!R^n \mid H_i^T z - g_i < 0\}, \tag{1}$$

where $H_i = [h_{i1} \ldots h_{ip_i}] \in I\!R^{n \times p_i}$, $g_i = [g_{i1} \ldots g_{ip_i}]^T \in I\!R^{p_i}$. Moreover the sets $\mathcal{R}_i$ partition a subset of the state space $\mathcal{X} \subset I\!R^n$ such that $\cup_{i=1}^M \overline{\mathcal{R}}_i = \mathcal{X}$, $\mathcal{R}_i \cap \mathcal{R}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{R}}_i$ denotes the closure of $\mathcal{R}_i$. Within each cell the dynamics are affine of the form

$$\dot{z}(t) = A_i z(t) + b_i + B_i u(t), \tag{2}$$

where $z(t) \in I\!R^n$, $u(t) \in I\!R^m$ and $b_i \in I\!R^n$. For system (2), we adopt the following definition of solutions.

Definition 1 Johansson (2003) *Let $z(t) \in \mathcal{X}$ be an absolutely continuous function. Then $z(t)$ is a trajectory of the system (2) on $[t_0, t_f]$ if, for almost all $t \in [t_0, t_f]$ and Lebesgue measurable $u(t)$, the equation $\dot{z}(t) = A_i z(t) + b_i + B_i u(t)$ holds for all i such that $z(t) \in \overline{\mathcal{R}}_i$.* □

Any two cells sharing a common facet will be called *level-1* neighboring cells. Let $\mathcal{N}_i = \{$*level*-1 neighboring cells of $\mathcal{R}_i\}$. Vectors $c_{ij} \in I\!R^n$ and scalars d_{ij} will then exist such that the facet boundary between cells $\mathcal{R}_i$ and $\mathcal{R}_j$ is contained in the hyperplane described by $\{z \in I\!R^n \mid c_{ij}^T z - d_{ij} = 0\}$, for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$. A parametric description of the boundaries is

$$\overline{\mathcal{R}_i} \cap \overline{\mathcal{R}_j} \subseteq \{z = l_{ij} + F_{ij} s \mid s \in I\!R^{n-1}\} \tag{3}$$

for $i = 1, \ldots, M$, $j \in \mathcal{N}_i$, where $F_{ij} \in I\!R^{n \times (n-1)}$ (full rank) is the matrix whose columns span the null space of c_{ij}^T and $l_{ij} \in I\!R^n$ is given by $l_{ij} = c_{ij} \left(c_{ij}^T c_{ij}\right)^{-1} d_{ij}$. It is further assumed that matrices E_i and f_i exist such that $\mathcal{R}_i \subseteq \varepsilon_i$,

$$\varepsilon_i = \{z \mid \|E_i z + f_i\| \leq 1\}. \tag{4}$$

This ellipsoidal covering is especially useful in the case where $\mathcal{R}_i$ is a slab because in this case the matrices E_i and f_i are guaranteed to exist and the covering (having one degenerate ellipsoid ε_i) is exact, *i.e.,* $\varepsilon_i \subseteq \mathcal{R}_i$ and $\mathcal{R}_i \subseteq \varepsilon_i$. More precisely, if $\mathcal{R}_i = \{z \mid d_1 < c_i^T z < d_2\}$, then the degenerate ellipsoid is described by $E_i = 2c_i^T/(d_2 - d_1)$ and $f_i = -(d_2 + d_1)/(d_2 - d_1)$. Finally it is assumed, without loss of generality, that the control objective is to stabilize the system to the origin.

3 Stability of Sampled-Data PWA Systems

In this section a stability result is presented for the closed-loop sampled-data system that is obtained when a continuous-time state feedback controller is implemented on a digital computer. It is assumed that a continuous-time state feedback controller parameterized by $K_i \in \mathbb{R}^{m \times n}$ and $m_i \in \mathbb{R}^m$ in the form

$$u = K_i z + m_i, \quad z \in \mathcal{R}_i \tag{5}$$

has already been designed such that the continuous-time closed-loop system is exponentially stable.[1] It is also assumed that the state z of the system is measured at a sampling rate $f_s = T^{-1},\ T > 0$, and that the controller in the feedback loop appears between a sampler and a zero-order-hold. At the sampling instants, the plant state and the sampled state overlap and therefore the sampled-data system is described by

$$\dot{z} = A_j z + b_j + B_j K_j z(kT) + B_j m_j, \tag{6}$$

for $z(t) \in \mathcal{R}_j,\ z(kT) \in \mathcal{R}_j$. However, for a given time t that is not a sampling instant, the general situation that should be considered is the one in which the state of the plant is in region $\mathcal{R}_i$ and the most recently sampled state is in region $\mathcal{R}_j$ with possibly $i \neq j$. The system is then described by the differential equation

$$\dot{z} = A_i z + b_i + B_i K_j z(kT) + B_i m_j, \tag{7}$$

for $z(t) \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$. This equation can be rewritten in the perturbed form

$$\dot{z} = \bar{A}_i z + \bar{b}_i + B_i \delta_{ij}, \tag{8}$$

for $z(t) \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$, where $\bar{A}_i = A_i + B_i K_i,\ \bar{b}_i = b_i + B_i m_i$ and

$$\delta_{ij} = K_j \left(z(kT) - z(t)\right) + \left(K_j - K_i\right) z(t) + \left(m_j - m_i\right). \tag{9}$$

Note that the first term in (9) represents the perturbation due to the error between the last available sample of the state and its current value. The second and third terms are associated with the perturbation due to the state and its most recent sample being possibly in different regions. The second term represents the perturbation due to a difference in the gain matrices in regions $\mathcal{R}_i$ and $\mathcal{R}_j$ and the third term represents the perturbation due to a difference in the affine control terms. Given a continuous-time controller of the form (5), the first step in the procedure outlined in this paper is to search for a quadratic Lyapunov function of the form

$$V(z) = z^T P z \tag{10}$$

that proves stability of the continuous-time closed-loop system. This can be done by solving for fixed $\alpha \geq 0$ the following set of LMIs (see for example Rodrigues & Boyd (2005) for details on the derivation of these conditions):

$$P = P^T > 0,\ \lambda_i < 0,\ i = 1, \ldots, M, \tag{11}$$

$$\begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i + \alpha P + \lambda_i E_i^T E_i & P\bar{b}_i + \lambda_i E_i^T f_i \\ (P\bar{b}_i + \lambda_i E_i^T f_i)^T & -\lambda_i \left(1 - f_i^T f_i\right) \end{bmatrix} < 0$$

The results that follow assume that such a Lyapunov function can be found. Note however that not all continuous-time PWA systems that are stable admit a globally quadratic Lyapunov function (see Johansson (2003) for counter-examples).

3.1 Conditions Independent of the Sampling Period

We now present the first result of this section. It gives conditions under which the trajectories of the sampled-data system (8) converge to a region around the closed-loop equilibrium point. Furthermore, it relates the size of this region to a measure of the perturbation term in the closed-loop system. In what follows, unless otherwise indicated, the time dependence of the indices $i,\ j$ will be omitted for simplicity.

Theorem 2 *Assume a Lyapunov function of the form (10) is found and is defined in $\mathcal{X} \subseteq \mathbb{R}^n$. Let the condition number of P be $\chi_P = \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}$. Assume there are finite constants $N_{ij} > 0,\ \Delta_{K_{ij}} \geq 0$, such that $\|\delta_{ij}\| \leq N_{ij} + \Delta_{K_{ij}}\|z\|,\ i, j = 1, \ldots, M$. Let $N = \max_{i,j=1,\ldots,M}\left(N_{ij}\right),\ \Delta_K = \max_{i,j=1,\ldots,M}\left(\Delta_{K_{ij}}\right),\ B = \max_{i=1,\ldots,M}\|B_i\|$. Define*

$$\mu_\theta = \frac{2\chi_P B}{\alpha\theta - 2\chi_P B \Delta_K} N$$

and the region

$$\mathcal{S}_\theta = \{z \in \mathcal{X} \mid \|z\| \leq \mu_\theta\}$$

for any positive constant $\theta < 1$ that verifies

$$\Delta_K < \frac{\alpha\theta}{2\chi_P B} \tag{12}$$

Then, if (12) is verified, the trajectories of the closed-loop sampled-data system (8) converge exponentially to the set

$$\Omega = \left\{z \in \mathcal{X} \mid V(z) \leq \sigma_{max}(P)\mu_\theta^2\right\}$$

Proof: For $z(t) \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$, using the dynamics (8), the derivative of the candidate Lyapunov function

[1] For optimization programs whose solution (when it exists) yields exponentially stabilizing PWA controllers see Rodrigues & How (2003); Rodrigues & Boyd (2005).

(10) along the trajectories of the system is

$$\dot{V}(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} + 2z^T P B_i \delta_{ij}$$

However, note that if a quadratic Lyapunov function is found by solving (11), using the $\mathcal{S}$-procedure (see Rodrigues & Boyd (2005) for details) it can be shown that for $z \in \mathcal{R}_i$

$$\begin{bmatrix} z \\ 1 \end{bmatrix}^T \begin{bmatrix} \bar{A}_i^T P + P\bar{A}_i & P\bar{b}_i \\ (P\bar{b}_i)^T & 0 \end{bmatrix} \begin{bmatrix} z \\ 1 \end{bmatrix} < -\alpha z^T P z.$$

Therefore, for $z \in \mathcal{R}_i,\ z(kT) \in \mathcal{R}_j$ it follows that

$$\dot{V}(z) < -\alpha z^T P z + 2z^T P B_i \delta_{ij}$$

Taking norms and using the bounds

$$\|\delta_{ij}\| \leq N_{ij} + \Delta_{K_{ij}}\|z\| \leq N + \Delta_K \|z\|$$

and $-z^T P z \leq -\sigma_{min}(P)\|z\|^2$ yields

$$\dot{V}(z) < -\alpha\sigma_{min}(P)\|z\|^2 + 2\|z\|\sigma_{max}(P)B\left(N + \Delta_K\|z\|\right)$$

or, for any positive constant $\theta < 1$

$$\dot{V}(z) < -(1-\theta)\alpha\sigma_{min}(P)\|z\|^2 - \theta\alpha\sigma_{min}(P)\|z\|^2 + 2\|z\|\sigma_{max}(P)B\left(N + \Delta_K\|z\|\right).$$

Therefore, for $0 < \theta < 1$, we have

$$\dot{V}(z) < -(1-\theta)\alpha\sigma_{min}(P)\|z\|^2 \leq -(1-\theta)\chi_P^{-1}\alpha V(z) \tag{13}$$

for

$$\|z\| > \frac{2\chi_P B}{\alpha\theta - 2\chi_P B \Delta_K} N$$

provided

$$\Delta_K < \frac{\alpha\theta}{2\chi_P B}. \tag{14}$$

Note that although condition (13) is only valid inside each region of the partition of the state space, it also guarantees that no unstable sliding modes can be generated at the boundaries because the Lyapunov function is of class $\mathcal{C}^1$ (see Samadi & Rodrigues (2007) for more details). As a result of (13), for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$V(z(t)) < V(z(t_0))e^{-(1-\theta)\chi_P^{-1}\alpha(t-t_0)}$$

Using the relation $\sigma_{min}(P)\|z\|^2 \leq V(z) \leq \sigma_{max}(P)\|z\|^2$ we can conclude that for $z \in I\!R^n \setminus \mathcal{S}_\theta$,

$$\|z(t)\| \leq \|z(t_0)\|\chi_P^{\frac{1}{2}} e^{-0.5(1-\theta)\chi_P^{-1}\alpha(t-t_0)}$$

Thus, there will be a positive and finite time t_1^θ such that $z(t_1^\theta) \in \mathcal{S}_\theta$ for any positive constant $\theta < 1$ that verifies (14). Note that $\mathcal{S}_\theta \subseteq \Omega$. This can be proved by contradiction. Assume that it is not true that $\mathcal{S}_\theta \subseteq \Omega$. Then, there exists at least one $z_0 \in \mathcal{S}_\theta$ for which $z_0^T P z_0 > \sigma_{max}(P)\mu_\theta^2$, a contradiction. Since $\dot{V} \leq 0$ at the boundary of Ω, Ω is an invariant set for system (8). Consequently, since $z(t_1^\theta) \in \mathcal{S}_\theta \subseteq \Omega$, $z(t) \in \Omega$ for all $t \geq t_1^\theta$ and for all $0 < \theta < 1$ that verifies (14). □

Remark 3 *This result relates the size of the region to which the trajectories converge to the size of the perturbations. The size of the region decreases with the size of the perturbations, as expected. Note that for the case where $K_i = K_j,\ \Delta_K = 0$ and (14) is automatically verified.*

Remark 4 *Bounds on δ_{ij} can be easily obtained in the case where all polytopic regions are bounded, by noticing that $\|z(kT) - z(t)\| \leq \max_{x \in \mathcal{R}_i, y \in \mathcal{R}_j}\|x - y\|$. These bounds are however potentially conservative and better ways of obtaining them should be investigated. In particular, the bound should depend on the sampling period T.*

The next section relates the bound on $\|z(kT) - z(t)\|$ to the sampling period T and offers a less conservative result that enables us to prove that if the sampling period converges to zero then the system is practically exponentially stable to the origin and the continuous-time behavior is recovered.

3.2 Conditions Dependent of the Sampling Period

Integrating equation (7) for $t \in [kT, (k+1)T]$ yields

$$z(t) - z(kT) = \int_{kT}^{t} A_{i(\tau)} z(\tau)d\tau + \int_{kT}^{t} b_{i(\tau)}d\tau + \int_{kT}^{t} B_{i(\tau)}d\tau\left(K_j z(kT) + m_j\right) \tag{15}$$

Thus, letting $A = \max_{i=1,\ldots,M}\|A_i\|$, $b = \max_{i=1,\ldots,M}\|b_i\|$, $B = \max_{i=1,\ldots,M}\|B_i\|$ yields

$$\|z(t) - z(kT)\| \leq A\int_{kT}^{t}\|z(\tau)\|d\tau + (t - kT)\left(b + B\|K_j z(kT) + m_j\|\right) \tag{16}$$

Since all possible dynamics in a PWA system with coefficients independent of the partition are affine, finite escape times cannot occur and therefore there will be a finite constant $Z(k,T) = \sup_{kT \leq t \leq kT+T}\|z(t)\|$ such that

$$\|z(t)\|_{kT \leq t \leq kT+T} \leq Z(k,T) \tag{17}$$

Using the bound (17) in expression (16) leads to

$$\|z(t) - z(kT)\| \leq (t - kT)(AZ(k,T) + b + B\|K_j z(kT) + m_j\|) \quad (18)$$

Remark 5 *Note that the bound $Z(k,T)$ gets smaller as the sampling time decreases and when $T \to 0$, $Z(k,T) \to \|z(kT)\|$. Note further that the Euler approximation for integration would lead to $Z(k,T) = \|z(kT)\|$ because $\int_{kT}^{t} \|z(\tau)\| d\tau \simeq \|z(kT)\|(t - kT)$.*

Letting $K = \max_{i=1,\dots,M} \|K_i\|$, $m = \max_{i=1,\dots,M} \|m_i\|$,

$$\|z(t) - z(kT)\| \leq (t - kT)(AZ(k,T) + b + BK\|z(kT)\| + Bm) \quad (19)$$

The worst possible (highest) bound is the one corresponding to $t = (k+1)T$, which leads to

$$\|z(t) - z(kT)\| \leq T\left(AZ(k,T) + b + BK\|z(kT)\| + Bm\right) \quad (20)$$

Recall that the expression for the perturbations developed in (9) was

$$\delta_{ij} = K_j\left(z(kT) - z(t)\right) + \left(K_j - K_i\right)z(t) + \left(m_j - m_i\right). \quad (21)$$

Let now $\Delta_{K_{ij}} = \|K_j - K_i\|$, $\Delta_{m_{ij}} = \|m_j - m_i\|$. Then we can write

$$\|\delta_{ij}\| \leq K\|z(t) - z(kT)\| + \Delta_{K_{ij}}\|z(t)\| + \Delta_{m_{ij}} \quad (22)$$

and therefore using (17) and (20) this finally yields

$$\|\delta_{ij}\| \leq N_{ij}(k,T) + \Delta_{K_{ij}}\|z\|, \; i,j = 1,\dots,M \quad (23)$$

where

$$N_{ij}(k,T) = \Delta_{m_{ij}} + KT\left(\bar{A}Z(k,T) + \bar{b}\right) \quad (24)$$

and $\bar{A} = A + BK$, $\bar{b} = b + Bm$. Using this bound and Theorem 2 the following result can now be stated.

Corollary 6 *Assume a Lyapunov function of the form (10) is found and is defined in $\mathcal{X} \subseteq \mathbb{R}^n$. Let the condition number of P be $\chi_P = \frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}$. Let $N_{ij}(k,T)$ be defined as in (24) where $Z(k,T) = \sup_{kT \leq t \leq kT+T} \|z(t)\|$. Define*

$$\bar{N}_{ij}(T) = \Delta_{m_{ij}} + KT\left(\bar{A}\bar{Z}(T) + \bar{b}\right) \quad (25)$$

where

$$\bar{Z}(T) = \lim_{L \to \infty} \max_{k \in \{0,\dots,L\}} Z(k,T) \quad (26)$$

and

$$A = \max_{i=1,\dots,M} \|A_i\|, \; b = \max_{i=1,\dots,M} \|b_i\|,$$
$$B = \max_{i=1,\dots,M} \|B_i\|, \; \Delta_K = \max_{i,j=1,\dots,M} \|K_j - K_i\|.$$

Furthermore, let $N(T) = \max_{i,j=1,\dots,M}\left(\bar{N}_{ij}\right)$. Define

$$\mu_\theta(T) = \frac{2\chi_P B}{\alpha\theta - 2\chi_P B \Delta_K} N(T)$$

and the region

$$\mathcal{S}_\theta(T) = \{z \in \mathcal{X} \mid \|z\| \leq \mu_\theta(T)\}$$

for any positive constant $\theta < 1$ that verifies

$$\Delta_K < \frac{\alpha\theta}{2\chi_P B} \quad (27)$$

Then, in the absence of sliding modes, if (27) is verified it follows that:

(1) The trajectories of the closed-loop sampled-data system (8) converge exponentially to the set

$$\Omega(T) = \left\{z \in \mathcal{X} \mid V(z) \leq \sigma_{max}(P)\mu_\theta^2(T)\right\}.$$

(2) When $T \to 0$, the trajectories of the closed-loop sampled-data system (8) are practically exponentially stable to the origin. By this it is meant that $z(t) \to 0$ exponentially a.e when $T \to 0$.

Proof: Result 1) follows directly from the proof of Theorem 2. Result 2) follows from the facts that:

In the absence of sliding modes, chattering phenomena is ruled out in closed-loop. Therefore, for any finite $t^\star$, there will be a finite number $N^\star(t^\star, T)$ of switchings in the time period $[0, t^\star]$.

At $t = kT$ the dynamics of the system will be governed by (6) where j is the index of the region where the state lies at or immediately after kT. Notice that (6) is an affine differential equation with constant coefficients and finite escape times are not possible, so $z(t)$ is bounded. Notice also that until a switch occurs, the solution of (6) is continuous and given by the variation of constants

$$z(t) = \Phi_j(t - kT)z(kT) + \Gamma_j(t - kT)u_j$$
$$u_j = b_j + B_j\left(K_j z(kT) + m_j\right)$$

where $\Phi_j(t - kT) = e^{A_j(t-kT)}$ and $\Gamma_j(t - kT) = \int_0^{t-kT} e^{A_j\tau} d\tau$. Note also that

$$\|z(t) - z(kT)\| \leq \|\Phi_j + \Gamma_j B_j K_j - I\|\bar{Z}(T) + \|\Gamma_j\|\|\bar{b}_j\|,$$

and thus $\|z(t)-z(kT)\|$ is also bounded since $\bar{Z}(T)$ must stay bounded due to the impossibility of finite escape times for PWA systems with constant coefficients not dependent on the partition. Furthermore, the first switching outside of region j will only occur when $w_{ji}(z) = h_{ji}^T z - g_{ji} = 0$ for some $i \in \mathcal{N}_j$. Since $z(t)$ and $w_{ji}(z)$ are continuous and $w_{ji}(z) \neq 0$ right after the update of the state at $t = kT$, $w_{ji}(z) \neq 0$ for at least a time interval with some positive measure $\epsilon_{ij} \leq T$ until the occurence of the next switch. Because of this fact, $N^\star(t^\star, T)$ cannot grow unbounded as $T \to 0$. Until the occurence of a switch, $\Delta_{m_{ij}} = 0,\ \Delta_{K_{ij}} = 0$.

From the previous point, we conclude that for any $T > 0,\ k \geq 0$ the Lebesgue measure of the set

$$S_{kT}^{i,j} = \{t \in [kT, (k+1)T] \mid \Delta_{K_{ij}}(t),\ \Delta_{m_{ij}}(t) > 0\}$$

verifies

$$\mu\left(S_{kT}^{i,j}\right) \leq T - \epsilon_{ij}(T) < T$$

Thus, for any finite $t^\star$ the set $\mathcal{S} = \cup_{k:\ kT<t^\star} \mathcal{S}_{kT}^{i,j}$ will have bounded Lebesgue measure $\mu(\mathcal{S}) \leq N^\star T$. As $T \to 0$, the Lebesgue measure of this set will also converge to zero because $N^\star(t^\star, T)$ cannot grow unbounded as $T \to 0$.

Therefore $\Delta_{K_{ij}} \to 0,\ \Delta_{m_{ij}} \to 0$ as $T \to 0$ except possibly on a set of time instants that has Lebesgue measure converging to zero. Thus, the set of times t for which $i(t), j(t)$ are different converges to zero almost everywhere as $T \to 0$ so $i = j$ $a.e$ when $T \to 0$.

$Z(k,T) \to \|z(kT)\|$ when $T \to 0$ and, as seen before, $\Delta_{m_{ij}} \to 0$ $a.e$ as $T \to 0$. This together with the fact that $\|z(kT)\|$ is bounded for any $k \geq 0$ implies by (25) and (26) that $\bar{N}_{ij}(T) \to 0$ $a.e$ when $T \to 0$. Thus $\mu_\theta(T) \to 0$ $a.e$ when $T \to 0$.

Following the rationale in the proof of Theorem 2, the previous points show that the closed-loop system trajectories converge exponentially to the set $\Omega(T)$ whose size converges to zero as $\mu_\theta(T) \to 0$ $a.e$ when $T \to 0$. In fact, following the arguments of the proof of Theorem 2, when $T \to 0$ the Lyapunov function $V(z)$ decreases exponentially except possibly for a set of times whose Lebesgue measure converges to zero as $T \to 0$. However, for this set of times, as discussed, $z(t)$ remains bounded and, by expression (20), $z(t) - z(kT)$ remains bounded and is small for small T. Since the Lyapunov function is of class $\mathcal{C}^2$ on z, the Lyapunov function also remains bounded. Given that the Lyapunov function remains bounded for these sets of time and these time instants form a set whose Lebesgue measure converges to zero as $T \to 0$, $V(t) < V(t_0)e^{-(1-\theta)\chi_P^{-1}\alpha(t-t_0)}$, $a.e.$ and the result of the theorem is established by the fact that $\sigma_{min}(P)\|z(t)\|^2 \leq V(z) \leq \sigma_{max}\|z(t)\|^2$. □

Remark 7 *This result formally establishes the very important and desired property that a sampled-data PWA system converges to a closed-loop continuous-time PWA system when the sampling period converges to zero. As desired, all the stability guarantees for the closed-loop continuous-time system can be recovered.*

Remark 8 *The result assumes the absence of sliding modes. Sliding modes can indeed be ruled out in feedback for PWA systems with hyperplane boundaries if the component of the vector fields perpendicular to the boundaries is continuous across the boundaries. This idea was first suggested for PWA systems in Rodrigues & How (2003) to avoid the generation of sliding modes in closed-loop. If the feedback construction suggested in Rodrigues & How (2003) is used, it can be shown following the reasoning explained in Rodrigues & How (2003) that sliding modes are still ruled out in feedback for sampled-data PWA systems if the additional constraints* $B_i = B_j = B,\ c_{ij}^T B\,(K_j - K_i + m_j - m_i) = 0,\ \forall i = 1, \ldots, M, \forall j \in \mathcal{N}_i$ *are verified. Notice that these constraints are linear in the controller parameters and can easily be included in the optimization procedure suggested in Rodrigues & How (2003) for systems with a constant input matrix* B *(such as the one presented in the example of the next section).*

Remark 9 *Note that for the case of continuous PWA systems, the continuous vector field from the state equation (2) given by* $f(z,u) = A_i z + b_i + B_i u$ *and* $f(z(kT), u(kT)) = A_j z(kT) + b_j + B_j u(kT)$ *is locally Lipschitz in* z *with Lipschitz constant* $L = \max_{i=1,\ldots,M} \|A_i\|$. *In this case, following the ideas presented in Khalil (2004), the Gronwall-Bellman inequality applied to the integral of the dynamical equation (2) between* kT *and* $t \leq kT + T$

$$z(t) = z(kT) + (t - kT) f(z(kT), u(kT)) + \int_{kT}^{t} \left[f(z(\tau), u(kT)) - f(z(kT), u(kT))\right] d\tau$$

enables us to show that

$$\|z(t) - z(kT)\| \leq \frac{1}{L}\left[e^{(t-kT)L} - 1\right] \cdot \|A_j z(kT) + b_j + B_j u(kT)\|,\ kT \leq t \leq kT + T$$

When the control input is replaced by its value $u(kT) = K_j z(kT) + m_j$, *it finally yields the bound*

$$\|z(t) - z(kT)\| \leq \frac{1}{L}\left[e^{TL} - 1\right]\left[\|\bar{A}_j\|\|z(kT)\| + \|\bar{b}_j\|\right]. \tag{28}$$

where we have used the fact that $t - kT \leq T$ *for* $kT \leq t \leq kT + T$ *and* $\bar{A}_j,\ \bar{b}_j$ *are defined as before. Following the reasoning leading to (24) a new value for* $N_{ij}(k,T)$

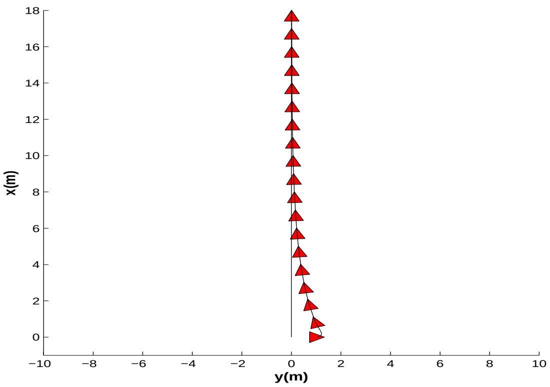

Fig. 1. $x-y$ trajectory with continuous-time controller, $\psi_0 = \frac{\pi}{2}$, $r_0 = 0$ rad/s, $y_0 = 1$ m

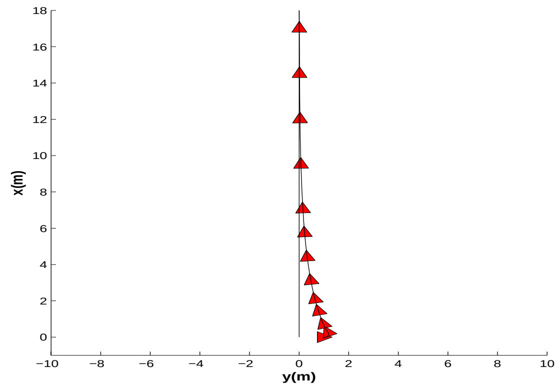

Fig. 2. $x-y$ trajectory for a sampling period of $T = 0.05s$

can be found as

$$N_{ij}(k,T) = \Delta_{m_{ij}} + \frac{K}{L}\left[e^{TL}-1\right]\left[\|\bar{A}_j\|\|z(kT)\| + \|\bar{b}_j\|\right] \tag{29}$$

Note that (24) and (29) become very similar ($\bar{A}$, $\bar{b}$ are replaced by $\|\bar{A}_j\|$, $\|\bar{b}_j\|$) for very small T if the Euler approximation is used in (24) to approximate $Z(k,T)$ by $\|z(kT)\|$. Note however that (24) is more general and less restrictive than (29) because it is valid even for discontinuous PWA systems that are therefore not locally Lipschitz. The important point to make is that from expression (29) when $T \to 0$, $\bar{N}_{ij}(T) \to 0$ a.e. since $\|z(kT)\|$ is bounded and $z(kT) \to z(t)$ so that i, j become the same, except on a set of measure zero. This leads to the same result obtained in Corollary 6 when one replaces (24) by (29) for the special case of continuous PWA systems.

4 Example

The objective of this example is to design a controller that forces a cart on the $x-y$ plane to follow the straight line $y = 0$ with a constant velocity $U_0 = 1$ m/s. It is assumed that a controller has already been designed to maintain a constant forward velocity. The cart's path is then controlled by the torque u about the z-axis according to the following dynamics:

$$\begin{bmatrix}\dot{\psi}\\ \dot{r}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{k}{I} & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\psi\\ r\\ y\end{bmatrix} + \begin{bmatrix}0\\ 0\\ U_0\sin(\psi)\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{I}\\ 0\end{bmatrix}u \tag{30}$$

where ψ is the heading angle with time derivative r, $I = 1$ Kgm^2 is the moment of inertia of the cart with respect to the center of mass and $k = 0.01$ Nms is the damping coefficient. Note that for this example $B_i = B_j = B$, $c_{ij}^T = [1\ 0\ 0]$, $c_{ij}^T B = 0$. The state of the system is $(z_1, z_2, z_3) = (\psi, r, y)$. Assume the trajectories can start from any possible initial angle in the range $\psi_0 \in [-\frac{3\pi}{5}, \frac{3\pi}{5}]$ and any initial distance from the line. The function $\sin(\psi)$ is approximated by a PWA function (see Rodrigues & How (2001)) yielding

$$\mathcal{R}_1 = \left\{z \in I\!R^3 \mid z_1 \in \left(-\frac{3\pi}{5}, -\frac{\pi}{5}\right)\right\},$$
$$\mathcal{R}_2 = \left\{z \in I\!R^3 \mid z_1 \in \left(-\frac{\pi}{5}, -\frac{\pi}{15}\right)\right\},$$
$$\mathcal{R}_3 = \left\{z \in I\!R^3 \mid z_1 \in \left(-\frac{\pi}{15}, \frac{\pi}{15}\right)\right\},$$

and $\mathcal{R}_4$ is symmetric to $\mathcal{R}_2$ and $\mathcal{R}_5$ is symmetric to $\mathcal{R}_1$, all with respect to the origin. A controller was designed to stabilize the origin (inside region $\mathcal{R}_3$) yielding

$$K_1 = [\,-49.908\ \ -9.467\ \ -13.926\,], \quad m_1 = 2.70 \times 10^{-6}$$
$$K_2 = [\,-48.316\ \ -9.330\ \ -13.812\,], \quad m_2 = 3.75 \times 10^{-7}$$
$$K_3 = [\,-50.148\ \ -9.468\ \ -13.742\,] \quad m_3 = 0.00 \times 10^{0}$$
$$K_4 = [\,-48.316\ \ -9.330\ \ -13.812\,] \quad m_4 = -m_2 \times 10^{0}$$
$$K_5 = [\,-49.908\ \ -9.468\ \ -13.926\,] \quad m_5 = -m_1 \times 10^{0}$$

The trajectory in the $x-y$ plane using this controller is shown in figure 1 where it is clear that the controller makes the cart trajectory converge to the desired straight line. For a sampling period of $T = 0.05s$ the same controller was emulated in discrete-time between a sampler and a zero-order-hold and the results of the corresponding $x-y$ trajectory are shown in figure 2. It can be seen that the trajectory still follows approximately the one obtained with the continuous-time controller. When the sampling period is further increased to $T = 0.2s$ the simulation of the $x-y$ trajectory close to the line is zoomed in figure 3. It is clear that the trajectory converges to a region around the desired straight line, as predicted by the results of this paper.

5 Conclusions

This paper has presented stability results for closed-loop sampled-data PWA systems under state feedback. It was shown that the emulation of a state feedback controller designed in continuous-time to exponentially stabilize the system to a target point would still exponentially stabilize the system to a region around the target point. The size of this region was related to the sampling period. It was shown that when the sampling period converges to zero the exponential stability results for the closed-loop continuous-time system are recovered.

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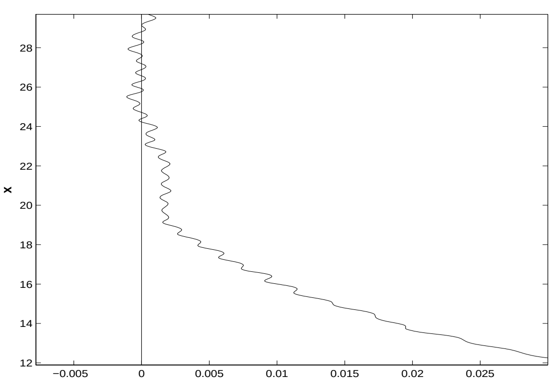

Fig. 3. $x - y$ trajectory zoomed for a sampling period of $T = 0.2s$

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