Failure Analysis in Multi-Agent Networks: A Graph-Theoretic Approach

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ABSTRACT

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Mohammad Amin Rahimian

A multi-agent network system consists of a group of dynamic control agents which interact according to a given information flow structure. Such cooperative dynamics over a network may be strongly affected by the removal of network nodes and communication links, thus potentially compromising the functionality of the overall system. The chief purpose of this thesis is to explore and address the challenges of multi-agent cooperative control under various fault and failure scenarios by analyzing the network graph-topology. In the first part, the agents are assumed to evolve according to the linear agreement protocol. Link failures in the network are characterized based on the ability to distinguish the agent dynamics before and after failures. Sufficient topological conditions are provided, under which dynamics of a given agent is distinguishable for distinct digraphs. The second part of this thesis is concerned with the preservation of structural controllability for a multi-agent network under simultaneous link and agent failures. To this end, the previously studied concepts of link and agent controllability degrees are first exploited to provide quantitative measures for the contribution of a particular link or agent to the controllability of the overall network. Next, the case when both communication links and agents in the network can fail simultaneously is considered, and graphical conditions for preservation of controllability are investigated.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS AND SYMBOLS

$\varnothing$	empty set
$\mathbb{N}_k$	set of integers $\{1,2,\ldots,k\}$
$\lvert\mathscr{X}\rvert$	cardinality of set $\mathscr{X}$
$\mathbb{N}$	set of all natural numbers
$\mathbb{W}$	set of all natural numbers and 0
$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
$\mathscr{X}\setminus\mathscr{Y}$	set of element in $\mathscr{X}$ that are not in $\mathscr{Y}$
$\binom{m}{k}$	number of $k-$combinations of a set of m elements
Ξ_Θ	finite group formed by the $(\lvert\Theta\rvert)!$ permutations on the set Θ
Ξ_n	finite group formed by the $n!$ permutations on the set $\mathbb{N}_n$
$\iota(\psi)$	length of permutation ψ, number of pairs $i<j, \psi(i)>\psi(j)$
$[M]_{ij}$	$ij-$th entry of matrix M is at its $i-$th row and $j-$th column
$\det(M)$	determinant of (square) matrix M
e^M	matrix exponential of (square) matrix M
$\mathscr{G}$	directed graph (digraph)
$\mathscr{V}$	set of vertices
$\mathscr{E}$	set of edges

(τ, ν)	a directed edge from τ (tail) to ν (head)
$\partial^{+}_{\mathscr{G}} \mathscr{X}$	set of all edges whose tails belong to $\mathscr{X}$ but heads do not
$\partial^{-}_{\mathscr{G}} \mathscr{X}$	set of all edges whose heads belong to $\mathscr{X}$ but tails do not
$d^{-}_{\mathscr{G}}\{\nu\}$	in-degree of ν in digraph $\mathscr{G}$
$d^{+}_{\mathscr{G}}\{\nu\}$	out-degree of ν in digraph $\mathscr{G}$
$\Gamma((\nu_i, \nu_j))$	a (square) matrix with a 1 at its $ji-$th entry and 0 elsewhere
$\sigma(\nu_i)$	a (column) vector with a 1 at its $i-$th row and 0 elsewhere
$A(\mathscr{G})$	adjacency matrix of $\mathscr{G}$
$\Delta(\mathscr{G})$	degree matrix of $\mathscr{G}$
$\mathscr{L}(\mathscr{G})$	in-degree graph Laplacian of $\mathscr{G}$
$\mathrm{c}_k(\tau, \nu)$	number of paths from τ to ν
$\mathrm{w}_k(\tau, \nu)$	number of walks from τ to ν
$\mathrm{d}(\tau, \nu)$	length of the shortest (path) walk from τ to ν
$\mathscr{K}^{\nu}_{\mathscr{G}}$	shortest paths subgraph of $\mathscr{G}$ w.r.t. ν
$x^{\varepsilon}_i(t)$	$x_i(t)$ calculated in digraph $(\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$
$x^{\mathscr{G}}_i(t)$	$x_i(t)$ calculated in digraph $\mathscr{G}$
$\mathscr{C}_p$	a critical link-set of cardinality p
$\mathscr{C}_q$	a critical agent-set of cardinality q
$\mathscr{C}_{rs}$	a critical agent-link set of r vertices and s edges
$lc(\mathscr{G};\mathscr{R})$	link controllability degree of digraph $\mathscr{G}$ with root-set $\mathscr{R}$

$ac(\mathscr{G};\mathscr{R})$	agent controllability degree of digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$jc(\mathscr{G};\mathscr{R})$	joint controllability degree of digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\sigma(\mathscr{G},\varepsilon;\mathscr{R})$	link controllability index of ε in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\rho(\mathscr{G},\varepsilon;\mathscr{R})$	agent controllability index of ε in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\delta(\mathscr{G},\nu;\mathscr{R})$	agent criticality index of ν in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\theta(\mathscr{G},\nu;\mathscr{R})$	link criticality index of ν in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\gamma(\mathscr{G},\nu;\mathscr{R})$	critical link index of ν in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\psi(\mathscr{G},\nu;\mathscr{R})$	uncritical link index of ν in digraph $\mathscr{G}$ with root-set $\mathscr{R}$
$\mathscr{W}_n(s)$	set of polynomials in s with degrees less than or equal to n

Chapter 1

Introduction

The past decade has seen a growing interest in the control of multi-agent networks [1, 2]. Multi-agent network systems consist of a group of dynamic agents, which interact according to a given information flow structure. The dynamic control law for each agent is a function of its state and the states of a few of its local neighboring agents, with which it can communicate [3, 4]. Distributed and cooperative control for these networked dynamic systems employs various concepts from different fields including parallel processing, distributed algorithms, control, and estimation [2].

Multi-agent systems have found many promising applications in diverse areas such as sensor networks, motion coordination of robots, automated highway systems, air traffic control, formation control of satellite clusters [5–11]. An important class of multi-agent systems is the one with leader-follower architecture [12]. Popular research problems related to multi-agent network control and leader-follower systems include connectivity, containment, consensus, rendezvous, formation, and flocking [13–18]. Preservation of network connectivity throughout the system's evolution is a major area of research that has attracted much attention [5, 19–28]. In the consensus problem, it is desired that the states of all of the agents in the network converge to a common value [29–32]. In the containment problem, on the other hand, a group of agents are designated as followers and their states are required to converge to the convex hull of the states of the leader agents [33–36]. Specification of the required configurations in terms of the relative position of the agents leads to a different class of problems known as formation control [37, 38]. Particular areas of application that have attracted much focus amongst the research community include the formation flight of unmanned aerial vehicles [39–43] and mobile robotic networks [44–47], and they in turn have lead to the emergence of interesting analytical problems such as collision and obstacle avoidance [48, 49].

1.1 Motivation and Related Work

The cooperative multi-agent dynamics over a network may be strongly affected by the removal of agents and communication links, thus potentially compromising the functionality of the overall system. On the other hand, it is known that some faults in the multi-agent systems can affect part of the network, containing a number of links and agents. This type of faults in multi-agent systems, where terrain properties or hardware faults affect a number of agents and limit their ability to communicate, provides motivation for the analysis of failures in multi-agent networks. The chief purpose of this thesis is to explore and address the challenges of multi-agent cooperative control under various fault and failure scenarios by analyzing the graph-topology of the information flow networks.

The effects of the removal of links or agents on a multi-agent system are investigated in terms of its information flow graph in [50–52], where graph-theoretic conditions are provided for the preservation of structural controllability. Moreover, the concepts of link and agent controllability degrees are proposed to provide quantitative measures for the reliability of networks subject to failures, and various ideas from the network flow problem [53] are exploited to design polynomial-time algorithms for the computation of these reliability measures.These concepts are then extended to the case of simultaneous link and agent failures in [54]. Two conceptually related issues are the fault tolerance of networks and connectivity of their interconnection digraphs, as discussed in Section 1.5 of [55].

In a very recent work, Liu, Slotine and Barabási [56] investigate complex real-world networks in terms of their structural controllability and exploit the graph-theoretic technique of maximum matching to determine the minimum number of inputs for maintaining full control over the network. The resulting unmatched nodes are subsequently proposed as the choice of input nodes for injecting the control signals and making the whole network controllable. The issue of robustness of control against link failures is then considered by introducing critical, redundant and ordinary links, depending on the minimum number of necessary inputs for controllability in the face of link loss, and the participation of the link in the maximally matched sets. Section 5.2 of [1] offers a probabilistic treatment of the link failures in multi-agent networks. A probability for any link failure is first assumed, and using the stochastic version of LaSalle's invariance principle the convergence of the agreement protocol over the resultant random graph is shown. Then in Subsections 5.2.2 and 5.2.3, some important concepts from the spectral theory of random matrices and stochastic Lyapunov theory are exploited to analyze the rate of convergence for the agreement protocol over random networks. In particular, it is demonstrated that with a fixed probability for link failures, larger networks have better convergence rates.

1.2 Thesis Objectives and Outline

After introducing some preliminaries on sets and graph theory as well as multi-agent dynamics in Chapter 2, the main contributions of the thesis are provided in Chapters 3 and 4.

In Chapter 3, the focus is on the cooperative control of a multi-agent system under the linear agreement protocol and subject to multiple communication link failures. The question of detectability of link failures is addressed and topological conditions are provided under which distinct digraphs result in distinguishable dynamics in the state of an agent. The mathematical characterization of simultaneous link failures in this chapter leads to useful design guidelines for realization of reliable and fault-tolerant multi-agent networks. Detection and isolation of faults are crucial to the cooperative and reliable control of multi-agent networks, and the chief aim of this chapter is to address these two important concepts. The results are therefore of both theoretical and practical interest. Geographical barriers, spatial requirements and scarcity of observation equipment may limit the designers' access to multiple observing agents; therefore, the study of distinguishable dynamics in this chapter is restricted to the viewpoint of a single agent.

Next, in Chapter 4 the problem of controllability preservation in the face of simultaneous link and agent failures is considered. Previous studies consider the case of failures in links or agents only and the chief aim of Chapter 4 is to extend the available results to the case where both links and agents can fail. Moreover, the existing controllability measures are used to quantify and compare the importance of different links and agents to the controllability of the overall network.

Finally, the contributions of the thesis are summarized in Chapter 5 and suggestions for further research directions are provided.

Chapter 2

Preliminaries

This chapter gives some preliminaries on sets and graph theory, and introduces the notation that is used throughout the thesis. In the sequel, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{W} = \{0\} \cup \mathbb{N}$, $\mathbb{R}$ denotes the set of all real numbers, and $\mathbb{C}$ denotes the set of all complex numbers. Also, the set of integers $\{1, 2, \ldots, k\}$ is denoted by $\mathbb{N}_k$, and any other set is represented by a curved capital letter. The cardinality of a set $\mathscr{X}$, which is the number of its elements, is denoted by $|\mathscr{X}|$. The difference of two sets $\mathscr{X}$ and $\mathscr{Y}$ is characterized as $\mathscr{X} \backslash \mathscr{Y} = \{x; x \in \mathscr{X} \wedge x \notin \mathscr{Y}\}$. Matrices are represented by capital letters, vectors are expressed by boldface lower-case letters, and the superscript ^{T} denotes the matrix transpose. Moreover, I denotes the identity matrix with proper dimension, and the determinant of a matrix M is denoted by $\det(M)$, while $[M]_{ij}$ indicates the element of M which is located at its $i-$th row and $j-$th column.

The remainder of this chapter is organized as follows. A brief introduction to the theory of directed graphs is provided in Section 2.1, followed by the presentation of the system model and dynamical evolution for the multi-agent systems under the linear agreement protocol in Section 2.2. Finally, Section 2.3 introduces the concept of structurally of controllability for multi-agent systems and reviews some results from [50] and [52].

2.1 Directed Graphs and the Associated Algebraic Entities

A directed graph or *digraph* is defined as an ordered pair of sets: $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{v_1, \ldots, v_n\}$ is a set of $n = |\mathscr{V}|$ vertices and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is a set of directed edges. The digraph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ is a subgraph of $\mathscr{G}$ if $\mathscr{V}_1 \subseteq \mathscr{V} \wedge \mathscr{E}_1 \subseteq \mathscr{E}$. In the graphical representations, each edge $\varepsilon := (\tau, v) \in \mathscr{E}$ is denoted by a directed arc from vertex $\tau \in \mathscr{V}$ to vertex $v \in \mathscr{V}$. Vertices v and τ are referred to as the *head* and *tail* of the edge ε, respectively. Given a set of vertices $\mathscr{X} \subset \mathscr{V}$, the set of all edges for which the tails belong to $\mathscr{X}$ but the heads do not, is termed the out-cut of $\mathscr{X}$, and is denoted by $\partial^+_{\mathscr{G}} \mathscr{X} \subset \mathscr{E}$. Similarly, the set of all edges for which the heads belong to $\mathscr{X}$ but the tails do not, is referred to as the in-cut of $\mathscr{X}$, and is denoted by $\partial^-_{\mathscr{G}} \mathscr{X} \subset \mathscr{E}$. The in-degree and out-degree of $\mathscr{X}$ are given by $d^-_{\mathscr{G}} \mathscr{X} = |\partial^-_{\mathscr{G}} \mathscr{X}|$ and $d^+_{\mathscr{G}} \mathscr{X} = |\partial^+_{\mathscr{G}} \mathscr{X}|$, respectively. Notice that the definition of $\mathscr{E}$ does not allow for the existence of parallel arcs in the graphical representation. In other words, if two edges share the same pair of head and tail, then they are identical.

Given an integer $k \in \mathbb{N}_{n-2}$, a set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} = \mathbb{N}_k$ and two vertices $\tau, v \in \mathscr{V}$, a sequence of distinct edges of the form $\mathscr{P} := (\tau, v_{\alpha_1}), (v_{\alpha_1}, v_{\alpha_2}), \ldots, (v_{\alpha_{k-1}}, v_{\alpha_k})(v_{\alpha_k}, v)$ is called a τv *path* with *length* $k+1$ if for any two edges $(\bar{\tau}, \bar{v}), (\hat{\tau}, \hat{v})$ of this sequence, $\bar{v} \neq \hat{v} \longleftrightarrow \bar{\tau} \neq \hat{\tau}$. For any $\mathscr{R} \subset \mathscr{V}$, a τv path is called $\mathscr{R}-$rooted if $\tau \in \mathscr{R}$. The set $\mathscr{R}$ associated with an $\mathscr{R}-$rooted τv path is referred to as the root-set, and a vertex $v \in \mathscr{V} \backslash \mathscr{R}$ is called reachable from the root-set $\mathscr{R}$ if there exists an $\mathscr{R}-$rooted τv path, for some $\tau \in \mathscr{R}$. Two distinct τv paths are called edge-disjoint if they do not share any edges. Two edge-disjoint τv paths are called disjoint if τ and v are the only vertices that are common to both of them. Furthermore, the number of τv paths with length k, denoted by $\mathrm{c}_k(\tau, v)$, is called the *k−th order topological connectivity* of τ to v. Likewise, the shortest length for all τv paths is referred to as the *distance from* τ *to* v and is denoted by $\mathrm{d}(\tau, v)$. Notably, $\mathrm{c}_k(\tau, v) = 0$ for $k < \mathrm{d}(\tau, v)$, and by convention $\mathrm{d}(v, v) = 0$, $\mathrm{c}_0(v, v) = 1$, and if $\forall k \in \mathbb{N}, \mathrm{c}_k(\tau, v) = 0$, then $\mathrm{d}(\tau, v) = \infty$. For a given vertex $v \in \mathscr{V}$, a vv path is called a *directed circle*, and a *self-loop* on vertex v signifies a directed circle of length 1, whose only edge is (v, v).

Similarly, for an integer $k \in \mathbb{N}$, a set of (possibly repeated) indices $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \mathbb{N}_{|\mathscr{V}|}$ and two vertices $\tau, v \in \mathscr{V}$, a τv *walk* with *length* $k+1$ signifies an ordered sequence of edges of the form $\mathscr{W} := (\tau, v_{\alpha_1}), (v_{\alpha_1}, v_{\alpha_2}), \ldots, (v_{\alpha_{k-1}}, v_{\alpha_k})(v_{\alpha_k}, v)$. Moreover, the number of τv walks with length k, denoted by $\mathrm{w}_k(\tau, v)$, is called the *k−th connectivity* of τ to v, and by convention, $\mathrm{w}_0(\tau, v) = 0$ if $\tau \neq v$, while $\mathrm{w}_0(v, v) = 1$. The following lemma follows from the fact that a given τv walk can be reduced to a τv path upon the removal of its directed circles. In particular every shortest walk from τ to v is also a shortest path from τ to v and vice versa.

Lemma 1. *For a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and vertices $\{v_i, v_j\} \subset \mathscr{V}$, $\{i, j\} \subset \mathbb{N}_{|\mathscr{V}|}$, the following relations hold:*

$$\mathrm{d}(v_j, v_i) = \min_{k \in \mathbb{W}, \mathrm{w}_k(v_j, v_i) \neq 0} \{k\}, \tag{2.1}$$

and

$$\mathrm{w}_{\mathrm{d}(v_j, v_i)}(v_j, v_i) = \mathrm{c}_{\mathrm{d}(v_j, v_i)}(v_j, v_i). \tag{2.2}$$

If $\forall \{\tau, v\} \subset \mathscr{V}$ there exists a τv walk in $\mathscr{G}$, then digraph $\mathscr{G}$ is strongly connected. A subgraph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ of $\mathscr{G}$ is a strongly connected component of $\mathscr{G}$ if $\forall v \in \mathscr{V} \setminus \mathscr{V}_1$, the digraph $\mathscr{G}_2 = (\{v\} \cup \mathscr{V}_1, (\partial^+_{\mathscr{G}}\{v\} \cap \partial^-_{\mathscr{G}} \mathscr{V}_1) \cup (\partial^-_{\mathscr{G}}\{v\} \cap \partial^+_{\mathscr{G}} \mathscr{V}_1) \cup \mathscr{E}_1)$ is not strongly connected. Two distinct vertices $\tau, v \in \mathscr{V}$ are siblings if they belong to the same strongly connected component of $\mathscr{G}$. Also, if τ and v are not siblings and there exists a τv walk in $\mathscr{G}$, then τ is an ancestors of v and v is a descendant of τ.

For a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and a particular edge $\varepsilon := (v_i, v_j) \in \mathscr{E} \cup \{\varepsilon\}$, where $v_i, v_j \in \mathscr{V}$, the *edge-index* of ε is defined as a $|\mathscr{V}| \times |\mathscr{V}|$ matrix whose only non-zero element is 1, which is located at its $j-$th row and $i-$th column. This matrix is represented by $\Gamma(\varepsilon) = \Gamma((v_i, v_j))$. Similarly, the *vertex-index* of any $v_i \in \mathscr{V}$ is defined as a $|\mathscr{V}| \times 1$ column vector whose only non-zero element is 1, which is located at its i-th row. This vector is denoted by $\sigma(v_i)$. The adjacency matrix of $\mathscr{G}$ is given by $A(\mathscr{G}) = \sum_{\varepsilon \in \mathscr{E}} \Gamma(\varepsilon)$, its *degree matrix* is defined as $\Delta(\mathscr{G}) = \sum_{v \in \mathscr{V}} d^{-}_{\mathscr{G}}\{v\}\Gamma((v,v))$, and the corresponding in-degree graph Laplacian is given by $\mathscr{L}(\mathscr{G}) = \Delta(\mathscr{G}) - A(\mathscr{G})$. The following lemma is a standard result in algebraic graph theory [57, 58], and it relates the powers of the adjacency matrix to the length and number of walks in the digraph.

Lemma 2. *Consider a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, an integer $k \in \mathbb{N}$ and vertices $\{v_i, v_j\} \subset \mathscr{V}$, $\{i, j\} \subset \mathbb{N}_{|\mathscr{V}|}$, the $k-$th connectivity of v_j to v_i is given by:*

$$\mathrm{w}_k(v_j, v_i) = \left[A(\mathscr{G})^k\right]_{ij}. \tag{2.3}$$

Given an integer $k \in \mathbb{N}_{|\mathscr{V}|}$ and two vertices $\{v_i, v_j\} \subset \mathscr{V}$, $|\{v_i, v_j\}| = 2$, the $k-$th order topological connectivity of v_j to v_i can be expressed in terms of $[\mathscr{L}(\mathscr{G})]_{mn} = L_{mn}, \{m,n\} \subset \mathbb{N}_{|\mathscr{V}|}$, as follows:

$$\begin{aligned} &\mathrm{c}_k(v_j, v_i) = (-1)^k \times \\ &\sum_{\Theta \in \Pi_{k-1}} \left(\sum_{\psi \in \Xi_{k-1}} L_{i\theta_{\psi(1)}} \left(\prod_{l=1}^{k-2} L_{\theta_{\psi(l)}\theta_{\psi(l+1)}} \right) L_{\theta_{\psi(k-1)}j} \right), \end{aligned} \tag{2.4}$$

where Ξ_{k-1} is the finite group formed by the $(k-1)!$ permutations on the set $\mathbb{N}_{k-1}$, $\Theta = \{\theta_1, \theta_2, \ldots, \theta_{k-1}\} \subset \mathbb{N}_{|\mathscr{V}|} \backslash \{i, j\}$, and $\Pi_{k-1} = \{\Theta; \Theta \subset \mathbb{N}_{|\mathscr{V}|} \backslash \{i, j\} \wedge |\Theta| = k-1\}$, so that:

$$|\Pi_{k-1}| = \frac{(|\mathscr{V}| - 2)!}{(k-1)!(|\mathscr{V}| - k - 1)!}. \tag{2.5}$$

For a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, a vertex $\tau \in \mathscr{V}$ is called an *out-branching root* if there exists a $\tau\nu$ path for every $\nu \in \mathscr{V} \backslash \{\tau\}$. Furthermore, a digraph $\tilde{\mathscr{G}} = (\mathscr{V}, \tilde{\mathscr{E}}), \tilde{\mathscr{E}} \subseteq \mathscr{E}$ with an out-branching root $\tilde{\tau} \in \mathscr{V}$ is a $\tilde{\tau}$-*rooted spanning out-branching* of $\mathscr{G}$ if $\tilde{\mathscr{G}}$ does not contain any directed circle and $\bar{\tau} \neq \hat{\tau} \longrightarrow \bar{\nu} \neq \hat{\nu}$ for any $\{(\bar{\tau}, \bar{\nu}), (\hat{\tau}, \hat{\nu})\} \subset \tilde{\mathscr{E}}$.

Remark 1. *For a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and two vertices $\{\nu_o, \nu_k\} \subset \mathscr{V}$, if ν_o is an out-branching root and there exists a $\nu_k \nu_o$ path in $\mathscr{G}$, then ν_k is also an out-branching root of $\mathscr{G}$.*

The next lemma is used in the proof of one of the main results in Subsection 3.1.1.

Lemma 3. *For a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and a vertex $\nu_o \in \mathscr{V}$, if ν_o is an out-branching root, then the number of ν_o-rooted spanning out-branchings of $\bar{\mathscr{G}} = (\mathscr{V}, \mathscr{E} \cup \{\varepsilon\}), \varepsilon := (\tau, \nu) \notin \mathscr{E}, \nu \neq \nu_o$, is more than the number of ν_o-rooted spanning out-branchings of $\mathscr{G}$.*

Proof: Since every ν_o-rooted spanning out-branching of $\mathscr{G}$ is also that of $\bar{\mathscr{G}}$, it suffices to show that there exists a ν_o-rooted spanning out-branching of $\bar{\mathscr{G}}$ that includes the edge ε. Let $\tilde{\mathscr{G}} = (\mathscr{V}, \tilde{\mathscr{E}} \subseteq \mathscr{E})$ be a ν_o-rooted spanning out-branching of $\mathscr{G}$. The digraph $\tilde{\mathscr{G}}$ contains a unique $\nu_o \tau$ path denoted by $\mathscr{P}_1$, and a unique $\nu_o \nu$ path denoted by $\mathscr{P}_2$. Let $\hat{\varepsilon}$ be the last edge in the sequence of edges representing $\mathscr{P}_2$, and note that ν is the head of $\hat{\varepsilon}$. For all $\tilde{\nu} \in \mathscr{V} \backslash \{\nu_o\}$, $\tilde{\mathscr{G}}$ contains a unique $\nu_o \tilde{\nu}$ path denoted by $\tilde{\mathscr{P}}$. Next, let $\hat{\mathscr{G}} = (\mathscr{V}, \tilde{\mathscr{E}} \cup \{\varepsilon\} \backslash \{\hat{\varepsilon}\})$ and note that if ν does not belong to $\tilde{\mathscr{P}}$, then $\tilde{\mathscr{P}}$ is a unique $\nu_o \tilde{\nu}$ path in $\hat{\mathscr{G}}$. On the other hand, if ν belongs to $\tilde{\mathscr{P}}$, then one can write $\tilde{\mathscr{P}} = \mathscr{P}_2 \mathscr{Q}$, where $\mathscr{Q}$ is a unique $\nu \tilde{\nu}$ path in $\tilde{\mathscr{G}}$.

The proof follows upon noting that $\hat{\mathscr{P}} = \mathscr{P}_1 \varepsilon \mathscr{Q}$ is a unique $v_o\tilde{v}$ path in $\hat{\mathscr{G}}$, and hence $\hat{\mathscr{G}}$ is a v_o-rooted spanning out-branchings of $\bar{\mathscr{G}}$ that includes the edge ε. ∎

The following lemma is a well-known generalization of the Matrix Tree Theorem for the case of directed graph and it is due to Tutte [59].

Lemma 4. *For a given digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and a vertex $v_i \in \mathscr{V}$, the number of v_i-rooted spanning out-branchings of $\mathscr{G}$ is equal to any cofactor in the $i-th$ row of $\mathscr{L}(\mathscr{G})$.*

2.2 Multi-Agent Systems under the Agreement Protocol

Consider a multi-agent system comprised of a set $\mathscr{S} = \{x_1, x_2, \ldots, x_n\}$ of n single integrator agents, where x_i, $i \in \mathbb{N}_n$ is the state of agent i, which is assumed to be scalar. Here, $x_i(t)$ is set to be its absolute position w.r.t. an inertial reference frame, and the agent dynamics is assumed to be decoupled along each axis of the frame. Further, assume that the control input of each agent is constructed according to the following nearest neighbor law:

$$\dot{x}_i(t) = \sum_{j,(v_j,v_i)\in\partial^-_{\mathscr{G}}\{v_i\}} (x_j(t) - x_i(t)),\ t > 0,\ i \in \mathbb{N}_n. \tag{2.6}$$

The interaction structure between the agents in (2.6) can be described by a directed information flow graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where each vertex $v \in \mathscr{V}$ corresponds to an agent $x \in \mathscr{S}$ and $|\mathscr{V}| = n$. A directed edge from vertex v_k to vertex v_i implies that the term $x_k(t) - x_i(t)$ appears in the control law of the agent x_i given by (2.6). As discussed in Section 3.2 of [1], the existence of an out-branching root $\tau \in \mathscr{V}$ is a necessary and sufficient condition for the states in (2.6) to converge to a common value. This correspondence is often referred to as the *agreement protocol*. Agreement protocol has been extensively investigated in the recent literature as a fundamental evolution law for multi-agent networks in both continuous and discrete-time, using probabilistic and deterministic models [12, 60–64], while some earlier results on adjacency-based agreement rules can be traced back to Vicsek's model [65].

For a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, the in-degree graph Laplacian can be used to represent the dynamic equations in (2.6) in a matrix form as follows:

$$\dot{\mathbf{x}}(t) = -\mathscr{L}(\mathscr{G})\mathbf{x}(t),\ t > 0, \tag{2.7}$$

where $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$. The matrix exponential solution to (2.7) is then expressed as:

$$\mathbf{x}(t) = e^{-\mathscr{L}(\mathscr{G})t}\mathbf{x}(0),\ t > 0, \tag{2.8}$$

and for a particular agent $x_i \in \mathscr{S}$ represented by the vertex $v_i \in \mathscr{V}$, the temporal evolution of its state is given by:

$$x_i(t) = \sigma(v_i)^T e^{-\mathscr{L}(\mathscr{G})t}\mathbf{x}(0),\ t > 0. \tag{2.9}$$

2.3 Structural Controllability of Multi-Agent Systems

Consider a team of n single integrator agents given by:

$$\dot{x}_i(t) = u_i(t),\ i \in \mathbb{N}_n, \tag{2.10}$$

where the first $n-m$ agents are followers, and the last m agents are leaders, with the following control inputs:

$$u_i(t) = \begin{cases} u_{ext}^i(t), & i \in \mathbb{N}_n \backslash \mathbb{N}_{n-m} \quad (2.11a) \\ \sum\limits_{j \in \mathbb{N}_n} \alpha_{ij} x_j(t), & i \in \mathbb{N}_{n-m} \quad (2.11b) \end{cases}$$

where $\alpha_{ij} \in \mathbb{R}$ and $\alpha_{ii} \neq 0$ in (2.11b). Note that the leaders are influenced by external control inputs, whereas the followers are governed by a control law which is the linear combination of the states of neighboring agents as given by (2.11b). The interaction structure between the agents in (2.10) can be described by a directed information flow graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where each vertex represents an agent, and a directed edge from vertex v_j to vertex v_i indicates that $x_j(t)$ is transmitted to agent i and $\alpha_{ij} \neq 0$ in (2.11b). Moreover, the condition $\alpha_{ii} \neq 0$ in (2.11b) implies the existence of a self-loop on each follower vertex of $\mathscr{G}$; however, the self-loops are omitted to simplify the graphical representations. In a digraph representing a leader-follower multi-agent system, the root-set $\mathscr{R}$ consists of all leaders, with $|\mathscr{R}| = m$. The state of each agent $x_i(t)$ is its absolute position w.r.t. an inertial reference frame, and the agent dynamics is assumed to be decoupled along each axis of the frame.

Remark 2. *Consider a leader-follower multi-agent system represented by the information* *flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R}$. *The control laws in* (2.11) *imply that no edges enter the root-set, i.e.* $\partial^-_{\mathscr{G}}\mathscr{R} = \varnothing$.

Definition 1. *The information flow digraph* $\mathscr{G}$ *corresponding to the leader-follower multi-agent system* (2.10) *is called controllable if the non-zero coefficients* α_{ij} *in* (2.11b) *can be chosen such that by properly moving the leaders, the followers would assume any desired configuration in an arbitrary time* $T > 0$.

The above definition of controllability, where the choices of non-zero parameters are left free is closely related to the study of controllability for linear structured systems [66–70]. The following theorem from [52] provides a necessary and sufficient condition for the controllability of an information flow digraph as defined above.

Theorem 1. *The information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R} \subset \mathscr{V}$ *is controllable if and only if every vertex* $v \in \mathscr{V} \backslash \mathscr{R}$ *is reachable from the root-set* $\mathscr{R}$.

The next subsection summarizes the main results of [50] and [52], upon which Chapter 4 expands.

2.3.1 Link and Agent Controllability Degrees

Link and agent controllability degrees provide quantitative insight into the reliability of a leader-follower multi-agent system in the face of agent and link failure, as investigated in [50] and [52] for a single leader and multiple leaders, respectively. A conceptually related issue is the fault tolerance of networks and the connectivity of their interconnection digraphs, as discussed in Section 1.5 of [55].

In Section 1.7 of [53], the results obtained by using the max-flow min-cut theorem for a single source and a single sink are extended to the flow networks with multiple sources and sinks by adding two new nodes [53]. The work [52] exploits a similar technique to extend the results of [50] to a digraph $\mathscr{G}$ with multiple leaders designated as the root-set $\mathscr{R}$, by using the expansion of $\mathscr{G}$ w.r.t. $\mathscr{R}$, defined bellow.

Definition 2. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, the expansion of $\mathscr{G}$ w.r.t. $\mathscr{R}$ is denoted by $\mathscr{G}'$, and is defined as $\mathscr{G}' = (\mathscr{V}', \mathscr{E}')$, where for a given vertex $r \notin \mathscr{V}$, $\mathscr{V}' = \{r\} \cup \mathscr{V}$ and $\mathscr{E}' = \{(r, v); v \in \mathscr{R}\} \cup \mathscr{E}$.*

The link controllability degree of an information flow digraph is defined as follows [52].

Definition 3. *An information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be $p-$*link controllable *if p is the largest number such that the controllability of the digraph is preserved after removing any group of at most $p-1$ edges. Moreover, a minimal set of p edges, whose removal makes $\mathscr{G}$ uncontrollable is referred to as a* critical link-set *and is denoted by $\mathscr{C}_p \subset \mathscr{E}$ and a link is said to be* critical *if it belongs to a critical link-set and* uncritical *otherwise. The number p is referred to as the* link controllability degree *of the digraph $\mathscr{G}$ w.r.t. the root-set $\mathscr{R}$, and is denoted by $lc(\mathscr{G};\mathscr{R})$. Moreover, for any $v \in \mathscr{V} \backslash \mathscr{R}$, the minimum number of edges of $\mathscr{G}$ whose removal makes the vertex v unreachable from the set $\mathscr{R}$ is denoted by $lc(\mathscr{G}, v; \mathscr{R})$.*

The following theorem from [52] provides a necessary and sufficient condition for the $p-$link controllability of an information flow digraph.

Theorem 2. *The information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is $p-$link controllable if and only if*

$$\min_{\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}} d^{+}_{\mathscr{G}} \mathscr{X} = p. \tag{2.12}$$

The agent controllability degree of an information flow digraph is defined as follows [52].

Definition 4. *An information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is said to be $q-$*agent controllable *if q is the largest number such that the controllability of the digraph is preserved after removing any group of at most $q-1$ non-root vertices. Moreover, a minimal set of q non-root vertices whose removal makes $\mathscr{G}$ uncontrollable is referred to as a* critical agent-set*, and is denoted by $\mathscr{C}_q \subseteq \mathscr{V} \backslash \mathscr{R}$ and an agent is said to be* critical *if it belongs to a critical agent-set and* uncritical *otherwise. The number q is referred to as the* agent controllability degree *of the digraph $\mathscr{G}$ w.r.t. the root-set $\mathscr{R}$ and is denoted by $ac(\mathscr{G};\mathscr{R})$. Furthermore, for any $\nu \in \mathscr{V} \backslash \mathscr{R}$, the minimum number of non-root vertices of $\mathscr{G}$ whose removal makes the vertex ν unreachable from the root-set $\mathscr{R}$ is denoted by $ac(\mathscr{G}, \nu; \mathscr{R})$.*

Remark 3. *An information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is not controllable if and only if $ac(\mathscr{G};\mathscr{R}) = lc(\mathscr{G};\mathscr{R}) = 0$.*

In Section 1.11 of [53], a technique involving duplication of nodes in a digraph is used to extend the results of the problem of finding a maximal flow from one set of nodes to another, to the information flow digraphs subject to both arc and node capacity bounds. The corresponding technique employs the max-flow min-cut theorem with some constraints on the maximum arc flow. The work [50] exploits a similar technique termed node-duplication to relate the link and agent controllability degrees of a given information flow digraph. The node-duplicated version of an information flow digraph $\mathscr{G}$ is defined as follows.

Definition 5. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, replace every non-root vertex $\nu \in \mathscr{V} \backslash \mathscr{R}$ with two vertices $\tilde{\nu}_1$ and $\tilde{\nu}_2$, which are connected together by an intermediate edge $\tilde{\varepsilon}_\nu = (\tilde{\nu}_1, \tilde{\nu}_2)$. The resulting digraph $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}}, \tilde{\mathscr{E}})$ is called the* node-duplicate *of $\mathscr{G}$. All edges in $\mathscr{G}$, whose heads are ν, have $\tilde{\nu}_1$ as their heads in the resultant digraph $\tilde{\mathscr{G}}$, and all edges of $\mathscr{G}$, whose tails are ν, have $\tilde{\nu}_2$ as their tails in $\tilde{\mathscr{G}}$.*

Remark 4. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, its node-duplicate $\tilde{\mathscr{G}} = (\tilde{\mathscr{V}}, \tilde{\mathscr{E}})$ does not possess any anti-parallel edges, i.e. for any $\tilde{\tau}, \tilde{\nu} \in \tilde{\mathscr{V}}$, if $(\tilde{\tau}, \tilde{\nu}) \in \tilde{\mathscr{E}}$, then $(\tilde{\nu}, \tilde{\tau}) \notin \tilde{\mathscr{E}}$.*

The following lemma from [50] describes the relationship between the agent controllability degree of a digraph $\mathscr{G}$ and the link controllability degree of its node-duplicate $\tilde{\mathscr{G}}$.

Lemma 5. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$, let its node-duplicate be denoted by $\tilde{\mathscr{G}}$. For all $\nu \in \mathscr{V} \backslash \mathscr{R}$, if $\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{\nu\} = \varnothing$, then $ac(\mathscr{G}, \nu; \mathscr{R}) = lc(\tilde{\mathscr{G}}, \tilde{\nu}_1; \mathscr{R})$.*

Remark 5. *Let $\mathscr{G}$, $\mathscr{V}$ and $\mathscr{R}$ be given as in Lemma 5. For all $\nu \in \mathscr{V} \backslash \mathscr{R}$, if $\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{\nu\} \neq \varnothing$, then the follower agent ν remains reachable from the root-set $\mathscr{R}$ after the removal of any set* *of follower agents that does not include* v*. For such an agent* v*, the relation* $ac(\mathscr{G},\mathrm{v};\mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$ *holds.*

The following theorem from [50] provides lower bounds on the number of edges in a $p-$link or $q-$agent controllable digraph.

Theorem 3. *For an information flow digraph* $\mathscr{G} = (\mathscr{V},\mathscr{E})$*, the following statements are true:* (a) *If* $\mathscr{G}$ *is* $p-$*link controllable, then* $|\mathscr{E}| \geqslant (|\mathscr{V}|-1)p$*.* (b) *If* $\mathscr{G}$ *is* $q-$*agent controllable, then* $|\mathscr{E}| \geqslant |\mathscr{V}|+q-2$.

Chapter 3

Digraphs with Distinguishable Dynamics

In this chapter, the ability to distinguish digraphs from the output response of a single observing agent in a multi-agent network under the agreement protocol has been studied. Given a fixed observation point, it is desired to find sufficient graphical conditions under which the failure of a set of edges in the network information flow digraph is distinguishable from another set. When the latter is empty, this corresponds to the detectability of the former link set with respect to the response of the observing agent. In developing the results, a powerful extension of the all-minors matrix tree theorem in algebraic graph theory is proved which relates the minors of the transformed Laplacian of a directed graph to the number and length of the shortest paths between its vertices. The results reveal an intricate relationship between the ability to distinguish between the responses of a healthy and a faulty multi-agent network and the inter-nodal paths in their information flow digraphs. Simulation studies at the end of the chapter reveal that the obtained analytical results hold for randomly initialized network configurations. The results have direct implications for the operation and design of multi-agent systems subject to multiple link losses.

The remainder of this chapter is organized as follows. The background provided in Chapter 2 is used in Section 3.1 to develop the main results. Multiple link failures are characterized through the concept of distinguishable digraphs and sufficient conditions are provided. The detectability of link failures is then considered as a special case. In Section 3.2, the results are illustrated and discussed using sample graphs, and finally concluding remarks are provided in Section 3.3.

3.1 Distinguishable Dynamics with the Agreement Protocol

Throughout this chapter, the temporal evolution of the state of the agents is assumed to be given by (2.6). The notions of failed response and distinguishable digraphs are formally defined next, and they provide a mathematical characterization of link failures according to the state of an agent after the failure of a certain set of links in the information flow structure [71].

Definition 6. *Given a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, the failed response of an agent $x \in \mathscr{S}$ w.r.t. a given edge $\varepsilon \in \mathscr{E}$ is denoted by $x^{\varepsilon}(t)$, and it is defined as the state of the agent x calculated in the digraph $\bar{\mathscr{G}} = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$ for some set of initial values $\mathbf{x}(0) \in \mathbb{R}^{|\mathscr{V}|}$.*

The notion of failed response will be used later to represent the state of an agent after the failure of a certain link in the information flow structure.

Definition 7. *Consider a multi-agent system $\mathscr{S}$ and two distinct digraphs $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E}_2)$ associated with it. $\mathscr{G}_1$ and $\mathscr{G}_2$ are said to be distinguishable from an agent $x \in \mathscr{S}$ if there exists $\mathbf{x}(0) \in \mathbb{R}^{|\mathscr{V}|}$ such that $x^{\mathscr{G}_1}(t) - x^{\mathscr{G}_2}(t) \not\equiv 0$, where $x^{\mathscr{G}_1}(t)$ and $x^{\mathscr{G}_2}(t)$ denote the state of the agent x calculated in the digraphs $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively, with the same initial condition $\mathbf{x}(0)$.*

Remark 6. *Definition 7 is motivated by the fact that different link failures in the original digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}_1 \cup \mathscr{E}_2)$ can lead to distinct digraphs $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E}_2)$ which share the same set of vertices $\mathscr{V}$ but have different sets of edges $\mathscr{E}_1$ and $\mathscr{E}_2$.*

Next, an extension of the all-minors matrix tree theorem that considers the Laplace-transformed Laplacian of a digraph is proved and is subsequently used to provide sufficient conditions for designating two link sets as distinguishable from a given agent.

Lemma 6. *Given a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, let $H = sI + \mathscr{L}(\mathscr{G})$, $s \in \mathbb{C}$. For any $i, j \in \mathbb{N}_{|\mathscr{V}|}$ and $i \neq j$, define C_{ij} as the matrix that results from removing the $i-$th row and $j-$th column of H. Then for all $k \in \mathbb{N}_{\mathrm{d}(v_i, v_j)}$, the following limit holds:*

$$\lim_{s \to +\infty} \left| \frac{\det(C_{ij})}{s^{|\mathscr{V}|-1-k}} \right| = \mathrm{c}_k(v_i, v_j). \tag{3.1}$$

Proof: $\det(C_{ij})$ can be written as a summation over the product of all possible permutations on the choice of $|\mathscr{V}|-1$ matrix elements [72], as given below:

$$\det(C_{ij}) = \sum_{\psi\in\Xi_{|\mathscr{V}|-1}} (-1)^{\iota(\psi)} \prod_{k=1}^{|\mathscr{V}|-1} \left[C_{ij}\right]_{k\psi(k)}, \tag{3.2}$$

where $\Xi_{|\mathscr{V}|-1}$ is the finite group formed by the $(|\mathscr{V}|-1)!$ permutations on the set $\mathbb{N}_{|\mathscr{V}|-1}$. Moreover, for a permutation $\psi(.)$ on the set $\mathbb{N}_{|\mathscr{V}|-1}$, $\iota(\psi)$ denotes its length and is defined as the number of pairs $i, j \in \mathbb{N}_{|\mathscr{V}|-1}$, $i<j$, such that $\psi(i)>\psi(j)$. Consider the case where $\mathrm{d}(v_i,v_j)=1$, which implies that there exists an edge $(v_i,v_j)\in\mathscr{E}$. Hence, $[\mathscr{L}(\mathscr{G})]_{ji}=-1$ and $\det(C_{ij})$ is a polynomial of degree $|\mathscr{V}|-2$ in s. Moreover, the only term in $\det(C_{ij})$ which includes $s^{|\mathscr{V}|-2}$ in the right-hand side of (3.2) is the one corresponding to the elements $[H]_{rr}, r\in\mathbb{N}_{|\mathscr{V}|}\backslash\{i,j\}$ and $[H]_{ji}=[\mathscr{L}(\mathscr{G})]_{ji}$. Accordingly, (3.2) can be rewritten as:

$$\begin{aligned}\det(C_{ij}) &= (-1)^{\kappa}[\mathscr{L}(\mathscr{G})]_{ji}\left(\prod_{v\in\mathscr{V}\backslash\{v_i,v_j\}} (s+d^{-}_{\mathscr{G}}\{v\})\right)+w_1(s)\dots\\ &= -(-1)^{\kappa}s^{|\mathscr{V}|-2}+w_2(s),\end{aligned} \tag{3.3}$$

for some $\{w_1(s),w_2(s)\}\subset\mathscr{W}_{|\mathscr{V}|-3}(s)$, where $\mathscr{W}_{|\mathscr{V}|-3}(s)$ denotes the set of all polynomials in s with degrees less than or equal to $|\mathscr{V}|-3$ and a permutation length $\kappa\in\mathbb{N}$ that depends on i, j and $|\mathscr{V}|$. This corroborates (3.2) for $\mathrm{d}(v_i,v_j)=1$. Next, let $[\mathscr{L}(\mathscr{G})]_{mn}=L_{mn}$, for $\{m,n\}\subset\mathbb{N}_{|\mathscr{V}|}$, and consider the expression for $\det(C_{ij})$ in the general case of $\mathrm{d}(v_i,v_j)\in\mathbb{N}_{|\mathscr{V}|-1}\backslash\{1\}$.

It follows from the inequality $\mathrm{d}(v_i, v_j) > 1$ that $[\mathscr{L}(\mathscr{G})]_{ji} = 0$ and (3.2) can be expanded as:

$$\begin{aligned}\det(C_{ij}) =& (-1)^{\kappa_2} \sum_{r=1, r\neq i,j}^{|\mathscr{V}|} \left(L_{jr} L_{ri} \prod_{k=1, k\neq i,j,r}^{|\mathscr{V}|} (s + d_{\mathscr{G}}^{-}\{v_k\}) \right) + \\ & (-1)^{\kappa_3} \sum_{r=1, r\neq i,j}^{|\mathscr{V}|} \sum_{u=1, u\neq i,j,r}^{|\mathscr{V}|} \left(L_{jr} L_{ru} L_{ui} \prod_{k=1, k\neq i,j,r,u}^{|\mathscr{V}|} (s + d_{\mathscr{G}}^{-}\{v_k\}) \right) + \dots \end{aligned} \tag{3.4}$$

for some κ_l, $l \in \mathbb{N}_{|\mathscr{V}|-2} \backslash \{1\}$, which depend on the indices i and j and vary with the length of the permutations. To express (3.3) in a more compact form, let $\Theta = \{\theta_1, \theta_2, \dots, \theta_{|\Theta|}\}$ and $\Pi = \{\Theta; \Theta \subseteq \mathbb{N}_{|\mathscr{V}|} \backslash \{i, j\} \wedge \Theta \neq \varnothing\}$, so that $|\Pi| = 2^{|\mathscr{V}|-2} - 1$. Let also Ξ_Θ be the finite group formed by the $(|\Theta|)!$ permutations on the set Θ. Furthermore, for $\Theta \in \Pi$ and $\psi \in \Xi_\Theta$ define:

$$S(\Theta) = \prod_{k \in \mathbb{N}_{|\mathscr{V}|} \backslash (\{i,j\} \cup \Theta)} (s + d_{\mathscr{G}}^{-}\{v_k\}), \tag{3.5a}$$

$$P(\Theta, \psi) = L_{j\theta_{\psi(1)}} \left(\prod_{l=1}^{|\Theta|-1} L_{\theta_{\psi(l)}\theta_{\psi(l+1)}} \right) L_{\theta_{\psi(|\Theta|)}i}. \tag{3.5b}$$

Using (3.5), for $\mathrm{d}(v_i, v_j) > 1$, (3.4) can be rewritten as:

$$\det(C_{ij}) = \sum_{\Theta \in \Pi} (-1)^{\kappa_{|\Theta|}} \left(\sum_{\psi \in \Xi_\Theta} P(\Theta, \psi) \right) S(\Theta), \tag{3.6}$$

for some $\kappa_{|\Theta|} \in \mathbb{N}$ that depend on i and j. Breaking the first summation over $|\Theta| = k$ for

$k = 1, \ldots, |\mathscr{V}| - 2$ and using the notation $\Pi_k = \{\Theta; \Theta \subset \mathbb{N}_{|\mathscr{V}|} \backslash \{i, j\} \wedge |\Theta| = k\}$ in (3.6) yields:

$$\det(C_{ij}) = \sum_{k=1}^{|\mathscr{V}|-2} (-1)^{\kappa_k} \sum_{\Theta \in \Pi_k} \left(\sum_{\psi \in \Xi_k} P(\Theta, \psi) \right) S(\Theta), \tag{3.7}$$

where $\kappa_k \in \mathbb{N}$ depend on i and j. Next, it follows from (2.4) and (3.5b) that:

$$\mathrm{c}_k(v_i, v_j) = (-1)^k \sum_{\Theta \in \Pi_{k-1}} \left(\sum_{\psi \in \Xi_{k-1}} P(\Theta, \psi) \right). \tag{3.8}$$

The fact that $S(\Theta)$ is a polynomial of degree $|\mathscr{V}| - |\Theta| - 2$ in s with leading coefficient 1, together with (3.3), (3.7) and (3.8), leads to:

$$\det(C_{ij}) = K + \sum_{k=1}^{|\mathscr{V}|-2} (-1)^{\kappa_k} \mathrm{c}_k(v_i, v_j) w_{|\mathscr{V}|-k-1}(s), \tag{3.9}$$

where $w_l(s)$, $l \in \mathbb{N}_{|\mathscr{V}|-2}$, are degree-$l$ polynomials in s whose leading coefficients are unity, $K \in \mathbb{N}$ is a constant, and $\kappa_k \in \mathbb{N}$ depend on i and j. The proof follows immediately from (3.9) and upon noting that $\mathrm{c}_k(v_i, v_j) = 0$ for $k < \mathrm{d}(v_i, v_j)$. ∎

Remark 7. *Lemma 6 is a significant extension to the all-minors matrix tree theorem in algebraic graph theory [73]. This lemma, together with* (3.9)*, relates the minors of the Laplace-transformed graph Laplacian to the inter-nodal paths and distances in the digraph. According to Lemma 6, for* $\mathrm{d}(v_i, v_j) \in \mathbb{N}_{|\mathscr{V}|-2}$*,* $\det(C_{ij})$ *is a polynomial of degree* $|\mathscr{V}| - \mathrm{d}(v_i, v_j) - 1$ *in* s *whose leading coefficient has an absolute value equal to* $\mathrm{c}_{\mathrm{d}(v_i, v_j)}(v_i, v_j)$.

Theorem 4. *Given a multi-agent system $\mathscr{S}$ and two distinct digraphs $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E}_2)$ associated with it, consider a vertex $v_i \in \mathscr{V}$ corresponding to agent $x_i \in \mathscr{S}$. If $\mathscr{G}_1$ and $\mathscr{G}_2$ are not distinguishable from x_i, then $\mathrm{d}_1(v, v_i) = \mathrm{d}_2(v, v_i)$ for all $v \in \mathscr{V} \backslash \{v_i\}$, and $\mathrm{c}_k^1(v, v_i) = \mathrm{c}_k^2(v, v_i)$ for $k = \mathrm{d}_1(v, v_i) = \mathrm{d}_2(v, v_i)$, where $\mathrm{c}_k^l(v, v_i)$ and $\mathrm{d}_l(v, v_i)$ denote $\mathrm{c}_k(v, v_i)$ and $\mathrm{d}(v, v_i)$, respectively, for digraph $\mathscr{G}_l$, $l = 1, 2$.*

Proof: If $\mathscr{G}_1$ and $\mathscr{G}_2$ are not distinguishable from x_i, then $x^{\mathscr{G}_1}(t) - x^{\mathscr{G}_2}(t) \equiv 0$ for any $\mathbf{x}(0) \in \mathbb{R}^{|\mathscr{V}|}$, and it follows from (2.9) that:

$$\sigma(v_i)^T e^{-\mathscr{L}(\mathscr{G}_1)t} = \sigma(v_i)^T e^{-\mathscr{L}(\mathscr{G}_2)t},\ t > 0. \tag{3.10}$$

Let $H_l = sI + \mathscr{L}(\mathscr{G}_l)$, $l = 1, 2$, where $s \in \mathbb{C}$ is the Laplace variable. Taking the Laplace transform of (3.10):

$$\sigma(v_i)^T {H_1}^{-1} = \sigma(v_i)^T {H_2}^{-1}, \tag{3.11}$$

or equivalently:

$$\left[{H_1}^{-1}\right]_{ip} = \left[{H_2}^{-1}\right]_{ip},\ p \in \mathbb{N}_{|\mathscr{V}|}. \tag{3.12}$$

The adjoint of H_l, $l = 1, 2$, can now be used to compute the inverse matrices as follows:

$$\left[{H_l}^{-1}\right]_{nm} = \frac{(-1)^{m+n} \det\left(C_{mn}^l\right)}{\det\left(H_l\right)},\ m, n \in \mathbb{N}_{|\mathscr{V}|},\ l = 1, 2, \tag{3.13}$$

where C_{mn}^l, $l = 1, 2$, is the matrix obtained by removing the $m-$th row and $n-$th column of

$H_l,\ l=1,2$. It follows from (3.12) and (3.13) that:

$$\frac{\det\left(C^1_{ji}\right)}{\det\left(H_1\right)}=\frac{\det\left(C^2_{ji}\right)}{\det\left(H_2\right)},\ j\in\mathbb{N}_{|\mathscr{V}|}, \tag{3.14}$$

or:

$$\det\left(C^1_{ji}\right)\det\left(H_2\right)=\det\left(C^2_{ji}\right)\det\left(H_1\right),\ j\in\mathbb{N}_{|\mathscr{V}|}. \tag{3.15}$$

Next, $\det\left(H_l\right), l=1,2$, in (3.15) can be written as a summation over the product of all possible permutations on $|\mathscr{V}|$ matrix elements [72], as given below:

$$\det\left(H_2\right)=\sum_{\psi\in\Xi_{|\mathscr{V}|}}(-1)^{\iota(\psi)}\prod_{j=1}^{|\mathscr{V}|}\left[H_2\right]_{j\psi(j)}, \tag{3.16a}$$

$$\det\left(H_1\right)=\sum_{\psi\in\Xi_{|\mathscr{V}|}}(-1)^{\iota(\psi)}\prod_{j=1}^{|\mathscr{V}|}\left[H_1\right]_{j\psi(j)}, \tag{3.16b}$$

where $\Xi_{|\mathscr{V}|}$ is the finite group formed by the $(|\mathscr{V}|)!$ permutations on the set $\mathbb{N}_{|\mathscr{V}|}$. Moreover, for a permutation $\psi(.)$ on the set $\mathbb{N}_{|\mathscr{V}|}$, $\iota(\psi)$ denotes its length. Note that the only term in $\det\left(H_l\right)$, $l=1,2$, which includes $s^{|\mathbb{V}|}$ (and $s^{|\mathbb{V}|-1}$) in the right-hand sides of (3.16) is the one corresponding to the diagonal elements of H_l, and is given by:

$$\prod_{k=1}^{|\mathscr{V}|}\left[H_l\right]_{kk}=\prod_{v\in\mathscr{V}}\left(s+d^-_{\mathscr{G}_l}\{v\}\right)=s^{|\mathscr{V}|}+\left(\sum_{v\in\mathscr{V}}d^-_{\mathscr{G}_l}\{v\}\right)s^{|\mathscr{V}|-1}+\ldots+\prod_{v\in\mathscr{V}}d^-_{\mathscr{G}_l}\{v\}. \tag{3.17}$$

The power of s in any other term in the right-hand sides of (3.16) is less than $|\mathbb{V}|-1$. The proof follows now from Lemma 6 and upon equating the leading coefficients and the polynomial degrees in the left-hand side and right-hand side of (3.15) for $j \in \mathbb{N}_{|\mathscr{V}|} \backslash \{i\}$. ■

The following proposition demonstrates that having different in-degrees for the vertex corresponding to the observing agent is a sufficient condition for two digraphs to be distinguishable.

Proposition 1. *Given a multi-agent system $\mathscr{S}$, an agent $x \in \mathscr{S}$, and two distinct digraphs $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E}_2)$ that are both associated with $\mathscr{S}$, let $v_i \in \mathscr{V}$ be the vertex that corresponds to x, for some $i \in \mathbb{N}_{|\mathscr{V}|}$. If $\mathscr{G}_1$ and $\mathscr{G}_2$ are not distinguishable from x, then $d^-_{\mathscr{G}_1}\{v_i\} = d^-_{\mathscr{G}_2}\{v_i\}$.*

Proof: Let H_l and C^l_{mn}, $l = 1, 2$, $\{m, n\} \subset \mathbb{N}_{|\mathscr{V}|}$, be defined as in the proof of Theorem 4. Rewriting (3.15) with $j = i$ yields:

$$\det\left(C^1_{ii}\right)\det\left(H_2\right) = \det\left(C^2_{ii}\right)\det\left(H_1\right). \tag{3.18}$$

Similarly to (3.16), $\det\left(C^l_{ii}\right), l = 1, 2$ can be written as a summation over the product of all possible permutations on $|\mathscr{V}| - 1$ matrix elements [72], as given below:

$$\det\left(C^1_{ii}\right) = \sum_{\psi \in \Xi_{|\mathscr{V}|-1}} (-1)^{\iota(\psi)} \prod_{j=1}^{|\mathscr{V}|-1} \left[C^1_{ii}\right]_{j\psi(j)}, \tag{3.19a}$$

$$\det\left(C^2_{ii}\right) = \sum_{\psi \in \Xi_{|\mathscr{V}|-1}} (-1)^{\iota(\psi)} \prod_{j=1}^{|\mathscr{V}|-1} \left[C^2_{ii}\right]_{j\psi(j)}, \tag{3.19b}$$

where $\Xi_{|V|-1}$ is the finite group formed by the $(|V|-1)!$ permutations $\psi(.)$ on the set $\mathbb{N}_{|V|-1}$ and for a permutation $\psi \in \Xi_{|\mathscr{V}|-1}$, $\iota(\psi)$ denotes its length. Note that the only term in $\det\left(C_{ii}^{1}\right)$ which includes $s^{|\mathbb{V}|-1}$ or $s^{|\mathbb{V}|-2}$ in the right-hand side of (3.19a) is the one corresponding to the diagonal elements of C_{ii}^{1}, and is given by:

$$\prod_{j=1}^{|\mathscr{V}|-1} \left[C_{ii}^{1}\right]_{jj} = \prod_{v \in \mathscr{V} \setminus \{v_i\}} (s + d_{\mathscr{G}_1}^{-}\{v\}) \dots$$
$$= s^{|\mathscr{V}|-1} + \left(\sum_{v \in \mathscr{V} \setminus \{v_i\}} d_{\mathscr{G}_1}^{-}\{v\} \right) s^{|\mathscr{V}|-2} + \dots + \prod_{v \in \mathscr{V} \setminus \{v_i\}} d_{\mathscr{G}_1}^{-}\{v\}. \tag{3.20}$$

The power of s in any other term in the right-hand side of (3.19a) is less than $|\mathbb{V}|-2$. A similar argument applies to the other three determinants in (3.16) and (3.19). Denote by $\mathscr{W}_n(s)$ the set of all polynomials in s with degrees less than or equal to n, for any $n \in \mathbb{N}$. One can then write:

$$\det\left(C_{ii}^{1}\right) = s^{|\mathscr{V}|-1} + \left(\sum_{v \in \mathscr{V} \setminus \{v_i\}} d_{\mathscr{G}_1}^{-}\{v\} \right) s^{|\mathscr{V}|-2} + w_1(s), \tag{3.21a}$$

$$\det(H_2) = s^{|\mathscr{V}|} + \left(\sum_{v \in \mathscr{V}} d_{\mathscr{G}_2}^{-}\{v\} \right) s^{|\mathscr{V}|-1} + w_2(s), \tag{3.21b}$$

and

$$\det\left(C_{ii}^{2}\right) = s^{|\mathscr{V}|-1} + \left(\sum_{v \in \mathscr{V} \setminus \{v_i\}} d_{\mathscr{G}_2}^{-}\{v\} \right) s^{|\mathscr{V}|-2} + w_3(s), \tag{3.22a}$$

$$\det(H_1) = s^{|\mathscr{V}|} + \left(\sum_{v \in \mathscr{V}} d_{\mathscr{G}_1}^{-}\{v\} \right) s^{|\mathscr{V}|-1} + w_4(s), \tag{3.22b}$$

for some $w_1(s), w_3(s) \in \mathscr{W}_{|\mathscr{V}|-3}(s)$ and $w_2(s), w_4(s) \in \mathscr{W}_{|\mathscr{V}|-2}(s)$. Multiplication of the expressions in (3.21) and (3.22) leads to:

$$\det\left(C_{ii}^{1}\right)\det\left(H_2\right) = s^{2|\mathscr{V}|-1} + \left(\sum_{v\in\mathscr{V}} d_{\mathscr{G}_2}^{-}\{v\} + \sum_{v\in\mathscr{V}\setminus\{v_i\}} d_{\mathscr{G}_1}^{-}\{v\}\right) s^{2|\mathscr{V}|-2} + w_5(s), \quad (3.23)$$

and

$$\det\left(C_{ii}^{2}\right)\det\left(H_1\right) = s^{2|\mathscr{V}|-1} + \left(\sum_{v\in\mathscr{V}} d_{\mathscr{G}_1}^{-}\{v\} + \sum_{v\in\mathscr{V}\setminus\{v_i\}} d_{\mathscr{G}_2}^{-}\{v\}\right) s^{2|\mathscr{V}|-2} + w_6(s), \quad (3.24)$$

respectively, for some $w_5(s), w_6(s) \in \mathscr{W}_{2|\mathscr{V}|-3}(s)$.

The proof follows by substituting (3.23) and (3.24) in (3.15), and upon equating the coefficients of $s^{2|\mathscr{V}|-2}$ in the right-hand sides of (3.23) and (3.24). ■

Remark 8. *The condition in Theorem 4 are derived by equating the degrees and coefficients of s at either side of* (3.15) *for any $j \in \mathbb{N}_{|\mathscr{V}|}\setminus\{i\}$, while the same procedure for $j = i$ leads to the in-degree condition in Proposition 1.*

3.1.1 Jointly Detectable Link Failures

In this section, the notion of distinguishable digraphs from Definition 7 is used to define and characterize the concept of *joint detectability* for a multi-agent system subject to simultaneous link failures. The following definition is motivated by the need to detect the failure of multiple links in the network based on the observed response of an agent [74].

Definition 8. *Given a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, a subset of edges $\mathscr{E}_1 \subset \mathscr{E}$ is said to be jointly detectable from the agent $x \in \mathscr{S}$ if $\mathscr{G}$ and $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \mathscr{E}_1)$ are distinguishable from x.*

The next corollary follows directly from (2.6) and Definition 8.

Corollary 1. *Consider a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, as well as an agent $x \in \mathscr{S}$ with its corresponding vertex $\nu \in \mathscr{V}$. For all $\varepsilon \in \partial^-_{\mathscr{G}}\{\nu\}$, the set $\{\varepsilon\}$ is jointly detectable from the agent x.*

The next proposition exploits Theorem 4 to provide sufficient graphical conditions for joint detectability of a link-set from an observing agent.

Proposition 2. *Given a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E})$, consider a vertex $\nu_i \in \mathscr{V}$ corresponding to agent $x_i \in \mathscr{S}$, and a subset of edges $\mathscr{E}_1 \subset \mathscr{E}$. If there exists an edge $\varepsilon := (\nu_j, \nu_k) \in \mathscr{E}_1$ such that $\mathrm{d}(\nu_j, \nu_i) > \mathrm{d}(\nu_k, \nu_i)$, then $\mathscr{E}_1$ is jointly detectable from x_i.*

Proof: To prove the proposition by contradiction, suppose that $\mathscr{E}_1$ is not jointly detectable from x_i and define $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E} \backslash \mathscr{E}_1)$, such that $\mathscr{G}_1$ and $\mathscr{G}_2$ are not distinguishable from x_i.

It follows from Theorem 4 that for some $\{\hat{d}_1, \hat{d}_2\} \subset \mathbb{N}_{|\mathscr{V}|-2}$:

$$\hat{d}_1 = \mathrm{d}_1(v_k, v_i) = \mathrm{d}_2(v_k, v_i), \tag{3.25a}$$

$$\hat{d}_2 = \mathrm{d}_1(v_j, v_i) = \mathrm{d}_2(v_j, v_i), \tag{3.25b}$$

and

$$\mathrm{c}^1_{\hat{d}_1}(v_k, v_i) = \mathrm{c}^2_{\hat{d}_1}(v_k, v_i), \tag{3.26a}$$

$$\mathrm{c}^1_{\hat{d}_2}(v_j, v_i) = \mathrm{c}^2_{\hat{d}_2}(v_j, v_i), \tag{3.26b}$$

where for $l = 1, 2$ and any $\{\tau, v\} \subset \mathscr{V}$, the numbers $\mathrm{c}^l_k(\tau, v)$ and $\mathrm{d}_l(\tau, v)$ denote $\mathrm{c}_k(\tau, v)$ and $\mathrm{d}(\tau, v)$, respectively, for digraph $\mathscr{G}_l$.

Let $\mathscr{P}$ be a $v_k v_i$ path of length $\hat{d}_1 = \mathrm{d}_2(v_k, v_i)$ in $\mathscr{G}_2$. Since the sequence $\varepsilon\mathscr{P}$ is a $v_j v_i$ path of length $1 + \hat{d}_1$ in $\mathscr{G}_1$, one has that:

$$\mathrm{d}_1(v_j, v_i) \leqslant 1 + \mathrm{d}_2(v_k, v_i), \tag{3.27}$$

On the other hand, the assumption $\mathrm{d}_1(v_k, v_i) < \mathrm{d}_1(v_j, v_i)$, together with (3.25a), leads to:

$$\mathrm{d}_2(v_k, v_i) < \mathrm{d}_1(v_j, v_i). \tag{3.28}$$

The relations (3.27) and (3.28) imply that:

$$\mathrm{d}_1(v_j, v_i) = 1 + \mathrm{d}_2(v_k, v_i), \tag{3.29}$$

Next, it follows from (3.25) and (3.29) that $\hat{d}_2 = 1 + \hat{d}_1$. Also, note that since $\mathscr{E}\backslash\mathscr{E}_1 \subset \mathscr{E}$, every $v_j v_i$ path of length $\hat{d}_2$ in $\mathscr{G}_2$ is a $v_j v_i$ path of length $\hat{d}_2$ in $\mathscr{G}_1$ as well, while the sequence $\varepsilon\mathscr{P}$ is a $v_j v_i$ path of length $\hat{d}_1 + 1 = \hat{d}_2 = \mathrm{d}_1(v_j, v_i)$ in $\mathscr{G}_1$ that does not exist in $\mathscr{G}_2$. Hence, $c^1_{\hat{d}_2}(v_j, v_i) \geqslant 1 + c^2_{\hat{d}_2}(v_j, v_i)$, which is in contradiction with (3.26b). ∎

Remark 9. *Under the conditions specified in Proposition 2, the vertex v_j is located at a greater distance from the observing vertex v_i as compared to the vertex v_k. Hence, the* "information" *concerning the state of v_j will* "reach" *the observing agent* "faster" *when passed through the failed edge (v_j, v_k).*

The following proposition offers a different set of sufficient conditions under which any combination of links sharing the common head vertex v_k is jointly detectable from the agent x_i.

Proposition 3. *Consider a multi-agent system $\mathscr{S}$, its associated digraph $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E})$, an agent $x_i \in \mathscr{S}$ with its corresponding vertex $v_i \in \mathscr{V}$, a vertex $v_k \in \mathscr{V}$, and a subset of edges $\mathscr{E}_2 \subset \partial^-_{\mathscr{G}_1}\{v_k\}$. If there exists an out-branching root $v_o \in \mathscr{V}\backslash\{v_k\}$ as well as a $v_k v_o$ path in $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E}\backslash\mathscr{E}_2)$, then $\mathscr{E}_2$ is jointly detectable from x_i.*

Proof: The result for $k = i$ follows from Corollary 1. For $k \neq i$, suppose that Proposition 3 does not hold, i.e., v_o is an out-branching root of $\mathscr{G}_2$ and there exists a $v_k v_o$ path in $\mathscr{G}_2$, but $x^{\mathscr{G}_1}(t) - x^{\mathscr{G}_2}(t) \equiv 0$ for all $\mathbf{x}(0) \in \mathbb{R}^{|\mathscr{V}|}$, where $x^{\mathscr{G}_1}(t)$ and $x^{\mathscr{G}_2}(t)$ denote the state of the agent x calculated in the digraphs $\mathscr{G}_1$ and $\mathscr{G}_2$, respectively, with the same initial condition $\mathbf{x}(0)$. Let $H_l = sI + \mathscr{L}(\mathscr{G}_l)$, $l = 1,2$, where $s \in \mathbb{C}$ is the Laplace variable. Moreover, denote by C^l_{mn}, $l = 1,2$, the $(|\mathscr{V}|-1) \times (|\mathscr{V}|-1)$ sub-matrix that results from removing the $m-$th row and $n-$th column of H_l. Since $x^{\mathscr{G}_1}(t) - x^{\mathscr{G}_2}(t) \equiv 0$ for all $\mathbf{x}(0) \in \mathbb{R}^{|\mathscr{V}|}$, (3.10) to (3.13) still hold, and it follows from the equality:

$$\mathscr{L}(\mathscr{G}_2) = \mathscr{L}(\mathscr{G}_1) - |\mathscr{E}_2|\Gamma((v_k, v_k)) + \sum_{\varepsilon \in \mathscr{E}_2} \Gamma(\varepsilon), \tag{3.30}$$

that the matrices H_l, $l = 1,2$, will only differ at $[H_l]_{kk}$ and $[H_l]_{kj}$ for all $j \in \{n \in \mathbb{N}_{|\mathscr{V}|}; (v_n, v_k) \in \mathscr{E}_2\}$. Therefore, after removing the $k-$th row from H_l, $l = 1,2$, one has that $C^1_{ki} = C^2_{ki}, s \in \mathbb{C}$, and consequently $\det(C^1_{ki}) \equiv \det(C^2_{ki}) \not\equiv 0$, where the inequality follows from Lemma 4, since $\det(C^2_{ki})$ at $s = 0$ is equal to the number of v_k-rooted spanning out-branchings of $\mathscr{G}_2$. On the other hand, it follows from (3.12) that $\left[{H_1}^{-1}\right]_{ik} \equiv \left[{H_2}^{-1}\right]_{ik}$. Now, substituting from (3.13) and using $\det(C^1_{ki}) \equiv \det(C^2_{ki}) \not\equiv 0$, one arrives at the relation $\det\left(sI + \mathscr{L}(\mathscr{G}_1)\right) \equiv \det\left(sI + \mathscr{L}(\mathscr{G}_2)\right)$. Similarly, the equality $\left[{H_1}^{-1}\right]_{io} \equiv \left[{H_2}^{-1}\right]_{io}$ along with the preceding result yields $\det\left(C^1_{oi}\right) = \det\left(C^2_{oi}\right), s \in \mathbb{C}$. According to Lemma 4, for $s = 0$ this means that $\mathscr{G}_1$ and $\mathscr{G}_2$ have the same number of v_o-rooted spanning out-branchings. This, however, is in contradiction with the result of Lemma 3 and the fact that $v_o (\neq v_k)$ is an out-branching root of $\mathscr{G}_2$. ■

Remark 10. *If all of the links in the set* $\partial^{-}_{\mathscr{G}_1}\{v_k\}$ *in Proposition 3 are simultaneously removed to form the digraph* $\mathscr{G}_2$*, then the vertex* $v_o \neq v_k$ *is not an out-branching root of* $\mathscr{G}_2$*. Therefore, the set* $\partial^{-}_{\mathscr{G}_1}\{v_k\}$ *does not satisfy the sufficient conditions offered by Proposition 3 for joint detectability from agent* x_i*.*

The following are direct consequences of Proposition 3 and Remark 1.

Corollary 2. *Consider a multi-agent system* $\mathscr{S}$*, its associated digraph* $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E})$*, a vertex* $v_k \in \mathscr{V}$*, and a subset of edges* $\mathscr{E}_2 \subset \partial^{-}_{\mathscr{G}_1}\{v_k\}$*. If* v_k *is an out-branching root of* $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E} \backslash \mathscr{E}_2)$ *and there exists a second out-branching root* $v_o \in \mathscr{V} \backslash \{v_k\}$ *in* $\mathscr{G}_2$*, then the edge-set* $\mathscr{E}_2$ *is jointly detectable from agent* x*,* $\forall x \in \mathscr{S}$*.*

Corollary 3. *Consider a multi-agent system* $\mathscr{S}$ *and its associated digraph* $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E})$*. If* $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$ *is strongly connected for all* $\varepsilon \in \mathscr{E}$*, then the set* $\{\varepsilon\}$ *is jointly detectable from agent* x*,* $\forall \varepsilon \in \mathscr{E}$ *and* $\forall x \in \mathscr{S}$*.*

Remark 11. *It follows from Remark 1 that under the conditions of Proposition 3,* v_k *is also an out-branching root of* $\mathscr{G}$*. Hence, there exists a* $v_k v_i$ *path, from the head vertex of the failed links to the vertex corresponding to the observing agent* x_i*. Consequently, the* “information" *concerning the link failures can be assumed to* “reach" *the observing agent.*

Remark 12. *The proof for the sufficient conditions in Proposition 2 relies on the highest power of* s *and the leading coefficients in the minors of the transformed graph Laplacian. According to Lemma 6, these parameters depend on the number of the shortest paths between the graph* *vertices, as well as their lengths. On the other hand, the proof for Proposition 3 revolves around the minors of the transformed graph Laplacian at* $s = 0$*, which according to Lemma 4, are related to the number of rooted spanning out-branchings in the digraph. The two sets of results can therefore be interpreted as the two extremes corresponding to* $s \to \infty$ *and* $s = 0$ *in the minors of the transformed graph Laplacian.*

The next subsection offers a complete graphical representation for the class of jointly detectable links that satisfy the condition of Theorem 4.

3.1.2 Shortest Paths Subgraph

The next definition and the corollaries which follow provide a complete characterization of the class of detectable links that are introduced by the sufficient condition given in Theorem 4.

Definition 9. *Given a digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *and a vertex* $\nu \in \mathscr{V}$*, suppose that* $\forall \mu \in \mathscr{V} \backslash \{\nu\}$*, there exists a* $\mu\nu$ *path in* $\mathscr{G}$*. The shortest paths subgraph of* $\mathscr{G}$ *w.r.t.* ν *is identified by* $\mathscr{K}_{\mathscr{G}}^{\nu} = (\mathscr{V}, \mathscr{E}_{\mathscr{K}})$*, where* $\mathscr{E}_{\mathscr{K}} \subset \mathscr{E} \backslash \partial_{\mathscr{G}}^{+}\{\nu\}$ *and* $\forall \varepsilon := (\tau, \vartheta) \in \mathscr{E} \backslash \partial_{\mathscr{G}}^{+}\{\nu\}$*,* $\varepsilon \in \mathscr{E}_{\mathscr{K}} \longleftrightarrow \exists k \in \mathbb{N}_{|\mathscr{V}|} \backslash \{1\}$*,* $\mathrm{d}(\tau, \nu) = \mathrm{d}(\vartheta, \nu) + 1 = k$*.*

Remark 13. *Given a digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *and a vertex* $\nu \in \mathscr{V}$*, suppose that* $\forall \mu \in \mathscr{V} \backslash \{\nu\}$*, there exists a* $\mu\nu$ *path in* $\mathscr{G}$*. Let* $\mathscr{K}_{\mathscr{G}}^{\nu}$ *be the shortest paths subgraph of* $\mathscr{G}$ *w.r.t.* ν*. For all* $\mu \in \mathscr{V} \backslash \{\nu\}$*, the following equality holds:* $\mathrm{c}_{\mathrm{d}(\mu,\nu)}(\mu, \nu) = \partial_{\mathscr{K}_{\mathscr{G}}^{\nu}}^{+}\{\mu\}$*.*

The following corollary is a restatement of Proposition 2 in terms of Definition 9.

Corollary 4. *Given a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, consider a vertex $\nu \in \mathscr{V}$ corresponding to agent $x \in \mathscr{S}$, and a subset of edges $\mathscr{E}_f \subset \mathscr{E}$. Assume also that for any $\mu \in \mathscr{V} \setminus \{\nu\}$, there exists a $\mu\nu$ path in $\mathscr{G}$, and let $\mathscr{K}_{\mathscr{G}}^{\nu} = (\mathscr{V}, \mathscr{E}_{\mathscr{K}})$ denote the shortest paths subgraph of $\mathscr{G}$ w.r.t. ν. If $\mathscr{E}_f \cap \mathscr{E}_{\mathscr{K}} \neq \varnothing$, then $\mathscr{E}_f$ is jointly detectable from x.*

The following corollary exploits Remark 13 to translate the sufficient condition of Theorem 4 into a set of sufficient conditions in terms of the shortest paths subgraphs, which in turn help determine if the removal of two distinct link-sets leads to two distinct digraphs with distinguishable dynamics for the multi-agent system.

Corollary 5. *Given a multi-agent system $\mathscr{S}$ and its associated digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, consider a vertex $\nu \in \mathscr{V}$ corresponding to agent $x \in \mathscr{S}$, and two distinct subsets of edges $\mathscr{E}_1^f \subset \mathscr{E}$ and $\mathscr{E}_2^f \subset \mathscr{E}$. Suppose that for any $\mu \in \mathscr{V} \setminus \{\nu\}$, there exists a $\mu\nu$ path in $\mathscr{G}$, and let $\mathscr{K}_{\mathscr{G}}^{\nu} = (\mathscr{V}, \mathscr{E}_{\mathscr{K}})$ denote the shortest paths subgraph of $\mathscr{G}$ w.r.t. ν. If $\exists \mu \in \mathscr{V} \setminus \{\nu\}$, $|(\mathscr{E}_1^f \setminus \mathscr{E}_2^f) \cap \partial^+_{\mathscr{K}_{\mathscr{G}}^{\nu}}\{\mu\}| \neq |(\mathscr{E}_2^f \setminus \mathscr{E}_1^f) \cap \partial^+_{\mathscr{K}_{\mathscr{G}}^{\nu}}\{\mu\}|$, then $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \setminus \mathscr{E}_1^f)$ and $\mathscr{G}_2 = (\mathscr{V}, \mathscr{E} \setminus \mathscr{E}_2^f)$ are distinguishable from agent x.*

Remark 14. *Setting $\mathscr{E}_2^f = \varnothing$ in the condition of Corollary 5 leads to $\exists \mu \in \mathscr{V} \setminus \{\nu\}$, $|\mathscr{E}_1^f \cap \partial^+_{\mathscr{K}_{\mathscr{G}}^{\nu}}\{\mu\}| \neq 0$ which is equivalent to the condition stated in Corollary 4. Hence, Corollary 4 can be regarded as a special case of Corollary 5.*

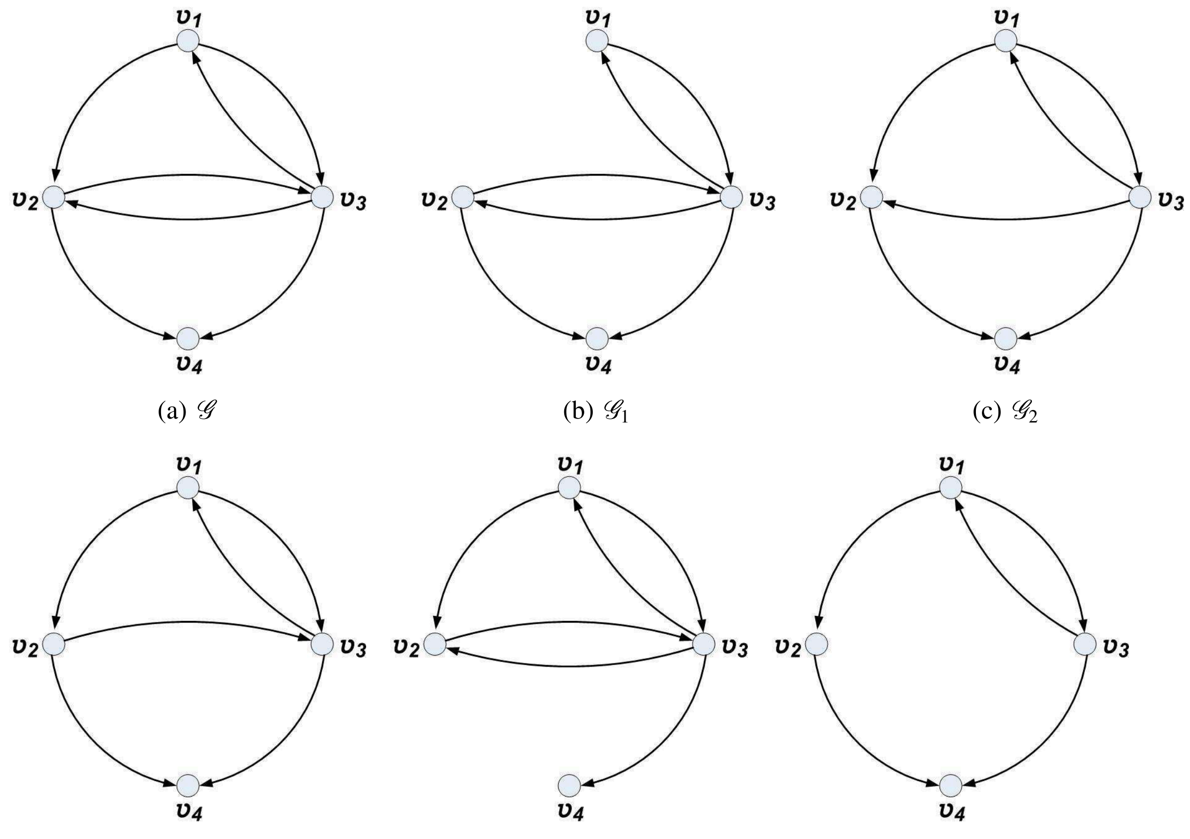
3.2 Examples and Discussion

Consider a multi-agent system $\mathscr{S} = \{x_1, x_2, x_3, x_4\}$ with the digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{v_1, v_2, v_3, v_4\}$ and $\mathscr{E} = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4), (v_3, v_1), (v_3, v_2), (v_3, v_4)\}$, as depicted in Fig. 3.1(a).

The digraphs resulting from the failure of each of the links $\varepsilon_1 := (v_1, v_2)$, $\varepsilon_2 := (v_2, v_3)$, $\varepsilon_3 := (v_3, v_2)$, $\varepsilon_4 := (v_2, v_4)$ are $\mathscr{G}_l = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon_l\})$, $l \in \mathbb{N}_4$, and are depicted in Figs. 3.1(b)-3.1(e). The failed responses of x_1 with respect to each of the edges ε_l, $l \in \mathbb{N}_4$ are depicted in Figs. 3.2(a)-3.2(d), with the initial states set randomly. It is straightforward to verify that only ε_4 is not detectable from x_1, and that is because v_4 is not an out-branching root of $\mathscr{G}_4$ and there does not exists a $v_4 v_1$ path in $\mathscr{G}_4$.

As a second example, consider a multi-agent system $\hat{\mathscr{S}} = \{\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4\}$ and suppose that its associated digraph is given by $\mathscr{G}_2$ in Fig. 3.1(c), with v_l, $l \in \mathbb{N}_4$, corresponding to $\hat{x}_l$, $l \in \mathbb{N}_4$.

The failure of the edge $\varepsilon_3 = (v_3, v_2)$ in $\mathscr{G}_2$ leads to the digraph $\hat{\mathscr{G}}$, which is depicted in Fig. 3.1(f). Since there does not exist a $v_2 v_1$ path in $\mathscr{G}_2$, the conditions of Propositions 2 or 3 are not satisfied for ε_3 and $\hat{x}_1$. Fig. 3.2(e) shows the failed response of $\hat{x}_1$ w.r.t. ε_3. Accordingly, it is the case that $\hat{x}_1(t) - \hat{x}_1^{\varepsilon_3}(t) \equiv 0$ and ε_3 is not detectable from $\hat{x}_1$. On the other hand, the failed response of $\hat{x}_2$ w.r.t. ε_3 in Fig.3.2(f) indicates that ε_3 is detectable from $\hat{x}_2$, as predicted by Corollary 1.



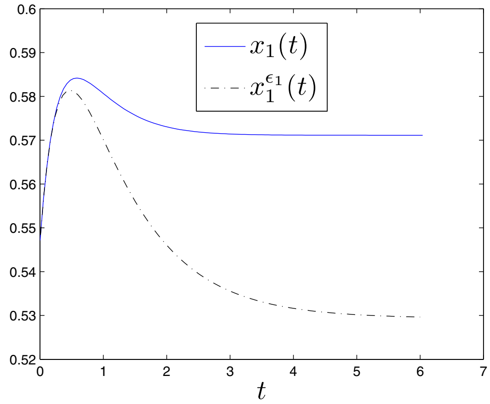
(a) $\mathscr{G}$ (b) $\mathscr{G}_1$ (c) $\mathscr{G}_2$

(d) $\mathscr{G}_3$ (e) $\mathscr{G}_4$ (f) $\hat{\mathscr{G}}$

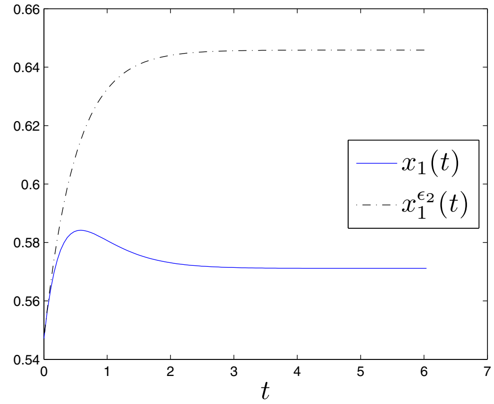
Figure 3.1: The digraphs for examples of Section 3.2.

3.3 Conclusions

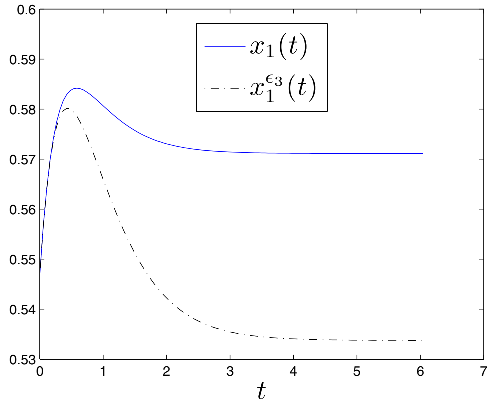
The analytical developments in this chapter include a powerful extension to the all-minors matrix tree theorem that relates the polynomial degree and leading coefficient of the minors of the transformed Laplacian of a digraph to the length and number of shortest paths connecting its vertices. Moreover, the analytical results reveal an intricate relationship between the ability to distinguish between the dynamic response of two multi-agent systems and the inter-nodal distances and number of shortest paths in their digraphs. Detectability of link failures is studied as a special case, and sufficient conditions indicate that, in order for a group of links to be jointly detectable from a given agent, it suffices to design the information flow digraph in such a way that for one of the links in the group, the head vertex is at a shorter topological distance from the observing agent, as compared to the tail vertex. Alternatively, a group of links with a common head are jointly detectable from any agent in the network, provided the common head vertex is an out-branching root and there exists another out-branching root in the network information flow digraph. Illustrative examples in the chapter demonstrate the applicability of the proposed concepts.



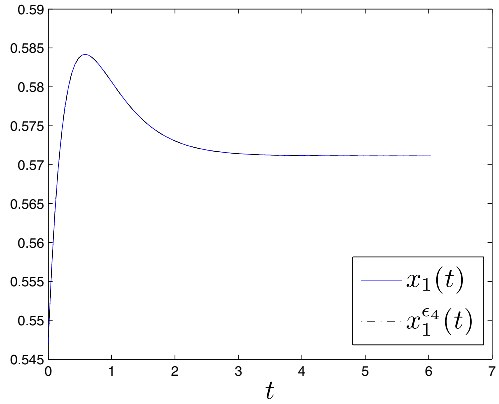
(a) $x_1^{\varepsilon_1}(t) - x_1(t) \not\equiv 0$



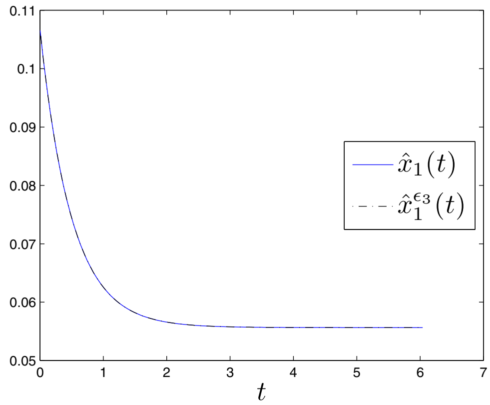
(b) $x_1^{\varepsilon_2}(t) - x_1(t) \not\equiv 0$



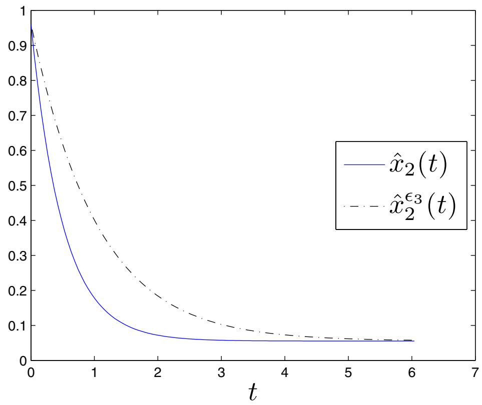
(c) $x_1^{\varepsilon_3}(t) - x_1(t) \not\equiv 0$



(d) $x_1^{\varepsilon_4}(t) - x_1(t) \equiv 0$



(e) $\hat{x}_1^{\varepsilon_3}(t) - \hat{x}_1(t) \equiv 0$



(f) $\hat{x}_2^{\varepsilon_3}(t) - \hat{x}_2(t) \not\equiv 0$

Figure 3.2: The failed responses for different agents and links with random initial conditions.

Chapter 4

Preservation of Network Controllability

In this chapter, structural controllability of a leader-follower multi-agent system with multiple leaders is studied from a graph-theoretic point of view. The problem of preservation of structural controllability under simultaneous failures in both the communication links and the agents is investigated. The effects of the loss of agents and communication links on the controllability of an information flow graph are previously studied. In this work, the corresponding results are exploited to introduce some useful indices and importance measures that help characterize and quantify the role of individual links and agents in the controllability of the overall network. Existing results are then expanded by considering the effects of losses in both links and agents at the same time. To this end, the concepts of joint $(r,s)-$controllability and joint $t-$controllability are introduced as quantitative measures of reliability for a multi-agent system, and their important properties are investigated. Lastly, the class of jointly critical digraphs are introduced and it is stated that if a digraph is jointly critical, then joint $t-$controllability is a necessary and sufficient condition for remaining controllable following the failure of any set of links and agents, with cardinality less than t. Various examples throughout the chapter are exploited to elaborate on the analytical findings.

The remainder of this chapter is organized as follows. The background and motivation of the study are explained in Section 4.1. The tools and concepts introduced in Chapter 2 are then used in Sections 4.2 and 4.3 to characterize and quantify the importance of every link and agent in the controllability of the overall digraph. In Section 4.4, first the notion of joint controllability is offered for the characterization of simultaneous failures in the network, and then the class of jointly critical digraphs are introduced and their favorable properties are pointed out. The analytical results of Sections 4.2 to 4.4 are illustrated and discussed using several examples throughout the text, and concluding remarks are provided in Section 4.5.

4.1 Background and Motivation

The problem of controllability in a Laplacian-based leader-follower multi-agent system with consensus-like interaction rules is first formulated by Tanner [75], where necessary and sufficient conditions for controllability are presented in terms of eigenvectors and eigenvalues of a sub-matrix of graph Laplacian corresponding to the follower nodes. The importance of a graph theoretic characterization for controllability is also pointed out in [75]. The authors in [76] propose a sufficient condition for controllability of the network that is based on the null-spaces of the leader and follower incidence matrices. The condition is then restated in terms of the first homology of the network graph and its quotient over follower nodes. In a different attempt, Rahmani and Mesbahi [77] use Tanner's results to establish a relation between the notion of graph symmetry and the system theoretic concept of controllability by stating that symmetry with respect to follower nodes results in the uncontrollability of the network. The work [78] applies the concept of controllability to a network represented by a weighted directed graph and provides an interpretation for the controllability matrix in terms of the gains of fixed-length paths originated from the input node. Further results by Ji and Egerstedt [79] show that the existence of a common eigenvalue between the Laplacians of the original graph and the graph corresponding to the follower nodes is a necessary and sufficient condition for uncontrollability. This result is subsequently used to develop a sufficient condition based on the graph-theoretic concept of equitable partitions [80], which is then used to obtain a necessary and sufficient condition using relaxed equitable partitions [81]. Moreover, while the results relating graph symmetry to uncontrollability in [80] are shown to be explicable using equitable partitions, the relaxed equitable partitions in [81] provide a graph-theoretic characterization of the controllable and uncontrollable subspaces in the network. Another recent result shows that a multi-agent system is controllable if and only if no eigenvector of graph

Laplacian takes a zero value on the elements corresponding to the leaders [82]. Other challenging scenarios involving dynamic topologies and time-delay have also been investigated in the literature. For example, it is shown in [83] that switching between fixed uncontrollable topologies can lead to a controllable system. In a recent work, the structural controllability of complex real-world networks is investigated by [56] and network links are categorized based on the robustness of control against link failures.

4.1.1 Controllability in the Face of Time-Delay

The question of investigating the effect of time-delay on network controllability is intriguing indeed. The earliest results on the extension of the concept of controllability to time-delay systems can be traced back to 1960s and 1970s, where system properties such as $\mathbb{R}^n$-controllability, point-wise completeness, and spectral controllability are studied and their satisfiability under various algebraic conditions is investigated [84–87]. It is pointed out in [88] that the concept of controllability for delay differential equations is fundamentally different from that for ordinary differential equations, since the issue of state-reachability for the former would involve functions over time-intervals as opposed to points in the state-space. Moreover, the delay introduces additional requirements on the minimum reaching time, and the design efforts are no longer restricted to the memoryless feedback control laws [88]. Historically, relying on Gramian rank conditions that make use of the symbolic solutions for delay differential equations as in [89], has proven less useful due to the difficulties in calculating the fundamental matrix of the delay differential equations [90]. Therefore, alternative conditions that involve the equation coefficients have been sought after as in [84, 91]. Another category of results make use of a commensurate delay model along with a delay operator to exploit the available tools from polynomial rings algebra as in [86]. The only available result on *structural controllability* subject to time-delay is of the same nature [92]. Most recently, the use of matrix Lambert W function has lead to an analytical solution for LTI systems with single time-delay [93]. This solution, which is expressed as an infinite sum, has facilitated the calculation of the Gramians and the corresponding controllability rank conditions of time-delay systems in both time and Laplace domains [94].

4.1.2 Controllability in the Face of Failures

The papers [50–52] approach the problem of link and agent failures by deriving graphical conditions for the preservation of structural controllability in the face of such failures. While existing results on the controllability of multi-agent systems provide an important measure of reliability of network to faults, they cannot handle the important problem of simultaneous failure of communication links and agents.

The chief aim of this chapter is to expand on the results of [50–52] by considering the case when communication links and agents in the network are both prone to failure. Moreover, the concepts of link and agent controllability degrees introduced in [50] and [52] are exploited to provide quantitative measures for the importance of individual links and agents in preserving the controllability of the overall digraph. To this end, the problem of preservation of structural controllability under link and agent failures is considered as a gateway for evaluation and comparison of the contributions made by various links in the network. In this venue, to evaluate the role of a link in the network the controllability of the multi-agent system with and without that link is investigated and conclusions are drawn accordingly. It is further stated that the importance of agents are reflected in their outgoing links and quantitative measures are subsequently offered to characterize the importance of individual agents in the network. So as to quantify the resilience of a network against multiple simultaneous failures, the notions of joint $(r,s)-$controllability and joint $t-$controllability are proposed and the latter is shown to be computable in polynomial-time. Next, a class of digraphs are investigated, for which joint $t-$controllability is a necessary and sufficient condition for remaining controllable after the failure of any set of links and agents with size less than t.

Identification and characterization of key link-points are important requirements in the cooperative and reliable control of multi-agent networks. Furthermore, the comparative study of the importance of individual agents in the network is key to the design of reliable fault-tolerant multi-agent systems, by providing guide-lines on where to focus the recovery operations and which agents to prioritize in the case of a network-wide failure. The results are therefore of both theoretical and practical interest. On the other hand, the study of simultaneous failures is important in light of the fact that in real-world multi-agent systems, some faults can affect part of the network, containing a number of links and agents. This type of failure in multi-agent systems, where terrain properties or hardware faults disable a number of agents and limit the ability of others to communicate, motivates the study of controllability under simultaneous failure of links and agents.

4.2 Importance of Links to Network Controllability

In Subsection 4.2.1, which succeeds this paragraph, the concept of link controllability degree (Definition 3) is employed to quantify the "contribution" of an *uncritical* link to the controllability of the overall network. Next, in Subsection 4.2.2 the agent controllability degree (Definition 4) is used to introduce a second measure of importance, namely the *agent controllability index*, that would apply to both *critical* and *uncritical* links. A relevant discussion on the robustness of control law against link failure and the subsequent classification of links can be found in [56].

4.2.1 Link Controllability Index

The next definition provides a means to determine which uncritical links take precedence in terms of their importance in the network.

Definition 10. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $\varepsilon \in \mathscr{E}$ be an uncritical link in $\mathscr{G}$. Let also $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$ be the digraph that results from the removal of the edge ε in $\mathscr{G}$. Denote the number of critical links in $\mathscr{G}$ and $\mathscr{G}_1$ by n and*

n_1, respectively. The link controllability index *of the edge ε is characterized as $\sigma(\mathscr{G},\varepsilon;\mathscr{R}) = n_1 - n$.*

Remark 15. *Given an information flow digraph $\mathscr{G} = (\mathscr{V},\mathscr{E})$ with the root-set $\mathscr{R}$, for all uncritical edges $\varepsilon \in \mathscr{E}$, the inequalities $0 \leqslant \sigma(\mathscr{G},\varepsilon;\mathscr{R}) \leqslant |\mathscr{E}| - p' - 1$ hold, where $p' \in \mathbb{N}$ is the cardinality of the union of all critical link-sets in $\mathscr{G}$.*

Remark 16. *Using Definitions 3 and 10, all links in a multi-agent network can be categorized based on their importance. Accordingly, the role of the critical links is superior to the uncritical ones. Moreover, amongst the uncritical links, those with higher link controllability indices are more important.*

The examples that follow, shed light upon the characteristics of link controllability index and how it reflects the “importance" of each uncritical link in the overall network. In all of the examples herein, nodes belonging to the root-set (leaders) are represented by colored vertices.

Example 1. *Uncritical links and link controllability indices.*

Digraph $\bar{\mathscr{G}}_1$ of Fig. 4.1 is $2-$link and $2-$agent controllable. The set of edges in this digraph can be partitioned into two disjoint sets comprised of critical and uncritical links. It has seven critical links which are represented by solid edges. These seven critical links join each other in five different critical link-sets, each of which has a cardinality of two. The link controllability indices of the uncritical links in this digraph take the values zero (single-dotted edges) and one (fully-dotted edges). The two links which connect the root-set to vertex v_4

have $\sigma = 1$. This is because the cardinality of the out-cut of the root-set is three, and as any of the two uncritical links are removed from the out-cut of the root-set, the remaining one forms a new critical link-set with the critical link which has v_5 as its tail. A similar argument applies to the in-cut of the vertex v_3, which again has three links: one critical and two uncritical with $\sigma = 1$. The two links shown by single-dotted edges have $\sigma = 0$, and they are the least significant links in terms of the preservation of controllability in face of link failures. The next example demonstrates a case where the link controllability indices exceed the link controllability degree of the digraph.

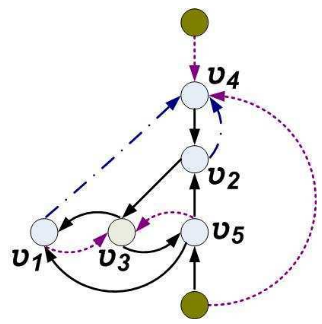


Figure 4.1: The digraph $\bar{\mathscr{G}}_1$ of Example 1 is 2−link and 2−agent controllable and has seven critical links (solid edges), two uncritical links with $\sigma = 0$ (single-dotted edges), and four uncritical links with $\sigma = 1$ (fully-dotted edges).

Example 2. *A digraph with link controllability indices greater than link controllability degree.*

Digraph $\bar{\mathscr{G}}_2$ of Fig. 4.2 is 2−link and 1−agent controllable. It has six critical links which are depicted by solid edges. These six critical links join each other in three mutually disjoint critical link-sets, each of which has a cardinality of two. The uncritical links in this digraph can be divided into two groups. The first group is comprised of the three links represented by fully-dotted edges. The link controllability indices of these three links are equal to four and they play a more important role in the preservation of controllability as compared to the remaining 12 uncritical links, which are denoted by single-dotted edges and have $\sigma = 2$. It is worth highlighting that as the removal of uncritical links causes new disjoint critical link-sets to be formed from the previously uncritical links, the value of the link controllability index for any uncritical link equals an integer multiple of the link controllability degree of the digraph. This property, however, does not always hold as indicated in Example 1.

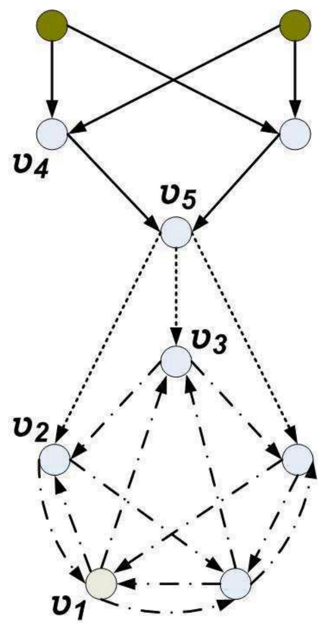


Figure 4.2: Digraph $\bar{\mathscr{G}}_2$ of Example 2 is $2-$link and $1-$agent controllable and has six critical links (solid edges), three uncritical links with $\sigma = 4$ (fully-dotted edges) and 12 uncritical links with $\sigma = 2$ (single-dotted edges).

In the next subsection, the effect of a link loss on the agent controllability degree of a digraph is utilized to quantify the importance of each link in the network, through the notion of *agent controllability index*.

4.2.2 Agent Controllability Index

Definition 11. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$ and let $\varepsilon \in \mathscr{E}$ be a given edge in $\mathscr{G}$, and $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$. The* agent controllability index *of the edge ε is denoted by $\rho(\mathscr{G}, \varepsilon; \mathscr{R})$ and is given by $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = ac(\mathscr{G}; \mathscr{R}) - ac(\mathscr{G}_1; \mathscr{R})$.*

Remark 17. *Unlike the link controllability index that is restricted to the case of uncritical links, agent controllability index provides a measure of importance that can be applied to every link in the network. Similarly to the link controllability index, the higher the agent controllability index of a link, the more superior is the role that it plays in the preservation of controllability throughout the network. However, as the uncritical links are already characterized through their link controllability indices, agent controllability index can be principally used to determine the relative importance of critical links.*

Remark 18. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, for any edge $\varepsilon \in \mathscr{E}$, it is straightforward to show that $0 \leqslant \rho(\mathscr{G}, \varepsilon; \mathscr{R}) \leqslant ac(\mathscr{G}; \mathscr{R}) \leqslant |\mathscr{V}| - |\mathscr{R}|$.*

Lemma 7. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$ and let $\varepsilon \in \mathscr{E} \backslash \partial^+_{\mathscr{G}} \mathscr{R}$ be a given edge in $\mathscr{G}$, whose head is the vertex $v \in \mathscr{V} \backslash \mathscr{R}$. If $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$, then $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = ac(\mathscr{G}, v; \mathscr{R}) - ac(\mathscr{G}_1, v; \mathscr{R})$.*

Proof: The proof follows from the fact that the removal of the edge $\varepsilon \in \mathscr{E} \backslash \partial^+_{\mathscr{G}} \mathscr{R}$ can affect the agent controllability degree of digraph $\mathscr{G}$ only by altering the available paths that connect the root-set $\mathscr{R}$ to the head vertex v, through other agents in the network. On the other hand, if $\varepsilon \in \partial_{\mathscr{G}}^{+}\mathscr{R}$, then ε provides a direct path from $\mathscr{R}$ to v without involving any other agents in the network, and according to Remark 5, $ac(\mathscr{G}, v; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$, regardless of the available paths. ■

Theorem 5. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$. If there exist a vertex $v \in \mathscr{V}$ and an edge ε such that $\{\varepsilon\} \subseteq \partial_{\mathscr{G}}^{+}\mathscr{R} \cap \partial_{\mathscr{G}}^{-}\{v\} \neq \varnothing$, then $\forall \hat{\varepsilon} \in \partial_{\mathscr{G}}^{-}\{v\} \backslash \{\varepsilon\}$, $\rho(\mathscr{G}, \hat{\varepsilon}; \mathscr{R}) = 0$.*

Proof: Let $\hat{\mathscr{G}} = (\mathscr{V}, \mathscr{E} \backslash \{\hat{\varepsilon}\})$. For the case where $\hat{\varepsilon} \notin \partial_{\mathscr{G}}^{+}\mathscr{R} \cap \partial_{\mathscr{G}}^{-}\{v\}$, the proof follows from Remark 5 and Lemma 7 upon noting that since the edge ε connects the vertex v directly to the root-set, $ac(\mathscr{G}, v; \mathscr{R}) = ac(\hat{\mathscr{G}}, v; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$, and hence $ac(\mathscr{G}, v; \mathscr{R}) - ac(\hat{\mathscr{G}}, v; \mathscr{R}) = 0$. On the other hand, if $\hat{\varepsilon} \in \partial_{\mathscr{G}}^{+}\mathscr{R} \cap \partial_{\mathscr{G}}^{-}\{v\}$, then the proof follows from the fact that both ε and $\hat{\varepsilon}$ are providing a direct connection from the root-set $\mathscr{R}$ to their common head vertex v that does not rely on any other follower agents in the network. Therefore, as long as this direct connection from the root-set $\mathscr{R}$ to the head vertex v exists, removing either one of the edges ε or $\hat{\varepsilon}$ has no effects on the agent controllability degree of digraph $\mathscr{G}$. Hence, if $\{\varepsilon, \hat{\varepsilon}\} \subseteq \partial_{\mathscr{G}}^{+}\mathscr{R} \cap \partial_{\mathscr{G}}^{-}\{v\}$, then $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = \rho(\mathscr{G}, \hat{\varepsilon}; \mathscr{R}) = 0$. ■

Theorem 5 facilitates the characterization of the agent controllability index for those edges, whose heads are directly connected to the root-set. In the special case that there exist multiple edges connecting the root-set to a vertex, Theorem 5 reduces to the following corollary.

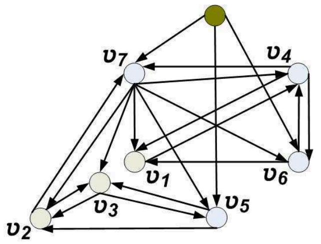
Corollary 6. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$ and a vertex $v \in \mathscr{V} \backslash \mathscr{R}$, if $|\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{v\}| > 1$, then $\forall \varepsilon \in \partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{v\}, \rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 0$.*

Remark 19. *Let $\mathscr{G}$, $\mathscr{V}$ and $\mathscr{R}$ be given as in Corollary 6. It is notable that increasing the cardinality of $\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{v\}$ can improve the link controllability degree of digraph $\mathscr{G}$, while according to Corollary 6, $|\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{v\}|$ has no impact on the agent controllability degree of $\mathscr{G}$ as long as $|\partial^+_{\mathscr{G}} \mathscr{R} \cap \partial^-_{\mathscr{G}} \{v\}| > 1$.*

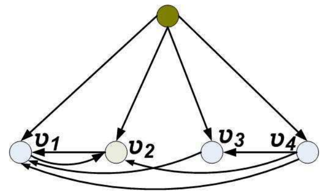
Example 3. *The range of values for agent controllability index.*

The digraph $\bar{\mathscr{G}_3}$ in Fig. 4.3(a) is 2−agent and 3−link controllable, and all of its links are critical. However, only those links that belong to the out-cut of the root-set have $\rho = 1$, and the rest have $\rho = 0$. Next, consider the digraph $\bar{\mathscr{G}_4} = (\bar{\mathscr{V}_4}, \bar{\mathscr{E}_4})$ depicted in Fig. 4.3(b), and assume that the colored vertex $r \in \bar{\mathscr{V}_4}$ at the top is designated as the root. It can be easily verified that $ac(\bar{\mathscr{G}_4}; \{r\}) = 4$ and $lc(\bar{\mathscr{G}_4}; \{r\}) = 1$. Moreover, since all non-root vertices are connected directly to the root-set $\{r\}$, it follows from Theorem 5 that $\forall \varepsilon \in \bar{\mathscr{E}_4} \backslash \partial^+_{\bar{\mathscr{G}_4}} \{r\}$, $\rho(\bar{\mathscr{G}_4}, \varepsilon; \{r\}) = 0$. On the other hand, for any $i \in \mathbb{N}_4$, if $\varepsilon_i = (r, v_i)$, then $\rho(\bar{\mathscr{G}_4}, \varepsilon_i; \{r\}) = i$. Therefore, the agent controllability indices of the edges in this digraph take all possible values between 0 and $ac(\bar{\mathscr{G}_4}; \{r\}) = |\bar{\mathscr{V}_4}| - |\{r\}| = 4$. It is also notable that ε_4 is the only critical link in $\bar{\mathscr{G}_4}$ and its agent controllability index is the highest of all links in $\bar{\mathscr{G}_4}$. The next example, however, indicates that the criticality of a link does not necessarily imply a higher agent controllability index.

Example 4. *Criticality of the links and their agent controllability indices.*



(a) $\bar{\mathscr{G}}_3$



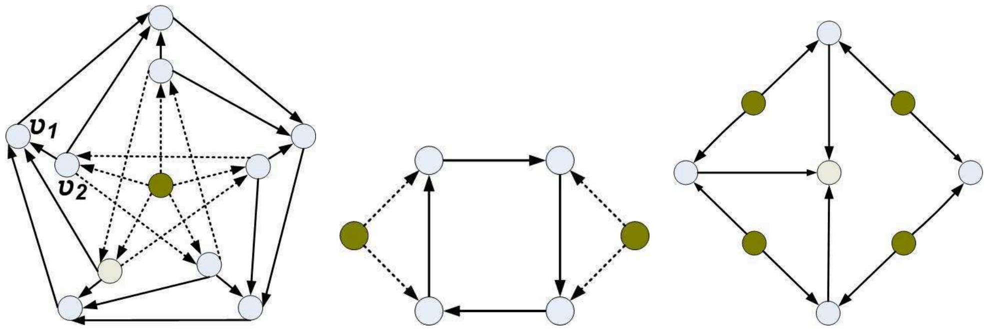
(b) $\bar{\mathscr{G}}_4$

Figure 4.3: (a)The digraph $\bar{\mathscr{G}}_3$, which is $2-$agent and $3-$link controllable. (b) The $1-$link and $4-$agent controllable digraph $\bar{\mathscr{G}}_4$ of Example 3, in which the agent controllability indices take all possible values from 0 to 4.

Consider the digraph $\bar{\mathscr{G}}_5$ of Fig. 4.4(a). Every dotted link in this digraph is critical (and vice versa) and it follows from Theorem 5 that those critical links which do not belong to the out-cut of the root-set have zero agent controllability index, while all other critical links have $\rho = 2$. On the other hand, the solid links are all uncritical with unity agent controllability index. Moreover, the link controllability index of every uncritical link in $\bar{\mathscr{G}}_5$ is equal to two as the removal of any of the solid links in $\bar{\mathscr{G}}_5$ causes two of the remaining solid links to form a new critical link-set together.

As a second example, consider the digraph $\bar{\mathscr{G}}_6$ of Fig. 4.4(b). This digraph is $2-$link and $4-$agent controllable, and although every link in $\bar{\mathscr{G}}_6$ is critical, according to Theorem 5 the links depicted by solid edges have zero agent controllability index. On the other hand, the agent controllability index of the dotted edges in $\bar{\mathscr{G}}_6$ is equal to three, pointing to their relatively superior role in the preservation of the flow of information from the root-set, and the controllability of the network. The digraph $\bar{\mathscr{G}}_7$ in Fig. 4.4(c) offers a third example, for which Corollary 6 indicates that all critical links have $\rho = 0$, while the uncritical ones have $\rho = 1$.

Overall, these examples indicate that, in general, the criticality of a link is not necessarily correlated with its agent controllability index. In particular, it is possible to have critical links with zero agent controllability indices and uncritical links with non-zero agent controllability indices. The first two examples further demonstrate that, as suggested in Remark 17, agent controllability index can be employed as an effective tool for the characterization of the relative importance of critical links in the network.



(a) $\bar{\mathscr{G}}_5$ (b) $\bar{\mathscr{G}}_6$ (c) $\bar{\mathscr{G}}_7$

Figure 4.4: (a)The digraph $\bar{\mathscr{G}}_5$ of Example 4, for which critical and uncritical links are denoted by dotted and solid links, respectively. (b) The 2-link and 4-agent controllable digraph $\bar{\mathscr{G}}_6$ of Example 4, in which every link is critical. (c)The digraph $\bar{\mathscr{G}}_7$ is $3-$agent and $2-$link controllable. It has eight critical links with $\rho = 0$, and three uncritical links with $\rho = 1$.

The next lemma and the theorem which follows provide a full characterization of the agent controllability index for those edges, whose heads are not directly connected to the root-set.

Lemma 8. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $\tau, \nu \in \mathscr{V}$ be two vertices in $\mathscr{G}$ such that $\varepsilon := (\tau, \nu) \in \mathscr{E}$. Denote $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$. If there exists a critical agent-set $\mathscr{C}_q^1 \subset \mathscr{V} \backslash \mathscr{R}$ of $\mathscr{G}_1$ such that $\tau \in \mathscr{C}_q^1$ or $\nu \in \mathscr{C}_q^1$, then $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 0$.*

Proof: The proof follows upon noting that $\mathscr{C}_q^1$ is a critical agent-set of $\mathscr{G}$ as well. ■

Theorem 6. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $\tau, \nu \in \mathscr{V} \backslash \mathscr{R}$ be two vertices in $\mathscr{G}$ such that $\varepsilon := (\tau, \nu) \in \mathscr{E} \backslash \partial_{\mathscr{G}}^{+} \mathscr{R}$ and $\partial_{\mathscr{G}}^{-}\{\nu\} \cap \partial_{\mathscr{G}}^{+} \mathscr{R} = \varnothing$. If $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) \neq 0$, then $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 1$ and there exists a critical agent-set $\mathscr{C}_q \subset \mathscr{V} \backslash \mathscr{R}$ of $\mathscr{G}$ such that $\tau \in \mathscr{C}_q$.*

Proof: Let $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$ and consider a critical agent-set $\mathscr{C}_q^1 \subset \mathscr{V} \backslash \mathscr{R}$ of $\mathscr{G}_1$. Since $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) \neq 0$, it follows from Lemma 8 that $\tau \notin \mathscr{C}_q^1$. The proof follows upon noting that $\mathscr{C}_q = \mathscr{C}_q^1 \cup \{\tau\}$ is a critical agent-set of $\mathscr{G}$ and $\tau \in \mathscr{C}_q$. ■

Remark 20. *Theorems 5 and 6 address two mutually exclusive cases: the former applies to the incoming edges of the vertices that are directly connected to the root-set, while any other edge in the network is addressed by the latter.*

In Section 4.3, which succeeds this paragraph, the notions of agent and link controllability degrees are exploited to judge the importance of an agent as reflected through its outgoing links. The *agent* and *link criticality indices* are defined in Subsections 4.3.1 and 4.3.2, respectively, and the former is shown to distinguish between critical and uncritical agents based solely on their outgoing links. Next, the *critical link index* is defined in Subsection 4.3.3 as the number of the outgoing critical links for an agent. Finally, in Subsection 4.3.4, the effect of an agent on the uncritical links of the network is captured through the so-called *uncritical link index*, and this, together with the three aforementioned indices, is offered as a design tool for ordering and prioritizing agents of a network.

4.3 Importance of Agents to Network Controllability

To evaluate the role of an agent in the network, one may remove that agent from the network and compare the controllability of the resultant network with the original one. By the same token, it is notable that the effect of the removal of an agent on the remaining agents is equivalent to the effect of the removal of all incoming and outgoing links of the agent in question. Hence from this viewpoint, the importance of an agent in the network is reflected in its incoming and outgoing links. However, the simple counterexample of a pathological branch out of the network indicates that they are indeed the outgoing links upon which the importance of an agent is primarily reflected. The link and agent controllability degrees of the digraph $\bar{\mathscr{G}}_8$ in Fig. 4.5 are both equal to 1, and (v_2, v_1) is the only critical link in this digraph. Nonetheless, despite being the head to the only critical link of $\bar{\mathscr{G}}_8$, vertex v_1 contributes the least to the flow of information and the preservation of network controllability.

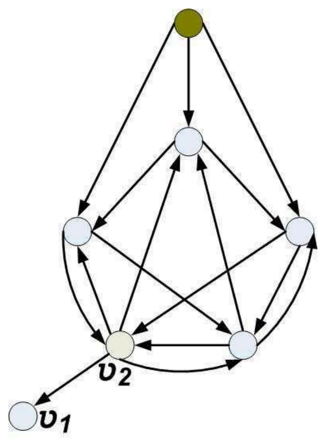


Figure 4.5: The digraph $\bar{\mathscr{G}}_8$ with a pathological branch out of the network.

4.3.1 Agent Criticality Index

The following measure can be used to relate the criticality of an agent, as given in Definition 4, to its outgoing links.

Definition 12. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $v \in \mathscr{V} \backslash \mathscr{R}$ be an arbitrary non-root vertex in $\mathscr{G}$. Let also $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \partial^+_{\mathscr{G}}\{v\})$. The* agent criticality index *of vertex v is denoted by $\delta(\mathscr{G}, v; \mathscr{R})$, and is given by $\delta(\mathscr{G}, v; \mathscr{R}) = ac(\mathscr{G}; \mathscr{R}) - ac(\mathscr{G}_1; \mathscr{R})$.*

Lemma 9. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, if $ac(\mathscr{G}; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$, then $\forall v \in \mathscr{V} \backslash \mathscr{R}$, $\partial^-_{\mathscr{G}}\{v\} \cap \partial^-_{\mathscr{G}} \mathscr{R} \neq \varnothing$ and $\delta(\mathscr{G}, v; \mathscr{R}) = 0$. Conversely, if for all $v \in \mathscr{V} \backslash \mathscr{R}$, $\partial^-_{\mathscr{G}}\{v\} \cap \partial^-_{\mathscr{G}} \mathscr{R} \neq \varnothing$, then $ac(\mathscr{G}; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$ and $\forall v \in \mathscr{V} \backslash \mathscr{R}$, $\delta(\mathscr{G}, v; \mathscr{R}) = 0$.*

Proof: The proof for the first part follows by contradiction, since if $ac(\mathscr{G}; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$, then all follower agents are critical and $\mathscr{C}_q = \mathscr{V} \backslash \mathscr{R}$ is the only critical agent-set. Now, if there exists a vertex $\hat{v}$ that is not the head of a link belonging to the out-cut of the root-set $\mathscr{R}$, then the removal of the agent-set $\mathscr{C}_q \backslash \{\hat{v}\}$ will make $\hat{v}$ unreachable from the root-set $\mathscr{R}$, which is in contradiction with $\mathscr{C}_q$ being a critical agent-set. By the same token, to prove the converse suppose that $ac(\mathscr{G}; \mathscr{R}) < |\mathscr{V}| - |\mathscr{R}|$. Then there exist a critical agent-set $\mathscr{C}^1_q \subsetneq \mathscr{V} \backslash \mathscr{R}$ and an agent $\tilde{v} \in \mathscr{V} \backslash \left(\mathscr{R} \cup \mathscr{C}^1_q\right)$ such that the removal of $\mathscr{C}^1_q$ will make $\tilde{v}$ unreachable from the root-set $\mathscr{R}$. This, however, is also a contradiction since $\tilde{v}$ is the head of a link belonging to the out-cut of the root-set $\mathscr{R}$. In both cases, the equality $\delta(\mathscr{G}, v; \mathscr{R}) = 0$ for all $v \in \mathscr{V} \backslash \mathscr{R}$ follows from the fact that $\forall v \in \mathscr{V} \backslash \mathscr{R}, ac(\mathscr{G}; \mathscr{R}) = ac(\mathscr{G}_1; \mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$, where $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \partial^+_{\mathscr{G}}\{v\})$. ■

Remark 21. *Digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with root-set* $\mathscr{R}$*, for which* $ac(\mathscr{G};\mathscr{R}) = |\mathscr{V}| - |\mathscr{R}|$*, corresponds to a pathological case where all agents are critical and they receive their "information" from the root-set "directly". In such a case, measures other than the agent criticality index are used to distinguish between the follower agents in terms of their significance in the network. Figs. 4.3(b) and 4.4(b) of Section 4.2 offer instances of such digraphs.*

Theorem 7. *Given an information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R}$*, suppose that* $ac(\mathscr{G};\mathscr{R}) < |\mathscr{V}| - |\mathscr{R}|$*. For all* $v \in \mathscr{V} \backslash \mathscr{R}$*,* $\delta(\mathscr{G}, v; \mathscr{R}) = 1$ *if and only if* v *is critical, and* $\delta(\mathscr{G}, v; \mathscr{R}) = 0$ *otherwise.*

Proof: Since $ac(\mathscr{G};\mathscr{R}) < |\mathscr{V}| - |\mathscr{R}|$, the pathological case set forth in Lemma 9 does not apply. Let $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \partial^+_{\mathscr{G}}\{v\})$, and suppose that $\mathscr{C}_q$ is an arbitrary critical agent-set of $\mathscr{G}$. The removal of $\mathscr{C}_q$ will make $\mathscr{G}_1$ uncontrollable, and if $v \in \mathscr{C}_q$, then $\mathscr{C}_q^1 = \mathscr{C}_q \backslash \{v\}$ is a critical agent-set of $\mathscr{G}_1$. Hence, $\delta(\mathscr{G}, v; \mathscr{R}) = |\mathscr{C}_q| - |\mathscr{C}_q^1| = 1$. This proves that if v is critical, then $\delta(\mathscr{G}, v; \mathscr{R}) = 1$. On the other hand, if v is uncritical, then every critical agent-set of $\mathscr{G}$ is a critical agent-set of $\mathscr{G}_1$ and vice-versa. Hence, $\delta(\mathscr{G}, v; \mathscr{R}) = ac(\mathscr{G};\mathscr{R}) - ac(\mathscr{G}_1;\mathscr{R}) = 0$, which completes the proof. ■

Remark 22. *Theorem 7 indicates that the criticality of an agent in any digraph (except for the pathological case described in Lemma 9 and Remark 21) is completely characterized by its outgoing links and through its agent criticality index given in Definition 12.*

4.3.2 Link Criticality Index

The agent criticality index offers the most conclusive importance measure for agents in a network, and role of agents with $\delta = 1$ is always superior to that of agents with $\delta = 0$. Similarly to the agent criticality index, which considers the effect of the removal of the outgoing links on the agent controllability degree of the digraph, link criticality index is offered in the sequel to capture how the link controllability degree of the digraph is affected by the removal of the outgoing links. This index can be used as a measure to compare the importance of agents with the same agent criticality index.

Definition 13. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $v \in \mathscr{V} \backslash \mathscr{R}$ be an arbitrary non-root vertex in $\mathscr{G}$. Let also $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \partial^+_{\mathscr{G}}\{v\})$. The* link criticality index *of vertex v is denoted by $\theta(\mathscr{G}, v; \mathscr{R})$, and is given by $\theta(\mathscr{G}, v; \mathscr{R}) = lc(\mathscr{G}; \mathscr{R}) - lc(\mathscr{G}_1; \mathscr{R})$.*

4.3.3 Critical Link Index

As a third measure of importance, the designer is advised to compare the number of critical links amongst the outgoing links of each agent as captured by the following index.

Definition 14. *Consider an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, and let $v \in \mathscr{V} \backslash \mathscr{R}$ be an arbitrary non-root vertex in $\mathscr{G}$. The* critical link index *of vertex v is denoted by $\gamma(\mathscr{G}, v; \mathscr{R})$, and is given by $\gamma(\mathscr{G}, v; \mathscr{R}) = |\{\varepsilon \in \partial^+_{\mathscr{G}}\{v\}; lc(\mathscr{G}; \mathscr{R}) - lc(\mathscr{G}_1; \mathscr{R}) \neq 0\}|$, where $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \{\varepsilon\})$.*

Remark 23. *The expression for* $\gamma(\mathscr{G},\nu;\mathscr{R})$ *in Definition 14 indicates the number of critical outgoing links of agent* ν *because for any edge* $\varepsilon \in \mathscr{E}$ *and digraph* $\mathscr{G}_1 = (\mathscr{V},\mathscr{E}\backslash\{\varepsilon\})$ *associated with it,* $lc(\mathscr{G};\mathscr{R}) - lc(\mathscr{G}_1;\mathscr{R})$ *is equal to one if and only if* ε *is critical, and is zero otherwise.*

The next corollary is a direct consequence of Definitions 13 and 14.

Corollary 7. *Given an information flow digraph* $\mathscr{G} = (\mathscr{V},\mathscr{E})$ *with the root-set* $\mathscr{R}$ *and a vertex* $\nu \in \mathscr{V}\backslash\mathscr{R}$*, if* $\gamma(\mathscr{G},\nu;\mathscr{R}) > 0$*, then* $\theta(\mathscr{G},\nu;\mathscr{R}) > 0$.

4.3.4 Uncritical Link Index

For the cases, where the comparison of the three indices introduced in Subsections 4.3.1 to 4.3.3 are inconclusive in evaluating the importance of an agent, one can consider the effect of the agent on the number of critical links in the network to characterize its role in maintaining the "flow of information" in the network. To this end, only the uncritical links are removed from the outgoing links of the agent and the resultant effect on the number of critical links in the network is considered. The procedure is discussed in the sequel.

Definition 15. *Consider an information flow digraph* $\mathscr{G} = (\mathscr{V},\mathscr{E})$ *with the root-set* $\mathscr{R}$*, and let* $\nu \in \mathscr{V}\backslash\mathscr{R}$ *be an arbitrary non-root vertex in* $\mathscr{G}$*. Let also* $\mathscr{E}_1 = \{\varepsilon \in \partial^+_{\mathscr{G}}\{\nu\}; lc(\mathscr{G};\mathscr{R}) - lc((\mathscr{V},\mathscr{E}\backslash\{\varepsilon\});\mathscr{R}) = 0\}$ *and* $\mathscr{G}_1 = (\mathscr{V},\mathscr{E}\backslash\mathscr{E}_1)$*. Denote the number of critical links in* $\mathscr{G}$ *and* $\mathscr{G}_1$ *by* n *and* n_1*, respectively. The* uncritical link index *of the agent* ν *is characterized as* $\psi(\mathscr{G},\nu;\mathscr{R}) = n_1 - n$.

The following proposition and the remark which follows justifies the removal of just the *uncritical* outgoing links when calculating the uncritical link index according to Definition 15.

Proposition 4. *Given an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R}$, let $\mathscr{C}_1$ be a set of critical links such that for some critical link-set $\mathscr{C}_p$ of $\mathscr{G}$, one has $\mathscr{C}_1 \subsetneq \mathscr{C}_p$. Every uncritical link of $\mathscr{G}$ is an uncritical link of $\mathscr{G}_1 = (\mathscr{V}, \mathscr{E} \backslash \mathscr{C}_1)$.*

Proof: The proof follows by contradiction. Suppose that $\varepsilon_1 \in \mathscr{E} \backslash \mathscr{C}_1$ is an uncritical link of $\mathscr{G}$ that is critical in $\mathscr{G}_1$. Since $lc(\mathscr{G}_1; \mathscr{R}) = lc(\mathscr{G}; \mathscr{R}) - |\mathscr{C}_1|$, there exist a set $\mathscr{C}_p^1$ of $lc(\mathscr{G}; \mathscr{R}) - |\mathscr{C}_1| - 1$ links in $\mathscr{G}_1$ such that the removal of $\{\varepsilon_1\} \cup \mathscr{C}_p^1$ from $\mathscr{G}_1$ leads to an uncontrollable digraph $\mathscr{G}_2$. Therefore, $\{\varepsilon_1\} \cup \mathscr{C}_p^1 \cup \mathscr{C}_1$ is a set of $lc(\mathscr{G}; \mathscr{R})$ links whose removal reduces $\mathscr{G}$ to the uncontrollable digraph $\mathscr{G}_2$. This in turn implies that $\{\varepsilon_1\} \cup \mathscr{C}_p^1 \cup \mathscr{C}_1$ is a critical link-set of $\mathscr{G}$ which is a contradiction since ε_1 is an uncritical link of $\mathscr{G}$. ∎

Remark 24. *In Proposition 4, the converse does not hold. In particular, the removal of a critical link can make some critical links uncritical in the resultant network. As an example, consider the digraph in Fig. 4.2. Following the removal of the critical link (v_4, v_5), every other critical link in the network will become uncritical except for the critical link that is the symmetric image of (v_4, v_5).*

Remark 25. *The four indices offered in this section can be used in their order of appearance to compare and prioritize the follower agents in the network. The primary influence on the information flow structure is captured by the agent and link criticality indices δ and θ. Of the* *two agents with the same criticality indices δ and θ, the one with the higher critical link index γ has a more prominent role, due to its effect in maintaining the critical links in the network. Finally, for the agents that cannot be distinguished based on the three indices δ, θ and γ, one can use the uncritical link index ψ to compare the agents based on their contributions to the flow of information across the uncritical links of the network.*

To demonstrate the applicability of the proposed indices for characterizing the importance of various agents in a network, the values of the indices are calculated for some of the digraphs introduced so far, and the results are listed in Table I. The following digraphs are considered: digraph $\bar{\mathscr{G}}_1$ of Fig. 4.1, digraph $\bar{\mathscr{G}}_2$ of Fig. 4.2, digraphs $\bar{\mathscr{G}}_3$ and $\bar{\mathscr{G}}_4$ of Fig. 4.3, and digraph $\bar{\mathscr{G}}_5$ of Fig. 4.4(a). Each row corresponds to one agent of a given digraph, and the rows corresponding to each digraph are in the order of the importance of the agents. In the cases, where due to the network symmetry two or more agents take identical roles, only one of them is represented in the table.

The results show that the proposed indices can be effectively combined to order the vertices of the digraph $\bar{\mathscr{G}}_1$ based on their priority. Accordingly, the critical agents are v_5 and v_4, which are distinguished through their non-zero agent criticality indices. Amongst the uncritical agents, v_3 has a higher γ and hence a more prominent role, followed by v_2 and then v_1. On the other hand, amongst the critical agents, v_5 with the higher γ takes priority. This example indicates that once the critical and uncritical agents of the network are determined, then the critical link index can be effectively employed to distinguish between the agents in

Table 4.1: Summary of the Results

Vertex − Digraph	δ	θ	γ	ψ
$v_5 - \bar{\mathscr{G}}_1$	1	1	2	1
$v_4 - \bar{\mathscr{G}}_1$	1	1	1	0
$v_3 - \bar{\mathscr{G}}_1$	0	1	2	0
$v_2 - \bar{\mathscr{G}}_1$	0	1	1	0
$v_1 - \bar{\mathscr{G}}_1$	0	0	0	1
$v_5 - \bar{\mathscr{G}}_2$	1	2	0	12
$v_4 - \bar{\mathscr{G}}_2$	0	1	1	0
$v_3 - \bar{\mathscr{G}}_2$	0	0	0	4
$v_2 - \bar{\mathscr{G}}_2$	0	0	0	4
$v_1 - \bar{\mathscr{G}}_2$	0	0	0	4
$v_7 - \bar{\mathscr{G}}_3$	1	2	6	0
$v_6 - \bar{\mathscr{G}}_3$	1	1	2	0
$v_5 - \bar{\mathscr{G}}_3$	1	1	2	0
$v_4 - \bar{\mathscr{G}}_3$	0	1	3	0
$v_3 - \bar{\mathscr{G}}_3$	0	1	2	0
$v_2 - \bar{\mathscr{G}}_3$	0	1	2	0
$v_1 - \bar{\mathscr{G}}_3$	0	1	1	0
$v_4 - \bar{\mathscr{G}}_4$	0	0	0	1
$v_3 - \bar{\mathscr{G}}_4$	0	0	0	0
$v_2 - \bar{\mathscr{G}}_4$	0	0	0	0
$v_1 - \bar{\mathscr{G}}_4$	0	0	0	0
$v_2 - \bar{\mathscr{G}}_5$	1	1	1	4
$v_1 - \bar{\mathscr{G}}_5$	1	0	0	2

each subcategory.

In the digraph $\bar{\mathscr{G}}_2$, the existence of a network bottleneck is effectively reflected in the agent and link criticality indices of v_5, whereas the superiority of v_4 is portrayed by its link criticality index. For the digraph $\bar{\mathscr{G}}_3$, all of the links are critical. Therefore, the uncritical link index ψ is zero for all agents in this digraph. However, the agent and link criticality indices indicate that agent v_7 plays the most prominent role in this network, as it maintains "the flow of information" to the two network sections formed by the agent subsets $\{v_1, v_3, v_5\}$ and $\{v_2, v_4, v_6\}$.

From Table I, it is clear that the proposed indices do not distinguish between the vertices v_1 to v_3 in the digraph $\bar{\mathscr{G}}_4$, while vertex v_4 of that digraph is prioritized. The importance of v_4 in $\bar{\mathscr{G}}_4$ does not come from the fact that it is the head of the only critical link in the network. Rather, its importance is due to the uncritical link (v_4, v_3) and the role of v_4 in maintaining the flow of information from the root vertex to vertex v_3. The digraph $\bar{\mathscr{G}}_4$ is an example of the pathological case described by Lemma 9, in which case the agent controllability degree of the links belonging to the out-cut of the root-set may provide the designer with some clues as to the importance of the head vertices involved. Finally, in the case of the digraph $\bar{\mathscr{G}}_5$, the fact that all follower agents are critical means that the agent criticality index δ cannot distinguish between the vertices. The link criticality index θ, however, points to the superior role of v_2 in this digraph.

This section is concluded by two remarks, which point out the polynomial-time computability of the proposed importance measures in Sections 4.2 and 4.3, and the possibility of considering higher order indices for a refined characterization of the uncritical links.

Remark 26. *The authors in [50, 52] provide polynomial-time algorithms for computation of agent and link controllability degrees in a given digraph. On the other hand, Remarks 21 to 23 together with Theorem 7 indicate that the criticality of a link/agent is related to the difference of two link/agent controllability degrees, and is therefore decidable in polynomial-time. This in turn implies that the critical link index introduced in Definition 14 is computable in polynomial-time. By the same token, the notions of agent controllability index and agent/link criticality index are all given by differences between controllability degrees in Definitions 11-13, and thus they are all computable in polynomial-time. Lastly, if for a particular uncritical link or set of links the criticality of other links with and without those links is determined, then Definitions 10 and 15 can be used to compute the link controllability index and uncritical link index of various links and agents in polynomial-time. In conclusion, all of the notions introduced in this chapter for the characterization and quantification of the relative importance of agents/links, are computable and/or decidable in polynomial-time using the previously developed algorithms.*

Remark 27. *From the examples provided thus far, it is clear that the uncritical link index fails as a measure to distinguish between the agents v_1 to v_3 in the digraph $\bar{\mathscr{G}}_2$. Similarly, the link controllability index for all but one of the uncritical links in the digraph $\bar{\mathscr{G}}_4$ is zero,* *which means that the link controllability index does not provide an effective measure for the prioritization of the uncritical links in this network. This and other similar cases suggest that higher-order importance measures for the uncritical links can offer a refined characterization of the information flow across the network. Accordingly, one may investigate the impact of the removal of a certain uncritical link or the outgoing uncritical links of a given agent together with any set of m other uncritical links in the network, and calculate the maximum possible increment in the number of critical links of the network after such removals. This provides an $m-th$ order link controllability index or uncritical link index that can facilitate the ordering and prioritization of the uncritical links and follower agents in the network. In particular, the polynomial-time calculation of such indices is an interesting direction for future research.*

The following section and the subsequent definitions, lemmas, and theorems provide useful tools for investigating the effects of simultaneous link and agent failures on the controllability of an information flow digraph.

4.4 Joint Controllability

The concept of joint controllability degree parallels the notions of agent and link controllability degrees in Subsection 2.3.1, and facilitates the extension of the previous results to the cases where multiple simultaneous failures occur, affecting both links and agents in the network.

4.4.1 Joint Controllability Degree

Definition 16. *An information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R} \subset \mathscr{V}$ *is said to be* joint $(r,s)-$controllable *if it remains controllable in the case of simultaneous failure of any set of links of size* $u \leqslant r$ *and any set of non-root vertices of size* $v \leqslant s$*, where* $u+v < r+s$ *(note the strict inequality in the last expression).*

The next lemma follows immediately from Definitions 3, 4 and 16.

Lemma 10. *The following statements hold:*

a) If $\mathscr{G}$ *is joint* $(r,s)-$*controllable, then* $\forall u \leqslant r$ *and* $\forall v \leqslant s$*,* $\mathscr{G}$ *is joint* $(u,v)-$*controllable.*

b) If $\mathscr{G}$ *is joint* $(r,s)-$*controllable, then* $r \leqslant lc(\mathscr{G};\mathscr{R})$ *and* $s \leqslant ac(\mathscr{G};\mathscr{R})$*.*

c) If $\mathscr{G}$ *is joint* $(r,s)-$*controllable and* $lc(\mathscr{G};\mathscr{R}) = r$*, then* $s = 0$*.*

d) If $\mathscr{G}$ *is joint* $(r,s)-$*controllable and* $ac(\mathscr{G};\mathscr{R}) = s$*, then* $r = 0$*.*

Definition 17. *An information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R} \subset \mathscr{V}$ *is said to be* joint $t-$controllable *if* t *is the largest number such that* $\mathscr{G}$ *is joint* $(u,v)-$*controllable for all* $u+v \leqslant t$*. Moreover, a minimal set of* r *vertices and* $s = t-r$ *edges whose removal makes* $\mathscr{G}$ *uncontrollable is referred to as a* critical agent-link set*, and is denoted by* $\mathscr{C}_{rs} \subset (\mathscr{V} \cup \mathscr{E} \backslash \mathscr{R})$*. The number* t *is called the* joint controllability degree *of the digraph* $\mathscr{G}$ *w.r.t. the root-set* $\mathscr{R}$*, and is denoted by* $jc(\mathscr{G};\mathscr{R})$*.*

From Definitions 16 and 17, it follows that a sufficient condition for the preservation of controllability in the face of simultaneous failure in links and agents is that the total number of failed links and agents is less than the joint controllability degree of the underlying information flow digraph.

Theorem 8. *Given an information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R}$*,* $jc(\mathscr{G};\mathscr{R}) = \min\{lc(\mathscr{G};\mathscr{R}), ac(\mathscr{G};\mathscr{R})\}$.

Proof: The proof follows by contradiction. Let $lc(\mathscr{G};\mathscr{R}) = p$, $ac(\mathscr{G};\mathscr{R}) = q$ and $jc(\mathscr{G};\mathscr{R}) = t$, and suppose that $t < \min(p,q)$. From Definition 17, for some $\{r,s\} \subset \mathbb{N}_t$ and $r+s = t$, there exists a critical agent-link set $\mathscr{C}_{rs}$ which can be partitioned as $\mathscr{C}_{rs} = \mathscr{A} \cup \mathscr{L}$, where $\mathscr{A} \subset \mathscr{V}$, $\mathscr{L} \subset \mathscr{E}$, $|\mathscr{A}| = s$, and $|\mathscr{L}| = r$. Moreover, the removal of $\mathscr{C}_{rs}$ from $\mathscr{G}$ leads to an uncontrollable digraph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$, where $\mathscr{V}_1 = \mathscr{V} \backslash \mathscr{A}$. Let $\mathscr{B}$ denote the set of agents corresponding to the heads of the links in $\mathscr{L}$. It follows that $|\mathscr{B}| \leqslant r$ and $\mathscr{B} \cap \mathscr{A} = \varnothing$, because otherwise the links whose heads belong to $\mathscr{B} \cap \mathscr{A}$ can be deleted from $\mathscr{C}_{rs}$ leading to a smaller agent-link set whose removal makes $\mathscr{G}$ uncontrollable, which contradicts with Definition 17 and $\mathscr{C}_{rs}$ being a critical agent-link set. Next, it follows from $\mathscr{B} \cap \mathscr{A} = \varnothing$ and $|\mathscr{B}| \leqslant r$ that $|\mathscr{A} \cup \mathscr{B}| \leqslant r+s = t < q$. Thus, the deletion of the agent-set $\mathscr{A} \cup \mathscr{B}$ from $\mathscr{G}$ leads to a controllable digraph $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, where $\mathscr{V}_2 = \mathscr{V}_1 \backslash \mathscr{B}$. This in turn implies that every agent belonging to $\mathscr{V}_1 \backslash \mathscr{B}$ in the digraph $\mathscr{G}_1$ is reachable from $\mathscr{R}$ and vice versa. The latter (the converse statement) follows from the fact that if there exists a vertex $v \in \mathscr{B}$ that is reachable from the root-set in $\mathscr{G}_1$, then the corresponding link in $\mathscr{L}$ whose head is v can be deleted from

$\mathscr{C}_{rs}$, leading to a contradictorily smaller critical agent-link set. Next for digraph $\mathscr{G}_1$, note that since every vertex in $\mathscr{B}$ is unreachable from $\mathscr{R}$ while every vertex in $\mathscr{V}_1 \backslash \mathscr{B}$ is reachable from $\mathscr{R}$, one should have $\partial^+_{\mathscr{G}_1}\mathscr{V}_1 \cap \partial^-_{\mathscr{G}_1}\mathscr{B} = \varnothing$ or equivalently $\partial^+_{\mathscr{G}}\mathscr{V}_1 \cap \partial^-_{\mathscr{G}}\mathscr{B} = \mathscr{L}$, because otherwise those vertices in $\mathscr{B}$ which are the heads of some links in $\partial^+_{\mathscr{G}_1}\mathscr{V}_1 \cap \partial^-_{\mathscr{G}_1}\mathscr{B}$ will be reachable from the root-set in $\mathscr{G}_1$. Now, consider an arbitrary vertex $\hat{v}$ belonging to the agent-set $\mathscr{B}$ in the digraph $\mathscr{G}$. Such a vertex is the head of some l links in $\mathscr{L}$, where $l \leqslant |\mathscr{L}|$. In addition to these l links, $\hat{v}$ can only be the head of some links whose tails are either any of the $m \leqslant |\mathscr{L}| - l = r - l$ agents in $\mathscr{B}$ or any of the s agents in $\mathscr{A}$. Hence, $|\partial^-_{\mathscr{G}}\{\hat{v}\}| \leqslant l + m + s \leqslant r + s$. On the other hand, the removal of $\partial^-_{\mathscr{G}}\{\hat{v}\}$ will make $\hat{v}$ unreachable from any vertex in $\mathscr{G}$, and therefore renders the digraph uncontrollable. This, however, is in contradiction with Definition 3 and the fact that $|\partial^-_{\mathscr{G}}\{\hat{v}\}| \leqslant r + s = t < p$. ■

The next corollary is a direct consequence of Theorems 3 and 8, and provides a necessary condition on the number of edges of a joint $t-$controllable digraph.

Corollary 8. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is joint $t-$controllable, then $|\mathscr{E}| \geqslant \max\{(|\mathscr{V}| - 1)t, |\mathscr{V}| + t - 2\}$.*

The next definition, and the theorem which follows, provide a mechanism to transform the problem of joint $t-$controllability of a given digraph into $q-$agent controllability of another digraph. This will, in turn, enable the multi-agent control system designer to take advantage of the polynomial-time algorithms developed in [50] and [52] for specifying the critical agent-link sets of a given digraph.

Definition 18. *Given a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, replace every edge $\varepsilon \in \mathscr{E}$ with two edges $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ in the same direction as ε, and connect them through an intermediate vertex $\hat{v}_\varepsilon$, termed a* black vertex. *The resulting digraph $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$ is called the* edge-duplicate *of $\mathscr{G}$. Every vertex of $\hat{\mathscr{G}}$ that is not a black vertex is referred to as a* white vertex.

Remark 28. *Given a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ and its edge-duplicate $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$, the following equalities hold for the number of vertices and edges: $|\hat{\mathscr{V}}| = |\mathscr{V}| + |\mathscr{E}|$ and $|\hat{\mathscr{E}}| = 2|\mathscr{E}|$. Moreover, every white vertex $\hat{v}_v \in \hat{\mathscr{V}}$ corresponds to one vertex $v \in \mathscr{V}$ and every black vertex $\hat{v}_\varepsilon \in \hat{\mathscr{V}}$ corresponds to one edge $\varepsilon \in \mathscr{E}$. There exists a one-to-one correspondence between the sets $\mathscr{V} \cup \mathscr{E}$ and $\hat{\mathscr{V}}$.*

Theorem 9. *Consider a digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ and its edge-duplicate $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\mathscr{E}})$. The digraph $\mathscr{G}$ is joint $t-$controllable if and only if $\hat{\mathscr{G}}$ is $t-$agent controllable.*

Proof: The proof follows by construction, from the fact that Definition 18 specifies a bijection between the sets $\mathscr{V} \cup \mathscr{E}$ and $\hat{\mathscr{V}}$. Using this bijection, any critical agent-link set of $\mathscr{G}$ can be transformed into a critical agent-set of $\hat{\mathscr{G}}$ and vice versa. ■

A digraph $\mathscr{G}$ and its node-duplicate $\tilde{\mathscr{G}}$ and edge-duplicate $\hat{\mathscr{G}}$ (constructed according to Definitions 5 and 18) are depicted in Figs. 4.6(a)$-$(c). The digraph in 4.6(a), with the upper-most vertex as the root, is $2-$link and $2-$agent controllable. The digraph is also joint $(1,1)-$controllable. It is jointly critical as well as joint $2-$controllable. According to Theorem 9, the latter is tantamount to $2-$agent controllability of the digraph in Fig. 4.6(c). If the node-duplication process of Definition 5 is applied to every white vertex in an edge-duplicated digraph $\hat{\mathscr{G}}$, a new digraph is generated, which can be used to investigate the agent controllability of $\hat{\mathscr{G}}$ efficiently. The new digraph is termed the *node-edge-duplicate* of $\mathscr{G}$, and is denoted by $\breve{\mathscr{G}}$. The node-edge-duplicate of the digraph in Fig. 4.6(a) is depicted in Fig. 4.6(d). Lemma 5 provides a method for converting the problem of agent controllability in $\hat{\mathscr{G}}$ to link controllability in $\breve{\mathscr{G}}$. Polynomial-time algorithms presented in [50, 52] for investigating the link controllability of a given digraph can now be applied to $\breve{\mathscr{G}}$, which, in fact, gives the joint controllability degree of the original digraph $\mathscr{G}$.

In the next subsection, the important class of *jointly critical digraphs* is introduced and their role in characterizing robustness against simultaneous failures in both links and agents is highlighted.

4.4.2 Jointly Critical Digraphs

The notion of agent controllability index from Subsection 4.2.2 can be exploited to characterize and compare the relative susceptibility of digraphs with regard to agent or link failures. Accordingly, in Lemmas 11 and 12, as well as Theorem 11 which follow, three classes of digraphs, termed as *agent-critical*, *link-critical* and *jointly critical*, are introduced and some of their important characteristics are pointed out.

Lemma 11. *For an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with root-set $\mathscr{R} \subset \mathscr{V}$, if $\forall \varepsilon \in \partial^+_{\mathscr{G}}\mathscr{R}$, $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 1$, then $jc(\mathscr{G};\mathscr{R}) = ac(\mathscr{G};\mathscr{R})$. Such a digraph for which the aforementioned assumption holds will be referred to as* agent-critical.

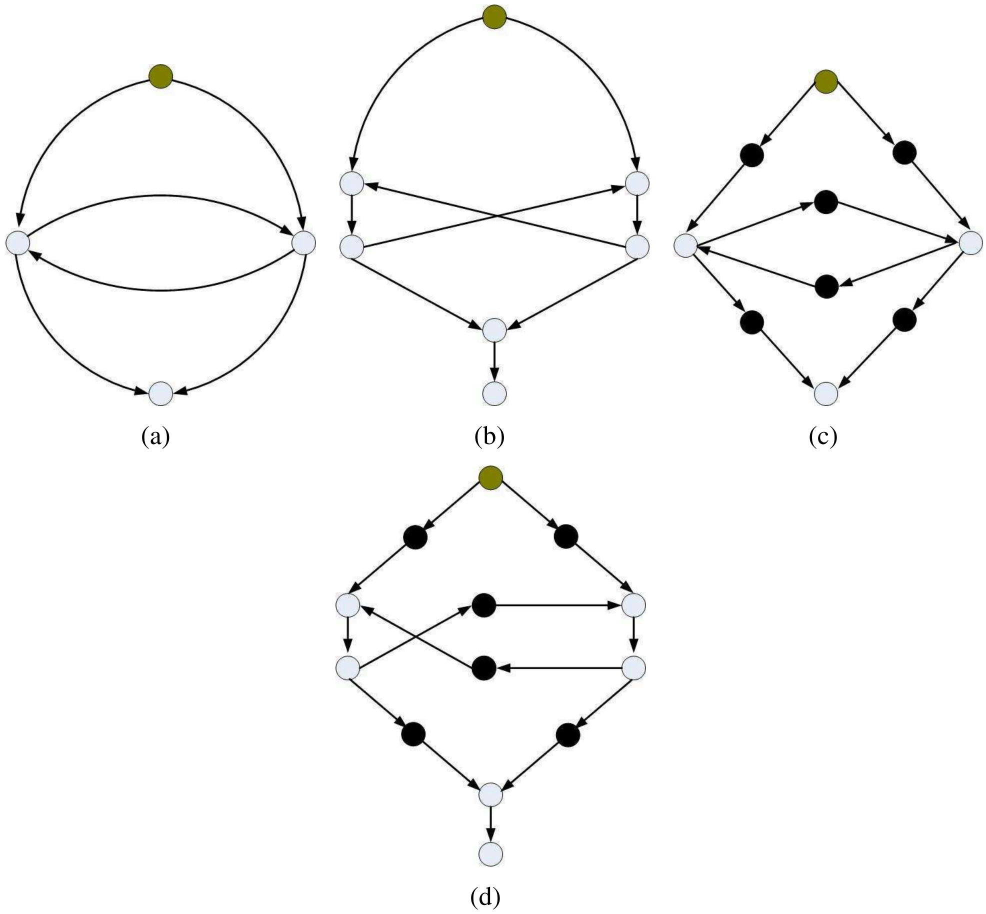


Figure 4.6: A digraph $\mathscr{G}$, its node-duplicate $\tilde{\mathscr{G}}$, its edge-duplicate $\hat{\mathscr{G}}$, and its node-edge-duplicate $\breve{\mathscr{G}}$ are depicted in (a), (b), (c) and (d), respectively. The uppermost vertex in all four digraphs is assumed to be the root.

Proof: Using Theorem 8, it suffices to introduce a set $\mathscr{A} \subset \mathscr{V} \backslash \mathscr{R}$ with the property $|\mathscr{A}| \leqslant lc(\mathscr{G};\mathscr{R})$, whose removal makes $\mathscr{G}$ uncontrollable. To this end, consider a solution $\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}$ to the minimization problem in (2.12), which means that $|\partial^+_{\mathscr{G}} \mathscr{X}| = lc(\mathscr{G};\mathscr{R})$. The following routine utilizes $\partial^+_{\mathscr{G}} \mathscr{X}$ to generate one such set $\mathscr{A}$ with the desired characteristics.

■

Remark 29. *When applying Routine 1 to an agent-critical digraph, it is notable that the*

Routine 1

1: $\mathscr{A} = \varnothing$
2: **for all** $(\tau, \nu) \in \partial^{+}_{\mathscr{G}} \mathscr{X}$ **do**
3:    **if** $\tau \notin \mathscr{R}$ **then**
4:       $\mathscr{A} = \mathscr{A} \cup \{\tau\}$
5:    **else**
6:       $\mathscr{A} = \mathscr{A} \cup \{\nu\}$
7:    **end if**
8: **end for**
9: **return** $\mathscr{A}$

assumption in Lemma 11 together with Corollary 6 ensures that step 6 *will not be executed more than once for a given vertex* ν.

Fig. 4.3(a) shows the case of an agent-critical digraph, for which the agent and link controllability degrees are given by $q = 2$ and $p = 3$, respectively, and they satisfy the relation $q \leqslant p$, as suggested by Lemma 11.

Lemma 12. *Consider an information flow digraph* $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ *with the root-set* $\mathscr{R} \subset \mathscr{V}$*. If there exists a critical agent-set* $\mathscr{C}_q \subset \mathscr{V} \backslash \mathscr{R}$ *of* $\mathscr{G}$ *such that* $\forall \nu \in \mathscr{C}_q, \exists \varepsilon \in \partial^{+}_{\mathscr{G}}\{\nu\}$*, for which* $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) \neq 0$*, then* $jc(\mathscr{G};\mathscr{R}) = lc(\mathscr{G};\mathscr{R})$*. Such a digraph for which the aforementioned assumption holds will be referred to as* link-critical.

Proof: The proof follows upon application of Theorem 8 and introducing a set $\mathscr{B} \subset \mathscr{E}$ with the property $|\mathscr{B}| \leqslant ac(\mathscr{G};\mathscr{R})$, whose removal makes $\mathscr{G}$ uncontrollable. Let $\mathscr{C}_q$ be a critical agent-set satisfying the condition of Lemma 12. The following routine utilizes $\mathscr{C}_q$ to generate one such set $\mathscr{B}$ with the property that $\mathscr{B} \subset \mathscr{E} \backslash \partial^{+}_{\mathscr{G}} \mathscr{R}$ and $|\mathscr{B}| = |\mathscr{C}_q| = ac(\mathscr{G};\mathscr{R})$. ■

Routine 2

1: $\mathscr{B} = \varnothing$
2: **for all** $v \in \mathscr{C}_q$ **do**
3:   **for all** $\varepsilon \in \partial^+_{\mathscr{G}}\{v\}$ **do**
4:     **if** $\mathscr{B} \cap \partial^+_{\mathscr{G}}\{v\} = \varnothing$ **and** $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 1$ **then**
5:       $\mathscr{B} = \mathscr{B} \cup \{\varepsilon\}$
6:     **end if**
7:   **end for**
8: **end for**
9: **return** $\mathscr{B}$

Remark 30. *When applying Routine 2 to a link-critical digraph, it is notable that with $\mathscr{C}_q$ satisfying the conditions of Lemma 12, step 5 will be executed exactly once for every vertex $v \in \mathscr{C}_q$.*

Remark 31. *Using Theorem 6, it can be stated that digraph $\mathscr{G}$ in Lemma 12 is link-critical if there exists a critical agent-set $\mathscr{C}_q \subset \mathscr{V} \backslash \mathscr{R}$ of $\mathscr{G}$ such that $\forall v \in \mathscr{C}_q$, $\exists \varepsilon \in \partial^+_{\mathscr{G}}\{v\}$, for which $\rho(\mathscr{G}, \varepsilon; \mathscr{R}) = 1$.*

The digraphs $\bar{\mathscr{G}_5}$ and $\bar{\mathscr{G}_7}$ in Fig. 4.4 are both 3-agent and 2-link controllable. These digraphs are link-critical and they satisfy the condition of Lemma 12.

Theorem 10. *Consider a joint $(r,s)-$controllable digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with root-set $\mathscr{R} \subset \mathscr{V}$. If $\mathscr{G}$ is agent-critical or link-critical, then $r + s \leqslant \max\{lc(\mathscr{G}; \mathscr{R}), ac(\mathscr{G}; \mathscr{R})\}$.*

Proof: According to the results of Lemmas 11 and 12, it suffices to prove that if $\mathscr{G}$ is agent-critical, then $r + s \leqslant lc(\mathscr{G}; \mathscr{R})$, and if $\mathscr{G}$ is link-critical, then $r + s \leqslant ac(\mathscr{G}; \mathscr{R})$. For an agent-critical digraph $\mathscr{G}$, consider a solution $\mathscr{R} \subseteq \mathscr{X} \subset \mathscr{V}$ to the minimization problem in (2.12). According to Theorem 2, the link controllability degree of $\mathscr{G}$ is equal to the out-degree of $\mathscr{X}$, i.e. $|\partial^+_{\mathscr{G}}\mathscr{X}| = lc(\mathscr{G};\mathscr{R})$, and it follows from Lemma 10(b) that $r \leqslant lc(\mathscr{G};\mathscr{R})$. If $r = lc(\mathscr{G};\mathscr{R})$, then Lemma 10(c) requires that $s = 0$, and hence the statement of the above theorem holds. If on the other hand $r < lc(\mathscr{G};\mathscr{R})$, then choose a set of edges $\mathscr{Z}_r \subset \mathscr{E}$ such that $\mathscr{Z}_r \subset \partial^+_{\mathscr{G}}\mathscr{X}$ and $|\mathscr{Z}_r| = r$. Use Routine 1 after replacing $\partial^+_{\mathscr{G}}\mathscr{X}$ with $\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r$ to generate a set $\mathscr{A} \subset \mathscr{V}$. Now, $\mathscr{A} \cup \mathscr{Z}_r$ is a set of $|\mathscr{A}|$ vertices and $|\mathscr{Z}_r| = r$ edges, for which $\mathscr{G}_{\mathscr{A},\mathscr{Z}_r} = (\mathscr{V}\backslash\mathscr{A}, \mathscr{E}\backslash\mathscr{Z}_r)$ is uncontrollable. However, $\mathscr{G}$ is joint $(r,s)-$controllable, which implies that $s \leqslant |\mathscr{A}|$. On the other hand, it follows from Routine 1 that $|\mathscr{A}| \leqslant |\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r|$. Hence $s \leqslant |\partial^+_{\mathscr{G}}\mathscr{X}\backslash\mathscr{Z}_r|$ or $s \leqslant lc(\mathscr{G};\mathscr{R}) - r$, which completes the proof for an agent-critical digraph. A similar argument can be utilized to prove the statement for the case where $\mathscr{G}$ is link-critical. From Lemma 10(b), it is clear that $s \leqslant ac(\mathscr{G};\mathscr{R})$. Now, starting from a critical agent-set $\mathscr{C}_q$ that satisfies the condition of Lemma 12, select an arbitrary subset $\mathscr{Z}_s \subset \mathscr{C}_q$ of $s = |\mathscr{Z}_s|$ agents. If $s = ac(\mathscr{G};\mathscr{R})$, then Lemma 10(d) requires that $r = 0$ and the statement for the link-critical case holds. If $s < ac(\mathscr{G};\mathscr{R})$, then applying Routine 2 to $\mathscr{C}_q\backslash\mathscr{Z}_s$ yields a set $\mathscr{B}$ of $|\mathscr{C}_q\backslash\mathscr{Z}_s| = ac(\mathscr{G};\mathscr{R}) - s = |\mathscr{B}|$ links, whose deletion together with the s agents in $\mathscr{Z}_s$ will render $\mathscr{G}$ uncontrollable. The proof for a link-critical digraph $\mathscr{G}$ follows now upon the realization that since $\mathscr{G}$ is joint $(r,s)-$controllable, one should have $|\mathscr{B}| = ac(\mathscr{G};\mathscr{R}) - s \geqslant r$. ■

Remark 32. *Using the pathological class of digraphs described in Lemma 9 and Remark 21, it is straightforward to construct a joint $(r,s)-$controllable digraph $\mathscr{G}$ with the root-set $\mathscr{R}$, such that $r+s > \max\{ac(\mathscr{G};\mathscr{R}), lc(\mathscr{G};\mathscr{R})\}$. Non-trivial counterexamples are also possible* *and important. The joint $(3,2)-$controllable digraph in Fig. 4.7 is $4-$link and $3-$agent controllable.*

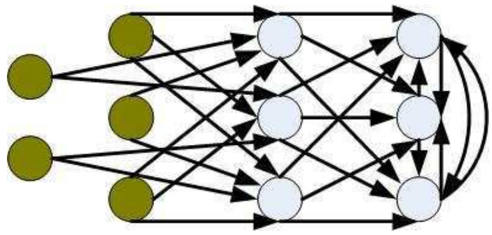


Figure 4.7: An example of a joint $(r,s)-$controllable digraph $\mathscr{G}$ which does not satisfy the relation $r+s \leqslant \max\{ac(\mathscr{G};\mathscr{R}), lc(\mathscr{G};\mathscr{R})\}$.

The next corollary is a direct consequence of Theorems 3 and 10, and it presents a necessary condition on the number of edges in the agent-critical or link-critical digraphs for joint $(r,s)-$controllability. The result can be used in the design of reliable multi-agent control systems.

Corollary 9. *Consider a joint $(r,s)-$controllable information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. The following statements hold: if $\mathscr{G}$ is agent-critical, then $|\mathscr{E}| \geqslant (|\mathscr{V}|-1)(r+s)$. Moreover, if $\mathscr{G}$ is link-critical, then $|\mathscr{E}| \geqslant |\mathscr{V}|+r+s-2$.*

The next theorem and the remark that follows capture the significance of joint controllability degree for the so-called *jointly critical* digraphs.

Theorem 11. *If an information flow digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ with the root-set $\mathscr{R} \subset \mathscr{V}$ is both agent-critical and link-critical, then $jc(\mathscr{G};\mathscr{R}) = ac(\mathscr{G};\mathscr{R}) = lc(\mathscr{G};\mathscr{R})$. Moreover, for every $(r,s) \in \mathbb{W} \times \mathbb{W}$, the digraph $\mathscr{G}$ is joint $(r,s)-$controllable if and only if $r+s \leqslant jc(\mathscr{G};\mathscr{R})$. Such a digraph, which is both agent-critical and link-critical will be referred to as* jointly critical.

Proof: From Lemmas 11 and 12, it is immediate that $jc(\mathscr{G};\mathscr{R}) = ac(\mathscr{G};\mathscr{R}) = lc(\mathscr{G};\mathscr{R})$. The rest of the proof, also, can be sketched as a combination of the proofs of Lemmas 11 and 12. Starting from a critical link-set $\mathscr{C}_p \subseteq \mathscr{E}$, one can use Routine 1 to transform any set of links $\mathscr{L} \subseteq \mathscr{C}_p$ into a set of agents $\mathscr{A}$, where $|\mathscr{A}| \leqslant |\mathscr{L}|$, such that the removal of $\mathscr{A}$ along with the links in $\mathscr{C}_p \backslash \mathscr{L}$ renders the digraph uncontrollable. Moreover, $|\mathscr{A}| = |\mathscr{L}|$ because the inequality $|\mathscr{A}| < |\mathscr{L}|$ contradicts the fact that $ac(\mathscr{G};\mathscr{R}) = lc(\mathscr{G};\mathscr{R}) = |\mathscr{C}_p|$. On the other hand, starting from a critical agent-set $\mathscr{C}_q \subseteq \mathscr{V} \backslash \mathscr{R}$ that satisfies the condition of Lemma 12, one can use Routine 2 to transform any set of agents $\mathscr{B} \subseteq \mathscr{C}_q$ into a set of $|\mathscr{B}|$ links, which if removed together with the agents in $\mathscr{C}_q \backslash \mathscr{A}$, then the digraph becomes uncontrollable. ■

Remark 33. *If a digraph $\mathscr{G}$ with the joint controllability degree t is jointly critical, then for all $(r,s) \in \mathbb{W} \times \mathbb{W}$ satisfying the inequality $r+s > t$, $\mathscr{G}$ is not joint $(r,s)-$controllable. Hence, the joint controllability degree alone completely characterizes the controllability preservation properties of the digraph $\mathscr{G}$. In other words, if the values of $(r,s) \in \mathbb{W} \times \mathbb{W}$ for which $\mathscr{G}$ is joint $(r,s)-$controllable are depicted as discrete points in the plane, then a pair of non-negative integers belongs to the jointly controllable set if and only if the corresponding point in the (r,s)-plane lies in the region $r+s \leqslant t$.*

Three special cases of interest are addressed in the sequel.

Complete Digraphs

As a special case, a digraph $\mathscr{G}_{cn} = (\mathscr{V}_{cn}, \mathscr{E}_{cn})$ is called *complete* if $\mathscr{E}_{cn} = \mathscr{V}_{cn} \times \mathscr{V}_{cn}$. Select a vertex r in a complete digraph as the root, and remove the $|\mathscr{V}_{cn}| - 1$ edges headed by the vertex r. Then the resultant information flow digraph is $(|\mathscr{V}_{cn}| - 1)-$link controllable [50]. This is the maximum value for link controllability degree in an information flow digraph with $n = |\mathscr{V}_{cn}|$ vertices, because a complete digraph possesses the maximum possible number of edges per a given number of vertices. The following proposition suggests that the joint controllability degree of a complete information flow digraph $\mathscr{G}_{cn}$ is also $n-1$.

Proposition 5. *Given a complete digraph $\mathscr{G}_{cn} = (\mathscr{V}_{cn}, \mathscr{E}_{cn})$ with $|\mathscr{V}_{cn}| = n$, choose a vertex r as the root and remove the $n-1$ edges which are headed by r. The resulting information flow digraph is jointly critical and joint $(n-1)-$controllable.*

Proof: The proof follows from the fact that $\mathscr{G}_{cn}$ has exactly $n-1$ disjoint rv paths for every $v \in \mathscr{V}_{cn} \setminus \{r\}$. ■

Remark 34. *It is to be noted that $jc(\mathscr{G}_{cn}; \{r\}) = n-1$ is the highest attainable joint controllability degree for a digraph with n vertices. This explains the desirable controllability preservation properties of the complete digraphs in the face of simultaneous link and agent failures.*

Kautz Digraphs

Kautz digraphs are introduced and discussed in Section 3.3 of [55]. Accordingly, a Kautz digraph $\mathscr{G}_k = (\mathscr{V}_k, \mathscr{E}_k)$ with $|\mathscr{V}_k| = n$ is given by:

$$\mathscr{V}_k = \{v_1, \ldots, v_n\}, \tag{4.1}$$

$$\mathscr{E}_k = \left\{(v_i, v_j) | i, j \in \mathbb{N}_n \wedge j \equiv (-id - \tau) \bmod n,\ \tau \in \mathbb{N}_d\right\},$$

for some $d \in \mathbb{N}\backslash\{1\}$ and $\kappa \in \mathbb{N}$, such that $d^\kappa + d^{\kappa-1} = n$. The following proposition gives the joint link controllability degree of an information flow digraph derived from a Kautz digraph.

Proposition 6. *Consider a Kautz digraph $\mathscr{G}_k = (\mathscr{V}_k, \mathscr{E}_k)$ where $\mathscr{V}_k$ and $\mathscr{E}_k$ are given by* (4.1)*. Choose a vertex r as the root and remove all edges which are headed by r. The resulting information flow digraph is jointly critical and joint $d-$controllable.*

Proof: The proof follows upon noting that $\mathscr{G}_k$ has exactly d disjoint rv paths for every $v \in \mathscr{V}_k \backslash \{r\}$. ■

Circulant Digraphs

Circulant digraphs are introduced and discussed in Section 3.4.5 of [55]. Accordingly, a circulant digraph $\mathscr{G}_c = (\mathscr{V}_c, \mathscr{E}_c)$ with $|\mathscr{V}_c| = n$ is given by:

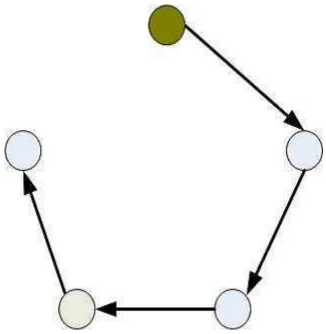
$$\begin{aligned} \mathscr{V}_c &= \{v_1, \ldots, v_n\}, \\ \mathscr{E}_c &= \left\{(v_i, v_j) | i, j \in \mathbb{N}_n \wedge j - i \equiv b \bmod n,\ b \in \mathscr{B}\right\}, \end{aligned} \tag{4.2}$$

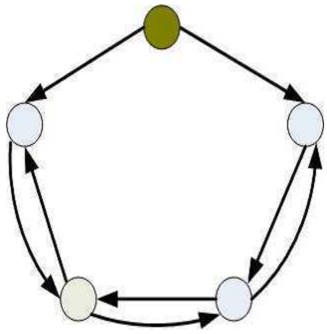
for some $\mathscr{B} \subseteq \mathbb{N}_{n-1}$. Choose a vertex $r \in \mathscr{V}_c$ as the root, and remove every edge whose head is r. Then in the resulting information flow digraph $\mathscr{G}_c$, $lc(\mathscr{G}_c; \{r\}) = |\mathscr{B}|$. This is due to the fact that $\mathscr{G}_c$ has exactly $|\mathscr{B}|$ edge-disjoint rv paths for every $v \in \mathscr{V}_c \setminus \{r\}$. For $|\mathscr{V}_i| = 5$, $i \in \mathbb{N}_3$, the choices of $\mathscr{B}_1 = \{1\}$, $\mathscr{B}_2 = \{1, n-1\}$, and $\mathscr{B}_3 = \{1, n-2\}$ correspond to a simple loop $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$, a distributed double-loop $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, and a daisy chain loop $\mathscr{G}_3 = (\mathscr{V}_3, \mathscr{E}_3)$, respectively. These digraphs are introduced in Section 3.4.1 of [55], and they are depicted in Figs. 4.8(a)$-$(c). For a simple loop $lc(\mathscr{G}_1; \{r\}) = ac(\mathscr{G}_1; \{r\}) = jc(\mathscr{G}_1; \{r\}) = 1$, while for the other two cases $lc(\mathscr{G}_i; \{r\}) = ac(\mathscr{G}_i; \{r\}) = jc(\mathscr{G}_i; \{r\}) = 2$, $i = 2, 3$. These three digraphs have the additional property that for any r, s satisfying the inequality $r + s > jc(\mathscr{G}_i; \{r\})$, $i \in \mathbb{N}_3$, $\mathscr{G}_i$ is not joint $(r, s)-$controllable. Accordingly, the joint controllability degree alone completely characterizes the controllability preservation properties for $\mathscr{G}_i$, $i \in \mathbb{N}_3$. This is due to the fact that $\mathscr{G}_i$, $i \in \mathbb{N}_3$ are jointly critical.

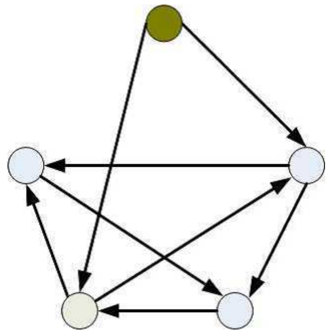
On the other hand, for the circulant digraph $\mathscr{G}_4$ with $|\mathscr{V}_4| = 6$, $\mathscr{B}_4 = \{2, 3, 5\}$ and the uppermost vertex selected as the root r, the resulting information flow digraph, shown in Fig. 4.8(d), is $3-$link and $2-$agent controllable [50]. This digraph is neither agent-critical nor link-critical, and hence is not jointly critical. The joint controllability degree for $\mathscr{G}_4$ is 2, and unlike $\mathscr{G}_i,\ i \in \mathbb{N}_3,\ jc(\mathscr{G}_4;\{r\}) = 2$ does not proffer a full characterization of the controllability preservation properties for $\mathscr{G}_4$. Accordingly, $\mathscr{G}_4$ is joint $(r,s)-$controllable for $(r,s) \in \{(2,1),(3,0)\}$, although $r+s > jc(\mathscr{G}_4;\{r\})$. To clarify this point, let the values of $(r,s) \in \mathbb{W} \times \mathbb{W}$ for which $\mathscr{G}_i$ is joint $(r,s)-$controllable, $i \in \mathbb{N}_4$, be shown as discrete points in the plane. For $i \in \mathbb{N}_3$, the line $r+s = jc(\mathscr{G}_i;\{r\})$ divides the first quadrant of the (r,s)-plane into two regions, where for $r+s \leqslant jc(\mathscr{G}_i;\{r\})$, $\mathscr{G}_i$ is joint $(r,s)-$controllable and otherwise it is not. This is depicted in Fig. 4.9(a) for $\mathscr{G}_2$ and $\mathscr{G}_3$, where the closed shaded region contains all pairs of integers belonging to the joint controllability set (the points associated with these pairs are shown by black circles). This property, however, does not hold for $\mathscr{G}_4$; it is evident from Fig. 4.9(b) that there exist two points above the line $r+s = jc(\mathscr{G}_4;\{r\})$ representing the pairs for which $\mathscr{G}_4$ is still joint $(r,s)-$controllable.

4.5 Conclusions

Structural controllability of a network of single-integrator agents with leader-follower architecture was investigated. The notions of agent and link controllability index were defined to characterize and quantify the importance of individual links to the controllability of the overall network. Similarly, the notions of agent and link criticality index as well as the critical and


(a) $\mathscr{G}_1$


(b) $\mathscr{G}_2$


(c) $\mathscr{G}_3$

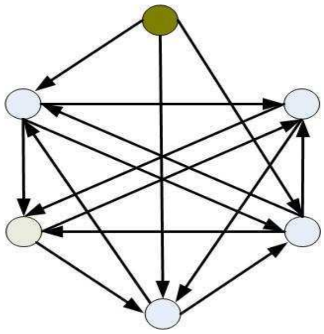
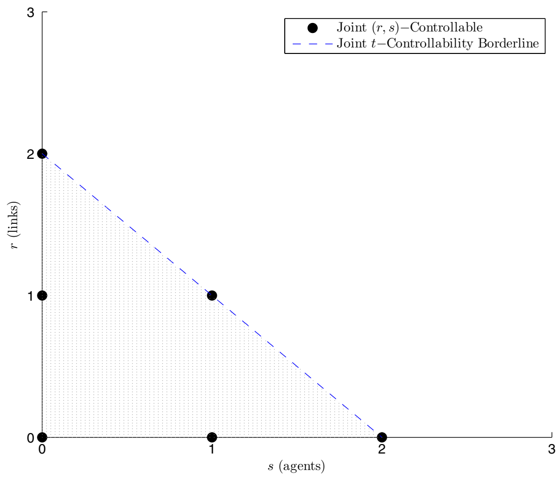

(d) $\mathscr{G}_4$

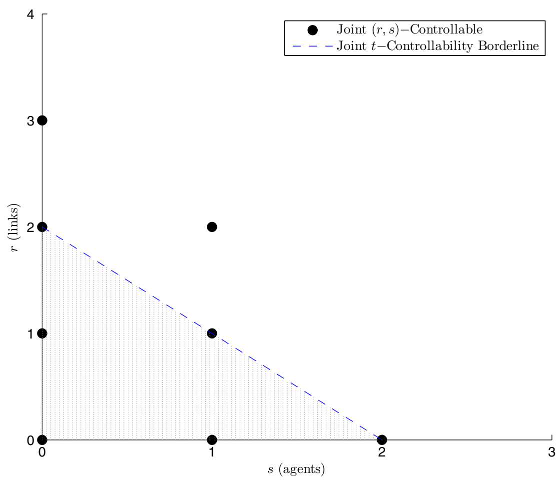
Figure 4.8: Circulant Digraphs

uncritical link index were offered as quantitative measures for the relative importance of individual agents to the controllability of the overall network. It was stated that the the proposed indices should be applied in their order of precedence, starting with the agent criticality index and ending with the uncritical link index. The results provide the designer of a multi-agent system with useful tools to evaluate (and enhance) the reliability of the network by deciding on which links and agents to prioritize for fault management and recovery operations.

In the next step, the concepts of joint $(r,s)-$controllability and joint $t-$controllability were proposed as quantitative measures of reliability in a multi-agent system subject to simultaneous failure of communication links and agents. It was noted that joint $t-$controllability is a *conservative* requirement which provides a *sufficient* condition for remaining controllable following the removal of any set of links and agents with size less than t. Nonetheless, for the important class of jointly critical digraphs, the joint controllability degree t proffers a *necessary and sufficient* condition, which fully characterizes the controllability preservation properties of the digraph. By and large, a digraph remains controllable after the removal of any u links and v agents *if and only if* there exists a pair $(r,s) \in \mathbb{W} \times \mathbb{W}$ such that the digraph is joint $(r,s)-$controllable and $u \leqslant r \wedge v \leqslant s \wedge u+v < r+s$. However, the authors' ongoing research indicates that for some digraphs, which are neither agent nor link critical, determining *all* (r,s) pairs for which the digraph is joint $(r,s)-$controllable may not be tractable in polynomial-time, and future research on this topic is of much interest. The presented results provide design guidelines for improving the network robustness against simultaneous failure of multiple links and agents. Several examples were offered to elucidate the results.



(a)



(b)

Figure 4.9: (a) The joint controllability of the circulant digraphs $\mathscr{G}_2$ and $\mathscr{G}_3$, given in Figs. 4.8(b), 4.8(c). (b) The joint controllability of the circulant digraph $\mathscr{G}_4$. The filled circles, which represent the pairs of integers belonging to the jointly controllable set are all in the shaded area in the case of (a), but not in the case of (b).

Chapter 5

Summary and Extensions

This chapter consists of a summary of the thesis contributions in Section 5.1, followed by some suggestions for future research in Section 5.2.

5.1 Summary of Contributions

The effect of link and agent failures on the dynamics and control of multi-agent networks was investigated in this thesis. In the first part, the focus was on whether the response that is observed from the state of a single agent will change by the removal of a set of links in the network. To this end, the linear agreement protocol was adopted as the dynamic interaction law, according to which the network evolves, and topological conditions were offered for detectability of a subset of links from the response of a given agent. The mathematical characterization of link failures is important in light of the crucial role that is played by the detection and isolation of faults in the management and recovery operations. Accordingly, the results provide useful design guidelines for realization of reliable and fault-tolerant multi-agent networks, and they have effectually opened a new horizon, where implications of network structure on the systems response under various fault and failure scenarios can be further explored and investigated. The contributed results are therefore of both theoretical and practical interest.

In the second part, the preservation of network controllability in the face of simultaneous link and agent failures was investigated and sufficient conditions were proffered that would ensure that the controllability of a network is preserved following the removal of any set of links and agents, whose size is smaller than a given number t. The study of simultaneous failures is important because in real-world multi-agent networks, various terrain properties or hardware faults may disable a number of control agents and limit the ability of others to communicate. The idea of considering the agent and link controllability degrees in a two dimensional plane is of particular significance in the study of multi-agent control systems and its study has lead to the provision of importance measures for comparing the role of different links and agents in the network.

5.2 Suggestions for Future Work

For Chapter 3, an analytical treatment of the effect of initial conditions is of theoretical interest. Based on the simulation results, it is conjectured that the derived results hold for almost all initial conditions. The extension of the available results to a control law that is less restrictive than (2.6) provides another window for future research. Also, one may consider the case when the designer has access to more than a single observation points and extend the results accordingly.

In Chapter 4, the coefficients in (2.10) are effectually required to satisfy various restrictions. These include the restrictions on their signs or their sums as considered in [95]. A question of interest for future research is if and how the derived results will continue to hold under such restrictions. Furthermore, it will be of practical significance to propose fixed α_{ij} coefficients for which the system remains controllable after the failures, without the need to update the system coefficients. It is also important to investigate the probability of remaining (structurally) controllable given a probability for link or agent failures. In general, it is not true that a $(p+1)-$link controllable digraph has a higher probability of remaining controllable than a $p-$link controllable digraph. Because such a probability depends on the number of critical links in the network and how they join each other to form the critical link-sets. As a closing remark, it should be highlighted that structural controllability is a systems property that yields itself to topological analysis much too easily, and it is therefore well-suited for a network-wise study. Despite this inherent convenience, the study of structural controllability in networks has received some criticisms as well. The authors in [96], for example, point out the issues of nodal and self-dynamics, as well as the lack of Gramian-link measures that are crucial in practical settings.

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