Connectivity Preservation in Distributed Control of Multi-Agent Systems

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ABSTRACT

Connectivity Preservation in Distributed Control of Multi-Agent Systems

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The problem of designing bounded distributed connectivity preserving control strategies for multi-agent systems is studied in this work. In distributed control of multi-agent systems, each agent is required to measure some variables of other agents, or a subset of them. Such variables include, for example, relative positions, relative velocities, and headings of the neighboring agents. One of the main assumptions in this type of systems is the connectivity of the corresponding network. Therefore, regardless of the overall objective, the designed control laws should preserve the network connectivity, which is usually a distance-dependent condition. The designed controllers should also be bounded because in practice the actuators of the agents can only handle finite forces or torques. This problem is investigated for two cases of single-integrator agents and unicycles, using a novel class of distributed potential functions. The proposed controllers maintain the connectivity of the agents that are initially in the connectivity range. Therefore, if the network is initially connected, it will remain connected at all times. The results are first developed for a static information flow graph, and then extended to the case of dynamic edge addition. Connectivity preservation for problems involving static leaders is covered as well. The potential functions are chosen to be smooth, resulting in bounded control inputs. These functions are subsequently used to develop connectivity preserving controllers for the consensus and containment problems. Collision avoidance is investigated as another relevant problem, where a bounded distributed swarm aggregation strategy with both connectivity preservation and collision avoidance properties is presented. Simulations are provided throughout the work to support the theoretical findings.

To my parents

for their love and sacrifice

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TABLE OF CONTENTS

List of Figures

Chapter 1

Introduction

Cooperative control of a group of autonomous agents has been extensively studied in the past few years. This relatively new line of research has been motivated by the increasing application of multi-agent systems such as mobile robots, formation flying of UAVs, deep-space missions and spacecraft formation, automated highway systems, air traffic control, and mobile sensor networks [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. The main goal in such applications is to find distributed control paradigms satisfying a global objective defined over the entire network. Examples of such an objective include flocking, consensus, rendezvous, containment, and formation [13, 14, 15, 16, 17, 18, 19, 20]. For instance, in the flocking problem it is aimed to achieve the convergence of the velocity and orientation of every agent to a common value [13, 14], whereas in the consensus and rendezvous problems it is desired that all the agents in the group reach a single point in the state space [15, 16, 17, 18]. In the formation control problem, on the other hand, the agents attain a desirable configuration specified by their relative positions [19]. In the containment problem, it is desired that a subset of the agents, called followers, converge to the convex hull formed by the rest of the agents, called leaders, which could be stationary or moving [20].

Early work on the consensus problem can be traced back to the field of computer science and distributed computations [21, 22, 23, 24]. In the classical consensus problem, it is desired to find a state update rule for the agents such that some quantity of interest in every agent converges to a common value in the steady state. Further results on this subject are presented in the literature in the past few years; e.g., see [15, 16, 25]. The work [15] shows that the alignment of all agents in the presence of time-varying communication topology can be achieved using the nearest-neighbor rule. In [16], linear time-invariant consensus protocols are proposed for multi-agent systems subject to switching communication topologies and time-delay. The work [25] proposes both discrete and continuous time consensus protocols for a group of agents which exchange information over limited and unreliable communication links with time-varying topology. Recently, some algorithms have been proposed in the literature which guarantee the connectivity of the underlying network of agents [26, 27, 28, 29, 30, 31, 32]. Collision avoidance is another important problem concerning the consensus algorithms, and has been addressed in a number of papers [33, 34, 35, 13, 14, 29].

In many of the above-mentioned algorithms, the stability of the system under some control strategy is to be determined, typically by finding an appropriate Lyapunov function. However, constructing a proper Lyapunov function is known to be cumbersome, in general. Motivated by this shortcoming, some recent papers consider the stability of general distributed consensus algorithms [36, 37, 38, 39, 40]. Graphical conditions are presented in [36] for the exponential stability of a class of continuous linear time-varying (LTV) systems whose state-space matrix is Metzler with zero row sums. In [37], the convergence of discrete-time nonlinear consensus algorithms with time-dependent communication links is shown under a convexity assumption and some conditions on the communication graph. [38] generalizes the results of [37] to the case where the agents move towards the relative interior of a set that is a function of the present and past states of the neighboring agents (not necessarily the convex hull of them). As the continuous-time counterpart of [37], the work [39] studies the state agreement for coupled nonlinear differential equations with switching vector fields and topology. It is shown that under a strict sub-tangentiality condition and uniformly quasi-strongly connectivity of the interaction digraph, the system has the property of asymptotic state agreement. Somewhat relaxed conditions for the case of a static interaction digraph are presented in [40]. Nonlinear consensus algorithms arise in applications where other design criteria such as connectivity preservation and collision avoidance are to be satisfied during the convergence to consensus [28, 29, 41].

Chapter 2 studies the convergence of a class of continuous-time nonlinear consensus algorithms for single-integrator agents. The information flow graph of the agents is assumed to be static and directed. The control input of each agent is considered as a state-dependent combination of the relative positions of its neighbors in the information flow graph. Sufficient conditions are provided which guarantee the convergence of the agents to a common point for this class of consensus algorithms. It is shown that under some mild conditions, the convex hull of the agents has a contracting property. This property is used later to prove the convergence of the agents to a common point. The proposed convergence conditions are more general than the ones reported in [40, 39] under the additional assumption that the weights are analytic for a static interaction graph. The results are later used in Chapters 3 and 4 to carry out stability analysis for the consensus application of the proposed connectivity preserving control strategies.

In cooperative control of multi-agent systems, each agent is required to measure some variables of other agents, or a subset of them. Such variables include, for example, relative positions, relative velocities, and headings of the neighboring agents. One of the main assumptions in the distributed control of multi-agent systems is the connectivity of the corresponding network. Therefore, regardless of the overall objective, the designed control laws should preserve the network connectivity, which is usually a distance-dependent condition. The problem of maintaining network connectivity has been extensively studied in the literature for different agent dynamics and various applications such as consensus, flocking, containment and formation control.

For the agents with single-integrator dynamics, this issue has been investigated in several recent papers. A localized notion of connectedness is introduced in [42], and it is shown that under certain conditions the global connectedness of the network is also guaranteed. Connectivity of the graph of a network is also related to the second smallest eigenvalue of the corresponding Laplacian matrix [43, 44, 45, 46]. Centralized and decentralized approaches are proposed in [47, 48, 49] to maximize the second smallest eigenvalue of the state-dependent Laplacian of the graph of the network in order to maintain connectivity. [50, 30] use a decentralized power iteration algorithm to estimate the eigenvector corresponding to the second smallest eigenvalue of the Laplacian matrix of the graph. They subsequently obtain an estimate of the algebraic connectivity of the network, and a control input to keep the algebraic connectivity positive over time. [51, 31] present a leader to follower ratio that ensures connectivity preservation in a leader-follower multi-agent network. In order to maintain the existing links in the network, the papers [52, 26, 27, 29, 53, 54] use some potential fields that "blow up" whenever a link in the network is losing connectivity. In [28, 55, 56], appropriate nonlinear weights are designed for the edges of the interaction graph to ensure network connectivity. However, these weights tend to infinity when a pair of agents forming an edge approach a critical distance at which they lose connectivity. These techniques may not be effective in practice since the actuators of the agents can only handle finite forces or torques. To the best of the author' knowledge, the only bounded control law reported in the literature for single-integrator agents so far is the one proposed in [57, 58], where connectivity is claimed to maintain for a distributed navigation function which was used earlier in [59, 60, 61] for collision avoidance concerning robot navigation, and in [62] for formation stabilization.

As for double-integrator agents, [63] uses the same ideas as [27] for connectivity preservation of single-integrator agents, to develop a hybrid control strategy which yields velocity alignment while maintaining connectivity and ensuring collision avoidance. The above paper utilizes local estimates of the network topology in order to preserve connectivity, and allows edge deletions using a distributed market-based control strategy. More recently, a cohesive overview of the main results of [48, 49, 26, 27, 28, 63] is presented in a unified framework in [32]. For unicycles, [53] proposes a discontinuous and time-invariant feedback control strategy to reach consensus in both positions and headings, while maintaining the connectivity of those neighbors which are initially in the connectivity range. However, the translational velocity of an agent may tend to infinity when it is about to lose connectivity from a neighbor. Thus, this technique may not be effective in practice since the actuators of the agents can only handle finite forces or torques.

As for the containment problem, a hybrid Stop-Go policy is presented in [20] for single-integrator agents. It is shown that under this policy the convergence of agents is guaranteed if the leaders are stationary and the interaction graph is connected. The containment problem has also been studied in [64] for a team of single-integrator agents. Three cases of multiple static leaders, multiple dynamic leaders, and containment control with swarming behavior are considered in the above work. For the latter case, it proposes a distributed algorithm to move the followers toward the convex hull of the leaders with bounded containment control error, while preserving the connectivity of the agents and avoiding collision. The containment problem for double-integrator agents for both cases of static and dynamic leaders is investigated in [65]. A distributed attitude containment control problem for a team of rotating rigid bodies is provided in [66]. Each leader is to converge to a prescribed relative orientation with respect to the rest of the leaders, and the followers' orientations are to be contained within the convex hull of the leaders' orientations. The work [67] proposes a containment control strategy for unicycle agents where the leaders are desired to converge to a predefined formation. However, to the best of the author's knowledge, connectivity preservation has not been studied for the containment problem of unicycles or double-integrators.

One of the unprecedented contributions of this dissertation is to address this shortcoming by providing bounded distributed control strategies for connectivity preservation of multi-agent systems for two cases of single-integrator and unicycle agents. In Chapter 3, a general class of distributed potential functions is introduced with the connectivity preserving property for single-integrator agents. The main idea of the proposed approach is to design the potential functions in such a way that when an edge belonging to the information flow graph is about to lose connectivity, the gradient of the potential function lies in the direction of that edge, aiming to shrink it. The results are presented for a static information flow graph first, and are then extended to the case of dynamic edge addition. The topology of the agents that may stay fixed under the proposed control strategy is properly characterized with the purpose of extending the strategy to problems involving static leaders in which the agents assigned as leaders are to stay fixed. This is another advantage of the control scheme presented here over existing connectivity preserving approaches. The potential functions are chosen to be smooth, resulting in bounded control inputs. Additional constraints may be imposed on the potential functions to meet other design specifications such as consensus, containment, and formation convergence. It is to be noted that although the connectivity preserving control law proposed in [57, 58] are also bounded, the corresponding framework can be regarded as a subcase of the one in this chapter. Furthermore, [57, 58] do not consider the case where some of the edges of the information flow graph start exactly at the critical distance. Consequently, [57, 58] cannot be used in the case of static leaders. The proposed connectivity preserving controllers are then used to design connectivity preserving control strategies for the consensus and containment applications, where the results developed in Chapter 2 along with some novel lyapunov functions are used to carry out the stability analysis.

Designing bounded connectivity preserving controllers for the case of unicycle agents is discussed in Chapter 4. In this chapter, a class of bounded distributed controllers is proposed that maintains the connectivity of those agents that are initially in the connectivity range. Therefore, if the network is initially connected, it will remain connected at all times under the controller provided in this work. Connectivity preservation is guaranteed even if some of the agents, namely static leaders, are to remain fixed. The main idea here is to design the local controllers in such a way that when an agent is about to lose connectivity with a neighbor, it is forced to move with an acute angle with respect to the corresponding edge. If the heading of the agent is perpendicular to this edge, then under the proposed control law the velocity of the agent is zero, the acceleration of the agent is perpendicular to this edge, and the derivative of the acceleration makes an acute angle with this edge, aiming to shrink it. The results are primarily developed for a static information flow graph, but are shown to also hold for the case of dynamic edge addition. Smooth potential functions are used in order to obtain bounded control inputs. The results are then used to design bounded connectivity preserving control strategies for containment and consensus, both being novel and unprecedented contributions of the present work with respect to the existing literature.

Collision avoidance is another important specification in distributed control of multi-agent systems, which is known to be closely related to the connectivity preservation property from the design point of view. This problem is thoroughly investigated for both cases of single-integrator agents (e.g., see [33, 34, 35, 13, 14, 29]) and unicycles (e.g., see [68, 69, 70, 29, 71]). The connectivity preservation and collision avoidance problems are also studied simultaneously in several works in the literature. The papers [26, 27, 29] use the idea of unbounded potential functions to avoid collision between agents besides the connectivity preservation. A containment control strategy for a team of single-integrator agents, while preserving connectivity and avoiding collision between them, is proposed in [64]. However, when two agents approach each other or reach the boundary of connectivity range, their control inputs become unbounded. Connectivity preserving control strategies for double-integrator agents are proposed in [54, 63]. In [54], using unbounded potential functions for double-integrator agents, a connectivity preserving controller is designed for flocking of the agents while avoiding collision among them. In [63] a hybrid control strategy is developed which yields velocity alignment while maintaining connectivity and ensuring collision avoidance. The potential functions used in the controller design tend to infinity when two agents are about to collide or to lose connectivity. For unicycles, A connectivity preserving collision-free aggregation control strategy is designed in [29] using potential functions that tend to infinity when two agents are about to collide or to lose connectivity.

Bounded distributed connectivity preserving control strategies for aggregation of a swarm of agents for two cases of single-integrator and unicycle dynamics with collision avoidance property is presented in Chapter 5. The main contribution of this chapter is to add collision avoidance feature to the results presented in Chapters 3 and 4 on bounded connectivity preservation of multi-agent systems. The proposed control strategy preserves the connectivity in the sense that if two agents enter the connectivity range at some point in time, they will stay in the connectivity range thereafter. The agents are shown to finally aggregate, while avoiding collision among themselves, in such a way that the average distance between the neighboring agents eventually falls below a pre-specified threshold. The control inputs of the agents stay bounded even if two agents are about to collide, or to leave or enter the connectivity range.

The results of this dissertation are published (or submitted for publication) in a number of journals and conference proceedings ([72, 73, 74, 41, 75, 76, 77, 78, 79, 80, 81, 82]). These publications are listed below for different chapters.

- Chapter 2

    1. A. Ajorlou, A. Momeni, and A. G. Aghdam, "Sufficient conditions for the convergence of a class of nonlinear distributed consensus algorithms," *Automatica*, vol. 47, no. 3, pp. 625-629, 2011.
    2. A. Ajorlou, A. Momeni, and A. G. Aghdam, "Relaxed convergence conditions for multi-agent systems under a class of consensus algorithms," in *Proceedings of American Control Conference*, 2011, pp. 2795-2800.
    3. A. Ajorlou, A. Momeni, and A. G. Aghdam, "Convergence analysis for a class of nonlinear consensus algorithms," in *Proceedings of American Control Conference*, 2010, pp. 6318-6323.

- Chapter 3

    1. A. Ajorlou, A. Momeni, and A. G. Aghdam, "A class of bounded distributed control strategies for connectivity preservation in multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2828-2833, 2010.
    2. A. Ajorlou, A. Momeni, and A. G. Aghdam, "A connectivity preserving containment control strategy for a network of single integrator agents," in *Proceedings of American Control Conference*, 2011, pp. 499-501.

3. A. Ajorlou, A. Momeni, and A. G. Aghdam, "Connectivity preservation in a network of single integrator agents," in *Proceedings of the 48th IEEE Conference on Decision and Control*, 2009, pp. 7061-7067.

- Chapter 4

1. A. Ajorlou and A. G. Aghdam, "Connectivity preservation in nonholonomic multi-agent systems: A bounded distributed control strategy," *IEEE Transactions on Automatic Control*, 2013 (in press).

2. A. Ajorlou, A. G. Aghdam, and A. Jadbabaie, "A connectivity preserving containment control strategy for unicycles with static leaders," in *Proceedings of American Control Conference*, 2012, pp. 4186-4191.

3. A. Ajorlou and A. G. Aghdam, "Convergence analysis for a class of bounded distributed connectivity preserving consensus algorithms for unicycles," in *Proceedings of American Control Conference*, 2012, pp. 1585-1590.

4. A. Ajorlou and A. G. Aghdam, "A class of bounded distributed controllers for connectivity preservation of unicycles," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 3072-3077.

- Chapter 5

1. A. Ajorlou and A. G. Aghdam, "A bounded distributed connectivity preserving aggregation strategy with collision avoidance property," *Systems & Control Letters*, 2013 (conditionally accepted).

2. A. Ajorlou and A. G. Aghdam, "A bounded connectivity preserving aggregation strategy with collision avoidance property for single-integrator agents," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 4003-4008.

The author has also published (or submitted) the following papers related to multi-agent control systems ([83, 84, 85, 86, 87]).

1. A. Ajorlou, K. Moezzi, A. G. Aghdam, and S. G. Nersesov, "Two-stage time-optimal formation reconfiguration strategy," *Systems & Control Letters*, vol. 62, no. 6, pp. 496-502, 2013.

2. A. Ajorlou, K. Moezzi, A. G. Aghdam, S. Tafazoli, and S. G. Nersesov, "Two-stage energy-optimal formation reconfiguration strategy," *Automatica*, vol. 48, no. 10, pp. 2587-2591, 2012.

3. A. Ajorlou, M. M. Asadi, A. G. Aghdam, and S. Blouin, "A consensus control strategy for unicycles in the presence of disturbances," in *Proceedings of American Control Conference*, 2013 (to appear).

4. A. Ajorlou, K. Moezzi, A. G. Aghdam, and S. G. Nersesov, "Two-stage time-optimal formation reconfiguration strategy under velocity and acceleration constraints," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 7455-7460.

5. A. Ajorlou, K. Moezzi, A. G. Aghdam, and S. Tafazoli, "Energy and time efficient formation reconfiguration strategies," in *Proceedings of the 48th IEEE Conference on Decision and Control*, 2009, pp. 3081-3086.

Finally, the author collaborated with other researchers on the following relevant papers during the course of this study ([88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98]).

1. S. Jafari, A. Ajorlou, and A. G. Aghdam, "Leader localization in multi-agent systems subject to failure: a graph-theoretic approach," *Automatica*, pp. 1744-1750, 2011.

2. M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, "Digraphs with distinguishable dynamics under the multi-agent agreement protocol," 2013 (submitted to a journal).

3. M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, "Necessary and sufficient conditions for detectability and identifiability of link failures in multi-agent systems under the agreement protocol," in *Proceedings of American Control Conference*, 2013 (to appear).

4. M. M. Asadi, A. Ajorlou, and A. G. Aghdam, "Distributed control of multi-agent systems with rotating field of view," in *Proceedings of American Control Conference*, 2013 (to appear).

5. M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, "Detectability of multiple link failures in multi-agent systems under the agreement protocol," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 118-123.

6. M. Tousi, A. Ajorlou, H. Mahboubi, and A. G. Aghdam, "Decentralized pole-placement using generalized sampled-data hold functions," in *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012, pp. 6921-6925.

7. M. A. Rahimian, A. Ajorlou, and A. G. Aghdam, "Characterization of link failures in multi-agent systems under the agreement protocol," in *Proceedings of American Control Conference*, 2012, pp. 5258-5263.

8. M. M. Asadi, A. Ajorlou, and A. G. Aghdam, "Cooperative control of multi-agent systems with limited angular field of view," in *Proceedings of American Control Conference*, 2012, pp. 2388-2393.

9. F. Salehisadaghiani, A. Ajorlou, and A. G. Aghdam, "Distributed connectivity preservation of a team of single integrator agents subject to measurement

error," in *Proceedings of IEEE Multiconference on Systems & Control*, 2011, pp. 626-631.

10. S. Jafari, A. Ajorlou, and A. G. Aghdam, "Leader selection in multi-agent systems subject to partial failure," in *Proceedings of American Control Conference*, 2011, pp. 5330-5335.

11. S. Jafari, A. Ajorlou, and A. G. Aghdam, "On the structural controllability of multi-agent systems subject to failure: A graph-theoretic approach," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 4564-4570.

12. S. Jafari, A. Ajorlou, and A. G. Aghdam, "Distributed control of formation flying spacecraft using deterministic communication schedulers," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 3084-3090.

Chapter 2

Sufficient Conditions for the Convergence of a Class of Nonlinear Distributed Consensus Algorithms

This chapter studies the convergence of a class of continuous-time nonlinear consensus algorithms for single-integrator agents. In the consensus algorithms studied here, the control input of each agent is assumed to be a state-dependent combination of the relative positions of its neighbors in the information flow graph. Using a novel approach based on the smallest order of the nonzero derivative, it is shown that under some mild conditions the convex hull of the agents has a contracting property. A set-valued LaSalle-like approach is subsequently employed to show the convergence of the agents to a common point. The results are shown to be more general than the ones reported in the literature in some cases.

The remainder of this chapter is organized as follows. The problem is formulated in Section 2.1, where some useful notations and definitions are also introduced.

Sufficient conditions for the convergence of the consensus algorithms introduced in Section 2.1 are presented in Section 2.2. Finally, the verification of the proposed convergence conditions is illustrated in Section 2.3.

2.1 Problem Formulation

Definition 2.1. *The function $f : \mathbb{R} \to \mathbb{R}^m$ is said to be of class C^k if the derivatives $f^{(1)}, \ldots, f^{(k)}$ exist and are continuous ($f^{(k)}$ is the k^{th} derivative of f). The function f is said to be of class C^∞ (or smooth) if it has derivatives of all orders.*

Definition 2.2. *For a smooth function $f : \mathbb{R} \to \mathbb{R}^m$, the index of f at time t, denoted by $\rho(f(t))$, is defined as the smallest natural number n for which $f^{(n)}(t) \neq 0$.*

Definition 2.3. *For a smooth function $f : \mathbb{R} \to \mathbb{R}^m$, the extended index of f at time t, denoted by $\tilde{\rho}(f(t))$, is defined as the smallest nonnegative integer n for which $f^{(n)}(t) \neq 0$, where $f^{(0)}(t)$ is defined to be $f(t)$.*

Definition 2.4. *A function $f : \mathbb{R}^m \to \mathbb{R}$, is called analytic on $\mathbb{R}^m$, written $f \in C^\omega(\mathbb{R}^m)$, if for any $\alpha \in \mathbb{R}^m$ the function f may be expressed as a convergent power series in some neighborhood of α (see [99]).*

Definition 2.5. *For a set of points $Q = \{q_1, \ldots, q_n\}$, $q_i \in \mathbb{R}^m$, $i \in \mathbb{N}_n := \{1, \ldots, n\}$, the convex hull of Q is defined as*

$$Conv(Q) = \{p | \exists \lambda_1, \ldots, \lambda_n \geq 0 : \sum_{i=1}^{n} \lambda_i = 1, p = \sum_{i=1}^{n} \lambda_i q_i\}$$

Definition 2.6. *A set-valued function $S(\cdot)$ is said to be nested if for every $t_1, t_2 \in \mathbb{R}$, where $0 \leq t_1 \leq t_2$, the relation $S(t_2) \subseteq S(t_1)$ holds.*

Definition 2.7. *In a digraph G, a vertex v is said to be reachable from a vertex u, if there is a directed path from u to v. The set of all reachable vertices from the vertex u in G is denoted by $R_u(G)$.*

Definition 2.8. *A digraph G is said to be quasi-strongly connected if for every two distinct vertices u and v of G, there is a vertex from which both u and v are reachable (see [100]).*

Definition 2.9. *A group of agents $1, ..., n$ is said to converge to consensus if $q_i(t) \to \bar{q}$ as $t \to \infty$ for any $i \in \mathbb{N}_n$, where $q_i(t) \in \mathbb{R}^m$ denotes the state of agent i at time t, and $\bar{q}$ is a constant.*

Definition 2.10. *For a function $q : \mathbb{R} \to \mathbb{R}^m$, the point $\bar{p} \in \mathbb{R}^m$ is said to be a positive limit point of $q(\cdot)$ if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $q(t_n) \to \bar{p}$ as $n \to \infty$. The set of all positive limit points of $q(\cdot)$ is called the positive limit set of $q(\cdot)$.*

Definition 2.11. *A family $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ of subsets of a set X is said to have the finite intersection property if every finite sub-family $\{A_1, A_2, \ldots, A_n\}$ of $\mathcal{A}$ satisfies $\bigcap_{i=1}^n A_i \neq \emptyset$ (see [101]).*

Consider a set of n agents in the 2D plane with single-integrator dynamics, i.e.

$$\dot{q}_i(t) = u_i(t), \quad i \in \mathbb{N}_n \tag{2.1}$$

where $q_i(t) \in \mathbb{R}^2$ represents the position of agent i at time t, and u_i is the corresponding control signal. The present work is concerned with those control signals under which the agents converge to consensus. Note that for brevity, the time argument is omitted hereafter in all time-dependent functions, wherever it is not necessary. Denote by $G = (V, E)$ the information flow graph, with $V = \{1, \ldots, n\}$ representing the set of n vertices (associated with the n agents), and $E \subseteq V \times V$ representing the corresponding edges. The information flow graph G is assumed to be static and directed. There is a directed edge from vertex j to vertex i in G if and only if $(j, i) \in E$. The set of neighbors of vertex i in G is defined as $N_i = \{j|(j, i) \in E\}$, and its indegree is denoted by $d_i = |N_i|$. Each agent is only allowed to incorporate its own position and the position of its neighbors in its control law. In this chapter, the distributed control laws of the following form are considered

$$u_i = -\sum_{j \in N_i} \beta_{ij}(q_i - q_j) \ , \quad i \in \mathbb{N}_n \tag{2.2}$$

where the coefficients $\beta_{ij} : \mathbb{R}^{2(d_i+1)} \to \mathbb{R}$, $i \in \mathbb{N}_n$, $j \in N_i$, are state-dependent. More specifically, each coefficient β_{ij} is a function of the position of agent i and the positions of the neighbors of agent i in G. The main contribution of this chapter is to present sufficient conditions on the coefficients β_{ij} in (2.2), which guarantee the convergence of the agents to consensus.

2.2 Sufficient Conditions for Convergence

Consider again a set of n agents in the 2D plane with the dynamics of the form (2.1), and let them evolve according to the control laws given by (2.2). The aim of this section is to show that under the following assumptions on the coefficients β_{ij} in (2.2), the agents converge to consensus.

Assumption 2.1. *The state-dependent coefficients β_{ij} in (2.2) are analytic, real and nonnegative for any $i \in \mathbb{N}_n$ and $j \in N_i$.*

Assumption 2.2. *The system (2.1) with the control law of the form (2.2) has no solution in which the convex hull of the agents is not a singleton and is fixed, with at least one agent being fixed at each vertex.*

Denote by $S(t)$ the convex hull of the agents at time t, i.e.

$$S(t) = \mathrm{Conv}\left(\{q_i(t) | i \in \mathbb{N}_n\}\right) \tag{2.3}$$

In what follows, a few lemmas are presented first in order to prove the nestedness property for $S(t)$. Using this property, a LaSalle-like approach is subsequently taken to prove the convergence of the agents to consensus.

Lemma 2.1. *Consider a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f \in C^{k+1}$, with the property that $f^{(1)}(t) = \ \ldots = f^{(k)}(t) = 0$ and $f^{(k+1)}(t) > 0$, for some t, where k is some positive integer. Then, there exists $\delta > 0$ such that*

$$f(t) < f(t+\tau) \ , \quad \forall \tau \in (0, \delta] \tag{2.4}$$

Proof. Since $f^{(k+1)}(t) > 0$, thus $f^{(k)}(t+\tau)$ is monotonically increasing for $\tau \in [0, \delta]$, for some $\delta > 0$. On the other hand $f^{(k)}(t) = 0$, which implies (along with the above result) that $f^{(k)}(t+\tau) > 0$ for any $\tau \in (0, \delta]$. Hence, $f^{(k-1)}(t+\tau)$ is monotonically increasing for $\tau \in [0, \delta]$. Using a similar argument iteratively, one arrives at the conclusion that $f^{(0)}(t+\tau)$ (which is by definition equal to $f(t+\tau)$) is monotonically increasing in the closed interval given above. Therefore, $f(t+\tau) > f(t)$ for any $\tau \in [0, \delta]$ and this completes the proof. ■

Remark 2.1. *if $f^{(k+1)}(t) < 0$, one can similarly show that there exists $\delta > 0$ for which*

$$f(t) > f(t+\tau) \ , \quad \forall \tau \in (0, \delta] \tag{2.5}$$

In order to show the nestedness property for the set $S(t)$, it is required to investigate the behavior of the agents on the boundary of the set. Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Note that this intersection will be on the boundary of $S(t)$, i.e., either an edge or a vertex of $S(t)$ (see Fig. 2.1 for the case when the intersection is an edge). Denote by e_l the unit vector perpendicular to l, in the direction of the half-plane containing $S(t)$. Define $f_l : \mathbb{R}^2 \rightarrow \mathbb{R}$ as $f_l(x) =< x, e_l >$, i.e., the projection of x on e_l. Let agent i be on l at time t. Denote by $N_i^l(t)$ the set of those neighbors of i lying on l, and with $\bar{N}_i^l(t)$ the set of those neighbors not lying on l. Now, define $\eta_{i1}^l(t)$ and $\eta_{i2}^l(t)$ as follows:

$$\eta_{i1}^l(t) = \begin{cases} \min_{j \in N_i^l(t)} \{\tilde{\rho}(\beta_{ij}) + \rho(f_l(q_j))\}, & N_i^l(t) \neq \emptyset \\ \infty, & N_i^l(t) = \emptyset \end{cases} \tag{2.6}$$

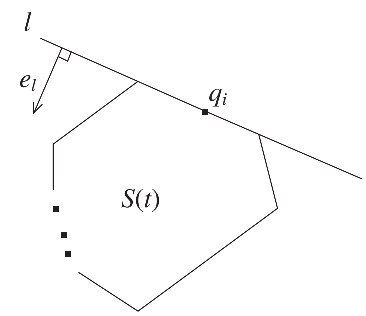

Figure 2.1: $S(t)$ is the convex hull of the agents at time t, q_i is the position of an agent on l, and e_l is the unit vector perpendicular to l in the direction of the half-plane containing $S(t)$.

and

$$\eta_{i2}^l(t) = \begin{cases} \min_{j\in \bar{N}_i^l(t)}\{\tilde{\rho}(\beta_{ij})\}, & \bar{N}_i^l(t) \neq \emptyset \\ \infty, & \bar{N}_i^l(t) = \emptyset \end{cases} \tag{2.7}$$

where in calculating $\tilde{\rho}(\beta_{ij})$, β_{ij} is regarded as an implicit function of time. It is straightforward to verify that $\eta_{i1}^l(t) \geq 1$ and $\eta_{i2}^l(t) \geq 0$. Define also

$$\eta_i^l(t) = \min\{\eta_{i1}^l(t), \eta_{i2}^l(t)\} \tag{2.8}$$

Lemmas 2.2-2.4 will enable us in the sequel to fully describe the behavior of the agents on the boundary of $S(t)$.

Lemma 2.2. *Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. Then, the following statements are true:*

i) If $\eta_i^l = 0$, then $f_l(\dot{q}_i) > 0$.

ii) If $\eta_i^l \geq 1$, then $f_l(q_i^{(k)}) = 0$, for $k = 1, \ldots, \eta_i^l$.

Proof.

Part (i): First, note that $f_l(q_j - q_i)$ is equal to zero for any $j \in N_i^l$, and is strictly positive for any $j \in \bar{N}_i^l$. Also, $\beta_{ij} \geq 0$ for any $j \in N_i$, according to Assumption 2.1. The relation $\eta_i^l = 0$ yields $\eta_{i2}^l = 0$, which implies that $\bar{N}_i^l \neq \emptyset$, and that there exists an agent $v \in \bar{N}_i^l$ for which $\beta_{iv} > 0$. Therefore, using (2.1) and (2.2) one can write

$$\begin{aligned} f_l(\dot{q}_i) &= \sum_{j \in \bar{N}_i^l} \beta_{ij} f_l(q_j - q_i) \\ &\geq \beta_{iv} f_l(q_v - q_i) \\ &> 0 \end{aligned} \tag{2.9}$$

Part (ii): It is straightforward to show that

$$f_l(q_i^{(k+1)}) = \sum_{j \in N_i} \sum_{r=0}^{k} \beta_{ij}^{(k-r)} (f_l(q_j^{(r)}) - f_l(q_i^{(r)})) \begin{pmatrix} k \\ r \end{pmatrix} \tag{2.10}$$

where $\beta_{ij}^{(k-r)}$ is the $(k-r)^{th}$ derivative of β_{ij} with respect to time (note that β_{ij} is an implicit function of time). Assume now $k < \eta_i^l$; this means that $k - r < \eta_i^l \leq \eta_{i2}^l$, and hence $\beta_{ij}^{(k-r)} = 0$ for $j \in \bar{N}_i^l$. On the other hand, since $k < \eta_i^l \leq \eta_{i1}^l$, one can easily show that $\beta_{ij}^{(k-r)} f_l(q_j^{(r)}) = 0$, for $j \in N_i^l$ and $1 \leq r \leq k$. Using these results along with the fact that $f_l(q_j - q_i) = 0$ for $j \in N_i^l$, equation (2.10) reduces to

$$f_l(q_i^{(k+1)}) = -\sum_{j \in N_i^l} \sum_{r=1}^{k} \beta_{ij}^{(k-r)} f_l(q_i^{(r)}) \begin{pmatrix} k \\ r \end{pmatrix} \tag{2.11}$$

The rest of the proof follows by a simple induction. ■

Lemma 2.3. *Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. If $\rho(f_l(q_i)) < \infty$, then $f_l(q_i^{(\rho(f_l(q_i)))}) > 0$.*

Proof. Since $\rho(f_l(q_i)) < \infty$, thus it is implied from Lemma 2.2 that $\eta_i^l < \infty$. Before getting to the proof, first some important properties of $f_l(q_i^{(\eta_i^l+1)})$ are characterized assuming $1 \leq \eta_i^l < \infty$. Using Lemma 2.2 and taking an approach similar to the one used to derive (2.11) from (2.10), one can show that

$$\begin{aligned} f_l(q_i^{(\eta_i^l+1)}) &= \sum_{j \in N_i^l} \sum_{r=1}^{\eta_i^l} \beta_{ij}^{(\eta_i^l-r)} f_l(q_j^{(r)}) \begin{pmatrix} \eta_i^l \\ r \end{pmatrix} \\ &+ \sum_{j \in \bar{N}_i^l} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \end{aligned} \tag{2.12}$$

There are three possible cases for η_i^l, η_{i1}^l, and η_{i2}^l:

Case (i): $\eta_i^l = \eta_{i2}^l < \eta_{i1}^l$. In this case, (2.12) reduces to

$$f_l(q_i^{(\eta_i^l+1)}) = \sum_{\substack{j \in \bar{N}_i^l \\ \tilde{\rho}(\beta_{ij})=\eta_i^l}} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \tag{2.13}$$

On the other hand, the relation $\tilde{\rho}(\beta_{ij}) = \eta_i^l \geq 1$ implies that $\beta_{ij} = 0$. If $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} < 0$, then it results from Remark 2.1 that β_{ij} is negative in a right-sided vicinity of t (β_{ij} is regarded here as an implicit function of time, as noted earlier). However, this is in contradiction with Assumption 2.1; therefore $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$, and it results from (2.13) that $f_l(q_i^{(\eta_i^l+1)}) > 0$.

Case (ii): $\eta_i^l = \eta_{i1}^l < \eta_{i2}^l$. In this case, (2.12) reduces to

$$f_l(q_i^{(\eta_i^l+1)}) = \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij})+\rho(f_l(q_j))=\eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \begin{pmatrix} \eta_i^l \\ \tilde{\rho}(\beta_{ij}) \end{pmatrix} \tag{2.14}$$

If $\beta_{ij} \neq 0$, then $\tilde{\rho}(\beta_{ij}) = 0$ and $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} = \beta_{ij} > 0$. If on the other hand $\beta_{ij} = 0$, the inequality $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$ still holds as shown in case (i).

Case (iii): $\eta_i^l = \eta_{i1}^l = \eta_{i2}^l$. It results from (2.12) in this case that

$$\begin{aligned} f_l(q_i^{(\eta_i^l+1)}) &= \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij})+\rho(f_l(q_j))=\eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \begin{pmatrix} \eta_i^l \\ \tilde{\rho}(\beta_{ij}) \end{pmatrix} \\ &\quad + \sum_{\substack{j \in \bar{N}_i^l \\ \tilde{\rho}(\beta_{ij})=\eta_i^l}} \beta_{ij}^{(\eta_i^l)} f_l(q_j - q_i) \\ &> \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij})+\rho(f_l(q_j))=\eta_i^l}} \beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} f_l(q_j^{(\rho(f_l(q_j)))}) \begin{pmatrix} \eta_i^l \\ \tilde{\rho}(\beta_{ij}) \end{pmatrix} \end{aligned} \tag{2.15}$$

(note that the inequalities $\beta_{ij}^{(\tilde{\rho}(\beta_{ij}))} > 0$ and $f_l(q_j - q_i) > 0$, $\forall j \in \bar{N}_i^l$, are used in deriving (2.15)).

From the results presented in cases (ii) and (iii), one can easily conclude that if $\eta_i^l = \eta_{i1}^l$, then

$$f_l(q_i^{(\eta_i^l+1)}) \geq \sum_{\substack{j \in N_i(l) \\ \tilde{\rho}(\beta_{ij})+\rho(f_l(q_j))=\eta_i^l}} \alpha_{ij} f_l(q_j^{(\rho(f_l(q_j)))}) \tag{2.16}$$

where α_{ij}'s are positive coefficients.

It is desired now to use induction on $\rho(f_l(q_i))$ together with the results developed thus far to prove the lemma. For $\rho(f_l(q_i)) = 1$, if $\eta_i^l \geq 1$ then it results from Lemma 2.2 that $f_l(\dot{q}_i) = 0$, which is a contradiction; therefore, $\eta_i^l = 0$, and hence according to Lemma 2.2 $f_l(\dot{q}_i) > 0$. Assume now that the statement of the lemma holds for $\rho(f_l(q_i)) \leq k$, for some $k \geq 1$; The objective is to prove that it holds for $\rho(f_l(q_i)) = k+1$ as well. Note first that Lemma 2.2 implies $1 \leq \eta_i^l < k+1$. If $\eta_{i2}^l < \eta_{i1}^l$, then it results from case (i) as well as Lemma 2.2 that $\rho(f_l(q_i)) = \eta_i^l + 1$ and $f_l(q_i^{(\eta_i^l+1)}) > 0$. If on the other hand $\eta_{i2}^l \geq \eta_{i1}^l$ (i.e. $\eta_i^l = \eta_{i1}^l$), then (2.16) holds. Moreover, for any j in the summation domain of (2.16), the relation $\rho(f_l(q_j)) \leq \eta_i^l < k+1$ holds, and hence the assumption of induction yields $f_l(q_j^{(\rho(f_l(q_j)))}) > 0$. It is concluded from this along with (2.16) that $f_l(q_i^{(\eta_i^l+1)}) > 0$, from which it is also implied (using Lemma 2.2) that $\rho(f_l(q_i)) = \eta_i^l + 1$. This completes the proof. ■

Corollary 2.1. *Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Assume that $q_i(t) \in l$, for some $i \in \mathbb{N}_n$. Then, $\rho(f_l(q_i)) = \eta_i^l + 1 = \min\{\eta_{i1}^l, \eta_{i2}^l\} + 1$, where η_{i1}^l and η_{i2}^l are defined in (2.6) and (2.7).*

Proof. This follows directly from Lemma 2.3 (as its by-product). ■

Lemma 2.4. *Consider a line l which intersects $S(t)$ at some time $t \geq 0$, but does not pass through it. Given $q_i(t) \in l$, if $f_l(q_i(t))$ has a finite index, then there exists $\delta_i > 0$ such that for any $\tau \in (0, \delta_i]$ the inequality $f_l(q_i(t)) < f_l(q_i(t+\tau))$ holds; otherwise, $f_l(\dot{q}_i) \equiv 0$.*

Proof. If $\rho(f_l(q_i)) < \infty$, then according to Lemma 2.3, $f_l(q_i^{(\rho(f_l(q_i)))}) > 0$. Therefore, it results from Lemma 2.1 that there exists $\delta_i > 0$ such that for any $\tau \in (0, \delta_i]$ the inequality $f_l(q_i(t)) < f_l(q_i(t+\tau))$ holds. This means that agent i will move towards the interior of the half plane (defined by l) containing $S(t)$.

Now, consider the case where $\rho(f_l(q_i)) = \infty$. Since β_{ij}'s are analytic, according to Theorem 39.12 in [102], q_i is also analytic, implying that $f_l(q_i)$ is analytic as well.

Therefore, $\rho(f_l(q_i)) = \infty$ implies that $f_l(q_i) \equiv f_l(q_i(t))$, meaning that q_i has been on l from the beginning and will stay on it at all times. ■

Theorem 2.1. *Under Assumption 2.1, the convex hull of the agents is nested.*

Proof. Consider the agents at any arbitrary time $t \geq 0$. By applying Lemma 2.4 to all edges on the boundary of $S(t)$, one can easily show that there exists $\delta(t) > 0$ such that

$$q_i(t+\tau) \in S(t), \quad \forall i \in \mathbb{N}_n, \quad \forall \tau \in [0, \delta(t)] \tag{2.17}$$

implying that $S(t+\tau) \subseteq S(t)$, for any $\tau \in [0, \delta(t)]$. For an arbitrary $t \geq 0$, define $T = \sup\{\Delta | \forall \tau \in [0, \Delta] : S(t+\tau) \subseteq S(t)\}$. It is desired now to show by contradiction that $T = \infty$. To this end, assume T is finite and note that $S(t+T) \subseteq S(t)$. Note also that the relation $S(t+T+\tau) \subseteq S(t+T)$ holds for any $\tau \in [0, \delta(t+T)]$, and hence $S(t+T+\tau) \subseteq S(t)$. Thus, the relation $S(t+\tau) \subseteq S(t)$ holds for any $\tau \in [0, T+\delta(t+T)]$, which is in contradiction with the definition of T. This implies that $T = \infty$, and as a result, $S(t)$ is nested; i.e. $S(t_2) \subseteq S(t_1)$, for any $t_1 \geq 0$ and $t_2 \geq t_1$. ■

The following lemma is borrowed from [103].

Lemma 2.5. *If a solution $q(t)$ of $\dot{q} = f(q)$ belongs to a bounded domain D for $t \geq 0$, then its positive limit set L^+ is nonempty, compact, and invariant. Moreover, $q(t)$ approaches L^+ as $t \to \infty$.*

The following result from [101] will also be used in the proof of main theorem.

Theorem 2.2. *A topological space is compact if and only if each family of closed sets which has the finite intersection property has a non-void intersection.*

In the sequel, sufficient conditions are provided for convergence to consensus, as the most important contribution of this chapter.

Theorem 2.3. *Consider a set of n agents in the 2D plane with the dynamics of the form (2.1), evolved under the local control laws given by (2.2). Under Assumptions 2.1-2.2, the agents converge to consensus.*

Proof. Since $S(t)$ is nested, the agents remain in $S(0)$ at all times. Define $\mu_1(q(t))$ and $\mu_2(q(t))$ as the area and the diameter of $S(t)$, respectively, where $q(t) = (q_1(t), \ldots, q_n(t))$. Clearly, μ_1 and μ_2 are bounded and decreasing (note that $S(t)$ is nested) but not necessarily differentiable. Let $\lim_{t\to\infty} \mu_1(q(t)) = a_1$ and $\lim_{t\to\infty} \mu_2(q(t)) = a_2$. Let also L^+ denote the positive limit set of $q(t)$. For any $p \in L^+$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $q(t_n) \to p$ as $n \to \infty$. It follows immediately from the continuity of μ_1 and μ_2, that $\mu_1(p) = a_1$ and $\mu_2(p) = a_2$.

It is desired now to show that $a_1 = 0$. If $a_1 > 0$, the invariance property of L^+ (see Lemma 2.5) along with the fact that $\mu_1(p) = a_1$ for any $p \in L^+$ and the nestedness property of the convex hull of the agents, yields that starting from any $p(0) = (p_1(0), \ldots, p_n(0)) \in L^+$, the convex hull $S(t)$ will remain fixed, i.e. $S(t) \equiv S(0)$. Consider an agent, say agent i, at a vertex of $S(0)$, and let l_1 and l_2 be the two lines obtained by extending the two edges connected to this vertex on the boundary of $S(0)$. Now, it results from Lemma 2.4 (once with $l = l_1$ and then with $l = l_2$) that either agent i moves away from this vertex, or $f_{l_1}(\dot{p}_i) \equiv f_{l_2}(\dot{p}_i) \equiv 0$; the latter case implies that agent i remains fixed at that vertex. Thus, in order for $S(t)$ to remain fixed, there should be at least one fixed agent at each vertex of $S(0)$, which contradicts Assumption 2.2. This contradiction yields $a_1 = 0$, i.e. if $p = (p_1, \ldots, p_n)$ is a positive limit point, then p_i's are collinear. Using this property and following an argument similar to the one given above, it is concluded that $a_2 = 0$, i.e. $p_1 = \ldots = p_n$ for any $p = (p_1, \ldots, p_n) \in L^+$. To complete the proof, note that since $S(t)$ is nested, it satisfies the finite intersection property, and hence according to Theorem2.2, $\bigcap_{t\geq 0} S(t) = Q \neq \emptyset$. On the other hand, $a_2 = 0$ implies that the diameter of $S(t)$ approaches 0 as $t \to \infty$, which means that Q is a single point. Furthermore, $Q \in S(t)$ yields $\|q_i(t) - Q\| \leq \mu_2(q(t))$ and this, in turn, implies that $q_i(t) \to Q$ as $t \to \infty$ because $\mu_2(q(t)) \to 0$ as $t \to \infty$. This completes the proof of the convergence of the agents to a fixed single point. ∎

Assumption 2.2 is essential in the above theorem, but it is not straightforward to verify it, in general. The following proposition will prove useful in verifying the conditions of this assumption.

Proposition 2.1. *Let the conditions of Assumption 2.1 hold, and assume the convex hull of the agents is fixed. Then for a fixed agent, say agent i, at a vertex of this convex hull, and for every $j \in N_i$, either $q_j \equiv q_i$ or $\beta_{ij} \equiv 0$.*

Proof. First note that under Assumption 2.1, Lemmas 2.2-2.4 and Theorem 2.1 still hold. Consider the agents at some $t \geq 0$, and let l_1 and l_2 be the two lines passing through the two edges on the boundary of the convex hull connected to the vertex at which q_i is fixed. Using Corollary 2.1 for both l_1 and l_2 leads to $\tilde{\rho}(\beta_{ij}) = \infty$ for $j \in \bar{N}_i(l_1) \cup \bar{N}_i(l_2)$, implying that β_{ij} is identically zero because it is analytic. The only remaining neighbors that are not in $\bar{N}_i(l_1) \cup \bar{N}_i(l_2)$ are those for which $q_j(t) = q_i(t)$. For such a neighbor, if $\tilde{\rho}(\beta_{ij}) = \infty$ then $\beta_{ij} \equiv 0$ similarly; if on the other hand $\tilde{\rho}(\beta_{ij})$ is finite, then $\rho(f_{l_1}(q_j(t))) = \rho(f_{l_2}(q_j(t))) = \infty$, and consequently $f_{l_1}(\dot{q}_j) \equiv f_{l_2}(\dot{q}_j) \equiv 0$. This implies that $\dot{q}_j \equiv 0$, which means that $q_j \equiv q_i$. ∎

The main advantage of this work over [40, 39] is described in the next proposition.

Proposition 2.2. *Consider a set of n agents in the 2D plane with the dynamics of the form (2.1), with a quasi-strongly connected information flow graph. Let the control law be of the form (2.2), where the corresponding coefficients are assumed to meet the conditions of Assumption 2.1. Define $Q_i = \{q_j | j \in N_i \cup \{i\}\}$, and assume that if agent i is at a vertex of Conv(Q_i) and Q_i is not a singleton, then $\dot{q}_i \not\equiv 0$. Then the agents converge to consensus.*

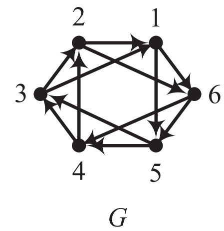

Figure 2.2: The information flow graph G for the case of $n = 6$, in Example 2.1.

Proof. It suffices to show that the conditions of the proposition imply that Assumption 2.2 holds. Suppose that there is a solution for which Assumption 2.2 does not hold, and let agent i be a fixed agent at a vertex of the convex hull for such a solution. Clearly, q_i is also a vertex of $\mathrm{Conv}(Q_i)$ at all times. This, along with the fact that $\dot{q}_i \equiv 0$, implies that Q_i should be a singleton at all times, and hence $q_j \equiv q_i$ for all $j \in N_i$. Repeating the same argument, one can conclude that $q_j \equiv q_i$ for any agent j from which i is reachable in G. Now, consider two fixed agents i_1 and i_2 at two distinct vertices of the convex hull. Since G is quasi-strongly connected, there exists an agent from which both i_1 and i_2 are reachable in G, implying that $q_{i_1} \equiv q_{i_2}$. This contradicts the assumption that agents i_1 and i_2 are located at two distinct vertices of the convex hull, and hence completes the proof. ■

Remark 2.2. *The results in [40, 39] do not guarantee the convergence to consensus under the setting of Proposition 2.2. More precisely, [40, 39] require $\dot{q}_i \neq 0$ instead of $\dot{q}_i \not\equiv 0$ (in the statement of the proposition) to deduce the convergence to consensus, while the above proposition allows agent i at a vertex of Conv(Q_i) to attain zero velocity (even if Q_i is not a singleton) as long as it is not fixed. The only limitation here, however, is that β_{ij}'s need to be analytic, while there is not such constraint in [40, 39] (it is only required there that the u_i's are continuous functions of the states).*

2.3 Simulation Results

Example 2.1. *Consider a swarm of n agents in a 2D plane with the dynamics of the form (2.1) and the control inputs given by*

$$\begin{aligned} u_i &= -\|q_i - q_{i+1}\|^2 (q_i - q_{i+1}) \\ &\quad -(1 - \|q_i - q_{i+2}\|^2)^2 (q_i - q_{i+2}) \end{aligned} \tag{2.18}$$

where $i \in \mathbb{N}_n$, $q_{n+1} = q_1$, and $q_{n+2} = q_2$. Clearly, Assumption 2.1 holds for the above control law. Therefore, to show the convergence of the agents to consensus, it suffices to show that Assumption 2.2 holds. Suppose that there is a solution with the control inputs given by (2.18), for which Assumption 2.2 does not hold. Assume also that agent i is fixed at a vertex of the fixed convex hull corresponding to this solution, for some $i \in \mathbb{N}_n$. Proposition 2.1 implies that either $\|q_i - q_{i+1}\|^2 \equiv 0$ or $q_{i+1} \equiv q_i$, either case yielding that $q_{i+1} \equiv q_i$. Similarly, one can conclude that $q_{i+2} \equiv q_{i+1}$. Repeating the same argument, it can be shown that all the agents should coincide with agent i, which is a contradiction because a solution which does not satisfy Assumption 2.2 should not be a singleton. Therefore, both Assumptions 2.1-2.2 hold and the convergence to consensus is deduced from Theorem 2.3. It is straightforward to verify that the convergence to consensus for this example cannot be deduced from [40, 39].

The information flow graph G and the trajectories of the agents under the given control law for the case of $n = 6$ are depicted in Figs. 2.2 and 2.3, respectively. The convex hull of the agents at three time instants $t_0 = 0$ sec, $t_1 = 0.3$ sec, and $t_2 = 1.25$ sec are also drawn in Fig. 2.3. It can be observed from this figure that $S(t_2) \subseteq S(t_1) \subseteq S(t_0)$. This is in accordance with the nestedness property of $S(t)$ as shown in Theorem 2.1. The norms of the control inputs u_i, $i \in \mathbb{N}_6$ are also plotted in Fig. 2.4.

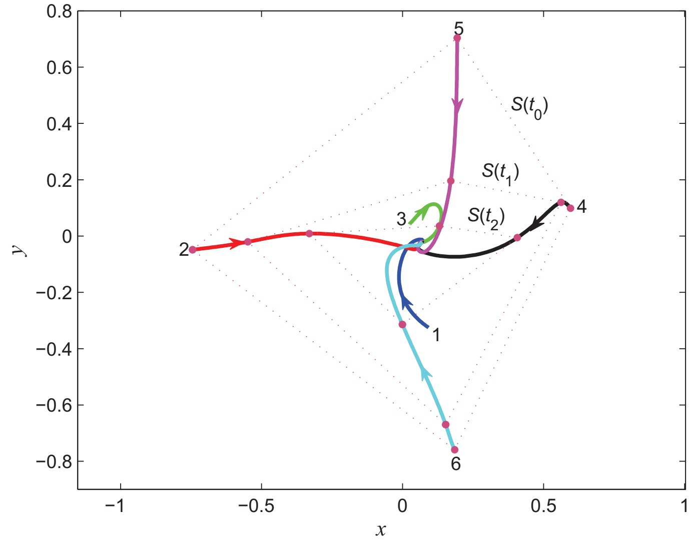

Figure 2.3: The agents' planar motion for the case of $n = 6$, in Example 2.1.

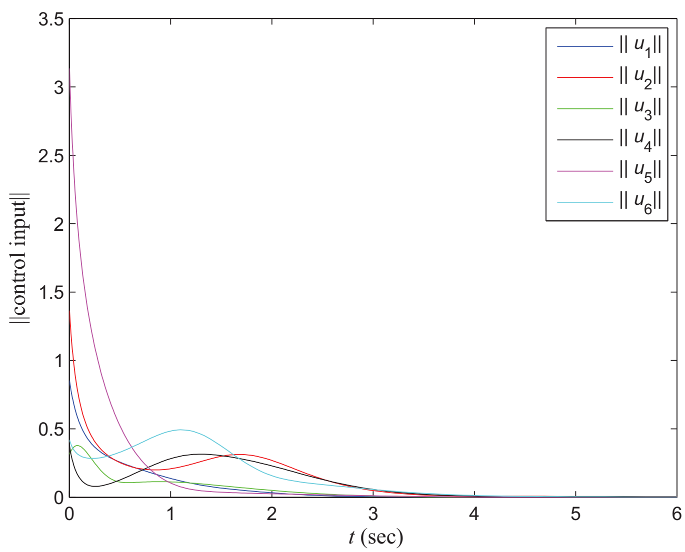

Figure 2.4: The norms of the control inputs u_i, $i \in \mathbb{N}_6$, in Example 2.1.

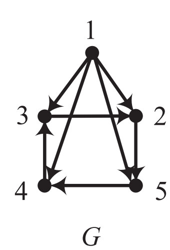

Figure 2.5: The information flow graph G for the case of $n = 5$, in Example 2.2.

Example 2.2. *Consider n agents with the dynamics of the form (2.1) moving in a 2D plane with local control laws given by*

$$\begin{aligned} u_i &= -(\|q_i - q_1\|^2 - {c_i}^2)^2(q_i - q_1) \\ &\quad -(\|q_i - q_{i+1}\|^2 - {c_1}^2)^2(q_i - q_{i+1}), \quad 2 \le i \le n \end{aligned} \tag{2.19}$$

where $q_{n+1} = q_2$, and c_i's, $i \in \mathbb{N}_n$, are distinct nonnegative constants satisfying $0 \le c_i < \frac{c_1}{2}$, for $i = 2, \ldots, n$. Assume also that agent 1 is a static leader, i.e. $u_1 \equiv 0$. Assumption 2.1 is clearly satisfied for the coefficients corresponding to the given control law. Hence, to prove the convergence of the agents to consensus, it suffices to show that Assumption 2.2 holds.

Suppose that there exists a solution with the given control law for which Assumption 2.2 does not hold. In other words, consider a solution where the corresponding convex hull of the agents is not a singleton and is fixed, with at least one agent being fixed at each vertex. Denote by $\mathcal{I}$ the set of fixed agents at the vertices of the convex hull. Proposition 2.1 implies that for any $i \in \mathcal{I}$, if q_i is not at q_1 then $\|q_i - q_1\| \equiv c_i$. Let d denote the diameter of the convex hull. Then,

$$\begin{aligned} d &= \max_{r,s\in\mathcal{I}}\{\|q_r - q_s\|\} \\ &\le \max_{r,s\in\mathcal{I}}\{\|q_r - q_1\| + \|q_s - q_1\|\} \\ &< \frac{c_1}{2} + \frac{c_1}{2} = c_1 \end{aligned} \tag{2.20}$$

Now, consider an agent $i \in \mathcal{I}$ for which $q_i \not\equiv q_1$. The relation $\|q_i - q_{i+1}\| \le d < c_1$ along with Proposition 2.1 yields $q_{i+1} \equiv q_i$. This means that q_{i+1} is also fixed and $q_{i+1} \not\equiv q_1$; hence as shown earlier $\|q_{i+1} - q_1\| \equiv c_{i+1}$. This is a contradiction since $\|q_{i+1} - q_1\| \equiv \|q_i - q_1\| \equiv c_i$ and $c_i \ne c_{i+1}$. Therefore, both Assumptions 2.1-2.2 hold and the convergence to consensus is deduced from Theorem 2.3. It is easy to verify that the convergence to consensus for this example cannot be deduced from [40, 39].

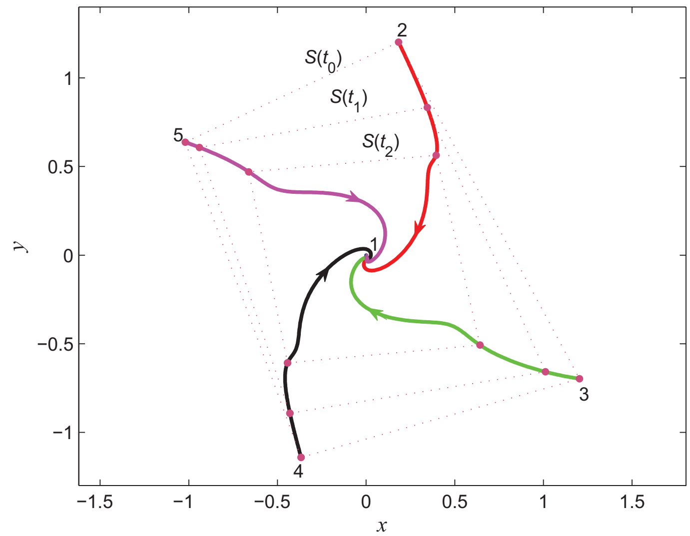

Figure 2.6: The agents' planar motion for the case of $n = 5$, in Example 2.2.

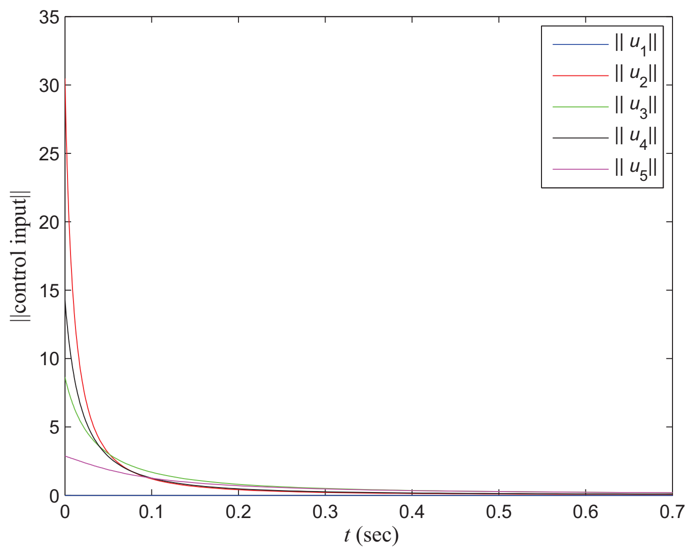

Figure 2.7: The norms of the control inputs u_i, $i \in \mathbb{N}_5$, in Example 2.2.

The information flow graph G and the trajectories of the agents under the given control law for the case of $n = 5$ are depicted in Figs. 2.5 and 2.6. The corresponding values of $c_1, \ldots, c_5$ are chosen to be 1, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{5}$, and $\frac{1}{5}$, respectively. The convex hull of the agents at three time instants $t_0 = 0$ sec, $t_1 = 0.03$ sec, and $t_2 = 0.43$ sec are also drawn in Fig. 2.5. It can be observed from this figure that $S(t_2) \subseteq S(t_1) \subseteq S(t_0)$. This confirms the nestedness property of $S(t)$ as shown in Theorem 2.1. The norms of the control inputs u_i, $i \in \mathbb{N}_5$, are also plotted in Fig. 2.7.

Chapter 3

A Class of Bounded Distributed Control Strategies for Connectivity Preservation of Single-Integrator Agents

In this chapter, a general class of distributed potential-based control laws with the connectivity preserving property for single-integrator agents is proposed. The potential functions are designed in such a way that when an edge in the information flow graph is about to lose connectivity, the gradient of the potential function lies in the direction of that edge, aiming to shrink it. The results are developed for a static information flow graph first, and then are extended to the case of dynamic edge addition. Connectivity preservation for problems involving static leaders is covered as well. The potential functions are chosen to be smooth, resulting in bounded control inputs. Other constraints may also be imposed on the potential functions to satisfy various design criteria such as consensus, containment, and formation convergence. The proposed control schemes are subsequently used to develop connectivity preserving controllers for the consensus and containment applications, where the stability analysis is also provided.

The remainder of this chapter is organized as follows. In Section 3.1, some notations and definitions are introduced which will prove convenient in presenting the main results, and also the problem statement is provided. The connectivity preserving control design is elaborated in Section 3.2. The extension of the results to the case of dynamic information flow graph and problems involving static leaders is presented in Sections 3.3 and 3.4. Stability analysis and simulation results for the examples of consensus and containment are presented in Section 3.5 to illustrate the effectiveness of the proposed control strategy.

3.1 Problem Formulation

Definition 3.1. *Multinomial coefficients are defined by*

$$\binom{k}{r_1, r_2, \ldots, r_\mu} := \frac{k!}{r_1! r_2! \ldots r_\mu!}$$

where $r_1, r_2, \ldots, r_\mu$ are nonnegative integers, and $k = r_1 + r_2 + \ldots + r_\mu$. In the special case of $\mu = 2$, the corresponding coefficients are called the binomial coefficients, and are shown by $\binom{k}{r_1, r_2} = \binom{k}{r_1} = \binom{k}{r_2}$.

Notation 3.1. *For any given function $h(x, y)$, by $\frac{\partial h}{\partial y}(x, 0)$ we mean $\frac{\partial h}{\partial y}(x, y)_{|y=0}$ (and similarly, $\frac{\partial h}{\partial x}(0, y) = \frac{\partial h}{\partial x}(x, y)_{|x=0}$). Notice that while this may be considered standard notation, it is emphasized here for the sake of clarity, and to avoid possible confusion.*

Consider a set of n single-integrator agents in a plane with a control law of the form

$$\dot{q}_i(t) = u_i = -\frac{\partial h_i}{\partial q_i} \tag{3.1}$$

where $q_i(t)$ denotes the position of agent i in the plane at time t, and h_i's are distributed potential functions. Denote by $G = (V, E)$ the information flow graph, with $V = \{1, \ldots, n\}$ its vertices, and with $E \subset V \times V$ its edges. It is assumed that the information flow graph G is connected and undirected, and that each agent can only use the relative position of its neighbors in its control law. Denote the set of the neighbors of agent i in G by $N_i(G)$, and the degree of agent i in G with $d_i(G)$. Two agents i and j are said to be in the connectivity range if $\|q_i - q_j\| \leq d$, for a pre-specified positive real number d, where $\|\cdot\|$ denotes the Euclidean norm. It is assumed that all agents in $N_i(G)$ are initially located in the connectivity range of agent i. The goal is to design a class of distributed potential functions that preserve connectivity. More precisely, it is desired to find a control scheme such that if $\|q_i(0) - q_j(0)\| \leq d$ for all $(i, j) \in E$, then $\|q_i(t) - q_j(t)\| \leq d$, for all $(i, j) \in E$ and all $t \geq 0$.

3.2 Connectivity Preserving Controller Design

For every agent i, define

$$\sigma_i(t) := \frac{1}{2} \sum_{j \in N_i(G)} \|q_i(t) - q_j(t)\|^2 \tag{3.2}$$

$$\pi_i(t) := \frac{1}{2} \prod_{j \in N_i(G)} (d^2 - \|q_i(t) - q_j(t)\|^2) \tag{3.3}$$

$$\pi_{ij}(t) := \prod_{\substack{k \in N_i(G) \\ k \neq j}} (d^2 - \|q_i(t) - q_k(t)\|^2) \tag{3.4}$$

Consider a set of distributed smooth potential functions of the form $h_i(\sigma_i, \pi_i)$ with the following properties

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, 0) = 0, \ \frac{\partial h_i}{\partial \pi_i}(\sigma_i, 0) < 0, \qquad \forall \sigma_i \in \mathbb{R}^+ \tag{3.5}$$

Intuitively, under these conditions when agent i is about to lose connectivity ($\pi_i = 0$), changes in h_i is only affected by changes in π_i and if the agents move in a direction such that h_i decreases, then the connectivity will improve (i.e., π will increase). On the other hand, when π_i becomes zero, changes in it is only affected by changes in q_i and q_j, where j is the agent which is exactly at distance d from agent i; therefore, only q_i and q_j can influence h_i. Agent i is clearly moving in a direction which tends to decrease h_i, according to (3.1). It can be shown that agent j also moves in a direction which tends to decrease h_i (although its corresponding potential function is different from h_i). This argument is valid only for the case when agent i is at distance d from only one neighbor. For the general case, one should look at higher-order derivatives (not just the gradient).

It is desired now to show that using this type of potential functions, the control law (3.1) is connectivity preserving. Using the equality $\frac{\partial h_i}{\partial q_i} = \frac{\partial h_i}{\partial \sigma_i}\frac{\partial \sigma_i}{\partial q_i} + \frac{\partial h_i}{\partial \pi_i}\frac{\partial \pi_i}{\partial q_i}$, one can rewrite the control law (3.1) as

$$\dot{q}_i = -\sum_{j \in N_i(G)} (q_i - q_j)(\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij}) \tag{3.6}$$

Define $\mathcal{T}$ to be the set of those time instants $t \geq 0$ at which $\|q_i(t) - q_j(t)\| \leq d$, for all $(i,j) \in E$. For any $t \in \mathcal{T}$, construct a graph $G_d(t) = (V_d(t), E_d(t))$ as the union of those edges $(i,j) \in E$ for which $\|q_i(t) - q_j(t)\| = d$. Define $s_{ij}(t) = \|q_i(t) - q_j(t)\|^2$, for $(i,j) \in E_d$. The following lemmas are key to the proof of the main results.

Lemma 3.1. *Consider a real-valued function f for which $f^{\rho(f(t))}(t) < 0$, for some t, where $\rho(f(t))$ denotes the index of f at time t as defined in Definition 2.2; then f is monotonically decreasing in the interval $[t, t+\epsilon]$, for some $\epsilon > 0$.*

Proof. Let $k = \rho(f(t))$; since $f^{(k)}(t) < 0$, the function $f^{(k-1)}$ is monotonically decreasing in the interval $[t, t+\varepsilon]$, for some $\varepsilon > 0$. On the other hand $f^{(k-1)}(t) = 0$, which implies (along with the above result) that $f^{(k-1)} < 0$ in $(t, t+\varepsilon]$, and hence $f^{(k-2)}$ is monotonically decreasing in $[t, t+\varepsilon]$. Using a similar argument iteratively, one arrives at the conclusion that $f^{(0)}$ (which by definition is equal to f) is monotonically decreasing in the above closed interval, and this completes the proof. ■

Lemma 3.2. *Suppose that* $q_i^{(r)}(t) = q_j^{(r)}(t) = 0$*, for all* $r \in \{1, \dots, k-1\}$ *and some* t*; then*

$$s_{ij}^{(k)}(t) = 2(q_i(t) - q_j(t))^T (q_i^{(k)}(t) - q_j^{(k)}(t)) \tag{3.7}$$

Proof. The proof follows directly from the fact that

$$\frac{d^k}{dt^k}(x^T x) = \sum_{r=0}^{k} x^{(r)T} x^{(k-r)} \begin{pmatrix} k \\ r \end{pmatrix} \tag{3.8}$$

■

Lemma 3.3. *Consider an agent* i *in* $G_d(t)$ *for some* $t \in \mathcal{T}$*, and assume that* $\eta = \min_{j \in N_i(G)}\{\rho(\pi_{ij})\}$*. Assume also that* $d_i(G_d) \geq 2$*; then the following statements hold:*

i) $\pi_{ij}^{(r)} = 0$*, for* $0 \leq r \leq \eta - 1$*, and* $j \in N_i(G)$*.*

ii) $\pi_i^{(r)} = 0$*, for* $0 \leq r \leq \eta - 1$*.*

iii) $(\frac{\partial h_i}{\partial \sigma_i})^{(r)} = 0$*, for* $0 \leq r \leq \eta - 1$*.*

iv) $\rho(q_i) \geq \eta + 1$*.*

Proof.

Part (i): Since $d_i(G_d) \geq 2$, one can easily verify that $\pi_{ij} = 0$. The rest of the proof follows immediately from the definition of the index of a function.

Part (ii): Since $\pi_i = \frac{1}{2}\pi_{ij} \times (d^2 - \|q_i - q_j\|^2)$ for any j in $N_i(G)$, therefore

$$\pi_i^{(r)} = \frac{1}{2} \sum_{m=0}^{r} \pi_{ij}^{(m)} (d^2 - \|q_i - q_j\|^2)^{(r-m)} \begin{pmatrix} r \\ m \end{pmatrix} \tag{3.9}$$

The proof follows directly by applying the result of part (i) to the above equation.

Part (iii): From (3.5), $\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, 0) = 0$. Now, using the fact that

$$\frac{\partial^{(r+1)} h_i}{\partial \sigma_i^{(r+1)}}(\sigma_i, 0) = \lim_{\Delta\sigma_i \to 0} \frac{\frac{\partial^r h_i}{\partial \sigma_i^r}(\sigma_i + \Delta\sigma_i, 0) - \frac{\partial^r h_i}{\partial \sigma_i^r}(\sigma_i, 0)}{\Delta\sigma_i} \tag{3.10}$$

it can be shown recursively that

$$\frac{\partial^r h_i}{\partial \sigma_i^r} = 0, \qquad \forall r \in \mathbb{N}$$

Using induction on r, one can express $(\frac{\partial h_i}{\partial \sigma_i})^{(r)}$ in the form of

$$(\frac{\partial h_i}{\partial \sigma_i})^{(r)} = \sum_{m \leq r+1} \frac{\partial^m h_i}{\partial \sigma_i^m} a_m(\sigma_i) + \sum_{m \leq r} b_m(\sigma_i, \pi_i)\pi_i^{(m)} \tag{3.11}$$

The first term in the right side of (3.11) is zero as noted above. Hence, the proof is completed by noting that $\pi_i^{(m)} = 0$ for $m \leq r$ (from the result of part (ii)).

Part (iv): By differentiating k times both sides of (3.6), one arrives at

$$q_i^{(k+1)} = -\sum_{j \in N_i(G)} \sum_{r=0}^{k} (q_i - q_j)^{(r)} (\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij})^{(k-r)} \binom{k}{r} \tag{3.12}$$

The right side of the above equation is equal to zero for all $k \in \{0, \ldots, \eta - 1\}$, as a consequence of parts (i)-(iii). This implies that $\rho(q_i) \geq \eta + 1$. ■

Remark 3.1. *In the case when $d_i(G_d) = 1$, it is straightforward to show that $\dot{q}_i = \frac{\partial h_i}{\partial \pi_i}\pi_{ij}(q_i - q_j)$, where j is the neighbor for which $\|q_i - q_j\| = d$.*

Remark 3.2. *If $\rho(\pi_{ij})$ is not the same for all $j \in N_i(G_d)$, then part (ii) of Lemma 3.3 also holds for $r = \eta$. Consequently, part (iii) also holds for $r = \eta$.*

Lemma 3.4. *Consider agent i in $G_d(t)$, $t \in \mathcal{T}$, and let ν be one of the (possibly multiple) neighbors of i in $G_d(t)$ for which $\rho(q_\nu) = \max_{j \in N_i(G_d)}\{\rho(q_j)\}$. Then*

$$\rho(q_i) \geq 1 + \sum_{\substack{j \in N_i(G_d) \\ j \neq \nu}} \rho(q_j) \tag{3.13}$$

Proof. The proof is trivial for the case when $d_i(G_d) = 1$. Hence, consider the case $d_i(G_d) \geq 2$; for any $j \in N_i(G)$, by differentiating (3.4) k times, one can show that

$$\pi_{ij}^{(k)} = \sum_{\substack{r_1 + \ldots + r_\mu = k \\ r_1, \ldots, r_\mu \geq 0}} \binom{k}{r_1, \ldots, r_\mu} \prod_{s=1}^{\mu} (d^2 - \|q_i - q_{i_s}\|^2)^{(r_s)} \tag{3.14}$$

where $\{i_1, \ldots, i_\mu\} = N_i(G) - \{j\}$. Let $k \leq \eta$; then, on using Lemma 3.2 and noting (from Lemma 3.3) that $\rho(q_i) > k$, one can easily verify that the term corresponding to $(r_1, ..., r_\mu)$ in the above summation is nonzero only if $r_s \geq \rho(q_{i_s})$ for every $i_s \in N_i(G_d) - \{j\}$. On the other hand

$$k = \sum_{s=1}^{\mu} r_s \geq \sum_{\substack{i_s \in N_i(G_d) \\ i_s \neq j}} r_s \tag{3.15}$$

Therefore, a necessary condition for $\pi_{ij}^{(k)}$ to be nonzero can be obtained as

$$k \geq \sum_{\substack{i_s \in N_i(G_d) \\ i_s \neq j}} \rho(q_{i_s}) \tag{3.16}$$

Now, choose $k = \eta$; since $\eta = \min_{j \in N_i(G)}\{\rho(\pi_{ij})\}$, thus $\pi_{ij}^{(\eta)} \neq 0$ for at least one $j \in N_i(G)$. Hence, (3.16) should hold for $k = \eta$ and at least one $j \in N_i(G)$. Clearly, the right side of this inequality is minimized when $\rho(q_j)$ is maximized (i.e., when $j = \nu$). This fact along with part (iv) of Lemma 3.3 results in (3.13). ■

Lemma 3.5. *Let $\rho_l(q_i)$ be the lower bound for $\rho(q_i)$ given in Lemma 3.4, i.e.*

$$\rho_l(q_i) = 1 + \sum_{\substack{j \in N_i(G_d) \\ j \neq \nu}} \rho(q_j) \tag{3.17}$$

where $\nu = argmax_{j \in N_i(G_d)}\{\rho(q_j)\}$. If ν is unique, then

i) $\pi_{i\nu}^{(\rho_l(q_i)-1)} = \tilde{\pi}_{i\nu} \prod_{\substack{j \in N_i(G_d) \\ j \neq \nu}} (q_i - q_j)^T q_j^{(\rho(q_j))}$, *where* $\tilde{\pi}_{i\nu} > 0$.

ii) $q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i} \tilde{\pi}_{i\nu} (q_i - q_\nu) \prod_{\substack{j \in N_i(G_d) \\ j \neq \nu}} (q_i - q_j)^T q_j^{(\rho(q_j))}$.

Proof.

Part (i): Let (3.14) be revisited for $k = \rho_l(q_i) - 1$. It results from the uniqueness of ν that (3.16) holds only for $j = \nu$; hence, $\pi_{ij}^{(k)} = 0$ for $j \neq \nu$. Also, $\pi_{i\nu}^{(k)}$ has only one possibly nonzero term as follows

$$\begin{aligned}
\pi_{i\nu}^{(k)} &= \binom{k}{r_1,\ldots,r_\mu} \prod_{\substack{j\in N_i(G_d)\\ j\neq\nu}} -2(q_i-q_j)^T(q_i^{(\rho(q_j))}-q_j^{(\rho(q_j))}) \\
&\times \prod_{\substack{j\in N_i(G)\\ j\notin N_i(G_d)}} (d^2-\|q_i-q_j\|^2) \\
&= \tilde{\pi}_{i\nu} \prod_{\substack{j\in N_i(G_d)\\ j\neq\nu}} (q_i-q_j)^T q_j^{(\rho(q_j))} \qquad (3.18)
\end{aligned}$$

Note that in obtaining the above relation the result of Lemma 3.2 and the fact that $q_i^{(\rho(q_j))}=0$ (because $\rho(q_j)\leq k<\rho(q_i)$) are used. Note also that

$$\tilde{\pi}_{i\nu} = 2^{d_i(G_d)-1}\binom{k}{r_1,\ldots,r_\mu} \prod_{\substack{j\in N_i(G)\\ j\notin N_i(G_d)}} (d^2-\|q_i-q_j\|^2) > 0 \qquad (3.19)$$

The corresponding values of $(r_1,\ldots,r_\mu)$ are

$$r_s = \begin{cases} \rho(q_{i_s}) & i_s\in N_i(G_d)-\{\nu\} \\ 0 & i_s\notin N_i(G_d)-\{\nu\} \end{cases} \qquad (3.20)$$

Part (ii): Consider (3.12) for $k=\rho_l(q_i)-1$. Using the fact that $\pi_{ij}^{(k)}=0$ (for $j\neq\nu$) along with Lemma 3.3 and Remark 3.2, one can conclude that $q_i^{(\rho_l(q_i))}$ has only one possibly nonzero term as

$$q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{i\nu}(q_i-q_\nu) \prod_{\substack{j\in N_i(G_d)\\ j\neq\nu}} (q_i-q_j)^T q_j^{(\rho(q_j))} \qquad (3.21)$$

This completes the proof. ■

Lemma 3.6. *Define the subgraph $G_d^{<\infty}(t)$ of $G_d(t)$ as the union of those edges $e=(i,j)\in E_d(t)$ for which $\min(\rho(q_i),\rho(q_j))<\infty$; denote its set of edges by $E_d^{<\infty}(t)$, and its set of vertices by $V_d^{<\infty}(t)$. Then, for any $(i,j)\in E_d^{<\infty}(t)$, the relations $\rho(s_{ij})=\min\{\rho(q_i),\rho(q_j)\}$ and $s_{ij}^{(\rho(s_{ij}))}<0$ hold.*

Proof. One can prove this lemma by induction on $\min(\rho(q_i),\rho(q_j))$. Start with $\min(\rho(q_i),\rho(q_j))=1$, and without loss of generality assume that $\rho(q_i)=1$. If

$\rho(q_j) > 1$, then $\dot{q}_j = 0$, and hence from Remark 3.1

$$\begin{aligned} \dot{s}_{ij} &= 2(q_i - q_j)^T(\dot{q}_i - \dot{q}_j) \\ &= 2(q_i - q_j)^T \frac{\partial h_i}{\partial \pi_i} \pi_{ij}(q_i - \ q_j) \\ &= 2d^2 \pi_{ij} \frac{\partial h_i}{\partial \pi_i} < 0 \end{aligned} \tag{3.22}$$

Similarly, if $\rho(q_j) = \rho(q_i) = 1$

$$\begin{aligned} \dot{s}_{ij} &= 2(q_i - q_j)^T(\frac{\partial h_i}{\partial \pi_i}\pi_{ij} + \frac{\partial h_j}{\partial \pi_j}\pi_{ji})(q_i - \ q_j) \\ &= 2d^2(\frac{\partial h_i}{\partial \pi_i}\pi_{ij} + \frac{\partial h_j}{\partial \pi_j}\pi_{ji}) < 0 \end{aligned} \tag{3.23}$$

Now, suppose that the lemma holds for $\min(\rho(q_i), \rho(q_j)) < k$. To prove the lemma for $\min(\rho(q_i),\, \rho(q_j)) = \ k$, assume without loss of generality that $\rho(q_i) = k$. Since $\rho(q_i) \leq \rho(q_j)$, using Lemma 3.4 one can easily show that $\text{argmax}_{\omega \in N_i(G_d)}\{\rho(q_\omega)\}$ is unique, and is, in fact, equal to j. As another consequence of Lemma 3.4, $\rho(q_\omega) < \rho(q_i)$ for $\omega \in N_i(G_d)$, $\omega \neq j$. Therefore, $\min(\rho(q_i), \rho(q_\omega)) = \rho(q_\omega) < k$ and hence $\rho(s_{i\omega}) = \rho(q_\omega)$ and $s_{i\omega}^{(\rho(s_{i\omega}))} < 0$. This along with Lemma 3.2 and Lemma 3.5 yields that

$$\begin{aligned} q_i^{(\rho_l(q_i))} &= \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij}(q_i - q_j) \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} (q_i - q_\omega)^T q_\omega^{(\rho(q_\omega))} \\ &= \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij}(q_i - q_j) \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} -\frac{1}{2} s_{i\omega}^{(\rho(s_{i\omega}))} \end{aligned} \tag{3.24}$$

Thus,

$$(q_i - q_j)^T q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i}\tilde{\pi}_{ij} d^2 \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} -\frac{1}{2} s_{i\omega}^{(\rho(s_{i\omega}))} < 0 \tag{3.25}$$

from which one can conclude that $\rho(q_i) = \rho_l(q_i)$. On the other hand,

$$\begin{aligned} s_{ij}^{(\rho(q_i))} &= 2(q_i - q_j)^T(q_i^{(\rho(q_i))} - q_j^{(\rho(q_i))}) \\ &= 2(q_i - q_j)^T q_i^{(\rho(q_i))} + 2(q_j - q_i)^T q_j^{(\rho(q_i))} \end{aligned} \tag{3.26}$$

If $\rho(q_j) > \rho(q_i)$, the second term in the last equation vanishes and it follows from (3.25) that $s_{ij}^{(\rho(q_i))} < 0$. If $\rho(q_j) = \rho(q_i)$, the same inequality as (3.25) holds for $\rho(q_j)$. Therefore, both terms in (3.26) are less than zero, and hence $s_{ij}^{(\rho(q_i))} < 0$. ■

Remark 3.3. *From the proof of Lemma 3.6, it can be easily seen that for every edge in $E_d^{<\infty}(t)$ the movement of the agent with lower (or equal) index is in the direction of the other agent, which results in shrinking of the edge.*

Lemma 3.7. *Consider a system of differential equations of the form*

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{3.27}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and f and g are C^1 functions. Assume that for some $y_0 \in \mathbb{R}^n$, $g(x, y_0)$ is equal to zero for every $x \in \mathbb{R}^m$. Now, suppose that $y(t_0) = y_0$ for some $t_0 \in \mathbb{R}$. Then, $y(t) = y_0$ for all $t \in \mathbb{R}$.

Proof. In order to prove this lemma, we first show that there exists $\epsilon > 0$ so that $y(t) = y_0$, for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. Denote the initial condition $x(t_0)$ by x_0, and let $\tilde{x}(t)$ be a solution of the differential equation $\dot{\tilde{x}} = f(\tilde{x}, y_0)$ satisfying the initial condition $\tilde{x}(t_0) = x_0$. Define also $\tilde{y}(t) = y_0$; it is straightforward to show that $[\tilde{x}\ \tilde{y}]$ is a solution of (3.27) satisfying the initial condition $[\tilde{x}(t_0)\ \tilde{y}(t_0)] = [x_0\ y_0]$. According to Theorem 1 in [104] (pages 162-163), there exists $\epsilon > 0$ so that the solution for (3.27) under the initial condition $[x(t_0)\ y(t_0)] = [x_0\ y_0]$ is unique over the time interval $(t_0-\epsilon, t_0+\epsilon)$. Particularly, $y(t) = \tilde{y}(t) = y_0$ for all $t \in (t_0-\epsilon, t_0+\epsilon)$. Now, define $\mathcal{E} = \{\epsilon > 0 | \forall t \in (t_0 - \epsilon, t_0 + \epsilon) : y(t) = y_0\}$. Let ϵ^+ be the supremum of $\mathcal{E}$, and assume that $\epsilon^+ < \infty$. It yields from the continuity of the solution that $y(t_0 + \epsilon^+) = y(t_0 - \epsilon^+) = y_0$. Now, applying the result obtained above with $t_0 + \epsilon^+$ and $t_0 - \epsilon^+$ instead of t_0 leads to a contradiction. Therefore $\epsilon^+ = \infty$, which completes the proof. ■

Lemma 3.8. *Consider the partition* $E_d(t) = E_d^{\infty}(t) \cup E_d^{<\infty}(t)$. *Define the graph* $G_d^{\infty}(t)$ *as the union of the edges in* $E_d^{\infty}(t)$, *and denote its set of vertices by* $V_d^{\infty}(t)$. *Then, for every* $t \in \mathcal{T}$ *and every* $i \in V_d^{\infty}(t)$,

i) $d_i(G_d^{\infty}) \geq 2$.

ii) $q_i(\tau) = q_i(t)$, *for* $\tau \geq 0$.

Proof.

Part (i): If $d_i(G_d^{\infty}) = 1$, then there exists a unique $j \in V_d^{\infty}$ for which $\rho(q_i) = \rho(q_j) = \infty$. This implies that $\operatorname{argmax}_{\omega \in N_i(G_d)}\{\rho(q_\omega)\}$ is unique and is equal to j. Hence, one can use Lemma 3.5 to obtain

$$q_i^{(\rho_l(q_i))} = \frac{\partial h_i}{\partial \pi_i} \tilde{\pi}_{ij}(q_i - q_j) \prod_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} (q_i - q_\omega)^T q_\omega^{(\rho(q_\omega))} \tag{3.28}$$

which according to Lemma 3.6 is nonzero for any pair (i, ω), $\omega \in N_i(G_d)$, $\omega \neq j$. This yields that

$$\rho(q_i) = \rho_l(q_i) = 1 + \sum_{\substack{\omega \in N_i(G_d) \\ \omega \neq j}} \rho(q_\omega) < \infty \tag{3.29}$$

, which is a contradiction; hence, $d_i(G_d^{\infty}) \geq 2$.

Part (ii): Choose an arbitrary $t \in \mathcal{T}$. Let $y(\tau)$ represent the positions of the agents belonging to $V_d^{\infty}(t)$, and $x(\tau)$ represent the positions of all other agents. Since $d_i(G_d^{\infty}) \geq 2$, one can conclude that if $y(\tau) = y(t)$ for some $\tau \geq 0$, then $\pi_{ij}(\tau) = 0$, for any $i \in V_d^{\infty}(t)$ and $j \in N_i(G)$. Using this argument, it is easy to show that x and y satisfy the conditions of Lemma 3.7, and as a result $q_i(\tau) = q_i(t)$ for $\tau \geq 0$ and $i \in V_d^{\infty}(t)$. ■

Lemma 3.9. *Under the conditions given in (3.5), the control law (3.1) is connectivity preserving.*

Proof. Assume that $\|q_i(0) - q_j(0)\| \leq d$ for all $(i,j) \in E$ (i.e. $0 \in \mathcal{T}$), and let $t_0 = \inf\{t | \ \exists (i,j) \in E : \|q_i(t) - q_j(t)\| > d\}$. Clearly, any $t \leq t_0$ belongs to

$\mathcal{T}$. Therefore, to prove the lemma it suffices to show that there is a neighborhood of t_0 in which for every $(i,j) \in E_d(t_0)$, s_{ij} is either decreasing or fixed. It follows from Lemmas 3.6 and 3.1 that s_{ij} is decreasing in a neighborhood of t_0 for any $(i,j) \in E_d^{<\infty}(t_0)$. Also, from Lemma 3.8, s_{ij} is fixed for any $(i,j) \in E_d^{\infty}(t_0)$. The proof is completed on noting that $E_d(t_0) = E_d^{\infty}(t_0) \cup E_d^{<\infty}(t_0)$. ■

Lemma 3.10. *Suppose that* $\|q_i(0) - q_j(0)\| \leq d$ *for all* $(i,j) \in E$ *(i.e.* $0 \in \mathcal{T}$*). Then,*

i) $G_d^{<\infty}(t) = \emptyset$ *for* $t > 0$.

ii) $G_d^{\infty}(t) = G_d^{\infty}(0)$ *for* $t \geq 0$.

iii) $G_d^{\infty}(0)$ *is the maximal induced subgraph of* $G_d(0)$ *with the property that the degree of each vertex in it is at least 2.*

Proof.

Part (i): Note that since $0 \in \mathcal{T}$, it results from Lemma 3.9 that $\mathcal{T} = \mathbb{R}^+ \cup \{0\}$. Hence $G_d(t)$ is well-defined for $t \geq 0$, and so are $G_d^{<\infty}(t)$ and $G_d^{\infty}(t)$. Now, assume that $G_d^{<\infty}(t) \neq \emptyset$ for some $t > 0$, and let $u = \text{argmin}_{i \in V_d^{<\infty}(t)}\{\rho(q_i(t))\}$. Lemma 3.4 implies that $d_u(G_d) = 1$, and consequently from Remark 3.1, $\rho(q_u(t)) = 1$. Let $v \in G_d(t)$ be the neighbor of u. According to Lemma 3.6, $\dot{s}_{uv}(t) < 0$, implying that $\|q_u - q_v\| > d$ in the interval $(t-\epsilon, t)$ for some $\epsilon > 0$, which contradicts Lemma 3.9.
Part (ii): This part is a straightforward consequence of part (ii) of Lemma 3.8.
Part (iii): Let $G_M = (V_M, E_M)$ be the maximal induced subgraph of $G_d(0)$ such that $d_i(G_M) \geq 2$ for $i \in V_M$. From part (i) of Lemma 3.8, $G_d^{\infty}(0) \subset G_M$. Therefore, it suffices to show that $G_M \subset G_d^{\infty}(0)$. Every $i \in V_M$ has at least two neighbors located at a distance d from it, yielding that $\pi_{ij} = 0$ for any $i \in V_M$ and $j \in N_i(G)$. Similar to the approach used in the proof of Lemma 3.8, one can use Lemma 3.7 to deduce that $q_i(t) = q_i(0)$ for any $t \geq 0$ and $i \in V_M$. Therefore, $\rho(q_i(0)) = \infty$ for $i \in V_M$, which implies that $G_M \subset G_d^{\infty}(0)$. This completes the proof. ■

As the main contribution of this chapter, the following theorem states that under certain boundary conditions, the control law (3.1) is connectivity preserving. In fact, connectivity preservation is strict for all pairs of agents forming an edge in the information flow graph, except those edges whose ends stay fixed over time under the control law (3.1). Moreover, the theorem precisely characterizes the topology of such fixed edges.

Theorem 3.1. *Consider a set of n agents in the plane with the dynamics of the form (3.1), and assume the conditions given in (3.5) hold. Assume also that $\|q_i(0)-q_j(0)\| \leq d$ for all $(i,j) \in E$. Then, the control law (3.1) is connectivity preserving. Moreover, Let $G_M = (V_M, E_M)$ be the maximal induced subgraph of $G_d(0)$ such that $d_i(G_M) \geq 2$ for every $i \in V_M$. Then, at any time $t \geq 0$, $q_i(t) = q_i(0)$ for $i \in V_M$, and $\|q_i(t) - q_j(t)\| < d$ for $(i,j) \in E - E_M$.*

Proof. The proof follows directly from Lemmas 3.9 and 3.10. ■

Remark 3.4. *It results from Theorem 3.1 (as a special case of practical interest) that if $\|q_i(0) - q_j(0)\| < d$ for all $(i,j) \in E$, then the connectivity preservation is strict, meaning that $\|q_i(t) - q_j(t)\| < d$, at all times $t > 0$, and for all $(i,j) \in E$. In other words, if two agents connected by an edge in the information flow graph are initially located at a distance less than the connectivity threshold distance, their distance stays below this threshold at all times.*

3.3 Dynamic Information Flow Graph

The results presented so far can be easily extended to the case of dynamic edge addition, where new edges may be added to the information flow graph once two agents enter the connectivity range. Suppose that new edges are added to the information flow graph at time instants t_k, $k = 1, 2, \ldots$, and denote by $G^{(k)}$ the resultant information flow graph at time t_k. Note that the two agents associated with a newly added edge to the information flow graph at time t_k should be in the connectivity range at the time of addition. Clearly, according to Theorem 3.1 the proposed control law preserves the connectivity of the agents connected in $G^{(k)}$ during the time interval $[t_k, t_{k+1}]$. This implies that for any edge added to the information flow graph, the connectivity of the corresponding agents will be preserved at all times, provided they are in the connectivity range at the time of addition.

Adding new edges to the information flow graph may result in more fixed agents since it may change the structure of G_M defined in Theorem 3.1. To avoid this problem, an additional constraint is imposed that at the time of adding a new edge, the corresponding agents should be in the strict connectivity range. Under this condition, the addition of new edges will not affect G_M, and hence the structure of the fixed agents can be determined from G_M.

3.4 Connectivity Preservation for Problems Involving Static Leaders

Consider the case in which some of the agents, called static leaders, are required to stay fixed. In this case, even if conditions given in (3.5) hold for the rest of the agents, called followers, one cannot directly deduce connectivity preservation from Theorem 3.1. In this section, it is shown how by using a simple trick connectivity preservation can be guaranteed assuming conditions (3.5) hold for the followers. Denote the set of static leaders by $\mathcal{L} \subset V(G)$; thus, $\dot{q}_i(t) = 0$ for every $i \in \mathcal{L}$ and $t \geq 0$. Assume that control laws of the form (3.1) are applied to the followers, where h_i's satisfy conditions given in (3.5). Construct a new graph $\bar{G}$ from G as follows. For any $i \in \mathcal{L}$, consider two *virtual agents* i_1 and i_2, initially located at distance d from each other and from i. Add the two new vertices i_1 and i_2 to $V(G)$, and all the possible edges between i, i_1, and i_2 to $E(G)$. Choose any h_i, h_{i_1}, and h_{i_2} satisfying conditions (3.5); then connectivity preservation is guaranteed for $\bar{G}$ according to Theorem 3.1. Clearly $i, i_1, i_2 \in \bar{G}_M$, and hence the corresponding agents remain fixed as desired. Therefore, connectivity preservation for the case of static leaders is deduced.

3.5 Simulation Results

3.5.1 Consensus Example

Consider 4 single-integrator agents moving in a two-dimensional space with the information flow graph G depicted in Fig. 3.1. The agents are to aggregate while preserving connectivity. This can be achieved by using the control law (3.1) with an appropriate choice of h_i's. Assume that in addition to the conditions in (3.5), h_i's also satisfy the following constraints

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, \pi_i) > 0,\ \frac{\partial h_i}{\partial \pi_i}(\sigma_i, \pi_i) \leq 0, \qquad \forall \sigma_i \geq 0,\ \forall \pi_i > 0 \tag{3.30}$$

Let d be equal to 1, and the initial position of each agent be marked by its index as shown in Fig. 3.2. As depicted in Fig. 3.1, $G_d(0)$ is a tree and hence $G_M = \emptyset$. Therefore, it results from Theorem 3.1 that $\|q_i(t) - q_j(t)\| < d$ for all $(i, j) \in E(G)$ and $t > 0$. Now, (3.30) yields that for any $i \in V(G)$ and $j \in N_i(G)$

$$\beta_{ij} := \frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij} > 0, \qquad \forall t > 0 \tag{3.31}$$

The above inequality along with (3.6) implies that the velocity of each agent points toward the convex hull of its neighbors. Define $Q_i = \{q_j | j \in N_i \cup \{i\}\}$, and assume that q_i is a vertex of $\mathrm{Conv}(Q_i)$. If Q_i is not a singleton, then the above inequality along with (3.6) implies that $\dot{q}_i \neq 0$. Using this property and Proposition 2.2, it is straightforward to show the convergence of the agents to a single point.

The above discussion shows that if h_i's satisfy conditions given by (3.5) and (3.30), then the agents reach consensus while preserving connectivity. There are a

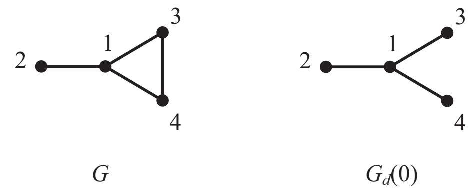

Figure 3.1: The information flow graph G and the graph $G_d(0)$ for the consensus example.

variety of functions satisfying these conditions, including the one used in [57]. The following function will be used in the simulation

$$h_i(\sigma_i, \pi_i) = \frac{\sigma_i}{\sigma_i + \pi_i + {\pi_i}^2} \tag{3.32}$$

It is desired now to verify the results obtained in this work by simulation. To this end, the planar motion of the agents is shown in Fig. 3.2. Denote the relative distance between agent i and its neighbor j by d_{ij} (i.e., $d_{ij} := \|q_i - q_j\|$). The relative distances d_{12}, d_{13}, and d_{34} are depicted in Fig. 3.3. Although d_{12}, d_{13} and d_{14} are initially equal to d ($d_{12} = d_{13} = d_{14} = 1$ at $t = 0$), the proposed controller ensures that $d_{ij} < d$ for all $(i, j) \in E(G)$, while the agents converge to consensus. Furthermore, the norms of the control inputs u_1, u_2 and u_3 are bounded, as depicted in Fig. 3.4. It is to be noted that in this example d_{13} and d_{14} are almost the same, and so are u_3 and u_4.

3.5.2 Containment Example

For this example, a team of 3 static leaders and 3 followers is considered, where the followers are desired to converge to the triangle of the leaders while preserving the connectivity of the information flow graph G given in Fig. 3.5. Consider the following potential function

$$h_i(\sigma_i, \pi_i) = -\pi_i \tag{3.33}$$

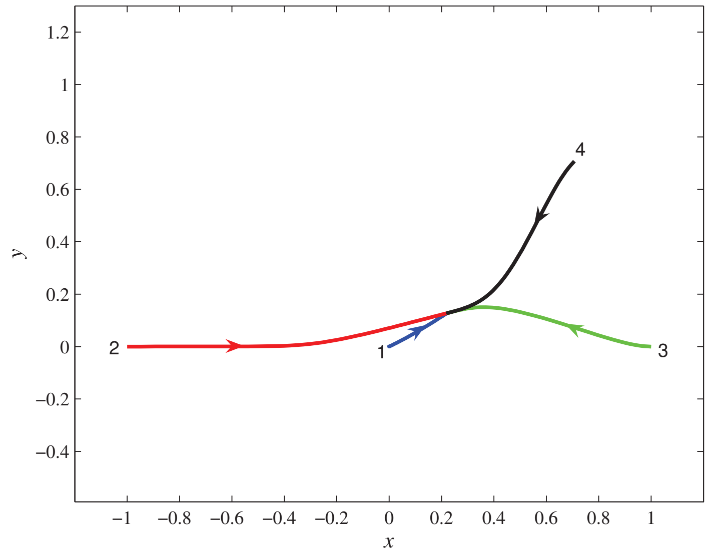

Figure 3.2: The agents' planar motion in the consensus example.

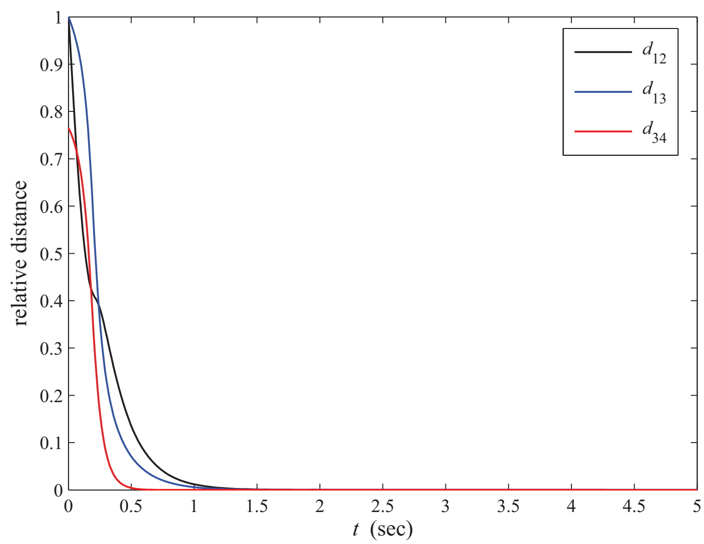

Figure 3.3: The relative distances d_{12}, d_{13} and d_{34} in the consensus example.

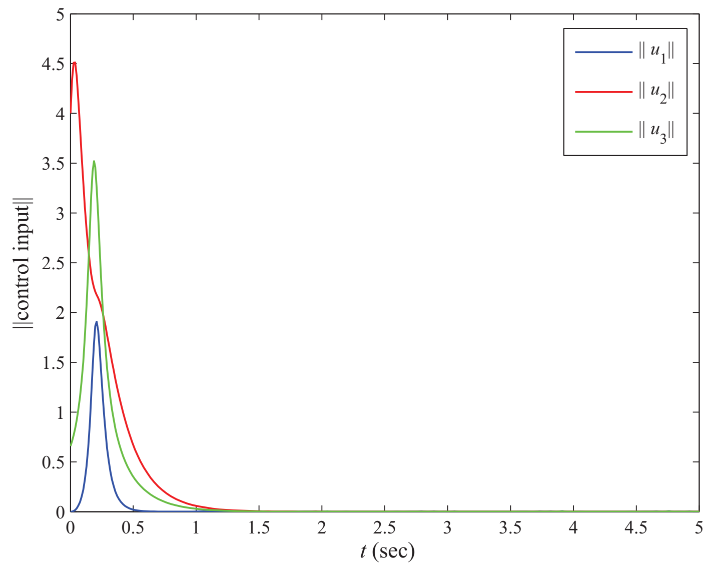

Figure 3.4: The norms of the control inputs u_1, u_2 and u_3 in the consensus example.

It can be easily verified that the function given above satisfies the conditions in (3.5), which means that the corresponding control law is connectivity preserving. Let d in this example also be equal to 1, and the initial position of each agent be marked by its index, as shown in Fig. 3.6. The graphs $\bar{G}$ (obtained by adding the virtual agents to G), $\bar{G}_d(0)$, and $\bar{G}_M$ are depicted in Fig. 3.5. According to Theorem 3.1, for all $(i,j) \in E(\bar{G}) - E(\bar{G}_M) = E(G)$, the inequality $\|q_i(t) - q_j(t)\| < d$ holds for any $t > 0$. To prove the convergence of the followers to the convex hull of the leaders, consider the function $\pi(t)$ defined by

$$\pi(t) = \prod_{\substack{(i,j)\in E(G) \\ i<j}} (1 - \|q_i(t) - q_j(t)\|^2)$$

Note that $\dot{\pi} = \sum_{i=4}^{6} \dot{q}_i{}^T \frac{\partial \pi}{\partial q_i} = \sum_{i=4}^{6} \dot{q}_i{}^T \frac{\partial \pi_i}{\partial q_i} \bar{\pi}_i = \sum_{i=4}^{6} \bar{\pi}_i \|\dot{q}_i\|^2$, where $\bar{\pi}_i$ is the product of those terms in π which do not appear in π_i (i.e. $\pi = \pi_i \bar{\pi}_i$). It results from strict connectivity preservation that $\bar{\pi}_i > 0$ for $t > 0$, and hence $\dot{\pi} \geq 0$ for $t > 0$. On the other hand, $0 < \pi < 1$ for $t > 0$; therefore, using LaSalle's invariance principle [103] one can conclude the convergence of the agents to the largest invariant set in $\dot{\pi} = 0$, which is $\dot{q}_i = 0$ for $i = 4, 5, 6$, i.e. the equilibrium set of (3.1).

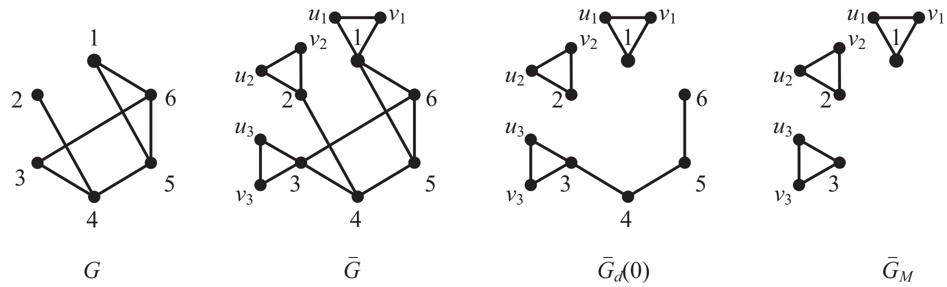

Figure 3.5: The information flow graph G along with the graphs $\bar{G}$, $\bar{G}_d(0)$, and $\bar{G}_M$ for the containment example.

Moreover, it yields from (3.6) that in the equilibrium set $\sum_{j\in N_i(G)} \pi_{ij}(q_i - q_j) = 0$ for each follower i. Therefore, $q_i = \sum_{j\in N_i(G)} \alpha_{ij} q_j$, where $\alpha_{ij} = \frac{\pi_{ij}}{\sum_{j\in N_i(G)} \pi_{ij}}$. Clearly, $0 < \alpha_{ij} < 1$ and $\sum_{j\in N_i(G)} \alpha_{ij} = 1$. This means that at equilibrium each follower i is in the convex hull of its neighbors. Thus, for q_i to be at a vertex of the convex hull of the agents, it should coincide with all of its neighbors in $N_i(G)$. Repeating the same argument, one can conclude that q_i should coincide with the agents reachable from i in G. This is a contradiction as every leader is reachable from i since G is connected. This completes the proof of the convergence of the followers to the convex hull of the leaders.

The motion of the agents is depicted in Fig. 3.6, and the relative distances are sketched in Fig. 3.7. The control input norms $\|u_4\|$, $\|u_5\|$ and $\|u_6\|$ are plotted in Fig. 3.8. This figure shows the boundedness of the control inputs, although some of the agents are initially about to lose connectivity.

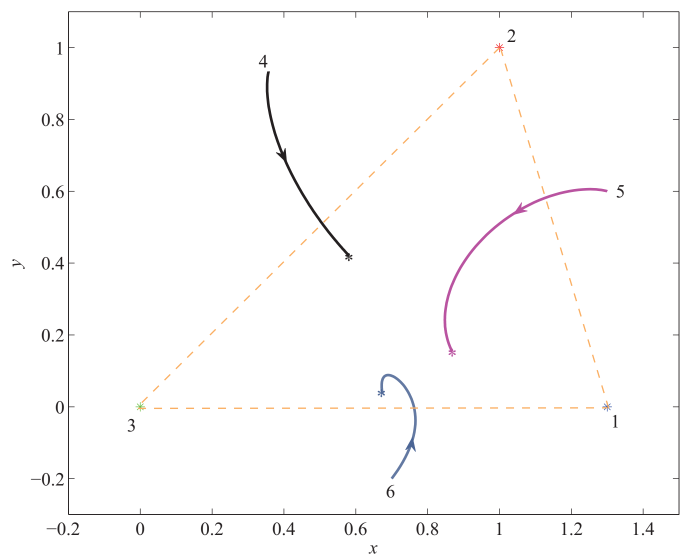

Figure 3.6: The agents' planar motion in the containment example.

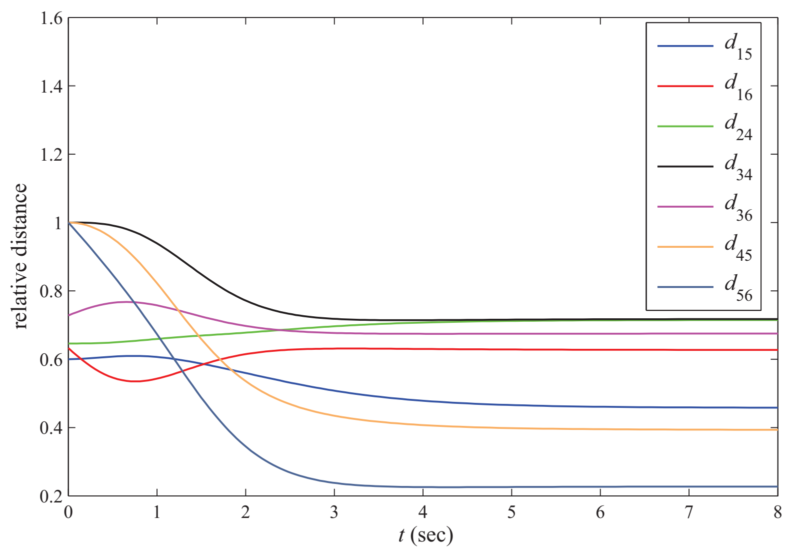

Figure 3.7: The relative distances in the containment example.

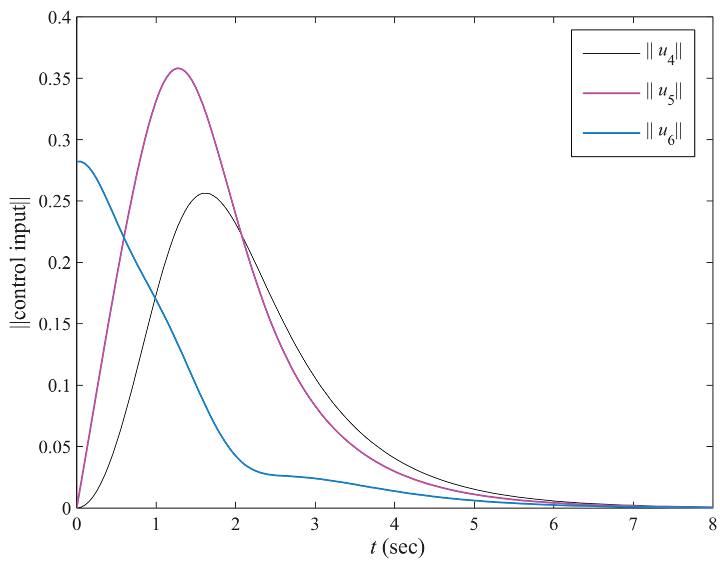

Figure 3.8: The norms of the control inputs in the containment example.

Chapter 4

A Class of Bounded Distributed Connectivity Preserving Control Strategies for Unicycles

This chapter is concerned with the connectivity preservation of a group of unicycles using a novel distributed control scheme. The proposed controllers are bounded, and are capable of maintaining the connectivity of those pairs of agents which are initially within the connectivity range. This means that if the network of agents is initially connected, it will remain connected at all times under this control law. Each local controller is designed in such a way that when an agent is about to lose connectivity with a neighbor, the lowest-order derivative of the agents position that is neither zero nor perpendicular to the edge connecting the agent to the corresponding neighbor makes an acute angle with this edge, which is shown to result in shrinking the edge. The results are first developed for a static information flow graph and are then shown to remain valid for the case of dynamic edge addition. The proposed methodology is then used to develop bounded connectivity preserving control strategies for the consensus and containment control problems as the novel and unprecedented contributions of this work.

The remainder of this chapter is organized as follows. The problem statement is presented in Section 4.1. The connectivity preserving control law is developed in Section 4.2, which is used later in Section 4.3 to derive connectivity preserving controllers for consensus and containment applications. Finally, simulation results are presented in Section 4.4.

4.1 Problem Formulation

Consider a set of n nonholonomic agents in a plane. Let $q_i = [x_i\ y_i]^T$ and θ_i denote the position and heading of agent i, respectively ($i \in \mathbb{N}_n$). The dynamics of each agent is of the form

$$\dot{x}_i = v_i \cos\theta_i \tag{4.1a}$$

$$\dot{y}_i = v_i \sin\theta_i \tag{4.1b}$$

$$\dot{\theta}_i = \omega_i \tag{4.1c}$$

where v_i and ω_i are the translational and angular velocities of agent i, respectively. Each agent is assumed to be capable of measuring the relative positions and relative velocities of its neighbors (as defined later). Denote by $G = (V, E)$ the information flow graph, where $V = \{1, \ldots, n\}$ is the set of vertices, and $E \subset V \times V$ is the set of edges. The information flow graph G is assumed to be connected, undirected, and static (the case of dynamic information flow graph is addressed later in Remark 4.1). Denote the set of neighbors of agent i in G by $N_i(G)$, and the degree of agent i in G with $d_i(G)$. Two agents i and j are said to be in the connectivity range if $\|q_i - q_j\| < d$, for a pre-specified positive real number d, where $\|\cdot\|$ denotes the Euclidean norm. It is assumed that all the agents in $N_i(G)$ are initially located in the connectivity range of agent i, for all $i \in \mathbb{N}_n$. It is also assumed that each agent belongs to either the set of leaders $\mathcal{L}$ or the set of followers $\mathcal{F}$, and that the leaders are static, i.e. $v_i \equiv 0$, $\omega_i \equiv 0$ for all $i \in \mathcal{L}$. The main objective is to design a class of distributed controllers for the followers to preserve connectivity. More precisely, it is desired to find a control scheme for the followers such that if the inequality $\|q_i(t) - q_j(t)\| < d$ holds for all $(i, j) \in E$ at $t = 0$, then it holds at any $t > 0$ as well. The proposed controllers are then used to develop connectivity preserving control strategies for the well-known applications of consensus and containment.

Notation 4.1. *For every agent i, the following functions are introduced, similar to those defined for single-integrator agents in Chapter 3:*

$$\begin{aligned}
\sigma_i(t) &:= \frac{1}{2} \sum_{j \in N_i(G)} \|q_i(t) - q_j(t)\|^2 \\
\pi_i(t) &:= \frac{1}{2} \prod_{j \in N_i(G)} (d^2 - \|q_i(t) - q_j(t)\|^2) \\
\pi_{ij}(t) &:= \prod_{\substack{k \in N_i(G) \\ k \neq j}} (d^2 - \|q_i(t) - q_k(t)\|^2)
\end{aligned}$$

4.2 Connectivity Preserving Controller Design

Analogous to Chapter 3, consider a set of distributed smooth potential functions of the form $h_i(\sigma_i, \pi_i)$, $i \in \mathcal{F}$, with the following properties for all $\sigma_i \in \mathbb{R}^+$:

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, 0) = 0 \tag{4.2a}$$

$$\frac{\partial h_i}{\partial \pi_i}(\sigma_i, 0) < 0 \tag{4.2b}$$

Define $r_i = -\frac{\partial h_i}{\partial q_i}$, and denote by θ_i^* the angle of r_i, i.e. $\theta_i^* = \text{atan2}(r_{iy}, r_{ix})$, where $r_i = [r_{ix}\ r_{iy}]^T$. For every agent $i \in \mathcal{F}$, consider a controller of the form

$$v_i = \|r_i\| \cos(\theta_i - \theta_i^*) \tag{4.3a}$$

$$\omega_i = \dot{\theta}_i^* - (\theta_i - \theta_i^*) \tag{4.3b}$$

Calculating r_i and θ_i^* requires only the relative positions of the neighbors of agent i. It is straightforward to show that calculating $\dot{\theta}_i^*$ also requires the relative velocities of the neighbors of agent i. The aim here is to show that the distributed controller given by (4.3) preserves connectivity (as defined in Section 4.1).

Define $\mathcal{T} = \{t|\ \exists(i,j) \in E : \|q_i(t) - q_j(t)\| \geq d\}$, i.e. the set of those time instants at which the connectivity preservation is violated. In order to prove that the controller given by (4.3) is connectivity preserving, it suffices to show that $\mathcal{T} = \emptyset$. Assume that $\mathcal{T} \neq \emptyset$, and let $t_0 = \inf_{t \in \mathcal{T}} t$. This implies that $\|q_i(t) - q_j(t)\| \leq d$, for all $(i,j) \in E$ and $t \leq t_0$, where the equality holds for at least one edge at $t = t_0$. Construct a graph $G_d = (V_d, E_d)$ as the union of those edges $(i,j) \in E$ for which $\|q_i(t_0) - q_j(t_0)\| = d$, i.e. those edges that are at the critical distance at $t = t_0$. Define

$$s_{ij}(t) = \|q_i(t) - q_j(t)\|^2, \quad \forall (i,j) \in E_d \tag{4.4}$$

Now, assume that s_{ij} is decreasing for some $(i,j) \in E_d$, in an open interval (t_a, t_b), where $t_a < t_0 < t_b$. For such an edge and for every $t_a \leq t < t_0$, the inequality $\|q_i(t) - q_j(t)\| > \|q_i(t_0) - q_j(t_0)\| = d$ holds which is in contradiction with the fact that $\|q_i(t) - q_j(t)\| \leq d$, for all $(i,j) \in E$ and $t \leq t_0$. This rejects the assumption of $\mathcal{T} \neq \emptyset$, and hence the control law given by (4.3) is connectivity preserving. Thus, in order to prove the connectivity preservation for the proposed controller, it suffices to show that the edge described above exists. In the sequel, some important properties of the graph G_d are presented, which will be used later in Theorem 4.1 for finding an edge with this property.

Define the rotation matrix

$$\mathrm{Rot}(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{4.5}$$

where α is the rotation angle in radians. It is straightforward to verify that $\frac{d}{dt}\mathrm{Rot}(\alpha) = \dot{\alpha}\mathrm{Rot}(\alpha + \frac{\pi}{2})$. Consider an agent $i \in \mathcal{F}$; from (4.1a), (4.1b), (4.3a), and on noting that $r_i = \|r_i\|[\cos\theta_i^* \;\; \sin\theta_i^*]^T$, one can obtain

$$\begin{aligned}
\dot{q}_i &= v_i \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \\
&= \mathrm{Rot}(\theta_i - \theta_i^*) r_i \cos(\theta_i - \theta_i^*) \\
&= \frac{1}{2} \begin{bmatrix} 1 + \cos 2(\theta_i - \theta_i^*) & -\sin 2(\theta_i - \theta_i^*) \\ \sin 2(\theta_i - \theta_i^*) & 1 + \cos 2(\theta_i - \theta_i^*) \end{bmatrix} r_i \\
&= \frac{1}{2}(\mathrm{Rot}(2\alpha_i) + I_2) r_i
\end{aligned} \tag{4.6}$$

where $\alpha_i = \theta_i - \theta_i^*$, and I_2 is the 2×2 identity matrix. It results from (4.3b) that $\dot{\alpha}_i = -\alpha_i$. Furthermore, since $\frac{\partial h_i}{\partial q_i} = \frac{\partial h_i}{\partial \sigma_i}\frac{\partial \sigma_i}{\partial q_i} + \frac{\partial h_i}{\partial \pi_i}\frac{\partial \pi_i}{\partial q_i}$, one can rewrite r_i as

$$r_i = -\sum_{j \in N_i(G)} \left(\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij}\right)(q_i - q_j) \tag{4.7}$$

The following lemma shows that G_d is a union of trees, with at least one follower as a leaf. This is used later in this section to prove connectivity preservation by showing that for at least one of these leafs, s_{ij} as defined in (4.4) is decreasing for the edge connected to this leaf in an open interval around t_0.

Lemma 4.1. *The graph G_d is acyclic, and there exists at least one leaf in G_d which is a follower.*

Proof. Suppose that G_d contains a cycle $\mathcal{C}$. Let $y(t)$ represent the positions of the agents belonging to this cycle, and $x(t)$ represent the positions of the rest of the agents along with the headings of all agents. If $y(t) = y(t_0)$ for some $t \geq 0$, one can easily show that $\pi_{ij}(t) = 0$ for any $i \in \mathcal{F}$ on $\mathcal{C}$ and $j \in N_i(G)$. This is due to the fact that every agent on $\mathcal{C}$ is at distance d from its two neighbors on this cycle. As a result, $r_i(t) = 0$ and hence $v_i(t) = 0$ for any $i \in \mathcal{F}$ on $\mathcal{C}$. Using this argument, it is easy to show that x and y satisfy the conditions of Lemma 3.7, and hence $y(t) = y(t_0)$ for all $t \geq 0$. In particular $y(0) = y(t_0)$, implying that some of the agents are initially located at distance d, which contradicts the assumption that $\|q_i(0) - q_j(0)\| < d$, for all $(i, j) \in E$. This proves that G_d is acyclic.

Now, let $\mathcal{P}$ be the longest path in G_d and denote by u and v the vertices at the two ends of this path. Clearly $d_u(G_d) = d_v(G_d) = 1$, i.e. u and v are two leafs of G_d. Assume that both u and v are static leaders. Then, every agent $i \in \mathcal{F}$ on $\mathcal{P}$ has two neighbors on this path located at distance d from it. Therefore, an argument similar to the one given above results that the agents on this path have been fixed from the beginning, which again contradicts the assumption that $\|q_i(0) - q_j(0)\| < d$, for all $(i,j) \in E$. This implies that at least one of the two leafs u and v is a follower, which completes the proof. ∎

The next 3 lemmas concern the follower leafs of G_d. They will be used later in Theorem 4.1 to find the derivative of s_{ij} for an edge connected to a follower leaf.

Lemma 4.2. *Consider an agent $i \in \mathcal{F}$ in G_d with $d_i(G_d) = 1$, and let agent j be the one for which $\|q_i - q_j\| = d$. If $\alpha_i \neq \pm\frac{\pi}{2}$, then $(q_i - q_j)^T \dot{q}_i < 0$.*

Proof. It is straightforward to show that for such an agent $r_i = \frac{\partial h_i}{\partial \pi_i}\pi_{ij}(q_i - q_j)$. Therefore,

$$
\begin{aligned}
(q_i - q_j)^T \dot{q}_i &= \frac{r_i^T}{2\frac{\partial h_i}{\partial \pi_i}\pi_{ij}}(\mathrm{Rot}(2\alpha_i) + I_2) r_i \\
&= \frac{(1+\cos 2\alpha_i)}{2\frac{\partial h_i}{\partial \pi_i}\pi_{ij}}\|r_i\|^2 \qquad (4.8)
\end{aligned}
$$

The proof follows on noting that $1 + \cos 2\alpha_i > 0$ for $\alpha_i \neq \pm\frac{\pi}{2}$, and that $\frac{\partial h_i}{\partial \pi_i} < 0$ (from (4.2b)). ∎

Lemma 4.3. *Consider an agent $i \in \mathcal{F}$ in G_d with $d_i(G_d) = 1$, and let agent j be the one for which $\|q_i - q_j\| = d$. If $\dot{q}_i = \dot{q}_j = 0$, then $\dot{r}_i = \frac{d}{dt}(\frac{\partial h_i}{\partial \pi_i}\pi_{ij})(q_i - q_j)$.*

Proof. Since $d^2 - \|q_i - q_j\|^2 = 0$, for any $l \in N_i - \{j\}$:

$$
\begin{aligned}
\dot{\pi}_{il} &= \frac{d}{dt}(d^2 - \|q_i - q_j\|^2) \prod_{\substack{k \in N_i(G) \\ k \neq l,j}} (d^2 - \|q_i(t) - q_k(t)\|^2) \\
&= -2(q_i - q_j)^T(\dot{q}_i - \dot{q}_j) \prod_{\substack{k \in N_i(G) \\ k \neq l,j}} (d^2 - \|q_i(t) - q_k(t)\|^2) \\
&= 0 \qquad (4.9)
\end{aligned}
$$

This also implies that $\dot{\pi}_i = 0$ since $\pi_i = \frac{1}{2}\pi_{il}(d^2 - \|q_i - q_l\|^2)$. It is now desired to show that $\frac{d}{dt}(\frac{\partial h_i}{\partial \sigma_i}) = 0$. One can write:

$$\frac{d}{dt}(\frac{\partial h_i}{\partial \sigma_i}) = \frac{\partial^2 h_i}{\partial \sigma_i^2}\dot{\sigma}_i + \frac{\partial^2 h_i}{\partial \pi_i \partial \sigma_i}\dot{\pi}_i \tag{4.10}$$

From (4.2a), it is straightforward to show that $\frac{\partial^2 h_i}{\partial \sigma_i^2} = 0$. On the other hand, $\dot{\pi}_i = 0$ as substantiated above. These two results imply that $\frac{d}{dt}(\frac{\partial h_i}{\partial \sigma_i}) = 0$. The proof follows now on noting that

$$\begin{aligned} \dot{r}_i &= -\sum_{l \in N_i(G)} (\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{il})(\dot{q}_i - \dot{q}_l) \\ &\quad - \sum_{l \in N_i(G)} \frac{d}{dt}(\frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{il})(q_i - q_l) \end{aligned} \tag{4.11}$$

■

Lemma 4.4. *Consider an agent $i \in \mathcal{F}$ in G_d with $d_i(G_d) = 1$, and let agent j be the one for which $\|q_i - q_j\| = d$. Also, assume that $\alpha_i = \pm\frac{\pi}{2}$. Then,*

a) $(q_i - q_j)^T \ddot{q}_i = 0$

b) If $\dot{q}_j = 0$, then $(q_i - q_j)^T q_i^{(3)} < 0$

Proof.

Part (a). Note that $\mathrm{Rot}(2\alpha_i) = \mathrm{Rot}(\pm\pi) = -I_2$. Thus,

$$\begin{aligned} \ddot{q}_i &= \frac{1}{2}(\mathrm{Rot}(2\alpha_i) + I_2)\dot{r}_i + \dot{\alpha}_i \mathrm{Rot}(2\alpha_i + \frac{\pi}{2})r_i \\ &= -\alpha_i \mathrm{Rot}(-\frac{\pi}{2})r_i \end{aligned} \tag{4.12}$$

Hence,

$$\begin{aligned} (q_i - q_j)^T \ddot{q}_i &= -\frac{\alpha_i}{\frac{\partial h_i}{\partial \pi_i}\pi_{ij}} r_i^T \mathrm{Rot}(-\frac{\pi}{2})r_i \\ &= 0 \end{aligned} \tag{4.13}$$

Part (b). If $\dot{q}_j = 0$, then Lemma 4.3 yields

$$\begin{aligned}(q_i - q_j)^T \mathrm{Rot}(-\frac{\pi}{2})\dot{r}_i &= \frac{d}{dt}(\frac{\partial h_i}{\partial \pi_i}\pi_{ij})(q_i - q_j)^T \mathrm{Rot}(-\frac{\pi}{2})(q_i - q_j) \\ &= 0 \end{aligned} \tag{4.14}$$

On the other hand, one can easily find the third derivative of q_i as

$$q_i^{(3)} = (\alpha_i \mathrm{Rot}(-\frac{\pi}{2}) + 2\alpha_i^2 I_2)r_i - 2\alpha_i \mathrm{Rot}(-\frac{\pi}{2})\dot{r}_i \tag{4.15}$$

Therefore,

$$\begin{aligned}(q_i - q_j)^T q_i^{(3)} &= \frac{r_i^T}{\frac{\partial h_i}{\partial \pi_i}\pi_{ij}}(\alpha_i \mathrm{Rot}(-\frac{\pi}{2}) + 2\alpha_i^2 I_2)r_i \\ &= \frac{2\alpha_i^2}{\frac{\partial h_i}{\partial \pi_i}\pi_{ij}}\|r_i\|^2 \\ &< 0 \end{aligned} \tag{4.16}$$

■

It follows from the above lemmas that for a follower leaf i in G_d, if the heading of the agent is perpendicular to r_i, then $\rho(q_i) = 2$; otherwise, $\rho(q_i) = 1$ (see Definition 2.2 for the definition of $\rho(\cdot)$). Also, note that since the leaders are assumed to be static, for every $i \in \mathcal{L}$, the index $\rho(q_i)$ is ∞. The following three lemmas provide a lower bound for the index of a non-leaf follower in G_d, which will be used in Theorem 4.1 to find the derivatives of s_{ij} for an edge connected to a leaf whose other end is a non-leaf follower.

Lemma 4.5. *Consider an agent $i \in \mathcal{F}$ in G_d. Assume that $\eta = \min_{j\in N_i(G)}\{\rho(\pi_{ij})\}$, and that $d_i(G_d) \geq 2$. Then, $\rho(q_i) \geq \eta + 1$.*

Proof. The proof is similar to that of Lemma 3.3, using the relation $\rho(q_i) \geq \rho(r_i) + 1$. ■

Lemma 4.6. *Consider an agent $i \in \mathcal{F}$ in G_d, and let ν be one of the (possibly multiple) neighbors of this agent in G_d for which $\rho(q_\nu) = \max_{j\in N_i(G_d)}\{\rho(q_j)\}$. Then*

$$\rho(q_i) \geq 1 + \sum_{\substack{j\in N_i(G_d) \\ j\neq\nu}} \rho(q_j) \tag{4.17}$$

Proof. The proof is similar to that of Lemma 3.4. ■

Lemma 4.7. *Consider an agent $i \in \mathcal{F}$ in G_d with $d_i(G_d) \geq 2$. If $\rho(q_j) = 2$ for any agent $j \in V_d \cap \mathcal{F}$ with $d_j(G_d) = 1$, then $\rho(q_i) \geq 4$.*

Proof. If $d_i(G_d) \geq 3$, then Lemma 4.6 yields $\rho(q_i) \geq 1 + 2 + 2 = 5$, and hence the statement of the present lemma holds in this case. Now, for the case when $d_i(G_d) = 2$, Lemma 4.6 implies that $\rho(q_i) \geq 1 + 2 = 3$. Let $N_i(G_d) = \{j, k\}$. Using the equality $\pi_{ij} = d^2 - \|q_i - q_k\|^2$, it can be easily shown that $\dot{\pi}_{ij} = 0$ and $\ddot{\pi}_{ij} = (q_i - q_k)^T \ddot{q}_k$. It is now aimed to prove that $\ddot{\pi}_{ij} = 0$. If $k \in \mathcal{L}$, then $\rho(q_k) = \infty$. Also, if $k \in \mathcal{F}$ and $d_k(G_d) \geq 2$, then $\rho(q_k) \geq 3$. Hence, in these two cases $\ddot{q}_k = 0$ and subsequently $\ddot{\pi}_{ij} = 0$. On the other hand, if $k \in \mathcal{F}$ and $d_k(G_d) = 1$, then from the assumption of the lemma $\rho(q_k)$ equals 2, implying that $\alpha_k = \pm\frac{\pi}{2}$. Thus, Lemma 4.4 yields $(q_i - q_k)^T \ddot{q}_k = 0$. It follows from this argument that $\dot{\pi}_{ij} = \ddot{\pi}_{ij} = 0$. Similarly, $\dot{\pi}_{ik} = \ddot{\pi}_{ik} = 0$. Therefore, $\eta = \min\{\rho(\pi_{ij}), \rho(\pi_{ik})\} \geq 3$, and hence it is concluded from Lemma 4.5 that $\rho(q_i) \geq 4$. ■

The above lemmas will now be used to prove one of the main results of this chapter, which is given below.

Theorem 4.1. *Consider a set of n nonholonomic agents in a plane with dynamics of the form (4.1), and assume that the leaders are static. Assume also that h_i's satisfy the conditions given by (4.2). Then, the distributed controller (4.3) for the followers is connectivity preserving.*

Proof. As stated earlier, to prove this theorem it suffices to show that for some $(i, j) \in E_d$, the function s_{ij} defined by (4.4) is decreasing in an open interval around t_0. It is shown in the sequel that any edge connected to a follower leaf of index one is an appropriate candidate. In case that all the follower leafs are of index 2, any edge connected to any follower leaf can be selected here. The proof is carried out considering two cases:

i) *G_d has at least one follower leaf of index 1.* Denote one of such vertices by i, and let j be the vertex for which $\|q_i - q_j\| = d$. Then,

$$\dot{s}_{ij} = 2(q_i - q_j)^T \dot{q}_i + 2(q_j - q_i)^T \dot{q}_j \tag{4.18}$$

Since $\rho(q_i) = 1$, thus $\alpha_i \neq \pm\frac{\pi}{2}$, and Lemma 4.2 implies that $(q_i - q_j)^T \dot{q}_i < 0$. If $\rho(q_j) \geq 2$, then $\dot{q}_j = 0$, and hence $\dot{s}_{ij} < 0$. If $\rho(q_j) = 1$, then j is also a follower leaf of G_d, and similarly $(q_j - q_i)^T \dot{q}_j < 0$, which results that $\dot{s}_{ij} < 0$. It follows from this inequality that s_{ij} is decreasing in an open interval around t_0, which completes the proof for this case.

ii) *The index of every follower leaf in G_d is 2.* Consider a leaf $i \in \mathcal{F}$ of G_d, and let j be the vertex for which $\|q_i - q_j\| = d$. Clearly, $\dot{s}_{ij} = 0$. Moreover,

$$\ddot{s}_{ij} = 2(q_i - q_j)^T \ddot{q}_i + 2(q_j - q_i)^T \ddot{q}_j \tag{4.19}$$

Lemma 4.4 implies that $(q_i - q_j)^T \ddot{q}_i = 0$. Similarly, if j belongs to $\mathcal{F}$ and is a leaf, then $(q_j - q_i)^T \ddot{q}_j = 0$. If, however, j is a static leader or is a follower but not a leaf, then $\ddot{q}_j = 0$. Therefore, regardless of j being a leaf or not, the equality $\ddot{s}_{ij} = 0$ holds. To find the third derivative of s_{ij}, note that since the index of every follower in G_d is assumed to be 2,

$$s_{ij}^{(3)} = 2(q_i - q_j)^T q_i^{(3)} + 2(q_j - q_i)^T q_j^{(3)} \tag{4.20}$$

From Lemma 4.4, $(q_i - q_j)^T q_i^{(3)} < 0$. If j belongs to $\mathcal{F}$ and is a leaf, then it can be concluded in a similar way that $(q_j - q_i)^T q_j^{(3)} < 0$, which along with the above inequality yields $s_{ij}^{(3)} < 0$. If $j \in \mathcal{F}$ and $d_j(G_d) \geq 2$, then Lemma 4.7 implies that $\rho(q_j) \geq 4$ and hence $q_j^{(3)} = 0$, resulting in $s_{ij}^{(3)} < 0$. The same result holds also if j is a static leader. Now, it is deduced from $\dot{s}_{ij} = \ddot{s}_{ij} = 0$ and $s_{ij}^{(3)} < 0$ that s_{ij} is decreasing in an open interval around t_0, which completes the proof. ■

Remark 4.1. *The connectivity preservation results presented so far can be easily extended to the case of dynamic edge addition, where new edges may be added to the* *information flow graph once two agents enter the connectivity range. Suppose that new edges are added to the information flow graph at the time instants $t_1, t_2, \ldots$, and denote by $G(t_k)$ the resultant information flow graph at $t = t_k$, $k = 1, 2, \ldots$. For any edge $e \in E(t_k)$, the corresponding agents remain in the connectivity range during the time interval $[t_k, t_{k+1}]$ according to Theorem 4.1. This, along with the fact that $E \subseteq E(t_1) \subseteq E(t_2) \subseteq \ldots$, implies that for any edge of the information flow graph, the corresponding agents remain in the connectivity range at all times after the edge creation.*

4.3 Applications

The controllers proposed in the previous section will now be used to develop connectivity preserving control strategies for the consensus and containment applications.

4.3.1 A Bounded Connectivity Preserving Consensus Algorithm for Unicycles

Consider a team of n unicycles with the dynamics of the form (4.1) in the plane and assume that the information flow graph G is static and is a tree. The objective of this subsection is to use the connectivity preserving controllers developed in the previous section to design a control strategy such that all agents converge to consensus while preserving connectivity.

Assume that h_i's are analytic functions and, in addition to (4.2), they satisfy the following constraints

$$\frac{\partial h_i}{\partial \sigma_i}(\sigma_i, \pi_i) > 0, \ \frac{\partial h_i}{\partial \pi_i}(\sigma_i, \pi_i) \leq 0, \qquad \text{for } \sigma_i \geq 0 \text{ and } \pi_i > 0 \tag{4.21}$$

Then, it is claimed here that using the controller given by (4.3), the agents converge to consensus while preserving connectivity. Connectivity preservation follows directly from condition (4.2) and the results of the previous section. To prove the convergence to consensus, a few lemmas and theorems are presented in the sequel.

It follows from connectivity preservation that σ_i and π_i in (4.7) are bounded, and so are $\frac{\partial h_i}{\partial \sigma_i}$ and $\frac{\partial h_i}{\partial \pi_i}$ as analytic functions of σ_i and π_i. Thus, there exists a positive real number r_M such that $\|r_i(t)\| \leq r_M$, for all $t \geq 0$ and all $i \in \mathbb{N}_n$. For a fixed point $P \in \mathbb{R}^2$, define $R^P(t) = \max_{i \in \mathbb{N}_n} \|P - q_i(t)\|$. Denote by $\frac{d^+}{dt} R^P(t)$ the right derivative of $R^P(t)$ with respect to t. The next lemma will be used to find an upper bound for $\frac{d^+}{dt} R^P(t)$.

Lemma 4.8. *Let i be an agent for which $\|P - q_i(t)\| = R^P(t)$ (i.e., the farthest agent from P at time t). Also, assume that $|\alpha_i(t)| \leq \frac{\pi}{2}$. Then*

$$\frac{d}{dt}\|P - q_i(t)\| \leq r_M |\alpha_i(t)| \tag{4.22}$$

Proof. Denote by γ_i the angle between $P - q_i$ and r_i, i.e. $\gamma_i = \measuredangle(r_i, P - q_i)$. Also, let $\beta_{ij} = \frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij}$. Then, (4.7) can be written as

$$r_i = -\sum_{j \in N_i(G)} \beta_{ij}(q_i - q_j) \tag{4.23}$$

It follows from (4.21) and connectivity preservation that $\beta_{ij} > 0$. Moreover, since q_i is the farthest point from P, the circle centered at P with the radius $\|P - q_i\|$ contains all agents, and hence $(P - q_i)^T (q_j - q_i) \geq 0$, for all $j \in \mathbb{N}_n$. Therefore,

$$\begin{aligned}
(P - q_i)^T r_i &= \sum_{j \in N_i(G)} \beta_{ij}(P - q_i)^T (q_j - q_i) \\
&\geq 0
\end{aligned} \tag{4.24}$$

which in turn implies that $|\gamma_i| \leq \frac{\pi}{2}$. On the other hand,

$$\begin{aligned}
\frac{d}{dt}\|P - q_i(t)\| &= -\frac{(P - q_i)^T}{\|P - q_i\|}\dot{q}_i \\
&= -\frac{(P - q_i)^T}{\|P - q_i\|}\mathrm{Rot}(\alpha_i) r_i \cos\alpha_i \\
&= -\|r_i\| \cos(\alpha_i + \gamma_i) \cos\alpha_i \\
&= \|r_i\| \sin(|\alpha_i + \gamma_i| - \frac{\pi}{2}) \cos\alpha_i
\end{aligned} \tag{4.25}$$

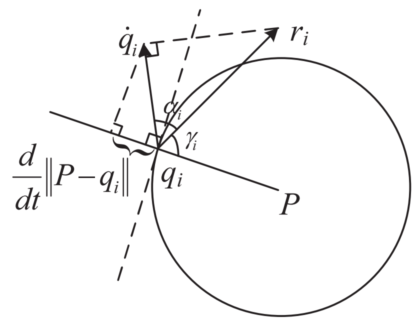

Figure 4.1: Configuration described in Lemma 4.8 for the case when $|\alpha_i + \gamma_i| > \frac{\pi}{2}$.

This is illustrated in Fig. 4.1 for the case where $|\alpha_i + \gamma_i| > \frac{\pi}{2}$. It follows from $|\alpha_i| \le \frac{\pi}{2}$ and $|\gamma_i| \le \frac{\pi}{2}$ that $-\frac{\pi}{2} \le |\alpha_i + \gamma_i| - \frac{\pi}{2} \le \frac{\pi}{2}$. If $-\frac{\pi}{2} \le |\alpha_i + \gamma_i| - \frac{\pi}{2} \le 0$, then it is concluded from (4.25) that $\frac{d}{dt}\|P - q_i(t)\| \le 0$ and the proof is complete. If $0 < |\alpha_i + \gamma_i| - \frac{\pi}{2} \le \frac{\pi}{2}$, then on noting that $\sin x < x$ for any $x \in (0, \frac{\pi}{2}]$, (4.25) yields

$$\begin{aligned}
\frac{d}{dt}\|P - q_i(t)\| &\le \|r_i\|(|\alpha_i + \gamma_i| - \frac{\pi}{2}) \cos \alpha_i \\
&\le r_M(|\alpha_i + \gamma_i| - \frac{\pi}{2}) \\
&\le r_M(|\alpha_i| + |\gamma_i| - \frac{\pi}{2}) \\
&\le r_M|\alpha_i| \qquad (4.26)
\end{aligned}$$

which completes the proof. ■

Lemma 4.9. *Let* $\alpha_M = \max_{i \in \mathbb{N}_n} |\alpha_i(0)|$, *and* $T_M = \max\{\ln \frac{2\alpha_M}{\pi}, 0\}$. *Then, for any* $t_1 \ge T_M$ *and any* $t_2 > t_1$, $R^P(t_2) \le R^P(t_1) + r_M \alpha_M e^{-t_1}$.

Proof. Let I denote the set of all agents at distance $R^P(t)$ from P at time t (i.e., the set of farthest agents from P at time t, which can, in general, have more than one element). Then, it can be easily shown that

$$\frac{d^+}{dt} R^P(t) = \max_{i \in I} \frac{d}{dt}\|P - q_i(t)\| \qquad (4.27)$$

To find an upper bound for $\frac{d^+}{dt} R^P(t)$, first note that $\alpha_i(t) = \alpha_i(0)e^{-t}$ (since $\dot{\alpha}_i = -\alpha_i$), and hence $|\alpha_i(t)| \le \frac{\pi}{2}$ for $t \ge T_M$. Now, using Lemma 4.8 along with (4.27) yields

$$\frac{d^+}{dt} R^P(t) \leq r_M \alpha_M e^{-t} \tag{4.28}$$

for any $t \geq T_M$. By integrating (4.28) from t_1 to t_2, one can obtain

$$\begin{aligned} R^P(t_2) - R^P(t_1) &\leq r_M \alpha_M (e^{-t_1} - e^{-t_2}) \\ &\leq r_M \alpha_M e^{-t_1} \end{aligned} \tag{4.29}$$

which completes the proof. ■

The immediate result of the above lemma is that under the proposed control law the agents evolve in a bounded region of the plane. Note, however, that unlike the case of single-integrator agents (e.g., see Chapter 2), the convex hull of the agents in the case of unicycles is not necessarily contracting. This is clearly due to the fact that when the heading of agent i is not exactly in the same direction as r_i (i.e., the angle α_i is nonzero), then the agent may not move toward the convex hull of its neighbors. Therefore, the methods used in [72, 39, 36], as well as Chapter 2, which are mainly based on the contracting property of the convex hull of the agents, cannot be directly employed here to deduce the convergence of the agents to consensus. However, it is shown in the sequel that by applying these results to the positive limit set of the closed-loop system (see Definition 2.10 for the definition of positive limit point and positive limit set), it is possible to deduce convergence to consensus.

The dynamics of the agents under the proposed control strategy can be written as

$$\begin{aligned} \dot{q}_i &= \frac{1}{2}(\mathrm{Rot}(2\alpha_i) + I_2) r_i \\ \dot{\alpha}_i &= -\alpha_i \end{aligned} \tag{4.30}$$

Denote by L^+ the positive limit set for a solution $[q^T(t)\ \alpha^T(t)]^T$ of (4.30), where $q(t) = [q_1^T(t) \dots q_n^T(t)]^T$ and $\alpha(t) = [\alpha_1(t) \dots \alpha_n(t)]^T$. Note that, according to Lemma 2.5, L^+ is nonempty, compact, and invariant, since the solution of (4.30) evolves in a bounded region of the plane (as shown earlier). Moreover, $[q^T(t)\ \alpha^T(t)]^T$ approaches L^+ as $t \to \infty$. For any $[p^T\ \beta^T]^T \in L^+$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $q(t_n) \to p$ and $\alpha(t_n) \to \beta$. This implies that $\beta = 0$ because $\alpha(t_n) = e^{-t_n}\alpha(0) \to 0$ as $t_n \to \infty$. Therefore, for a solution $p(t) = [p_1^T(t) \dots p_n^T(t)]^T$ starting in L^+ (and hence staying in L^+ as this set is invariant), (4.30) reduces to

$$\dot{p}_i = r_i = -\frac{\partial h_i}{\partial p_i} \tag{4.31}$$

This is the same connectivity preserving control law developed for single-integrator agents in Chapter 3.

Lemma 4.10. *For any $[p^T\ 0^T]^T \in L^+$ and any $(i,j) \in E$, the inequality $\|p_i - p_j\| < d$ holds.*

Proof. By definition, for any $[p^T\ 0^T]^T \in L^+$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $q(t_n) \to p$. Since $\|q_i(t_n) - q_j(t_n)\| < d$ (because of connectivity preservation), hence $\|p_i - p_j\| \leq d$. Now, choose an arbitrary $\tau > 0$ and let $p^\tau(t)$ be a solution of (4.31) which passes through p at time τ, i.e. $p^\tau(\tau) = p$. It follows from the invariance property of L^+ that $[(p^\tau(t))^T\ 0^T]^T \in L^+$ for all $t \geq 0$. In particular, $[(p^\tau(0))^T\ 0^T]^T \in L^+$ implies that $\|p_i^\tau(0) - p_j^\tau(0)\| \leq d$. Let $G_d(0)$ be the union of those edges of G for which $\|p_i^\tau(0) - p_j^\tau(0)\| = d$. Let also $G_M = (V_M, E_M)$ be the maximal induced subgraph of $G_d(0)$ such that $d_i(G_M) \geq 2$ for every $i \in V_M$. However, since G is a tree, $G_d(0)$ is acyclic and thus G_M is empty. Therefore, Theorem 3.1 yields $\|p_i^\tau(t) - p_j^\tau(t)\| < d$ for all $(i,j) \in E$ and $t > 0$. The proof follows now on noting that $\|p_i - p_j\| = \|p_i^\tau(\tau) - p_j^\tau(\tau)\|$. ∎

Theorem 4.2. *Consider a team of n unicycle agents in a plane with the dynamics of the form (4.1), and the control law (4.3). Consider also a set of analytic functions h_i, $i \in \mathbb{N}_n$, satisfying the conditions given by (4.2) and (4.21), which are used to obtain the control parameters in (4.3) as discussed in the previous section. Moreover,* *assume that the information flow graph is a static tree. Then, the agents converge to consensus while preserving connectivity.*

Proof. The first step is to show that there exists a constant vector $\bar{p}$, for which $\bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_n$, and $[\bar{p}^T \ 0^T]^T \in L^+$. To this end, let $p(t)$ be a solution to (4.31) starting from a point $p(0)$, where $[p^T(0) \ 0^T]^T \in L^+$. Since L^+ is invariant, hence $[p^T(t) \ 0^T]^T \in L^+$ for all $t \geq 0$. On the other hand, (4.31) can be written as

$$\dot{p}_i = -\sum_{j \in N_i(G)} \beta_{ij}(p_i - p_j) \tag{4.32}$$

where $\beta_{ij} = \frac{\partial h_i}{\partial \sigma_i} - \frac{\partial h_i}{\partial \pi_i}\pi_{ij}$. Now, note that according to Lemma 4.10, $\|p_i(t) - p_j(t)\| < d$ for all $(i,j) \in E$ and $t \geq 0$, which in turn implies that $\pi_i, \pi_{ij} > 0$. This along with (4.21) yields $\beta_{ij} > 0$ for all $(i,j) \in E$.

The stability of the system governed by (4.32) is extensively studied in the literature (see Chapter 2 and the references therein). Define $\mathcal{P}_i = \{p_j | j \in N_i \cup \{i\}\}$, and assume that p_i is a vertex of $\mathrm{Conv}(\mathcal{P}_i)$. If $\mathcal{P}_i$ is not a singleton, then the above discussion implies that $\dot{p}_i \neq 0$. Using this property and Proposition 2.2, it is straightforward to show the convergence of $p(t)$ to a point $\bar{p}$ for which $\bar{p}_1 = \bar{p}_2 = \ldots = \bar{p}_n := \mathfrak{p}$. Now, one can conclude $[\bar{p}^T \ 0^T]^T \in L^+$ on noting that L^+ is a closed set according to Lemma 2.5.

To complete the proof, it suffices to show that for the solution $[q^T(t) \ \alpha^T(t)]^T$ of (4.30), $q(t)$ converges to $\bar{p}$, or equivalently $R^{\mathfrak{p}}(t) = \max_{i \in \mathbb{N}_n} \|\mathfrak{p} - q_i(t)\| \to 0$ as $t \to \infty$. Since $[\bar{p}^T \ 0^T]^T \in L^+$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $q(t_n) \to \bar{p}$, implying that $R^{\mathfrak{p}}(t_n) \to 0$ as $n \to \infty$. For an arbitrary $\epsilon > 0$, choose a sufficiently large number n so that $t_n > T_M$, $r_M \alpha_M e^{-t_n} < \frac{\epsilon}{2}$, and $R^{\mathfrak{p}}(t_n) < \frac{\epsilon}{2}$. Then, it results from Lemma 4.9 that for every $t > t_n$, $R^{\mathfrak{p}}(t) \leq R^{\mathfrak{p}}(t_n) + r_M \alpha_M e^{-t_n} < \epsilon$, which completes the proof of convergence of q_i's to $\mathfrak{p}$. ■

4.3.2 A Bounded Connectivity Preserving Containment Algorithm for Unicycles with Static Leaders

The objective of this section is to design a control strategy for the team of agents described in Section 4.1, such that while preserving connectivity, the followers converge to the convex hull of the leaders, i.e.

$$\lim_{t\to\infty} q_i(t) \in \mathrm{Conv}(\{q_j | j \in \mathcal{L}\}), \quad \forall i \in \mathcal{F} \tag{4.33}$$

Consider a potential function of the form

$$h_i(\sigma_i, \pi_i) = -\pi_i \tag{4.34}$$

and the controller given by (4.3), where $r_i = -\frac{\partial h_i}{\partial q_i} = \frac{\partial \pi_i}{\partial q_i}$. It is straightforward to verify that the above function satisfies the conditions given by (4.2). Thus, the connectivity preservation for the resultant control strategy is deduced from Theorem 4.1.

Using the connectivity preservation property, it is desired now to show the convergence of the followers to the convex hull of the leaders under the proposed control strategy. The dynamics of the agents with this control law can be written as

$$\dot{q}_i = \begin{cases} \frac{1}{2}(\mathrm{Rot}(2\alpha_i) + I_2) r_i, & i \in \mathcal{F} \\ 0, & i \in \mathcal{L} \end{cases} \tag{4.35a}$$

$$\dot{\alpha}_i = \begin{cases} -\alpha_i, & i \in \mathcal{F} \\ 0, & i \in \mathcal{L} \end{cases} \tag{4.35b}$$

where $r_i = \frac{\partial \pi_i}{\partial q_i}$, and α_i is defined identically zero for all $i \in \mathcal{L}$. Also, define $q(t) = [q_1^T(t) \dots q_n^T(t)]^T$ and $\alpha(t) = [\alpha_1(t) \dots \alpha_n(t)]^T$. It follows from the connectedness of the graph G and the connectivity preservation property of the team that for any static leader $i \in \mathcal{L}$ and any $j \in \mathcal{F} \cup \mathcal{L}$, the inequality $\|q_i - q_j\| < (n-1)d$ holds. This result along with (4.35b) guarantees the boundedness of the solutions of (4.35). Denote by L^+ the positive limit set for a solution $[q^T(t)\ \alpha^T(t)]^T$ of the nonlinear system described by (4.35). For any $[p^T\ \beta^T]^T \in L^+$, it can be easily shown that $\beta = 0$, and $\|p_i - p_j\| \leq d$ for all $(i,j) \in E(G)$ (similar results are proved in Subsection 4.3.1). Next lemma shows that in the above relation equality cannot hold and it is, in fact, strict inequality.

Lemma 4.11. *For any $[p^T\ 0^T]^T \in L^+$ and any $(i,j) \in E$, the inequality $\|p_i - p_j\| < d$ holds.*

Proof. Consider the function $\pi(q(t))$ defined by

$$\pi(q(t)) = \prod_{\substack{(i,j)\in E(G) \\ i<j}} (d^2 - \|q_i(t) - q_j(t)\|^2) \tag{4.36}$$

Note that

$$\begin{aligned}
\dot{\pi} &= \sum_{i\in\mathcal{F}} (\frac{\partial \pi}{\partial q_i})^T \dot{q}_i \\
&= \sum_{i\in\mathcal{F}} \bar{\pi}_i (\frac{\partial \pi_i}{\partial q_i})^T \dot{q}_i \\
&= \frac{1}{2}\sum_{i\in\mathcal{F}} \bar{\pi}_i r_i^T (\text{Rot}(2\alpha_i) + I_2) r_i \\
&= \frac{1}{2}\sum_{i\in\mathcal{F}} \bar{\pi}_i (1 + \cos 2\alpha_i) \|r_i\|^2
\end{aligned} \tag{4.37}$$

where $\bar{\pi}_i = \frac{\pi}{\pi_i}$ (i.e., $\bar{\pi}_i$ contains those product terms in π which do not appear in π_i, and hence is independent from q_i). It results from the connectivity preservation property that $\bar{\pi}_i > 0$ for $t \geq 0$, and hence (4.37) yields $\dot{\pi} \geq 0$, implying that π is a non-decreasing function of time. On the other hand, $0 < \pi < d^{|E(G)|}$. Therefore, $\pi(q(t))$ has a limit, say a, as $t \to \infty$. Note that $a > 0$ because $\pi(q(t)) \geq \pi(q(0)) > 0$. For any $[p^T\ 0^T]^T \in L^+$, there is a sequence $\{t_n\}$ such that as $n \to \infty$, $t_n \to \infty$ and $q(t_n) \to p$. As a result, $\pi(p) = a$ since $\pi(q(t_n)) \to a$ as $t_n \to \infty$. This, along with the relations $a > 0$ and $\|p_i - p_j\| \leq d$ for all $(i,j) \in E(G)$, implies that $\|p_i - p_j\|$ is, in fact, strictly less than d for all $(i,j) \in E(G)$. ∎

Theorem 4.3. *Consider a team of n nonholonomic agents in the plane with the dynamics of the form (4.1). Assume that each agent either belongs to the set of followers $\mathcal{F}$ or the set of static leaders $\mathcal{L}$ as described in Section 4.1, and that the information flow graph G is static and connected. Then, the controller given by (4.3), with h_i's of the form (4.34), results in convergence of the followers to the convex hull of the leaders while preserving connectivity.*

Proof. Consider the solution $[p^T(t)\ 0^T]^T$ of (4.35) starting from a point $[p_0^T\ 0^T]^T \in L^+$. The invariance property of L^+ (see Lemma 2.5) yields $[p^T(t)\ 0^T]^T \in L^+$ for all $t \geq 0$. This yields $\pi(p(t)) \equiv a$ and hence $\dot{\pi}(p(t)) \equiv 0$. Using (4.37), one arrives at

$$\dot{\pi}(p(t)) = \sum_{i\in\mathcal{F}} \bar{\pi}_i \|r_i\|^2 \tag{4.38}$$

Note that $\bar{\pi}_i > 0$ because according to Lemma 4.11, $\|p_i(t) - p_j(t)\| < d$ for all $(i, j) \in E(G)$. Therefore, the relation $\dot{\pi}(p(t)) \equiv 0$ results that starting from $[p_0^T\ 0^T]^T \in L^+$, $r_i \equiv 0$ for all $i \in \mathcal{F}$. On the other hand, r_i can be written as

$$\begin{aligned} r_i &= \frac{\partial \pi_i}{\partial p_i} \\ &= -\sum_{j\in N_i(G)} \pi_{ij}(p_i - p_j) \end{aligned} \tag{4.39}$$

Setting r_i to zero in the above equation and solving for p_i yields

$$p_i = \sum_{j\in N_i(G)} \alpha_{ij} p_j, \quad i \in \mathcal{F} \tag{4.40}$$

where $\alpha_{ij} := \frac{\pi_{ij}}{\sum_{j\in N_i(G)} \pi_{ij}}$. Clearly, $0 < \alpha_{ij} < 1$ and $\sum_{j\in N_i(G)} \alpha_{ij} = 1$. This means that for any $[p^T\ 0^T]^T \in L^+$, every follower is in the convex hull of its neighbors. It is claimed now that for any $[p^T\ 0^T]^T \in L^+$, no follower can be at a vertex of the convex hull of the team unless all agents coincide. Assume that one of the followers, say follower $i \in \mathcal{F}$, is at a vertex of the convex hull. Then, it results from (4.40) that p_i should coincide with all of its neighbors in $N_i(G)$. Repeating the same argument, one concludes that p_i should coincide with all the agents reachable from vertex i in G, which in turn means that all the agents should coincide since G is connected. This proves the above claim, and implies that for any $[p^T\ 0^T]^T \in L^+$ and any $i \in \mathcal{F}$, p_i belongs to $C_{\mathcal{L}}$, where $C_{\mathcal{L}}$ denotes the convex hull of the static leaders. The convergence of the q_i's to $C_{\mathcal{L}}$ for any $i \in \mathcal{F}$ is implied from the above result because $q(t)$ approaches L^+ as $t \to \infty$, according to Lemma 2.5. ■

Remark 4.2. *Note that due to the connectivity preservation property, one can write $E(t_1) \subseteq E(t_2)$ for any $t_2 > t_1 \geq 0$, where $E(t)$ denotes the set of edges of the information flow graph at time t. This, together with the fact that the number of the edges that can be added to the information flow graph is finite, implies that there exists a time T after which no more edge is added to the information flow graph. Hence, the stability analysis in this case becomes equivalent to that in the case of static information flow graph.*

4.4 Simulation Results

4.4.1 A Connectivity Preserving Consensus Example

To verify the controller proposed in Subsection 4.3.1, consider 6 unicycle agents with dynamics of the form (4.1) moving in a 2D plane, with the information flow graph G_1 depicted in Fig. 4.2, and assume $d = 1$. Let also the initial position and heading of each agent be as shown in Fig. 4.2. Suppose that agent i is using a controller of the form (4.3), and that

$$h_i(\sigma_i, \pi_i) = -\frac{\pi_i}{1+\sigma_i}, \quad i \in \mathbb{N}_6 \tag{4.41}$$

It can be easily shown that this function satisfies the conditions given in (4.2) and (4.21), and hence the resultant controller is connectivity preserving and leads to consensus, according to Theorem 4.2.

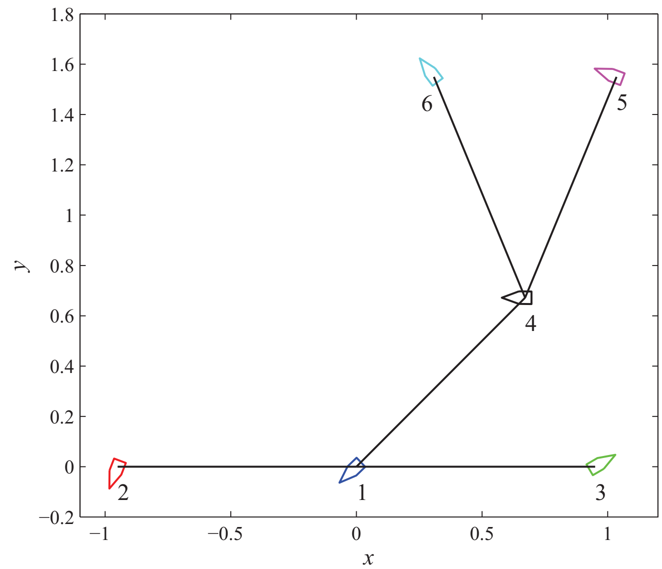

Figure 4.2: The information flow graph G_1 along with the initial positions and headings of the agents for the consensus example.

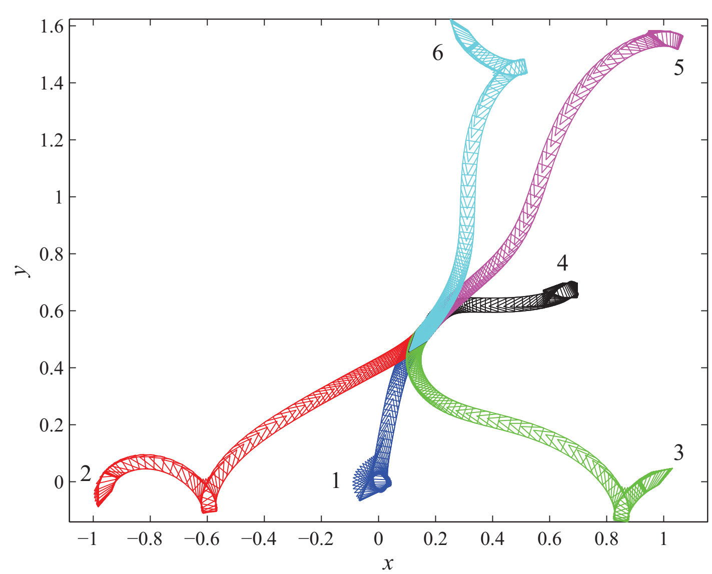

Figure 4.3: The agents' planar motion in the consensus example.

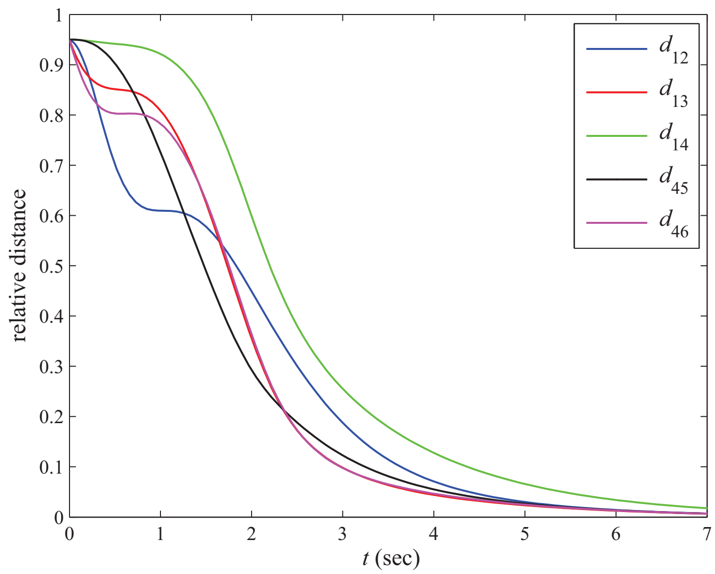

Figure 4.4: The distances between the neighboring agents in the consensus example (d_{ij} represents the distance between neighboring agents i and j).

The planar motion of the agents under the proposed controller is shown in Fig. 4.3. Denote the distance between agent i and its neighbor j by d_{ij} (i.e., $d_{ij} := \|q_i - q_j\|$). This distance is depicted in Fig. 4.4 for different agents as a function of time. While all initial distances are relatively close to d, the proposed controller keeps them less than d for every $(i, j) \in E(G_1)$ at all times, as the agents converge to consensus. The translational and angular velocities of the agents are depicted in Figs. 4.5 and 4.6, respectively.

4.4.2 A Connectivity Preserving Containment Example

It is desired now to validate the controller designed in Subsection 4.3.2 by simulation. Consider a team of 3 static leaders and 3 followers with unicycle dynamics given by (4.1), and let the connectivity range be $d = 1$. The information flow graph G_2 and initial positions and headings of agents are depicted in Fig. 4.7, where the static leaders are marked by an asterisk.

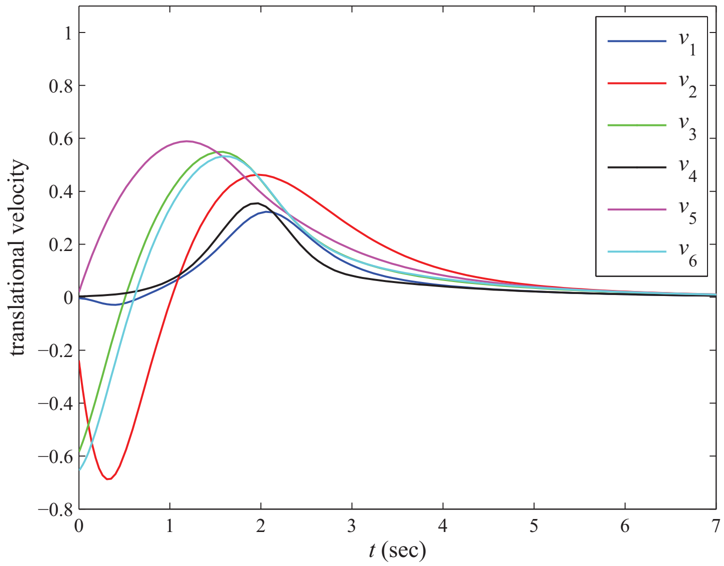

Figure 4.5: The translational velocities of the agents in the consensus example.

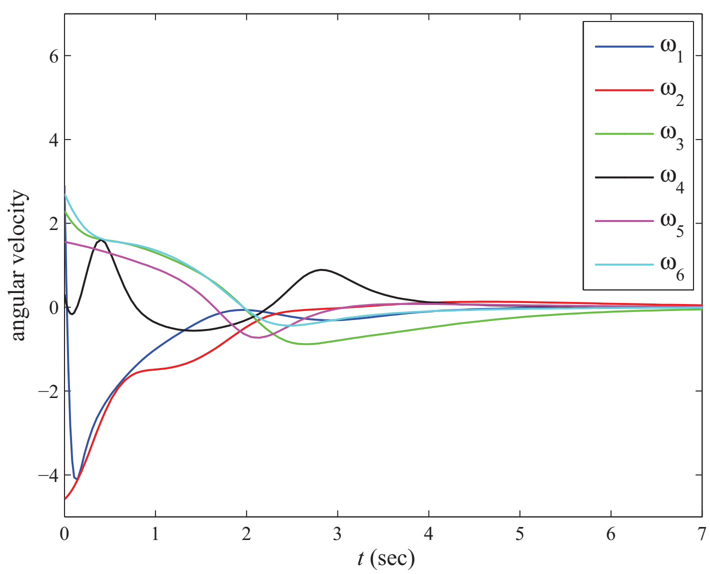

Figure 4.6: The angular velocities of the agents in the consensus example.

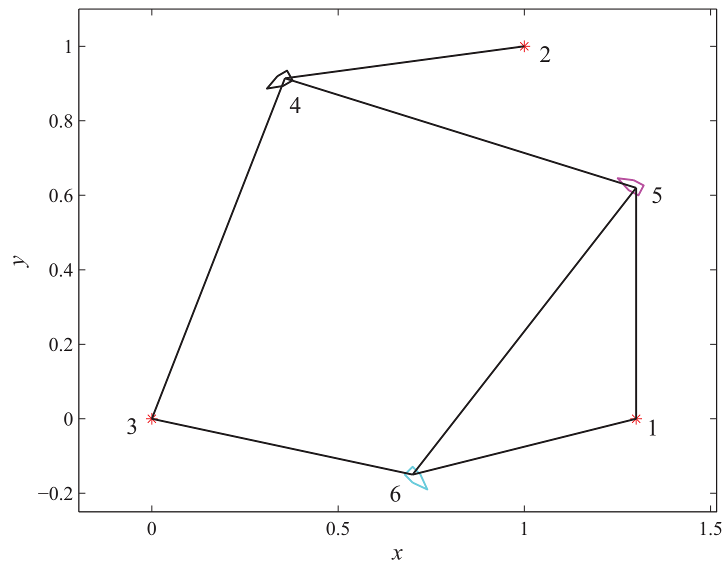

Figure 4.7: The information flow graph G_2 along with the initial positions and headings of the agents for the containment example.

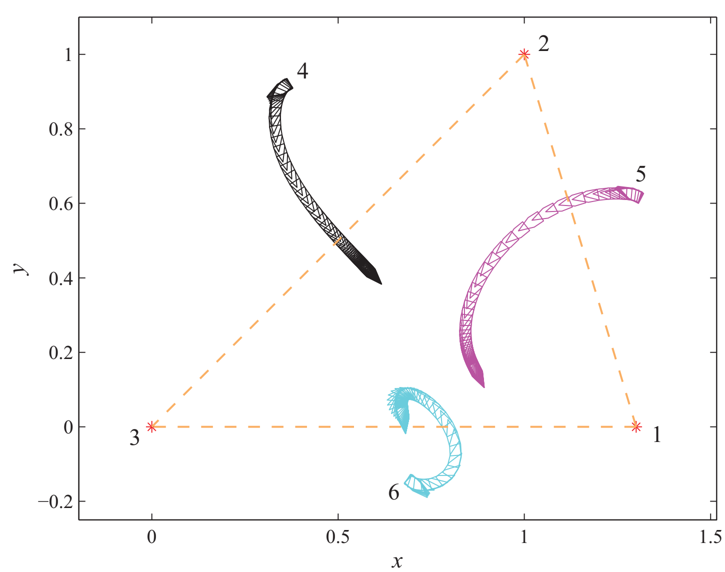

Figure 4.8: The followers' planar motion in the containment example.

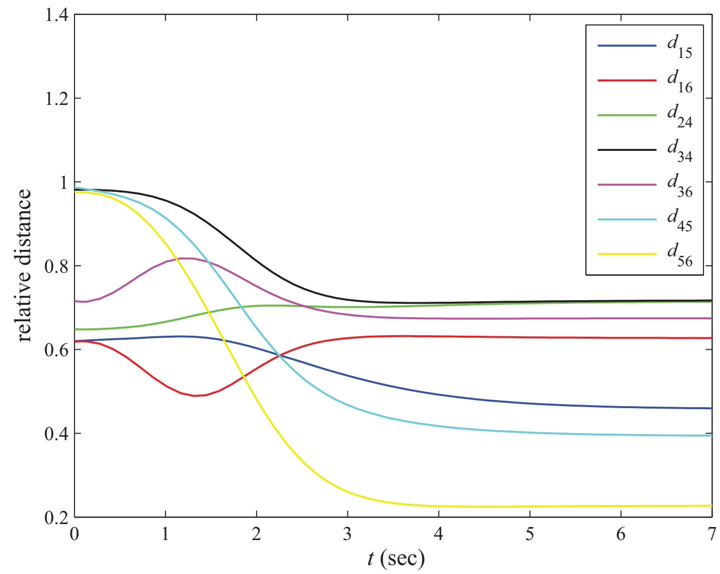

Figure 4.9: The distances between the neighboring agents in the containment example.

The planar motion of the agents under the controller proposed in Subsection 4.3.2 is shown in Fig. 4.8. Similar to the previous example, let the distance between agent i and its neighbor j be denoted by d_{ij}. The distances for different agents are depicted in Fig. 4.9. Although some of the distances are initially close to d, under the proposed controller d_{ij} remains less than d for every $(i, j) \in E(G_2)$ at all times, while the agents are converging to the convex hull of the static leaders. Figs. 4.10 and 4.11, respectively, demonstrate the translational and angular velocities of the followers.

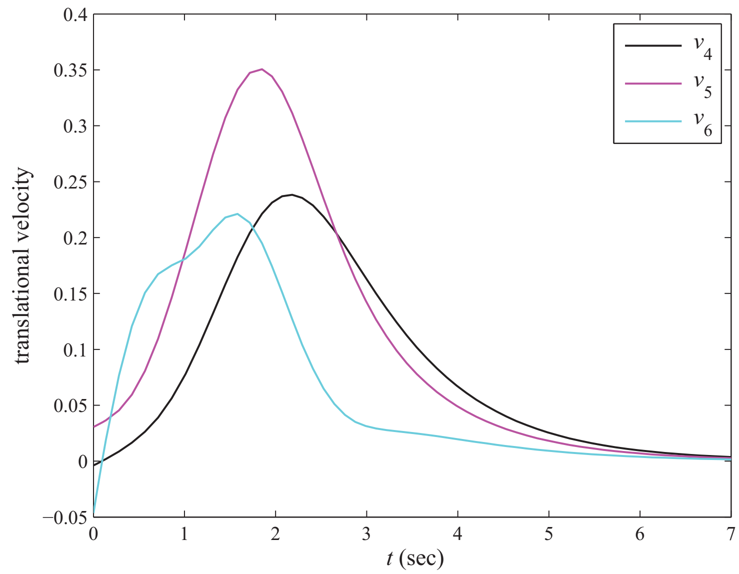

Figure 4.10: The translational velocities of the followers in the containment example.

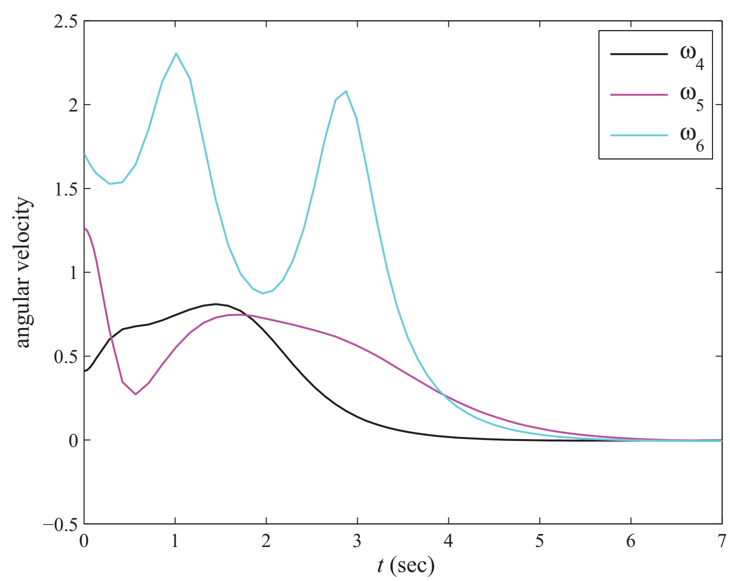

Figure 4.11: The angular velocities of the followers in the containment example.

Chapter 5

A Bounded Distributed Connectivity Preserving Aggregation Strategy with Collision Avoidance Property

This chapter presents a potential-based bounded distributed connectivity preserving control strategy for the aggregation of multi-agent systems. The problem is investigated for two cases of single-integrator agents and unicycles. Under the proposed control strategy, if two agents are in the connectivity range at some point in time, they will stay connected thereafter. The agents finally aggregate while avoiding collision in such a way that the average of the distances between every pair of neighboring agents is bounded by a pre-specified positive real number, which can be chosen arbitrarily small. The results are developed based on some important characteristics of the positive limit set of the closed-loop system under the proposed control strategy and a fundamental property of convex real functions.

The remainder of this chapter is organized as follows. The problem statement is presented in Section 5.1. Section 5.2 includes the details of the controller design and the proofs of the connectivity preservation and collision avoidance properties and the aggregation of the agents for the case of single-integrator agents. The case of unicycle agents is studied in Section 5.3. Finally, simulation results are presented in Section 5.4.

5.1 Problem Statement

Consider a set of n single-integrator agents in a 2D plane, and let the dynamics of each agent be described by

$$\dot{q}_i(t) = u_i(t) \tag{5.1}$$

where $q_i(t) \in \mathbb{R}^2$ and $u_i(t)$ represent the position and control input of agent i at time t. Each agent is assumed to be capable of measuring the relative positions of a subset of agents which are in its connectivity range, i.e. any agent within a pre-described distance d from it. More precisely, agent i is capable of measuring the relative position of agent j at time t if and only if $\|q_i(t) - q_j(t)\| < d$, where $\|\cdot\|$ denotes the Euclidean norm. This information flow structure is represented by an information flow graph $G(t) = (V, E(t))$, where $V = \{1, \ldots, n\}$ is the set of vertices, and $E(t) = \{(i, j) | i, j \in V, i \neq j, \|q_i(t) - q_j(t)\| < d\}$ is the set of edges. Denote the set of neighbors of agent i in $G(t)$ by $N_i(G(t))$, and the degree of agent i in $G(t)$ with $d_i(G(t))$. It is assumed that the initial positions of the agents are such that the initial information flow graph $G(0)$ is connected. The main objective of this chapter is to design a bounded distributed controller is such a way that

1. connectivity is preserved; in other words, if $(i, j) \in E(t_0)$ for some $t_0 \geq 0$, then under the proposed control strategy $(i, j) \in E(t)$ for all $t \geq t_0$.

2. collision among the agents is avoided in the sense that $q_i(t) \neq q_j(t)$ for every $t > 0$ and all $i \neq j$, assuming $q_i(0) \neq q_j(0)$.

3. the agents finally aggregate so that the average of the distances among neighboring agents is bounded by a pre-specified positive real number r. More precisely, there exists $T > 0$ such that for every $t \geq T$,

$$\frac{1}{|E(t)|} \sum_{(i,j) \in E(t)} \|q_i(t) - q_j(t)\| \leq r \tag{5.2}$$

5.2 Control Design for Single-Integrator Agents

For every agent i, define

$$\pi_i(t) := \frac{1}{2} \prod_{j \in N_i(G(t))} (d^2 - \|q_i(t) - q_j(t)\|^2)^m \|q_i(t) - q_j(t)\|^2 \tag{5.3}$$

where m is a natural number which satisfies $m \geq \frac{d^2 - r^2}{r^2}$, and consider a control law of the form

$$u_i = \frac{\partial \pi_i}{\partial q_i} \tag{5.4}$$

The aim of this section is to show that this control law satisfies the design specifications as described in Section 5.1. In order to express (5.4) in a more explicit form, define:

$$\pi_{ij}(t) := \prod_{\substack{k \in N_i(G(t)) \\ k \neq j}} (d^2 - \|q_i(t) - q_k(t)\|^2)^m \|q_i(t) - q_k(t)\|^2 \tag{5.5}$$

Then,

$$u_i = \sum_{j \in N_i(G(t))} \pi_{ij}(q_i(t) - q_j(t))(d^2 - \|q_i(t) - q_j(t)\|^2)^{m-1}(d^2 - (m+1)\|q_i(t) - q_j(t)\|^2) \tag{5.6}$$

The next lemma proves the collision avoidance property of the proposed controller.

Lemma 5.1. *Under any controller of the form (5.4) the agents will not collide. More precisely, if $q_i(0) \neq q_j(0)$ for all $i, j \in V$ $(i \neq j)$, then $q_i(t) \neq q_j(t)$ for every $t > 0$.*

Proof. Consider the function $\pi(t)$ defined by

$$\pi(t) = \prod_{\substack{(i,j)\in E(t)\\ i<j}} (d^2 - \|q_i(t) - q_j(t)\|^2)^m \|q_i(t) - q_j(t)\|^2 \tag{5.7}$$

Note that

$$\begin{aligned}
\dot{\pi} &= \sum_{i\in V} (\frac{\partial \pi}{\partial q_i})^T \dot{q}_i \\
&= \sum_{i\in V} \bar{\pi}_i (\frac{\partial \pi_i}{\partial q_i})^T \dot{q}_i \\
&= \sum_{i\in V} \bar{\pi}_i \|\dot{q}_i\|^2 \\
&\geq 0
\end{aligned} \tag{5.8}$$

where $\bar{\pi}_i = \frac{\pi}{\pi_i}$ (i.e., $\bar{\pi}_i$ contains those product terms in π which do not appear in π_i, and hence is independent from q_i). Note that $\pi, \pi_i, \bar{\pi}_i \geq 0$, and if no collision has happened at time t, then these three functions will all be strictly positive.

Now, assume that a collision can happen under a controller of the form (5.4). Let t_0 be the first time instant at which two agents, say i and j, collide. It follows from the assumption of the lemma that $t_0 > 0$. Since $G(t)$ is piece-wise constant, one can choose $t_0^- < t_0$ in such a way that the topology of the network of the agents represented by $G(t)$ stays fixed for any $t \in [t_0^-, t_0)$. Also, since $q_i(t_0) = q_j(t_0)$, t_0^- can be chosen sufficiently close to t_0 such that $\|q_i(t) - q_j(t)\| < d$ for every $t \in [t_0^-, t_0)$, and hence one may assume $(i,j) \in E(t)$ for every $t \in [t_0^-, t_0)$. The equality $q_i(t_0) = q_j(t_0)$ implies that $\lim_{t\nearrow t_0} \|q_i(t) - q_j(t)\|^2 = 0$. This along with the fact that all product terms in (5.7) are bounded by either d^{2m} or d^2 yields

$$\lim_{t\nearrow t_0} \pi(t) = 0 \tag{5.9}$$

On the other hand, since $G(t)$ is fixed for $t \in [t_0^-, t_0)$, one can conclude from (5.8) that $\pi(t) \geq \pi(t_0^-)$ for all $t \in [t_0^-, t_0)$. This is clearly a contradiction with (5.9) on noting that $\pi(t_0^-) > 0$ (since no collision has happened before t_0), which completes the proof. ∎

The connectivity preservation property of the controller (5.4) is justified in the next lemma.

Lemma 5.2. *The controller given by (5.4) is connectivity preserving. In other words, if* $(i,j) \in E(t_0)$ *for some* $t_0 \geq 0$*, then* $(i,j) \in E(t)$ *for all* $t \geq t_0$

Proof. The proof is similar to that of Lemma 5.1. First, note that according to collision avoidance property, for every $i \in V$, all the three functions π, π_i, and $\bar{\pi}_i$ are positive. Assume that connectivity is not preserved for an edge $(i,j) \in E(t_0)$ for some $t_0 \geq 0$, and let $t_1 > t_0$ be the time instant at which the corresponding agents i and j lose connectivity, i.e. $\|q_i(t_1) - q_j(t_1)\| = d$. Since $G(t)$ is piece-wise constant, there exists $t_1^- < t_1$ such that $G(t)$ stays fixed for any $t \in [t_1^-, t_1)$. The equality $\|q_i(t_1) - q_j(t_1)\| = d$ implies that $\lim_{t \nearrow t_1} d^2 - \|q_i(t) - q_j(t)\|^2 = 0$, which in turn yields

$$\lim_{t \nearrow t_1} \pi(t) = 0 \tag{5.10}$$

On the other hand, since $G(t)$ is fixed for $t \in [t_1^-, t_1)$, one can conclude from (5.8) that $\pi(t) \geq \pi(t_1^-)$ for all $t \in [t_1^-, t_1)$ which clearly contradicts (5.10) on noting that $\pi(t_1^-) > 0$. This completes the proof. ∎

It follows from the connectivity preservation property that for every $t_1 < t_2$, $E(t_1) \subseteq E(t_2)$. This along with the fact that the number of the edges that can be added to a graph with n vertices is finite, implies that there exists $T_f > 0$ such that $G(t)$ is fixed for $t \geq T_f$. Therefore, to prove the third property for the proposed controller, it is assumed in the reminder of the chapter that the agents have reached their fixed topology $G_f = (V, E_f)$.

Denote by L^+ the positive limit set for a solution $q(t) = [q_1^T(t) \dots q_n^T(t)]^T$ of the closed-loop system under the controller given by (5.4) (see Definition 2.10 for the definition of positive limit point and positive limit set). It is important to note that every positive limit set is invariant. Note also that $q(t)$ approaches L^+ as $t \to \infty$

(see [103]). An important property of L^+ for the proposed controller is stated in the next lemma.

Lemma 5.3. *For any $p = [p_1^T(t) \dots p_n^T(t)]^T \in L^+$ the following relation holds:*

$$\frac{1}{|E_f|} \sum_{(i,j)\in E_f} \frac{\|p_i - p_j\|^2}{d^2 - \|p_i - p_j\|^2} = \frac{1}{m} \tag{5.11}$$

Proof. Since the graph $G(t)$ is fixed for all $t \geq T_f$, hence there is no discontinuity in the function $\pi(t)$ for any $t \geq T_f$. Thus, it follows from (5.8) that $\pi(t)$ is non-decreasing over the time interval $t \geq T_f$, resulting in $\pi(T_f) \leq \pi(t)$. On the other hand, $\pi(t) < d^{(m+1)|E_f|}$. This means that $\pi(t)$ is a non-decreasing bounded function of time, and hence it has a limit, say a, as $t \to \infty$. One can conclude from the relation $a \geq \pi(T_f)$ that $a > 0$ because $\pi(T_f) > 0$. Now, since for any $p \in L^+$ there is a sequence $\{t_n\}$ such that as $n \to \infty$, $t_n \to \infty$ and $q(t_n) \to p$, it results that $\pi \equiv a$ for any solution belonging to L^+. Therefore, for any solution $p(t)$ starting in L^+ (and hence staying in L^+) the relation $\dot{\pi} \equiv 0$ holds, which implies $\dot{p}_i \equiv 0$ for every $i \in V$ because analogous to (5.8), $\dot{\pi} = \sum_{i \in V} \bar{\pi}_i \|\dot{p}_i\|^2$. Now, it results from (5.6) that

$$\sum_{j \in N_i(G_f)} \pi_{ij}(p_i - p_j)(d^2 - \|p_i - p_j\|^2)^{m-1}(d^2 - (m+1)\|p_i - p_j\|^2) = 0 \tag{5.12}$$

for every $i \in V$. Dividing both sides by $2\pi_i$ leads to

$$\sum_{j \in N_i(G_f)} (p_i - p_j) \frac{d^2 - (m+1)\|p_i - p_j\|^2}{\|p_i - p_j\|^2(d^2 - \|p_i - p_j\|^2)} = 0 \tag{5.13}$$

Multiplying (5.13) by p_i^T from the left and taking the summation over all $i \in V$, one arrives at

$$\sum_{(i,j)\in E_f} \frac{d^2 - (m+1)\|p_i - p_j\|^2}{d^2 - \|p_i - p_j\|^2} = 0 \tag{5.14}$$

from which (5.11) can be easily obtained. ■

Now, it is desired to show that a controller of the form (5.4) results in aggregation of the agents as defined in Section 5.1.

Lemma 5.4. *Consider a controller of the form (5.4), and assume that $m \geq \frac{d^2-r^2}{r^2}$. Then the agents aggregate as t increases, such that the average of the distances between the neighboring agents is bounded by r. In other words, there exists $T > 0$ such that for every $t \geq T$,*

$$\frac{1}{|E(t)|} \sum_{(i,j)\in E(t)} \|q_i(t) - q_j(t)\| \leq r \tag{5.15}$$

Proof. Define $f(x) = \frac{x^2}{d^2-x^2}$. It is straightforward to show that f is convex over the interval $x \in (0, d)$. This along with Lemma 5.3 implies that for every $p \in L^+$

$$\begin{aligned} \frac{1}{m} &= \frac{1}{|E_f|} \sum_{(i,j)\in E_f} f(\|p_i - p_j\|) \\ &\geq f(\frac{1}{|E_f|} \sum_{(i,j)\in E_f} \|p_i - p_j\|) \end{aligned} \tag{5.16}$$

It can be easily verified that if $f(x) \leq \frac{1}{m}$, then $x \leq \frac{d}{\sqrt{m+1}}$. Therefore, it follows from (5.16) that

$$\begin{aligned} \frac{1}{|E_f|} \sum_{(i,j)\in E_f} \|p_i - p_j\| &\leq \frac{d}{\sqrt{m+1}} \\ &\leq \frac{d}{\sqrt{\frac{d^2-r^2}{r^2}+1}} \\ &\leq r \end{aligned} \tag{5.17}$$

on noting that $m \geq \frac{d^2-r^2}{r^2}$. The proof follows now from the fact that $G(t) = G_f$ for $t \geq T_f$, and that $q(t)$ approaches L^+ as $t \to \infty$. ■

The main results of this section are summarized in the next Theorem.

Theorem 5.1. *Consider a team of agents in a 2D plane with the dynamics of the form (5.1). Assume that the control input of each agent is given by (5.4), where π_i is defined in (5.3) and m is a natural number satisfying $m \geq \frac{d^2-r^2}{r^2}$. Then,* *under the proposed control strategy, the agents will finally aggregate while preserving connectivity and avoiding collision such that the average of the distances among the neighboring agents is bounded by r.*

Proof. The proof follows directly from Lemmas 5.1-5.4. ■

Note that in the above theorem the average is taken over the edges of the information flow graph and not necessarily all pairs of agents. However, by choosing a sufficiently small value for r, the distances among the neighboring agents can be made arbitrarily small. The next proposition presents a sufficient condition on m which guarantees the convergence of all pairs of agents to the connectivity range. Therefore, for any m satisfying that condition, the above-mentioned average is taken over all pairs of agents.

Proposition 5.1. *If $m \geq \frac{n^2(n-1)^2}{4} - 1$, then G_f is a complete graph, i.e., $(i,j) \in E(t)$ for all $i, j \in V$ and $t \geq T_f$.*

Proof. It follows from (5.17) that for every $p \in L^+$,

$$\sum_{(i,j)\in E_f} \|p_i - p_j\| \leq \frac{d|E_f|}{\sqrt{m+1}} \tag{5.18}$$

From the connectivity preservation property and the assumption that $G(0)$ is connected, it results that G_f is also connected. Assume that G_f is not complete. Then there exist $u, v \in V$, for which $(u,v) \notin E_f$. Since G_f is connected, there is a path in G_f from u to v. Denote this path by $\mathcal{P}$ and the set of its edges with $E_{\mathcal{P}}$. Then, it is straightforward to show that

$$\|p_u - p_v\| \leq \sum_{(i,j)\in E_{\mathcal{P}}} \|p_i - p_j\| \tag{5.19}$$

Clearly, $E_{\mathcal{P}} \subseteq E_f$, and hence (5.18) and (5.19) yield $\|p_u - p_v\| \leq \frac{d|E_f|}{\sqrt{m+1}}$. Now, since G_f is assumed not to be complete, $|E_f| < \frac{n(n-1)}{2} \leq \sqrt{m+1}$. This implies that $\|p_u - p_v\| < d$ which is in contradiction with the initial assumption of $(u,v) \notin E_f$, and this completes the proof. ■

It is to be noted that the above proposition provides a sufficient condition in terms of m (i.e., it grows with the quadruple of the swarm size) to ensure that all pairs of agents will eventually enter the connectivity range. The above result is very conservative in practice, and usually a much smaller m can also fulfill this.

5.3 Control Design for Unicycle Agents

This section uses an approach analogous to the one presented in the previous section to design a controller for the case of unicycles. The dynamics of each unicycle agent is given by

$$\dot{x}_i = v_i \cos\theta_i \tag{5.20a}$$

$$\dot{y}_i = v_i \sin\theta_i \tag{5.20b}$$

$$\dot{\theta}_i = \omega_i \tag{5.20c}$$

where $q_i = [x_i \ y_i]^T$ and θ_i denote the position and heading of agent i, and v_i and ω_i are its translational and angular velocities, respectively. For every agent $i \in V$, consider a controller of the form

$$v_i = \|u_i\| \cos(\theta_i - \theta_i^*) \tag{5.21a}$$

$$\omega_i = \dot{\theta}_i^* - \kappa_i(\theta_i - \theta_i^*) \tag{5.21b}$$

where $u_i = [u_{ix} \ u_{iy}]^T$ is the same control input designed for single-integrator agents, θ_i^* denotes the angle of u_i (i.e. $\theta_i^* = \text{atan2}(u_{iy}, u_{ix})$), and $\kappa_i > 0$ is a constant gain. The objective of this section is to show that this controller satisfies the design specifications described in Section 5.1.

From (5.20) and (5.21), one can obtain

$$
\begin{aligned}
\dot{q}_i &= \|u_i\| \cos(\theta_i - \theta_i^*) \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} \\
&= \frac{1}{2}\|u_i\| \begin{bmatrix} \cos(2\theta_i - \theta_i^*) + \cos\theta_i^* \\ \sin(2\theta_i - \theta_i^*) + \sin\theta_i^* \end{bmatrix} \\
&= \frac{1}{2}(\mathrm{Rot}(2(\theta_i - \theta_i^*)) + I_2)u_i
\end{aligned}
\tag{5.22}
$$

where I_2 is the 2×2 identity matrix, and $\mathrm{Rot}(\cdot)$ is the rotation matrix defined as

$$
\mathrm{Rot}(\varphi) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \tag{5.23}
$$

Let α_i denote the deviation of the heading of agent i from u_i, i.e. $\alpha_i = \theta_i - \theta_i^*$. It follows from (5.20) and (5.21) that $\dot{\alpha}_i = -\kappa_i\alpha_i$. Therefore, the dynamics of q_i and α_i under the controller given by (5.21) can be written as

$$
\begin{aligned}
\dot{q}_i &= \frac{1}{2}(\mathrm{Rot}(2\alpha_i) + I_2)u_i \\
\dot{\alpha}_i &= -\kappa_i\alpha_i
\end{aligned}
\tag{5.24}
$$

The main result of this section is presented in the next theorem.

Theorem 5.2. *Consider a team of unicycles with the dynamics of the form (5.20) moving in a 2D plane. Assume that the translational and angular velocities of each agent are as in (5.21), where u_i is given in (5.4) and m is a natural number satisfying $m \geq \frac{d^2-r^2}{r^2}$. Then, under the proposed control strategy, the agents will finally aggregate while preserving connectivity and avoiding collision such that the average of the distances among the neighboring agents is bounded by r.*

Proof. For the function $\pi(t)$ defined by (5.7), it is straightforward to show that

$$
\begin{aligned}
\dot{\pi} &= \sum_{i\in V}(\frac{\partial \pi}{\partial q_i})^T \dot{q}_i \\
&= \sum_{i\in V}\bar{\pi}_i(\frac{\partial \pi_i}{\partial q_i})^T \dot{q}_i \\
&= \frac{1}{2}\sum_{i\in V}\bar{\pi}_i u_i^T(\mathrm{Rot}(2\alpha_i)+I_2)u_i \\
&= \frac{1}{2}\sum_{i\in V}\bar{\pi}_i(1+\cos 2\alpha_i)\|u_i\|^2 \\
&\geq 0 \qquad (5.25)
\end{aligned}
$$

The collision avoidance and connectivity preserving properties for the proposed controller can now be proved using the above relation and following an approach similar to the ones used in the proofs of Lemmas 5.1 and 5.2.

Denote by L^+ the positive limit set for a solution $[q^T(t)\ \alpha^T(t)]^T$ of the closed-loop system given in (5.24), where $q(t) = [q_1^T(t)\dots q_n^T(t)]^T$ and $\alpha(t) = [\alpha_1(t)\dots\alpha_n(t)]^T$. For any $[p^T\ \beta^T]^T \in L^+$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $q(t_n) \to p$ and $\alpha(t_n) \to \beta$. This yields $\beta = 0$ because $\alpha_i(t_n) = e^{-\kappa_i t_n}\alpha_i(0) \to 0$ as $t_n \to \infty$, for all $i \in V$. Using (5.25) and an approach similar to the one used in the proof of Lemma 5.3, it can be shown that for any solution $[p^T(t)\ \beta^T(t)]^T$ starting in L^+ (and hence staying in L^+) the relation $\dot{\pi} \equiv 0$ holds. This, along with (5.25) and the fact that $\beta_i \equiv 0$, implies that for any such solution and all $i \in V$, the relation $u_i \equiv 0$ holds. Using this, it is straightforward to verify that the proofs of Lemmas 5.3 and 5.4, and hence the result on the aggregation of the agents, are still valid for the case of unicycle agents. This completes the proof. ■

5.4 Simulation Results

Example 5.1. *To verify the theoretical results obtained for the single-integrator agents, consider a team of 5 agents with the dynamics of the form (5.1), and let the*

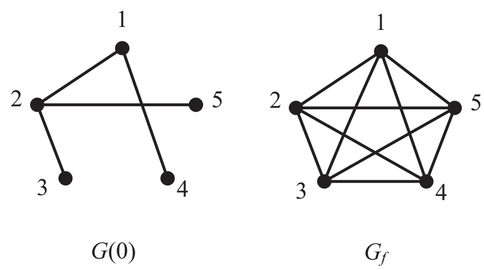

Figure 5.1: The initial and final information flow graphs for both Examples 5.1 and 5.2.

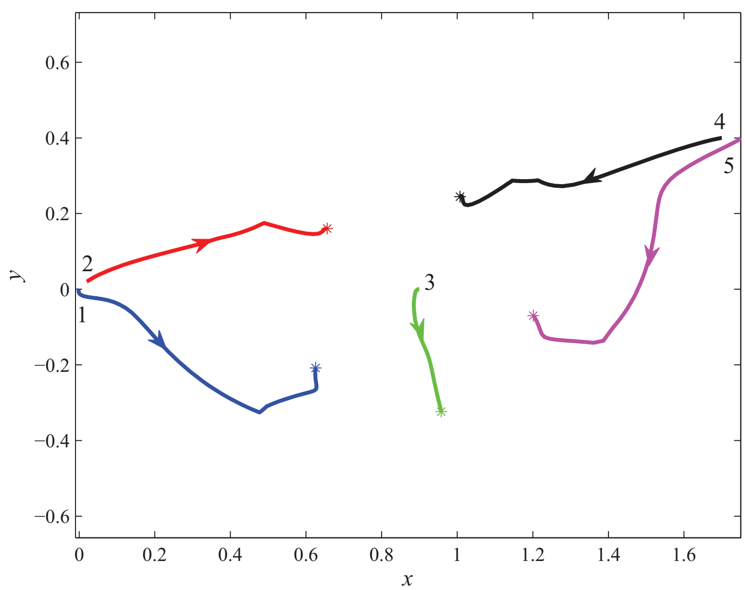

Figure 5.2: The planar motion of the agents in Example 5.1.

connectivity range be specified by $d = 1$. *The initial information flow graph* $G(0)$ *is shown in Fig. 5.1. Assume* $r = 0.5$ *and choose* $m = 3$, *which satisfies the condition of Theorem 5.1. Therefore, using a controller of the form (5.4), the agents are expected to aggregate while preserving connectivity and avoiding collision, in such a way that the average of the distances among the neighboring agents finally falls below* $r = 0.5$. *The trajectories of the agents are depicted in Fig. 5.2, where the initial position of agent* i *is marked by* i, *for* $i = 1, 2, \ldots, 5$.

Denote by d_{ij} *the relative distance between two agents* i *and* j *(i.e.* $d_{ij} = \|q_i -$

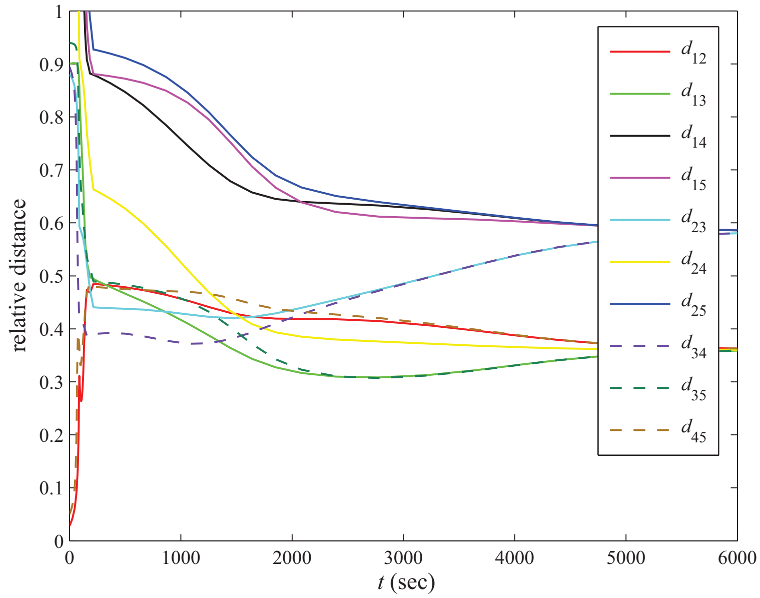

Figure 5.3: The relative distances between the agents in Example 5.1 (d_{ij} represents the distance between agents i and j).

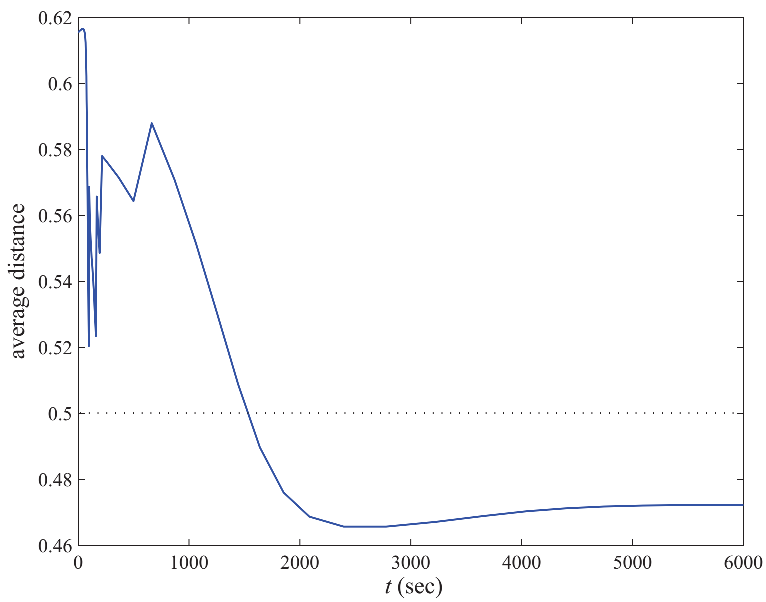

Figure 5.4: The average of the distances between every pair of neighboring agents in Example 5.1. The dotted line represents the reference distance $r = 0.5$.

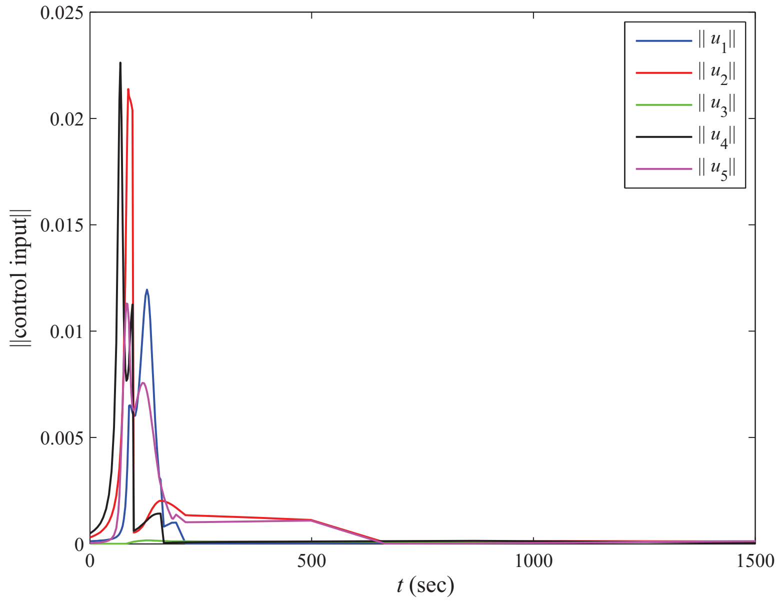

Figure 5.5: The norms of the control inputs in Example 5.1.

$q_j\|$). A new edge is added to the information flow graph as soon as $d_{ij} < d$. However, considering the fact that this inequality provides an open set, in the simulation an edge is added to the information flow graph when $d_{ij} \leq d - \epsilon$, where ϵ is chosen to be $0.1d = 0.1$. The relative distance between every pair of neighboring agents is shown in Fig. 5.3, confirming the connectivity preservation and collision avoidance properties of the proposed controller. As can be seen from this figure, when two agents enter the connectivity range, their relative distance stays less than d at all times thereafter. Also, all relative distances are nonzero, which confirms that no collision occurs. The average of the distances between every pair of neighboring agents is depicted in Fig. 5.4, which eventually falls below $r = 0.5$ as expected. It is worth mentioning that even though m does not satisfy the sufficient condition of Proposition 5.1, the final topology of the network under the proposed controller represented by G_f is a complete graph as can also be inferred from Fig. 5.2. The boundedness of the control inputs of the agents is also demonstrated in Fig. 5.5.

Example 5.2. *To verify the results of Section 5.3, consider 5 unicycles described* *by (5.20), moving in a 2D plane. The connectivity range d and the initial positions of the agents are chosen to be the same as Example 5.1, resulting in the same initial information flow graph $G(0)$ shown in Fig. 5.1. Also, assume $r = 0.5$ and choose $m = 3$ and $\kappa_i = 0.05$ for $i = 1, 2, \ldots, 5$. This choice of m clearly satisfies the condition of Theorem 5.2, and hence a controller of the form (5.21) is expected to fulfill the three design specifications described in Section 5.1. The trajectories of the agents are depicted in Fig. 5.6, where the initial position of agent i is marked by i, for $i = 1, 2, \ldots, 5$. The relative distance between every pair of neighboring agents is shown in Fig. 5.7, from which the connectivity preservation and collision avoidance properties of the proposed controller can be easily verified similar to Example 5.1. As can be seen from this figure, when two agents enter the connectivity range, their relative distance stays less than d at all times thereafter. Also, all the relative distances are nonzero, which means that no collision occurs. The average of the distances between every pair of neighboring agents is depicted in Fig. 5.8, which eventually falls below $r = 0.5$ as expected. Headings of the agents and their translational and angular velocities are also demonstrated in Figs. 5.9-11 respectively.*

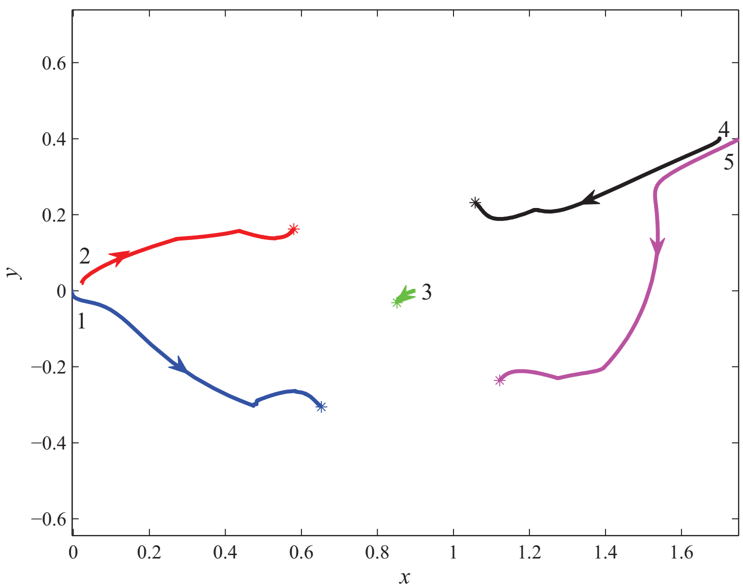

Figure 5.6: The planar motion of the agents in Example 5.2.

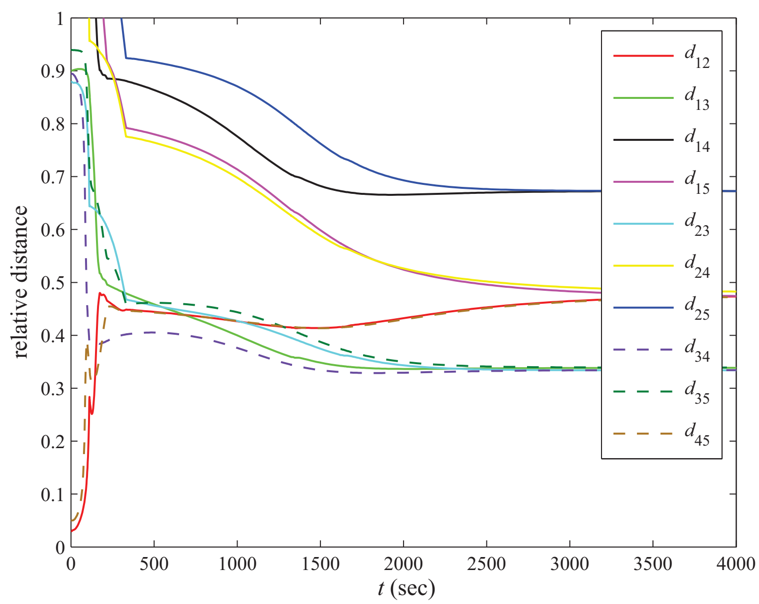

Figure 5.7: The relative distances between the agents in Example 5.2.

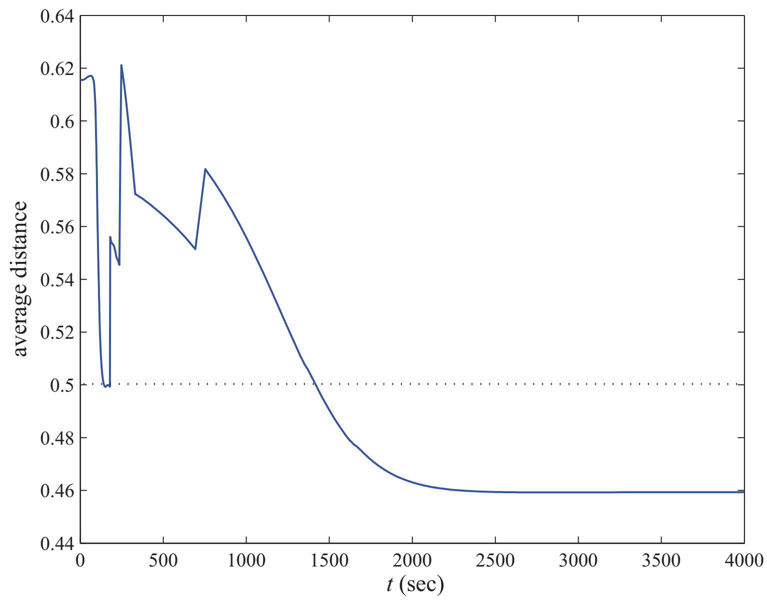

Figure 5.8: The average of the distances between every pair of neighboring agents in Example 5.2. The dotted line represents the reference distance $r = 0.5$.

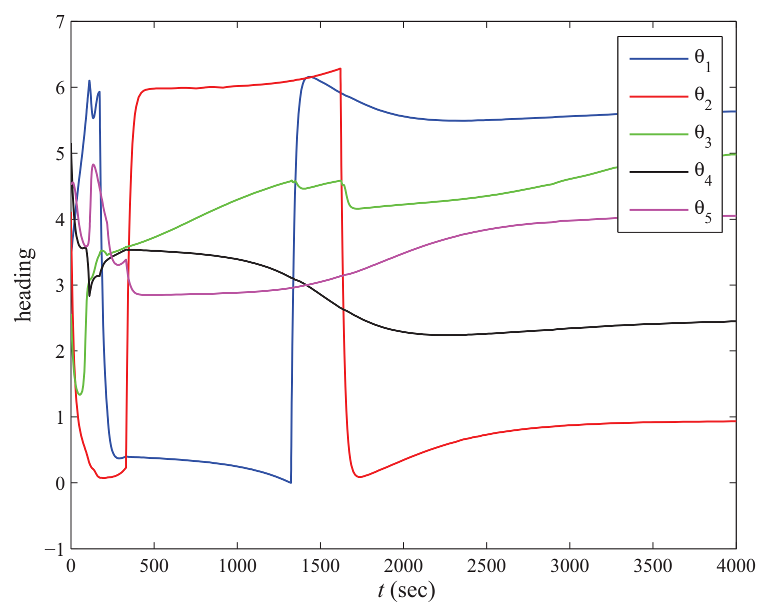

Figure 5.9: The headings of the agents in Example 5.2.

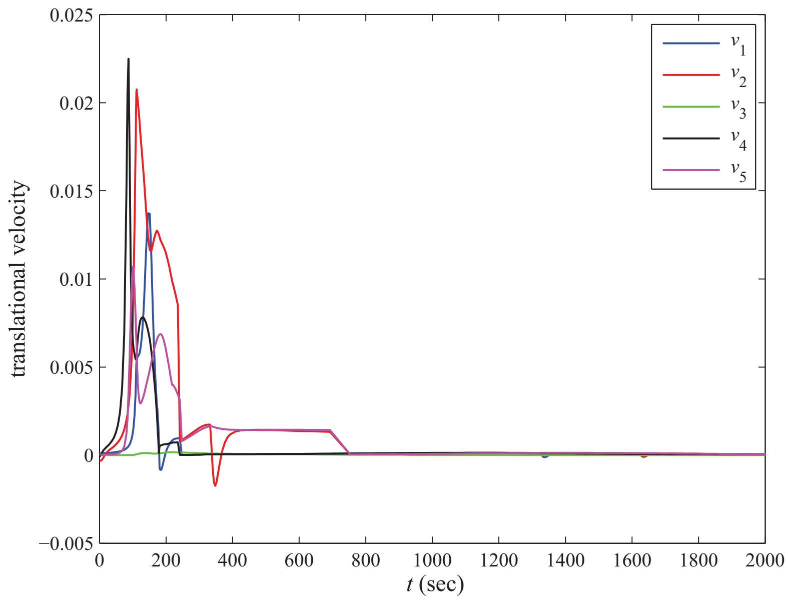

Figure 5.10: The translational velocities of the agents in Example 5.2.

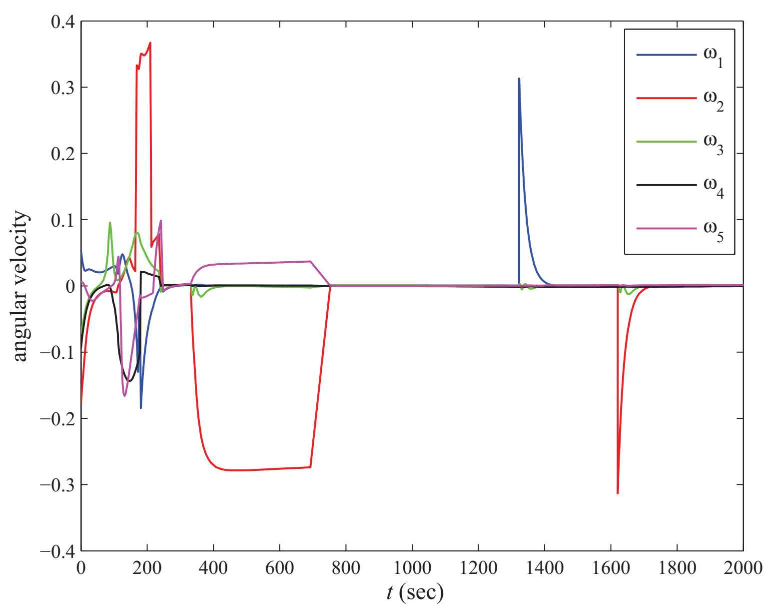

Figure 5.11: The angular velocities of the agents in Example 5.2.

Chapter 6

Conclusions

6.1 Summary

The results developed in this dissertation can be summarized as follows.

Chapter 2 deals with a class of continuous-time nonlinear consensus algorithms for single-integrator agents. It is assumed that the information flow graph is static and directed. It is also assumed that the control input of each agent is a state-dependent combination of the relative positions of its neighbors in the information flow graph. Sufficient conditions are then derived for the convergence of the agents to a common point. Under these conditions, it is shown that the convex hull of the agents has a contracting property. The convergence is subsequently proved using a LaSalle-like approach as well as the finite intersection property of the convex hull. The criteria obtained are shown to be more general than the existing results in the literature.

In Chapter 3, a class of distributed potential functions is proposed which guarantee the connectivity preservation of the resultant control laws for the single-integrator agents. The main idea behind the proposed technique is that when two agents are about to lose connectivity, the gradients of their corresponding potential fields lie in the direction of the edge connecting the two agents, aiming to shrink it. When an agent is at a critical distance from more than one agent, this gradient vanishes. To handle the problem in this case, the lowest order nonzero derivative of the agent's position at any given time (referred to as *index* of the function) is used in the analysis. Shrinking of the edge is performed by moving the agent with lower index towards the agent with higher index. The results are valid for both static and dynamic information flow graphs, and are also extended to cover the problems involving static leaders. Unlike many existing connectivity preserving control strategies proposed in the literature, the potential functions here are designed in such a way that the corresponding control inputs are bounded, making them more practical (as far as the actuators are concerned). The proposed controllers are applied to consensus and containment examples.

Chapter 4 extends the results of Chapter 3 to the case of unicycle agents. If two agents are initially located in the connectivity range, under the proposed control strategy they will remain connected at all times. This implies that the connectivity of an initially connected network is guaranteed. The controller is designed in such a way that when an agent is about to lose connectivity with a neighbor, the lowest order derivative of the agents position which is neither zero nor perpendicular to the corresponding edge makes an acute angle with this edge, aiming to shrink it. The results are shown to be valid for both cases of static and dynamic information flow graphs, and also in the presence of static leaders. Designing bounded connectivity preserving controllers for consensus and containment applications using the proposed method is the novel and unprecedented contribution of this chapter. Detailed stability analysis using some important properties of the positive limit sets of nonlinear systems is carried out for both consensus and containment problems.

A bounded distributed control strategy for aggregation of a swarm of agents for two cases of single-integrator and unicycle dynamics is presented in Chapter 5. The proposed controller is connectivity preserving in the sense that if two agents enter the connectivity range at some point in time, they will stay in the connectivity range thereafter. It is shown that under this controller the agents will aggregate while avoiding collision such that the average distance among the neighboring agents eventually falls below a pre-specified threshold. The control inputs of the agents stay bounded at all times, even if two agents are about to collide or lose connectivity. This is, in fact, one of the important advantages of the work presented in this chapter over existing results in the literature.

6.2 Suggestions for Future Work

In what follows, some of the possible extensions of the results obtained in this dissertation as well as some relevant problems for future study are presented.

- In the class of consensus algorithms studied in Chapter 2, communication and computational delays are not considered. The proof of convergence to consensus is based on the contracting property of the convex hull of the agents; a property that does not necessarily hold in the presence of delay. Therefore, deriving convergence conditions in the presence of delay is a relevant problem. Also, the results of Chapter 2 are developed for a static information flow graph. Considering networks with switching topology is another possible extension.

- The results of Chapters 3 and 4 are developed for an undirected information flow graph. It would be interesting to design a connectivity preserving control law for the case where the information flow graph is directed. It would also be of special interest to solve the problem for the case where connectivity is not distant-based. This is important for example when the sensors of the agents have limited field of view (e.g., camera-based sensors). Moreover, only static leaders are considered in this work. Thus, as a possible future extension one can develop bounded distributed connectivity preserving control strategies for the case where the leaders are moving (e.g., with fixed but unknown velocities).

- This work studies the bounded distributed connectivity preserving controller design only for agents with single-integrator and unicycle dynamics. As a natural extension of this work, it would be interesting to study the problem for agents with other types of dynamics (e.g., double-integrator agents).

- Another interesting extension of the problem investigated in this work is the case where the agents move in a 3D space instead of a flat plane. One can study the problem of designing a bounded connectivity preserving controller, and also find sufficient conditions for the convergence to consensus for the class of controllers studied in Chapter 2.

- Calculating the control inputs for the angular velocities for the unicycle agents in both Chapters 4 and 5 requires the relative velocities of the neighbors. One possible future work is to refine the controllers such that only the relative positions and headings of the neighbors are used in calculating the angular velocities.

- In the connectivity preserving swarm aggregation strategy developed in Chapter 5, the collision avoidance property only holds for point agents. Developing a similar scheme for a more general case where each agent has a known shape would be of more practical interest.

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