STUDENTS' MODELS OF THE KNOWLEDGE TO BE LEARNED ABOUT LIMITS IN COLLEGE LEVEL CALCULUS COURSES. THE INFLUENCE OF ROUTINE TASKS AND THE ROLE PLAYED BY INSTITUTIONAL NORMS

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ABSTRACT

Students' models of the knowledge to be learned about limits in college level Calculus courses. The influence of routine tasks and the role played by institutional norms

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This thesis presents a study of instructors' and students' perceptions of the knowledge to be learned about limits of functions in a college level Calculus course, taught in a North American college institution. I have modeled these perceptions using a theoretical framework, which combines elements of the Anthropological Theory of Didactics, developed in mathematics education, with a framework for the study of institutions – the Institutional Analysis and Development framework – developed in political science. I describe the models and illustrate them with examples from the empirical data, on which they have been built: final examinations from the past six years (2001-2007), used in the studied College institution, and specially designed interviews with 28 students. While a model of the instructors' perceptions could be formulated mostly in mathematical terms, a model of the students' perceptions had to include an eclectic mixture of mathematical, social, cognitive and didactic *norms*. The analysis that I carry out shows that these students' perceptions have their source in the institutional emphasis on routine tasks and on the norms that regulate the institutional practices. Finally, I describe students' thinking about various tasks on limits from the perspective of Vygotsky's theory of concept development. Based on the 28 interviews that I have carried out, I will discuss the role of institutional practices on students' conceptual development.

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to Laura and Lucas

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INTRODUCTION

For the overwhelming majority of students, the calculus is not a body of knowledge, but a repertoire of imitative behavior patterns. (Moise, 1984; cited in Tall, 1996)

This thesis focuses on the teaching and learning of the notion of limit of a function in college level Calculus courses. Research in mathematics education has addressed the same topic at different levels of education[1], based mainly on Piagetian perspectives, and the focus was on how a student constructs his or her own concepts and how instruction could help students to overcome cognitive and epistemological obstacles. With the incorporation of other psychological perspectives – in particular that of L. S. Vygotsky – and of socio-cultural and anthropological frameworks to the field of mathematics education, there was a turn to consider teaching and learning practices as social and cultural practices embedded in societal, cultural and institutional contexts. Social interactions as constitutive factors of the learning process attracted more and more attention. In this shift of focus, from a constructivist or individualistic approach to a socio-cultural approach of the development of mathematics concepts, several authors pointed out the influence of institutions in the teaching and learning practices. For example, Chevallard's theory of didactic transposition (Chevallard, 1985) highlights the institutional relativity of knowledge. The framework which emerged from the theory of didactic transposition – the Anthropological Theory of Didactics or "ATD" (Chevallard,

[1] A literature review is presented in Chapter 1.

1999; 2002) – proposed that the object of research in mathematics education is institutionalized mathematical activity (Bosch et al., 2005).

1. RESEARCH QUESTIONS

My research questions originated in the hypothesis – born from personal experience as a teacher and discussions with colleagues – that "institutional" practices (where "institution" could mean school boards, mathematics departments, classroom, etc.), in the form of definitions, properties, examples and exercises appearing in textbooks and examinations, strongly influence what students learn about limits at college level. Recently, Barbé et al. (2005) have discussed the institutional restrictions imposed on the teacher's practices in the classroom in relation to the teaching of limits in Spanish high schools. My most general goal was to understand how these institutional restrictions reached the students – independently of the personal mediation of a teacher.

While my research questions guided my choice of theoretical framework, they also mutated with the gradual incorporation of these frameworks and the posterior analysis of the gathered data.

First, I considered the phenomenon through the lens of the ATD and I formulated my research question as "how institutional practices influence students' conceptions of limits of functions". My intention was to describe the practice of teaching limits of functions in a North American college[2] institution in terms of "mathematical

[2] "College" refers here to an educational institution situated between high school and university. In the frame of this jurisdiction, students finish high school at the age of 16. The high school curriculum in mathematics does not include Calculus. A first one-variable calculus course is taught only at the college

praxeologies", that is, organizations of typical mathematical tasks, techniques for solving them and discourses used to produce and justify the techniques (Chevallard, 2002: 3). My typology of tasks was based on a classification of questions on limits of functions used in the studied institution over several years, in its final examinations for the calculus course. The classification was guided by a generalization of the mathematical features of these tasks. I was interested in identifying the influence of these mathematical praxeologies on students' conceptions of limits of functions. For this, I designed a "task-based interview" (Goldin, 1997) that consisted of three parts. Because the typical mathematical tasks that I have identified when characterizing the institutional praxeologies are all of the type "find the limit" of some given function, the focus of the interview was on limit finding tasks. In the first part, the students were asked to classify 20 limit expressions according to a rule of their choice. In the second part, the students were asked to find limits that resembled those appearing in final examinations – routine tasks – but differed from them on the conceptual level. In the third part, the students were asked to find limits that did not resemble the routine tasks. All along the interview, the students were asked to think aloud and I followed a flexible script containing different questions and interventions that aimed at better understanding what students' techniques and justifications were. While conducting these interviews with college level students, I realized that their implicit models were quite different from the institution's praxeologies that I have identified. They were not so "purely mathematical"; their structure was much more complex and eclectic.

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level, in academically oriented (as opposed to vocational) programs leading to studying science, engineering, mathematics or computer science at the university level.

In my research, I was expecting, assuming even, that there is difference between the scholarly knowledge related with limits of functions and the knowledge to be taught, and that the knowledge to be learned is usually only a subset of the knowledge to be taught, which is yet distinct from the knowledge actually learned by the students. These distinctions are predicted in the theory of didactic transposition (Chevallard, 1985) and its subsequent refinements (see for example Barbé et al., 2005, and Bosch et al., 2005). I intended to explore the interactions amongst these different types of knowledge. However, looking at these interactions through the lens of institutional praxeologies, brought out the difference between the mathematical perspective on what is to be learned and the learners' perspective on what is to be learned. Epistemologically, "knowledge to be learned" may be a well-defined object. From an anthropological point of view, however, its unity breaks down into distinct praxeologies, different for students and for teachers. These differences are not only structural: while it seemed possible to label the institutional praxeologies as "mathematical", the students' praxeologies were of much more heterogeneous nature, involving a mixture of mathematical, social, cognitive and didactic norms.

After several transformations through the interactions with the theory and the observed data, the main research question took its final form:

> How institutional practices influence students' perceptions of the knowledge to be learned about limits of functions at college level?

In order to address this question, I had to consider other associated questions:

- What is an institution and what does it mean that a practice is institutionalized?
- What is the institutional model of the knowledge to be learned about limits of functions at college level?
- What are the students' models of the knowledge to be learned? How are these models related with students' *mathematical* capabilities to deal with the task of finding limits?

To address these questions, I incorporated a framework for institutional analysis developed in political science – the Institutional Analysis and Development framework ("IAD"; Ostrom, 2005; Sierpinska, 2008) – and Vygotsky's theory of concept development. In the IAD framework it is clearly defined what an "institution" is, together with the notions of "participant", "position", "rules", "norms" and "strategies". From this perspective, I was able to distinguish *institutionalized* practices from practices in general. Based on this distinction, I was able to conceptualize the differences between instructors' and students' spontaneous models of the knowledge to be learned about limits at college level. My analysis shows that students' models are based on norms and habit rather than on mathematical rules and strategies and that this situation is not challenged by the instructors' models. Using Vygotsky's theory, I was able to characterize the modes of thinking that students were using when dealing with the task of finding limits. These modes of thinking were a way of measuring students' cognitive development in the context of limit finding tasks. The Vygotskyan perspective allowed me to present a discussion of how social interactions – in this case, students' interactions with the institution, mediated by the institutional praxeologies – may interfere with students' acquisition and consolidation of the limit concept.

Among the many questions and reflections that were triggered by the obtained results and the relation of these results with previous research, I would like to highlight two of them – I discuss them in detail in Chapter 6.

- If students' spontaneous models of the knowledge to be learned about limits do not correspond purely to mathematical knowledge, to what other *types* of knowledge do they correspond?
- Within the framework provided by Vygotsky's theory of concept development the interviewed students were using mostly a complexive mode of thinking, which is not scientific thinking, and thus, it is not mathematical thinking. What would be the types of tasks that could foster the conceptual mode of thinking in relation to the type of task "find the limit"?

2. THE STRUCTURE OF THE THESIS

This thesis is structured as follows. The next chapter (Chapter 1) contains a review of the literature related with the present research. In Chapter 2, I present the theoretical framework used in the research. In Chapter 3, I describe the methodology and research procedures. In Chapter 4, a model of instructors' spontaneous models of the knowledge to be learned about limits is described. Chapter 5 is divided into four sections. In the first three sections, I analyze the three different parts of the interviews and build partial models of students' spontaneous models of the knowledge to be learned about limits. In section four, these partial models are combined to derive a general model of students' spontaneous models of the knowledge to be learned about limits at college level. The last chapter (Chapter 6) is divided into four sections. In the first one, I present the conclusions derived from the models that I have built of instructors and students' spontaneous models. In the second section, I discuss the results in the context of previous research. In the third section, the conclusions are summarized. Finally, I discuss some perspectives for future research.

CHAPTER 1

LITERATURE REVIEW

> *S1: Ok, well. Because if I put it in I get nothing, well, I get twenty one over zero. Then if I factor it out it doesn't give me anything different, like you can't cross anything out. But then... I am trying to remember, back to Cal I, all the different steps you could do. [I am trying to remember] the methods. There was always first you try to factor and cross out anything that you can. Then... (Student S1 thinking aloud while dealing with the task of finding* $\lim_{x \to 5} \frac{x^2 - 4}{x^2 - 25}$*.)*

Literature on teaching and learning of Calculus, often related to the notion of limit, is vast and it would not be possible to review all of it within the frame of this thesis. Comprehensive reviews of research concerning the teaching and learning of post-secondary mathematics, with references to Calculus, have been regularly published in handbooks of research in mathematics education (e.g., Tall, 1992a; 1996; Harel et al., 2006; Artigue et al., 2007).

Therefore, this review will focus on situating my research within the evolution of the mathematics education research domain. In the second section, I discuss in more depth the theoretical perspectives that have been used in the past to study teaching and learning of limits. In the last section, recent research that is of particular relevance to this thesis is presented.

1.1 THIS THESIS IN THE CONTEXT OF THE EVOLUTION OF THEORETICAL PERSPECTIVES IN THE MATHEMATICS EDUCATION DOMAIN

In the Second Handbook of Research on Mathematics Teaching and Learning, Artigue et al. (2007) distinguish four perspectives in research on teaching and learning of post-secondary mathematics:

- cognitivist perspectives;
- socio-historical or epistemological perspectives;
- embodied cognition perspectives; and
- institutional practices perspectives.

Research on teaching and learning of post-secondary mathematics during the 1980s and early 1990s was influenced by psychology (especially Piaget's work and cognitive constructivism with its focus on *cognitive conflicts*) and epistemology (especially the notion of *epistemological obstacle* proposed by Bachelard (1938), and discovered for mathematics education by Brousseau (1997)). The book *Advanced Mathematical Thinking*, published in 1991, is a good reflection of the state of the domain at the time. The focus of research on teaching and learning mathematics at the post-secondary level was then on identifying cognitive conflicts and epistemological obstacles and characterizing the mental processes by which mathematical concepts are conceived and learned. Learning post-secondary mathematics was studied in those aspects that made it similar to (a) the spontaneous cognitive development of intelligence from childhood to adolescence, and (b) the work of research mathematicians. Therefore, learning a particular mathematical concept was described in terms of "developmental stages" and the various tentative conceptions that appeared in the history of mathematics. These notions of "development" and "tentative conception" suggest progress and overcoming the limitations of a previous conception to go further towards a better conception, perhaps more general, more applicable, without some unnecessary assumptions. Whence the importance of the concepts of "cognitive conflict" and "epistemological obstacle" in this perspective of learning.

Different theoretical frameworks supported research in this direction, e.g., the notions of *concept image* and *concept definition* (Tall and Vinner, 1981), the *process-object duality* (Dubinsky, 1991; Sfard, 1991), or the notion of *epistemological obstacle* (Brousseau, 1983; Cornu, 1983; Sierpinska, 1985; 1990).

The 1990s brought about the realization of the importance of the social, cultural and institutional aspects of learning mathematics at school. There was a shift from paradigms of constructivist cognition towards socio-cultural and anthropological ones (Lerman and Sierpinska, 1996). Cognitive and epistemological analyses could not always explain students' behavior in front of mathematical tasks. These perspectives were efficient to describe students' behavior in situations of discovery: students were taken out of the classroom and given problems that required mathematical concepts that they had not been taught at school yet. Cognitive and epistemological factors become insufficient to describe students' behavior in relation with mathematical tasks and mathematical knowledge that have already been presented to them in a social, cultural and institutional context. In the cognitivist perspectives, students are considered only as cognitive subjects. Most of the time, however, even when learning mathematics, people respond to the world as subjects of various institutions, participating in practices they only partially understand, seeking to survive and make the best of their living. One could say that cognitive and epistemological obstacles afflict the theoretical mind, the scholar seeking the truth (in brief, the *Homo Sapiens*). But if we don't take into account the fact that the learner is not only *Homo Sapiens*, but also *Homo Economicus* and *Homo Institutionalis*, then his or her behavior may appear inconsistent to us. For example, students asked to compare 0.9999... and 1, and then to calculate the sum $\sum_n 9/10^n$, would answer that 0.9999... < 1, but these same students would correctly answer that the sum is 1 (Artigue et al., 2007: 1014). Different explanations have been proposed for this inconsistent behavior. These explanations often rely on the process/object duality: Students are bound to a process view of the symbolic notation 0.9999..., and this view prevents them from seeing beyond the infinite process whose terms are all less than 1. When asked the second question, they *recognize* the geometric series and apply the formula they know to calculate the sum, thus getting the right answer. These explanations are given, however, in abstraction from the socio-cultural or institutional context in which the questions were posed. A reasonable research question would be, for example, whether these students' recognition of the geometric series is determined *only* by the individual mental process put in motion when learning the associated mathematical concepts.

Within this context, some authors presented theoretical constructs that combined cognitive psychology with socio-cultural frameworks (e.g., Cobb and Yackel, 1996). In describing the necessity of a synthesis of socio-cultural and cognitive perspectives, Wood et al. (1995) wrote: "It is useful to see mathematics as both cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals" (p. 402). Other authors developed theoretical frameworks taking into account the socio-cultural context but distancing themselves from the cognitive and socio-constructivist perspectives. This is the case of Godino's onto-semiotic approach (Godino et al., 2005), or the work of Cantoral and Farfán (2003). These authors were most likely inspired by Chevallard's theoretical frameworks, starting from his theory of didactic transposition (Chevallard, 1985) and developing into a broad framework for thinking about and studying institutionalized mathematics teaching and learning, now called Anthropological Theory of Didactics (Chevallard, 1999; 2002).

Despite their differences, all these approaches share the common view that mathematical objects emerge from human practices, and that these practices are institutional and socio-cultural. Cognition is seen as emerging from these practices, and understanding learning processes cannot be achieved without analyzing these institutional practices and identifying the underlying norms and values (Artigue et al., 2007: 1016).

In this shift of focus from cognitive constructivism to socio-cultural and anthropological perspectives, references to Vygotsky started to replace references to Piaget[3]. The Vygotskyan approach assumes that mathematical concepts have already been historically and culturally constructed. Hence, the role of mathematical instruction does not consist in helping the student to construct his or her own concepts – as

[3] It was only in the late 1970s that Vygotsky's work become widely available to non-Russian speaking readers. It is undoubtedly the availability of his work in the West that has triggered in part the shift from the cognitivist perspective to a socio-cultural perspective in the domain of mathematics education.

constructivist theory proposes – but in mediating the already culturally constructed concepts to the students. From this perspective, development cannot be separated from social contexts. Thus, Vygotsky's psychology provided a framework whereby the socio-cultural roots of thought become internalized in the individual (Lerman, 2001: 89).

At the turn of the present century, the embodied cognition perspective took relevance in the domain of mathematics education (e.g., Bazzini, 2001; Rassmussen et al., 2004; Brown and Reid, 2006; Edwards et al., 2009). The main references in cognitive science for mathematics educators working in this domain became the works of Lakoff & Johnson (1980); Johnson (1987), and Varela et al. (1993) on cognition in general, and those specific to mathematical cognition by Lakoff and Nuñez (2000) and Nuñez (2000). Attention shifted to the dependence of learning processes on the biological condition of human beings. From this perspective, cognition is considered as a physically embodied phenomenon realized via a process of codetermination between the organism and the medium in which it exists (Artigue et al., 2007: 1023). Although this perspective is not related to the work done in the present thesis, I mention it because it forms part of a larger movement in the mathematics education community to seek for or develop theoretical frameworks that could support research on *teaching and learning mathematics in context*.

The beginning of the present century has also seen an increase of research in mathematics education based on approaches that take into account the institutional contexts (e.g., Praslon, 2000; Castela, 2004; Barbé et al., 2005; Bosch et al., 2005; Sensevy et al., 2005; Sierpinska et al., 2008). This is where my research is situated. Concern with institutional practices in the domain of mathematics education comes in the wake of the realization, that the evolution of post-secondary education obliges our research community to reconsider answers that have been traditionally given to fundamental epistemological issues about the nature of mathematics, and the nature of mathematical learning and thinking (Artigue et al., 2007: 1044).

From the institutional perspective, mathematics learning is enabled and constrained not only by the human ability to develop complex mental structures and by what and how the human body allows us to perceive and communicate, but also by the social and cultural fact that this learning, today, takes place in educational *institutions* and within their special *institutional practices*. What I have learned from my research, however, is that not all institutional practices are *institutionalized*, and that the non-institutionalized aspects of these practices may have a very real, and not necessarily positive, impact on students' mathematics learning. This realization required a more precise conceptualization of "institution". In the existing literature in mathematics education, the term "institution" has been mostly used in a very wide sense, including any kind of formal or informal structure that organizes or conditions our social and cultural activities (ibid., p. 1025). Castela (2004) acknowledged this fact and discussed some issues related to the notions of institution, position, and regulation, without, however, providing a theoretical framework. It is in the work of Sierpinska et al. (2008) – reporting on the institutional constraints implicated in the sources of students' frustration in pre-requisite mathematics courses – that a theoretical framework for institutional analysis in mathematics education was proposed. The framework contains a definition of the notion of institution and a description of its regulatory mechanisms. I have used this framework in my research, showing, in particular, how an investigation of the mechanisms that regulate institutional practices may contribute to understanding the role played by these practices in students' learning of mathematics.

In the next section, I briefly overview some of the past research on the teaching and learning of the mathematical topic with which this thesis is concerned, namely finding limits of functions.

1.2 COGNITIVE AND EPISTEMOLOGICAL PERSPECTIVES ON LEARNING LIMITS

Students' difficulties with limits have been the object of many studies in mathematics education. Some researchers have revealed students' spontaneous representations of limits (e.g., Hitt and Lara-Chavez, 1999; Fischbein, 2001; Mammona-Downs, 2001; Przenioslo, 2004; Hitt, 2006; Hah Roh, 2008); others couched their findings in terms of cognitive and epistemological obstacles (e.g., Sierpinska, 1985; Davis and Vinner, 1986; Sierpinska, 1987; Sierpinska, 1990; Cornu, 1991). Sources of students' difficulties with limits have been sought in the logical intricacies of the definitions of limits (e.g., Dubinsky and Yiparaki, 2000), as well as in language and semiotic representations (e.g., Monaghan, 1991; Richard, 2004; Hähkiöniemi, 2006), etc. Some studies tried to capitalize on the accumulated knowledge about learning limits and experimented with developing teaching strategies to help students overcome some of the identified obstacles or common misconceptions (e.g., Tall and Schwarzenberger, 1978; Mammona-Downs, 2001; Kidron and Zehavi, 2002; Przenioslo, 2005; Grugnetti et al., 2006).

With a few exceptions (see Selden and Selden, 2005), a large amount of the research in the teaching and learning of limits has been conducted from cognitive and epistemological perspectives (Harel et al., 2006). Different theoretical constructs supported research in this direction, e.g. *procept* theory (Gray and Tall, 1994), *APOS* theory (Asiala et al., 1996; 1997; Cottrill et al., 1996), or the notion of *epistemological obstacle* (e.g., Sierpinska, 1990; Cornu, 1991).

Within the cognitive or epistemological perspectives, authors investigating the teaching and learning of limits were concerned with students' intuitions of limits, prior to formal teaching and learning, that is, with students' concept acquisition in abstraction from the institutional context. For example, Sierpinska (1990) proposes a method of performing an epistemological analysis of a given mathematical concept:

1. Take one or more sentences that define the concept.
2. Ask for their *sense*: WHAT DO THE SENTENCES SAY?
3. Ask for their *reference*: WHAT DO THEY TALK ABOUT?
4. This will give a map of things to understand.
5. Then define acts of understanding / overcoming obstacles required to understand those things.

The method itself exemplifies this idea of studying "concept acquisition in abstraction from the institutional context". Using this method on the example of the concept of convergent sequence, Sierpinska identifies 25 acts of understanding, which are, for the most part, also acts of overcoming relevant epistemological obstacles. For example, one of the acts of understanding is:

> Synthesis of discussion around the problem of reaching the limit in the light of the formal definition of limit; awareness that the formal definition of limit avoids raising this problem and is acceptable within many different conceptions of infinity. (ibid., p. 35)

This last condition of understanding limits shows the difference between Sierpinska's approach to epistemological obstacles and some other researchers'. For example, Cornu (1991) mentions the conception "limit cannot be reached" as an obstacle, which can be overcome by realizing that "sometimes limit can be reached". In the analysis of Sierpinska, both of these conceptions are obstacles. Overcoming both consists in the realization that from the presently accepted definition of limit it doesn't even make sense to raise the question of "reaching" or "not reaching" because all the dynamics and motion have been evacuated from the concept of variable in its formulation.

The perspective taken by Cornu (1991) places the source of epistemological obstacles concerning limits in the historic development of the limit concept and then he addresses the problem of tracing their transmission in the educative practices. Sierpinska (1990; 1994), on the other hand, investigates the cultural roots of epistemological obstacles. For this, she considers the three types of cultural consciousness introduced by Hall (1981): "formal", "informal", and "technical".

> The 'formal' level is the level of traditions, conventions, unquestioned opinions, sanctioned customs and rites that do not call for justification. The transmission of this level of culture is based on direct admonition, explicit correction of errors without explanation (don't say "I goed", say "I went"). [...] The informal level is the level of the often unarticulated schemes of behavior and thinking. Our knowledge of typing or skiing [...] belongs to this culture if we do not happen to be instructors of these skills. This level of culture is acquired through imitation, practice and participation in a culture, and not by following a set of instructions. (Sierpinska, 1994: 161)

In the context of mathematical culture, the "formal" level corresponds to beliefs, convictions and traditions about the nature and subject of mathematics, and the legitimate tools and methods for mathematical work. The "informal" level corresponds to schemes of action and thought, unspoken ways of doing things or thinking that result from experience and practice: this is often called "tacit knowledge".

> At the 'technical' level, knowledge is explicitly formulated. This knowledge is analytical, aimed to be logically coherent and rationally justified. (Sierpinska, ibid., p. 162)

Hence, this "technical" level corresponds, in the context of mathematical culture, to the explicit knowledge of techniques mathematically explained by theories widely accepted by the community of mathematicians. In a "formal" or "technical" way, we can acquire certain knowledge about mathematics, e.g., algorithms, methods of proof, solving some typical problems. It is only on the "informal" level, however, working with mathematicians, through imitation and practice, that we can become creative in mathematical thinking – learning to pose sensible questions, to propose generalizations, to synthesize concepts, to explain and to prove (ibid., p. 165).

Epistemological obstacles are culturally rooted in the "formal" and the "informal" levels. At the "formal" level, our understanding is grounded in beliefs; at the "informal" level – in schemes of actions and thought; at the "technical" level – in rationally justified, explicit knowledge.

In Chapter 6, I discuss a relation between these three levels of consciousness in the context of mathematical culture and my findings concerning students' behavior in front of typical finding limit tasks.

As an introductory remark for the next section, let me observe that research on the teaching and learning of limits based on cognitive perspectives has highlighted a conceptual-procedural dichotomy (Cottrill, 1996; Tall, 2006) in relation with the mental processes involved in the acquisition of the limit concept. In the section that follows, I overview research that shows a conceptual-procedural dichotomy in the institutional practices related to the teaching of limits.

1.3 A CONCEPTUAL-PROCEDURAL DICHOTOMY IN EDUCATIVE PRACTICES RELATED TO LIMITS

At the college level, topics related to limits are not necessarily associated with the limit concept or its formal definition. On the one hand, there is dissociation among intuitive ideas, the formal definition and the techniques in the college level Calculus textbooks (Lithner, 2004; Raman, 2004). In many of these textbooks, the ε-δ (ε-N) definition is presented in a different section than that in which the intuitive ideas about limits are discussed, and that in which the algebraic calculations for finding limits are presented[4]. The sections of these textbooks that present algebraic techniques for finding limits are self-contained. This phenomenon extends to other areas of Calculus in which the limit concept is essential. For example, Raman (2004) observes the non-use of the ε-δ definition of limits in the treatment of continuity in college level Calculus textbooks. At college level, the components textbook – curricula – exams can be taken as a reflection of what students are studying and what educational institutions are expecting students to learn. In this sense, exercise sets in textbooks contain exercises that students are expected to solve. Lithner's research (2004) shows that at least 90% of those exercises can be done

[4] See for example: Larson, R., Hostetler, R. P. and Edwards, B. H. 'Calculus of a single variable', 8th Edition, Houghton Mifflin Company; and Stewart, J. 'Calculus. Early transcendentals. Single variable', 5th Edition, Thomson – Brooks/Cole. Lithner (2004) analyzed other three college level Calculus textbooks.

by searching the text for methods. Therefore, students may develop strategies where the question "which method should be applied?" is immediately asked, instead of first trying to reach a qualitative representation of the task and base the solution attempt on the intrinsic mathematical properties of the components involved[5] (ibid. p. 426). Lithner discusses the impact that these routine practices have on students' problem solving behavior. He argues that these practices may foment a narrow type of reasoning in which the only strategy to tackle a problem is to search for similarities with other problems already seen and the resources that may be developed are restricted to surface mathematical areas (ibid. 424). These assertions are based on a previous empirical study (Lithner, 2000) in which students were asked to solve two problems that were neither purely routine nor completely non-routine tasks. Lithner (2000; 2004) considers the four categories provided by Schoenfeld (1985) to analyze students' problem solving behavior: resources, heuristics, control, and beliefs. The study presented in this thesis has points of contact with Lithner's research in the sense that it studies the influence that routine practices on final examinations have on students' learning and understanding (of limits). In chapter 6, I discuss the relations between Lithner's findings and mine.

On the other hand, research indicates that (at high-school and pre-university level) the teaching of the formal definition and its uses is dissociated from the teaching of "finding" limits (Barbé et al., 2005). In their paper, Barbé et al. discuss the restrictions imposed by an atomized curriculum on the teacher's practice. That is, they investigate the restrictions imposed by *the knowledge to be taught* as defined in curricular documents on *the knowledge actually taught* by the teacher in the classroom. Using the Anthropological

[5] Lithner (2004) characterized 598 exercises taken from three different Calculus textbooks.

Theory of Didactics as theoretical framework, they show that, on the one hand, the didactic organization of the teaching of the limit definition consists only of a theoretical block; a practical block consisting of tasks and techniques is missing. On the other hand, the didactic organization of the teaching of the algebra of limits consists only in a practical block – tasks and techniques – and the corresponding theoretical block is missing. In my research, I analyze a different type of knowledge, namely *knowledge to be learned*[6]. One of my goals is to identify the theoretical blocks and the practical blocks pertaining to this type of knowledge. For this, I considered problems traditionally proposed in final examinations, and I characterize knowledge to be learned based on these problems and the solutions that instructors expect students to provide. In Chapter 6 I discuss the relations between Barbé et al.'s results and mine.

As pointed out above, at the college level, the components textbook – curricula – exams can be taken as a reflection of what students are studying and what educational institutions are expecting students to learn. Lithner's work focuses on textbook presentation of the limit concept. Barbé et al.'s work addresses issues concerning the curricular presentation of limits. My research focuses on the tasks proposed in final examinations together with the types of solutions that students are expected to present. As it will be discussed in Chapter 6, the combined results present strong evidence of the negative influence of institutional practices – observed in three different countries – on students' and teachers' practices related to the limit concept.

[6] Definitions of these types of knowledge are given in the next Chapter; let me only briefly say here that *knowledge to be learned* refers to the knowledge that the institution expects students to know.

CHAPTER 2

THEORETICAL FRAMEWORK

> *S5: Well, when I was kind of in the Cal I type of thinking mode, I would see that minus one is like a barrier, like the limit is a barrier. But, now, that I haven't touched it for a couple of months, I see it as a very mechanical thing: find the answer, cancel out, plug in. I don't see it visually anymore because I am not in the Cal I mode. (Student S5 explaining what the expression* $\lim_{x\to 2}\frac{(x-3)(x-1)}{x^2-9}=-1$ *means for him.)*

This chapter starts with an overview of theories that have been used to conceptualize the research described in this thesis, and interpret its results.

From an epistemological point of view, according to the theory of didactic transposition (Chevallard, 1985), any didactic phenomenon involves the production, teaching, learning and practice of some mathematical activities. The form of these activities depends on the process of didactic transposition, that is, the changes that a body of knowledge has to go through to become knowledge that can be taught and learned at school. Considering Chevallard's original distinctions and subsequent refinements (e.g. Barbé et al., 2005, and Bosch et al., 2005), we can analyze school mathematics as composed of several kinds of knowledge:

scholarly knowledge, understood as knowledge produced by professional mathematicians;

knowledge to be taught, described in curricular documents;

knowledge actually taught which can be gleaned from the teachers' classroom discourse and the tasks he or she prepares for the students;

knowledge to be learned, which can be a subset of the knowledge to be taught or of the knowledge actually taught and whose minimal core can be deduced from the assessment instruments; and

knowledge actually learned, which can be derived, to a certain extent, from students' responses to tasks, clinical interviews, observations of students' behavior in and out of class in specially designed problem solving situations.

This framework, called the Anthropological Theory of Didactics (ATD; Chevallard, 1999; 2002) provides an epistemological model to describe mathematical knowledge as one human activity among others, as it is practiced in various institutions (research mathematics; applied mathematics; engineering; school mathematics at different educational levels; mathematics teacher training institutes, etc.). The model proposed by ATD states that any mathematical knowledge can be described in terms of a mathematical organization, or a praxeological organization of mathematical nature also called "mathematical praxeology". Mathematical praxeology is a special case of the praxeology of any activity, which is defined as a system made of four main components:

a collection T of types of tasks which define (more or less directly) the nature and goals of the activity;

a corresponding collection τ of techniques available to accomplish each type of tasks;

a technology θ that justifies these techniques; and

a theory Θ that justifies the technology.

The term "technology" is understood as the *logos* or the discourse about the techniques, which allows the practitioners to think of, about, and out the techniques. A technology can be a framework of concepts, procedures and rules for applying them. The theory Θ provides a coherent system in which these concepts are defined and rules and procedures are justified. The subsystem [T, τ] corresponds to the know-how, and is called the *practical block* of the praxeology; the *theoretical block* [θ, Θ] describes, explains and justifies the practical block. The theoretical block makes it possible to preserve the activity as a practice and to communicate it to others, so that they, too, can participate in it. This suggests that there is a didactic intention in any cultural practice (if there are no means to teach and therefore perpetuate an activity, it cannot become part of a practice), whence the word "didactic" in the name of the theory.

From the perspective of ATD, the object of research in mathematics education is *institutionalized* mathematical activity (Bosch et al., 2005). This implies the need to define clearly the institutions taken into account in a study of a didactic phenomenon. For example, in studying the teaching of limits of functions in Spanish secondary schools, Barbé et al. (2005) consider the following institutions: the mathematical community, the educational system, and the classroom. To this list, in their exposition of the ATD perspective, Bosch et al. (2005) add "the community of study" (e.g. students of all sections of a course) whose status as an institution is perhaps less obvious. Yet, my own research made me realize how very real this institution can be.

My research questions, originally triggered by my experience of teaching college level Calculus courses, took their actual form based on the notions defined and discussed by the ATD framework (see the introduction, page 5). To answer these research questions, my first idea was to describe instructors' and students' *mathematical* praxeologies for finding limits of functions. An analysis of students' behavior in the task-based interviews made me realize, however, that students' praxeologies do not qualify as *mathematical* praxeologies. They were made of heterogeneous elements, some mathematical and some not. The techniques, technologies and theories incorporated a mixture of social, cognitive, didactic and mathematical rules and norms.

In ATD, the term "institution" is treated as a "primitive term" and is therefore not defined. This may not be a problem in research where the institutional status of the studied social practices is not questioned. Such is not necessarily the case in my research, and I felt the need to base my claims about this or that practice being institutionalized, and distinct from another institutionalized practice, on some theoretical foundations. In a tentative manner, I have tried to combine ATD with a framework for "institutional analysis and development" (known under its acronym IAD, Ostrom, 2005). IAD has already been used in combination with ATD in the study of students' frustration in prerequisite mathematics courses (Sierpinska et al., 2008). In that work, it was assumed that an institution is an organized action whose aim is to achieve certain outcomes (an IAD term) or fulfill certain tasks (an ATD term). The organization defines who are the participants, what positions they can occupy with respect to the tasks and outcomes, and what rules, norms and strategies[7] (IAD terms) or techniques (ATD term) will regulate and make possible the accomplishment of the tasks. These means of regulation require a specific set of discourses to be conceived of and communicated to others. The discourses can be analyzed into "technologies" and "theories", as in ATD.

ATD, with its lack of precision regarding the notion of institution, is unable to make an explicit distinction between institutionalized practices and practices in general. It does not distinguish between rules and norms. Both these regulatory mechanisms are covered under the term "technology". In my study, however, this particular distinction turned out to be very important. It became important because it allowed me to conceptualize the difference between the students' models of the knowledge to be learned and models of this knowledge constructed from other positions in the College Calculus institution (classroom instructors, members of the curriculum committee, members of the final examination committee, etc.).

It is precisely the non-institutionalized or weakly institutionalized layer of *norms* that generates the variety of the spontaneous models[8] co-existing within an institution.

The distinction between rules and norms, afforded by the IAD framework, allowed me to explain, in the particular case of the institution I was studying, the difference

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[7] In section 2.1 I present the exact definition of rules, norms and strategies from the perspective of the IAD framework.

[8] In this research, the terms 'spontaneous' or 'implicit' models are used in the sense given to these terms in the Institutional Analysis and Development framework (Ostrom, 2005) and refers to the 'spontaneous' model of behavior developed by a participant of an institution (see Section 2.1).

between the mathematical praxeologies and the students' praxeologies representing the knowledge to be learned.

Therefore, I felt compelled to use a combination of the ATD and IAD frameworks, in a complementary way, since the two do not contradict each other, but rather throw light on different aspects of the institution I was looking at.

However, while analyzing students' behavior in the task-based interviews, from the perspective of the ATD and IAD frameworks, I arrived at the conjecture that institutional practices were, in some sense, inhibiting students' *development* of mathematical thinking. This idea is related to cognition and so I turned to theories of conceptual development, looking for analytical tools that could help in framing my conjecture. Vygotsky's theory of concept development (1987) seemed appropriate because it focuses on the development of scientific (and not everyday) concepts, which is certainly the case of limits, and (unlike the Piagetian theory) it attributes a primary role to the socio-cultural factors in the development of scientific concepts. According to Vygotsky, the development of *scientific* concepts does not happen naturally, but must be *pulled* by instruction. In the case I am studying, the notion of limit is introduced to students explicitly by name, definitions and properties, and not through spontaneous interactions with an environment. The level of cognitive sophistication at which students' learn this concept depends on the tasks they are challenged to engage with.

Vygotsky's theory of concept development describes the genesis of concepts from early childhood to adolescence. It distinguishes several stages in the development of concepts, each characterized by a specific *mode of thinking* (Vygotsky, 1987: 134-166).

This theory functioned as a structuring framework in my analysis of students' cognitive behavior in the interviews: I would be describing a student's thinking about various tasks on limits in terms of the Vygotskyan modes of thinking. In discussing the role of institutional practices in students' conceptual development, another Vygotskyan concept became quite useful, namely the concept of "zone of proximal development" (ZPD; Vygotsky, 1987: 212). ZPD is the domain of potential development, or the range of tasks the learner can perform with some help of the teacher. The often-quoted related Vygotsky's words are the following:

> Instruction is only useful when it moves ahead of development. When it does, it impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development. This is the major role of instruction in development. (Vygotsky, 1987: 212)

In mathematics education, researchers taking the socio-cultural perspective on the processes of learning and teaching assume that the general pattern of learning a new mathematical concept or domain is similar to the Vygotskyan pattern of stages of conceptual development. They also assume that instruction, to be effective, must go ahead of this development, stretching, as it were, the boundaries of the learner's zone of proximal development (Sierpinska, 1994: 143). However, when Vygotsky's theory of stages of development is applied to studying a mathematical concept in secondary school students and older, the model will apply only to the stages of development of this particular concept, and not to the general cognitive abilities of the students. The students may be capable of conceptual thinking at the highest stage in one domain, but not in another domain that they only start to study.

In the following two sections of this chapter, I outline the details of the IAD framework and Vygotsky's theory of concept development, as they have been applied in my research.

2.1 A FRAMEWORK FOR INSTITUTIONAL ANALYSIS

The IAD framework defines institutions as prescriptions humans used to organize all forms of repetitive and structured interactions. Individuals interacting within *rule*-structured situations face choices regarding the actions and strategies they take, leading to consequences for themselves and for others (Ostrom, 2005: 3). In this sense, institutions are *rule*-structured organizations within which humans *repeatedly* interact to achieve certain goals. Thus, individuals are the basic component of an institution, but on top of these individuals are structures composed of multiple individuals, and these structures may be composed of many parts and, in turn, parts of still larger structures. We have institutions within institutions. What is a whole system at one level is part of a system at another level. For example, a Mathematics Department is an institution within a larger institution which can be the faculty of Arts and Sciences, itself a sub-institution in a university. Another example is a committee that prepares the final examinations of a Calculus course as an institution within the larger institution of a Mathematics Department within a College institution.

The definition of institution proposed by the IAD framework highlights the importance of the notion of *rule*. IAD sharply distinguishes between rules, norms and strategies. It is the existence of rules, which contain sanctions against those who break them, that distinguishes *institutionalized practices* from practices in general. Practice, in general, is based on norms and strategies. Norms do not have to be precise or even explicit, like rules. Newcomers into a practice get to know there is a norm, when they inadvertently transgress it and experienced practitioners tell them, "that's not how we normally do things here".

Formally, I use the following criteria, extracted from Ostrom (2005: 16, 17), to distinguish among rules, norms and strategies.

Strategy denotes an individual's plan of action for accomplishing a task or achieving a goal; an example of strategy can be: "when trying to find $\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x^2-4}$ start by multiplying and dividing by the conjugate of $\sqrt{x+7}-3$".

Norms are used as precepts for prudent or moral behavior; they are part of the generally accepted moral fabric of the community. As it will be discussed later, the Final-Examination institution I have been studying has not – not even once in the last six years – included a problem of finding a limit involving radicals in which the rationalization technique mentioned above would not apply. Thus, the sentence "to find the limit of a function involving radicals, start by multiplying and dividing by the conjugate" represents an implicitly accepted norm in this institution.

Rules denote regulation and they are established by a recognized (legal) authority. Thus, mathematical theorems can be interpreted as 'rules' in the institutional sense. Techniques to find limits follow such mathematical rules; for example, the rule "the limit of the product of a function tending to zero by a bounded function is zero" is a basis for certain techniques of finding limits.

Rules can be put into effect, enforced, rescinded, broken, disobeyed, changed, etc. None of these actions makes sense in relation to norms or strategies; norms are *followed* or not, strategies may be *used* or not.

Participants of an institution are assigned – or assign themselves – to different available *positions*. These positions are associated with different *actions* to be taken to achieve certain goals. Because an institution is, by definition, an organization of repetitive interactions, by observing these repetitive interactions and usual outcomes participants construct implicit or spontaneous models for *acting* in the institution. These models are different from those constructed by an external analyst whose intention is to have theoretical models that would allow him or her to *predict* interactions and outcomes (Ostrom, 2005: 33).

A major challenge in institutional analysis is to choose the appropriate level of analysis relevant in addressing a particular phenomenon. That level of analysis is called the *focal unit of analysis*. Once this focal unit is chosen, the analyst can zoom out from it to have a better understanding of the phenomenon. Zooming out could mean, for example, looking at the exogenous factors that affect the structure of the considered institutions, but also looking at how the institutions in focus are linked together either sequentially or simultaneously.

In this research, I am looking at an educational system where there is an intermediate institution between high school and university, called "college". I am particularly interested in one of the subjects taught in college, namely Calculus. The teaching of this subject is an institution in itself, which I call "College-Calculus". The course is prepared and coordinated collectively by a group of instructors, who teach the individual sections of the courses in classes of 25-35 students, and then collectively write and administer the common final examination. I could have looked at what is going on in particular sections and then the "classroom" would have been one of the institutions involved in the study. Instead, as I explain below, I decided to focus my analysis on two sub-institutions of the College-Calculus institution, which I call, "Final-Examination" and "Community-of-study".

The College-Calculus institution that I study is regulated by

- an official outline of the course, which must be endorsed by the instructors of all sections of the course;
- an official textbook that most students use to practice for tests and final examinations (following the recommendations in the outline of the course);
- a common final examination for all students enrolled in these courses prepared by an ad-hoc committee which consists of all the instructors teaching the course in the given semester.

Each of these regulation mechanisms functions in the frame of an organized action, which is an institution in itself: a sub-institution of the College-Calculus. The institution that decides about the contents and competencies to be taught is not to be confused with the institution that prepares the final examination, or with the classroom institution. Members of the mathematics department are participants in all these institutions, but they occupy different positions in each of them (Ostrom, 2005: 18, 40). They also abide by different rules in each of these institutions. Being a member of the curriculum committee imposes other loyalties relative to mathematical knowledge than being a member of the final examination committee or being a section instructor who has to cope, on a daily basis, with many students' lack of basic mathematical skills and who is interested in obtaining a good average grade for his or her students.

Using IAD terminology, I might rephrase my main research question as, "How institutional practices influence students' spontaneous models of the knowledge to be learned". From the perspective of institutional analysis, however, the notion "knowledge to be learned" becomes relative to the institution defining it. Thus, the definition of "knowledge to be learned" could be different for the curriculum institution, the classroom institution, or the institution in charge of preparing the final examination. Assuming that a core of the knowledge to be learned, as defined by the College-Calculus institution, could be conjectured from final examinations, I focused on analyzing the institutions that are immediately linked to the situation "final examination". The institutions in the focal level of analysis are the "Final-Examination" institution, whose participants are instructors of the mathematics department and students, and the "Community-of-study" institution, whose participants are students enrolled in all sections of the Calculus course. These students indeed form a community: they exchange information about what is going on in their respective sections; they work on assignments in small groups of students from different sections; they study for the common final examination together. I chose to focus on the Final-Examination institution because the only control that the College-Calculus institution has over the knowledge to be learned is through the common final examinations. Hence, I decided to characterize institutional tasks related to limits of functions according to the tasks proposed in final examinations. I chose to focus on the Community-of-study institution because I wanted to characterize students' interpretation of the knowledge to be learned, but not in relation to a particular situation such as passing the final examination, or passing the class tests (for this I would have to consider students as participants of the Final-Examination institution or the classroom institution, respectively). Students are the only participants of the Community-of-study institution and their choice of being in this or that position is independent of individuals that, in other institutions, are in a position of power with respect to them.

Sierpinska et al. (2008) considered four positions that a student could occupy in the institution Pre-requisite Mathematics Courses: *Student* – subject of a school institution who has to abide by its rules and norms; *Client* of this institution who pays for services and has the right to evaluate their quality; *Person* – member of the society at large, and *Learner* – cognitive subject. Based on these ideas and students' behavior in the interviews, I identified three possible students' positions in the institution Community-of-study. I decided to use the same names for the positions as Sierpinska et al. (2008), as I believe they reflect the same behavior.

Student: the participant in this position abides by the rules and norms of the College-Calculus institution addressed to its students; in particular, he or she studies because there is a test coming, not out of disinterested curiosity or passion for the subject.

Client: a student takes the position of *Client* when he or she considers the final examination as a price to pay to attain other goals – passing the course, graduating, having a high grade point average, etc.

Learner: a student is in the position of a *Learner* when he or she behaves as a cognitive subject interested in knowing Calculus; his or her goal in the institution is to learn.

My goal then is to build (a) a theoretical model of instructors' spontaneous models of the knowledge to be learned about limits – instructors as participants of the Final-Examination institution, and not, for example, as participants of the curriculum or classroom institutions – and (b) a theoretical model of students' spontaneous models of the knowledge to be learned about limits – students as participants of the Community-of-study institution and not, for example, as participants of the classroom institution.

I surmise that, for the participants of the Final-Examination institution, knowledge to be learned is identified with the knowledge that students explicitly have to prove that they know, this is, knowledge that is *tested* in the final examination. This knowledge, however, is not regulated by the institution: the contents of the final examination are not fully institutionalized. There are some unwritten norms, some traditions, because the final examinations do not change much over the years. Of course, some things change from one examination to the next, like the formulas of the functions whose limits are to be calculated. Some things, however, remain invariant, like the type of the function. These "invariants" point to the existence of norms. One can glean these norms from empirical data, such as, for example, the texts of the past final examinations, textbooks, or interviews with students. In my research, I tried to show that it is precisely the existence of these *norms* that is determinant in students' construction of spontaneous models of the knowledge to be learned.

2.2 MODES OF THINKING

In the development of conceptual thinking from early childhood to adolescence, Vygotsky distinguishes three stages: syncretic images, complexes and concepts (1987: 134-166), each with several phases. The most mature phase of complexes is pseudoconcepts. Each stage is characterized by a *mode of thinking*, which "leads to the formation of connections, ... relationships among different concrete impressions, the unification and generalization of separate objects, and the ordering and systematization of the whole of child's experience" (Vygotsky, 1987: 135). At each stage, the basis, on which these connections are made, is of quite different nature. Here is a selection of excerpts from Vygotsky's explanations of the nature of these stages.

> **Thinking in syncretic heaps or images**: "Faced with a task that an adult would generally solve through the formation of a new concept, the child... isolates an unordered heap of objects... that are unified without sufficient internal foundation and without sufficient internal kinship or relationships.... [The objects] are externally connected in the [subjective] impression they have had on the child but not unified internally among themselves." (Vygotsky, 1987: 134)

Thinking in complexes (or complexive thinking): "[In contrast with syncretic thinking] generalizations created on the basis of this mode of thinking are complexes of distinct, concrete objects or things that are united on the basis of objective connections, connections that actually exist among the objects involved.... Complexive thinking is thinking that is both connected and objective.... At this stage..., word meanings are best characterized as family names of objects that are united in complexes or groups. What distinguishes the construction of the complex is that it is based on connections among the individual elements that constitute it as opposed to abstract logical connections. It is not possible to decide whether a given individual belonging to the Smith family can properly be called by this name if our judgment must be based solely on logical relationships among individuals.... The foundation of the complex lies in empirical connections that emerge in the individual's immediate experience. A complex is first and foremost a concrete unification of a group of objects based on empirical similarity of separate objects to one another." (Vygotsky, 1987: 136-7) "In accordance with some associative feature, the object is included in the complex as a particular, concrete object which retains all its features rather than as the carrier of a single feature which defines the object's membership in the complex. No single feature abstracted from others plays a unique role. The significance of the feature that is selected is essentially functional in nature. It is an equal among equals, one feature among others that define the object." (ibid., p. 140)

Thinking in pseudoconcepts: [T]he adult cannot transfer his own mode of thinking to the child. Children acquire word meanings from adults, but they are obliged to represent these meanings as concrete objects and complexes.... [They are] obtained through entirely different intellectual operations. This is what we call a pseudoconcept. In its external form, it appears to correspond for all practical purposes with adult word meanings. However, it is profoundly different from these word meanings in its internal nature [which is closer to a complex than to a concept].' (ibid., p. 143)

Vygotsky strongly advocated the use of artificial, experimental situations, where children are given classification tasks, to study their cognitive development: "The experiment uncovers the real activity of the child in forming generalizations, activity that is generally masked from casual observation." (Vygotsky, 1987: 143) This encouraged me to use a classification task in my research as well.

In this research, Sierpinska's (1994: 142-159) interpretation of Vygotsky's theory of concept development will be followed. Thus, it will be assumed that syncretic thinking is characterized by loose criteria; objects are placed together on the basis of subjective, often affective, impressions of contiguity or closeness. In the complexive mode of thinking, impressions of kinship are replaced by connections that actually exist between the objects. In a concept, these relations are, logically, of the same type. In a complex, these connections are factual. Any connection between the object and the model suffices to include the former into the complex. A symptom of complexive thinking is that, in classifying objects, there is a lack of a homogeneous set of criteria. Objects are put together in classes based on some resemblances that may differ from one class to another. Another strongly discriminating characteristic of complexive thinking is that no feature plays a unique role. While a concept is based on a hierarchy of connections and a hierarchy of relations between features that create a new object, which is more than the union of its elements, a complex is a conglomerate of its elements and relations with other conglomerates are not relevant.

A particular form of complexization is that of forming "chains of complexes". The phenomenon refers to the event in which the subject, in classifying objects, focuses on the last object classified and is satisfied by any link between this last object and the new one, ignoring any contradictions with previously classified objects that might follow from this (Sierpinska, 1994: 147).

The transition from complexes to concepts takes the form of pseudoconcepts. In an experimental situation of classification, pseudoconceptual thinking and conceptual thinking produce the same classes. The difference lies in the type of criteria used to decide whether an object belongs to a certain class. While conceptual thinking is guided by abstract and logically coherent criteria, pseudoconceptual thinking is guided by concrete factual features and connections, as in complexive thinking. The referential meaning (the name of the class) is that of a concept, but the categorical meaning (the criterion to decide whether an object belongs to a class or not) is that of a complex.

2.3 COMBINING THE INSTITUTIONAL AND THE PSYCHOLOGICAL FRAMEWORKS

As mentioned before, the general pattern of the genesis of concepts in a child, from early childhood to adolescence, described by Vygotsky, seems to be recapitulated each time a person embarks on the project of understanding or building a mathematical concept (Sierpinska, 1994: 143). This application of a theory of *cognitive development* to the study of *concept development* in students learning particular mathematical concepts requires some explanation.

A theory of cognitive development speaks about *developmental stages* of an individual from birth to maturity. A theory of concept development speaks about *levels of thinking* an individual (mature or not) goes through when learning a particular concept. Thus, in a theory of concept development based on Vygotsky's theory of cognitive development, it is assumed that, when learning a new mathematical concept, the learner would go from thinking about it in syncretic images, to complexive thinking, to thinking in pseudoconcepts, and finally – in concepts.

This theory suggests some kind of linear progress in concept development: once an individual thinks of a concept at the level of, say, pseudoconcepts, he or she cannot go back to syncretic images, but must either stay at this level or go forward to conceptual thinking. The interviewed students' performance, however, appeared to contradict this presupposition of linear progress. They seemed to be using different modes of thinking with respect to the same concept, showing one mode of thinking at one moment and another immediately after.

How could this phenomenon be explained? An analysis of students' responses to various questions in the interview suggested an explanation taking into account the institutional context in which students were interpreting the mathematical tasks given to them. I realized that the mode of thinking that a student was using when approaching a task, or answering a particular question about a task, depended on what was required from him or her with respect to the concepts associated with that task. Thus, for example, in the language of ATD, when justifying the use of a particular technique, a student might use the complexive mode of thinking, and when supporting a technology, he or she might use the syncretic mode. A further explanation was needed at this point: why would students do this kind of switches between different modes of thinking in relation with moving between the different spheres of their praxeologies? A hypothesis will be given later on in the thesis. For now, let me only point to the fact that the possibility of such incongruence between modes of thinking in the different spheres of a praxeology has been predicted in the ADT theory.

In explaining the difference between "technology" and "theory", Chevallard (1999: 228), gives an example of different modes of thinking with respect to these two explanatory discourses:

> Soit ainsi le principe de récurrence: $P \subseteq N \wedge 0 \in P \wedge \forall n\ (n \in P \Rightarrow n+1 \in P) \Rightarrow P = N$. Pour justifier cet ingrédient technologique principal des démonstrations par récurrence, on peut, entre autres choses, soit se référer, comme le faisait Henri Poincaré, à «la puissance de l'esprit qui se sait

capable de concevoir la répétition indéfinie d'un même acte dès que cet acte est une fois possible» (Poincaré, 1902), soit admettre comme un axiome que toute partie non vide de N a un premier élément, et montrer alors que le principe de récurrence en découle.9

Using Vygotsky's categories of modes of thinking, we can say that a student, who justifies the use of induction as a technique of proof by stating the Principle of Mathematical Induction, gives evidence of thinking at the conceptual level, at the technology level of explanatory discourse. The same student, however, might justify this element of the technology – the Principle of Mathematical Induction – by stating the axiom mentioned in the above citation or, as Henri Poincaré did, by stating a belief in the capacity of the human mind to conceive of an infinite repetition of the same act, once this act can be realized at least once. In the first case, we would assess the student's thinking as conceptual at the Theory level of explanatory discourse; in the second case, we would say that, at the Theory level, he or she thinks in syncretic images.

As Chevallard (1999: 227) pointed out, we can imagine an infinite regression of explanatory discourses that starts with technology and theory, and goes on and on: the theory of the theory of the theory...[10] This means that what can be one level of

[9] "Consider the induction principle: $P \subseteq N \wedge 0 \in P \wedge \forall n\,(n \in P \Rightarrow n+1 \in P) \Rightarrow P = N$. To justify this technological component, essential in proofs by induction, we can, among other possibilities, refer, like Henri Poincaré did, to «the power of the human mind, capable of conceiving the infinite repetition of the same act once this act can be realized at least once» (Poincaré, 1902), or accept as an axiom that every non-empty subset of N has a first element, and then prove that the induction principle follows." (My translation.)

[10] Chevallard (1999: 227) claims that technology and theory suffice as a theoretical block to describe a mathematical praxeology. I interpret this in the sense that the theory level of justification is at the axiomatic level; anything beyond that does not necessarily correspond to mathematical concepts anymore.

justification for one individual, can be another level of justification for another individual. Each of these justifying discourses can reveal different modes of thinking in an individual. This seems to be the case of the college level Calculus students whose thinking I have studied. A question that arises from this analysis and that I address in the discussion (Chapter 6) is, *what is the part played by institutions, and, in particular, by their norms, rules and preferred strategies, in the modes of thinking that a student uses at the different levels of explanatory discourses?*

2.4 OPERATIONALIZATION OF VYGOTSKY'S DESCRIPTIONS OF MODES OF THINKING FOR THE PURPOSES OF DATA ANALYSIS

In order to be able to decide which mode of thinking a student is using at a particular moment of the interview, I had to operationalize the descriptions of the four modes of thinking in the form of sets of clear criteria, illustrated with examples of their application. The criteria and the examples had to be clear enough for another person to produce the same assessment, by applying them to analyze a transcript of the interviews. Here are the criteria, followed by examples illustrating their use.

Syncretic images: The subject classifies objects according to an affective relation he or she has with these objects.

Complexive thinking: The subject does not describe his or her classification in terms of a key feature or criterion that discriminates between the classes nor is he or she concerned about finding such key feature. Most importantly, the classes the subject forms are such that the classification rule cannot be, even theoretically, described using a unifying, key feature. One can say that the classes are "all over the place". Each class has its own rule or rules and the "name" of a class or the criterion to decide about the membership applies perhaps to some of the objects in the class but not necessarily to all of them. The hierarchy of features of an object changes from one class to another. Therefore, based on the subject's description of the class, another person may have trouble deciding whether an object different from those the subject has him or herself put in a class belongs to this class or not.

Conceptual thinking: The subject consciously searches for a classification key which, when found, is consistently applied in forming classes. The subject is not happy with the classification key unless it allows him or her unambiguously to decide whether a given object belongs to a class or not. There is a stable hierarchy in the features of classified objects.

Pseudoconcepts: The subject produces classes that could be produced using conceptual level. The criteria, however, the subject gives for putting an object in a class do not qualify as based on conceptual thinking: the subject does not give a unified classification key in his or her explanations, and his or her criteria may sound like those given at the complexive thinking stage. Theoretically, however, a conceptual classification key can be construed for the classes.

EXAMPLE 2.1. A classification task and possible outcomes in relation to different modes of thinking

Task: To classify 3 triangles, 3 squares and 3 discs. In each of these triplets, one object is yellow, one is red and the other is blue.

Syncretic images: The subject puts the 3 discs together in one class because he or she likes round things. The yellow triangle and the yellow square form a class of their own, because one can make a nice little yellow house with them. The last class is made of the red and blue triangles and the red and blue squares because he or she doesn't like them as much, and the blue square reminds him or her of having hit her head last week against a kitchen table, which was covered with a blue cloth.

Complexive thinking: the subject forms 4 classes which she describes as "round objects", "blue objects", "yellow objects" and "red objects". The first class contains the 3 discs, the second – the blue square and the blue triangle, the third – the yellow square and the yellow triangle, and the fourth – the red square and the red triangle.

Pseudoconceptual thinking: the subject makes 2 classes which she describes as "round objects" and "objects with three or four vertices". In the first class, she puts the discs, and in the second – the remaining objects.

Conceptual thinking, Case 1: the subject puts the 3 discs in one class and the other 6 objects in another class. The subject's classification key is whether objects have vertices or not.

Conceptual thinking, Case 2: the subject puts the discs in one class, the triangles in another, and the squares in the third one. The general key is the shape of the object.

Conceptual thinking, Case 3: the subject puts the three yellow objects together, the three red together, and the three blue together. The general key is the color of the object.

2.5 PREVIEW OF THE OUTCOMES OF THE RESEARCH

Using the above criteria, the analysis of the interviews carried out with college students will show that, on tasks related to finding limits of functions, most of these students' thinking did not go beyond complexive thinking at the technology level of explanatory discourses. An explanation of this phenomenon will be sought by examining it through the lens of the combined ATD-IAD framework. This will lead to the hypothesis that institutional practices do not fulfill the role of *pulling* students' development of concepts beyond the limits of their zone of proximal development, but rather generate an institutional environment where, in particular, the complexive thinking mode is not challenged.

CHAPTER 3

METHODOLOGY AND RESEARCH PROCEDURES

"[With the graphic calculator] I can find the limit, no problem, but as far as knowing why... I am trying to understand why it converges." (Student S2, trying to understand why in his graphic calculator it seemed that the limit of sin(1/x) as x tends to zero was convergent.)

In this chapter, I present my research instruments and analysis procedures. In particular, I describe the reference documents that I considered to characterize the institutional praxeologies, and the protocol of the interviews I carried out with college students. Then I explain how I combined the theoretical frameworks presented in the previous chapter to analyze the results. Finally, I discuss the objectivity measure used in analyzing the interviews with students.

3.1 RESEARCH INSTRUMENTS

As mentioned before, *knowledge to be learned* is understood as the knowledge that students explicitly have to prove that they know. In the context of this research, it has been assumed that it is the knowledge that is *tested* by the College-Calculus institution. Considering that the only control that the College-Calculus institution has over this type of knowledge is through the common final examinations, I focused my attention on how the types of tasks related to limits of functions proposed in final examinations influence students' spontaneous models of the knowledge to be learned about limits.

First, I identified the types of tasks that appeared in final examinations over the last six years. There appeared to be three types of tasks. I refer to these tasks as *routine* tasks and describe them below:

T_1: To find the limit of a rational function taken at a constant, such that it is a zero over zero type of indetermination and the polynomials in the rational expression can be factored by standard school Algebra techniques.

T_2: To find the limit of a rational expression involving square radicals taken at a constant, such that it is a zero over zero indetermination and the limit can be found by a rationalization technique.

T_3: To find the limit of a rational function taken at infinity, where the polynomials involved cannot be factored by standard school Algebra techniques.

Next, I constructed models of the mathematical praxeologies of the College-Calculus institution corresponding to these tasks. To do this, I used the following reference documents: the official textbook, the topics listed in the course outline, the final examinations from the last six years (2002-2008), and the solutions for these examinations, written by teachers and made available to students. The descriptions of the *techniques* corresponding to the types of tasks listed above were based on teachers' solutions and the solution methods presented in the textbooks. The description of the theoretical block [θ, Θ] was based on topics listed in the outline, properties and theorems used in textbooks to justify the techniques, and teachers' solutions.

Based on these mathematical praxeologies, I have built a model of instructors' spontaneous models of the knowledge to be learned. In the previous sentence, "instructors" refers to individuals as participants of the Final-Examination institution, and not as participants of, for example, the curriculum or the classroom institution. It is important to understand this difference because the model I constructed is related more to the positions that the participants occupy in the institution Final-Examination than to the individuals filling these positions; an instructor can be a member of the final examination committee for Calculus in one semester but not in the next one. However, when he or she becomes a participant in this institution, the model becomes his or her model. Furthermore, when an individual is new to this institution, the model is presented to him or her by the institution as the *normal* model to follow.

The mathematical praxeologies of the College-Calculus institution and the inferred model of instructors' spontaneous models of the knowledge to be learned are described in Chapter 4.

To construct a model of the students' praxeological organizations representing their spontaneous models of the knowledge to be learned, I conducted twenty eight (28) "task-based" interviews. A description of the methodology of task-based interviews can be found in Goldin (1997).

In this paper, Goldin emphasizes the differences between *structured clinical* interviews and *task-based* interviews. Structured clinical interviews have been used in research to observe "the mathematical behavior of children and adults, usually in an exploratory problem-solving context" and to draw "inferences from the observations to allow something to be said about the problem solver's possible meanings, knowledge structures, cognitive processes, affect, or changes in these in the course of the interview" (ibid., p. 40). The method is geared to the needs of research that focuses on conceptual understanding and students' internal constructions of mathematical meanings in place of, or in addition to, procedural and algorithmic learning (ibid.). In contrast, task-based interviews offer the possibility of obtaining information that directly bears on the goals of teaching and learning in the classroom or in other institutions, -and can help understanding, for example, what cognitive representational structures students are developing and what beliefs about mathematics they are acquiring in these contexts (ibid., p. 41).

In discussing research methods in mathematics education, many authors have addressed issues related to ethics, objectivity, reproducibility, and generalizability, among other concerns. Goldin (ibid.) refers to some of these issues; in particular, he discusses the replicability of results and the generalizability of findings when using task-based interviews as research methodology. For replicability, he argues that it is essential to distinguish *observations* from *inferences*. Inferences must be based on explicit criteria, so that the inferencing process itself becomes open to discussion (ibid., p. 53). In addition, task-based interviews have to be explicitly characterizable as research instruments, subject to reuse, refinement and improvement by different researchers.

In the context of my research, from *observing* students' behavior I had to *infer* their positioning in the Community-of-study institution and their modes of thinking. Hence, to address the issue of replicability, I made explicit the criteria that I have followed to make these inferences (see Section 2.4 and Section 3.3.4). As I explain at the end of this chapter, a triangulation method was used to ensure the objectivity of this inference process as well as the clarity of the established criteria – so that the method of analysis could be reused by other researchers.

In the sections that follow, I have made explicit the reasons I had and the goals I wanted to achieve when constructing each of the tasks used in the task-based interview. My intention here was not only to describe my research instruments but also to characterize these tasks as research instruments so that they could be eventually refined and improved by other researchers.

Task-based interviews have been recently used in studies concerning issues related to the teaching and learning of limits (e.g. Lithner, 2000; Alcock and Simpson, 2004, 2005; Hähkiöniemi, 2006; Hah Roh, 2008).

For the present research, subjects were recruited from among students enrolled in college level Calculus II courses in the winter semester of 2008. All subjects had successfully completed a Calculus I course in the previous semester, that is, in the fall of 2007. The distribution of their grades is presented in Table 3.1.

Grade range (in percentages)	71-75	76-80	81-85	86-90	91-95	96-100
Frequency (N=27)	11.1 (3)	18.5 (5)	11.1 (3)	14.8 (4)	18.5 (5)	26 (7)

TABLE 3.1. Number of students per grade range. One student of the 28 interviewed did not disclose his grade information[11].

Subjects were selected to represent a vast spectrum of the sections of the Calculus I course, taught by different teachers in the fall of 2007. In that semester, there were 19 sections taught by 14 different teachers; the sample of interviewed subjects covers at least 12 of these 14 teachers. Table 3.2 shows the number of interviewed students corresponding to each teacher. Five students could not remember the name of their Calculus I teacher; those are taken into account in the last column of the table.

Teacher's code	T1	T2	T3	T4	T5	T8	T9	T10	T11	T12	T13	T14	T?
# of students	1	1	2	2	4	2	1	1	3	2	2	2	5

TABLE 3.2. Number of students per teacher.

Interviews were aimed at characterizing the *mathematical* praxeologies of students and how the routine tasks proposed by the College-Calculus institution – those appearing in the final examinations – influence students' behavior when facing the task of finding

[11] The student was still considered in my analyses because students' grades have not been a variable in the research.

the limit of a function. This is why my design of the tasks for the interview was based on an analysis of the routine tasks.

3.1.1 Protocol of the interviews

The limit finding tasks that students are asked to solve in final examinations are such that only algebraic techniques are required. Therefore, my expectation was that students would classify the analytic tasks[12] of finding limits from an algebraic point of view. What I mean is that they would consider two such tasks as different if the algebraic manipulations necessary to solve them were different. In addition, I wanted to verify my conjecture that students' approaches to finding limits are strongly influenced by the algebraic techniques associated with tasks in the final examinations. Thus, since problems appearing on final examinations are mostly of the factoring or rationalization type (see Chapter 4), I expected students to factor and simplify common terms in a problem like "find $\lim_{x \to 2} \frac{x+1}{x^2-1}$", before trying direct substitution (of 2 for x in the expression, in the example). I also expected them to get stuck trying different algebraic techniques in a problem such as "find $\lim_{x \to 1} \frac{x+1}{x^2-1}$". The two examples visually resemble the typical final examination tasks but differ from them on the conceptual level. In this way, the interview was designed to reveal some of the reasons behind students' typical approaches to finding limits.

[12] i.e. tasks belonging to mathematical analysis.

Interviews consisted in individual encounters with each student. For the purposes of the analysis to be done in this research, I divide the interview into three "parts". The first part was based on a classification task. In the second part, students were asked to find limits that visually resembled tasks proposed in final examinations but differed from them on the conceptual level. In the third part, students were asked to find three limits that were essentially different from those appearing in final examinations. In each of the three parts, students were asked to think aloud while performing the given tasks. Then, students were questioned in the aim of obtaining more information about their thinking. Some questions were prepared in advance (see below) and posed to all students, while others varied according to a student's performance.

3.1.1.1 The first part of the interview: The classification task

In the first part of the interview, students were presented with twenty cards. Each card contained a written expression of the type $\lim_{x \to c} f(x)$, where c was either a constant or ∞, and $f(x)$ was a constant, a polynomial, a function involving a radical, a rational function, a quotient of functions involving radicals, or a function involving a trigonometric function. Students were asked to classify these twenty cards according to a rule of their choice. They were not asked to make the rule explicit before doing the classification. Once they had formed the classes, I asked them to "explain the rule" they had used for the classification. As I show in Chapter 5, most students could not state a unique rule applying to all the classes they had formed. Rather, students offered short phrases describing each of these classes. After this "naming process", I challenged the membership of some of the objects placed in this or that class.

Table 3.4 presents the twenty expressions appearing in the cards. Here, the expressions have been numbered for reference purposes. The cards and the expressions were not numbered when they were presented to the students.

1. $\lim\limits_{x\to 3}\dfrac{x^3-x^2-3x-9}{2x^2-4x-6}$	2. $\lim\limits_{x\to 1}\dfrac{x^2-1}{(x-1)(x-3)}$	3. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+2}-\sqrt{2}}{x}$
4. $\lim\limits_{x\to\infty}\dfrac{9x^3-x+2}{3x^3+1}$	5. $\lim\limits_{x\to\infty}\dfrac{x^2-25}{x^3-1}$	6. $\lim\limits_{x\to\infty} 7$
7. $\lim\limits_{x\to 5} 3$	8. $\lim\limits_{x\to 1} 4x^3+7x-9$	9. $\lim\limits_{x\to 2}\dfrac{x^2-4}{x^2-3x-4}$
10. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x+4}$	11. $\lim\limits_{x\to\infty}\dfrac{x^3+2x+1}{x^2+7x-1}$	12. $\lim\limits_{x\to\infty}\dfrac{x^2-1}{x-1}$
13. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{2x^2+1}}{3x-5}$	14. $\lim\limits_{x\to\infty}\sqrt{x^2+1}-x$	15. $\lim\limits_{x\to 1}\dfrac{x^2-x-2}{x-1}$
16. $\lim\limits_{x\to 1}\dfrac{x^2+6x+19}{x^3-3x+2}$	17. $\lim\limits_{x\to 5}\dfrac{x^2-4}{x^2-25}$	18. $\lim\limits_{x\to 5}\dfrac{\sqrt{x+20}-\sqrt{5}}{5-x}$
19. $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x}$	20. $\lim\limits_{x\to 0}\sin\left(\dfrac{1}{x}\right)$	

TABLE 3.3. Expressions given to students in the first part of the interview. The cards and the expressions were not numbered when presented to the students.

Expressions 1 to 4 are routine expressions in the sense of being instances of types of tasks appearing in final examinations. Expressions 1 and 2 are examples of type of tasks T_1. Expression 1 was chosen because exercises consisting in finding limits in which the factorization of a cubic expression is needed are common in final examinations (see Table 4.1). However, students might feel that factoring a quadratic polynomial is easier than factoring a cubic one, especially when the quadratic is a difference of squares that they usually recognize right away. To see how students react to these two versions of type of tasks T_1, I chose also to have expression 2. My assumption was that students would not put them together in the same group when doing the classification; they would prioritize the algebraic particularities of the difference of squares over any other characteristic.

Expression 3 is an example of a limit that can be found by rationalization – type of tasks T_2.

Expressions 4 and 5 are examples of limits of rational functions taken at infinity. Expression 4 is similar to those appearing in final examinations, but expression 5 contains the rational function $\frac{x^2-25}{x^3-1}$, which is factorable using standard techniques learned in school Algebra courses and thus it is not a routine expression. My expectation was that students would tend to classify expressions 2 and 5 together, because they both contain a difference of squares, ignoring the fact, relevant to Calculus, that the limit in expression 2 is taken at a constant and the limit in expression 5 is taken at infinity.

Expressions 5 to 20 are non-routine in the sense that they do not belong to any of the three types of tasks T_1, T_2 or T_3. Expressions 6 and 7 were chosen because, although they look very simple, in both, finding the limit requires either some conceptual understanding or a good memory (remembering by heart that the limit of a constant is the constant itself). The main idea in choosing the expressions 8 to 18 was to have examples, of which some would look algebraically similar to routine expressions, and some would not, but for all the corresponding techniques would appear in the outline of the course. These techniques would be:

- direct substitution (expressions 8, 9, and 10);

- dividing every term in the numerator and the denominator by the highest power of x in the rational function, or factoring out the highest power in the numerator and the highest power in the denominator, then cancelling out, and using the fact that, if c is a real number and r is a positive integer then $\lim_{x \to \pm\infty} \frac{c}{x^r} = 0$ (expressions 5 and 11 to 13);

- rationalizing technique (expression 14); and

- finding limits by inspection (expressions 15 to 18).

As for expressions 19 and 20, they contain trigonometric expressions. Students might not be familiar with techniques to find limits of this kind as the topic is listed as optional in the course outline. They were chosen to see how students consider expressions that are very different from routine ones.

3.1.1.2 The second part of the interview: Four routine looking limits tasks

In the second part of the interview, students were asked to find the four limits shown in Table 3.4.

2.1 $\lim_{x \to 1} \frac{x-1}{x^2+x}$	2.2. $\lim_{x \to 2} \frac{(x+3)(x-1)}{x^2-9}$
2.3. $\lim_{x \to 5} \frac{x^2-4}{x^2-25}$	2.4. $\lim_{x \to 1} \frac{x^3+4x^2+9}{x^2+2}$

TABLE 3.4. The four limits that students were asked to find in the second part of the interview.

Taking into consideration the length of the interview I decided to ask students to solve limits involving only rational functions. Therefore, the focus was on task of types T_1 and T_3. The tasks corresponding to T_1 that we can find in textbooks involve rational expressions that are easily factorable using algebraic techniques such as "difference of squares", "taking common factors", "factoring by grouping"[13], or simple cases of "undoing the distribution property"[14]. Hence, the polynomials in the rational expressions

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[13] The "factoring by grouping" technique can be summarized as: 1. Collect the terms into two groups so that each group has a common factor. 2. Factor out the greatest common factor from each group. 3. Factor a common binomial factor from the result of step 2. If step 2 does not result in a common binomial factor, try a different grouping. (Source: Lial, M., Hornsby, J. and McGinnis, T., *Beginning Algebra*, 9th Ed.)

[14] From the distribution property we get that $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$. "Undoing the distribution property" refers to finding the constants a, b, c and d. For example, to factor $8x^2 + 6x - 9$ we need to find constants a, b, c and d such that $ac = 8$, $bd = -9$ and $ad + bc = 6$. (Source: *ibid.*)

are usually of degree 2 or 3, and, on rare occasions, of degree 4. On the other hand, tasks in T_3 always involve polynomials that cannot be easily factored, or are not factorable at all. Problems 1, 2 and 4 above, can all be solved by direct substitution. Problem 3 cannot be solved algebraically, but by inspection or by making a table of values. I chose these four problems for the interview because the rational expressions in problems 1, 2 and 3 are (in the case of problem 2), or seem to be (in the case of problems 1 and 3), instances of rational expressions in type of task T_1, and the rational expression in problem 4 belongs neither to T_1 nor to T_3, as the involved polynomials are not factorable (T_3) and the limit is taken at a constant (T_1). The idea behind this choice was, in the case of problems 1, 2 and 3, to deceive students to engage into the factoring techniques typically used to solve a task of type T_1, and in the case of problem 4, to present students with a rational expression that is not factorable, to contrast their approach to problem 2. My expectation was that in problems 1 and 3 students would get frustrated and show difficulties in providing an answer, because of the lack of common factors, while in problem 2, they would be comfortable with cancelling the common factors and then using substitution to arrive at a correct answer, without noticing that the factoring was not necessary. My expectation for students' approach to problem 4 was that they would recognize right away that the technique to find the limit is direct substitution, because the involved polynomials are not (easily) factorable.

3.1.1.3 The third part of the interview: Non-routine limits tasks

In the third part of the interview, students were asked to find the three limits shown in Table 3.5. These are non-routine tasks. I refer to them as *essentially* non-routine because they are non-routine *and* they do not resemble routine tasks. The functions involved are different from the standard functions that students are given when asked to find limits, i.e., rational functions and functions involving radicals (compare with tasks in the second section of the interview). However, the interviewed students were taught, at some point, the graphs of the sine, cosine, and exponential functions. In addition, techniques to conjecture the value of a limit by means of a calculator were listed in the outline of Calculus I. In the case of $\lim_{x \to 0} \frac{\sin(x)}{x}$, either the students were told that this limit is 1, in the form of a theorem in Calculus I, or they had seen it as an example of L'Hôpital's rule in Calculus II. The purpose of this part was to test students' ability to combine the different things they have learned, such as using the calculator to conjecture limits, or reading limits from graphs, in dealing with more challenging limits. My expectation was that students will not make use of resources other than algebraic to find limits and hence, that they will not be able to find or make conjectures about the limits presented in this third section of the interview. Whenever this was the case during an interview, I proposed the student, after some time, to use other techniques to conjecture the value of the limits; in particular, I proposed using the calculator and graphing. My intention in proposing these approaches was to verify the conjecture that the problem was not in students not knowing any other approaches, but only in the institutional emphasis on algebraic techniques, which may have obscured any other methods students had an opportunity to learn. My expectation was that, given a slight instructional prompt, students would be able to *think in mathematical terms* about these limits.

3.1 Find $\lim_{x \to +\infty} e^x \cos(x)$	3.2 Find $\lim_{x \to -\infty} e^x \cos(x)$
3.3. Find $\lim_{x \to 0} \frac{\sin(x)}{x}$	

TABLE 3.5. The three limits that students were asked to find in the third part of the interview.

3.2 DATA ANALYSIS PROCEDURES

> *[T]hough this be method, yet there is madness in't. (Schoenfeld, 2002: 435, paraphrasing Polonius' comment in Hamlet)*

My conjectures for students' performance in the interviews were confirmed on a structural level. In most cases, students did distinguish two tasks if the algebraic manipulations necessary to solve them were different. Students did approach limits in the way I expected they will. The reasons, however, for this behavior, were not as mathematical as I first believed. The interviews revealed that students' understanding of limits of functions is strongly based on mathematical, social, cognitive and didactic norms (Hardy, 2009). Hence, although my a priori idea was to describe students' *mathematical* praxeologies for finding limits of functions, these praxeologies turned out to be of a more heterogeneous type. It was the necessity to analyze the interviews within this situation that motivated the incorporation of the IAD framework and Vygotsky's theory of concept development as part of my theoretical framework. The ATD, with its notion of praxeological organization, provided me with a structure in which I could place students' behavior, the IAD and the theory of concept development provided me with the means to *interpret* students' behavior and its relation with the institutional practices.

I consider each part of the interview as providing one *snapshot* of students' behavior in front of the task of finding limits of functions. From snapshot #1, I reconstruct the praxeology in relation of the task of finding limits of each of the 28 interviewed students and I surmise each student's mode of thinking. From snapshots #2 and #3 I make a general reconstruction of students' praxeologies in relation to the task of finding limits, and I try to derive the general – most frequent – positioning. These reconstructed praxeologies constitute three partial models of the students' models of the knowledge to be learned about limits. These models are about students' spontaneous models when dealing with the task of finding limits, but they are reconstructed from snapshots taken in very different situations and thus provide very different information about students' behavior. Then I combined these partial models to build a model of students' spontaneous models of the knowledge to be learned about limits in college level Calculus courses.

3.2.1 Analysis procedures for snapshot #1

The first snapshot, focused on the classification task, is analyzed from the perspective of ATD and Vygotsky's theory of concept development. This generates two levels of analysis that are brought together to build the first partial model of students behavior in front of the task of finding limits of functions. Using the notion of praxeology of ATD, I identified each student's technique to accomplish the classification task, and his or her corresponding technology and theory. Next, based on Vygotsky's theory of concept development, I characterized each student's mode of thinking in relation to the two levels of justification.

Based on each student's techniques, technology and theory related to the classification task, I reconstructed the student's praxeology in relation to the task of finding limits. Hence, the first partial model consists of this reconstructed praxeology together with the student's mode(s) of thinking.

3.2.1.1 Analysis procedures to reconstruct the students' praxeologies from snapshot #1

Based on the student's classification and explanatory discourse, I tried to understand what features of the limit expressions he or she was looking at to guide his or her classification. For this, I considered a scheme (see Figure 3.1) in which I represent the expressions in the cards and some additional information. The scheme has four "boxes". The symbol "lim" and boxes 1 and 2 represent the expressions that students could see on the cards. The third box corresponds to the arithmetic outcome of direct substitution (in $\boldsymbol{R} \cup \{+\infty, -\infty\}$). The fourth box corresponds to the value of the limit.

In terms of a praxeological organization, the features a student is considering and *how* he or she is considering them correspond to his or her technique to accomplish the classification task. These features could be of four kinds: arithmetic, algebraic, belonging to Calculus, or analytic (i.e. belonging to Mathematical Analysis). For example, a student who focused only on box 3 and made a classification forming four classes, named them "a number over zero", "infinity over infinity", "zero over a number", and "a number", would be characterized as using a technique that belongs to arithmetic in $\boldsymbol{R} \cup \{+\infty, -\infty\}$.

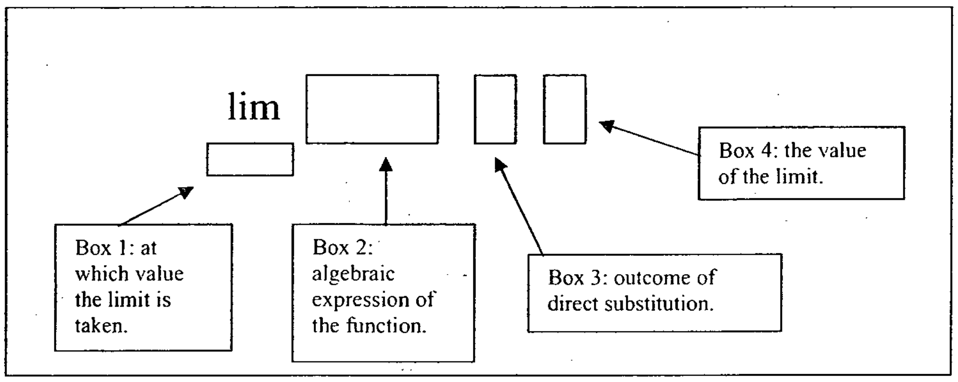


FIGURE 3.1. A scheme of limit expression.

A student who focused only on box 3 but made the classes: "infinity", "indetermination zero over zero", "indetermination infinity over infinity", and "a constant", would be described as using a technique that belongs to Calculus. A student who focused only on box 2 and made the classes: "rational functions", "expressions with radicals", "constants", and "trigonometric functions", would be characterized as using a technique that belongs to Algebra. I assumed that the features that are so relevant to the student that he or she chooses them as a basis to accomplish the classification task are also those that the student is likely to consider when he or she has to choose a technique to find the limit of a function.

I characterized the student's supporting discourses, the technology and the theory, based on the phrases he or she used to describe the classification. In the analysis, I have been consistent in considering as technology that discourse, which is the immediate justification of the technique, the *logos* about the technique. Hence, this level of justification corresponds to the phrases that a student used to "name" or describe the classes. Whenever the student had provided enough information, I conjectured what his or her theory – the discourse supporting the technology – could be. From these explanatory discourses, I inferred students' technologies and theories in the praxeological organization corresponding to the task of finding limits.

3.2.1.2 Analysis procedures to identify students' mode(s) of thinking from snapshot #1

The use of classification tasks to figure out a child level of thinking has become common practice in cognitive psychology. Vygotsky's experiments related to concept development were based on them, as mentioned in Chapter 2. To identify a student's mode of thinking based on a classification, we not only have to observe the classes he or she has made and the elements that he or she has decided to place in each class, but also the criteria with which he or she has made these decisions. Hence, to figure out the mode of thinking at the technology level, I took into consideration the classes with its members and the phrases that the students used to describe each class. Whenever the information provided by the student about his or her own reasoning was sufficient, I made conjectures about his or her mode of thinking at the level of theory. I used the criteria described in Chapter 2, Section 2.4.

An important observation to understand the analysis of Vygotskyan modes of thinking within the framework of the ATD is the following. If we accept that an infinite sequence of explanatory discourses is conceivable (Chevallard, 1999: 227), it might be the case that affect – characteristic of thinking in syncretic images – always plays a role at some level of the chain. For my research, it is essential to distinguish, whenever a student made a statement that seemed to reflect a syncretic mode of thinking, whether the student was thinking in syncretic images at the technology level, at the theoretical level, or at none of these. I considered that an individual is thinking syncretically at the technology level if, for classifying an object, he or she is considering features that are internal to him or her – but not intrinsic to the object being classified. In the other modes of thinking, the features considered relevant for the classification are external to the classifier. Thus, to decide whether the first level of justification, the technology level, is or is not syncretic, I analyzed *how* affect influenced the classification. Whenever affect influenced an individual's attention, but was not the classifying feature, I did not consider that he or she was thinking syncretically, although it might be the case that he or she was thinking in syncretic images at the theory level.

3.2.2 Analysis procedures for snapshot #2

Snapshot #2 corresponds to the second part of the interview. Here, students were asked to find four limits that resembled routine tasks. More precisely, they resembled instances of mathematical praxeologies associated with types of tasks T_1 and T_3, but differed from them on the conceptual level. In the analysis, I considered students' solutions to each of the problems and the justifications they gave of their approaches. In particular, I checked how many students were approaching the tasks as if they were routine ones, what were the problems they encountered because of this, and whether or not they were able to deal with their misperceptions and misconceptions.

In this study, I distinguish "miscalculations", "misperceptions" and "misconceptions". "Misperception" refers to situations where the student fails to see one or more aspects of a limit expression. "Misconception" applies to cases where the student uses a technique or refers to an aspect of a limit expression or an explanatory discourse in a way that reflects a conceptual error. "Miscalculation" corresponds to making a non-conceptual mistake in a calculation. For example, consider the task of finding $\lim_{x \to 1} \frac{x^2 - 1}{x^2 + 3x + 2}$. Suppose that a student given this task claims that he or she would need to factor the polynomials and cancel common factors, and justifies this claim by stating that the expression is an instance of the zero over zero indetermination. Then,

- the student has a "misperception" if he or she does not calculate the substitution but infers – from other features in the expression – that the given expression is an instance of that type of indetermination;
- the student does a "miscalculation" if he or she does calculate the result of the substitution but makes a mistake in the addition or multiplication of numbers so that both the denominator and the numerator result in zero.
- the student has a "misconception" if he or she calculates the substitution correctly (obtaining 0/5) but still claims that to find the limit he or she would have to factor the expression and cancel common factors.

The distinction between miscalculations, misperceptions and misconceptions is essential in this research. As it will be discussed later, some misperceptions and misconceptions could be the result of the influence of routine tasks. Whether it is a misperception or a misconception throws light on the character of this influence: a student with a misperception is a student who knows but cannot *see*.

Based on students' spontaneous approaches to the tasks proposed in the second part of the interview and their responses to my questioning, I tried to identify what students perceive as the different types of "find-the-limit" tasks, and what are their corresponding explanatory discourses. With this information, I reconstructed students' praxeologies with respect to the task of finding limits of functions. These reconstructed praxeologies constitute the second partial model of students' models of the knowledge to be learned about limits in college level Calculus courses.

3.2.3 Analysis procedures for snapshot #3

In the third section of the interview, the students were asked to find three limits that did not resemble the routine tasks – the *essentially* non-routine problems.

First, I evaluated students' spontaneous performance on the three problems and then I evaluated their performance following my intervention. From students' spontaneous behavior, I built the third partial model of student's models of the knowledge to be learned in relation to the task of finding limits of functions. As in the case of the analysis of snapshot #2, I present this model in the form of a praxeology. Then, I contrast this model of spontaneous behavior with students' performances after my intervention.

3.2.4 Analysis procedures to infer the students' positioning in the Community-of-study institution from snapshots #2 and #3

As it was pointed out in Chapter 2, I have identified three possible students' positionings in the Community-of-study institution. As a criterion to decide in which position a student was, I considered an extension of the descriptions given in Chapter 2. This extension had the sole purpose of simplifying the identification of the students' positioning from their behavior in the tasks.

Student: A participant in this position abides by the rules and norms of the College-Calculus institution addressed to its students; in particular, he or she studies because there is a test coming, not out of disinterested curiosity or passion for the subject. A student in this position would take it for granted that he or she would be given routine tasks to solve. When presented with a non-routine task, he or she would tend to believe that the skills with which the institution provided him or her would suffice to deal with the task. Hence, a participant in this position would have spontaneous models to behave that would allow him or her to deal successfully with routine tasks. Someone in the position of a Student might complain about unfairness, but only on moral terms.

Client: A student takes the position of *Client* when he or she considers the final examination as a price to pay to attain other goals – passing the course, graduating, having a high grade point average, etc. A student in this position would take it as "part of the deal (or contract)" to be given only routine tasks. When given a non-routine task, he or she would not attempt to deal with it, considering that it is not his or her obligation to do so. Someone in the position of a Client might complain about unfairness as a violation of norms, interpreted as legalized rules.

Learner: A student is in the position of a *Learner* when he or she behaves as a cognitive subject interested in knowing Calculus; his or her goal in the institution is to learn. He or she would approach non-routine tasks as learning opportunities and would try to take advantage of them, being critical of his or her previous knowledge and of his or her actual thinking. His or her spontaneous models would not be strictly associated with routine tasks but with abstraction and generalization of concepts involved in those tasks.

Thus, the spontaneous models of behavior built by a participant are themselves an indication of his or her position in the institution. A participant who has a model that allows him or her to deal only with routine tasks is in the position of Student or Client; the difference would be in the participant's attitude towards non-routine tasks.

A participant in the position of Learner has a spontaneous model of behavior that transcends the routine tasks; his or her model is flexible enough to allow him or her to incorporate strategies as needed by the given task.

At the level of justification, Students are confident that the truth and consistency of "theoretical explanations" are safeguarded by the authority represented by the instructors or the mathematical community. If any, their explanatory discourses would evoke the instructor's or the textbook's discourses. Clients do not have to provide any justification.

If questioned, for example, why a technique is valid they would answer that it is not their obligation to know the reasons. Learners would try to justify everything they do, because for them, theoretical justification is an essential component of doing mathematics; furthermore, they would question the institution's explanations and their own, because critical thinking is, for them, a learning strategy.

It was by looking for the above-described patterns of behavior that I concluded what was each student's position in the Community-of-study institution.

3.3 OBJECTIVITY MEASURE

In an attempt to have a measure of the objectivity of the data analysis, a triangulation method was used. I was one vertex of this triangulation; another vertex was occupied by my supervisor; the third one – by a researcher in mathematics education completely external to this research. The chosen task was to read part one of the interviews and classify students' mode(s) of thinking according to the criteria described in Chapter 2, Section 2.4.

My supervisor and I read the twenty eight (28) interviews independently, and classified the students' mode(s) of thinking. In most cases, her assessment matched mine; discrepancies were resolved through discussion.

The other researcher was asked to analyze 5 students. He analyzed the data on his own, and provided me with a written report. Our communication was only in writing.

I provided him with the following documents:

1. excerpts of transcripts of five interviews, the excerpts corresponded to the first part of the interviews – the classification task;
2. the twenty expressions that students were asked to classify;
3. a description of the Vygotskyan modes of thinking.

Then, I gave the researcher the following instructions:

1. Read the given descriptions of the Vygotskyan modes of thinking.
2. Read each interview excerpt and analyze it to decide which Vygotskyan mode of thinking the student is using.
3. Write a short justification of your decision in each case.

The researcher's assessment matches exactly the modes of thinking that I inferred for the same students at the technology level. In addition, the particular behavior, from which the researcher and I inferred these modes, was similar in each case. I believe that the researcher focused on the modes of thinking used by the students to build the classes and decide membership (technology level) instead of considering the modes of thinking used by the students to justify the choice of this or that feature to build a class (theory level). Thus, our classification of students' mode(s) of thinking coincided at the level of technology. However, as it can be seen in Appendix B, in the only case – among those that were given to the researcher – in which I identified different modes of thinking at the technology and the theory level – the researcher noted that there could be a different mode of thinking involved in the student's motivation for building certain classes. This mode of thinking, that he considered involved in the 'motivation', matched the mode of thinking that I identified for the theory level of explanatory discourse; it is the mode of thinking that the student would use to answer a question of the type "why did you consider this or that feature as a key to build a class?".

Appendix B includes the exact documents provided to this researcher together with his complete assessment and a detailed discussion of how his assessment matched my analysis.

CHAPTER 4

MODEL OF INSTRUCTORS' SPONTANEOUS MODELS OF THE KNOWLEDGE TO BE LEARNED

> *S18: Basically I look at a problem and the first thing I see... and I always assume it is factorable, I mean, they never gave me a problem that wasn't factorable, so I wouldn't even ask whether it's factorable. I'd say, ok, where can I factor it. And I'd say ok let's look at the different categories. If I see a trinomial or a difference of squares and the method to factor them, and so long and so forth, but if it wasn't factorable... I never came across a problem that wasn't factorable.*

In this chapter, based on the official textbook, topics listed in the course outline, past final examinations from the last six years (2001-2007), and solutions for these examinations written by teachers and made available to students, I build a model of instructors' spontaneous models of the knowledge to be learned[15]. This model corresponds to instructors as participants of the Final-Examination institution; as participants of other institutions, instructors may have other models.

I start by characterizing the mathematical praxeologies related with the task of finding the limit of a function. I characterize them from the perspective of the Final-

[15] Recall that in the context of this research, knowledge to be learned is as a subset of the knowledge to be taught or of the knowledge actually taught. Its minimal core can be deduced from the assessment instruments. From the perspective of the Final-Examination institution, knowledge to be learned is identified with knowledge that students have to prove that they have acquired; that is, it is the knowledge tested in the final examination (see Chapter 2).

Examination institution within the College-Calculus institution. Each praxeology corresponds to a type of tasks presented in final examinations. The types of tasks are characterized based on instances appearing in final examinations. Techniques are described following teachers' solutions and the techniques presented in the textbooks. The description of the theoretical block [θ, Θ] is based on topics listed in the outline, and properties and theorems used in textbooks to justify techniques.

Then, based on those mathematical praxeologies, I build a model of instructors' spontaneous models of the knowledge to be learned. Here "instructors" refers to participants of the Final-Examination institution.

4.1 MATHEMATICAL PRAXEOLOGIES RELATIVE TO TASKS APPEARING IN FINAL EXAMINATIONS IN THE COLLEGE-CALCULUS INSTITUTION

By analyzing the final examinations, I identified the following three types of tasks.

TASK TYPE T1: Evaluate the following limit: $\lim_{x\to c}\frac{P(x)}{Q(x)}$.

Description: c is a fixed constant; $P(x)$ and $Q(x)$ are polynomials such that the factor $x - c$ occurs in both $P(x)$ and $Q(x)$; $x - c$ has degree one in $Q(x)$.

TASK TYPE T2: Evaluate the following limit: $\lim_{x\to c}\frac{\sqrt{P(x)} - Q(x)}{R(x)}$.

Description: $P(x)$, $Q(x)$ and $R(x)$ are polynomials such that $\sqrt{P(c)} - Q(c) = 0$, $R(c) = 0$, and the factor $P(x) - [Q(x)]^2$ has degree one in $R(x)$.

TASK TYPE T3: Evaluate the following limit: $\lim_{x\to\infty}\frac{P(x)}{Q(x)}$.

Description: *P(x)* and *Q(x)* are polynomials such that *m*, the degree of *P(x)*, is less or equal to *n*, the degree of *Q(x)*.

This is a formalization of the tasks described in chapter 3, section 3.1; they correspond to *routine* tasks.

Table 4.1 shows the occurrence of the types of tasks described above in the final examinations that I have analyzed. If an instance of a type of task occurred, that is the only instance of that type that appeared on the final examination; there were no examples of final examinations where two or more instances of the same type would be given.

	Task of type T_1 occurred	Task of type T_2 occurred	Task of type T_3 occurred
2001 Fall & Winter	Yes, m=3, n=2(*)	Yes	Yes, m = n
2003 Fall	Yes, m=3, n=2	Yes	Yes, m = n
2003 Winter	Yes, m=3, n=2	Yes	Yes, m < n
2004 Fall & Winter	Yes, m=3, n=2	Yes	Yes, m = n
2005 Winter	Yes, m=3, n=2	Yes	Yes, m = n
2005 Fall	Yes, m=3, n=3	No	Yes, m = n
2006 Fall & Winter	Yes, m=3, n=2	Yes	Yes, m = n
2007 Fall	Yes, m=4, n=2	Yes	No

Note: (*) m = deg *P(x)*, n = deg *Q(x)*

TABLE 4.1. Occurrence of routine tasks in final exams, and some characteristics of the expressions involved.

To identify the techniques associated with these tasks I considered two sources: the sections of the textbook listed in the outline of the course and the solutions of past final examinations written by teachers and made available to the students. For tasks of type T1 and T2, the considered sources are consistent in the sense that they provide the same techniques. These techniques are described below and are labeled $\tau 1$ and $\tau 2$, respectively.

TECHNIQUE τ1: Substitute c for x and recognize the indetermination 0/0 [16]. Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

EXAMPLE:

Task: Evaluate the following limit: $\lim\limits_{x \to 1} \frac{x^3 - 6x + 5}{x^2 - 6x + 5}$.

Expected solution:

(Substitution of 1 for x in the expression to check whether the indetermination 0/0 is the case is not expected in students' written solutions. Students are not penalized if there are no traces on paper of this verification).

$$\lim_{x \to 1} \frac{x^3 - 6x + 5}{x^2 - 6x + 5} = \lim_{x \to 1} \frac{(x-1)(x^2 + x - 5)}{(x-1)(x-5)} = \lim_{x \to 1} \frac{x^2 + x - 5}{x - 5} =$$
$$= \frac{1^2 + 1 - 5}{1 - 5} = \frac{-3}{-4} = \frac{3}{4}$$

TECHNIQUE τ2: Substitute c for x and recognize the indetermination 0/0. Multiply and divide by the conjugate of $\sqrt{P(c)} - Q(c)$. Factor out $P(x) - [Q(x)]^2$ from $R(x)$. Simplify and substitute c for x. The obtained value is the limit.

[16] The first step in τ1 appears in the textbooks when strategies of calculating limits are described in general. However, this step is omitted in most worked out examples in the textbooks and in solutions written by teachers and made available to students. The same is true for τ2.

EXAMPLE:

Task: Evaluate the following limit: $\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16}$.

Expected solution:

(Substitute 4 for x and recognize the indetermination *0/0*.)

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x^2-16} = \lim_{x\to 4}\frac{(\sqrt{x}-2)}{(x^2-16)}\frac{(\sqrt{x}+2)}{(\sqrt{x}+2)} = \lim_{x\to 4}\frac{x-4}{(x-4)(x+4)(\sqrt{x}+2)} =$$
$$= \lim_{x\to 4}\frac{1}{(x+4)(\sqrt{x}+2)} = \frac{1}{(4+4)(\sqrt{4}+2)} = \frac{1}{32}$$

It is important to observe that in the textbook there is no formalization of the techniques using mathematical symbols expressing generality. The techniques are only shown on particular instances of the types of tasks.

The presentation of the technique for type T3 – in the textbook and in the teachers' solutions of final examinations – requires special consideration. First, the textbook presents the solutions for several examples using the following approach[17]:

[17] This is not only the case in the particular textbook assigned to the Calculus course in the studied college. I have verified that the same approach is given in other three recent editions of North American college level Calculus textbooks.

TECHNIQUE τ3a: Divide both *P(x)* and *Q(x)* by x^n. Simplify each term and then use the algebraic properties of limits and the fact that the limit of a constant over a power of x, as $x \to \infty$, is 0.

EXAMPLE

Task: Evaluate the following limit: $\lim\limits_{x\to\infty} \dfrac{5x^4 - 2x + 1}{6x - 2x^4}$.

Expected solution:

$$\lim_{x\to\infty} \frac{5x^4 - 2x + 1}{6x - 2x^4} = \lim_{x\to\infty} \frac{\dfrac{5x^4 - 2x + 1}{x^4}}{\dfrac{6x - 2x^4}{x^4}} = \lim_{x\to\infty} \frac{5 - \dfrac{2}{x^3} + \dfrac{1}{x^4}}{\dfrac{6}{x^3} - 2} =$$

$$= \frac{\lim\limits_{x\to\infty} 5 - \lim\limits_{x\to\infty} \dfrac{2}{x^3} + \lim\limits_{x\to\infty} \dfrac{1}{x^4}}{\lim\limits_{x\to\infty} \dfrac{6}{x^3} - \lim\limits_{x\to\infty} 2} = \frac{5 - 0 + 0}{0 - 2} = -\frac{5}{2}$$

After several examples are solved this way, the textbook provides "guidelines" to find limits of rational functions at infinity. These "guidelines" state that it suffices to compare m and n, the degrees of the numerator and the denominator. If $m = n$, the value of the limit is the quotient between the leading coefficients of *P(x)* and *Q(x)*. If $m < n$, the value of the limit is 0. The guidelines are a generalization of the examples shown before. The mathematical proof of the generalization is not given and it is not indicated that such a proof is needed. Furthermore, no application of these "guidelines" is shown[18]. This, as it will be discussed later on Chapter 6, disrupts the didactic flow of the textbook – although the "guidelines" are stated, they are not used on examples in the way that other techniques are used.

In the teachers' solutions of final examinations, made available to the students, I have found two different approaches. One is the same as the one found in the textbook: technique τ3a (although the step in which the limit is distributed is skipped). The other approach is described below.

TECHNIQUE τ3b: Factor x^m from *P(x)* and x^n from *Q(x)*, and simplify $\frac{x^m}{x^n}$ to get $\frac{1}{x^{n-m}}$. Use the fact that the limit of a constant over a power of x, as $x \to \infty$, is 0.

EXAMPLE

Task: Evaluate the following limit: $\lim_{x\to\infty} \frac{5x^4 - 2x + 1}{6x - 2x^4}$.

Expected solution:

[18] Only one of the other three textbooks presents these "guidelines" in the exact same way as described here. The other two only show the approach described as technique τ3a.

$$\lim_{x\to\infty}\frac{5x^4-2x+1}{6x-2x^4}=\lim_{x\to\infty}\frac{x^4\left(5-\frac{2}{x^3}+\frac{1}{x^4}\right)}{x^4\left(\frac{6}{x^3}-2\right)}=\lim_{x\to\infty}\frac{\left(5-\frac{2}{x^3}+\frac{1}{x^4}\right)}{\left(\frac{6}{x^3}-2\right)}=-\frac{5}{2}$$

From this, it is reasonable to infer that the "guidelines" provided by the textbook do not form part of the knowledge to be learned (from the perspective of the Final-Examination institution). It is not expected that students will use these guidelines in solving the limit tasks proposed in the final examination.

On the other hand, from the fact that those "guidelines" are in a section listed in the outline of the course, it is also reasonable to infer that they belong to the knowledge to be taught. In this research, I am not considering data that would allow me to say whether this knowledge is actually taught or not. It *is,* however, knowledge made available to the students by the College-Calculus institution (the section where these guidelines appear is listed in the outline of the course). Once the "guidelines" are known, the techniques τ3a and τ3b become proof techniques: they show in each particular case that the "guidelines" are valid. These techniques are a way of solving, for example, a task of the type *prove that* $\lim_{x\to\infty}\frac{5x^4-2x+1}{6x-2x^4}=-\frac{5}{2}$.

At it will be explained in more detail in Chapter 6, from a mathematical point of view, there is an essential difference between the techniques τ1/τ2 and the techniques τ3a/b. While the techniques τ1 and τ2 are essential to find the limit in any given instance of T1 and T2, respectively, the techniques τ3a/b are not necessary – once the guidelines are known.

Considering the knowledge made available to the students in the textbook (as indicated in the outline of the course), I surmise that, from the perspective of the College-Calculus institution, the technique to solve the type of tasks T3 corresponds to those "guidelines", while what was labeled above as τ3a and τ3b are part of the justification of the technique, i.e., part of the technology. To complete the explanatory discourses, technology and theory, I considered the justifications appearing in the textbook (there are no justifications provided in the teachers' solutions of final examinations).

Hence, from the perspective of the College-Calculus institution, I have characterized the following mathematical praxeologies in relation to the tasks appearing in final examinations.

Mathematical praxeology 1 (MP1)

TASK TYPE T1 – as described above.

TECHNIQUE τ1 – as described above.

TECHNOLOGY θ1: If two functions *f* and *g* agree in all but one value *c* then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$. If *r(x)* is a rational function and *c* is a real number such that *r(c)* exists, then $\lim_{x\to c} r(x) = r(c)$.

THEORY Θ1: A graph supports the fact that two functions agreeing in all but one point have the same limit behavior. The ε-δ definition of limits taken at a constant; ε-δ proofs of both statements in θ1. Algebraic properties of limits; proofs of these properties.

Mathematical praxeology 2 (MP2)

TASK TYPE T2 – as described above.

TECHNIQUE τ2 – as described above.

TECHNOLOGY θ2: If two functions f and g agree in all but one value c then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$. If n is a positive integer and c is a real number, then $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$ for all c if n is odd, and for all non-negative c if n is even.

THEORY Θ2: A graph supports the fact that two functions agreeing in all but one point have the same limit behavior. The ε-δ definition of limits taken at a constant; ε-δ proofs of both statements in θ2. Algebraic properties of limits; proof of these properties.

[Mathematical] praxeology 3 ([M]P3)

TASK TYPE T3 – as described above.

TECHNIQUE τ3: (The "guidelines" appearing in the textbook.) Let $P(x)$ be the numerator, with leading coefficient a_m, and $Q(x)$ be the denominator, with leading coefficient b_m. Compare m and n, the degrees of $P(x)$ and $Q(x)$, respectively. If $m = n$, the value of the limit is a_n/b_n. If $m < n$, the value of the limit is 0.

TECHNOLOGY θ3: The limit of a constant over a positive power of x, as $x \to \infty$, is 0. Several examples, employing technique τ3a. (This technique could be formalized into a mathematical proof of the "guidelines": Let $P(x) = \sum_{i=0}^{m} a_i x^i$ be the numerator of the rational function, such that $a_m \neq 0$, and let $Q(x) = \sum_{i=0}^{n} b_i x^i$ be the denominator, such that $b_n \neq 0$. Assume that $m \leq n$. Then

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{i=0}^{n} b_i x^i} = \lim_{x\to\infty} \frac{\sum_{i=0}^{m} \frac{a_i x^i}{x^n}}{\sum_{i=0}^{n} \frac{b_i x^i}{x^n}} = \lim_{x\to\infty} \frac{\frac{a_m}{x^{n-m}} + \varepsilon}{b_n + \delta} = \begin{cases} 0 & \textit{if } m < n \\ \frac{a_m}{b_n} & \textit{if } m = n \end{cases}$$

Where ε and δ tend to 0 by virtue of the fact that the limit of a constant over a positive power of x, as $x \to \infty$, is 0.)

THEORY Θ3: The ε-*N* definition of limits at infinity; ε-*N* proof of the fact that the limit of a constant over a positive power of x, as $x \to \infty$, is 0. Algebraic properties of limits; proofs of these properties.

In the case of MP1 and MP2, the technologies are presented in the textbook in the same section where the techniques are discussed. Furthermore, a fragment of the theories also appears in this section: the graph showing the intuitive idea that two functions agreeing in all but one point must have the same limit behavior, and the algebraic properties of limits. All this belongs to the knowledge to be taught. The rest of the theories are in sections that are not listed in the course outline; this portion of the theories belongs to scholarly knowledge.

In the case of [M]P3, the only fragments of the technology present in the textbook are: the statement that the limit of a constant over a positive power of x, as $x \to \infty$, is 0, and examples of applications of τ3a. This belongs to the knowledge to be taught. It is not said that the examples support the validity of the technique τ3, and it is not mentioned that this technique is a generalization of the examples. The formalization of τ3a into a proof of τ3 does not appear in the textbook. This constitutes a weak form of technology, from a mathematical point of view, because the explanatory discourse is not a *mathematical explanatory discourse*. This explains the brackets in the name of the praxeology. On the theory level, the formal definition of limits taken at infinity appears in the same section where the examples of instances of T3 are given with their solutions (using τ3a). The algebraic properties of limits are shown in the chapter where limits taken at a constant are presented (the section where T1, T2 and their respective technologies are shown). The rest of the theory appears in a section that is not listed in the course outline (scholarly knowledge).

4.2 A MODEL OF INSTRUCTORS' MODELS OF THE KNOWLEDGE TO BE LEARNED

Now I consider the knowledge to be learned from the point of view of the Final-Examination institution and derive a model of instructors' models of this knowledge.

Instructors' solutions contain no explanatory discourses. It can be concluded, therefore, that instructors' spontaneous models of the knowledge to be learned consist only of tasks and techniques, and the corresponding theoretical blocks are not included. They form part of the knowledge to be taught but not of the knowledge to be learned. From this, we can reconstruct a theoretical model of instructors' models of the knowledge to be learned (instructors as participants of the Final-Examination institution). This model consists of the types of tasks T1, T2 and T3 (*routine tasks*) and the corresponding techniques τ1, τ2, τ3a and τ3b (*routine techniques*). I refer to it as PBs – for "practical blocks" (fragments of praxeologies in which the theoretical blocks are missing) – and in particular, as PB1, PB2, PB3a and PB3b. In chapter 6, I discuss the possible implications of

(a) the differences between the mathematical praxeologies MP1, MP2 and [M]P3 and these practical blocks, and

(b) the institutional use of τ3a and τ3b as techniques to accomplish tasks of type T3.

As a preview of this discussion, I will make three key observations. Firstly, the model consisting of the types of tasks T1, T2 and T3 and the techniques τ1, τ2, τ3a and τ3b does not correspond to a complete praxeology anymore because the theoretical block is missing. Secondly, this model, in particular the block PB3a/b, hardly corresponds to a mathematical practice, as the techniques τ3a and τ3b are illustrations of mathematical explanatory discourses at the technology level, not mathematical techniques. Finally, the occurrence of tasks T1, T2 and T3 in final examinations is not institutionalized in the sense discussed in chapter 2, section 2.1. On the one hand, the College-Calculus institution does not have explicit rules stating that these types of tasks *have to* appear in final examinations. Their occurrence is based on tradition and the shared idea that this is the minimum knowledge that students should learn. Teachers usually refer to these types of tasks as "the least common denominator of what is taught in our courses". Thus, the occurrence of these tasks is the result of a practice regulated by norms, not by rules. On the other hand, although the committee preparing the final examination also prepares a grading scheme, there are no sanctions for not following it to the letter. It is only a suggestion and the final decisions about the grades are left to the discretion of the instructors. This implies that the Final-Examination institution considers the techniques as norms, not as rules. Therefore, the practical blocks (PBs) defined above, are regulated by norms, not by rules. Hence, they do not define an *institutionalized* practice. In other words, the model of the models of the knowledge to be learned of the instructors as participants of the Final-Examination institution is not a model of an institutionalized practice, but of a *normal* practice. I will argue later that this difference is quite sharp for the participants, and could be at the basis of students' positioning in the Community-of-study institution.

CHAPTER 5

MODELS OF STUDENTS' SPONTANEOUS MODELS OF THE KNOWLEDGE TO BE LEARNED

> *S7: Well... I do not know... for me... because most of the exercises that we were given, every time that you'd replace it'd give you zero over zero, so it is kind of a reflex. (Student S7 explaining why she factored an expression although later on she recognized that this factoring was not needed.)*

In this chapter, I analyze students' behavior in the tasks proposed in the interviews. As I explained in Chapter 3, I divided the interview into three parts corresponding to different tasks: a classification task, a set of typical looking limits tasks, and a set of non-routine limits task. Each part offered a snapshot of students' behavior in front of tasks related with limits. From the first snapshot, I infer partial, individual models for each student's spontaneous model of the knowledge to be learned in relation to the task of finding limits. From the second snapshot, I build a partial model of students' spontaneous models to deal with tasks that resembles routine tasks. From the third snapshot, I build a partial model of students' spontaneous models to deal with tasks that do not resemble routine tasks. Each of these models is presented in the form of a praxeology which I associate, in the case of snapshot 1, with the student's mode – or modes – of thinking and, in the cases of the second and third snapshot, with the general, most frequent, students' positioning in the institution Community-of-study. In section 5.4, I put the partial models together to build a general model of students' spontaneous models of the knowledge to be learned with respect to the task of finding limits of functions.

5.1 SNAPSHOT #1: PARTIAL MODEL BASED ON A CLASSIFICATION TASK

In this section, I present and analyze students' behavior in the classification task. From their techniques to accomplish this task and from their explanatory discourses, I inferred their praxeological organizations to deal with the task of finding limits of functions, their mode(s) of thinking about their techniques, and their positioning in the Community-of-study institution. These praxeologies constitute the first partial model of students' spontaneous models of the knowledge to be learned. They provide the first hint that students' models are not purely mathematical, but a mixture of social, cognitive, didactic and mathematical norms.

I briefly recapitulate what I explained in Chapter 3, *Methodology and research procedures*, to facilitate the reading and understanding of this section. The task that I analyze here was the same for all students: *Classify the* 20 *cards according to a rule of your choice*. To better understand students' techniques to accomplish this task, I consider the general scheme of the expressions in the cards plus two boxes, one of which (Box # 3) corresponds to the arithmetic outcome of direct substitution (in $\boldsymbol{R} \cup \{+\infty, -\infty\}$), and the other (Box # 4) – to the value of the limit (see Figure 5.1). The technique that a student uses to classify the expressions is based on where his or her focus of attention is placed with respect to this schema.

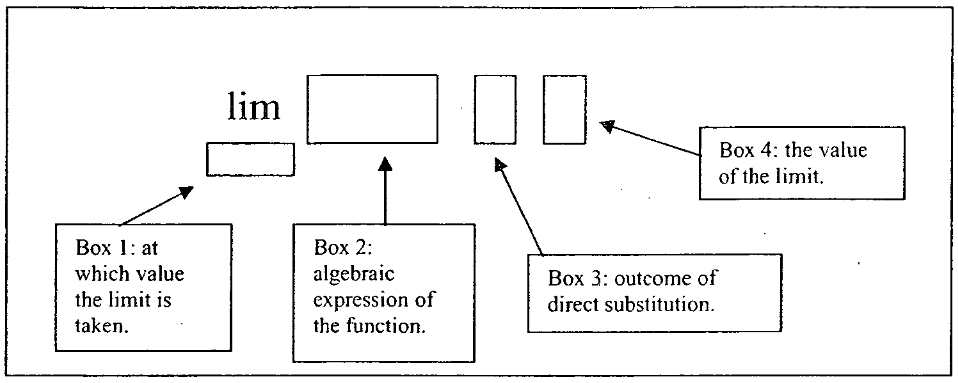


FIGURE 5.1 (Copy of Figure 3.1.). A scheme of limit expression.

I characterized students' techniques as belonging to Arithmetic, to Algebra, to Calculus, or to Analysis. An Arithmetic technique would often be associated with the student's focusing only on boxes 2 and 3, while a Calculus technique – with focusing only on box 1. There were many subtleties, however, and these could be only grasped when analyzing students' own explanations of their techniques. Hence, for example, a student who focused on box 4 could have been using a Calculus technique or an Analysis technique, and only the student's discourse could give an indication of which was the case. Furthermore, a student who focused, for example, on boxes 1 and 2, could have combined them to build a class or consider them independently to build different classes, and this gives information on the student's mode of thinking. For example, if a student was focusing on one box to build one class and on a different box to build another class, he or she was clearly employing a complexive mode of thinking.

In Chapter 2 (page 42) I described the criteria used in this research to decide which mode of thinking a student was using; these criteria have been derived from Sierpinska's (1994) interpretation of Vygotsky's theory of concept development.

In the next section, I detail the classification that each student proposed, with a verbatim reproduction of the phrases he or she used to describe the classes. Then, based on each student's classification, his or her techniques to accomplish the classification task and his or her explanatory discourse, I reconstruct the student's praxeological organization to deal with limits.

Of course, my analysis relied on the information that I could extract from the interviews. As it will be clear below, this information varied considerably from one student to another, and so did my level of analysis. In particular, for some students, I was able to conjecture on their theories, that is, the second level of justification, from the point of view of ATD. For many students, however, I did not have sufficient ground for conjecture, and then I could only discuss the first level of justification, the level of technology. A summary of the analysis of the students' performance in the classification task will follow in section 5.1.2.

5.1.1 Analysis of individual student's performance in the classification task

In this section, I present the reconstructed praxeologies of each of the 28 interviewed students with a discussion justifying each reconstruction. The presentation is divided into six categories, based on the mathematical domain to which the students' techniques apparently belonged: Algebra, Calculus, Arithmetic, a mixture of Arithmetic and Calculus, a mixture of Algebra and Calculus, and Analysis.

As it was defined in Chapter 4 (page 86) the notation PB1, PB2 and PB3a/b refers to the practical blocks that model the spontaneous models of the knowledge to be learned about limits of the instructors as participants of the Final-Examination institution. I use the notation "PBs" to refer to the model itself. To facilitate the reading of this section, I briefly recall each PB.

PB1

TASK TYPE T1: Evaluate the following limit: $\lim_{x \to c} \frac{P(x)}{Q(x)}$.

Description: c is a fixed constant; $P(x)$ and $Q(x)$ are polynomials such that the factor $x-c$ occurs in both $P(x)$ and $Q(x)$; $x-c$ has degree one in $Q(x)$.

TECHNIQUE τ1: Substitute c for x and recognize the indetermination 0/0 [19]. Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

[19] The first step in τ1 appears in the textbooks when strategies of calculating limits are described in general. However, this step is omitted in most worked out examples in the textbooks and in solutions written by teachers and made available to students. The same is true for τ2.

PB2

TASK TYPE T2: Evaluate the following limit: $\lim_{x \to c} \frac{\sqrt{P(x)} - Q(x)}{R(x)}$.

Description: *P(x)*, *Q(x)* and *R(x)* are polynomials such that $\sqrt{P(c)} - Q(c) = 0$, *R(c)* = 0, and the factor $P(x) - [Q(x)]^2$ has degree one in *R(x)*.

TECHNIQUE τ2: Substitute c in x and recognize the indetermination 0/0. Multiply and divide by the conjugate of $\sqrt{P(c)} - Q(c)$. Factor out $P(x) - [Q(x)]^2$ from *R(x)*. Simplify and substitute c for x. The obtained value is the limit.

PB3a/b

TASK TYPE T3: Evaluate the following limit: $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$.

Description: *P(x)* and *Q(x)* are polynomials such that m, the degree of *P(x)*, is less or equal to n, the degree of *Q(x)*.

TECHNIQUE τ3a: Divide both *P(x)* and *Q(x)* by x^n. Simplify each term and then use the algebraic properties of limits and the fact that the limit of a constant over a power of x, as $x \to \infty$, is 0.

TECHNIQUE τ3b: Factor x^m from *P(x)* and x^n from *Q(x)*, and simplify $\frac{x^m}{x^n}$ to get $\frac{1}{x^{n-m}}$. Use the fact that the limit of a constant over a power of x, as $x \rightarrow \infty$, is 0.

5.1.1.1 Students whose technique belongs to the domain of Algebra

In the analysis of student S1's performance in the classification task, I explain the way in which I have interpreted students' behavior and discourses to infer their praxeologies in front of the task of finding limits, their mode(s) of thinking and their positioning in the Community-of-study institution. In this sense, the reading of this first analysis is the key to understanding the other 27 analyses.

Student S1[20]

The student's classification of the limit expressions given in the first part of the interview is shown in Table 5.1a.

[20] Students were labelled S1 to S28; the numbers 1 to 28 were given at random, they reflect neither the order in which students were interviewed nor the order in which the interviews were analyzed.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?"
1	2, 5, 9, 12, 17	Difference of squares.
2	6, 7	Constants.
3	19, 20	With trigs. That confuses me.
4	3, 10, 13, 14, 18	With square roots.
5	1, 4, 8, 11, 15, 16	Polynomials.

TABLE 5.1a. Student S1's classification.

In describing class 1, the student used the phrase "difference of squares" to refer to rational functions that contained a difference of squares. When she said "with trigs" to describe class 3, she was referring to expressions containing the sine function. When she said "with square roots" to describe class 4, she was referring to rational expressions containing square radicals. Finally, she used the phrase "polynomials" to refer to rational functions.

Technique. The student focused on box 2 to make her classification. In particular, she considered some features of the algebraic form of the function to decide about the membership of an expression to one class or another. She was considering features that were purely algebraic and not related to the algebra of Calculus. Hence, I concluded that her technique belonged to the domain of Algebra.

Technology. The justification of the technique, i.e., the phrases that the student used to describe each class, evoked typical topics of school Algebra textbooks: constants, difference of squares, trigonometric functions, rational functions, rational expressions with radicals. The phrases that she used to "name" the classes constitute themselves the immediate explanatory discourse about her technique, i.e., her technology. I interpret the discourse as, "I placed expression 19 in class 5 because it has a trigonometric function in it"; or "I placed expression 13 in class 3 because there's a square root in it". I do *not* interpret it as, for example, "class 5 corresponds to quotients of polynomials such that none of them is a difference of squares". Although one might believe that this is what she meant, one can only infer it from looking at the members of the classes, not from the phrases or "names" she used; one would be changing the "name" or description of the class. Moreover, one might want to go further in completing the ellipsis in her discourse, saying, for example, that she meant, "class 5 corresponds to quotients of polynomials such that the numerator is not a constant and none of the polynomials is a difference of squares". However, the "name" or description is an essential component in Vygotsky's definition of stages of concept development and modes of thinking. The mode of thinking depends not only on how an individual sorts a certain collection of objects, but also on his or her criteria of this classification, and these criteria can be only inferred from his or her justifications. Another person given the same objects and asked to classify them based only on student S1's descriptions would not necessarily be able to complete the elliptic discourse and might struggle to decide on the membership of some of the objects. In the particular case of the classification done by student S1, another person might start by trying to classify expression 8 and place it in class 5 because it is a polynomial. Then this person might take expression 17 and place it in class 1 because both the numerator and the denominator are differences of squares. However, when considering, for example, expression 9 he or she might hesitate in prioritizing the feature "numerator is a difference of squares" over the feature "both numerator and denominator are polynomials". These considerations led me to claim that student S1's technology is based on complexive thinking.

Theory. While explaining what her classes were and why she had placed this or that object in a class, the student said, in reference to class 3: "Because that [trigonometric functions] confuses me, I put them together". I analyze the levels of her explanatory discourse as follows. The sentence "these are with trigs" is an answer to, "what is the description of the class containing objects 19 and 20?". This belongs to the level of technology. The sentence "because that confuses me, I put them together" is an answer to "why did you consider the feature 'trigonometric function' as a key to build a class?". This is an explanatory discourse at the next level of justification, the level of theory. Her statement is of syncretic nature, because it refers to an affective relation that she has with this particular feature. Her short statement about trigonometric functions could also point to a reason why she put the rational functions with a "difference of squares" apart from her "polynomials". First, the "difference of squares" is a topic given a specially emphasis in school Algebra textbooks and is treated separately from the chapter on polynomials in general and even separately from the chapter on quadratic functions. Moreover, if one considers these expressions as tasks to be performed, and the operation to be done is factoring, then this student could have taken into account the fact that a difference of squares is easier to factor than an arbitrary trinomial or polynomial. All this points to an affective context of learning. At the level of theory, she justified her behavior based on her affective relation with the tasks to be performed. Hence, I claim that student S1 was *thinking in syncretic images at the theory level of justification.*

Based on this analysis, I construct a model of student S1's praxeology in front of the task of finding limits of functions. I propose that the student's technique belongs to Algebra. This implies, in particular, that she would decide which technique to apply in a problem upon, exclusively, the algebraic form of the function. Next, I propose that the student's technology, or her explanatory discourse about the technique, is based on her previous algebraic knowledge with a vocabulary and system of concepts typical of high school Algebra. As for her theory, it appears to be based on her affective rapport with the different problems she has had to do as a student. It is important to notice that this praxeology does not qualify as a mathematical praxeology because the theoretical block is not of mathematical nature.

The above model is summarized in Table 5.1b.

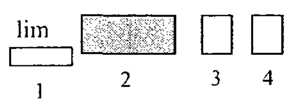
Technique	Technology	Theory
	The technology evokes school Algebra textbook categories – this might be reinforced by the PBs.	Affective context of learning.
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on syncretic thinking.

TABLE 5.1b. A model of student S1's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S7

Student S7's classification is very similar to that of student S1. Both students used the phrase "polynomials" to describe the class in which they had placed rational functions. In the class described by the phrase "by replacing", S7 put expressions with constant functions, 6 and 7, and the expression with a polynomial, 8.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	With sine.
2	3, 10, 13, 14, 18	Contain square roots.
3	1, 2, 4, 5, 9, 11, 12, 15, 16, 17	Polynomials.
4	6, 7, 8	By replacing.

TABLE 5.7a. Student S7's classification.

Technique. The student focused on box 2 to make her classification. This could mean that in front of the task of finding limits, the algebraic form of the function would be a decisive feature for her approach.

Technology. The first three classes evoke topics of a school Algebra textbook. The student chose to describe her fourth class using a phrase evoking the method of direct substitution. However, the name was not well chosen because the technique applies to only one of these items, namely 8. The expressions 6 and 7 correspond to limits of constant functions where the notion of "substitution" does not make sense; moreover, expression 6 is a limit taken at infinity. On the other hand, the technique applies to several other items, which she has not included in the "By replacing" class. The student may have failed to study the items deeply enough to notice these facts (misperception), but it may be also a result of some misconception in relation to the technique of substitution. Her classification is based on complexive thinking because the classifying features shifted from the algebraic form of the function (e.g. "contain square roots") to the technique to be used to find the limits (e.g. "by replacing").

As it was the case with student S1, we cannot ignore that S7 was aware that these were limit expressions. She used the phrase "by replacing" to describe class 4, showing that she recognized the expressions as tasks to be performed and giving some sense to the symbol $x \rightarrow a$. I believe that her misperceptions could have been based on her habits in dealing with routine problems, those in the PBs.

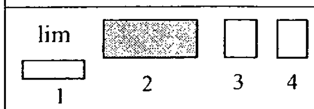
Technique	Technology	Theory
	The technology refers to school Algebra textbook categories and the technique of direct substitution, with misconceptions and, perhaps, misperceptions – influenced by the PBs.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.7b. A model of student S7's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S8

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 10, 13, 18	Rationalization. That's the first thing I think of. I have to do that first.
2	1, 2, 4, 5, 9, 11, 12, 15, 16, 17	These I have to factor first. Then see if something cancels out.
3	19, 20	These are trig ones.
4	6, 7, 8	These you can just plug in.
5	14	And this I am not sure.

TABLE 5.8a. Student S8's classification.

The phrase "these are trig ones" refers to the expressions containing the sine function. In her description of class 5, I believe she meant that she was not sure about the value of the limit and of how to find it.

Technique. The student focused her attention exclusively on box 2, the algebraic form of the functions, and, based on this form, classified the items according to the technique she believed was necessary to find the limit. Despite of this, her technique was purely algebraic: she failed to notice or to interpret the information in box 1.

Technology. It appears that, in the student's mind, these expressions are tasks to be done. From her experience with limits of functions, it seems that she retained two approaches: factoring-cancelling-plugging in and rationalization (these correspond to PB1 and PB2, respectively). Quotients with radicals were placed in class 1, the class of the rationalization technique, although for expressions 10, 13 and 18 this technique is not helpful at all. Rational functions were placed in class 2, the class of the factoring technique, which is not useful in items 4, 5, 11, and 12, where limits are taken at infinity. Item 9 was also put in class 2 although the limit can be found by direct substitution. Also items 15, 16 and 17 are in class 2; here, the factoring technique is not helpful because direct substitution gives a non-zero number over zero. Expressions 6, 7 and 8, in which there are no radicals (class 1) and there is no need to factor because there are no visible denominators – and hence nothing would cancel out (class 2) – were placed in the class whose name ("plug in") evokes the direct substitution technique. Student S7 also had class {6, 7, 8} and named it with a term evoking substitution. Class 3, which S7 named "polynomials", matches exactly student S8's class 2; S8 described it as "These I have to factor first...". It may be the case that when student S7 made her "polynomials" class she also had in mind, as S8, a common technique to find those limits. This would reinforce the conjecture that S7 misperceptions were based on her habits in dealing with routine problems, those in the PBs – see discussion above.

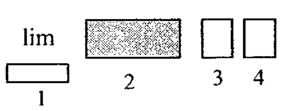
Technique	Technology	Theory
	The technology refers to routine techniques to calculate routine limits, with misconceptions and misperceptions – influenced by the PBs.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.8b. A model of student S8's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Expression 14 contains radicals but no visible denominator. This, I believe, is the reason why she did not place it in class 1, for her the rationalization technique does not apply in that case. Finally, she placed together, in a class of their own, the trigonometric functions, as many other students did.

From this interpretation of the student's behavior, I conclude that she was thinking in complexes. Firstly, to decide which technique should be applied to find the limit, she was using a feature – the information in box 2 – that is not *per se* informative about this matter. Secondly, another person would not be able to reconstruct her classification based on the phrases she provided. For example, one would place expressions 9 and 10 in class 4 because both limits can be found by "just plugging in", and would not know where to place expressions 4, 5, 11, and 12 as they correspond to limits of rational functions taken at infinity.

Student S9

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 4, 9, 17	Numerator and denominator have the same degree.
2	1, 11, 12, 15	Upper is more than lower.
3	5, 16	Upper is less than lower.
4	8, 14	Polynomial.
5	3, 10, 13, 18	Upper is less than lower and it's not an integer.
6	19, 20	Sines.
7	6, 7	Constants.

TABLE 5.9a. Student S9's classification.

The phrases "upper is more than lower" and "upper is less than lower" refer to the relation between the degrees of the polynomials in the numerator (upper) and the denominator (lower) of the rational expressions.

Technique. The student focused exclusively on box 2 to build her classes. Hence, it could be inferred that, if she has a technique for solving limits, it is likely to belong to Algebra.

Technology. The student first explanatory discourse refers to algebraic features. In this sense, she might be employing a conceptual mode of thinking – but only in relation to concepts of Algebra, not of Calculus.

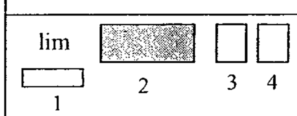
Technique	Technology	Theory
	The technology refers to algebraic features.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on conceptual thinking – in relation to concepts of Algebra, not of Calculus.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.9b. A model of student S9's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S10

Class	Members of class *(labels refer to Table 3.3, p. xx).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 6, 7, 8, 10, 12, 13, 18, 19	...easy...*
2	1, 2, 4, 11, 16	...factoring...
3	5, 9, 14, 15, 17, 20	...not sure...

(*) The student spoke a lot while doing the classification task; the words mentioned in the table are those he used many times in describing the elements of the class.

TABLE 5.10a. Student S10's classification.

The student took all the cards together in his hands and went through them one by one. He was then placing them in one of three piles: the pile of those where the limit is "easy" to find; the pile of the ones you have to factor to find the limit; and the pile of those where he was not sure how to find the limit. The phrases "easy", "factoring" and "not sure" that I wrote in Table 5.10a are those which appeared most often in the flow of his verbose speech (see page 296-299). Here are some excerpts of the interview showing the type of phrases he was using:

> S10: [Talking about expression 1.] The top is a little odd but you know... it's a little odd because there's usually an x square right in front but I'm pretty sure that yeah I think it would work if you have something that cancels out like a minus two x or something like that... S10: [Talking about expression 8.] [...] this one seems like a bit of a trick, as if it is too easy, like x goes to one you know and there's nothing even under it. It's just...
>
> S10: [Talking about expression 2.] Now this one is easy, because this would be x plus one times x minus one and you can already cancel something out... yeah, you would get something that would work fine.

S10: [Talking about expression 5.] I don't know, I haven't done enough problems of x cube minus one.

Technique. At first, the student considered the possibility of taking into account the information in box 1. He said: "But I don't know maybe I should take into account the limit, so x is going to infinity, x is going to three". But immediately he discarded this possibility: "[...] but like usually for something like that it's better just to try to... because everything here is kind of scattered so you know... It's probably best to simplify everything and then solve it, you know". Then he focused on the information given in box 2. Going through the different expressions, he was referring to the techniques he would use in each case to find the limits: factoring, substitution, partial fractions, long division, L'Hôpital's rule. His choice of these techniques was guided by the algebraic form of the function and recognition of familiarity with other limits that he remembered having dealt with. Since he was explicitly referring to the techniques he would use, I assume that his technique to find limits belongs to the domain of Algebra.

Technology. In the classification task, based on a recognition of familiarity with the content of box 2, student S10 made choices about the techniques he would use to find each limit. When he decided that the technique to be applied was factoring, he placed the cards in class 2, when the technique was not factoring, he mentioned some other techniques, such as partial fractions, long division, and L'Hôpital's rule, but he placed those cards in class 1 or 3. Some misconceptions can be found in his reference to using partial fractions to find limits (probably influenced by the fact that he was taking

Calculus II and partial fractions is a routine technique to find antiderivatives), or in his idea of using factoring for limits taken at infinity (e.g., for expression 4). His technology is based on syncretic thinking because he used affective aspects such as easiness and lack of certainty as his classifying features; these aspects are not intrinsic to the classified objects but to his personal appreciation.

From this student's classification, it is hard to understand what is "easy" and what is not for him. For example, he placed item 12 (an infinity over infinity indetermination) in class 1 (the "easy" class), but he put item 4 (another infinity over infinity indetermination) in class 2 (the "factoring" class), despite the fact that polynomials in expression 12 are a lot easier to factor (they are $x^2 - 1$ and $x - 1$) than those in expression 4 ($9x^3 - x + 2$ and $3x^3 + 1$).

However, while classifying, he was most of the time referring to standard Calculus techniques (though not all of them related to finding limits of functions). In addition, from some excerpts of his interview, it seems that he was forming chains of complexes. In some cases, his observations about an expression were first influenced by the expression he had just seen. It is not that his technology in the *classification task* is based on complexive thinking, but I conjecture that his technology on the *task of finding limits* would be based on complexive thinking, shifting from one feature of an expression to another and being influenced by the immediately preceding task.

Theory. The student's explanatory discourse would be reflecting thinking in syncretic images, as the shifting on his attention is based on affective impressions.

Technique	Technology	Theory
$\lim\limits_{\underset{1}{\square}} \underset{2}{\boxed{}}\ \underset{3}{\square}\ \underset{4}{\square}$	Calculus standard and non-standard techniques to find limits, with misconceptions.	Affective impressions about expression in box 2.
The technique belongs to Algebra.	The justification of the technique is based on syncretic images.	The justification of the technology is based on syncretic thinking.

TABLE 5.10b. A model of student S10's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S12

Class	Members of class *(labels refer to Table 3.3, p. xx).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Constants.
2	1, 2, 4, 5, 8, 9, 11, 12, 15, 16, 17	Polynomials. Either divide by x or cancel out. [...] These are the ones you have to divide by x. [...] These either you divide by x or cancel out.
3	3, 10, 13, 14, 18	Roots.
4	19, 20	[No description was given.]

TABLE 5.12a. Student S12's classification.

As many other students, S12 used the word "polynomials" to refer to rational functions. Then she used the word "roots" to refer to rational expressions with radicals. Expressions with the sine function were put a class of their own, and not described at all.

Technology. Hence, her first explanatory discourse evokes topics of school Algebra textbooks: constants, rational functions, rational expressions with radicals, trigonometric functions. In this sense, she might be employing a pseudoconceptual mode of thinking – but only in relation to concepts of Algebra, not of Calculus, as it was the case with student S9.

Technique. The student's technique was purely algebraic as she was considering only the information in box 2 to build her classification.

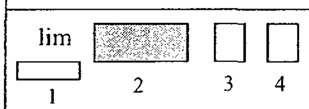
Technique	Technology	Theory
	The technology evokes topics on a school Algebra textbook.	Techniques for finding limits of routine functions, with misconceptions – influenced by the PBs.
The technique belongs to Algebra.	The justification of the technique is based on pseudoconceptual thinking – in relation to concepts of Algebra, not of Calculus.	The justification of the technology is based on complexive thinking.

TABLE 5.12b. A model of student S12's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Theory. I infer that, in her description of class 2, she was referring to techniques to find limits, such as those involved in the PBs. To "divide by *x*" would be a reference to the technique to find limits of rational functions at infinity, and "cancel out" would be a reference to the factoring-cancelling-substituting technique to find limits of rational functions that are instances of the zero over zero type of indetermination. From this perspective, her theory would contain several misconceptions because it would fail to explain membership of most of the objects placed in classes 2 and 3. At this level of explanatory discourse, she was employing a complexive mode of thinking. Although she was aware that the techniques she mentioned did not apply to some of the objects in class 2, and thus that her description applied only to some of the objects but not necessarily to all of them, she did not want to change her arrangement. This is a clear symptom of complexive thinking.

Student S15

Student S15's classification is almost the same as that made by S9 – the differences are that S15 considered expressions 8 and 14 as rational expressions while it seems that S9 did not, and that S15 did not distinguish between integer and fractional exponents as S9 did. Nevertheless, it is also the case with S15, as it was with S9, that the student's discourse does not allow me to infer what his technology or theory would be when dealing with the task of finding limits. I infer that if he has a technique for such task, it belongs to the domain of Algebra (see discussion for S9).

Class	Members of class *(labels refer to Table 3.3, p. xx).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 8, 11, 12, 15	The power is higher on top.
2	3, 5, 10, 14, 16, 18	The powers are higher at the bottom.
3	2, 4, 9, 13, 17	The powers are the same.
4	6, 7	There's no variable. No x, just a number.
5	19, 20	With trigs and there's no powers.

TABLE 5.15a. Student S15's classification.

Technique. The student focused exclusively on box 2 to build his classes – as S9 did. If he has a technique for solving limits, it is likely to belong to Algebra.

Technology. The student's first explanatory discourse refers to algebraic features. In this sense, he might be employing a conceptual mode of thinking – but only in relation to concepts of Algebra, not of Calculus.

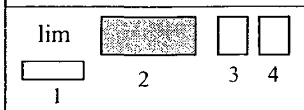
Technique	Technology	Theory
	The technology refers to algebraic features.	[The theory can't be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on conceptual thinking – in relation to concepts of Algebra, not of Calculus.	[The mode of thinking can't be inferred from the classification task.]

TABLE 5.15b. A model of student S15's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S16

The student focused exclusively on box 2 to construct his classes and decide membership. However, his use of the word "limit" in his description of class 4, suggests that he was aware that he was classifying limit expressions.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 5, 9, 12, 17	These have difference of squares.
2	19, 20	These are trig functions.
3	3, 10, 13, 14, 18	These have square roots.
4	6, 7	These are limit of constants.
5	1, 4, 8, 11, 15, 16	These are either quadratic or third degree.

TABLE 5.16a. Student S16's classification.

From the student's classification, it is not possible to construct a general conceptual key for classification. Furthermore, membership of, for example, expression 5 would be conflictive as it is a rational function with a cubic polynomial in the denominator and a difference of squares in the numerator. He was prioritizing the feature "has a difference of squares" over the feature "has a third degree polynomial". From this, I conclude that his mode of thinking was complexive.

Student S16's classes are the same as those of student S1, and they used almost the same phrases to describe their classes. My discussion of S1's behavior applies to S16 – except for the second level of justification, where I do not have sufficient information in the case of S16.

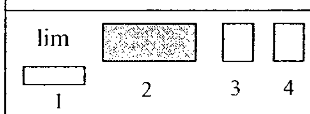
Technique	Technology	Theory
	The technology evokes topics on school Algebra textbooks.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.16b. A model of student S16's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S17

The student used the word "trinomial" to refer to expressions containing cubic polynomials. In his descriptions of classes 1, 4 and 5, he used the phrase "these are" referring to expressions that *contain* trigonometric functions, square roots, and the cubic power, respectively. The last two phrases in his description of class 6 were his answer to my question about what he meant by "regular".

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	These are trig.
2	6, 7	These are regular limits, there's not much to do.
3	8	There's no fraction.
4	3, 10, 13, 14, 18	These are square roots.
5	1, 4	These are trinomials.
6	2, 5, 9, 11, 12, 15, 16, 17	These are regular. Not that tricky. Factorable.

TABLE 5.17a. Student S17's classification.

Technique. Student S17 focused on box 2 to build his classification. His technique is purely algebraic.

Technology. However, he referred to the fact that the objects were limit expressions and not just algebraic expressions. His justification resembles topics in school Algebra textbooks (factorable polynomials, expressions with radicals, trigonometric functions). However, it is influenced by the fact that he was dealing with limit expressions: "regular limits" (for his description of class 2) and "regular [limits]" for his description of class 6. The link between the algebraic form of the function and the technique to be applied could be influenced by the routine tasks – those in the PBs.

At this level of justification, he was thinking in complexes. For example, when he said, "These are regular limits, there's not much to do" to describe class 2, another person could have decided that this description also applied to expression 8. Also, when he said, "There's no fraction" to describe class 3, another person could have decided that this description also applied to expression 14.

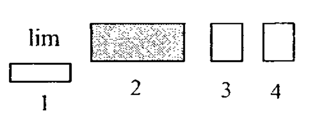
Technique	Technology	Theory
	The technology resembles topics in school Algebra textbooks, influenced by the fact that they were limit expressions and not just algebraic expressions.	The theory is based on the easiness of the application of the technique needed to find each of the limits.
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on syncretic images.

TABLE 5.17b. A model of student S17's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Theory. However, at the next level of justification, he was a syncretic thinker. His use of the adjective "regular" refers to the easiness – from his point of view – of the application of the technique he believed should be used to find the limit in each case. Hence, for example, although expression 1 is factorable, he placed it in class 5 ("cubic polynomials"). It could be, for example, that he was not familiar with techniques to factor cubic polynomials. Furthermore, it seems that the only technique he had in mind when dealing with limits of rational functions was the factoring-cancelling-substituting technique, as he did not distinguish limits taken at infinity from limits taken at a constant. When I asked him, how expression 4 (member of class 5) was different from expression 5 (member of class 6), he said, "This [expression 4] has two cubes". This is irrelevant from the point of view of the application of a technique to find the limit, as both were limits taken at infinity. He believed that he would have to factor these expressions to find the limit, and factoring expression 4 would be more difficult than factoring expression 5.

Student S18

It is quite difficult to understand what the student meant by "trinomials". Did she mean trinomials in the algebraic sense or polynomials of cubic degree, or perhaps both of them? When, in her description of class 3, she said "these have a trinomial in the denominator but difference of squares I kept them aside", she pointed to cards 5 and 9. In item 5 the numerator is a difference of squares and the denominator is a binomial of third degree. In item 9, again, the numerator is a difference of squares but the denominator is a trinomial of degree 2.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 4, 8, 11, 15, 16	These are all trinomials.
2	3, 10, 13, 14, 18	There's a square root.
3	2, 5, 9, 12, 17	These have a difference of squares. I like those. These have a trinomial in the denominator but difference of squares I kept them aside.
4	6, 7	Limits three and seven.
5	19, 20	Trigonometric.

TABLE 5.18a. Student S18's classification.

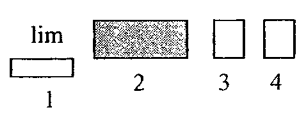
Technique	Technology	Theory
	The technology evokes topics of school Algebra textbooks.	The technology seems to be supported by his like or dislike for the algebraic expressions.
The technique belongs to Algebra.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on syncretic thinking.

TABLE 5.18b. A model of student S18's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

This student's classes are the same as those of student S1, and similar phrases were used to describe them. My discussion of student S1's behavior applies to S18 as well.

Technique. The student considered only information in box 2 to build her classes. Thus, her technique was purely algebraic.

Technology. From the student's reference to limits in her description of class 4, I infer that she was aware that these were limit expressions. With the statement "I kept them aside" she meant that although 5 and 9 were "trinomials" she prioritized the feature "difference of squares" to decide about the membership of these expressions in class 3. Hence, she was employing a complexive mode of thinking, and she was aware that she was not using consistent criteria to form disjoint classes.

Student S24

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Just a number in them.
2	2, 5, 9, 12, 15, 16, 17	These are with squares. Squares at the top.
3	3, 10, 13, 14, 18	These are roots. Roots at the top.
4	1, 4, 8, 11	Cubes. Cubes at the top.
5	19, 20	Sines.

TABLE 5.24a. Student S24's classification.

The phrase "just a number in them" that the student used to describe class 1 refers to the fact that the functions in this class are constants. The student's use of the phrase "the top" in her description of classes 2, 3 and 4, refer to the numerator of the objects in those classes.

Technique. The student's answer to the question "what was the rule for your classification" was: "I kind of went through a pattern". The pattern she followed was neither a characteristic of school Algebra textbooks nor of college level Calculus textbooks. She focused on algebraic features (box 2) in a way that she saw a pattern, which would not attract the attention of many Calculus students. None of the other interviewed students referred to the fact that the square roots in the rational expressions were in the numerator; they identified (usually incorrectly) these expressions with the technique of multiplying and dividing by the conjugate, but the fact that the radicals were in the numerator was not a relevant feature for them. Nevertheless, if she has a technique for finding limits, I surmise that this technique will belong to the domain of Algebra.

It is difficult to explain why this student's attention was caught by those features. It is also impossible to predict her behavior in front of limit finding tasks because her classification and explanatory discourse were disconnected from limit concepts. Therefore, I can't make inferences about her explanatory discourses in relation to the task of finding limits.

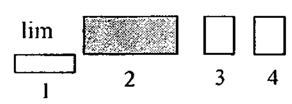
Technique	Technology	Theory
	[The technology cannot be inferred from the classification task.]	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	[The mode of thinking cannot be inferred from the classification task.]	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.24b. A model of student S24's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S26

The student's language of description could not be understood without his gestures. The phrase "the thing on the top and the bottom" appeared to refer to the relation between the degrees of the polynomials in numerator and denominator. In class 2 he placed rational functions with numerator and denominator of same degree; in class 3, those in which the degree was higher in the numerator than in the denominator, and in class 4, those in which the degree was higher in the denominator than in the numerator.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Constants.
2	2, 4, 9, 17	The thing on the top and the bottom.

3	1, 11, 12, 15	The thing on the top and the bottom. These the power is more in the top than the bottom.
4	5, 16	The thing on the top and the bottom. These the opposite.
5	3, 10, 13, 14, 18	Square roots.
6	8	[No description was given.]
7	19, 20	Sines.

TABLE 5.26a. Student S26's classification.

Technique. It follows from the observations above that the student obviously singled out all rational functions from the set and applied a consistent classification scheme to this subset, based on the relation between the degrees of the numerator and the denominator. In general, it seems that the student classified the items consistently according to the type of function in box 2: rational functions, functions with square roots, trigonometric functions and constants. In this sense, his classification resembles topics in a college level Calculus textbook. At a second level of classification, he analyzed the rational functions set into three subclasses according to the relation between the degrees of the numerator and the denominator; at this second level, his classification resembles topics in a school Algebra textbook.

Technology. As pointed above, his classification evokes topics in a college level Calculus textbook and topics in a school Algebra textbook. In the domain of Algebra, he was using a conceptual mode of thinking; but, from his discourse, I cannot infer what his mode of thinking in Calculus was.

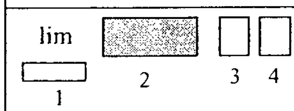
Technique	Technology	Theory
	The technology refers to algebraic features.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra.	The justification of the technique is based on conceptual thinking – in relation to concepts of Algebra, not of Calculus.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.26b. A model of student S26's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

5.1.1.2 Students whose technique belongs to the domain of Calculus

Student S2

Class	Members of class *(labels refer to Table3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?"
1	6, 7	Limit of a constant. You can do it right away.
2	3, 10, 18	Multiply top and bottom by the square root.
3	1, 2, 9, 15, 16, 17	Fractions.
4	4, 5, 11, 12	Fractions to infinity.
5	13, 14	These are square roots but to infinity.
6	8	[No description was given.]
7	19, 20	[No description was given.]

TABLE 5.2a. Student S2's classification.

When the student stated the phrase "multiply top and bottom by the square root" to describe class 2, he was referring to the rationalization method: multiplying and dividing by the conjugate of the expression with radicals. When he used the phrase "these are square roots but to infinity" to describe class 5, he was referring to expressions involving radicals where the limit was taken at infinity instead of at a constant, as was the case of class 2. I surmise that he made a first classification distinguishing "fractions" – rational expressions – from "square roots" – expressions with radicals. Then he separated each of these general classes into two classes depending on whether the limit was taken at a constant or at infinity. This hypothesis is supported by the fact that when I asked him why objects in class 4 were separated from those in class 3, he said: "those are to infinity". When I asked him what was in class 5, he said, "those are the same idea as the square root but to infinity, those [the ones in class 2] were to a number". The student ignored classes 6 and 7 when asked to describe each class.

Technique. The student combined information in box 1 – whether the limit was taken at a constant or at infinity – with the algebraic form of the function (box 2). This is a standard approach to finding limits of functions in Calculus.

Technology. His reference to the rationalization technique in the phrase that describes class 2 could be an indication that when he placed the "fractions" together [class 3], he was thinking of a common technique to find those limits. Within this interpretation, his discourse to justify this technique for classifying evokes typical topics of college level Calculus textbooks. He is deceived, however, by the algebraic form of the functions. In particular, the objects he placed in class 3 are very heterogeneous from the point of view of Calculus. While expressions 1 and 2 are indeterminations of the zero over zero type, expression 9 corresponds to a continuous function at the value at which the limit is taken. Expressions 15, 16 and 17 correspond to functions with vertical asymptotes at the values at which the limits are taken. There are no common techniques to find these three different types of limits. This misconception could be the result of the student's habit to deal with routine problems – the PBs.

At the technology level of explanatory discourse, this student is thinking in complexes. Another person would have problems, for example, to decide the membership of the item 13. Although the algebraic form of the function contains a square root and the limit is taken at infinity (class 5), it could also be placed in class 4, as the algebraic form of the function is a fraction and the limit is taken at infinity.

Theory. The student did not make any further statements, so I do not have any basis to make claims about his explanatory discourse at the level of theory.

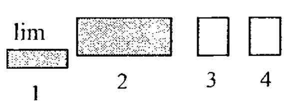
Technique	Technology	Theory
	The technology refers to typical topics of college level Calculus textbook, but the student might be deceived by the algebraic form of the functions – influenced by the PBs.	[The theory cannot be inferred from the classification task.]
The technique belongs to Calculus. The student combined the information in boxes 1 and 2.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.2b. A model of student S2's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S3

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13	These ones I organized them for the same technique to solve the problem [...] the infinity thing [...] taking the highest power and that would help me to find the limit.
2	1, 2, 8, 9, 10, 15, 16, 17, 18	This was basically how to solve the problem, like it's gonna be zero over zero type and I have to separate them and then find a way of doing it.
3	6, 7	This one is basically like wherever you are going the limit is going to be the same, it's constants.
4	3, 19, 20	Limit going to zero. With the sine and the trig thing. I put the zeros together.
5	14	This one I don't know.

TABLE 5.3a. Student S3's classification.

When the student said, "the infinity thing" and "taking the highest power", to describe class 1, she was referring to the standard technique used to find limits of rational expressions when the limit is taken at infinity: taking as a common factor the highest power of the variable, in both the numerator and the denominator.

When she said, "I have to separate them and then find a way of doing it", to describe class 2, I believe she was referring to the standard techniques used to find limits of rational expressions where the limit is taken at a constant and they are instances of zero over zero type of indetermination: factoring and rationalizing.

In class 4, she placed together the limits taken at zero, both involving trigonometric functions, and one a rational function.

Technique. The student focused on features that are fragments of an approach to finding limits typical in Calculus. However, the student failed in combining these pieces of information.

Technology. The student made explicit references to her criteria for classifying. In response to my question "can you explain what was your rule?", she said, "these ones [referring to objects in class 1] I organize them like for the same technique to solve the problem". Then, she said, "and this [referring to class 2] was basically how to solve the problem". This last statement does not reflect what she actually classified together in class 2, as some of the items required a factoring technique, some other – a rationalization technique, and some – substitution. Let us try to interpret her process of thinking. If we look at the objects that she placed in classes 1, 2 and 4, it seems that her rule for classification was whether the limit was taken at infinity, at a non-zero constant, or at zero. She was focusing on box 1. However, for her, this feature triggered the thought of the techniques she believed were necessary to find the respective limits. This made her switch from considering the classifying key "at where the limit is taken" to "what is the technique necessary to find the limit". This switching could have been reinforced when she went through the expressions 6 and 7 (the constant functions) and 14 (which does not resemble the routine tasks). This would explain why she built classes 3 and 5, as well as the phrases that she ended up using to describe the classes. This shifting of the focus of attention – between boxes 1 and 2 – is a symptom of complexive thinking.

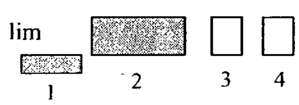
Technique	Technology	Theory
	The technology refers to standard techniques of finding limits of routine functions, but the student would be deceived by information in box 1 – influenced by the PBs.	[The theory can't be inferred from the classification task.]
The technique belongs to Calculus. Information in the boxes is not combined by the student.	The justification of the technique is based on complexive thinking.	[The mode of thinking can't be inferred from the classification task.]

TABLE 5.3b. A model of student S3's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Furthermore, this change in the classification key did not make her to go back on what she had already classified. It is very unlikely that the student miscalculated the direct substitution in expressions 8, 9, 10, 15, 16, 17, and 18. None of these correspond to a zero over zero indetermination. Thus, all limits in which x tended to a non-zero constant fell into the category "zero over zero type". It seems that she was misguided by the information in box 1, and she quickly took the items to be instances of routine tasks of the types in PB1 and PB2. She took the items in class 1 to be examples of tasks that can be solved using technique τ3b (which they are). It was only when she encountered the constant functions, or expression 14, that her attention moved from box 1 to box 2.

Student S13

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	[No description was given.]
2	1, 2, 3, 7, 8, 9, 10, 15, 16, 17, 18	x goes to a number.
3	4, 5, 6, 11, 12, 13, 14	x goes to infinity. There's one solution for all of them.

TABLE 5.13a. Student S13's classification.

With the sentence "there's one solution for all of them" the student meant that there was one technique to find the limit that applied to all objects in class 3. I provide evidence to support this claim below.

The student first tried a classification based on the algebraic form of the functions. However, he interrupted this process and switched to another approach whose result is presented in Table 5.13a. After he completed his final classification, the following exchange took place:

I: And why do you prefer this order better than the other one?

S13: Because in this there is one solution for all of them.

I: Do you mean one solution or one technique?

S13: Yes, one way of solving it.

Technique. The student was considering information that was meaningful from the point of view of Calculus. I surmise that, in tasks of finding limits of functions, he would take into consideration the same kind of information.

Technology. At the first level of justification, the student was thinking in complexes: he focused his attention on box 1 to build classes 2 and 3 but he focused on box 2 to build class 1. His phrases evoke the main topics of college level Calculus textbooks: limits taken at a constant and limits taken at infinity.

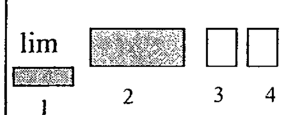
Technique	Technology	Theory
	The technology evokes main topics in college level Calculus textbook.	Techniques for finding limits of functions, with misconceptions.
The technique belongs to Calculus; the information in boxes 1 and 2 are not combined by the student.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on conceptual thinking.

TABLE 5.13b. A model of student S13's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Theory. However, the reason why he was doing this seems to be based on conceptual thinking, since he had a general key: *what technique should be used to find the limit*. This conceptual thinking was based on several misconceptions: the two limits involving trigonometric functions require radically different approaches to find their respective limits; the same is the case with many of the objects in class 2.

Student S14

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 3, 10, 13,18, 19	L'Hôpital's rule. Zero over zero or infinity over infinity.
2	14, 20	These ones I didn't know where to put them.
3	6, 7, 8	Straightforward. You just replace the x.
4	1, 4, 5, 9, 11, 12, 15, 16, 17	Divide by the highest degree.

TABLE 5.14a. Student S14's classification.

I believe that with the phrase "divide by the highest degree" that the student used to describe class 4, she was referring to the standard technique to find limits of rational expressions, at infinity, of dividing numerator and denominator by the highest power of the variable.

Technique. The student considered information typically taken into account in Calculus. I infer that this same information would guide her approach in front of the task of finding limits.

Technology. Student S14 focused on box 3 to build class 1, and on box 2 to build classes 2, 3 and 4. This is a symptom of complexive thinking. Her explanatory discourse of this technique evokes standard techniques of finding limits: L'Hôpital's rule, direct substitution, to divide every term in a rational expression by the highest degree of the variable. Nonetheless, her classification reveals misconceptions, or perhaps misperceptions, about the use of these techniques:

- limits 10 and 18, put in class 1, are not instances of zero over zero or infinity over infinity indeterminations, and thus, L'Hôpital's rule does not apply – a misperception if she took the expression to be instances of the zero over zero or infinity over infinity type, a misconception if she thought that L'Hôpital's rule applied even in the case where the expressions don't correspond to those indeterminations;
- in expression 13, L'Hôpital's rule does not help to find the limit because of its algebraic form and the repetitive use of the chain rule – this could just be a misperception based on the inexperience of the student with L'Hôpital's rule;
- limits 1, 9, 15, 16 and 17, classified in class 4, are taken at a constant and thus, the technique of dividing every term in numerator and denominator does not help to find the limit – a misconception if she thought that by using

that technique she would be able to find the limit, a misperception if she took them to be limits taken at infinity.

From her own explanation, it seems that the student did know the conditions under which L'Hôpital's rule can be applied. Therefore, it is possible that, in the case of items 10 and 18, the student was mislead by the algebraic form of the functions and she took them for instances of zero over zero indetermination because they resemble the typical tasks in MP2. In the case of item 13, she may have identified the expression as an instance of the infinity over infinity type of indetermination. She was, however, not sufficiently familiar with L'Hôpital's rule to anticipate that it would not help her to find the limit. However, it could also be the case that she repeated by heart the conditions for L'Hôpital's rule "zero over zero or infinity over infinity" but was not really considering them in classifying the items. It is hard to understand why she failed to notice the zero over zero or infinity over infinity indeterminations in expressions 1, 4, 5, 11 and 12 (which she classified in class 4). Another striking fact is that, except for item 2, all the other items in the "L'Hôpital's rule" class are rational expressions with radicals, which are algebraic forms to which L'Hôpital's rule is not typically applied (because of the chain rule). It could be that she was just learning L'Hôpital's rule in her Calculus II course and was shown an example with radicals, and inferred that the rule applied whenever there is a square root.

I believe that an explanation for this classification may be that the student was employing a complexive mode of thinking. I conjecture that while going through the cards, some feature of a rational expression containing a square root triggered in her the idea that L'Hôpital's rule was a feasible technique for the case. It could be that she recalled an example, or that she noticed a "zero over zero" or "infinity over infinity" indetermination, or something else. Based on that, she built a complexive chain, placing all rational expressions with square roots in the same class. A similar behavior could have been at the root of her formation of class 4. She encountered an expression with a rational function (just division of polynomials) that reminded her of the technique "divide by the highest degree". It could be, for example, that she encountered expression 11 or 12 (for which the technique applies), or that some other feature in the other expressions reminded her of the technique, and then built a complexive chain, placing all the expressions with rational functions in class 4. If this is how she was thinking, then the reason for putting expression 2 in class 1 could be that it was the first expression she classified in that class, immediately after she encountered a rational expression with a square root and started her complexive chain. Then, when she encountered expressions with rational functions again, one or more features in them captured her attention over the feature "zero over zero" or "infinity over infinity".

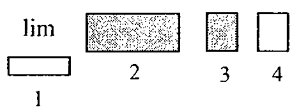
Technique	Technology	Theory
	The technology evokes standard techniques of find limits, with misconceptions and misperceptions.	[The theory can't be inferred from the classification task.]
The technique belongs to Calculus; the information in boxes 2 and 3 is not combined.	The justification of the technique is based on complexive thinking.	[The mode of thinking can't be inferred from the classification task.]

TABLE 5.14b. A model of student S14's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S20

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13, 14	With the infinity. I should take the biggest power up and down.
2	6, 7, 8, 9, 10	The answer is clear. These ones are solvable if you put in the number.
3	1, 2, 3, 15, 16, 17, 18	Indeterminate. If you substitute you get a number over zero.
4	19, 20	Exceptions.

TABLE 5.20a. Student S20's classification.

The student's phrase, "with the infinity", in the description of class 1, refers to the information in box 1. In the description of the same class, when he said "take the biggest power up and down", he was referring to the technique, typically used to find limits of rational expressions at infinity, of factoring out the highest power of the variable from the numerator and from the denominator.

Technique. This student was using and trying to combine the information in boxes 1, 2 and 3. In this sense, his technique is characteristic of Calculus.

Technology. The student focused on box 1 to build class 1 and on box 3 to build classes 2 and 3. He focused on box 2 to build class 4. This lack of common feature to define the classes is already a symptom of complexive thinking. It is not possible to provide a single conceptual classification key that would generate the classes he has built.

Furthermore, to describe class 1, the student said, "the first one [the first class] is with the infinity thing, because I know I should take the biggest power up and down". However, following my intervention, he proved to be aware that the technique of taking "the biggest power up and down" was of no use in the case of expression 14. Our exchange was:

> I: And what about this one [How the mentioned technique would be used in expression 14]?
>
> S20: This one is infinity minus infinity. I don't think it goes to infinity though... it is positive... so the answer is zero.
>
> I: But when you put them together in the same group [all the members of class 1], what were you thinking?
>
> S20: Just the infinity thing. I didn't look at one by one, I just thought I can take the biggest... like this one too [expression 11], over x should be infinity, and here too [expression 5], so this is one

over x. [Even if he did realize that this technique does not work in all the cases in class 1, he didn't change the class.]

He was aware of the fact that that it was an indetermination of the type ∞ - ∞ and that the limit was zero. But he decided about the membership based on the content of box 1, "$x\to\infty$" in this case.

Theory. From the paragraph above, it follows that the student was justifying his choice of the feature to decide membership to class 1 with a phrase that applied only to some of the objects in the class, also a symptom of complexive thinking.

For the four classes, it seems that the student's justification for choosing the features to decide membership is based on techniques to find limits of routine functions. However, whenever I asked him about the reasons why this or that expression was a member of a class, he was eager to discuss each limit and try to find their values, even if he decided not to use this information for the classification. In most cases, his reasoning was not only correct but also conceptual. For example, he said that the limit in expression 20 was divergent and he linked this with what he was learning at the time about convergence of series in the Calculus II course, and tried to prove (although he was not successful) that the limit in 19 was 1. I believe this is a hint that, although in the classification task he behaved as a complexive thinker, in front of the task of finding limits he would use a conceptual mode of thinking.

The student had the misconception of considering the form "a number over zero" as an indeterminate form, even in cases where the "number" was not zero. These misconceptions were extended to the techniques he would use to find the limits: "either multiply by the conjugate or factor" (page 363).

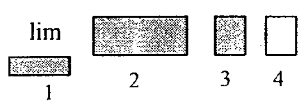
Technique	Technology	Theory
	The technology evokes topics on college level Calculus textbook, with misconceptions.	The theory refers to techniques for finding limits of typical functions, with misconceptions.
The technique belongs to Calculus.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on conceptual thinking.

TABLE 5.20b. A model of student S20's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S23

The phrase "multiplying by the conjugate" refers to the technique of multiplying and dividing by the conjugate of the expression with radicals. When the student used the phrase "more complicated" in his description of classes 4 and 5, he was referring not to the algebraic expressions alone but to the technique that he thought must be used to find the limit. Later in the interview, he said that, because of the cube, the technique to find the limit was not the same as the one used for, for example, a quadratic expression. By "simple cancelling out" I infer that he meant factoring and cancelling. Hence, I interpret that his reference to "simple" and "complicated" refers to the factoring technique; for him, to factor a quadratic polynomial is easier than to factor a cubic one.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 10, 18	Multiplying by the conjugate.
2	12	Simple cancelling out.
3	2, 9, 15, 17	Simple cancelling out.
4	4, 5, 11	More complicated, cubes, $x \to \infty$.
5	1, 16	More complicated, cubes. They look very similar to me [...] you'd use the same technique to solve them.
6	13, 14	No cubes and no multiplying by the conjugate.
7	8	Definite answer.
8	6, 7	There are no xs.
9	19, 20	They are both sines. There are some similarities.

TABLE 5.23a. Student S23's classification.

Technique. The student first focused on box 2 to build his classification and then used the information in box 1 to distinguish limits taken at infinity from limits taken at a constant (class 2 vs. class 3, class 4 vs. class 5, class 1 vs. class 6). Only for class 7 did he appear to consider the information in box 4. His approach belongs to Calculus.

Technology. The student's description of the classes is typical of complexive thinking, as he "named" them according to different features. For class 7 he chose the feature "there are no xs", while for class 8 he chose the feature "definite answer". For class 6 he used a phrase that explains why the objects are not members of other classes, in particular classes 1, 4 and 5. With this description, however, any object that is not a member of classes 1, 4 and 5 could be placed in class 6.

He had misconceptions of at least three very different kinds. Firstly, he had misconceptions related to the use of the typical techniques. He believed that the technique of multiplying and dividing by the conjugate applies only in the case of limits taken at a constant. Secondly, he believed that the fact that the sine function was in the expression was a unifying feature in terms of limits. Yet the two expressions, 19 and 20, have nothing in common from the point of view of the limits. Thirdly, he confused algebraic techniques to find limits with algebraic manipulations. He said that the technique to deal with cubic polynomials is "more complicated" than the technique to deal with quadratic polynomials.

He also had a misperception, related to the fact that he failed to see that direct substitution was possible also for expressions 9 and 10. He was aware of this technique. He applied it in expression 8, and thus classified it as "definite answer". However, the algebraic forms of the functions in 9 and 10 probably misled him. It did not even occur to him to try the substitution technique. Both expressions 9 and 10 resemble instances of tasks in PB1 and PB2, respectively.

Theory. The discourse supporting his technology seems to relate to how easy or how difficult it was for him to apply the techniques for finding limits. In this sense, he was thinking syncretically.

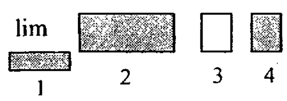
Technique	Technology	Theory
	The technology resembles routine techniques to find limits, with misconceptions and misperceptions – influenced by PBs.	The theory refers to affective impression about how easy or difficult are the techniques to find the limits.
The technique belongs to Calculus.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on syncretic thinking.

TABLE 5.23b. A model of student S23's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S27

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	The limit is to a number.
2	1, 2, 3, 8, 9, 10, 15, 16, 17, 18, 19, 20	The x is going to a number.
3	4, 5, 11, 12, 13, 14	Functions going to infinity.

TABLE 5.27a. Student S27's classification.

From the student's classification I infer that she used the phrase "the limit is to a number" to mean that the functions were constants or that the value of the limit was a constant, and she used the phrase "functions going to infinity" to mean that the limit was taken at infinity, i.e., referring to the symbol $x \to \infty$.

Technique. Her technique belongs to Calculus as she was considering either the value of the limit or the values at which the limits were being taken.

Technology. The student's answer to the question "what was the rule of your choice?" was: "what the limit was going to". Considering the classes she built, by that phrase she could have meant "at where the limit was being taken" or "what the value of the limit was". Thus, she stated a general key for classifying but she did not follow it for all her classes. It could be that the stage of her cognitive development in general corresponded to the stage of concepts. That is, in terms of scientific thinking she was a conceptual thinker. In particular, she knew that a general conceptual key is needed to accomplish a classification task. Her failure in providing a key that she could follow could be based on some misconceptions about limits. Nevertheless, her mode of thinking, to accomplish the classification tasks, was complexive, because for class 1 she focused on box 2 and for classes 2 and 3 she focused on box 1. The fact that in expressions 6 and 7 the functions where constants caught her attention over the fact that the limit in 6 is taken at infinity and in 7, it is taken at a constant.

The classes she made evoke main topics on college level Calculus textbooks: limits taken at a constant and limits taken at infinity. There was no indication in her discourse of what her theory for choosing those features for classification was.

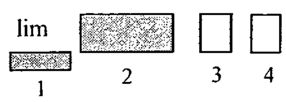
Technique	Technology	Theory
	The technology evokes main topics on college level Calculus textbooks.	[The theory cannot be inferred from the classification task.]
The technique belongs to Calculus.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.27b. A model of student S27's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

5.1.1.3 Students whose technique belongs to the domain of Arithmetic

Student S6

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 6, 7, 8, 14, 19	These ones I spotted them very quickly.
2	5, 11, 12, 13, 20	These ones are infinity or infinity over infinity.
3	1, 2, 3, 9, 10, 15, 16, 17, 18	They were going to zero or it was a zero over zero form .

TABLE 5.6a. Student S6's classification.

Technique. This student's technique for classifying was based mostly on the outcome of direct substitution in the set $\boldsymbol{R} \cup \{-\infty, +\infty\}$. At least, this is likely the case for classes 2 and 3, and for expression 8 and perhaps 14 in class 1. I infer that this type of arithmetic would be at the core of his technique in front of the task of finding limits. This might lead him into miscalculations such as the one I believe he made when he claimed that he spotted the limit for expression 14 "very quickly": $\infty - \infty = 0$.

Technology. To justify why expression 20 was placed in class 2, he said: "I saw the one over x and I knew it was infinity right away" (page 278). It seems then that he ignored completely the symbol *sin* – referring to the sine function – or he saw the symbol but considered, for the classification, the feature he was most familiar with: "one over x tends to infinity as x tends to zero".

Based on the assumption that the student's main technique was direct substitution and computation in the extended real domain, I presume that putting the expressions 15, 16, 17 and 18 into the class described as "going to zero or a zero over zero form" was the result of the misconception that, as far as Calculus is concerned, "a constant over zero is zero". In the items 15, 16, 17 and 18 the results of direct substitution are $-2/0$, 26/0, 21/0 and [5 – sqrt(5)]/0 respectively, and the limits are: undefined, infinity, undefined, and undefined, respectively.

His mode of thinking is syncretic as he was motivated by familiarity and easiness in defining his classes.

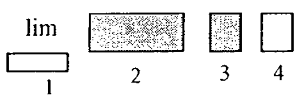
Technique	Technology	Theory
	The technology refers to some recognition of familiarity, probably with misconceptions	[The theory cannot be inferred from the classification task.]
The technique is based on an arithmetic in $\boldsymbol{R} \cup \{-\infty, +\infty\}$.	The justification of the technique is based on syncretic thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.6b. A model of student S6's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S19

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 2, 3, 15, 18, 19, 20	Zero over zero.
2	9, 10, 16, 17	A number over zero.
3	6, 7, 8	A definite number.
4	4, 5, 11, 12, 13, 14	Infinity over infinity.

TABLE 5.19a. Student S19's classification.

Student's phrases to describe his classes refer to the outcome of a direct substitution with values in $\boldsymbol{R} \cup \{-\infty, +\infty\}$. When asked why he put expression 16 in class 2, he said, "it is zero over a number". When I asked him about why expression 14 was in class 4, he said, "it went to infinity".

Technique. To build his classification and decide about the membership of each object, the student focused his attention on box 3. In particular – on the outcome of arithmetic operations on elements of $\boldsymbol{R} \cup \{-\infty, +\infty\}$.

Technology. The student's descriptions evoke the typical indeterminations treated in college level Calculus textbooks: zero over zero and infinity over infinity, plus the indefinite form "non-zero constant over zero". He had the misconception of considering the form "zero over a non-zero constant" as being of the same type as "non-zero constant over zero".

Expressions 15, 18 and 20, which he classified as "zero over zero", are instances of the form "non-zero constant over zero" (assuming that, for expression 20, he ignored the symbol of the sine function and considered only $1/x$, which happened for some other students as well). It may well be that, in his process of classification, he put the "zero over zero" forms together and then he encountered 15, 18 and 20 for which the denominator became zero and he decided to put them also in class 1. Then he found instances of the form "zero over a non-zero constant" (9 and 10) and made a new class. When, after this, he had to decide the membership of expressions 16 and 17 ("non-zero constant over zero") they seemed to him better described by his idea of class 2 ("a number over zero") than by his idea of class 1 ("zero over zero"). This would be an example of an individual forming a chain of complexes.

With respect to his last class, “infinity over infinity” I asked him why expression 14 was a member (as the value of the limit is zero and the outcome of this arithmetic in $\boldsymbol{R} \cup \{-\infty, +\infty\}$ would be $\infty - \infty$). His answer was: “it went to infinity” (see page 356). It could be the case that he actually thought that the outcome of this arithmetic technique was infinity. He might have shifted his attention to box 1 and his sentence “it went to infinity” referred to the limit being taken at infinity. This response and shift of attention might have been a result of a second thought, when asked questions about the results of his classification after the fact. Whatever was actually the case, his behavior supports the conjecture that he was thinking in complexes. The phrase he used to describe class 4 was a description of some of the objects on the class, but not of all of them (the same observation applies to his description of class 2) – a very clear symptom of complexive thinking.

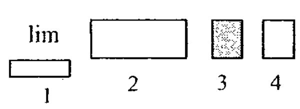
Technique	Technology	Theory
 	The technology refers to types of indeterminations typically analyzed in college level Calculus textbook, plus the non-zero constant over zero instances, with misconceptions.	[The theory can’t be inferred from the classification task.]
The technique belongs to an arithmetic in $\boldsymbol{R} \cup \{-\infty, +\infty\}$.	The justification of the technique is based on complexive thinking.	[The mode of thinking can’t be inferred from the classification task.]

TABLE 5.19b. A model of student S19’s praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

5.1.1.4 Students whose technique belongs to the domain of Arithmetic/Calculus

Student S4

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7, 8, 9, 10	These ones I just have to plug in the numbers.
2	4, 5, 11, 12, 13, 14	These ones are all limits where x goes to infinity.
3	1, 2, 3, 15, 16, 17, 18, 19, 20	These ones give zero at the bottom.

TABLE 5.4a. Student S4's classification.

The student's phrase to describe class 1 refers to limits that can be found by direct substitution. In class 3, he put together expressions in which direct substitution turns the denominator into zero.

Technique. The student focused on box 1 to form class 2 and on box 3 to form classes 1 and 3. Similarly as student S3, this student was considering fragments of features that belong to Calculus, but he was also considering a feature that belongs to Arithmetic: the outcome of direct substitution gives a number over zero. The Calculus approach to limits would distinguish between the case of a non-zero number over zero and an indetermination of the type zero over zero.

Technology. The student's classes resemble the way in which college level Calculus textbooks typically approach limits of functions: those that can be found by direct substitution, those in which the denominator gives zero (most textbooks would distinguish, as different topics, the zero over zero indetermination from the limits in which direct substitution give a non-zero constant over zero), and limits taken at infinity.

The switching from considering features corresponding to box 1 to features corresponding to box 2 is typical of complexive thinking. Furthermore, expressions 6 and 7, members of class 1, involve constant functions and thus the phrase used by the student to describe the class, "these ones I just have to plug in the numbers", does not apply to them. This is also a symptom of complexive thinking. However, it seems that, in class 1, the student put together limits that can be easily found: there is no need for an algebraic technique and the denominators are not zero. It is also important to mention that immediately after stating his phrase to describe class 2, the student said, "Although in the first group there are limits where something goes to infinity, the variable is not in the function" (see page 265). With these considerations, the classes are mutually exclusive. Hence, the student was starting to be, if not concerned about a general classification key, then at least concerned about a non-contradictory classification. This could be a sign that, in terms of cognitive development, the student was at the most mature phase of the complexive stage, namely the phase of pseudoconcepts. Nevertheless, his mode of thinking about limits was complexive and not pseudoconceptual.

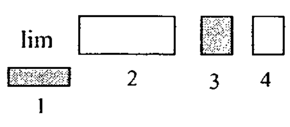
Technique	Technology	Theory
	The technology resembles college level Calculus textbook categories.	[The theory cannot be inferred from the classification task.]
The technique belongs to both Arithmetic and Calculus, but these two approaches are not combined.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.4b. A model of student S4's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S25

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 6, 7, 8, 10, 17	Real answer.
2	2, 9, 11, 12, 14, 15, 16, 18, 20	Answer is infinity.
3	1, 3, 13, 19	Answer is zero over zero or infinity over infinity.

TABLE 5.25a. Student S25's classification.

Technique. The student considered features characteristic of Calculus, looking at whether the value of the limit was a constant ("real answer") or infinity ("answer is infinity") in the first two classes, and the outcome of substitution in $\boldsymbol{R} \cup \{-\infty, +\infty\}$ ("zero over zero or infinity over infinity") in the third class. The student's evaluations were all correct for class 3, all but one correct in class 1; however, the evaluation of limit in class 2 was correct only in one third of the cases.

Technology. The student's verbal descriptions of the classes suggest that he based his classification on a single key, namely "the answer". A priori, a classification consistently following a single, objective feature of the items would qualify his mode of thinking as conceptual. However, looking at the context of his use of this word provides evidence of actual lack of consistency. The student's use of the word "answer" would imply that he focused solely on box 4, the value of the limit. However, while the descriptions of the first two classes indeed focused on the value of the limit, the description of the third class appeared to refer to box 3, the type of indeterminacy. It is not clear, from the student's discourse if he indeed referred to the type of indeterminacy in his description of class 3. Rather, it is quite likely, as conjectured in the above characterization of his technique, that the student focused on the outcome of direct substitution as the result of an arithmetic in $\boldsymbol{R} \cup \{-\infty, +\infty\}$ (box 3) and on his recognition of *normal* functions (box 2). Because his calculations were correct in most cases (13 out of 20) and his assessments of the items in class 1 were correct in 6 out of the 7 cases, it is likely that he knew some *normal* techniques to find limits. He probably knew how to find limits at infinity of rational functions. It seems that he knew, at least, how to decide about the limit based on the relation between the degrees of numerator and denominator (items 4 and 5 in class 1, and 11 and 12 in class 2). He probably also knew that a non-zero constant over zero gives infinity and that if the outcome of direct substitution is a real number, this is the value of the limit. Hence, when he could find the value of the limit by any of these methods, he placed the item in either the class 1 or class 2. If he could not do so, he placed the expression in class 3, mentioning only the result of direct substitution, as if referring to the type of indetermination. Hence, at the technology level he was a complexive thinker: for example, although expression 4 is an indetermination of the type infinity over infinity, he placed it in class 1 ("real answer"), the same would be the case for expressions 5 – also in class 1 – and 11 and 12, members of class 2. However, if the expression was an indetermination, for which he did not know the answer, he placed it in class 3.

Some of the student's mistakes could be attributed to miscalculation. The values of limits in expressions 2 and 9 are real numbers (–1 and 0, resp.); yet the student put them in class 2 ("answer is infinity"). The limit in expression 17 is infinity, yet it has been assigned to class 1 ("real answer"). I interpret these mistakes as miscalculations because the student succeeded in placing correctly (according to his classification) all the other expressions in class 1.

There are clear misconceptions in the classification of items 14 and 20. I conjecture that, in the case of item 20, the student disregarded the trigonometric function and considered only the expression $1/x$, which, with $\to \infty$,indeed diverges to infinity. It is difficult to understand what could be the source of his misconception in the case of expression 14, but the value of the limit is zero and not infinity. Maybe he inferred that the limit was infinity from the fact that $x \to \infty$ and there was no indetermination that he could recognize (the indetermination of type infinity minus infinity is not covered in the Calculus I course).

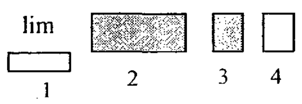
Technique	Technology	Theory
	The technology refers to the value of the limit or to types of indeterminations typically analyzed in college level Calculus textbook, with misconceptions and miscalculations.	[The theory can't be inferred from the classification task.]
The technique belongs to an arithmetic in ***R*** *U{-∞, +∞}* and to Calculus.	The justification of the technique *is based on complexive thinking.*	[The mode of thinking can't be inferred from the classification task.]

TABLE 5.25b. A model of student S25's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

5.1.1.5 Students whose technique belongs to the domain of Algebra/Calculus

Student S5

The student's phrase to describe class 3 refers to expressions with trigonometric functions. When he said "with the conjugate that you have to put on the bottom" to describe class 5, he might have been referring to the rationalization technique. But this is not a common figure of speech. It could be that either the student was just being sloppy or that he did not know what the rationalization technique was, but the sight of a rational expression with radicals made him recall *something* about this technique.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 4, 5, 9, 11, 12, 15, 17	Quadratics on top or whatever I saw, like this is a cube but since it is to infinity... The ones that are very mechanical.
2	1, 16	These are the same idea [as class 1] but require more work.
3	19, 20	Trig ones.
4	6, 7	The answer is already given.
5	3, 10, 13, 14, 18	With the conjugate that you have to put on the bottom.
6	8	[No description was given.]

TABLE 5.5a. Student S5's classification.

Technique. This student focused on boxes 1 and 2 to make his classification. In some cases, as when saying, "this is a cube but since it is to infinity" he was combining the information in these two boxes. In these cases, or, for example, when he described class 5, he was making reference of approaches to finding limits that are characteristic of Calculus. However, his paying attention to the degree of the polynomials refers to a technique, which is purely algebraic and unrelated to Calculus. Hence, I infer that his techniques to find limits belong to both Algebra and Calculus.

Technology. The student's phrase, "very mechanical", in the description of class 1, appears to refer to standard techniques, used in the PBs, in particular to the technique $\tau 3b$ (taking the highest power of the variable as a common factor in the numerator and denominator of rational expressions), and the technique $\tau 1$ (factoring and cancelling to get rid of the indetermination of type zero over zero). The two objects that he placed in class 2 suggest that he was mislead by the algebraic form of the functions. In the case of expression 16, the factoring technique would not be helpful at all, as the polynomials have no common factors. I infer that this misleading is a consequence of the student's habit to deal with routine tasks such as those in PB1, PB2 and PB3b.

I conjecture that the student's thinking in choosing the classifying features was of syncretic nature, because he classified as a separate class the rational functions involving cubic polynomials (class 2), with limits taken at a constant, likely because he usually had more trouble in factoring them.

Theory. The description made above points to an affective context of learning. Therefore, student S5's thinking at the theory level of justification is classified as belonging to the syncretic mode.

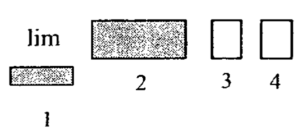
Technique	Technology	Theory
	The technology refers to techniques for finding limits of typical functions, but the student might be mislead by the algebraic form of the functions – influenced by the PBs.	Affective context of learning.
The technique belongs to Algebra and to Calculus.	The justification of the technique is based on syncretic thinking.	The justification of the technology is based on syncretic thinking.

TABLE 5.5b. A model of student S5's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Similarly as many other students (18 out of the 28), student S5 formed a separate class with the expressions involving the sine function. On the one hand, limits of trigonometric functions are discussed in a separate chapter in most college level Calculus textbooks. On the other hand, these limits are non-routine as they do not resemble the limits tasks in the institution's mathematical praxeologies. These could be the main reasons why students placed them in a separate class.

Student S11

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 2, 4, 9, 16, 17	These are fractions.
2	3, 5, 11, 12, 13, 14	These are going to infinity or zero.
3	6, 7, 8	Easy ones. You can do them right away.
4	15	Fraction that can't be divided.
5	19, 20	Sines.
6	10, 18	[No description was given.]

TABLE 5.11a. Student S11's classification.

With the sentence, "these are going to infinity or zero", the student was making references to the content of box 1, not to box 4.

During the classification process, the student questioned herself more than once whether she should put all the fractions together or not. Her final arrangement was as shown in the table 5.11a. However, it is hard to interpret what she meant by, "fractions that can't be divided", when she described class 4. In the particular case of expression 15, I believe that she meant that the denominator is not a divisor of the numerator, but this is also the case in expressions 9, 16 and 17, that are members of class 1.

I take it as the result of a misplacement (a careless mistake) the fact that, in the final arrangement, expression 4 ended up in class 1 rather than 2. In this part of the interview, the student appeared overwhelmed by the number of cards she had to consider. At some point, she sighed, saying, "There is a lot of them". I believe that the final classification is the result of an ongoing process that, at some point, she decided to interrupt.

Technique. For class 2, student S11 considered box 1 (Calculus); for classes 1 and 3 to 5, she seems to have considered box 2 (Algebra).

Technology. In her explanatory discourse, there is no indication of reasons why she chose to put fractions where x tended to zero or infinity in a different class than those in which x tended to a non-zero constant. She referred to some algebraic techniques but these did not evoke the standard techniques for finding limits. It is clear from her discourse that she was not confident in her ability to deal with the classification task, and she was not happy with the outcome of her work. In the beginning, she said, "I am not very good at this, I have to think a lot". At the end of this part of the interview, she said, "Maybe it [her classification] doesn't make any sense". I believe there is not enough ground to conjecture about her first level of justification, although, whatever it was, it seemed to be based on complexive thinking.

It is important to observe that her reference to "easiness" in class 3 is not an affective statement on her part. When I asked her about expression 7 being a member of class 3, the class she described as "easy ones" and "you can do them right away", her answer was:

> S11: This is just the limit of three going to five so you get three... or five, I am not sure. **I actually do not remember seeing those.**

"Easy" is for her a feature intrinsic to the object. It appears to be her way of saying that the limit, *in principle*, can be found "right away", even if she, herself, does not know how to find it. Her explanation for her own lack of knowledge makes reference to her unfamiliarity with the problem.

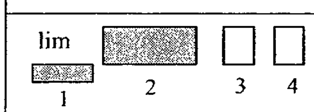
Technique	Technology	Theory
	Recognition of unfamiliarity.	[The theory cannot be inferred from the classification task.]
The technique draws partly on Algebra and partly on Calculus. The information in boxes 1 and 2 is not combined by the student.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.11b. A model of student S11's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

Student S21

Technique. The student considered pieces of information, some related to Calculus – such as "limit as x approaches infinity" – and others related exclusively to Algebra – such as "with sine function" or "with division".

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13, 14	Limit as x approaches infinity.
2	8	Simpler. There's no division or square roots.
3	6, 7	Also simpler.
4	1, 2, 3, 9, 10, 15, 16, 17, 18	With division.
5	19, 20	With sine function.

TABLE 5.21a. Student S21's classification.

Technology. The student considered the information in box 1 to build his first class, but then he focused exclusively on box 2 to build the other four classes. I surmise that he first distinguished limits taken at infinity from limits taken at a constant, and then, among the latter, he focused on box 2 to separate them into different classes. This lack of homogeneity in the classificatory criteria would be a symptom of complexive thinking. First, he used a Calculus feature and then – a purely algebraic one. His phrases evoke the way in which limits are presented in college level Calculus textbooks.

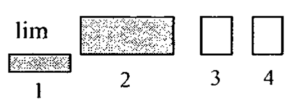
Technique	Technology	Theory
	The technology is a mixture of typical topics of college level Calculus textbooks.	[The theory cannot be inferred from the classification task.]
The technique belongs to Algebra and to Calculus; the student did not combine the information in boxes 1 and 2.	The justification of the technique is based on complexive thinking.	[The mode of thinking cannot be inferred from the classification task.]

TABLE 5.21b. A model of student S21's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

Student S22

The student's phrases "work out with rules" or "you have to rationalize" were all in reference to the techniques that she believed were needed to find the limits. When she said, "equal top and bottom" and "unequal top and bottom" in her descriptions of classes 1 and 6, respectively, she was referring to the degrees of the polynomials in the numerator and the denominator.

Technique. The student considered features related to Calculus, such as "substitution", and features related to Algebra, such as the relation between the degrees of the polynomials in a rational expression or "with square roots", that *per se* are not informative from the point of view of Calculus.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 9, 17	Equal top and bottom, the powers equal to two. Can be factored and simplified.
2	6, 7	Constants. ... well these are just constants, so the limit as x goes to infinity of seven so the answer is seven, 'cause these are just plain constants.
3	3, 10, 13, 14, 18	With square roots. You have to rationalize. Common methods.
4	4	You can just simplify it and it will be nine over three.
5	19, 20	Trigonometric.
6	1, 5, 11, 12, 15, 16	Unequal top and bottom. Work out with rules.
7	8	I just have to substitute the x.

TABLE 5.22a. Student S22's classification.

Technology. The student only considered the content of box 2 to build classes 1 to 4 and 7. However, she must have noticed $x \to \infty$ in box 1 when building class 6. Although the item she placed in that class contains a rational function with polynomials of equal degree, such as those in class 1, she decided to place it in a class of its own. The fact that she took into account box 1 for the construction of class 4, but put, in class 6, the expressions 5 and 11 which also correspond to limits of rational functions taken at infinity, could be a symptom of complexive thinking: she failed to notice the $x \to \infty$ in 5 and 11. It is also possible, however, that she knew by heart that the limit at infinity of a rational function with polynomials of the same degree is the quotient of the leading coefficients, but did not know the equivalent rules for the case of polynomials with different degrees, and thought that, in this case, she would have to factor the expressions. Nevertheless, her immediate level of justification was based on complexive thinking, since another person would have trouble deciding about the membership of some items. For example, expression 10 could be described, just as well, by the phrase "I just have to substitute the x".

Theory. On the level of theory, when the student was justifying her "names" for the classes, she referred to standard techniques for finding limits of functions. However, these references reflected misconceptions, or misperceptions, as she seemed to believe that whenever there is a radical, the technique to be applied is rationalization, or that whenever there is a rational function and the limit is taken at a constant, the technique to be applied is factoring-cancelling-substituting. These could be mere misperceptions; she was being mislead by the algebraic forms and taking the expressions to be instances of tasks in PB1 and PB2.

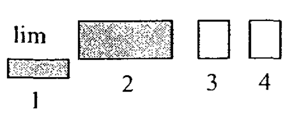
Technique	Technology	Theory
	The technology evokes a mixture of topics on school Algebra textbooks and on college level Calculus textbooks.	The theory evokes techniques for finding limits of typical functions, with misconceptions and/or misperceptions – influenced by the PBs.
The technique belongs to Algebra and Calculus.	The justification of the technique is based on complexive thinking.	The justification of the technology is based on complexive thinking.

TABLE 5.22b. A model of student S22's praxeology related to tasks of finding limits of functions, inferred from her behavior in the classification task.

5.1.1.6. Student whose technique belongs to the domain of Analysis

Student S28

Technique. Student S28 calculated each of the 20 limits, some mentally and others with pen and paper. Thus, he focused entirely on box 4. Furthermore, the techniques he used for some of the limits were characteristic more of Real Analysis than, Calculus. For example, he mentally calculated that the limit in item 14 ($\lim_{x\to\infty}\sqrt{x^2+1}-x$) was zero, using the following reasoning: "when this [pointing to x] tends to plus infinity, this [pointing to the term with the square root] is going to tend to x, because this is going to be very big and once you do the square root of this, the one will just vanish and you'll have minus x (the other term) so it will be zero" (page 411).

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	5, 7, 9, 14	The limit goes to zero.
2	11, 12, 15, 16, 17	The limit is plus/minus infinity.
3	1, 2, 3, 4, 6, 8, 10, 13, 18, 19	Finite numbers that are not zero.
4	20	Divergent.

TABLE 5.28a. Student S28's classification.

Technology. The student miscalculated some of the limits (item 7 in class 1, item 15 in class 2, and items 10 and 18 in class 3). He even acknowledged this possibility. In fact, he started his explanation by saying, "I may have made a mistake in the calculation...". He did not attach much importance to these mistakes; the only thing that seemed to matter for him was the classification key. The student considered also other criteria for the classification but discarded them as "not specific enough" (see page 411). He made his classification key explicit: "I decided to put them into something that already tells you if you are going to have an answer finite, zero, divergent or infinity".

Theory. The way, in which the student calculated limit 14 (see the paragraph corresponding to Technique) suggests that he was dealing with the task of finding the limits in a way that corresponds to the domain of Real Analysis.

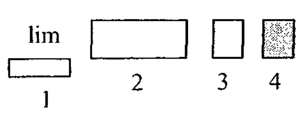
Technique	Technology	Theory
	The technology refers to the value of the limit.	The theory refers to analytic inferences about the behavior of functions.
The technique is analytical.	The justification of the technique is conceptual.	The justification of the technology is conceptual.

TABLE 5.28b. A model of student S28's praxeology related to tasks of finding limits of functions, inferred from his behavior in the classification task.

5.1.2 A summary of the individual students' performance on the classification task

In assessing students' behavior in the interview, it is important to take into account the fact that students were told – when they were recruited – that the interview would take about one hour, but they were not told at any point, before or during the interview, how many tasks there would be. Each task was given to them as if it was the last one. Hence, there is no ground to believe that students felt time-pressured while doing the classification task. The students' decision to calculate or not the limits in the cards they were classifying was a choice based on their own criteria to carry out the classification. The only student who calculated each of the 20 limits – some mentally and some with pen and paper – was student S28. Other students made references to the fact that they were considering the value of the limit for some expressions (for example S20); other students referred to the techniques they would use to find the limits (for example S10); and there were students who did not seem to be concerned at all with finding the limits (for example S9).

Of the twenty eight (28) interviewed students, twelve (12) focused exclusively on the algebraic features of the functions (box 2) to classify the twenty items. Seven (7) students considered only the Calculus features (in one way or another they considered information that is meaningful from the point of view of Calculus: the value at which the limit is being taken, the type of indetermination as a tool to decide which technique to apply, the actual value of the limit). Two (2) students based their classification on arithmetic computations in $\boldsymbol{R} \cup \{-\infty, +\infty\}$. Only one (1) student's classification applied criteria characteristic of mathematical analysis. Six (6) students used a combination of two approaches: two (2) combined Arithmetic and Calculus, and four (4) of them – Algebra and Calculus.

Among the twelve students who based their classification exclusively on the algebraic features of the functions, for eight students (S1, S7, S8, S10, S12, S16, S17 and S18) the expressions in the cards triggered a behavior that could be associated with a path of developing concepts about limits. These eight students either made an explicit reference to the fact that they were classifying limit expressions, or their classes seemed to be influenced by the institution's mathematical praxeologies related to limits tasks. This was not the case for the other four students (S9, S15, S24 and S26), who made no reference to limits in their descriptions or explanations and their classification features cannot be associated with the mathematical praxeologies.

The following tables summarize the analysis above.

Students' labels	Technique in limits tasks belongs to the domain of:	Mode of thinking about Technology	Mode of thinking about Theory	Number of students in category
1, 10, 17, 18	Algebra	Complexive	Syncretic	4
7, 8, 16	Algebra	Complexive	[unable to say]	3
12	Algebra	Pseudoconceptual	Complexive	1
5	Algebra/Calculus	Syncretic	Syncretic	1
11, 21	Algebra/Calculus	Complexive	Syncretic	2
22	Algebra/Calculus	Complexive	Complexive	1
6	Arithmetic	Syncretic	[unable to say]	1
19	Arithmetic	Complexive	[unable to say]	1
25	Arithmetic/Calculus	Complexive	Syncretic	1
4	Arithmetic/Calculus	Complexive	[unable to say]	1
23	Calculus	Complexive	Syncretic	1
20	Calculus	Complexive	Complexive	1
2, 3, 14, 27	Calculus	Complexive	[unable to say]	4
13	Calculus	Complexive	Conceptual	1
28	Analysis	Conceptual	Conceptual	1
9, 15, 24, 26	[unable to say]	[unable to say]	[unable to say]	4
TOTAL				28

TABLE 5.29. Students classified according to the domain of their techniques and their modes of thinking.

Domain of technique	Number of students & frequency	Mode of thinking about technique				Mode of thinking about theory				
		Syncr.	Compl.	Pseudo-concept.	Concept.	Syncr.	Compl.	Pseudo-concept.	Concept.	Un-known
Algebra	8 (33%)	0	7	1	0	3	1	0	0	3
Calculus	7 (29%)	0	7	0	0	1	1	0	1	4
Alg./Cal.	4 (17%)	1	3	0	0	4	1	0	0	0
Ar./Cal.	2 (8%)	0	2	0	0	1	0	0	0	1
Arithm.	2 (8%)	1	1	0	0	0	0	0	0	2
Analysis	1 (4%)	0	0	0	1	0	0	0	1	0
TOTALS	**24**	**2**	**20**	**1**	**1**	**9**	**3**	**0**	**2**	**10**
% (*N*=24)	**100 %**	**8%**	**83%**	**4%**	**4%**	**38%**	**12%**	**0%**	**8%**	**42%**

TABLE 5.30. Modes of thinking in the technology and theory levels vs. the domain of the techniques.

Several observations can be made based on Table 5.30. Complexive thinking prevails in justifications of techniques embedded in all domains except Analysis. Conceptual thinking in justification of techniques occurs only if the techniques belong to Analysis. Syncretic thinking in justification of techniques (technology) is rare. It is more frequent in justification of technology (theory). However, offering a syncretic justification of technology could be seen as better performance than offering no justification at this level at all: 50% of all 28 students did not offer any justification at the theory level.

The praxeologies of at least thirteen (13) students seemed to be influenced by the PBs (S1, S2, S3, S5, S7, S8, S12, S16, S17, S18, S21, S22 and S23). In addition, at least eight (8) students had a discourse that evoked topics or sections on college level Calculus textbooks (S2, S4, S13, S19, S20, S21, S25 and S27). The union of these two classes results in nineteen (19) students, out of the twenty eight, whose explanatory discourses were not related to concepts but to *how* these concepts are presented by the institution (either in the final exams or in the textbook).

Finally, it is interesting to observe that eighteen (18) students (S1, S2, S3, S5, S8, S9, S11, S12, S13, S15, S16, S17, S18, S21, S22, S23, S24, and S26) classified expressions 19 and 20 (involving the sine function) in a class of their own. I surmise that this behavior has two (related) sources. On the one hand, limits of trigonometric functions are treated in college level Calculus textbooks separately from limits of polynomials, rational functions and expressions with radicals (in a different chapter). On the other hand, limits of trigonometric functions do not belong to the institution's mathematical praxeologies. Thirteen students are in the intersection of the class of students who put the expressions 19 and 20 together, and the class of nineteen students mentioned in the previous paragraph. I surmise that, for these students, there are four types of tasks. Three types are defined by the techniques in PB1, PB2 and PB3a/b[21]. The fourth type consists of tasks that cannot be solved by any of these methods; the associated technology is based on the algebraic form of the function and evokes the typical examples in college level Calculus textbooks. These praxeologies are not of mathematical nature alone; they are a mixture of social and didactic *norms*. Norms are of social nature because students distinguish *normal* tasks from *non-normal* tasks. The distinction is not

[21] These are: tasks that can be solved by factoring, tasks that can be solved by rationalization, and tasks that can be solved by factoring out the highest power in the numerator and the highest power in the denominator or by dividing every term by the expression of the highest degree in the rational function. The last two techniques define only one type of task: a student who uses one, does not use the other.

based on mathematical rules but on institutional habit. The norms are of didactic nature because students base their approaches on those tasks and techniques that are the most emphasized by the institutions taking part in the teaching process.

5.2 SNAPSHOT #2: PARTIAL MODEL BASED ON A TASK THAT RESEMBLES ROUTINE LIMIT FINDING TASKS

In this section, I discuss students' behavior in front of the task of finding limits that resemble instances of tasks in MP1 and [M]P3[22] but differ from them on the conceptual level.

First, I recall the tasks, already presented in Chapter 3 (Table 3.4), in Figure 5.2:

2.1 $\lim_{x \to 1} \frac{x-1}{x^2+x}$	2.2. $\lim_{x \to 2} \frac{(x+3)(x-1)}{x^2-9}$
2.3. $\lim_{x \to 5} \frac{x^2-4}{x^2-25}$	2.4. $\lim_{x \to 1} \frac{x^3+4x^2+9}{x^2+2}$

Figure 5.2 (Copy of Table 3.4). The four limits that students were asked to find in the second part of the interview.

Limit tasks 2.1, 2.2, 2.3 and 2.4 resemble the types in T1 and T3 in that the functions, whose limits are calculated, are rational functions. However, all limits are taken at a constant, and there are, therefore, no tasks of type T3. The denominators of the

[22] As it was observed in Chapter 3, Section 3.1.1.2, the second part of the interview focuses on limits of rational functions.

functions in tasks 2.1, 2.2 and 2.3 have value different from 0 at the point where the limit is calculated, and therefore, none of them belongs to T1, either. Task 2.3 resembles T1 tasks the most, because the denominator has value 0 at the point where the limit is calculated; however, the numerator is not 0 at this point and this indetermination cannot be eliminated, as would be the case in T1 tasks. In fact, the limit in task 2.3 does not exist: the function tends to $-\infty$ for x approaching 5 on the left, and to $+\infty$ for x approaching 5 on the right. As could be anticipated, this task (2.3) produced the highest percentage of incorrect responses among the interviewees.

Table 5.33 presents the frequencies of correct, incorrect and no answer responses among the interviewed 28 students, for the four problems.

N=28	Correct answer	Incorrect answer	No answer
Problem 2.1	82.1 (23)	10.8 (3)	7.1 (2)
Problem 2.2	96.4 (27)	0 (0)	3.6 (1)
Problem 2.3	42.9 (12)	14.3 (4)	42.9 (12)
Problem 2.4	92.9 (26)	7.1 (2)	0

Table 5.31. Frequency of correct, incorrect, and lack of answer in the four problems given to students in the second part of the interview.

Both incorrect answers in problem 2.4 were due to miscalculations.

Prior to any intervention on my part, only by observing students' notes and listening to students' spontaneous talk, I noticed that, in problems 2.1 to 2.3, most students would factor the numerator and the denominator trying to find common factors, even in cases where they tried direct substitution first. When cancellation was not possible, as in problems 2.1 and 2.3, some students could not produce a final answer (two in problem 2.1, ten in problem 2.3; see Table 5.33). Students displayed similar behavior when approaching problems 2.1 and 2.3. Therefore, I discuss these two problems first. Then I discuss solutions to problems 2.2 and 2.4.

5.2.1 Description of students' solutions in part 2 of the interview (tasks resembling routine tasks)

5.2.1.1 Students' solutions to problem 2.1: $\lim_{x \to 1} \frac{x-1}{x^2+x}$

Of the twenty-eight interviewed students, twenty factored the numerator and the denominator in problem 2.1. These students can be divided into two groups: those who tried direct substitution first (seven students) and those who factored first (thirteen students). From the spontaneous talk, my first interpretation was that students, who used direct substitution as a first approach in problem 2.1, and then proceeded with factoring, did so because they were not sure about the value of 0 divided by 2. I was thus interpreting their behavior as lack of algebraic knowledge. For example, student S1 said:

> S1: Ok. The first thing I do when I see limits is to put in the number it goes to, to see what it gives. So in this case I do zero over two, right? [...] Then [...] what I would do is factor. Now I do not remember, if I factor a negative one, can I cross them out? [Student S1 took common factor x in the denominator but then got stuck at the fact that the other factor in the denominator was $x + 1$ and not $x - 1$ as he expected. He was then trying to factor out $- 1$ so as to have the same factors and cancel them out; something that cannot be done in this case.]

The analysis of students' explanations following my intervention, however, shows that what was triggering the factoring was a routine sequence of actions ("steps") of which, for these students, substitution was the first in line. Although S1 said he did substitution "to see what it gives", he disregarded the result and tried to factor. In the next minutes of the interview, there was this exchange:

> I: The first thing you did was to put in the one...
>
> S1: Yeah. The next step is to factor.

Thus, it seems that students were doing substitution not to find the limit, or to characterize an indetermination, but because that's "what you do first". They were not paying attention to the outcome of the substitution. Rather, they were engaging in a kind of *normal* behavior developed for the context of finding limits of rational functions: to find the limit of a rational function, some algebraic technique has to be applied (see discussion of problem 2.4). For example, realizing that there are no common factors in the expression in problem 2.1, student S2 tried another algebraic approach:

> S2: What I think might work is if I separate them into two different parts. If I have x over x squared plus x minus one over x squared plus x. Which follows the rules of how you are allowed... how you are allowed to solve for limits. Now I can easily put in the x values, so one over two minus one over two, so in this case the limit does equal to zero.
>
> I: Ok, and why do you think you have to go through this step?
>
> S2: Well, I guess I could just put in the one here, but I am used to have something divided by zero.

The last sentence of S2 makes explicit his expectations about the types of tasks he can be given in the context of limits of rational functions. Such expectations could also be found in other students' talk.

When questioned, many students realized that some of the "steps" were not necessary; they explained their behavior as the result of following perceived *norms*. For example, student S18 explained her behavior (she used the word "factorable" to mean that there are common factors in the numerator and the denominator):

> S18: Basically I look at a problem and the first thing I see... and I always assume it is factorable, I mean, they never gave me a problem that wasn't factorable, so I wouldn't even ask whether it is factorable. I'd say, ok, where can I factor it. And I'd say ok let's look at the different categories. If I see a trinomial or a difference of squares and the method to factor them, and so long and so forth, but if it wasn't factorable... I never came across a problem that wasn't factorable.
>
> I: And do you remember ever coming across something like this [2.1]?
>
> S18: Yes, it is one of the tricky ones. You have to think of a special... I do not know, I am not saying this would work, but you can multiply by negative one to inverse the signs of your equation and then you'd be able to simplify and it would work out.

Even in the interview, which was an event outside of the institution, student S18 thought she would not be given a problem that is not "factorable". For her, problem 2.1 is a problem to be solved by factoring. It is just that she does not know how to factor it: a "tricky" limit.

Student S3 seemed to hold the same assumptions as S18; he factored because he thought it was a zero over zero type of problem:

> I: But why did you factor here?

S3: Because I try the zero over zero thing not realizing...

I: When did you realize the denominator was not zero?

S3: I was substituting one on the bottom.

I: Did you do the substitution here or here [the initial form or the factored form]?

S3: No, here [the factored form].

Student S17 made a strong assertion to defend his factoring approach. When asked, "Was it necessary to factor?", he said:

S17: Was it necessary? No. But I was taught, if you can factor, factor.

In the same vein, student S15 answered this question by saying:

S15: Oh, no, no, I do that in every problem: I see if something would cancel. Even if nothing cancels I do it anyway, in case I miss something.

5.2.1.2 Students' solutions to problem 2.3: $\lim_{x \to 5} \frac{x^2-4}{x^2-25}$

Students displayed a very similar behavior in their approaches to problem 2.3. Of the twenty eight interviewed students, sixteen factored before trying direct substitution, seven factored after trying direct substitution. Their reasons for factoring were very similar to those expressed for problem 2.1: either their strategy is always to factor first or they expected the problem to be an indetermination of the type zero over zero. Again, their explanations refer to *norms*. For example, student S7 factored the numerator and the denominator right away, and when asked why, she said:

> S7: Well... I do not know... for me... because in most of the exercises that we were given, every time that you'd replace it'd give you zero over zero, so it is kind of a reflex.

Student S11 was frustrated by the fact that factoring would not lead to simplification, but he was convinced that there was "something else" to be done:

> S11: If I open them, there's nothing I can cancel... there must be something else. I cannot bring them up either, then I can't divide. I cannot pull any *x*s out. I'll open them [he means to factor] and I see after, maybe, but I do not think here anything will work.

The interview with student S7 also showed that she believed that some algebraic technique should be applied; factoring is the only technique she thought she remembered.

> I: What happens with that one? [She wrote that the limit equals 21/0.] Did you try anything in your mind?
>
> S7: No, nothing works out.
>
> I: Why do you say that nothing works out?
>
> S7: Ok, well. Because if I put it in I get nothing, well I get twenty one over zero. Then [...] if I factor it out it does not give me anything different, like you can't cross anything out. But then... I am trying to remember, back to Cal I, all the different steps you could do. [...] There was always first you try to factor and cross out anything that you can. Then...

5.2.1.3 Students' solutions to problem 2.2: $\lim_{x \to 2} \frac{x+3}{x^2-9}$

In problem 2.2, seventeen students were able to produce a correct answer by substitution only after factoring and cancelling out common terms. Student S6 gave an insight into the state of his mind, which suggests a psychological explanation why students were factoring and cancelling common terms before checking if this was necessary to find the limit:

> S6: [S6 tried substitution first in every problem except 2.2.] I think because I saw the top factors out... I think every time I see x square minus nine **I get mentally excited** and I want to factor out and cancel. And I knew I would be able to cancel so I was confident.

5.2.1.4 Students' solutions to problem 2.4: $\lim_{x \to 1} \frac{x^3+4x^2+9}{x^2+2}$

With respect to problem 2.4, two students "believed" the question was to find the limit as $x \to \infty$, and three tried long division or factoring. Twenty students solved problem 2.4 right away by direct substitution. Students who substituted right away observed that the problem was "too easy", but eight of these students did not do direct substitution in problem 2.2. I surmise that these eight students identified problem 2.2 as belonging to type of tasks T1, but they could immediately see that problem 2.4 did not belong to T1. The three students, who tried some algebraic technique to simplify the expression, focused on the fact that $x \to 1$ and classified problem 2.4 as belonging to T1 and then tried the techniques characteristic of 0/0 indeterminations. The two students, who "believed" that $x \rightarrow \infty$, considered problem 2.4 as an instance of type of tasks T3. One of them was student S12, whose self-analysis was particularly insightful with regard to students' thinking about the limit tasks. When given problem 2.4, she said:

> S12: [...] Isn't this one [the limit] just infinity too? I do not really remember if we are allowed, when you can like... let's say you divide [the numerator and the denominator] by x cube... Can I do it, even if I have just one x cube? Or is the rule like... because if I do it then I'll get one over zero which is infinity... **Oh, it is one, right?** [referring to the "1" in "$x \rightarrow 1$"] [...] Then it is just fourteen over three.
>
> I: Why did you think it was infinity?
>
> S12: Oh, because I like I saw... I did not look... I did not see it was one [in $x \rightarrow 1$]. So **if it would have been like as x goes to infinity** and you divide everything by x cube then here it is zero and zero [for the two last terms in the numerator and the two terms in the denominator] and here is one [for the first term in the numerator], so one by zero is infinity.
>
> I: And why do you think that... because you did not look at this [$x \rightarrow 1$]... you looked at this problem and you thought it was an infinity type of problem... did it remind you of problems you've seen before?
>
> S12: Well it is just because it is like **you can't factor this**. Can you? No, I do not think you can. **So the only thing they could ask us is [to] divide by x.**

Her explanation was key in the process of my understanding of what students believe to be the knowledge to be learned, and thus in the construction of a model of students' spontaneous models of the knowledge to be learned.

5.2.2 A model of students' praxeologies based on their performance in the second part of the interview (tasks resembling routine tasks)

It appears that students distinguish two types of limits. I describe them below:

Type 1. *When x tends to a constant, expressions are normally indeterminations of the type zero over zero, and they involve binomials or trinomials. The polynomials in these expressions can be easily factored using the standard algebraic identities learned in high school, such as difference of squares or the square of the sum, and techniques such as "undoing" the distribution property, factoring by grouping*[23]*. Limits involving expressions that contain polynomials in the "high school factoring categories" cannot be found by a straightforward application of direct substitution; something else must be done.*

Type 2. *If a rational expression involves polynomials that are not binomials or trinomials easily recognized as belonging to one of the "high school factoring categories" it must be a limit in infinity, or a limit that can be found by substituting a constant for x.*

[23] For an explanation of the factoring techniques such as "undoing the distribution property" and "by grouping" see footnotes 3 and 4 in Chapter 5.

In these types of tasks we see a mixture of mathematical, cognitive, social and didactic norms. For example, in Type 1, the sentence, "When x tends to a constant, expressions are *normally* indeterminations of the type zero over zero, and they involve binomials or trinomials", describes a *social norm*. The notion that these polynomials can be *easily* factored corresponds to a *cognitive norm*, while the techniques typically used for this factoring correspond to a *didactic norm*. These standard techniques (difference of squares, "undoing" the distribution property, factoring by grouping) follow *mathematical norms*.

Students appear to classify limits of rational expressions into different types of tasks according to their algebraic appearance, instead of using some Calculus criteria such as types of indeterminations, type of technique to be applied, convergence or divergence, etc. The technique to be used to accomplish a task is chosen based on the algebraic form of the expression. In problem 2.4, students applied direct substitution without hesitation; they took it for granted that the denominator was not zero because it did not look like the trinomials or binomials they were used to be given. Many of these same students did not check that the expression in problem 2.2 was not an indetermination; they thought it was because those polynomials do fall into the "high school factoring categories".

The technology, i.e. the discourse supporting the technique, seems to be that of *norm*: "we do this because that's what we usually do under the circumstances". This is an extrapolation of what the students actually said in the interviews:

S2: … I am used to having something divided by zero.

S18: ... they never gave me a problem that wasn't factorable.

S7: Most exercises that we were given... it'd give you zero over zero.

S12: I do not think [you can factor this]. So the only thing they could ask us is to divide by x.

I surmise that, in students' praxeology, the role of a theory justifying this technology is played by the students' trust in the authority of the teachers, the textbooks, and the solutions to past examinations. It is the institution, embodied in the persons responsible for the knowledge to be taught and learned, and in the official documents and texts, and not the students, who is responsible for the validity of this knowledge. Theory in the mathematical sense is not under the students' jurisdiction or responsibility (Chevallard, 1985: 75). This, combined with the observation that students took it for granted that the problems were routine tasks suggests that their positioning was that of *Students* or *Clients*. The difference is a subtle one. If the students' "trust in the authority" is rooted in their reliance on the integrity of the institution, then they position themselves as Students. On the other hand, if the students' trust is based on thinking that it is the institution's obligation to guarantee the validity of the theory – in the sense that society has entrusted the institution with mathematics theory whose duty is then to keep it safe – then students' positioning is that of the Client. Moreover, if students were "taking for granted" the validity of the theory because they expected the institution to follow its own norms, then they were positioning themselves as Students. If, on the other hand, their reason for taking the theory for granted was the conviction that an institution has an obligation to respect its own regulations, then they positioned themselves as Clients (as it was noted in Chapter 3, students in the position of Clients perceive some institutional norms as legalized rules, page 69).

5.3 SNAPSHOT #3: A PARTIAL MODEL BASED ON NON-ROUTINE TASKS

In this section, I analyze students' behavior in the third part of the interview.

5.3.1 The tasks used in the third part of the interview

The aim of this part of the interview was to reveal student's abilities to mobilize techniques and concepts – not exclusively related to limits – to find limits of functions or to make conjectures about them.

Students were asked to find three essentially non-routine limits (see Figure 5.3 below). As discussed in Chapter 3, Section 3.1.1.3, the purpose of asking the students to find these limits was to test the students' ability to combine the different things they have learned, such as using the calculator to conjecture limits, or reading limits from graphs, in dealing with more challenging limits. My expectation was that students would not make use of resources other than algebraic to find limits and hence, that they would not be able to find or make conjectures about the limits presented in this third section of the interview. I also wanted to verify, however, that it was not the case that students did not know any other approaches, but, rather, that the institutional emphasis on algebraic techniques had obscured any other methods that students might have learned. My expectation was that, given some instructional prompt, students would be able to *think in mathematical terms* about these limits.

3.1 Find $\lim_{x \to +\infty} e^{x} \cos(x)$	3.2 Find $\lim_{x \to -\infty} e^{x} \cos(x)$
3.3. Find $\lim_{x \to 0} \frac{\sin(x)}{x}$	

Figure 5.3 (Copy of Table 3.4). The three limits that students were asked to find in the third part of the interview.

The three limits are non-routine as they are not instances of types of tasks T1, T2 or T3. Furthermore, they are essentially non-routine because the algebraic forms of the functions do not resemble the algebraic forms of the functions in routine tasks.

The outline of the Calculus course includes the topic "to find limits numerically" and graphing exponential and trigonometric functions and compositions of functions involving them. However, the course does not cover limits of trigonometric functions; the limits 3.1 and 3.2 are essentially non-routine. Limit 3.3 is sometimes covered in the Calculus course to justify that the derivative of the sine function is the cosine function, but it is not a topic in the outline. However, it is a typical example of the application of L'Hôpital's rule, a topic covered in Calculus II, which the interviewed students had not completed at the time of the interview.

Limit 3.1 does not exist as one can find two sequences, a_n and b_n, such that $\lim_{x \to +\infty} e^{a_n} \cos(a_n) \neq \lim_{x \to +\infty} e^{b_n} \cos(b_n)$. To assess this limit, a mathematician may observe that the exponential function e^x tends to positive infinity and the cosine function oscillates between -1 and 1, hence, for the x values for which $cos(x) = 1$, $e^x cos(x)$ would tend to positive infinity, while for the x values for which $cos(x) = -1$, $e^x cos(x)$ would tend to negative infinity (or for the x values for which $cos(x) = 0$, $e^x cos(x)$ would equal 0), and from this they conclude that the limit does not exist. From the mathematical point of view, this argument is the "informal" version of the proof that the limit does not exist – the proof would consist in exhibiting two sequences a_n and b_n with the property mentioned above. This informal argumentation involves properties of functions and multiplication of functions that are known to college level Calculus students. Furthermore, as mentioned above, these students have studied (according to the outline of the course) techniques to find limits numerically (using tables of values). A problem such as 3.1, however, requires the construction of a sophisticated table of values, choosing "special" x values that can reflect the behavior of the function (this type of construction is not specially emphasized in the textbook).

Limit 3.2 has value 0. To find this limit, a mathematician may state that since e^x tends to 0 as $x \to -\infty$, and the cosine function is bounded between -1 and 1, the product of these two functions tends to zero as $x \to -\infty$. The argument is an "informal" version of a mathematical proof (using the squeeze theorem) that the limit is 0: $-1 \le cos(x) \le 1$, then $-e^x \le e^x cos(x) \le e^x$ and, using the property of monotonicity of limits, we obtain the inequalities $\lim\limits_{x\to-\infty} -e^x \le \lim\limits_{x\to-\infty} e^x \cos(x) \le \lim\limits_{x\to-\infty} e^x$. Since the rightmost and the leftmost limits are both 0, the limit in the middle must also be 0. Although college level students may not be familiar with the formalization, the "informal" argument combines features that they are supposed to know (the behavior of an exponential function, that of the cosine function, the product of functions).

The value of limit 3.3 is 1. To find this limit, a mathematician would may use of L'Hôpital's rule or of the squeeze theorem. At the time the interview took place, some students may have covered L'Hôpital's rule in their Calculus II courses and hence this technique was available to them. The squeeze theorem and its application to this particular limit are listed as optional in the outline of Calculus I. Even if some students had been exposed to this example, its reproduction requires them to handle properties of trigonometric functions in the unit circle; this knowledge cannot be assumed without caution about college level students[24]. A college level student could approach limit 3.3 by doing a table of values; plugging in a few values on the right side of 0 and a few on the left side of 0 would suggest that the limit must be 1.

5.3.2 The interview procedures: instructions and interventions

To facilitate the reading of this section I briefly recall the research procedures corresponding to this part of the interview (for a more detailed explanation see Chapter 3, page 67). I gave the students one page with the three problems and empty space below each of them. As in the other parts of the interviews, I asked the students to think aloud. If the student remained silent for more than 2 minutes, I would intervene, asking questions or suggesting different approaches to find the limits or to verify their affirmations/guesses/conjectures. If the student offered an answer that was correct, I

[24] College level students admitted into the Calculus courses are supposed to know these properties as they have studied them in their last mathematics course in high school. Experience shows that this is not the case; students have several difficulties and misconceptions in relation with trigonometric functions. The verification of this supposition was out of the scope of the present research.

would ask him or her about his or her methods and how could they be sure that the answer was right. These interventions varied according to the students' behavior.

My intervention consisted mostly in

- suggesting the student to consider the limits of e^x and $cos(x)$ independently and then to combine them to conjecture the limits of $e^x cos(x)$ at $+\infty$ and $-\infty$, and
- asking whether the student could use the calculator to conjecture the limits or to verify his or her answers.

As it can be seen in the transcription of the interviews (Appendix A) my interventions in this part were frequent but short and repetitive. I would repeat perhaps several times the question "could you use the calculator to check your answer?" until the student reacted to it, or I would insist on considering the limits of e^x and $cos(x)$ independently and then trying to combine the answers, but without giving hints on how to do this. In addition, students who eventually used the calculator tended to arrive to conclusions about the limits after doing only one substitution. In those cases, I would ask them if they were convinced by their own calculations, or if they thought that substituting with one value was sufficient, etc.

I observed that asking the students whether they could use the calculator to guess the limits or to verify the answers they had already conjectured would trigger a torrent of thoughts in the students. Independently of whether they ended up using the calculator or not, they seemed to interpret this offer as a permission to think outside the box. They would start questioning their thoughts and beliefs. Some would attempt to graph the functions involved in the problems and draw conclusions about the limits from the graphs. Although their reasoning was sometimes fraught with misconceptions, it was clear that they were able to think mathematically. Unlike the routine problems in part 2, the non-routine problems had the effect of revealing students' misconceptions; not only for me but also for the students.

5.3.3 Overview of students' performance on the tasks in the third part of the interview

Table 5.34 summarizes students' performance on the three problems before any intervention on my part. In the table, and in what follows, *correct answer* means an answer that contained not only a correct evaluation of the limit but also a reasonable mathematical justification. The category of *correct answers* excludes answers with justifications such as "zero times anything is zero", "does not exist times anything is does not exist", or "infinity times anything is infinity".

N=28	Correct answer	Incorrect answer	No answer
Problem 3.1	32.1 (9)	21.5 (6)	46.4 (13)
Problem 3.2	21.5 (6)	32.1 (9)	46.4 (13)
Problem 3.3	50 (14)	10.7 (3)	39.3 (11)

TABLE 5.32. Frequency of correct, incorrect, and lack of answer in the three problems given to students in the third part of the interview, before the interviewer's intervention.

The thirteen students in the first two entries of the third column are the same. The six students that got a correct answer for 3.2 are a subset of those that got a correct answer for 3.1. The six students who got an incorrect answer for 3.1 also got an incorrect answer for 3.2. The three other students who got a wrong answer for 3.2, got a correct answer in 3.1; these three students stated that the answers of both 3.1 and 3.2 were the same, or changed in sign ($+\infty$ to $-\infty$), thus getting an incorrect answer for 3.2.

Four students (S2, S16, S18 and S26) spontaneously suggested using the calculator to conjecture the limits or to verify their answers. These students did a table of values to infer the behavior of e^x in plus and minus infinity (all four students) and of *cos(x)* (only students S16, S18 and S26). All these students plugged in integers (9, 10, 100, -10, -100); students S16 and S18 did two different table of values, one for e^x and the other for *cos(x)*, they didn't multiply the values they had obtained for each function. Only student S26 built a table of values for the function $e^x cos(x)$. Two students, S9 and S28 neither asked for a calculator nor gave me a chance to offer it to them as they immediately provided correct answers for the three problems with mathematically valid justifications. Nine students refused to use the calculator in one way or another to calculate limits (see discussion below).

Table 5.35 summarizes students' performance on the three problems after my intervention.

In the following section, I present some more detailed information about students' behavior in this part of the interview.

N = 28	Correct answer	Incorrect answer	No answer
Problem 3.1	53.6 (15, +6)	17.9 (5, –1)	28.6 (8, –3)
Problem 3.2	60.7 (17, +11)	17.9 (5, –4)	21.4 (6, –6)
Problem 3.3	67.9 (19, +5)	7.1 (2, –1)	25 (7, –4)

TABLE 5.33. Frequency of correct, incorrect, and lack of answer in the three problems given to students in the third part of the interview, after the interviewer's intervention. The numbers preceded by signs + or – indicate how numbers of students in a given category changed with respect to performance before the interviewer's intervention (see Table 5.34).

5.3.4 Details of students' behavior in the third part of the interview

In this section, I first present students' behavior on tasks 3.1 and 3.2, which were often tackled by the students together, and then their behavior on task 3.3. In each case, I start by describing students' behavior before my intervention, which is then contrasted with their behavior after my intervention.

5.3.4.1 Tasks 3.1 and 3.2: $\rightarrow$ (;) $\rightarrow$ ( )

Students' behavior before the intervention

Of the 28 students, nineteen (67.8%) considered the limits of e^x and *cos(x)* independently and from that tried to derive a conclusion about the limits of $e^x cos(x)$ at $+\infty$ and $-\infty$. Some students knew the answers for the limits of e^x and *cos(x)* at $+\infty$ and $-\infty$; some knew the graphs and could infer the limits from them; others correctly guessed the limits by using the calculator.

In relation to problem 3.1, of the nineteen (19) students mentioned in the paragraph above, nine (9) (see Table 5.34) derived a correct answer with a mathematically valid justification (making use of the fact that the cosine function is bounded by 1 and –1, and reaches those bounds). Of the students who derived an incorrect answer (6), most argued that "plus infinity times anything is plus infinity"; one student (S22) used the argument "does not exist times anything is does not exist".

Of the nine (9) students who derived a correct answer for problem 3.1, six (6) derived also the correct answer for 3.2 (see Table 5.34), again using the fact that the cosine is a bounded function. The other three (3) students claimed that 3.2 had the same answer as 3.1, thus getting an incorrect answer for 3.2. Students who derived a wrong answer for 3.1 based on analyzing the behaviors of the cosine and the exponential functions separately, also derived a wrong answer for 3.2 (with the exception of student S18, who got a wrong answer in 3.1, plus infinity, but a correct answer in 3.2). In addition, students who could not find an answer for 3.1 could not do it for 3.2 either (thirteen students).

Of the thirteen (13) students who did not provide an answer for problem 3.1 before my intervention (see Table 5.34), three (3) remained silent in front of the problems. Four (4) elaborated on possible approaches, discussed the graphs and the behavior of the cosine and the exponential functions and tried reasoning on the behavior of their product. However, they would not find their own arguments sufficiently convincing and therefore refrained from reaching a conclusion. Of the remaining six (6) students, three (3) said they did not "remember" how to do problems like 3.1 and 3.2; two (2) said that they did not know how to do the tasks; one said that he did not "remember doing problems with the exponential function".

From students' behavior, I conclude that almost half of the interviewed students had immediate access to meaningful resources to approach problems 3.1 and 3.2; I refer to the nine students who actually found the right answers plus the four students who engaged in a meaningful discussion although they did not arrive at an answer. About 18% of the students based their answers in misconceptions, thus arriving at wrong conclusions. Finally, about 32% of the interviewed students were stuck because of the problems looked unfamiliar to them.

Students' behavior after the intervention

I intervened in interviews with students who obtained incorrect answers or who didn't provide any answers.

Of the six students who got a wrong answer for 3.1 before any intervention on my part, two ended up obtaining a correct answer. In the case of problem 3.2, the number of students changing from a wrong answer (8 students) to a correct one was six (6).

Of the thirteen (13) students who could not provide any answers for 3.1 and 3.2 before my intervention (see Table 5.34), three (3) were able to find an answer, and four succeeded in finding a correct answer for 3.2.

Despite the details described above, the increase in correct answers in 3.1 and 3.2 after my intervention shows that students did have resources to find the limits and to provide either theoretical or empirical valid justifications. It is just that these resources were not immediately available to them. Students needed to be slightly pushed in the direction of non-standard techniques, and then they were able to use them. This would suggest that these non-routine problems and the reasoning involved in solving them are within the "zone of proximal development" of these students. Moreover, students who got incorrect answers or no answers at all in the beginning showed later that they were capable of critical and scientific thinking. For example, the following exchange took place with student S14, who first got an incorrect answer for 3.2 and whom I asked if she could use the calculator to verify her answer (see pages 330-331):

S14: [Calculating.] Syntax error... [after more calculations] it is zero.

I: Is that just for *e* to the *x*?

S14: No, no, for *e* to the *x* times cosine of *x*. It gave me zero.

I: Which number did you plug in?

S14: Nine nine... ten nines, with a negative sign.

I: So you got zero. Are you convinced it should be zero instead of does not exist as you wrote before?

S14: I know the calculator lies sometimes [laughs]. But, yeah, I think so, it keeps getting to zero.

I: Now that the calculator told you that is zero, do you have an argument of why that's true?

S14: I know that *e* to the *x* as *x* goes to negative infinity is zero, and the cosine keeps going so there's no limit, and I guess that if you multiply them... I can't picture the graph on my head, I can picture them separately but I can't combine them. But I would think that it does not exist though... by multiplying it, it changes the amplitude... then it shouldn't exist because it keeps going anyway.

I: But the fact that this is going to zero is changing the amplitude...

S14: Oh, but it is getting smaller and smaller and smaller, so it would be zero. Yeah.

I: You are changing the amplitude but each time by a smaller factor.

S14: Yeah, so the limit is zero. That shows that the limit can be in the middle of the thing. [This last observation refers to a discussion we had before about whether a function could cross one of its asymptotes.]

Student S14 was critical about her own conclusion, inferred from the data obtained by using the calculator. She *understood* that independently of the exponential function, there is an infinite oscillation induced by the cosine function, but eventually she *saw* that the values were getting smaller and smaller – in absolute value – and could picture the situation in her mind: "the limit can be in the middle".

Student S20 did not give answers for 3.1 and 3.2 before my intervention; however, after my suggestion of considering the graphs of cosine and the exponential function, and the use of the calculator, the following exchange took place:

I: So what the limit would be when x increases?

S20: I am taking this in Cal II now, something about this. This [the cosine function] diverges, does not give... Oh, that's the topic. Convergence and divergence. If it gives a definite answer, then it converges. If it does not, it diverges. This is a divergent limit [the limit of $cos(x)$ as x tends to infinity]. So there's no proper answer for it.

I: Ok. And what about when you multiply by the exponential function.

S20: But this is unknown, I do not know what it is [3.1]. But this [3.2] is a number like say, between one and minus one, multiplying it by zero is gonna give zero, I think. This is my guess.

I: Ok. And what about this one? [3.1]

S20: This is gonna give plus or minus infinity. If it is cosine of a negative number, maybe it is going to give you something negative. The number x chosen will give a negative number, multiplied by a big number gives minus infinity.

Eventually, student S20 was able to reason out the behavior of the function $e^x cos(x)$ at $+\infty$ and $-\infty$ from the fact that the cosine function is bounded by 1 and -1 and the respective limits of the exponential function.

5.3.4.2 Task 3.3: $\lim_{x \to 0} \frac{\sin(x)}{x}$

Students' behavior before the intervention

Prior to my intervention, of the twenty eight (28) students, fourteen (14; 50%) gave a correct answer in task 3.3. Of these 14 students, five (5) used L'Hôpital's rule, one student (S2) used his calculator, seven (7) said they had memorized the answer, and one student (S28) sketched the geometric proof typically given in college level Calculus textbooks. Among the other fourteen (14) students, three (3) gave an incorrect answer and 11 did not give an answer (see Table 5.34).

Of the students who could not provide an answer, four recognized the problem as one they had seen before, but said that they did not remember how to find the limit, or what the answer was.

Students' behavior after the intervention

Of the three students who got incorrect answers before my intervention, one ended up finding the correct answer using L'Hôpital's rule and another realized that her answer could be wrong. Of the eleven students who did not give an answer, four were able to find the right answer (three using the calculator and one using L'Hôpital's rule).

The behavior of the function *sin(x)/x* around zero is quite difficult to *see*, and impossible to graph. Hence, rather than discussing the graph, students engaged in a discussion related to finding limits numerically. In this sense, this problem proved to be useful in showing some of students' misconceptions. For example, 18% (5) of the students said they could not use the calculator to find the limit because the only thing they could think of was plugging in zero, and thus getting zero over zero. Students said, for example:

> S18: I don't think [the limit] exists. Because you can't divide by zero, so I'd say the limit does not exist (page 354).
>
> S24: You get the limit when you put it in, when you make x equal whatever the number is here [in the expression $x \rightarrow c$] (page 393).

To my question "what this [the symbol $x \rightarrow 0$] means for you?" student S27 answered:

> S27: When x is exactly at zero (page 409).

In some cases, when students knew by heart that the answer for the limit was one or used L'Hôpital's rule to calculate it, I asked them if they would be able to find the limit if they couldn't remember the answer or if they knew another way to find the limit that wasn't using L'Hôpital's rule. For example, the following exchange took place with student S23 (page 386):

> I: If you didn't know the answer was one. If you did not remember, could you still figure it out?

S23: Probably plugging numbers?

I: Ok, how would you do that?

S23: Sine of five... and then closer to zero.

I: Which other numbers?

S23: What do you mean?

I: Well, you said you'll plug in five first, what you'd do next?

S23: Plug in one? First I plug in zero. Sine of zero is zero.

I: So what happens if you plug zero?

S23: I guess it would be one or zero. Well, zero over zero is a... what's the word in...

I: Indetermination?

S23: Yes, I don't know what I'd do.

Student S20 said that the limit could be found using the *squeeze* theorem and wrote a flawed argument using it. Student S25 said (page 398):

S25: Oh, I do not know, I thought that for some reason the answer is one, but if you plug in zero you get zero over zero.

In the next section, I describe the "third snapshot" of students' practices in relation with limits tasks.

5.3.5 A model of students' praxeologies based on the third part of the interview (non-routine tasks)

The preceding analysis shows that almost 50% of the interviewed students had immediate access to meaningful resources to tackle non-routine problems – thirteen students in problem 3.1, ten in 3.2 and fourteen in 3.3. From the remaining students, on average, 38% of the students were able to arrive at correct answers after the intervention, showing that they did have resources, but had not considered them without encouragement (four out of fifteen students in problem 3.1, nine out of eighteen in problem 3.2, and five out of fourteen in 3.3).

Independently of their performance, all students recognized problems in this part as non-routine ones. Their discourse related to this recognition was based on unfamiliarity: they claimed that these were problems that they have not often – or ever – done ("I don't remember doing limits with e^x", "I don't know", "do we cover this in Calculus I?"). In this sense, I surmise that, in the most general setting, students immediately classify tasks they are confronted with into those they have done ("familiar") and those they have not done (unfamiliar). The set of familiar tasks contains the set of routine tasks, but these sets are not identical. For example, for many students, 3.3 was a familiar task, although it does not qualify as a routine task (in this thesis) as it does not appear on final examinations.

These two types of tasks – "familiar tasks" and "unfamiliar tasks" – define two different praxeologies related to limit finding tasks. The techniques to tackle unfamiliar tasks include finding limits from graphs and numerically. It is my conjecture that the technology associated with these techniques is precisely the fact that the problems are of the unfamiliar type. The justification of the technique is that *routine techniques do not apply to these problems* – for example, the expressions cannot be factored or rationalized – and therefore one must resort to unorthodox means. This justification is a *cognitive norm*. On the theory level, the students made use of *mathematical rules* (e.g., the limit of a bounded function multiplied by a function that tends to zero is zero), and *mathematical strategies*: e.g., making a table of values to conjecture the value of a limit. Therefore, students' spontaneous models to tackle unfamiliar limits are based on cognitive norms and mathematical rules and strategies. When dealing with these types of problems, students eventually – before or after my intervention – positioned themselves as Learners, bringing together different techniques and concepts, challenging their own beliefs, their confidence in the calculator, and being suddenly aware of their misconceptions. Hence, there is a radical difference if we compare this partial model with the model inferred from snapshot #2, which was based on *social, didactic, cognitive and mathematical norms*. In the next section, I put together the three partial models in an effort of deriving a general model of students' spontaneous models in front of the task of finding limits.

5.4 PUTTING THE PICTURES TOGETHER: A MODEL OF STUDENTS' SPONTANEOUS MODELS OF THE KNOWLEDGE TO BE LEARNED

The three partial models described in sections 5.1, 5.2 and 5.3 complement each other in the sense that they are snapshots of students' behavior in front of different types of problems related to the same task: "find the limit of a given function".

From snapshot #1, I was able to construct a model of students' models to deal with routine limits (e.g., items 1, 3, 4), non-routine limits that resemble routine ones (e.g., items 5, 9, 12), and non-routine limits that vaguely resembled non-routine ones (e.g., items 13, 14). The inclusion of the two expressions involving trigonometric functions (items 19 and 20) in the classification task helped to throw light on how students perceive non-routine problems that do not resemble the routine ones in an essential way. Some students recognized them as such, but others identified 19, $\lim_{x\to 0}\frac{\sin(x)}{x}$, as a limit they were familiar with. This was also the case in the third part of the interview where they were explicitly asked to find that limit (see discussion at the end of section 5.3).

Based on a comparison of students' behavior in the three parts of the interview, I surmise that students identify as a routine problem any problem that resembles it – where the resemblance is based on the algebraic form of the function – and that students do not consider the possibility of using non-standard techniques on routine problems. These two situations combined make students unable to tackle problems that resemble routine tasks but differ from them on the conceptual level. For example, student S6 based his classification in part one on recognition of familiarity with the given expressions. Then he failed to find the limit 2.3, which resembled a routine problem, in part 2 of the interview. He was saying, when dealing with the limit 2.3. ($\lim_{x\to 5^-}\frac{x^2-4}{x^2-5}$):

> S6: My first reflex was, is always, I put the five here so I saw it was over zero. [...] Again I factor out but... [...] I can't remember how to treat it. [...]

However, he succeeded in finding the limit 3.2 ($\lim_{x\to-\infty} e^{x}\cos(x)$). First he said that the value of the limit was the same as the one he gave for 3.1 (infinity) but after I made a brief intervention he found the correct limit by reasoning about the behavior of the two functions involved in the expression and their product. Our exchange was the following:

> I: What about e to the x?
>
> S6: Oh, no, it would all go towards zero, yes, that's true. e to the x would be closer and closer to zero. So this would go towards... but I do not know how to treat the cos in that situation. **I know this goes towards zero** [for the exponential function] **and cos ... Because even if this is going towards zero, if my cos is getting bigger and smaller... but it is always the same sequence multiplied by that, I guess I could assume it is going towards zero.**

He even made the non-trivial reflection that, if a function is going towards zero and it is multiplied by a function that is getting bigger and smaller, that does not guarantee by itself that the limit would be zero. He convinced himself that the limit was zero when he realized that the behavior of cosine was "always the same sequence".

Students S16 and S18 made a classification based on the algebraic form of the functions, evoking topics in a school Algebra textbook in the first part of the interview. Both of them failed in finding the limit 2.3. When given 2.3, student S16 said:

> S16: I know there's a difference of squares but that wouldn't change much because nothing will cross out. If I put the five it'd become a number over zero. I guess there's no limit. I'm not sure.

Later on, when he and I we were discussing his approaches to all the problems in part two, I asked him why he thought that there was no limit. He said:

S16: Well, I'm not sure because there's nothing I can cancel out and I'm missing something...

Yet, when given problem 3.1, he spontaneously thought about using the calculator:

S16: Well, this [for the limit of the exponential function in 3.1] is infinity... [...] Can I use my calculator?

Student S18, like students S6 and S16, was stuck when trying to apply standard techniques. When given problem 2.3, she said:

S18: You are giving me the ones I like, I see difference of squares. The first thing I'd do is to break it out. But I can already tell by the twenty five, that the five would pose a problem at the bottom. So x plus five, x minus five. The problem is that I need to get rid of the x minus five or else my denominator will end up being zero, which isn't good at all. What can I do to get rid of the denominator? How can I do to get rid of the x minus five? I am stuck.

Yet when dealing with problems 3.1 and 3.2, not only did S18 spontaneously ask to use the calculator and used it to conjecture the limits, she was also able to reason. Her reasoning went along the following line: because the cosine function is bounded (she considered incorrect bounds, however, confusing the values of the angles with the value of the cosine), what matters for the limit is the behavior of the exponential function. When given task 3.3, the following exchange took place:

S18: Oh... I am not a big fan of trigonometry... How does cos look again? **I can't use a calculator to see the pattern, can I?**

I: Yes, you can. [...]

S18: Ok, this equals positive infinity... oh, no... Ok, I'd say positive infinity for now.

I: Which calculations you were doing?

S18: Oh, trial and error. As x gets bigger what's y. Cos of ten, cos of a hundred. Then I went to e to the x and it gets bigger, I should have remembered. So basically you multiply and just approaches infinity, from what I remember. [...] Here [3.2] I do more trial and error with negative numbers. In this case, as x approaches negative infinity... we are getting closer and closer to zero. So I would say zero.

I: So exactly which calculations were you doing?

S18: I was doing e to the negative ten, e to the negative a hundred, to see the pattern, and it gets smaller and smaller and smaller, approaching zero.

I: But you did not try the multiplication.

S18: No, because the way I see it is cosine, whether is a positive or a negative it keeps revolving around pi, pi over two, three quarters pi, around and around in the circle and it the number would never keep getting smaller or bigger, it just keeps revolving that circle. So basically what matters to me is e to the x.

Student S21 also failed in finding 2.3:

S21: Ok. So at the top and bottom there's a square term, like x square and four, two square, and x square and five square. If you put in the five I do not think it would be an indeterminate form. Twenty five minus four nineteen over zero. I think you have to fix that somehow.

I: Why?

S21: Because if it is over zero I think it could mean infinity, or it is not defined. So you have to... I think maybe L'Hôpital's rule... [...] At the top you could do also x plus two and x minus two. I think x plus two times x minus two over, it would not help though, x minus five times x plus five. It could be undefined? At five?

However, when given 3.1, he reasoned:

S21: [3.1] Well, there's an e, that would keep growing. Cosine goes between one and negative one. Well, the cosine you can't really find it, it keeps going up and down so the limit does not exist. For the e to the x it would be positive infinity. It might be that you have to do L'Hôpital's rule but I am not sure. Oh, no, because it is not a division. You have to make it a division. e x, because cosine is also one over secant. Then L'Hôpital's rule you can do the derivative of the one on top over the derivative of the one on bottom. The derivative of secant is secant x tan x, I think. I think the limit... **Well, since cos is either one or negative one it would be positive infinity or negative infinity. But since it is to infinity, it can't really exist.**

He was also stuck at the resemblance of 2.3 with routine problems and could not reason out the value of the limit. Yet, in 3.1, he considered different approaches and finally he reasoned that a function bounded between one and negative one, when multiplied by a function diverging to positive infinity, must oscillate infinitely.

These four students were unable to find limit 2.3. They could not overcome the fact that the typical techniques that apply to routine tasks did not seem to work in this case. They claimed that they did not remember how to deal with a limit of this kind or that they could not see how to use the standard techniques. Nevertheless, when given the essentially non-routine tasks, they proved to have other resources to deal with limits; they either spontaneously thought of using the calculator, or they reasoned about the behavior of a function from the behavior of its factors.

Furthermore, from students' discourses in the different parts of the interview, I surmise that when dealing with familiar tasks – parts 1 and 2, and problem 3.3 – they do not feel the necessity to support their techniques with a scientific conceptual discourse. They position themselves as Students or Clients. These positions make it natural to rely on (or refer to) the institution's authority when asked to justify. However, when dealing with unfamiliar tasks – problems 3.1 and 3.2 – students seem to switch to the position of Learners, responsible for the explanatory discourses.

In trying to combine the partial models inferred from snapshots #1, #2 and #3, I noticed that in some sense they define nested praxeologies. In Table 5.36 I tried to schematize the *picture* made out of the three snapshots. Of course, the picture is still (will always be) incomplete.

Students distinguish familiar limits from unfamiliar limits mostly based on the algebraic form of the function – rational functions and functions with radicals are familiar – and on their recall of having dealt with a given instance – for some students the limit $\lim_{x\to 0}\frac{\sin(x)}{x}$ is a familiar one. Within familiar limits, they distinguish routine from non-routine. These distinctions are not of mathematical nature, but of social, cognitive and didactic kind. Based on these differences, students decide which techniques to apply. The possible causes and implications of this model are discussed in the next chapter.

TASKS			TECHNIQUES	TECHNOLOGY	THEORY
T_1: Familiar tasks	T_{11}: Routine tasks	T_{111}: Type 1	τ_{111}: Techniques in PB1 and PB2.	θ_{111}: Based on the algebraic form of the function; evokes typical examples in college Calculus textbooks. Mixture of social, didactic, cognitive and mathematical norms. (Complexive thinking.)	Θ_{111}: Absent or of syncretic nature.
		T_{112}: Type 2	τ_{112}: Techniques in [M]P3 or direct substitution.	θ_{112}: Based on the algebraic form of the function; evokes typical examples in college Calculus textbooks. Mixture of social, didactic, cognitive and mathematical norms. (Complexive thinking.)	Θ_{112}: Absent or of syncretic nature.
	T_{12}: Tasks that don't resemble routine tasks (e.g. limit of $sin(x)/x$ as $x \to 0$)		τ_{12}: Memorizing or using non-standard techniques (finding limits by graphing or numerically).	θ_{12}: The problem cannot be tackled with standard techniques; a cognitive norm. (Complexive thinking.)	Θ_{12}: Absent or of syncretic nature.
	T_{13}: Tasks that resemble routine tasks		τ_{13}: Based on τ_{111} and τ_{112}.	θ_{13}: Based on the resemblance with T_{11}; a social norm. (Complexive thinking.)	Θ_{13}: Absent or of syncretic nature.
T_2: Unfamiliar tasks			τ_2: Finding limits by graphing and numerically.	θ_2: The problem cannot be tackled with standard techniques; a cognitive norm.	Θ_2: Based on mathematical rules and strategies (conceptual thinking).

TABLE 5.34. A model of students' spontaneous models of the knowledge to be learned about limits.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

S12: Well it's just because is like you can't factor this. Can you? No, I don't think you can. So the only thing they could ask us is to divide by x. (Student S12 explaining why she assumed that the expression $\lim_{x \to 1} \frac{x^3 + 4x^2 + 9}{x^2 + 2}$ *corresponded to a limit taken at infinity.)*

In this chapter, I present a discussion of the results described in the previous chapters. Then I discuss these findings in relation with previous literature about the teaching and learning of limits. In the third section, I summarize the conclusions of this thesis. The chapter ends with directions for future research.

6.1 DISCUSSION OF THE RESULTS

The following discussion is based on findings presented in Chapters 4 and 5. First, I discuss my theoretical model of instructors' spontaneous models of the knowledge to be learned about limits. Next, I discuss my theoretical model of students' spontaneous models of the knowledge to be learned, their modes of thinking and their positioning in the Community-of-study institution. Finally, I reflect on the possible relations between instructors' and students' models.

6.1.1 *Knowledge to be taught and knowledge to be learned: the institutional perspective*

In Chapter 4, I presented a characterization of the mathematical praxeologies MP1, MP2 and [M]P3 in the College-Calculus institution, relative to the "find the limit" tasks appearing in final examinations. From this characterization, and considering instructors' solutions to final examinations, a model of instructors' spontaneous models of the knowledge to be learned about limits was built (where instructors were taken as participants of the Final-Examination institution). This model was a compound of three praxeologies reduced to practical blocks (their theoretical blocks were virtually non-existent), labeled as PB1, PB2 and PB3a/b, corresponding to the routine types of tasks T1, T2 and T3, respectively.

As observed in Chapter 4, with respect to MP1 and MP2, the outline of the course indicates sections of the textbook in which explanatory discourses on both the technology and the theory levels are given. In particular, the technology (in both MP1 and MP2) refers to the fact that if two functions agree in all but one value *c*, their limits at *c* are the same. This statement is justified by a graph (second level of justification – theory level); a mathematical proof of the statement is presented in an appendix, which is not listed in the outline of the course. As it has been pointed out before, the fact that the technology and the graph justifying it belong to the topics listed in the outline indicates that this is knowledge to be taught from the perspective of the College-Calculus institution. The mathematical proofs (and definitions) belong to the scholarly knowledge. This means that an important fragment of explanatory discourses in MP1 and MP2 does belong to the knowledge to be taught[25]. The situation of the praxeology [M]P3 is strikingly different.

From a mathematical point of view, the techniques in MP1 and MP2 are unavoidable to solve tasks of types T1 and T2. Textbooks provide several examples of the applications of the techniques $\tau 1$ and $\tau 2$, and these examples can be used by students as "templates" to apply the techniques.

With respect to tasks of type T3 in [M]P3, textbooks provide several examples of applying techniques $\tau 3a$ or $\tau 3b$[26] (see Chapter 4). These techniques, however, are not as unavoidable as $\tau 1$ and $\tau 2$. I explain what I mean in more detail. Consider, for example, T1 and $\tau 1$, which I recall here.

TASK T1: Find $\lim_{x \to c} \frac{P(x)}{Q(x)}$

Description: c is a fixed constant; $P(x)$ and $Q(x)$ are polynomials such that the factor $x-c$ occurs in both $P(x)$ and $Q(x)$; $x - c$ has degree one in $Q(x)$.

TECHNIQUE τ1: (Substitute c for x and recognize the indetermination *0/0*.) Factor $P(x)$ and $Q(x)$ and cancel common factors. Substitute c for x. The obtained value is the limit.

[25] As it was mentioned before, in the context of this research, knowledge to be learned is understood as a subset of the knowledge to be taught whose minimum core can be deduced from tasks appearing in final examinations.

[26] Not only in the textbook assigned to the course in the studied institution. As observed in footnote 17 in Chapter 4, this is the typical approach to limits of rational functions at infinity in college level Calculus textbooks.

In mathematical symbols:

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \lim_{x \to c} \frac{(x-c)P_1(x)}{(x-c)Q_1(x)} = \lim_{x \to c} \frac{P_1(x)}{Q_1(x)} = \frac{P_1(c)}{Q_1(c)}$$

where $P(c) = 0$ and $Q(c) = 0$, but $x - c$ is a factor of neither $P_1(x)$ nor $Q_1(x)$.

To find the value of the limit, it is essential to know the form of $P_1(x)$ and $Q_1(x)$. In fact, the goal of the technique is to find these two polynomials, which are factors of $P(x)$ and $Q(x)$. A similar observation can be made with respect to T2 and τ2.

Now consider, for example, T3 and τ3a (technique τ3b is only a slight modification of τ3a):

TASK TYPE T3: Evaluate the following limit: $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$.

Description: $P(x)$ and $Q(x)$ are polynomials such that m, the degree of $P(x)$, is less or equal to n, the degree of $Q(x)$.

TECHNIQUE τ3a: Divide both $P(x)$ and $Q(x)$ by x^n. Simplify each term and then use the algebraic properties of limits and the fact that the limit of a constant over a power of x, as $x \to \infty$, is 0.

In mathematical symbols:

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\infty}\frac{\sum_{i=0}^{m}a_ix^i}{\sum_{i=0}^{n}b_ix^i}=\lim_{x\to\infty}\frac{\sum_{i=0}^{m}\frac{a_ix^i}{x^n}}{\sum_{i=0}^{n}\frac{b_ix^i}{x^n}}=\lim_{x\to\infty}\frac{\frac{a_m}{x^{n-m}}+\varepsilon}{b_n+\delta}=\begin{cases}0 & \text{if } m<n\\ \frac{a_m}{b_n} & \text{if } m=n\end{cases}$$

where ε and δ tend to *0* by virtue of the fact that the limit of a constant over a positive power of x, as $x\to\infty$, is *0*.

To find the value of the limit, it is irrelevant to know the forms of ε and δ, it suffices to know a_m, b_n, m and n, which are values explicitly given in the expression $P(x)/Q(x)$. Once this is understood, the use of technique τ3a can be avoided. Instead, for every instance of T3, the following technique, labeled τ3, can be applied:

> **TECHNIQUE τ3:** (as formulated in the "guidelines" in the textbook.) Let $P(x)$ be the numerator, with leading coefficient a_m, and $Q(x)$ be the denominator, with leading coefficient b_m. Compare m and n, the degrees of $P(x)$ and $Q(x)$, respectively. If $m = n$, the value of the limit is a_m/b_n. If $m < n$, the value of the limit is *0*.

For this technique τ3, the general description of τ3a would be a "technology" θ3: a proof of the validity of τ3. In the praxeology where τ3a is a "technique", for each particular task of type T3, students are performing an illustration of the proof, defying the "economy of mathematics" (Castela, 2004: 37), but satisfying the norms of the didactic contract (Brousseau, 1997: 31) suggested in the model solutions of final examination questions provided by instructors, and, to some extent, in the textbook.

Some textbooks (in particular the one assigned to the Calculus course in the studied institution) provide the technique $\tau 3$. In these textbooks, however, there are no examples of applications of this technique, and, there are no mathematically valid justifications of this technique: the technique is given after several examples of the application of $\tau 3a/b$ to different, particular instances of T3, without making explicit how this "new" technique is obtained. The absence of examples of applications of $\tau 3$ disrupts the didactic flow of the textbook. In the case of tasks of type T1 and T2 the worked out examples can be used by students as templates to apply $\tau 1$ and $\tau 2$. In the case of tasks of types T3, when the students search for templates in the textbook to apply technique $\tau 3$, they find applications of $\tau 3a/b$.

The sequence T3 – $\tau 3a$ – $\tau 3$ – $\theta 3$ is a very simple example of how techniques and proofs are sometimes developed in mathematics. A particular task is presented (for example, an instance of T3). Relevant features of this task are identified and a type of tasks is defined (e.g., T3). A technique is developed to tackle the particular instances that are given (for example, $\tau 3a$). It is noticed that a generalization of this technique can be obtained ($\tau 3$). This generalization becomes a technique for solving the type of tasks, and the previous method of solving the tasks (such as $\tau 3a$) is generalized to give a proof of the new, more efficient technique (part of $\theta 3$). It is hard to understand why the opportunity to show this construction is missed in the textbook, since the operations required are at the level of high school Algebra. Whether this opportunity is used in the classrooms or not, cannot be inferred from the available data. Nevertheless, as noted above, instructors' solutions to final examinations show the use of τ3a (or τ3b) to tackle instances of T3 (model PB3). Hence, the model of instructors' knowledge to be learned (relative to the Final-Examination institution) corresponds, in the case of tasks of type T3, to a Babylonian way of doing mathematics – do the same for every particular problem, again and again, without recognizing patterns in the techniques. The goal seems to be set in identifying patterns in the tasks – and so to have types of tasks – but the further step of recognizing patterns in techniques and theories seems to be avoided at college level – at least in relation to the topic "limits of functions".

Furthermore, in the three praxeologies, MP1, MP2 and [M]P3, the practical blocks belong to the knowledge to be taught, but (parts of) the theoretical blocks belong to this type of knowledge only in the cases of MP1 and MP2. In the case of [M]P3, the complete theoretical block is exclusively in the domain of the scholarly knowledge. There are no portions of the theoretical block of [M]P3 that belong to the knowledge to be taught, not even in the form of intuitions as it is the case for the theoretical blocks of MP1 and MP2. The institution's rationale for this choice remains to be investigated.

The model of the instructors' spontaneous models of the knowledge to be learned about limits built in Chapter 4 reveals that these spontaneous models correspond only to practical blocks (PB1, PB2 and PB3a/b), that is, to systems of tasks and techniques. In other words, instructors, as participants of the Final-Examination institution, do not expect students to reproduce the explanatory discourses corresponding to the knowledge to be taught about MP1 and MP2. Furthermore, as explained above, instructors do not expect students to learn, derive or even use, the technique τ3. On the contrary, they expect students to reproduce technique τ3a or τ3b whenever an instance of T3 is given. In the next section, I discuss how these expectations (the "PBs model") may influence students' spontaneous models of the knowledge to be learned.

6.1.2 Institutionalized and non-institutionalized practices

At the end of Chapter 4, it was observed that the occurrence of types of tasks T1, T2 and T3 is not institutionalized in the sense of the IAD framework – there are no explicit *rules* stating that these types of tasks have to appear in final examinations. On the contrary, their occurrence is the result of a tradition, i.e., of following certain *norms*. Furthermore, none of the techniques corresponding to these tasks are institutionally regulated, whether by the College-Calculus institution, or by the Final-Examination sub-institution. On the one hand, there are no explicit rules that would state which techniques students have to use. On the other, the "value" (in the form of a grade) that an instructor assigns to the use of a certain technique is left to his or her discretion[27].

It follows, from the observations above, that the PBs model of the knowledge to be learned about limits, from the perspective of the Final-Examination institution, is not regulated within the institution: neither the occurrence of the tasks nor the techniques to be used to accomplish these tasks obey some explicit rules. I do not mean to say, however, that I think these practices *should be institutionalized.* I only wish to highlight the fact that these practices are based on *norms* rather than rules. It is this, more than the absence of institutional rules, that allows students to base their models of the knowledge

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[27] For example, one teacher may give a higher grade to a student using an algebraic technique than to a student employing a numeric or a graphing technique, while in the same term and with respect to the same final exam, another teacher may assign the same grade to students using these two different approaches.

to be learned on social, cognitive and didactic norms rather than on mathematical rules. The "normative" character of instructors' models of the knowledge to be learned emphasizes learning on the plane of tradition rather than on the scientific, mathematical plane. It is as if the implicit institutional discourse was, "this technique is used to solve this problem because this is how things are usually done here" instead of, for example, "this technique is used to solve this problem because it is one of the (many) mathematical strategies to find the answer and because of this or that mathematical feature of this problem, it is an efficient strategy, better than...". This, as it is discussed below, may have the effect that students end up learning how to behave *normally* rather than how to behave *mathematically*.

6.1.3 Knowledge to be learned: the students' perspective

In Chapter 5, I have built partial models of students' spontaneous models of the knowledge to be learned about limits, which were then combined to outline a general theoretical model of students' spontaneous models of the knowledge to be learned.

Of course, any investigation of a (didactic) phenomenon is partial; we can only aim to take one or more "good" snapshots of it that would allow us to make inferences that reflect some relevant features of the phenomenon. In this sense, the three snapshots taken in this research are just pieces of a bigger picture that can never be completed. A combination of these pieces allowed me to build a model that might help explaining students' behavior in front of limit finding tasks (Chapter 5, Section 5.4).

In the following sections I reflect on the possible causes and consequences of the partial and the general models.

6.1.2.1 The didactic organization of concepts vs. a conceptual system

The first part of the interview (the classification task) provided me with a snapshot of students' *spontaneous* behavior when considering limit expressions. In the other two parts of the interview, there was a context that certainly influenced students' behavior. In the second part, students were asked to solve non-routine tasks that resembled routine tasks; in the third part they were asked to solve essentially non-routine tasks.

The classification task aimed at understanding how students contextualize the limit expressions. At least 68% (19 out of 28) of the students proposed a classification that evokes the *way* in which concepts are presented in textbooks (high school Algebra and college level Calculus textbooks). Of course, concepts and techniques *have to be presented in some way*. Students' behavior, however, suggests that, for many students, concepts have become closely associated with their didactic organization. Furthermore, because the presentation of techniques $\tau1$, $\tau2$ and $\tau3a/b$ in the textbooks and in the instructors' solutions to final examinations puts a strong emphasis on the algebraic aspects, students may have misinterpreted these techniques as purely algebraic. Thus, for example, students would consider the features "rational functions", "radicals", and "none of these" (see section 5.1.1.1), as their guiding features to choose a technique, instead of, for example, considering types of indeterminations. In fact, the routine tasks are such that interpreting them and the corresponding techniques from an algebraic perspective is likely to produce a correct solution. Students may take it as a fact that the algebraic approach always *gives* the right answer, and thus become unable to tackle problems for which this algebraic approach fails. This conjecture from students' behavior in the first part of the interview, has been confirmed by their behavior in the second part, where students seemed rather obsessed with using some algebraic technique (for example, 71.4% of the students tried technique $\tau 1$ in problem 2.1 in the second part of the interview).

The analysis of the first and second parts of the interview revealed that students have a "step by step" procedure to tackle limit finding tasks. This procedure can be inferred from students' own explanatory discourses, e.g. (see also Section 5.2):

> S1: Ok. **The first thing I do when I see limits is to put in the number it goes to** to see what it gives. So in this case [Task 2.1] I do zero over two, right? [...] **Then** [...] **What I would do is factor.** Now I don't remember [...]
>
> S1: Yeah. The methods. There was always **first you try to factor and cross out anything that you can. Then...** [...]
>
> S8: These... **basically rationalization** [explaining why certain items were put in class 1], like that's the first thing I think of. **I have to do that first. These I have to factor first** [class 2]. (Student S8's explanatory discourse in the classification task.)
>
> S8: Yes, I have to factor, **I have to see if I can factor first and then if something cancels out**. [...] First plug it in, I guess. This is so Cal one... There is no reason to cancel out if it doesn't give zero on the denominator [she still factored the expression]. (S8 explaining her general approach to problems in the second part of the interview.)

This procedure follows didactic norms: students are trying to recall the "steps" to follow in solving T1, T2 and T3 tasks as they are presented in the textbook. Concepts (and techniques) have not been integrated into a conceptual system; what students handle are fragments of the didactic presentation and their attention is focused on remembering the order of this presentation rather than on arranging their knowledge in a suitable way to deal with the task at hand.

The tasks appearing in final examinations, T1, T2 and T3, cannot help students in rearranging their knowledge, away from the didactic presentation and towards a conceptual system. The tasks themselves reproduce the way in which problems and techniques and technologies are presented in the textbooks.

6.1.2.2 Students' modes of thinking

The classification task has shown that 83% (see Table 5.30) of the students operated in the complexive mode of thinking when justifying techniques (technology part of their praxeology). These students were at the age and stage of their cognitive development which is propitious for the development of the conceptual mode of thinking. Yet, with respect to concepts relative to limits of functions, most of them were operating at the complexive level. Their attention would shift from one feature to another of a limit finding task, and they would fail to identify features relevant from the perspective of Calculus. Students operating at the complexive level of thinking would not identify the abstract logical connections among the individual instances of the limit finding tasks; they would only see empirical connections emerging from their individual, immediate experiences. Routine tasks T1, T2 and T3 could not challenge this mode of thinking. To solve them, it suffices to identify algebraic features with which students are familiar: division of polynomials (T1 and T3), the reducibility ("factorability" in students' language) of the polynomials (if not "factorable", then it is an instance of T3), and the presence of radicals (T2). This is an indication that instructors' models of the knowledge to be learned do not challenge students' complexive mode of thinking with respect to the limit finding tasks. Therefore, institutional practices do not fulfill the role of *pulling* students' cognitive development beyond their immediate individual capabilities – that is, they fall short of awakening "a whole series of functions that are in a stage of maturation lying in the zone of proximal development" (Vygotsky, 1987: 212).

As noted in Chapter 3, Section 3.2.1.2, a student's mode of thinking inferred from the classification task reflects thinking involved not only in forming the classes, but also in the criteria to decide which item goes into which class. Absence of the theoretical blocks in the instructors' models of the knowledge to be learned deprived many (26 of the 28 interviewed students – 93%) students of means to develop mathematical justifications at the conceptual level. Students had no mathematical theoretical resources to reflect on their own behavior and to justify it. Hence, their explanatory discourses about why they would use this or that technique, referred to social validations of the techniques, that is, to the institutional uses and not to why the techniques were mathematically valid in this or that situation. Analysis of students' behavior in the second part of the interview supported this interpretation: many have explained their "factoring" behavior by a habit ("this is what we usually do in this case"), rather than by reference to mathematical reasons (see Chapter 5, page 182).

Furthermore, the absence of the theoretical blocks may be one of the causes of students switching from the complexive mode of thinking at the technology level to the syncretic mode at the theory level. This kind of behavior affected at least 30% of the students. Although some fragments of the theoretical blocks belong to the knowledge to be taught (and perhaps to the knowledge to be learned in some sub-institutions of the College-Calculus institution), the fact that no part of them belongs to the knowledge to be learned in the Final-Examination institution may allow students to neglect (mathematical) theory. It is not their business. Hence, when required to provide deeper explanatory discourses, students would produce reasons based on affect (which represents thinking at the syncretic level).

As mentioned, according to Vygotsky, the development of the highest levels of thinking requires instruction. Thus, if adolescents are not challenged, through instruction, to think at the highest level of conceptual development, we might be missing the right moment to propitiate the development of some intellectual abilities in students. At the university, it may already be too late (Sierpinska 1994: 140). If it does in some domains but not in the mathematical courses, there is a risk they will never develop conceptual thinking in mathematics. The remedial courses in mathematics offered to mature entry candidates at the university may not be able to develop conceptual thinking in students whose ways of thinking in mathematics have stopped, at the crucial moment of their cognitive development, at the complexive level (Sierpinska 2000: 245).

6.1.2.3 The negative influence of the PBs model on students' spontaneous models

Perhaps one of the main contributions of the second part of the interview has been to show the negative influence of the PBs model on students' spontaneous models. The institutional practices, that make the tasks and techniques in this model routine, are such that they have conditioned students to expect only these tasks and to consider the possibility of applying only these techniques. This does not mean that students are doomed to fail when dealing with non-routine problems that resemble the routine ones (see Table 5.33). Nevertheless, while dealing with these problems students have shown that their thinking is not mathematical thinking.

The four tasks presented to the students in the second part of the interview belong to what can be identified as the knowledge to be taught: these are topics covered by the textbook in sections listed in the outline of the course. Tasks of these types, however, did not appear in the final examinations of the last 6 years. The first three problems (2.1, 2.2 and 2.3) do not look like the ones in the textbook but resemble the routine tasks in the sense that the polynomials involved can be easily factored; students approached these problems by way of algebraic techniques. It is problem 2.4 that they easily recognized as a non-routine but still familiar task; this task does resemble the ones that, occasionally, they had to deal with in the textbook (and probably also in the classroom). For a mathematician or for a Calculus teacher at the college level, problems 2.1, 2.2 and 2.4 are all of the same type. The interviews revealed, however, that students would rather treat problems 2.1, 2.2 and 2.3 as belonging to the same type, while problem 2.4 would be in a separate category. The reasons for these models are based on habit; habit induced by the *normal* practices of the institution. Of course, routine tasks might be chosen so as not to trick the students into using algebraic techniques when they are not needed. The analysis of the second part of the interviews shows, however, that they have a negative impact on students' generalizations: where a mathematician sees a limit that can be found by direct substitution (e.g. problem 2.2), the students see a limit that has to be found by an algebraic technique. Furthermore, when in a rational expression the polynomials could be factored using simple techniques but there were no common factors, as was the case in problems 2.1 and 2.3, this caused confusion and a significant number of students could not produce an answer (42.9% of the students in the case of problem 2.3; see Table 5.33). In the routine problems presented in textbooks and final examinations, students have identified patterns on which they have built their spontaneous models for practices related to limits of functions. These patterns, however, are not mathematical. Yet students' models are valid or, perhaps, more accurately, viable. They are viable because the College-Calculus institution to which they belong does not propose tasks that would challenge them.

6.1.2.4 Non-routine and essentially non-routine tasks; the Zone of Proximal Development

As noted in the previous section, in the second part of the interview students took it for granted that the problems were routine ones. They took it for granted not only because the problems resembled the routine ones but also because, as the students claimed, they have never been given non-routine problems (see Section 5.2.2). Because of this, students' spontaneous models are such that when a problem can be solved using a routine technique (problem 2.2) students fail to see that there are other, more efficient approaches[28]. If a problem does not admit a routine technique (problem 2.3), students are

[28] This type of behavior is well-known in psychology; it has been found to be very common in the classical research by Luchins & Luchins (1950: 281). In Luchins (1942) it was attributed to the mechanisms of "habituation": "Einstellung – habituation – creates a mechanized state of mind, a blind attitude towards problems; one does not look at the problem on its own merits but is led by a mechanical application of a used method" (Luchins, 1942: 15).

stuck on their approach[29]. The conditioning induced by the tasks in the final examinations has obscured other techniques that the students may have learned in the course (e.g., direct substitution or finding limits numerically). When dealing with this type of problems, students cannot think *outside the box*.

The third part of the interview presented a different scenario. Because the problems in this third part were essentially different from the routine ones, students discarded at once the possibility of using routine techniques. Students' failure in providing a correct answer, before any intervention of the interviewer, ranged from 50% (in problem 3.3) to 79% (in problem 3.2) and was much higher than the range 4% to 57% in the second part of the interview on problems 2.2 and 2.3, respectively. Nevertheless, at the slightest prompt from the interviewer, students proved to be able to think in mathematical terms[30].

Of course, both non-routine tasks that resemble routine tasks and essentially non-routine tasks are in the zone of proximal development of these students. When dealing with essentially non-routine limits, however, students proved to be able to combine their mathematical knowledge (not only about limits but also about operations with functions, graphs, and uses of the calculator). This is something that they seemed unable to do when

[29] This behavior has also been identified as common in classical psychological research (Duncker, 1945, called it "functional fixedness")

[30] Behavior of this kind – the ability to abandon habituation and solve non-routine problems – has also been studied in classical psychology (Maier, 1945). Mayer (1992) was trying to identify the differences between the experiments of Duncker (1945) and Maier, that could reconcile their apparently contradictory results. One of these factors was that, "the change from normal use to [a new use] was great in Maier's experiment but small in the experiments by Duncker" (Mayer, 1992: 62). In my study, this corresponds to the fact that tasks in part 3 of the interview were essentially non-routine.

dealing with non-routine tasks resembling the routine ones. Furthermore, students were critical about their own beliefs and the results they obtained from the calculator. This led them to take theoretical responsibility of their affirmations; they could no longer rely on arguments based on habit. All this suggests that essentially non-routine tasks are the type of tasks that may guide students to engage in more formal reasoning, in creative and critical thinking, and hence allow them to move towards the construction of a conceptual system.

6.1.2.5 Students' positioning in the institution Community-of-study

The analyses of the second and third parts of the interview have shown that students' positioning in the institution Community-of-study changes as a function of the tasks they are engaged with[31]. If the goal of mathematical instruction is to create conditions that would propitiate the development of mathematical concepts, it seems reasonable to say that these conditions should aim at *enticing* students to assume the position of *Learner*.

When students faced tasks that they interpreted as routine tasks (part 2) they seemed to take the position of Client or Student. From their perspective, the institution would give them only routine tasks (Clients misinterpret this as a rule; Students trust that the institution will follow its own norms[32]). When they found that the tasks were not routine, they seemed to lose interest in them. I surmise that, in the case of Clients, this

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[31] This may happen in other institutions as well, for example, on a market participants can be in the position of *seller* or *buyer* and they may change their position depending on the particular specificities of a given situation.

[32] See Chapter 3, page 68.

disinterest is the result of the fact that these tasks are not means to obtain their goals (passing the course, graduating, etc.). In the case of Students, the lack of interest could be the result of realizing that the institution has conditioned them in such a way that at the slightest perturbation on a problem, they are unable to tackle it with confidence. When explaining their behavior in the second part of the interview, students referred to "reflexes" they have developed by dealing with routine tasks – e.g., S6, S7 – and to the way they have been "trained" – e.g., S23.

In the position of Client or Student, a student relies on authority for justification; it is not his or her business to know technologies and theories. Clients do not need this to obtain their goals as they take it as their right never to be asked to produce discourses at the technology or theory level in Final-Examinations. Students abide by the norms of the institution: the institution does not ask them anything about technologies and theories; hence, they do not learn them.

When, however, students were asked to deal with essentially non-routine tasks, they behaved as cognitive subjects and they based their explanatory discourses on mathematical rules and strategies. In other words, students positioned themselves as Learners.

6.2 DISCUSSION OF THE RESULTS IN THE CONTEXT OF PREVIOUS RESEARCH

In this section, I discuss how the results obtained in this thesis are related with findings in previous literature.

6.2.1 Practical blocks and theoretical blocks in the knowledge to be taught and in the knowledge to be learned

In their paper, Barbé et al. (2005) present a model of the *scholarly knowledge* and of the *knowledge to be taught* about limits in Spanish high schools. In the knowledge to be taught, and in relation with limit finding tasks, they have identified only a practical block. The corresponding theoretical block – present in the scholarly knowledge – is absent. In other words, in relation with limit finding tasks, the curriculum proposes only a practical block. The types of tasks T1, T2 and T3 described in Chapter 4 and their respective technologies $\tau 1$, $\tau 2$ and $\tau 3$, are a subset of the practical block described by Barbé et al. (ibid., p. 244-245). In the context of Calculus taught in Spanish high schools, however, the above mentioned "absence" was different from the corresponding one in the College-Calculus institution I have observed. In the Spanish high school, the theoretical blocks are absent from the knowledge to be taught. In the College-Calculus institution, the theoretical blocks (or at least, parts of them) were *present* in the knowledge to be taught. They were *absent* from the knowledge to be learned.

From the perspective of an institutional analysis, this difference is essential. In the context of the College-Calculus institution that I study here, the theoretical blocks corresponding to techniques $\tau 1$, $\tau 2$ and $\tau 3$ form part of an institutionalized practice. In particular, they are institutionalized by the Curriculum institution. They do not, however, form part of any institutionalized practice of the Final-Examination institution: the model of the knowledge to be learned of the instructors participating in this institution follows norms, not rules. This tension between institutionalized and weakly – or not at all – institutionalized practices might play a significant role in the spontaneous models of instructors as participants of the Classroom institution; this remains to be investigated[33]. As discussed above (Section 6.1.2 and its Subsection 6.1.2.3), however, it definitely plays a role in shaping the spontaneous models of students as participants of the Community-of-study.

The combined results of Barbé et al. and mine imply that, at some point of the process of didactic transposition, the theoretical block corresponding to limit finding tasks *disappears*. The study of Barbé et al. indicates the negative influence of this absence in the teacher's practice. This thesis points to the negative influence of the absence in the students' practices. The origin of this disappearance can probably be found by studying the history of the complex processes of didactic transposition in the particular educational institutions (Barbé et al., 2005: 245). I surmise that this disappearance, in the language of the IAD framework, is related to the layer of norms in these institutions.

[33] Instructors have to "decode" the knowledge to be taught as stated in curricular documents to transform it into knowledge actually taught. I believe that the IAD framework, in which rules, norms, and strategies are well defined, can contribute to the understanding of this decoding process.

6.2.2 Routine and non routine tasks

Several authors have acknowledged the (negative) influence of routine tasks in students' practices. For example, Tall (1992) observes that specific examples are likely to dominate the learning and this could lead to misinterpretation of one's own experience. In particular, by repeatedly using examples of sequences in which the general term is given by an algebraic formula, we cause students to mistakenly assume that this formula is an essential part of the theory (ibid. pp. 501-502).

The work carried out by Lithner (2000; 2004) implies that the problem is not (only) with college level Calculus students' misinterpretation of their experience. The blame must be attributed, at least partly, to the experiences to which the institutions expose the students. As it was pointed out in Chapter 1, Lithner's research (2004) shows that 90% of the exercises in the Calculus level college textbooks can be done by searching the text for methods, that is, looking for template solutions. The results presented in this thesis have shown that, when the textbook is not available (the situation of the final examination or of the task-based interviews), students' intellectual effort is invested in trying to recall the "steps" in these templates. Furthermore, their choice of a method for a given problem is based on patterns and features that are not always relevant from the point of view of Calculus (e.g., they choose to apply a factoring technique based on the fact that the function is a rational expression without taking into account the types of indeterminations). This implies that students studying strategies may be based on memorizing "steps" and on identifying (non-relevant) patterns or features. For example, students have shown that, when dealing with the tasks in the first and second parts of the interview, they tend to evoke the didactic presentation of concepts in the textbooks, instead of the concepts themselves. The absence of a theoretical block in the worked out examples in the textbooks leaves the students with no means to figure out what is relevant and what is not. However, this does not appear as a problem, neither to the students nor to the institution, as the proposed tasks in the final examinations can be very well solved with these non-mathematical strategies.

A study of Calculus students dealing with non-routine tasks by Selden et al. (1999) describes students who, while able correctly to solve routine tasks, were helpless in front of the non-routine ones. The authors surmise that this inability is not exclusively the result of the students' lack of knowledge, but also of the students' difficulties in retrieving this knowledge and applying it to non-routine tasks. The analysis of part 3 of the interviews that I have carried out confirms these results. Selden et al. raise two questions. One is the question of the accessibility of knowledge: why is a student who has knowledge to tackle a given problem unable to retrieve it? The other question is: if a student does retrieve the knowledge, how does he or she do it? My research implies that, in the case of limit finding tasks, knowledge to tackle at least some essentially non-routine tasks seems to be on the "surface", retrievable at the slightest prompt; in Vygotsky's terms, it belongs to students' ZPD. The reasons why students cannot retrieve it on their own might lie in a combination of the way homework exercises are presented, the students' study habits (ibid., p. 18), the overwhelming presence of routine problems in the textbooks (Lithner, 2004), and the institutional norms regulating tasks in final examinations.

The main difference between routine tasks and non-routine tasks is given by the categorization itself: problems that are practiced all the time, and problems that are not.

Of course, the nature of the tasks is important – for example, as it was observed before, it is in the nature of the tasks T1, T2 and T3 that these cannot help students in the formation of a conceptual system. It would not suffice, however, to transform the non-routine tasks into routine tasks and train students in solving them. What might help is a change in some institutional educative habits. For example – by creating situations where students are given a chance to engage in creative, critical mathematical thinking (the type of thought that students have used when dealing with the tasks proposed in the third part of the interview).

6.2.3 Students' types of knowledge

In Chapter 5 I have built a model of students' spontaneous models of the knowledge to be learned and showed how this model does not follow exclusively mathematical rules but is strongly based on cognitive, didactic and social norms. The fact, however, that this model does not correspond (exclusively) to mathematical knowledge does not mean that is not some kind of knowledge. The question of the type of knowledge that students build in place of the one accepted by the community of mathematicians, in relation with a particular concept, has been formulated by different authors (e.g., Hitt, 2006). Referring to college Calculus students, Smith and Moore (1991) write:

> Much of what our students have actually learned….– more precisely, what they have invented for themselves – is a set of 'coping skills' for getting past the next assignment, the next quiz, the next exam. When their coping skills have failed them, they invent new ones. The new ones don't have to be consistent with the old ones; the challenge is to guess right among the available options and not to get faked out by the teacher's tricky questions… (ibid., p. 85, cited in Tall, 1996: 307)

The results obtained in my study suggest that students' "inventions" are emphasized and validated by the tasks proposed by the institution. They are rooted in the void left by the absence of a theoretical component in the knowledge to be learned as defined by some sub-institutions of the College-Calculus institution. Participants of these institutions fill this void with cognitive, didactic and social norms. As Sierpinska (2000) points out

> There is a clear-cut division of labor, whereby the student is given no control over the validity of the statements on which he or she bases the reasonings and calculations. This atmosphere is enhanced by the assignment of 'exercises' [...] where the student is not given a chance to choose a method of solution [...] Knowledge acquired in this way is very likely a 'school survival' knowledge, not a scientific knowledge of any kind. (Sierpinska, 2000: 245)

Despite the *normal* behavior that students have displayed in the first and second parts of the interviews, they do have some mathematical knowledge about limits. There is evidence for this in the third part of the interview, where students were making use of mathematical rules and strategies to tackle the proposed problems. These mathematical rules and strategies do not belong to the knowledge to be learned as defined by the Final-Examination institution. It remains to be investigated where this knowledge comes from and how students make it their own.

6.2.4 Students' "formal", "informal" and "technical" behavior

The analysis of the interviews shows that, in front of routine tasks and tasks that resemble routine tasks, students behave as if they were operating in E.T. Hall's "informal" and "formal" spheres of culture, rather than in the "technical" sphere (Hall, 1981; see also

Sierpinska, 1994: 161-169, for an adaptation to mathematics development, teaching and learning). On the one hand, students refer to having been "trained" for choosing this or that technique in front of a task, and "reflexes" developed out of this "training". This type of learning is in the informal level, similar to the way one learns a sport, like biking or skiing. It is not enough that someone tells you what to do; learning occurs through observation, imitation and repetition. On the other hand, students' justify their choice of a technique to tackle a problem by stating their beliefs and convictions that the technique indeed applies (their expectations about the tasks that the institution asks them to do). These beliefs and convictions are themselves based on explicitly communicated elements of tradition, i.e. part of the formal plane of culture, which is transmitted by admonition and does not require any justification: "this is how things are done, no ifs, ands or buts". Furthermore, students' use of the technique is an algorithmic use; it is based on a recall of a set of "instructions" or "steps" given by the textbook or the instructor. As, however, there is no *technical* level – no mathematical justification of the technique – the steps form an arbitrary list. A simple consequence of this arbitrariness is that students have difficulties in remembering the order. Thus, for example, they hesitate whether to do direct substitution first or factoring first.

When dealing with non-routine tasks, the interviewed students suddenly started behaving as if operating in the technical sphere of the mathematical culture. They felt the necessity of explicitly formulating their knowledge, they were cautious and critical about their own statements.

Sierpinska suggests that the cultural roots of epistemological obstacles are to be sought in the formal and informal spheres of culture (Sierpinska, 1994). As discussed in

Chapter 1, research within the epistemological perspectives was concerned with students' concept formation prior to any formal teaching and learning; in abstraction from the institutional context. On the other hand, my own research – framed within institutional practices perspectives – is concerned with students' behavior *after* formal teaching and learning has already started. This, combined with the discussion above, implies that the weight an institution gives to the informal and the formal layers of culture may facilitate or, on the contrary, hinder students' overcoming of epistemological obstacles.

6.3 CONCLUSIONS

In the interviews carried out in this study, students revealed that their spontaneous models to deal with limit finding tasks are not built using Calculus criteria. Their approaches to finding limits of rational expressions show that their models are grounded in high school Algebra, and in a type of strategic knowledge associated with succeeding on the final examination. Thus, their praxeologies are not exclusively mathematical but strongly based in social, cognitive and didactic norms. The analysis of instructors' spontaneous models of the knowledge to be learned about finding limits has shown that the institutional approach – as it is – cannot help students in "rebuilding" their models so that they eventually become models of *mathematical* behavior (instead of models of *normal* behavior). Furthermore, institutional practices, based on norms and emptied of theoretical content, do not challenge students' complexive mode of thinking.

The different notions of knowledge described in the process of didactic transposition highlight "the institutional relativity of knowledge and situate didactic problems at an institutional level, beyond individual characteristics of the institutions' subjects" (Bosch et al., 2005). In the present work, I tried to highlight the possible relativity of these notions when considered from an anthropological point of view. As it was pointed out in the introduction, from a strictly epistemological perspective, notions such as *knowledge to be taught* or *knowledge to be learned* might be quite well defined objects. From an anthropological perspective, they become relative to the institution to which they belong. Even this relativity is not subtle enough, however. The notion of praxeology does not distinguish between practices regulated by rules, and practices regulated by norms. Hence, although the ATD notion of praxeology can be used to build a theoretical epistemological model of knowledge, it may not be sufficiently sharp as a tool for describing the differences between the spontaneous models of knowledge that participants in different positions with respect to the institution may have. I found it helpful to complement the ATD framework with the IAD framework to clearly see and describe these differences.

6.4 FUTURE RESEARCH

Research aimed at understanding the relations among the different sub-institutions of the College-Calculus institution, what are the available positions in each of them, and based on which criteria participants are assigned – or assigned themselves – to these positions, will contribute to our understanding of institutionalized mathematical activity. In what follows I suggest possible directions for future research from an institutional perspective.

In Section 6.1.1 it was observed that the sequence T3 – τ3a – τ3 – θ3 provides a simple example of how proofs are sometimes developed in mathematics. The opportunity to show this is missed in the textbook. Furthermore, instructors, as participants of the

Final-Examination institution, expect students to use technique $\tau 3a$ (or $\tau 3b$) to tackle instances of T3. As it was mentioned before, the institution's rationale for this choice remains to be investigated. On the other hand, modeling instructors' implicit models – as participants of other institutions (e.g., the Curriculum institution or the Classroom institution) – of the knowledge to be learned about limits, would contribute to the understanding of students' behavior in front of limit finding tasks and the different status – relative to different institutions – of the knowledge to be learned.

In the context of the College-Calculus institution studied here, the Classroom, as an institution, interacts and shares participants with the Curriculum and the Final-Examination institutions. Barbé et al.'s (2005) paper focuses on some aspects of the interactions between the institutions Curriculum and Classroom. In particular, they analyze teachers' decoding of the curricular documents – the step in the process of didactic transposition that changes knowledge to be taught into knowledge actually taught – through the lens of ATD. Perhaps, an analysis of such decoding based on the combined framework ATD-IAD would allow us to characterize the institutional status of this decoding process, that is, to understand which are the mechanisms that regulate it. In relation with the Final-Examination institution, it was described above (Section 6.2.1) how instructors' spontaneous models about limits influence students' spontaneous models of the knowledge to be learned. To investigate the role played by these implicit models in the spontaneous models of instructors as participants of the Classroom institution, will contribute to the understanding of the Classroom – Final-Examination interactions.

Finally, as observed in Section 6.2.3, despite the negative influence of routine tasks, students were able to engage in mathematical – as opposed to 'normal' – thinking about limits of functions. This phenomenon requires an explanation. The fact of just engaging in mathematical thinking could be, simply, the effect of the realization that, in the interview, the institutional norms are not binding any more. This does not explain, however, why the students' mathematical thinking about the non-trivial non-routine tasks was also so often correct. They seem to have learned more about functions and limits in the Calculus course than what they thought they had to learn. How did they learn it and what exactly is the nature of this 'surplus' knowledge remains to be investigated. This will be the direction of my future research.

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APPENDIX A

This appendix contains the transcription of the 28 interviews with students. Students are identified as in the body of the thesis: S1 to S28. The tables A1 to A28 appearing here are the same as the tables 5.1a to 5.28a appearing in Chapter 5. The notation [...] is used to represent a long silence.

STUDENT S1

I: The first thing I'll ask you to do is to look at these twenty cards. I want you to classify them into groups, according to any rule that makes sense to you.

S: Ok. [...]

S: Something like that... there has to be three groups?

I: No, no, as many as you want.

S: Something like that.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 5, 9, 12, 17	Difference of squares.
2	6, 7	Constants.
3	19, 20	With trigs. That confuses me.
4	3, 10, 13, 14, 18	With square roots.
5	1, 4, 8, 11, 15, 16	Polynomials.

Figure A1 (Copy of Table 5.1a). Student S1's classification.

I: Can you explain me what was the rule of your choice?

S: Ok, here are all the... how do you call them? Difference of squares? I put those together [class 1]. These are...

I: Constants?

S: Yeah [class 2]. These are with trigs [class 3].

I: Ok.

S: Because that confuses me so I put them together. These are all ones with roots on them [class 4]. I don't know why I put them together. And these are all the polynomials [class 5]. You just like...

I: Let say you are working or studying with a friend and you have to read this [card 8] aloud, how do you read that?

S: How would I read that?

I: Yes.

S: The limit as x approaches one of four x cube plus seven x minus nine?

I: Ok. So now I am going to ask you to solve some limits. And as I told you, I am interested in the process of solving, so as much as you can, please think aloud.

S: Ok.

I: So I would know what comes to your mind first...

S: Ok. The first thing I do when I see limits is to put in the number it goes to to see what it gives. So in this case [2.1] I do zero over two, right? [...] Then [...] what I would do is factor. Now I don't remember if I factor a negative one, can I cross them out? [She meant whether factoring a negative one from the numerator will leave a factor x + 1 as the one appearing in the denominator]. [...] Can I do that? [...] I don't remember any of this. I don't know why.

I: Ok. Let's move on. We'll go back and forth between problems. So now I give you another one [2.2].

S: Ok. [...] So this is what I'd do in this case. So basically I just factored this out and crossed it out. [She factored the denominator, and cancelled the common factors on numerator and denominator. She stopped there].

I: And your final answer would be? [She wrote the equivalent expression without the common factors and substituted in her mind to obtain negative one, she wrote negative one as a final answer].

S: Ok?

I: Ok. What about this? [2.3]. [...]

I: What happens with that one? [She wrote 21/0]. Did you try anything in your mind?

S: No, nothing works up.

I: Why do you say that nothing works up?

S: Ok, well. Because if I put it in I get nothing, well I get twenty one over zero.

I: Ok.

S: Then [...] if I factor it out it doesn't give me anything different, like you can't cross anything out. But then... I am trying to remember, back to Cal I, all the different steps you could do.

I: Oh, so you are trying to remember the techniques? The methods?

S: Yeah. The methods. There was always first you try to factor and cross out anything that you can. Then... [...]

I: Ok.

S: I am sorry [because she couldn't get an answer].

I: It's ok; actually you don't need to get the right answer to help me on my research.

S: Ok, this one [2.4] [...]

S: I don't like the cube, I'll take that out. [She factored an x from the numerator, though she left the constant as 9, instead of as $9/x$]. Oh, I don't need to do that [she meant the factoring she just did].

I: Why not?

S: Ok, hold on. I'm just thinking, if I put in one... it gives an answer, right? Because then... [She wrote 14/3].

I: Ok. I'm just curious... say in this one [2.2], did you try replacing the x by two in the beginning or you just factored first?

S: No, I just factored first because I knew this was one of the factors of that [of $x^2 - 9$ in the denominator].

I: But do you think it was necessary?

S: No, I guess... [...] No, it wasn't necessary. But when I saw this... it just made more sense to me that way.

I: And for this one [2.1], the first thing you did was to replace by one, but when you saw that you got zero over two...

S: Yeah, the next step was to factor because they look very similar.

I: Ok, but why do you think you have to go into another step?

S: You mean why this wouldn't be the final answer?

I: Yes.

S: Eeeh...

I: You think this zero over two is like this one [pointing at the 21/0 she obtained in 2.3].

S: Yeah.

I: Ok. I have one more for you. If you can find these three limits... [...]

I: Do you remember what the graphs of e to the x and cosine x look like?

S: Cos x yeah, because... isn't it like this.

I: Yes. And e to the x?

S: Something like this. [She correctly sketched $y = e^x$ and $y = cos(x)$.]

I: Can you figure out from the graphs what the limit might be?

S: From the graphs? [...] Would it... [...] I am not sure.

I: If you were, let say, on an exam, and you need to figure out those limits and you have no clue what to do, would you think of using the calculator to try to figure them out?

S: Probably not.

I: Ok, and how about the last one?

S: The last one? [...] Looks like a problem I saw... [...] I think, probably, I would try to use L'Hôpital's rule. Which is the derivative of each one, right? [...] [She wasn't sure about the derivatives.]

I: And once you find the derivatives what would you do?

S: Then put in the values. That's what I would do with this one.

STUDENT S2

I: Ok, so the first thing is, these are twenty problems, I ask you to classify them according to any rule that makes sense to you.

S: As for deriving them? I mean, integrating them?

I: Any rule that makes sense to you.

S: Ok. [...]

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Limit of a constant. You can do it right away.
2	3, 10, 18	Multiply top and bottom by the square root.
3	1, 2, 9, 15, 16, 17	Fractions.
4	4, 5, 11, 12	Fractions to infinity.
5	13, 14	These are square roots but to infinity.
6	8	
7	19, 20	

Figure A2 (Copy of Table 5.2a). Student S2's classification.

S: This one is the limit of a constant so that you can do it right away [class 1]... Are there the ones you have to factor... no... You want them according to how to solve them?

I: Not necessarily, just according to a rule of your choice.

S: There's probably another way to get this...

I: So you have these groups.

S: These are the ones you multiply top and bottom by the square root [class 2].

I: And these ones [class 3]?

S: These... I think it would be sort of fractions.

I: Why these are separate from these ones?

S: I had a reason... I don't know. Maybe this was meant to be a whole group.

I: And these are separate because...?

S: Those are to infinity [class 4].

I: And these [class 5]?

S: Those are the same idea as the square root [class 2], but to infinity [class 5], those were to a number [class 2].

I: How would you read this [expression 8] aloud if someone asked you?

S: Limit of four times x cube plus seven x minus nine.

I: Ok.

S: Limit x to the one of all that.

I: Ok. Next, I ask you to solve some limits, and as much as you can, I ask you to talk aloud [2.1].

S: Ok, this one is limit of x to the one and the function is x minus one over x square plus x. Generally, if I saw this on a test, what I would do first would be to put in a number very close to one, a little bit bigger and a little bit smaller than one and get a number probably extremely close… Is it ok if I use my calculator.

I: Yes.

S: So something very close to one like zero point nine nine nine divided by the same number squared plus zero point nine nine nine. I get close to zero. So when I put something really close on the left side I get zero, something very close to zero. And now I will check if it is continuous. When I put something really close to the right side, so one point zero zero one minus one, divided by one point zero zero one squared plus one point zero zero one. So from the right I also get something very close to zero. So now I know it's continuous. At this point I think it's probably zero. If I wanted to find the limit by other way, I think I could use L'Hôpital's rule here? Which gives one over two x plus one, does that work? Oh, no, never mind, it's not infinity over infinity. What I think it might work is if I separate them into two different parts. If I have x over x square plus x minus one over x square plus x. Which follows the rules of how you are allow… how you are allow to solve for limits. Now I can easily put in the x values, so one over two minus one over two, so in this case the limit does equal to zero. So I can do it from left and right or just fooling around until you get something where you are not dividing by zero or anything.

I: Ok, and why do you think you have to go through this step?

S: Well, I guess I could just put in the one here, but I am use to have something divided by zero.

I: What about this one [2.2]?

S: For this one… now I am gonna try being very smart, I am gonna make sure it doesn't just go to an asymptote or anything. Here I don't think you need… you can just put the two I think. But you have to check right and left I think. So I put numbers really close, I'd put two point zero zero one… so when I approach from the right I get something very close to one. Now I do one point nine nine nine, once again I get something very close to one. So I know it should be continuous because I checked on both sides of the two, as far as I know I could do a final proof and put in the two because I just checked it is continuous. So I have two plus three… five over negative five so in this case I get negative one. Ok, so I think originally I got negative one.

S: [2.3] Oh, here you do have an asymptote, from the bottom part you get zero, and dividing by zero is a very bad thing. So in this case… well, I think right off the back it's going to give infinity because as you get closer and closer the numbers become larger and larger. I check by doing the same as for the last two problems. I'd take four point nine nine... I get two thousand ninety nine... which is a huge number, so it's approaching negative infinity on this side. On the other side I put five point zero zero one... In this case I get a positive number. [He made a sketch.] But technically you can't have a positive infinity and a negative infinity at the same time, so I'd put this one doesn't exist.

S: [2.4] I don't think there's any way to factor this which would be very handy. So I have to go to my method... one point zero zero one... Well I should also check for asymptotes before... but the bottom is not zero. Ok, I am approaching from the right and I get something like four point sixes. And zero point zero zero nine... from this side I get something similar, both sides something like four point five. So it seems it would be continuous. Roughly four point five. Now I check that the middle value is the same, just to check that the point doesn't jump away. So one cube plus four times one square... I this case I'm getting four point six to the infinity.

I: If you had.... If this was a test would you write something else?

S: Yes, I would write a lot more.

I: Like what, what would you write?

S: Generally, something like checking for a vertical asymptote, the denominator at one is not zero. Then I do checking... approaching from the right so in this direction, we can take a value close to one but larger and I just pick one point zero zero one, the limit as x approaches one from the right.... equal four point six. And then I do the same for the other side.

I: And you'd do this only for this value or you'd try with others?

S: I probably only do it with this value, but I know you are suppose to try others before jumping into a conclusion. And then I would check the middle value and if it works I would say that's the limit.

I: What would you think... of a problem you get different from the right than from the left?

S: Then I don't think the limit exists. We did this at the beginning in cal one. I don't remember a hundred percent... broken discontinuity?

S: Ok, I am not especially good at this but I think that as this goes to infinity you get... Ah, well for this one... right when I look at the cos, we are talking about it in radians, it doesn't setting upon anything because it's oscillating. So I don't think there would be a real limits... This is in radians right? As for e to the x... this is like two point seven, right? So the limit would be infinity but the cos of x doesn't mean anything because it's going up and down. So I don't think this would have a limit. It would just keep going up and down, so I don't think this exist. This one [3.2] I think it would be the same scenario... although... no, e to the minus x is getting smaller and smaller, if you have e to the minus a thousand [he checked in the calculator], so yes, it's approaching zero. So no matter what the cos is, as x approaches negative infinity, e to the x is getting closer to zero, so regardless of what cos is, e to the x would make it zero. So this is one of the functions that look like this [he got a very good sketch of the whole function]. Ok. Sine of zero... I think that's zero [he checked in the calculator] yes. Over zero, ok, so that's infinite. On either side... [he checked in the calculator]. Ok, even if at the point, when x

equals to zero it doesn't exist, on either side is equal to one. I just make sure it's equal to one [he checked again]. So even if it doesn't exist, sine of x is equal to one... no, yes, it's ok. [He did the calculation again in the calculator.] Ok, I'm just going to go with what the calculator says and say the limit is one, because that seems to be the case, even if it doesn't exist when x equals zero.

I: What is it that you were doing?

S: I kept trying numbers closer to zero.

I: I propose you one more [I wrote $\lim_{x\to\infty} \sin\left(\frac{1}{x}\right)$].

S: Sine of one over x... ok, that's different. Sine of infinity, at the point sine of infinity doesn't exist because it's up and down, up and down. Doesn't really exist. And you can't divide by zero, so there's nothing at this point. On either side of it... well, either is positive or negative, you are going to have sine of a number really really big so the value for one over x... the sine of infinity in either side. Hold on. Let me think. I could try my handy method where you try a number very close to zero. I'm thinking this would not exist.

I: Why would you say that?

S: Because I'm thinking the infinite value would be... as x approaches zero the infinite value approaches infinity, and sine of infinite doesn't really exist. When you go to either side you are gonna have a big number and a really big negative number. But I guess it would exist, because sine it doesn't matter... Oh, no, I don't think it exists. Because sine of a negative number if I'm doing it in radians and sine of a positive number should be different, so it would kind of hit like this on one side and down on the other side. Like this, I think. I have a graphing calculator, but I can't use it.

I: What would you do with the graphing calculator?

S: I'd just graph the function.

I: Ok, let's do that.

S: In general you are not allow to. It helps to picture it. Sine of one over x... I have to change the... I have to zoom in it. Oh, there is a limit, it's going to be zero. Because it exist on either side and the value is approaching zero on either side, so it does exist. Why does that happen? Because as it goes to zero you are going closer and closer to sine of infinity that doesn't exist. I am not quite sure why the limit would be zero.

I: Can you zoom more?

S: I can find the limit, no problem, but as far as knowing why... I am trying to understand why it converges. It's hard to work with sine because sine is a number that doesn't have like asymptotes... which would make it easy because I could say it goes to this number.

STUDENT S3

I: The first thing I ask you to do is, these are twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: What do you mean by making sense?

I: I give you all these and you put them...

S: Like in the same family?

I: Put them in groups...

S: Like if I know how to solve them?

I: Just into groups according to...

S: Ok, I group them.

I: ... what's more relevant for you or...

S: Ok. [...] Ok.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13,	These ones I organized them for the same technique to solve the problem. The infinity thing. Taking the highest power and that would help me to find the limit.
2	1, 2, 8, 9, 10, 15, 16, 17, 18	This was basically how to solve the problem, like is gonna be zero over zero type and I have to separate them and then find a way of doing it.
3	6, 7	This one is basically like whatever you are going the limit is going to be the same, it's constants.
4	3, 19, 20	Limit going to zero. With the sine and the trig thing. I put the zeros together.
5	14	This one I don't know.

Figure A3 (Copy of Table 5.3a). Student S 3's classification.

I: Can you explain me what was your rule?

S: These ones [class 1] I organize them like for the same technique to solve the problem. And...

I: What would be that technique?

S: Would be like the infinity thing, like taking the highest power and that would help me finding the limit. And this [class 2] was basically how to solve the problem, like is gonna be zero over zero type and I have to separate them and then I'd find a way of doing it.

I: And the others?

S: This one [class 3] is basically like whatever you are going the limit is going to be the same, it's constant. And this is the limit going to zero [class 4], with the sine and the trig thing [class 6] and this one [class 5] I don't know... but I put the zeros together.

I: And these are separate?

S: This is the same idea as this, solving.

I: And why did you put this and this [expressions 3 and 14] in different groups?

S: I don't know, basically I looked at limit goes to zero, that's what I see.

I: Now I ask you to solve some limits.

S: Ok. [...]

I: Can I ask you, if possible, would you think aloud? So I can keep track of what you are thinking.

S: Oh, sure.

I: What was that you were doing?

S: [2.1] The first thing, I checked if is the zero over zero type, because if it is zero over zero I would have to factor and cancel out and then find the limit. But if I remember zero over a number is just zero. So the top is going to zero over two, zero divided by two is going to be zero.

I: But why did you factor here?

S: Because I try the zero over zero thing not realizing...

I: When did you realize the denominator was not zero?

S: I was substituting one on the bottom.

I: Did you do the substitution here or here [the initial form or the factored form]?

S: No, here [the factored form].

I: Ok, what about this [2.2]?

S: This one you can substitute right away and you'll see the limit. Like two plus three minus one divided by four minus nine... minus five two.

S: [2.3] Ok. So the top you have a number over zero, so you have to factorize them so I can see if any can cancel out and then find the limit. Because at the bottom you have zero... so I'm basically gonna solve it. [...] Now I am stuck. [...] Isn't the limit going to infinity?

I: Why do you think it's infinity?

S: Now I'm guessing.

I: Are you guessing from calculation or...?

S: I am trying to remember from cal one, but I am thinking if this over this... could it be possible that the limit doesn't exist?

I: And when you say that you are trying to remember, are you trying to remember a rule or a calculation?

S: The rule that we learned. And I'm just thinking about the rules...

I: Let me give you another one [2.4].

S: I substitute...

I: Ok, and going back to this one [2.3] if you wanted... would it help you in any way to use the calculator to find the limit?

S: With the calculator? No. Because how could... because if I substitute x by five... I have twenty one over zero... If I remember, this means going to infinity.

I: Ok. And if someone asks you to read this [2.3] aloud, what would you say?

S: Limit of... limit of x going to... limit of the function x square minus four divided by... no, wait. It's the limit of x square minus four divided over x square minus twenty five going to... as limit going to five.

I: And when you read this, what do you have in mind?

S: When reading this I say for this function I am looking... x going to five would be my limit at that point.

I: And for example, for this one [2.1], when you say that this equals zero... what do you understand by that?

S: Equal zero. Is as the limit, when as this function going to one is going to be closer and closer to zero, it's not gonna be zero, but it's gonna be closer to zero. That's how making it limiting it.

I: Ok, let's finish with this. [...]

I: What did you try with the calculator?

S: Oh, I was just checking what cos of zero so to draw. [...]

S: This is always infinity [for the exponential function in 3.1]. [...]

I: What is it that you are thinking?

S: I am thinking about the limit of e to the x and cos of x as x goes to infinity is positive infinity, or maybe I would figure the graph first... How do we solve this problem?

I: If you have to look at them separately?

S: This is just infinity, and cos is just plus or minus one.

I: And from that can you figure out the limit of the multiplication?

S: But the only point at where they are meeting is at one. Can I say the limit is one? If I only look at this...

I: But what about this [the expression $x \to \infty$ in 3.1]?

S: I would say the limit doesn't exist.

I: What about this one [3.2]?

S: This one, as the limit goes to minus infinity this is going to zero but not zero and the cos is gonna be plus or minus one, so infinity... no.

I: Do you know any way in which you can use the calculator to help you find these limits?

S: No. Maybe with a graphing calculator.

I: How would you use a graphing calculator.

S: By graphing.

I: What about this one [3.3]?

S: Zero over zero. I would use the graph first... [She did a table of values to get the graph.] Sine pi over two divided pi over two.

I: Why did you choose pi over two.

S: Because sine of pi over two is one. So as it goes to zero... Now I am remembering the squeeze theorem...

I: How would you use the squeeze theorem? [...]

I: Could you use the squeeze theorem for this one [3.2]? [...]

S: I don't think so... How could I use the squeeze theorem there?

I: Why do you think you can use it here but not here [3.3 but not 3.2]?

S: Because of the sine... it was one of the examples we did, but it was with sine of x, but now I have sine of x divided by x. [...]

I: If you wanted to do the same here as in here...

S: This one would be zero, and this, what I'm thinking... doesn't exist.

I: Ok.

STUDENT S4

I: The first thing I ask you to do is to classify these cards into groups, as many as you need, according to any rule that makes sense to you.

S: In groups you said, right?

I: Yes. [...]

S: That's it, three groups.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7, 8, 9, 10	These ones I just have to plug in the numbers.
2	4, 5, 11, 12, 13, 14,	These ones are all limits where x goes to infinity.
3	1, 2, 3, 15, 16, 17, 18, 19, 20	These ones give zero at the bottom.

Figure A4 (Copy of Table 5.4a). Student S4'sclassification.

I: Can you explain me...

S: These ones I just looked [class 1]... I just have to plug in the numbers. These ones [class 2] are all limits where x goes to infinity. Although in the first group there are limits where something goes to infinity, the variable is not in the function [card 6]. These ones are all as x goes to infinity, and these ones [class 3] are all that give me zero at the bottom.

I: When you were doing this, was there a point when you consider a different arrangement?

S: Not really. I was just confuse, I just wanted to make sure some of them would give me zero.

I: So why this is in this group?

S: Because is sine of one over zero, whenever there was something over zero I put it here [class 3].

I: Ok. Let's say, this one, if you were studying with a friend and you have to read him this [card 8] on the phone, what would you say?

S: Limit as x goes to one of four x cube plus seven x minus nine.

I: Now I ask you to solve some limits, this is the first one [2.1]. If you can think aloud, so I can keep track of the order in which you are thinking...

S: Ok, this is, I just write zero. I plugged in the one, it gave me zero at the top so I don't care about the bottom.

I: What about this one [2.2]?

S: This one [...] I just plugged in the two and...

I: Ok [2.3].

S: These ones I don't remember how to do them.

I: When you say these ones, what do you mean?

S: The ones you get zero at the bottom, I don't remember how to do them. Maybe you have to cancel something, but I can't remember. [...] I can't remember, sorry.

I: What about this one [2.4]?

S: Ok. I just plug in the number and it gives me thirteen over three.

I: When you write, for example here, when you write that this limit equals negative one, what do you have in mind? Does it have any meaning for you?

S: Well, ok... well... I just, yes, it's like when x approaches two, what the function becomes. Here the function becomes minus one. And it doesn't approach it, it actually becomes minus one.

I: So when you say it becomes minus one you mean it is minus one.

S: It is minus one, it is not close to it, it is minus one.

I: Ok, let's move to the next one. [...]

S: Cosine of that... [...] I would say for the first one [3.1] infinity.

I: Why would you say infinity?

S: e to the infinity would be something like infinity... and here is the problem: I don't know the definition of cosine. I don't know what cosine of something very big would be... but don't matter how small the number here would be... multiply by something that is infinity would still give infinity.

I: Do you remember the graph of cosine?

S: Yes. Something... is that sine?

I: Well, they are shifted so they are very similar.

S: This is periodic so at infinity there's no real value because it continues doing that. [...]

I: What about this one [3.2]?

S: I would say... We don't have a value for that then I don't think we would have a value for this. My guess would be... my guess would be that they don't exist.

I: Both of them?

S: Yes, I am not sure what the cosine function does in the negative... but I am pretty sure is similar to what it does that way. So it wouldn't have a specific value as it approaches infinity or negative infinity so they would be pretty much the same.

I: What about the exponential function?

S: Ok. [...] Well, I guess minus infinity. I just don't see the difference between minus infinity from positive infinity.

I: If you could use your calculator to check if your guesses are right, could you do it? Could you use it to find what these limits are?

S: I would certainly plug in some numbers?

I: Do you have one?

S: Yes. Ok. So, *e* to the something...

I: You have to change it to radians.

S: Oh, ok. Something very very big would give me... oh, come on, a small value [he got an error]... Ok.

I: Do you know why you get the math error?

S: Well, I guess it doesn't have space for the number.

I: Let's try with a hundred.

S: Ok, *e* to the hundred... is something really big, so it is infinity.

I: You are checking with that... so you are sure it's infinity. Could you use the calculator to check the whole thing?

S: Well, cosine of something very big... ok, cosine of something very big becomes negative. Cosine... ok. If it always gives a negative number then I would guess that it gives negative infinity. If I were in a test I would do that. But then, I don't know when it goes negative and when it goes positive... [He tried more numbers.] Ok, no. Why it does that?

I: You got a positive number now?

S: Yes. This doesn't have a specific value when it goes to infinity...

I: And this one, can you check [3.2]?

S: It becomes a really really tiny number... What? Ok, as this goes to... I get very confused with minus infinity. If I do e to the minus a hundred...

I: Do you remember the graph of e to the x?

S: No.

I: When you did e to the minus a hundred you got?

S: A very very tiny number. So this [e^x] goes to zero? [...] And the cosine I wouldn't know. Cosine just goes up and down.

I: So what about the multiplication?

S: Well, if this approaches zero, then it [3.2] would approach zero. And this [3.1] infinity.

I: What about this one [3.3]?

S: Sine of zero is... zero [checking in the calculator]. [...] I am just guessing... [he wrote infinity].

I: Why would you guess infinity?

S: Oh, wait, sine of zero is zero. [...] I don't know.

I: Do you think in this case you can try using the calculator to guess what that is?

S: No.

I: Why not?

S: Because this is the kind of thing... I don't know how to ask the calculator that question. If I would plug zero it would give me error. I wouldn't know how to use my calculator for that. I don't think it would give me any information.

STUDENT S5

I: I ask you to do different things, the first one is... these are twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: Like difficulty?

I: Anything that makes sense to you.

S: Like trying to relate them. [...]

S: That's it.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 4, 5, 9, 12, 15, 16, 17	Quadratics on top or whatever I saw, like this is a cube but since it's to infinity... The ones that are very mechanical.
2	1, 11	These are the same idea [as class 1] but require more work.
3	19, 20	Trig ones.
4	6, 7	The answer is already given.
5	3, 10, 13, 14, 18	With the conjugate that you have to put on the bottom.
6	8	

Figure A5 (Copy of Table 5.5a). Student S5's classification.

I: Can you explain me the rule you use.

S: These ones are the quadratics on top or whatever I saw [class 1], like this is a cube but since it's to infinity... like I didn't do it... but these are the ones that are very mechanical. These [class 2], sort of the same idea but maybe they require more work, a couple more

xs to work with. These are the trig ones [class 3]. These [class 4] the answer is already given. And these ones [class 5] are with the conjugate that you have to put on the bottom.

I: And if you have to read this [card 8] aloud, say you are talking to someone on the telephone and you have to read this, what would you say?

S: What I am suppose to say or what would I say? I'd say four x three plus seven x minus nine, but I should say four x cube plus seven x minus nine.

I: But...

S: Maybe x arrow one.

I: If you read the whole thing what would you say?

S: That would be limit of four x cube plus seven x minus nine x to one.

I: Next I ask you to find some limits.

S: It's so long since I did limits... [...]

S: Ok, this is relatively easy [2.1] ... this is zero over two so I have to, I don't remember... so the limit is zero? Yes, right, is the zero over zero that I have to keep working.

I: Ok, what about this one [2.2]? [...]

S: Minus one.

I: What about this one [2.3]?

[...] [He factored numerator and denominator, then left that calculation and factored out an x from numerator and denominator and cancelled them.]

I: No, leave it, please [my intervention as he wanted to erase the first factoring].

S: Ok. This would be something over zero. Four point two over zero.

I: Ok, and the answer for the limit would be?

S: [Inaudible. He wrote 4,2/0 = 0.]

I: Ok, what about this one [2.4]? [...]

S: One plus... fourteen over three. Just plug in the one.

I: Why in here [2.4] you just plug in right away but in here [2.3, you factored]...

S: Because in here [2.3] I knew that five square minus twenty five would give me zero so I tried to do some more work to see if I could get out of it. But here [2.4] I thought that it would work, putting in the one.

I: But then in here [2.3] before doing this [factoring out an x from numerator and denominator] you did try to check [if something would cancel by factoring the difference of squares]...

S: Well, I guess kind of saw it automatically [the difference of squares].

I: And in here [2.2] you saw that two?

S: No, actually in here [2.2] I didn't see it right away, I just solve the x plus three and I saw the opportunity that they will cancel out.

I: But do you think in this case is necessary to do this operation?

S: Eh... no. Probably not... I made my life harder.

I: Do you think there could be a problem in which doing this way [factoring, cancelling and then substituting] or just substituting would give you a different answer?

S: You mean in the case when substituting would give you zero?

I: Well, just in general.

S: Well, I guess, if at the bottom you get zero and you keep doing it you get a completely different answer which is like the right answer, so yes, I guess.

I: And in this one [2.1] do you remember what you did?

S: I remember seeing one minus one and here was one plus one and I tried doing something different but I still get zero over a number so it's zero.

I: And let's say in this problem... for example [2.2], when you see that this equals negative one, what does it mean to you?

S: Well, when I was kind of in the cal one mode type of thing I would see that minus one is like a barrier, like the limit is a barrier. But now, that I haven't touch it for a couple of months, I see it as a very mechanical thing, find the answer, cancel out, plug in... I don't see it visually anymore because I am not in the cal one mode.

I: Ok, one more problem.

S: Oh, shit, I hate these. I don't remember. Sine of zero is zero, then, oh... I don't remember how to do these at all.

I: Could you use your calculator to try to somehow guess the answers?

S: Well for this [3.3] I just would plug in, sine of zero, zero over zero, and whenever you get zero over zero I know something is wrong. For these [3.1 and 3.2], if I had zero it would have been more encouraging, but I have infinity...

I: But you think there's any way in which you could use the calculator to try to guess those limits?

S: I wouldn't do that as a reflex. Maybe I just try a few things out of desperation. Sometimes when I don't know what to do I take out my calculator... so maybe I sort, maybe you don't see the logic in this, but I start switching things around, I put in three and that makes sense...

I: Do you remember the graph of e to the x, the graph of cosine?

S: I remember sine of one over x, that was crazy... Oh, cosine, yes. And sine was kind of the opposite, well, not the opposite but... Like one starts here and one starts here. And e to the x I just don't remember.

I: And would the calculator help you to figure out what the graph is?

S: Well, I don't really never use a graphic calculator.

I: No, I mean a regular calculator.

S: A normal one? I guess, what I would do is put e to the zero, e to the one, and so on. And I get a general idea and try to derive a general idea.

I: Do you think you can do one of those for me?

S: [He tried some numbers on his calculator.] When x equals zero you have a one, when x is two... Ok, I try a negative number I guess, so it gets smaller. From this I figure out that it looks like this, it never touches the x axis and keeps on getting... yes smaller.

I: And looking at this graph you did, if you had to figure out the limit as x goes to infinity [I wrote $\lim\limits_{x\to\infty} e^x$] and as x goes to negative infinity [I wrote $\lim\limits_{x\to-\infty} e^x$], just by looking at the graph.

S: I say the limit approaches towards this side...

I: So what the answer would be?

S: I choose this one [$\lim\limits_{x\to-\infty} e^x$].

I: Sorry? What the answer would be?

S: I say it's closer and closer to... zero, but I am not sure.

I: And for this one [$\lim\limits_{x\to\infty} e^x$] what would you say?

S: I would say also zero... No, no, it would be different. Y would be infinity, so f of x would be infinity.

I: And using this do you think you can figure out the answer for these [3.1 and 3.2]?

S: Well, now that I do have the graph, I would try to plug in numbers. Maybe I can see something from that... yeah. [He plugged numbers in the calculator.] e to the... it does get bigger... I didn't try any negative numbers. So it gets smaller towards here, and it gradually gets bigger. So as x approaches negative infinity it would be zero [3.2].

I: Using a similar idea can you try this one [3.3]?

S: Here I have zero over zero so like there's nothing, the equation is... sort of does not exist. Except for that I'd put infinity I guess.

I: And could you use the calculator in some way?

S: No, because it's just plug in the zero... it's more technical I guess...

STUDENT S6

I: This are twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: Ok. [...] Any way is right, right? [...]

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 6, 7, 8, 14, 19	These ones I spot them very quickly.
2	5, 11, 12, 13, 20	These ones are infinity or infinity over infinity.
3	1, 2, 3, 9, 10, 15, 16, 17, 18	They were going to zero or it was a zero over zero form .

Figure A6 (Copy of Table 5.6a). Student S6's classification.

I: What was your rule?

S: These ones I spot them very quickly [class 1], these ones are infinity or infinity over infinity [class 2], and these ones I put them if I knew they were going to zero or if it was a zero over zero form [class 3].

I: So this group was infinity over infinity or infinity... this for example [he had placed card 1 in class 2]?

S: No, right, this one would be here [class 3].

I: Why there?

S: I would have to calculate twenty seven minus nine... over... yes it's zero over zero.

I: And this one [he had placed card 19 in class 3]?

S: I don't remember how to treat this one, ok, it was one the answer right? So it would be here, with the ones I can do right away [class 1].

I: But in your classification you were thinking about the answer?

S: Yes, I was thinking how to take it apart to see if I could go to the answer quickly or if it was an indeterminate form.

I: And this one [card 20]?

S: This one I saw the one over x and I knew it was infinity right away.

I: Ok. Now I ask you to solve some limits [2.1]. And if you can think aloud.

S: Ok. I start, I do one minus one, I see it's zero over three and... I remember that it has to be going to infinity for me to take the leading coefficients. So I factor the bottom I guess to try to cancel the x minus one... Oh, how do you treat this? Is zero over three, should be treated as an indeterminate?

I: You can use your calculator if that helps you in any way.

S: I never use calculators. It might be useful but I feel it slows me down. Ok, so it's zero over three... [he decided to use the calculator and did some calculations].

I: What are the calculations you are doing?

S: I put zero point five and I will go towards one. [...] Ok, I think is going towards zero... but it's frustrating that I can't prove it, I know there's a way.

I: Ok, I'll give you another [2.2].

S: Ok, this one, this one I think I'm gonna factor the bottom, that's gonna to work out. [...] I think that's correct.

I: What about this one [2.3]?

S: My first reflex was, is always, I put the five here so I saw it was over zero. [...] Again I factor out but... [...] I can't remember how to treat it. [...]

I: If you have to read this [2.3] aloud, what would you say?

S: You mean like the numbers?

I: The whole expression.

S: I would say the limit of x square minus four over x square minus twenty five.

I: And what does it mean to you?

S: As my x goes towards five point... I always picture that big l that we see everywhere. I see the x axis... like a correlating point...

I: For example in here, when you say this equals negative one, what's the meaning of that for you?

S: So as this function moves towards x equals two, y moves towards negative one, like a boundary.

I: What moves to what.

S: The whole function moves towards x equals two, there would be a correlating point y equals one. That's how I see it. The limit of x square as x goes to two equals four, that's how I see it, that's how I see all of them.

I: Ok, what about this [2.4]?

S: I see this is very simple, I just plug in the numbers. [He went back to problem 2.1.]

I: Now, if you try to use this idea of the x going towards something, would that help you in any way to find the limit?

S: I could probably find it by trial and error.

I: How would that be?

S: I would try a bunch of values around that point.

I: Ok. Do you think that this trial and error method [plugging values in the calculator] is less accurate than doing this [direct substitution]?

S: No, I think it amounts to the same... but we look for the easy way out to do, and we are so used to having formulas for everything and like a method to do everything, we are not used to computational, I think.

I: And do you think the step that you did here [2.2] was necessary to solve the problem.

S: No it wasn't.

I: Why not?

S: Because there's nothing going zero over zero.

I: Why do you think you did it? Because in the other problems, the first thing you did was to substitute, but not here.

S: I think because I saw the top factors out... I think every time I see x square minus nine I get mentally excited and I want to factor out and cancel. And I knew I would be able to cancel so I was confident.

I: Ok, the last thing, to solve those three limits.

S: I know e to the x would go to infinity. Cosine of infinity... it can't exist because it's a harmonic motion.

I: What about this case [3.2], when it's going to minus infinity?

S: Oh, it doesn't matter, it's going to be the same thing.

I: What about e to the x?

S: Oh, no, it would all go towards zero, yes, that's true. e to the x would be closer and closer to zero. So this would go towards... but I don't know how to treat the cos in that situation. I know this goes towards zero [for the exponential function] and cos ... Because even if this is going towards zero, if my cos is getting bigger and smaller... but it's always the same sequence multiplied by that, I guess I could assume it's going towards zero.

I: Do you think you could use the calculator?

S: I could try, I could do minus ten... [he did it] Ok, so this is minus ten. I am doing minus twenty. Yes, I think is getting closer to zero. Yeah, it is going towards zero. And this one [3.1], I could test it but I am pretty sure it's going towards infinity. Yes, it goes to infinity. If you can treat it this way, I don't remember how to treat it. And this one [3.3], oh, yeah because sine goes... as sine goes towards zero... how I use to treat this... I just remember the graph getting smaller and smaller, and it goes towards zero.

I: But that is as x goes towards infinity.

S: Oh, yes, but as x goes towards zero... I remember it going towards one.

I: Why do you remember that?

S: Well, I remember memorizing it.

STUDENT S7

I: The first thing I ask you to do is... here are twenty cards, I ask you to classify them into groups, according to any rule that makes sense to you.

S: Ok. [...] Is there any limit on groups?

I: No.

S: I hope I don't get a grade on this...

I: No, this is not a test at all, there is no right answer or wrong answer. [...]

S: Ok.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	With sine.
2	3, 10, 13, 14, 18	Contain square roots.
3	1, 2, 4, 5, 9, 11, 12, 15, 16, 17	Polynomials.
4	6, 7, 8	By replacing.

Figure A7 (Copy of Table 5.7a). Student S7's classification.

I: What are your four groups?

S: With sine [class 1]. These are that contain square roots [class 2]. These are polynomials [class 3]. These are just by... [class 4]

I: Just by?

S: Replacing [class 4].

I: How do you read this [expression 8], if you have to read it aloud?

S: Limit as x approaches one of four x cube plus seven x minus nine.

I: Ok. Then I ask you to solve some limits, and if you can think aloud...

S: [2.1] Well, because it's a polynomial, the coefficient is higher, we can do the misse en evidence of the x. Because the first down has a minus and the one has a plus... unless I can take the minus here and cancel, so it be one x and I replace so it be one.

I: Ok. What about this one [2.2]?

S: This is a perfect square so we can do x plus three, x minus three. Cancel and replace two. So we have two minus one over one, so it'd be minus one.

I: What about this one [2.3]?

S: I can take an x square because it's up and down. It'd be x minus [inaudible] over one minus twenty five square, and this would go to five, well here the limit goes to zero, and here two. So one minus zero over one minus zero it would be one.

I: How about this [2.4]?

S: If we replace the one, it'd be one plus two. And here it'd be four plus nine which is fourteen. So it's not an indetermination as infinity over infinity or zero over zero. Is fifteen... no...

I: Ok. Let's say... in this problem here [2.2], do you think it was necessary this step that you did here? Could you solve the problem without doing this?

S: Yes, because if you replace the two it'd give four minus nine, so that's minus five, and here it'd be five times one, so it'd be minus five over five, so it'd be minus one.

I: And why do you think you did this anyway [the factoring]?

S: Well... I don't know... for me... because most of the exercises that we were given, every time that you'd replace it'd give you zero over zero, so it's kind of a reflex.

I: And what about this one [2.3]? Do you think you can solve the problem without doing this step?

S: This... you get twenty one over zero [as she's doing direct substitution]. I am not sure, because as you replace the five. it'd give twenty five over zero, which is impossible, it's not defined.

I: Ok.

S: I can't remember any other way.

I: Ok. Maybe in this example here [2.2]... when you see written that limit as x goes to two of this function equals negative one. What do you have in mind? What's the meaning of that for you?

S: It would be in the graph. If the value of x approaches two, then you'll approach minus one, the limit would approach minus one, it won't be higher than minus one.

I: Ok. When you say the limit would be minus one... what do you mean exactly?

S: Well, if x, for example, is two here, all the values that are approaching two on the x axis, they will be approaching minus one on the y axis. They wouldn't... between, I don't know from minus infinity to two, for example, there would be... there would always stop at minus one... they wouldn't cross, an asymptote?

I: Ok. The last one, I ask you to solve these limits. And again, if you can think aloud... [...]

I: If you have to solve them separately, just the limit of e to the x and the limit of cosine of x, as x goes to infinity, could you do it?

S: Well, the limit of e to the x, the graph of e to the x is like this, so it'd be plus infinity.

I: And for the cosine?

S: Is like this [he correctly sketched the graph of cosine]. I think the limit would go... I am not sure.

I: If you have to use your calculator... Could you use your calculator to guess those limits?

S: Well, I could start at... a value, plug in two and continue, and if was going towards the same... I think the limit would be one, because it continues to one.

I: What about this one [3.2]?

S: For e to the x, it'd go to zero, and if it's minus something so it would be one over e to the x, so that limit is zero, and then I am not really sure about the cosine x. I think is one.

I: Let's say is one... what this multiplication would give you?

S: Zero.

I: What about the last one [3.3]?

S: This I know there was a property with sine of x over x, but I can't remember exactly...

I: And why can't you just calculate it?

S: Because it's over zero, you can't divide by zero.

STUDENT S8

I: The first thing I ask you to do is, here are twenty cards, I ask you to classify them according to any rule of your choice, any rule that makes sense to you.

S: Ok. [...] Ok.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 10, 13, 18	Rationalization. I have to do that first.
2	1, 2, 4, 5, 9, 11, 12, 15, 16, 17	Factor. I have to do that first. Then see if something cancels out.
3	19, 20	Trig.
4	6, 7, 8	Just plug in.
5	14	Not sure [what to do].

Figure A8 (Copy of Table 5.8a). Student S8's classification.

I: Can you explain me what was the rule.

S: These... basically rationalization [class 1], like that's the first thing I think of. I have to do that first. These I have to factor first [class 2]. These are trig ones [class 3]. These you can just plug in [class 4]. And this I am not sure [class 5].

I: Let's say, if I ask to read this [card 8] aloud what would you say?

S: Four x cube plus seven x minus nine.

I: And the whole thing, the whole statement?

S: The limit as x approaches one of four x cube plus seven x minus nine.

I: Ok, and this group you said it was to factor?

S: Yes, I have to factor, I have to see if I can factor first and then if something cancels out.

I: Ok. Now I ask you to solve some limits. And as much as you can, think aloud so I can keep track of what you are thinking.

S: [2.1] Ok, the first thing I do is to see if the denominator is zero, I guess... Zero over two, zero. Then I factor, it doesn't work.

I: Why did you decide to factor?

S: Because... to see if something... Is forced pattern I guess. I just plug it in.

I: What about this one [2.2]? [...]

S: First plug it in, I guess. This is so cal one... There is no reason to cancel out if it doesn't give zero on the denominator [she still factored the expression].

I: Why do you think you did this?

S: Because, usually in cal one what the teacher gave us the denominator would equal zero, so therefore you had to find out the asymptote, or if there was a hole, discontinuous.

I: What about this one [2.3]?

S: Ok, this is fine, you plug it in, you get zero... [...] I don't know what I would do. Because nothing factors [she meant that there are no common factors]... no rationalization. I don't know. [...] Do I have to do it?

I: What about this one [2.4]? [...]

S: Fourteen over three.

I: Say... when you write this, the limit is fourteen over three, what's the meaning of that for you?

S: You mean graphically?

I: What do you think about when I say that the limit of this expression is fourteen over three? What does that bring to your mind?

S: First a blank. Then, well, I can see the function... as x approaches one from both sides, it approaches this number.

I: What is it that approaches one and what approaches that number?

S: Is... I can see x equals one and the function approaches fourteen over three, that point there.

I: Ok, the last thing. [...]

S: Ok. [...] Actually, I don't know.

I: If you have to calculate them separately, the limit of e to the x and the limit of cosine of x...

S: Well, e to the x would be infinity, and cos of x alternates... so it wouldn't exist?

I: And with those two ideas combined, can you figure out what the limit of the multiplication is?

S: Well, this one doesn't exist [for the cosine] and this one goes to infinity [for e to the x]. It doesn't exist?

I: If you could use the calculator to check this, would that help you in any way?

S: No.

I: You don't remember using the calculator in cal one to calculate limits?

S: No... unless for basic calculations. But not for theoretical, like finding the derivative or the limit.

I: How about this [3.2]?

S: Well... cos would be the same.

I: Do you remember the graph of e to the x.

S: Something like this, right?

I: Can you use the calculator to check your graph?

S: Well, I could, if I plug in... but for cos, I know it goes like this and it always alternate between one and negative one.

I: Could you use the calculator to check if what you remember about the graph of e to the x is right? [...]

S: Ok.

I: So what's the limit as x goes to minus infinity.

S: Zero.

I: And then combining the graph of cosine and this, can you figure out the limit of the multiplication?

S: I don't think it exists.

I: How about this one [3.3]? [...]

S: This looks very familiar. [...]

I: Are you trying any calculations in your mind?

S: I am trying to picture the graph... And then plugging in zero but that doesn't help.

I: And in this case do you think you could use the calculator?

S: No. Because I already know how the sine function looks like. Sine of zero equals zero.

I: If you just had to calculate the limit of this... [I wrote $\lim_{x\to 0}\frac{1}{x}$], what is that? [...]

S: I could use the calculator to guess... it would be infinity. Yes, definitely it would be infinity.

I: Why?

S: Well, actually, it goes like that.

I: Why from the positive side you say it goes up?

S: Because at zero point zero zero zero one it would be really really really big.

I: And when you said that in this case you would be able to use the calculator, what were you thinking? How would you use it?

S: Plugging in zero point zero zero one...

I: Can't you use that in here [3.3]?

S: Yes. Definitely.

I: Can you try it?

S: Ok. [...]

S: It goes to zero point zero five seven one... this doesn't make any sense at all.

I: Is the calculator in radians?

S: No! Ok. [...] It approaches one.

I: Now that you tried this... do you think you could use the calculator to try to check these ones [3.1 and 3.2]?

S: Ok, but what can I plug to equal infinity? I don't think I can.

I: When it says... like in here, it says x goes to zero, but you didn't plug in zero, right?

S: Right. Ok.

I: So when here says x to infinity...

S: I don't think you can. Because if you plug a very very very big number, it doesn't show where it is, unless you graph it.

I: Ok, let me understand this idea that you have. You say that if you plug a big number you are still far from infinity?

S: Well, if you plug a very big number, it shows error.

I: But, when you plug a very small number, like zero point zero zero one, do you think that's realistic for the limit?

S: Well, sort of... kind of... in a sense no, but you do get the right answer.

I: In which sense no.

S: In the sense that you really never reach the actual... what it is, because you can't.

I: Let's say you take a really big number, a trillion, and a very small number, zero point many many zeros and then a one, do you think that with this number you are closer to zero than with your trillion you are closer to infinity?

S: No. I think they are both as far.

STUDENT S9

I: Ok, the first thing I ask you to do, these are twenty cards, I ask you to classify them according to any rule of your choice... any rule that makes sense to you. [...]

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 4, 9, 17	Numerator and denominator have the same degree.
2	1, 11, 12, 15	Upper is more than lower.
3	5, 16	Upper is less than lower.
4	8, 14	Polynomial.
5	3, 10, 13, 18	Upper is less than lower and it's not an integer. [The power is not an integer.]
6	19, 20	Sines.
7	6, 7	Constants

Figure A9 (Copy of Table 5.9a). Student S9's classification.

I: Can you explain me what was your rule?

S: Numerator and denominator have the same degree [class 1]. This is upper is more than lower [class 2]. This is upper is less than lower [class 3]. This is polynomial [class 4]. And this is upper is less than lower and is not an integer [class 5]. These are sines [class 6]. These are constants [class 7].

I: If you have to read this [card 8] aloud to a friend, what would you say?

S: The limit of... when x is approaching one.

I: Now I ask you to solve some limits. If you can while you do it, tell me what you are thinking.

S: [2.1] Can I use what I learned in calculus two?

I: Yes, anything you know. [...]

[She found the limit but is not able to talk while doing them, so I gave her all remaining three limits in part 2.]

I: Ok, in this one [2.2], do you remember what was the first thing you tried?

S: I see I can cancel something. I know this can turn... you can cancel, it's more easy.

I: Do you think it's necessary?

S: I don't think it's necessary, but I think it's faster.

I: If not, what other thing could be done?

S: Just substitute the number.

I: Why is it easier to simplify? [...]

I: In the first one [2.1] did you try substitution right away?

S: I tried substitution and this is zero [the numerator] but if this was zero [the denominator] I would change it...

I: And in this one, when you write that the limit is one, what is the idea for you?

S: We learned the theorem... the limit is going to be...

I: But, say you have to explain the meaning of this to someone else, what would you say?

S: I don't know, just two numbers.

I: The last one... to calculate these three limits. [...]

I: Ok, why did you write here [3.1] that this doesn't exist?

S: Because this changes, when x approaching to infinity.

I: How about this one [3.2]?

S: This is going to infinity but this doesn't exist.

I: But what about in this case, when x is going to negative infinity?

S: Oh, this is zero, so this one would be zero.

I: Why would it be zero?

S: Because this [the cosine function] is always between one and minus one.

I: Ok. And here [3.3]?

S: This a theorem.

STUDENT S10

I: The first thing I'll ask you to do is... here are twenty cards, with different statements in each... I ask you to classify them into groups according to any rule that makes sense to you.

S: Ok, well, I am not sure exactly how to proceed with this... I guess well I can say this looks like a problem where you know they look like something that will break into [inaudible] you know, x minus three times x plus three or something like that, you know. They would just break up, so the first thing I would probably do is factoring and cancelling them out, but again I might need a pen and paper to see if it would work, probably it would though you know, but it could be one of those screwed problems that it's gonna drive me nuts. But it looks like a fairly simple problem where you just put some factors you know. And this one [card 1] looks pretty similar to that. The top is a little odd but you know... it's a little odd because there's usually an x square right in front but I'm pretty sure that yeah I think it would work if you have something that cancels out like a minus two x or something like that... But I don't know maybe I should take into account the limit, so x is going to infinity, x is going to three, but like usually for something like that it's better just to try to... because everything here is kind of scattered so you know... It's probably best to simplify everything and then solve it, you know. Again this [card 8] does look like... this looks like a bit similar so... oh oh, ok, this one seems like a bit of a trick, as if it is too easy, like x goes to one you know and there's nothing even under it. It's just...

I: So you'll put it on a different group?

S: Yeah. I'll throw that in here too. I don't know, maybe partial fractions could be useful but still I think this is a stage in which you can just factor you know, break it up into little bits you know. Leave the square... anything that would work out. This one [card 15] will definitely work, I can tell already. I can even calculate that... that's like x minus two times x plus one, yeah, or something like that. I don't know, it [card 15] seems kind of tricky actually because it's going to be x plus one times x minus two divided by x minus one so that's actually gonna be a bit tricky I guess... but again I think it can be break up because it's going to be two things multiplying each other so you can basically, so you can get x minus one divided by x plus one you know, and you can do some long division you know, something like that and you'll get a simple value, simple enough I think. And then you get probably two terms so you know it would work out I believe. This one looks... that's just gonna... I think, I don't know, again this [card 9] is one of the ones where it looks it's really really really simple, but I am pretty sure it's gonna go to something screwed because here you have x plus two times x minus two you know. Then you got x minus... x minus four times x minus... no, times x plus one. So yeah, that one would be a bit tricky but... you might want to break that into partial fractions... I am pretty sure... Yeah, you could do that, it's cool. Now this one [card 2] is easy, because this would be x plus one times x minus one and you can already cancel something out... yeah, you would get something that would work fine. You see... x plus two times... yeah I think this one would need partial fractions. Ok, this one is pretty insanely easy, so over there goes. I don't know, I haven't done enough problems of x cube minus one [card 5]. It might work as a difference of squares, I might... but even still it's a little tricky because there'd still be a different term up above there. It would be x minus five times x plus five and you also have... if like, if something like a difference of squares would work like x minus one times x plus one times x minus one or something like that, it still be a lot of terms down there and different ones would appear so I'll put that over here. This one... I'm pretty sure this one's, yeah you can just use L'Hôpital's rule on this [card 19] you know. Cosine x over one, that'll give you one you know, that's probably a basic rule, so that's easy. X goes closer to zero, this gets big... I'm not sure! Ok, let see. All the ones we did x's going to infinite so… I always had a hard time figuring out like the sign of infinite, you know that's a little, I don't know why I can't seem to figure that out but I can't so over here. This one [card 10] is really really really… yeah I'm pretty sure you gonna have zero over eight, wouldn't it? Isn't that allowed? Is not one in which zero is beneath so I'm pretty sure that would work. And even if it wasn't I'm pretty sure… no you can't break it up. But it's zero over eight so it goes over there. Let see square root of x plus two [card 3]… I don't know, something about it tells me that if I had pen and paper and I did this I might get an easy answer just because it looks like the one that's really complicated but it's actually pretty simple but on the top of my head I can't quite think of it and it does look like even if you do square roots… Let see, square root of five [card 18]… square root of… Again I'm not too sure about it, I can't think of it just on the [inaudible] so I'll put it on that pile. I think that's… this one, yeah. Let see, you can just use L'Hôpital's rule here [card 18]… you can do that here too because the base is just five minus x, the derivative of that is one, minus one, but that doesn't really matter, so then you are just left with two terms that you can deal with separately and you get something so that's actually pretty easy now that I think about it. So I'm putting it over there. Again [card 14] L'Hôpital's rule apparently, I think. Yeah, because you do L'Hôpital's rule, it will still be going to infinite, so it's just gonna be infinite, yeah, I think so. Seems really easy, seems really easy. It's just, it's going to infinite and there's no way… this isn't gonna be zero… I don't know, I think there actually might be a limit here, just because it'll get higher and higher so it's gonna be the square root and there's always gonna be this term you know, so… I think what I would do here is maybe, I don't know if you can, maybe square it, just square the whole thing you know, and then see what you get, I don't know. That looks like it might make things more complicate actually. I can't think of it right away so I'll put that there. And I'm not even saying anything about that [card 6].

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 6, 7, 8, 10, 12, 13, 18, 19	Easy.
2	1, 2, 4, 11, 16	Factoring.
3	5, 9, 14, 15, 17, 20	Not sure [how to find the limit].

Figure A10 (Copy of Table 5.10a). Student S10's classification.

I: Good. Let see, those were the factoring ones right?

S: Yeah.

I: Let say for this one [card 11], how would you approach it? How would you use a factoring technique to solve it?

S: I may have overestimated on that one. I suppose what I could do, and although I'm not sure this would get me anywhere, but I suppose I'll do it anyway, is something along the line of completing the square you know, because you got a seven x here you know, it's going to be three point five or something so you take three point five square and you add that and you subtract it you know, and you take this value out and you get the square you know, you get the little factor here. It might be something up there and if it isn't you know I'm not quite sure what you do next. I might have to restart the problem you know... just do some long division. Long division will be a lot simpler actually. Because with long division... No. but actually I was saying before, with completing the square, that would work for a bit. if it doesn't get rid of anything you can do long division, you'll be left with the remainder, that would be divided by that, but it would be a lot simpler, so yeah.

I: How about... A different question I'll ask you now, let say you are studying with a friend and you talk on the phone with him and you have to read this [card 2] to him. What would you say?

S: For this one? I guess it's you know x square minus one over x minus like x minus one together time x minus three together.

I: And if you had to read the whole expression? Everything that is here.

S: Well I say at first, you know I can say the limit as x goes to one but that's irrelevant you know this is the only complicated part, limit as x goes to one is pretty easy to understand.

I: Ok. Now I'll ask you to solve some limits.

S: All right.

I: And again, if you can sort of think aloud so I can have an idea of what's in your mind...

S: Well you know the first thing you'll do here [2.1] I guess would be to factor whether it worked or not. It doesn't quite work here because you got x minus one over x plus one, you can't deal with that. And you got... the denominator is bigger than the numerator so you can't really do long division but I'm pretty sure you can do partial fractions here. So you have to split it like this, you have x times x plus one you know. So that would be a over x plus b over x plus one, which gives you two terms and these ones would just be constants so that would be pretty easy you know. So you do the multiplication... x minus one equals a times x plus one plus b times x. And you know for that basically what you do is... you got to make one of this equal to zero, that way you can find the value of the other. I'll make x equal zero that will make life damn simpler so zero minus one equals a times zero plus one, which now gives you minus one equals a. So you got, let see. Negative one... sorry, I scratched a little but... you got a that is negative one so you can put it there and solve for the rest you know, which is basically equals negative one over yeah, x plus one. Yeah, I think that works you know. [...] I don't know, for some reason it looks something it's gonna go screwed and I'm not sure [...] Ok, yeah yeah...

I: You want to find b, right?

S: Yeah yeah, this is what confused me a little... I thought if I had an equal negative one I could just replace it and solve but it's not really working. But I'll just do the same thing, I'll make x equals negative one so I can get rid of a... make life even simpler you know. I don't know, for all the other problems I did with partial fractions the x just disappears you know, something like that... I can usually leave it like that but... So negative two equals a times zero plus negative b. So b is two. Well I am taking a fair amount of space, I'm thinking that's ok. Then this gives you limit x goes towards to one of negative one over x plus two over x plus one, and now this is going to be fairly simple, it's just because none of these values are going to get one over zero so it won't matter. It's going to be negative one plus two over two, just one, it's gonna be zero.

I: What about this one [2.2]?

S: This is gonna be nice and simple and easy. I don't even have to do all this, it's not gonna be something over zero. I can just... I don't even have to factor anything. I can just put it in right now. It's going to be five times one times four minus nine. I'm not sure what that would be. That would be negative one.

I: What about this one [2.3]?

S: Ok, that's not gonna be… I think what I'll do is… actually I think you can just use… no no I'll make life simpler, I'll just use partial fractions you know. Let see… I'll start by opening it up and giving some factors, you know, like this and… x plus five, like that. Just to make life easy I'm gonna take out one of this, that's gonna make x minus two… and I'm just gonna do here what I did before, again you know. [...] That's going to be b over x minus five and a over x plus five you know. I guess x equal to five and seven equals a time zero plus ten b. So be equals point seven. And I am not putting it in there, I'm just going to make x equals minus five. And it will be negative three over here, equals a times negative ten plus b zero… equals zero point three, again pretty simple. So now I'll put them all the way back here you know. I'll put a little line there, make it a little easier to read… I might have [inaudible] a little trap for myself, but I think it would be ok anyway, it's just because I'm gonna have to values here you know, I'm gonna have zero point seven over x minus five and zero point three over x plus five. Now this is just going to be me taking a bit of a hunch, it might be a terrible hunch but I'll do it anyway. Basically, x is going to be going to five, so this is going to be zero and that's going to be pretty crappy but if I have to actually… no actually, this is fine, what I was going to do was I just well there's another term here so it's going to be fine I'm going to get a value you know but just to be safe, because I'm not quite sure if that would be acceptable, I'm going to have to terms that are gonna to go limit to x to the five you know, it's going to be… I'll just write zero point seven times x minus two over x minus five plus limit of zero point three times x minus two over x plus five. Over here it doesn't matter, I can just put that in here and it won't make a difference. Over here, if I have to, I could use L'Hôpital's rule because this would be one down here and this would also be one, so that would be zero point seven… I guess… yeah… you know it seems ok. If it isn't I'll blame it all on L'Hôpital's you know. And over here it would be zero point three times three over ten, which is zero point zero… is that right? Yes, it's right. And I'll add them together.

I: Do you remember that there are some conditions under which you can apply L'Hôpital's rule?

S: Well, now that you mention it… but no, I don't remember them. No, I don't.

I: Ok, what about this one [2.4]?

S: Well, that's sweet, nothing is going to be zero here. It's just going to be one plus four plus nine over one plus two, which is fourteen over three. I'm not going to bother writing decimals that's gonna be a waste of time.

I: Let see, in the first one you did, that you started with the partial fractions method, do you think it was necessary to do these steps to get the answer or…

S: The problem is that right now partial fractions are a pretty simple thing to do, especially when is this simple, so I went for that one.

I: But once you have the problem written in this way, is it in any way simpler than this expression here? [I meant the expression obtained by partial fractions compare with the one given in the exercise.]

S: Well, now that you mention it, I accidentally forgot that this would be... that this would not be zero and that was dumb... yeah that was a bit of a waste of time but I guess I wanted to show off my use of partial fractions...

I: And let me ask you a question about this other exercise [2.2]... when you write here that this limit equals negative one, what's the meaning of that for you? What do you have in mind when you say the limit as x goes to two of this function equals negative one?

S: Well this is again thanks to my Cal I teacher, he explained a brilliant story about limits you know, although I'm not going to go into a lot of depth about it because he refer to it as almost almost almost almost having sex with somebody and then getting out at the last minute... basically the idea of the limit is that it gets closer and closer and closer to something, and sometimes it gets there, and sometimes it doesn't you know, this is something because nothing equals zero it gets to two and when it does is going to be negative one, if it would be something else, I think if it would be equal zero over here it would get closer and closer and closer to zero but it wouldn't get there... is that clear enough?

I: So the fact that nothing is zero here means that it actually will get to negative one.

S: It's really just the bottom [denominator], the top part can be zero you know, it's just the bottom because it doesn't make sense, for whatever reason.

I: Ok, ok. This is the last one.

S: Ok... well, this [3.1] is problematic because I'm not sure what the cosine as x goes to infinite is. What x is when it is infinite in cosine but let see... It's not quite accurate but that's never the point... Well ok, I don't quite remember the conditions for L'Hôpital's rule but in this moment of desperation I'm going to kind of cheat and use it because you know you are going to do the cosine of x over e to the negative x and that works you know. You can do the limit as x goes to [...] It works for positive infinite. So let's see, L'Hôpital's rule... again this doesn't quite help me because I'm not sure about cosine of infinite or sine of infinite...

I: Let's say you could use your calculator... would that help you?

S: I could write a really big number and then try cosine... the problem of that is that it's bouncing back and forth between whatever it is so you know. I could have x equals nine hundred and ninety seven and then nine hundred and ninety seven plus five and that kind of screws it all up you know.

I: What about the e to the x? What's the limit of e to the x as x tends to infinity?

S: Well, that's pretty much infinite because it's just going to get bigger and bigger and bigger, and the problem you know, and the problem is I would say it's infinite but I don't know if cosine of infinite is equal to zero or if it is equal to one or... well it is really whether it equal zero because if it equals zero then the whole thing equals zero, and if is anything above that, I'm just going to take a hunch and assume that a number times infinite is infinite.

I: And could you use the calculator to kind of guess what the answer would be?

S: Well the problem with that is... I just type a big number, I'm going to write it down so I don't forget it [...] Well this is in degrees...

I: Is in degrees?

S: Well it doesn't really matters I can add degrees… I'm just… It's now negative, and before it was positive [he tried two different big numbers]. Before it was point nine something but now is back down again. So I can't really use a big number because I used every single digit… I did nine nine nine nine… but again that probably wouldn't help me because as long as it shoots back and forth it doesn't approach anything I know. And I'm pretty sure there's an answer for what cosine of infinite is but I just don't know it. Is one of those things I haven't look at…

I: What about the last one [3.3]?

S: I'm pretty sure we covered that… This is the sort of thing where we can use L'Hôpital's rule again and I'm pretty sure this will work here because mainly because this is one of those fundamental rules of calculus, where sine of x over x equals one you know. Or is it the other way around? X over sine x equals one? Oh, it wouldn't matter, if it equals one it should work either way around. So it'd be cosine of zero is one, it's one.

I: What you just said… is it sine of x equals one? Or is it the limit as x goes to zero of sine of x over x that equals one?

S: Well I don't know, what I did here is I just used L'Hôpital's rule to convert it basically so that I could put an x and it give me and actual value of the limit…

I: But let's say that now I change it and I write sine of x over x equals… and I ask you what that equals… would you say one?

S: No, I don't think… I wouldn't be able to… no, because x could be anything in here, but here x is approaching zero… x is approaching zero so therefore I can use L'Hôpital's rule and punch that in there. But here sine of… here, this would just… this is one of those situations where I would try my calculator though I don't think it would help me would it?

I: What would you try to punch in there?

S: Well, it's not going to help me because…

[He tried a few numbers in the calculator for the expression sine of x over x.]

S: So yeah, is bouncing back and forth like with the cosine and I was pretty sure it would. But you don't know what x is. And x could be anything in here. It could be high and then down, and high and then down. So I don't know, if you say x goes to infinite I would say zero because this would never be bigger than one and this would be a lot bigger, and this would be divided, so it would be zero if x goes to infinite. But... here it's just that.

STUDENT S11

I: The first thing I'll ask you to do is... here are twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: Ok, so this are twenty of them... I am not very good at this, I have to think a lot. [...] Normal... this a normal one... It can be anything, right? Substitution...

I: Just a way of organizing these cards...

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 2, 4, 9, 16, 17	These are fractions.
2	3, 5, 11, 12, 13, 14	These are going to infinity or zero.
3	6, 7, 8	Easy ones. You can do them right away.
4	15	Fraction that can't be divided.
5	19, 20	Sines.
6	10, 18	

Figure A11(Copy of Table 5.11a). Student S11's classification.

S: Ok, ok. [...] I'll put fractions with fractions [class 1]. [...] These are going to infinity [class 2]. These are just a number [class 3]. These are fractions [class 1]. Ok, I think I am done.

I: Ok, let's see. You put these two [class 5] together because...?

S: I guess because they were sines [class 5].

I: Ok. Did you have a general rule when you were organizing?

S: There is a lot of them, but since I saw... this is radical one eh? Not really, maybe if they were fractions... like I can think of dividing them and then something that doesn't work if I put it up here...

I: And why are these in different groups [objects in classes 1 and 4]?

S: Well, these are fractions, but this I can't divide [card 15]. These I don't know. Maybe we should put all the fractions together, but the ones you can divide. This we can do something like that too. This we can multiply and put it down to get rid of the top.

I: And this [an object in class 1]?

S: This you can divide the top by the bottom.

I: But not this one [card 15]?

S: No. Not by an easy way I can think of. Not by long division anyway.

I: Ok, let's see the other groups. What are these?

S: These are going to infinity or zero [class 2], I think. Yeah.

I: And these ones [class 3]?

S: These ones are just the easy ones, you can do them right away [class 3].

I: Ok. And this [card 7]?

S: And this is just the limit of three going to five so you get three... or five, I am not sure. I actually don't remember seeing those. [...] Maybe it doesn't make any sense [she means the classification].

I: Well, if it does make sense to you... Then I ask you to solve some limits.

S: Ok. Anyway I want?

I: Yeah.

S: If I remember how to do it...

I: And while you are doing it, if you could think aloud so I can keep track of what you are thinking...

S: Ok. I don't remember how to do this [2.1]. I think I can pull an x out of here [taking common factor x from the denominator], like... On the top is the same thing and on the bottom is going to be x plus one, but then I cannot cancel anything. I mean that doesn't work. This is really really far in my head. [...] If I put one here this will be zero? I'd like it better if you give me antiderivatives. If I multiply top and bottom by x minus one... that's just make it longer...

I: Where did you get this part from?

S: No, I just put it away and then I said well if I plug in one, but then it gives zero [the numerator], and this is actually two [the denominator]. And zero divided by two I am not sure if it is zero or is infinity.

I: [Since she took her calculator in the beginning and put it next to her, I asked the following.] Do you think you can check with the calculator?

S: Ok... if it gives math error... Ok, zero, but then the derivative will be zero.

I: The derivative?

S: The limit, I mean. But I don't think of anything else that I can do, so I'll do it like that.

I: Ok, what about this one [2.2]?

S: My first impulse is to open this up to see what I can get in common. So I'll just do it like I always do it and then I just cancel things out. This stays the same. This is x to the two minus nine, this is x to the two minus plus two x minus three, divided x to the two minus nine, and then... If I do, can I erase? [She multiplied out the numerator to get $x^2 + 2x - 3$.]

I: No, please just leave like that.

S: Ok, because this is more complicated, I cannot cancel anything. But then if I use the top as it is and if I open the bottom [factor the bottom]... So x plus three and x minus three, I can cancel these two and is x minus one over x minus three and I can just plug in two. One... minus one, it's going to be minus one.

I: Ok, what about this [2.3]?

S: If I open them, there's is nothing I can cancel... there must be something else. I cannot bring them up either, then I can't divide. I cannot pull any xs out. I'll open them [factor them] and I see after, maybe, but I don't think here anything will work. I can put this to be u and this to be v and then I can do like quotient rule.

I: Quotient rule?

S: U minus, u v minus v u over v to the two.

I: To do the derivative?

S: Yeah, but that won't work because it's derivatives and this is limits. So, so, so [she tried substitution]... this is twenty one on zero... but then that would be infinity. Ok, I am not sure.

I: Ok, what about this one [2.4]?

S: This I think I can do long division, at least I'll try to. I have to complete the square [she means completing the missing terms so to have all the powers of x] or whatever is on the top. So zero x plus nine and then divide. [...] This doesn't work. But then I have a two x that I cannot take away because of the x to the two. This is not a minus b to the two, so it won't work. So I plug in one... fourteen over... three. I guess I can always do that, just plug it in. It's fourteen over three. Other things won't work, so this is the only thing I can do.

I: Now, if you look at the problems, say in these three problems... in here you did this factoring and then you end up replacing the x by five [2.3], do you think this step was necessary?

S: Not really. No.

I: And why do you think you did it?

S: Just to visualize better sometimes... but it's not completely... Because if you put it in here [before factoring] you get the same thing, will give the same twenty one over zero.

I: And what about in here [2.2]?

S: Then yes [in 2.2 the factoring led to cancellation], because if I put in two... would it work? And then I did all that for nothing? I didn't check actually, this would be five, this would be two minus one.... minus five, that's not minus one. So it would give the same thing.

I: Do you think there could be a problem where if you do it in this way [factoring] you get an answer and if you do it in this way [direct substitution] you get a different answer?

S: If it's a problem?

I: Same type of problem, could it be that when you replace the x you get a number and when you do the factoring and simplification you get a different number.

S: Well, it should've been the same thing.

I: Just a question, when you look at this expression [$\lim_{x \to -2} f(x) = 0$] here... what do you read? If you have to read it aloud, what would you say?

S: As x goes to minus two, f of x is zero.

I: And what's the meaning of that for you?

S: Like it approaches zero, so the more is going to minus two the more it's to zero.

I: What's approaching minus two, what's approaching zero?

S: The line is approaching zero when x is minus two. Something... I don't know from where it's approaching minus two. You know what I mean?

I: I am not sure.

S: Like if I draw a line like this, f of x is zero when x is approaching minus two, but it won't always be zero, so is something like that, and then when you get to zero, well, it's zero.

I: Does it get there?

S: Well, it's approaching... maybe, not... I really don't remember.

I: Ok, let's move to the last part.

S: I don't know if my things will give you a lot of insight... How lost I am... Oh, my gosh! If I plug in zero [in 3.3] it would be one. [She tried something on her calculator.] No, sine of zero is zero, so zero over zero. I think we are learning this on calculus two. Can I use that?

I: Yes, anything you know, you remember...

S: Ok. I can do then like a Hospital's rule, whatever. So I do the derivative of this, it would be... This is one, over one. Ok, that's not that bad. And the others... x is going to infinity, cos of infinity... These [meaning 3.1 and 3.2] I think we should treat them the same way, I just don't know how. I can always bring something down. I can do cos to the x over e to the minus x and then try to figure something out of the infinity over infinity.

I: If you have to think of them separately [I wrote $\lim_{x \to \infty} e^x$ and $\lim_{x \to \infty} \cos(x)$]...

S: Can I do graphs?

I: Yes, of course.

S: e to the x is like this, is going to infinity. And this... will be between one and minus one, but then it would keep going forever and ever, so it's going to be infinity too. Because it won't stop.

I: And what do you think it would happen when you multiply them?

S: Multiplying infinity by infinity it's just infinity.

I: What about this [3.2]?

S: If I do e to the minus infinity, e to the zero [she checked on her calculator]. I have one here, so it's going this way [she graphed] so it would be zero, it would just approach zero, it would never reach zero.

I: Can you use that to guess the answer for the multiplication here?

S: This will be for sure minus infinity [for the cosine function], it'd be like this but to the other way. Can we say it doesn't exist [she meant the multiplication]? Because it's not gonna touch.

I: Can you think of any way of using the calculator to check, to confirm your ideas?

S: I can put e to the x and then I can put like a huge number like three million, like one zero zero zero zero. That gives me error, it means it goes to infinity. Because it doesn't really shows me a number. And then if I do cos... Oh, it was minus infinity [so she trie again with the exponential] so minus a thousand, it gives zero, but it doesn't really touch. And cos of minus one zero zero zero, this would be infinity, so it would be zero.[She did not look at her answer in the calculator.]

STUDENT S12

I: The first thing I'll ask you to do is… these are twenty cards, with different problems in each of them, I ask you to classify them into groups according to any rule that makes sense to you.

S: Anything?

I: Yes, you look at those and say I'll put these together and these together…

S: Ah, ok. […]

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Constants.
2	1, 2, 4, 5, 8, 9, 11, 12, 15, 16, 17	Polynomials. The ones you have to divide by x or cancel out.
3	3, 10, 13, 14, 18	Roots.
4	19, 20	

Figure A12 (Copy of Table 5.12a). Student S12's classification.

S: These are constants… this is one group [class 1]. Polynomials [class 2], I guess. These are the ones that you divide by x. […] Roots [class 3]… and… just like that.

I: Ok. At some point, when you were organizing these [class 2], you mentioned a technique to solve… Do you think that technique applies to all of them?

S: No. [She looked at each of the limits in class 2 but didn't change the classification]

I: But anyway, if you want to group them you'll put all of them together? Like you did in the beginning?

S: Yes. I don't know… these either you divide by x or cancel out.

I: If you have to… let say you are studying with a friend and you have to read this [expression 8] aloud, what would you say?

S: Read that?

I: Yes.

S: Limit as x goes to one of four x cube plus seven x minus nine.

I: Ok. Then I'll ask you to solve some limits. And if you can think aloud so I can get an idea of what comes first to your mind… it'd be great.

S: [Staring at the exercise.] Is that like a Cal 1…?

I: Well, you can use anything you know to solve it.

S: Ok. [Inaudible.]

I: Yes.

S: Well I guess I'll take out the one x, so is x minus one on x plus one... which doesn't give me anything. Well it's just zero right? It's just zero. [She was referring to the limit in 2.1. She took x as a common factor in the denominator.]

I: Ok. How about this one [2.2]?

S: [...] So I'll just... I am [inaudible] I'll put the two in it? Because it's just five times minus one over minus five, it's one.

I: In the beginning that's the first thing you saw?

S: Yeah, because if it would have been another number, like it would have...

I: But do you remember, when you just looked at the problem, did it come first to you to replace by two?

S: No. First I saw that I could factor and cancel out x plus three, but then instead of doing that like I try putting in the two.

I: Ok.

S: Ok, so this one [2.3] I would factor out but then I'll try again [she meant trying the substitution technique]. Yeah, so it doesn't work, because here [the denominator] is zero. So I'll factor out which won't really help me [she didn't try the factoring]. Well it's just infinity right?

I: Ok. The last one [2.4].

S: [...] Isn't this one [the limit of the function] just infinity too? I don't really remember if we are allow, when you can like... let say you divide by x cube... Can I do it, even if I have just one x cube? Or is the rule like... because if I do it then I'll get one over zero which is infinity... Oh, it's one right? [For the 1 in $x \to 1$.]

I: Yeah.

S: Oh. [...] Then it's just fourteen over three.

I: Why did you think it was infinity?

S: Oh, because I like I saw... I didn't look... I didn't see it was one [in $x \to 1$]. So if it would have been like as x goes to infinity and you divide everything by x cube then here it's zero and zero [for the two last terms in the numerator and the two terms in the denominator] and here is one [for the first term in the numerator], so one by zero is infinity.

I: And why do you think that... because you didn't look at this [$x \to 1$]... you looked at this problem and you thought it was an infinity type of problem... did it remind you of problems you've seen before?

S: Well it's just because is like you can't factor this. Can you? No, I don't think you can. So the only thing they could ask us is divide by x.

I: So all the problems you remember where x was going to a number were problems in which you could factor?

S: Yes.

I: Ok. One more thing. I ask you to calculate these three limits.

S: [Staring at the paper] I don't remember. [...] Oh, it's the product rule, right? Is it?

I: Did you say power rule?

S: The product rule.

I: For derivatives?

S: Yes... but here I have to find the limit. Ok, so this is infinity and cos of... can't you say it's just infinity. Because this is infinity and cos is... bounded. So infinity... I don't remember... if that's how we do it... And... e to the negative infinity... is zero, I think. [She picked up her calculator and did a calculation in it.]

I: Which calculation are you doing?

S: Oh, just e to the negative nine. So e to the minus infinity, the limit of that is zero, so the limit will be zero because cos is again bounded by... a constant? And the limit as x approaches zero... can I use L'Hôpital's rule? Because this is zero over zero, then L'Hôpital's rule... like you just do the derivative of them. That's cos of x over one so it's just one.

I: In the first one here, do you know if your answer is plus infinity or minus infinity?

S: Plus, I guess. Because as x gets bigger, y gets bigger two.

I: Do you think you can use your calculator to check if your answer is right?

S: I guess... yes. I do e to the nine thousand [she punched the numbers in her calculator] and it doesn't work. e to the nine hundred? Oh, it doesn't work, maybe it doesn't exist. e to the hundred... well it's not infinity but it's very big...

I: And if you multiply by cosine of a hundred?

S: It doesn't change… the number [she seemed confused about the fact that the number didn't change].

I: If you were on an exam, would you think of using the calculator?

S: Yes, I double check everything.

I: And would this suffice? You will convince after doing this calculation?

S: Yes… well if I had time I would do it with other numbers maybe.

I: Which other numbers would you choose?

S: Well I'd do bigger ones… but nine hundred didn't work. Five hundred? [She did it.] Doesn't work. Two hundred? Yes, very big.

STUDENT S13

I: The first thing I ask you to do is, here are twenty cards, I ask you to classify them according to any rule that makes sense to you. [...]

I: Can you explain me your rule?

S: First of all these are all x cube.

I: They contain x cube.

S: So you can do long division.

I: Oh, so they have x cube in the numerator. And this one?

S: [Inaudible.]

I: That one?

S: These are all square roots and you are suppose to do one solution.

I: One...?

S: Because they are all square roots.

I: And this one, it has square root too [but it's not in the square root-class].

S: Because of the denominator.

I: Why these are in the same group?

S: Because they don't have denominator.

I: And this one?

S: These are all square in the numerator.

I: And these are?

S: Just one... I am going to change my idea. I am sorry.

I: What is the arrangement now?

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	
2	1, 2, 3, 7, 8, 9, 10, 15, 16, 17, 18	*X* goes to a number.
3	4, 5, 6, 11, 12, 13, 14	X goes to infinity. There's one way of solving for all of them.

Figure A13 (Copy of Table 5.13a). Student S13's classification.

S: These are the same thing because x goes to a number. These are [inaudible] to a number. These x goes to infinity and there is one solution for all of them.

I: And why do you prefer this order better than the other one?

S: Because in this there is one solution for all of them.

I: Do you mean one solution or one technique.

S: Yes, one way of solving it.

I: Let's say, if I ask you to read this [expression 8] aloud what would you say?

S: You mean like a question?

I: What it says here.

S: Limit of x goes to one of four x cube plus seven x minus nine. I am sorry I changed my mind before, but I didn't see the x before [in the symbol $x \to a$ in each of the 20 cards of the classification task].

I: It's ok. Now I ask you to solve some limits. If you can, while your working, think aloud.

S: [2.1] So the first thing I would plug in the number. One minus one over one plus one, and this is zero. We are fine because there is no infinity over infinity or zero over zero. This is good.

S: Here [2.2] first I plug the number but I know is not going to work, two plus two, two square... five over minus five, this is minus one. Again you plug in the number [2.3]...

twenty one over zero, it's infinity, because there's no infinity over infinity or zero over zero, it's fine. Here [2.4]... fourteen over three. Yeah. These are easy.

I: When you write here that this limit equals zero, what do you have in mind? What it means for you?

S: I think this is basically to plug in a number, for example if you have limit x goes to something, what happens if I plug x in the equation. Is it acceptable or not acceptable? If the equation works for this number or not.

I: Let me give you an extra problem. Let's say [$\lim_{x \to -3} \frac{x^2+x-6}{x^2-9}$]. [...]

S: This is zero over zero. So it's going to be... yes, zero over zero. So you do x plus three... over... Are you sure this is a two? [He was not sure about my writing.]

I: It is a one.

S: No, no, no. [He had trouble with the factoring.] So, x plus two... So the answer is five over [inaudible].

I: And in this case what is the meaning of the limit being five over six?

S: Excuse me?

I: Now when you say that this limit is five over six, what does it mean to you?

S: I mean the limit, when x goes to minus three of this equation is five over six, after I solve. I mean if you plug in directly isn't gonna work, you have to solve first.

I: And when you say the word limit, what do you have in mind?

S: Limit... basically... For any equation, if you want to take limit... what's the smallest number that works out for this equation. When x goes to... what's the best number you can get when you plug in x in the equation.

I: Next, the last. [...]

I: If you separate them [the two functions in 3.1], can you calculate the limit?

S: What's the cosine of plus infinity?

I: Sorry?

S: Is there something cosine plus infinity [he actually wrote $\cos(\infty)$]?

I: Well, infinity is not a number. [...]

I: Could you use somehow the calculator to find this limit [3.1]?

S: It could work if x goes to a number. But infinity... [...]

I: What about the last one [3.3]?

S: Sine of zero is zero, right? [...]

I: What is it that you wrote before... about the adding one or subtracting one?

S: I was trying to add something... to see...

STUDENT S14

I: The first thing I'll ask you to do is, here are twenty cards, I ask you to classify then according to any rule that makes sense to you. [...]

S: Like any rule that I can apply to solve them.

I: Anything that makes sense to you.

S: Can I say like they are similar, the same method I am using?

I: Yes, you can say that.

S: Do I include things we learn in cal two?

I: Yes, anything. [...]

S: I think I am good.

I: Can you explain me your groups?

S: These are the ones I divide by the highest degree. That I would solve it easily, I think. These ones you used, how do you call that, the division ones...

I: I don't know, what do you mean?

S: You got the product rule and the division one.

I: You mean for derivatives?

S: Oh, ok, hold on. [She re-did the exercise.] I don't remember... [...]

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 3, 10, 13,18, 19	L'Hôpital's rule. Zero over zero or infinity over infinity.
2	14, 20	These ones I didn't know where to put them.
3	6, 7, 8	Straightforward. You just replace the *x*.
4	1, 4, 5, 9, 11, 12, 15, 16, 17	Divide by the highest degree.

Figure A14 (Copy of Table 5.14a). Student S14's classification.

S: Ok, these [class 1] you have to apply L'Hôpital's rule to all of them, and these ones [class 2]... wait [...]. I am not sure how to classify them...

I: So what are these ones [class 3]?

S: These [class 3] are really straightforward.

I: Why is this also straightforward [card 8]?

S: You just replace the x.

I: And these [class 4] ones?

S: You just divide by the highest degree.

I: These are by L'Hôpital's rule [class 1]?

S: Yes, is zero over zero, infinity over infinity.

I: And these [class 2] ones?

S: These are the ones that I didn't know where to put them.

I: If you have to, say you are working with a friend and you have to read him or her this [expression 8], what would you say?

S: Limit as x goes to one, as x approaches one, of four x cube plus seven x minus nine.

I: When you say x goes to one or x approaches one, what's the image you have in mind? What's the idea you have of that.

S: Like the function approaches that but never reaches it.

I: But when you say approaches...

S: Getting closer to.

I: Now I will ask you to solve some limits. First this one [2.1]. [...] Oh, don't erase, just start again on the side.

S: I don't remember my limits.

I: But do you have anything on mind that you should do?

S: Yeah... because I get zero [on the numerator], my answer is zero.

I: Why?

S: Because if you directly plug it in, it's gonna be zero over two [she writes 0/2].

I: Did you try doing the substitution before doing this?

S: Oh, I started that, I didn't realize... because what we are doing now [in Calculus 2].

I: But do you know if you get zero over two what the answer would be?

S: I don't remember. [...] It's gonna be zero. I don't remember. Can I go studying and come back [laughing]?

I: What about this one [2.2]? [...]

S: I don't remember, plug in?

I: Do you remember some situations where you can plug in and others where you can't?

S: When it's improper, when you get over zero you can't plug in. You have to do L'Hôpital's rule.

I: So what about this?

S: I don't remember.

I: If you plug in what do you get?

S: Mmmm, negative one. I don't know if that's right.

I: Are you convinced that's the answer?

S: No, I don't remember anything from limits. I know it's wrong. I am sorry, I don't remember.

I: When you were doing the classification before, you told me that these were straight forward. Why you were so sure before and now... ?

S: I don't know. Because now I am just thinking I don't remember my limits.

I: When you say you don't remember...

S: I don't remember how to do them. Like if you ask me now, we are doing series and sequences, I can do that.

I: You don't remember the methods?

S: Yeah, like what the answers should come out to be. I don't remember all the rules too.

I: If you look at these one [2.3], what...

S: This one goes to infinity. Like it's zero over zero, that's not right. I don't remember how to fix that. I don't remember what you do when it's not right.

I: When you say not right...

S: Like it's over zero.

I: But in this case [2.2], is that right?

S: No, I don't know, because I don't remember my limits anymore, so I'm kind of thinking I'm not doing anything right.

I: If you have to explain what this expression means...

S: The limit as x approaches negative two of f of x is zero. So as it approaches negative two, on the x axis, as x approaches negative two, the limit is zero, y zero is here [she made an incorrect sketch]. It should be like that, I don't think that's right. I don't remember.

I: Ok, I ask you now to look at these three limits. [...]

S: I think the limit does not exist. But I am not sure. I use to know this stuff. I have no clue. Wait, cosine of zero is... one. Am I right? Yes [checking in the calculator].

I: Could you use the calculator to check if your answers are right or wrong?

S: Yes, I could try a very big number and see what happens.

I: Ok.

S: [Doing calculations in the calculator.] Yeah, doesn't exist [for the limit in 3.1].

I: What about this one [3.2]?

S: [Calculating.] Syntax error... it's zero.

I: Is that just for e to the x?

S: No, no, for e to the x times cosine of x. It gave me zero.

I: Which number did you plug in?

S: Nine nine... ten nines, with a negative sign.

I: So you got zero, are you convinced it should be zero instead of doesn't exist as you wrote before?

S: I know the calculator lies sometimes [laughs]. But, yeah, I think so, it keeps getting to zero.

I: Now that the calculator told you that is zero, do you have an argument of why that's true?

S: I know that e to the x as x goes to negative infinity is zero, and the cosine keeps going so there's no limit, and I guess that if you multiply them... I can't picture the graph on my head, I can picture them separately but I can't combine them. But I would think that it doesn't exist though... by multiplying it, it changes the amplitude... then it shouldn't exist because it keeps going anyway.

I: But the fact that this is going to zero is changing the amplitude...

S: Oh, but it's getting smaller and smaller and smaller, so it would be zero. Yeah.

I: You are changing the amplitude but each time by a smaller factor.

S: Yeah, so the limit is zero. That shows that the limit can be in the middle of the thing.

I: Having that idea, what do you think that would happen in this case [as x goes to positive infinity]. You are changing the amplitude.

S: Yeah, it would be... no, because x increases so it would be no limit. Yeah! I remember something [laughing].

I: When you say in here that this is one [3.3] do you mean that sine of x over x is one or...

S: Yes, the function... the limit as x approaches to zero.

STUDENT S15

I: The first thing I ask you to do is, these are twenty cards, I ask you to classify them according to any rule that makes sense to you. [...]

S: Ok.

Class	Members of class *(labels refer to Table3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 8, 11, 12, 15	The power is higher on top.
2	3, 5, 10, 14, 16, 18	The powers are higher at the bottom.
3	2, 4, 9, 13, 17	The powers are the same.
4	6, 7	There's no variable. No x, just a number.
5	19, 20	With trigs and there's no powers.

Figure A15 (Copy of Table 5.15a). Student S15's classification.

I: Tell me about the groups.

S: One group is where the power's higher on top [class 1]. The second group is where powers in the bottom are smaller [class 2; he probably meant that the powers in the bottom are higher]. And the third group is where powers are the same [class 3]. One where there's no like variable, no x, just a number [class 4]. And the other is with trigs and there's no powers [class 5].

I: Ok, if you have to read this [card 8] aloud what would you say?

S: I would tell them the limit as x approaches one of four x cube plus seven x minus nine.

I: Now I ask you solve some limits.

S: Oh... how do you do this [2.1]? [...] What if I screw up? [...] I would say... zero.

I: Why would you say zero?

S: Oh, I would check it first. [...] Ok. This one [2.2]? [...] I think... minus one.

I: Ok, what about this [2.3]?

S: Zero... so... [...] I am probably making this more complicate that what I have to... If I had a calculator, what I would do is plug in values from the right and I'd see where is going to and then try from the left and see where's going to.

I: Can you try? I have my calculator.

S: [He tried some numbers in the calculator.] Ok, to infinity... Then... negative infinity.

I: Ok, what about this [2.4]?

S: [...] Oh, one. So... fourteen over three.

I: Why in the first three problems... say why in the first one you factored the x out?

S: Oh, no, no, I do that in every problem to see if something would cancel. Even if nothing cancels I do it anyway, in case I miss something.

I: So that's the reason here too [2.2]?

S: Yes, even if nothing cancels I just do it anyway.

I: So if nothing cancels, do you think you still get the same answer if you don't factor?

S: I'd try... Yeah, you would get the same answer... wait, I am confused by my own writing... Yeah, you would get the same answer.

I: Could it be a problem where you get a different answer? I mean that with simplification and without simplification you get a different answer?

S: Not actually.

I: If you have to explain, say in this problem, that the limit as x goes to two of this function is negative one, what does that mean for you?

S: The answer? What does the answer means to me? I don't know how it looks like [he sketches a function]. But say this is negative one, this is two, so it's closer to this value... It means nothing important to me.

I: Ok, finally I ask you to calculate these three limits.

S: This is indeterminate [for the limit of the cosine function as x goes to positive or negative infinity], you don't know... how would you calculate this? This [3.3] is one, I think.

I: Do you remember the graph of cosine?

S: Yes, it's like this.

I: Does the graph help you in any way to figure out what the limit is?

S: No. There is no limit because it's up and down.

I: And what about the multiplication with the exponential?

S: With e to the x you mean?

I: Yes. I mean, it is true that the cosine doesn't have a limit as it goes up and down but when multiplied by e to the x, does it have a limit?

S: Maybe, I don't know. When you divide yes, for sure. But when you multiply...

I: Why when you divide you get a limit?

S: [3.3] Because the way I see it is like, sine over x right, the sine function, what the x does, because it's a straight line is mixed... I can't explain, it's the line right and the sine function is symmetrical, and what the line does is like is like cut the... the x in the bottom cuts the sine function and makes it smaller and smaller and smaller. So I believe that if you divide something it should like shrink it. But here [3.1], since this is infinity, regardless of what this is, it should be actually plus infinity.

I: Ok, what about the other one [3.2]?

S: As this approaches zero, it doesn't matter what this does, it should be zero.

I: Do you think you could use the calculator to check if your guesses are right?

S: Well, I do believe it [he believes in his guesses] say this is zero, it would be one times whatever number... if this is like a million, no matter what this is, it would be bigger than the previous number.

I: Could you check?

S: It could be wrong because cos varies up and down. [He tried some numbers in the calculator.] It is true cos goes up and down, is positive and negative positive and negative. Ok so... You know what I also know, I know cos goes only up to ninety nine million nine hundred... like nine nine nine. Because once I tried cos of one billion and it gave me an error, so I try to see what's the limit, what's the biggest number it goes to and I got ninety nine million nine nine nine nine... I want to know why. [He keeps trying numbers on the calculator.] You see, I get an error, it's very weird. Why is that, ok, anyway. [...] I would say you can't find the limit because it goes up and down all the time. I would say that for certain intervals, like pi over two intervals or something like that, like pi over two to pi over four it would go to infinity but then next pi over two, it would go the other way.

I: What about for this one [3.2]?

S: It should apply the same thing. It doesn't exist.

I: Why not in this case?

S: Because it doesn't matter if it goes to negative infinity or positive infinity [he means the values of x], because certain intervals will determine where it's going.

I: It doesn't matter that here [3.1] was infinity but here [3.2] is zero?

S: No, I don't think so... wait. This would be almost zero times whatever... it would be zero.

I: And why did you put one here [3.3]?

S: Last semester... actually in the summer I read the book and I remember this.

I: Do you know anyway of checking if that's right or wrong?

S: I would go right and left with the calculator.

I: Do you know any other way that is not using the calculator?

S: Maybe... maybe not. If it is for a million dollars maybe... No.

I: I didn't have a million dollars for you anyway...

STUDENT S16

I: These are twenty cards, the first thing I ask you to do is to classify these cards into groups according to any rule of your choice.

S: Make groups?

I: Yes, as many as you want. [...]

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 5, 9, 12, 17	Difference of squares.
2	19, 20	Trig functions.
3	3, 10, 13, 14, 18	These have square roots.
4	6, 7	Limit of constants.
5	1, 4, 8, 11, 15, 16	Quadratic or third degree.

Figure A16 (Copy of Table 5.16a). Student S16's classification.

S: Ok, I think that's it. All these have perfect squares... difference of squares [class 1]. These are trig functions [class 2]. These have square roots [class 3]. These are the limit of constants [class 4]. These are either of quadratic or third degree [class 5].

I: Ok. Let's say, how do you read this [expression 15] aloud if you have to read it to someone else.

S: The limit as x approaches one of the function x square minus x minus two all divided by x minus one.

I: Ok. Next I ask you to solve some limits. This is the first one [2.1]. And if you can think aloud so I can keep track of what you are thinking.

S: Ok. First I check... I put in the number... if the limit gives me a number over zero. But you get one minus one over one square plus one, so is zero.

I: Ok.

S: [2.2] At the bottom is a difference of squares which will cancel out with the top. Then I put in the two, which will become one over negative one so will become negative one.

S: [2.3] I know there's a difference of squares but that wouldn't change much because nothing will cross out. If I put the five it'd become a number over zero. I guess there's no limit. I'm not sure.

I: What about this one [2.4]?

S: In this one you can put in the one because it'd give us a number. So one cube... fourteen over three.

I: Do you think in this problem [2.2] was necessary this step?

S: No. No, I guessed it would be easier to simplify, to take out some terms.

I: But if you don't simplify what would you do instead?

S: I'll put in the two and calculate.

I: And will you still get the same answer?

S: Eeh... Can I try that? [He does direct substitution.] So yeah.

I: Do you think there could be a problem in which doing like this [direct substitution] and like this [simplifying first] you can get different answers?

S: Well, I guess it depends... I guess if it was limit as x goes to three that would give you zero so you'd have to take out... I mean negative three, that would become zero.

I: And for this one [2.3]. You were saying before that the limit doesn't exist?

S: Well, I'm not sure because there's nothing I can cancel out and I'm missing something...

I: When you say this doesn't exist, what do you have in mind? What does that mean to you? [...] Or, say, when you look at this [2.1] and you say the limit is zero, what do you have in mind? What's the meaning of that?

S: As we approach one, the value of the function approaches zero.

I: Ok, the last problem.

S: Well, this [for the limit of the exponential function in 3.1] is infinity... [...] Can I use my calculator?

I: Yes, yes.

[He did some calculations with the calculator.]

S: Ok, cos only goes from negative one to one. I'm not really sure, this will be zero?

I: What did you do with the calculator?

S: I try to put in cos big numbers but I kept getting bigger, I guess because cos is up and down, the two bounds.

I: And what about *e* to the *x*?

S: *e* to the *x* is infinity because it just keeps going up as it does the graph. I guess this is going to be infinity times negative infinity... I guess you can do L'Hôpital's rule, but that's not from cal one.

I: And what about this one [3.2]?

S: This one, you replace with negative infinity, *e*... one over *e* to the infinity which will become zero. Zero times cos will become zero.

I: Could you use your calculator to check if your answer is right or wrong?

S: [He spoke as he did the calculations on the calculator.] One over *e* to a very big number... I do one divided by *e* two hundred times cos two hundred and it becomes six point seven four times ten to the negative eighty eight so it becomes closer to zero.

I: And you think you can do something similar to figure out this one [3.1]?

S: *e* two hundred times cos two hundred. It's point five times ten to the eighty six so it keeps getting bigger and bigger.

I: So what is this limit?

S: The limit as x approaches positive infinity is positive infinity.

I: Are you convinced?

S: Well, because this goes from one to negative one, and this will always get bigger. But then if that's negative [the cosine] it'd make it negative. I'd say it goes to infinity.

I: And what about this one [3.3]?

S: If you plug in zero you get zero over zero, which is undefined. So that doesn't work. [...] I remember from cal one there were special limits with sine or cosine, but I don't remember what they were.

I: And if you wanted to try something with your calculator?

S: Well, sine of zero... makes an error, because divided by zero is undefined.

I: But could you use it in any other way to figure out what the limit might be? Something that is not replacing by zero?

S: Eh... Well, I know that in cal two you can take the derivative of top and bottom. Cos of zero is one, so it'd be one over one which is equal to one.

I: And can you think a way of using the calculator to check if this is right?

S: [...] Unless... you can use a number really really close to zero, like point zero zero one. Gives zero point nine nine nine, which is really close to one.

STUDENT S17

I: I will ask you to do different tasks... the first is to organize these twenty cards into groups according to any rule that makes sense to you. [...]

S: Like polynomials?

I: It can be any rule that makes sense to you.

S: Ok. [...]

S: There's no specific number of piles?

I: No. You can do as many as you want.

S: Ok. [...] Ok.

I: Can you explain me the rule you had in mind?

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	19, 20	These are trig.
2	6, 7	These are regular limits, there's not much to do.
3	8	There's no fraction.
4	3, 10, 13, 14, 18	These are square roots.
5	1, 4	These are trinomials.
6	2, 5, 9, 11, 12, 15, 16, 17	These are regular. Not that tricky. Factorable.

Figure A17 (Copy of Table 5.17a). Student S17's classification.

S: These are trig. These are regular limits, there's not much to do. Here there's no fraction. These are square roots. These are trinomials. These are regular.

I: When you say regular what do you mean?

S: Not that tricky.

I: Things you've seen before?

S: Yes. Factorable.

I: And for you how this [expression 4] is different from this [expression 5]?

S: This [expression 4] has two cubes.

I: Ok. Now I will ask you to solve a few limits.

[2.1] [...]

I: Ok, what about this one [2.2]? If it's not too hard for you, if you could think aloud so I can keep an idea... a track of the way you are thinking the problem.

S: Ok. [...] [What follows is about 2.3.] Do I have to get an answer? Or can I just leave what I wrote?

I: Why?

S: Because I got zero.

I: How... what was the order in which you thought...

S: *X* plus five, *x* minus five, *x* minus four, *x* plus four.

I: Why you didn't write...

S: If I plug in now it can't be done, it's zero [he meant the denominator is zero]. So I figure there's must be another way to do it.

I: So you did the long division, and you got this... Ok. What about this [2.4]? [...]

S: Can I leave my answer like this?

I: Yes. Why did you do long division here?

S: I am forgetting if I have to divide the top by the bottom or the bottom by the top.

I: But why did you think you have to do something here?

S: Because the top is larger than the bottom. There's an x square in the bottom and an x cube [in the top]. When I see that, I think I have to divide.

I: Ok. Let's say if we look at the problems. Do you think it was necessary this [factoring] to get the answer?

S: I think it makes it easier, if you can cross things out.

I: Ok. But could you have found the answer without doing that?

S: Probably not. Maybe if you multiply this out and then do long division.

I: Ok. And here was it necessary to do the factoring?

S: Yes, because you can't divide them.

I: Do you think you can get the answer without doing that?

S: Not in a way I can think of.

I: And what was the next step from here to here?

S: I simplified, cancelled out and plugged in x approaches two.

I: And why couldn't you do it in here?

S: I assumed it wouldn't work but now I see that it would work.

I: What about in here?

S: This one now because twenty five minus twenty five would give you zero so you have to simplify.

I: How do you read this [$\lim_{x\to 7} f(x) = 2$]?

S: Limits as x approaches seven for the function f of x equals two.

I: And when you say that, what do you have in mind?

S: Well, the graph would be approaching seven, well, x would be approaching seven.

I: If someone asks you what this mean, what would you say?

S: I would say, y equals two. [...] Like [...].

I: Ok. Now I give you these problems. [...]

S: I don't remember doing limits with e x.

I: Not even if you look at this problem separately? [The limit of e^x and the limit of the cosine function.] [...]

I: If you could use the calculator to find or guess these limits, would that help in any way?

S: Well, this is infinity. So it would just be infinity, the answer.

I: What would be infinity?

S: If you make x infinity, it would just be infinity.

I: What is it that would be infinity?

S: The limit approaches infinity. If you make e to the infinity it would just be infinity.

I: And what about the other one?

S: Can I use the calculator?

I: Yes.

S: It wouldn't be infinity.

I: For the cosine? Do you remember the graph of the cosine function?

S: Yes, it looks like that.

I: And by looking at the graph, can you tell what the limit would be?

S: One? Negative one?

I: And considering this, what about the multiplication of the two functions?

S: It would still be infinity.

I: What about this one [3.2]?

S: It'd be negative infinity.

I: Negative infinity for which one?

S: For this [the exponential]. This [the cosine function] would still be one and negative one.

I: And what would be the answer for the multiplication?

S: Negative infinity.

I: Do you remember the graph of e to the x? [...]

S: Like this... or... not really. I don't remember.

I: Can you use the calculator to verify if this is right or wrong?

S: If I put e to a high number. Say a thousand. It'd give me an error, so it's infinity. And if I put a negative number it would be negative infinity.

I: Do you want to try?

S: It would give me the same error... No, it's zero. And for the cos it doesn't matter, it's the same thing. So this would be zero, if you multiply anything by zero, it's zero.

I: Ok. What about this one [3.3]? [...]

I: What rule are you using in there?

S: A trig identity. [...]

S: Well, I don't know... this is one over one, so it would be zero.

I: Which trig identity are you using?

S: Sine x equals one over cos x.

I: Do you remember any rule for this that you learned in cal one.

S: To solve the limit of this?

I: Or maybe, would the calculator help you in any way...?

S: I am not remembering what to do.

I: When you did cal one, did you use calculators to solve limits?

S: After we simplified.

I: So you could always simplify.

S: Yes.

STUDENT S18

I: So the first thing I ask you to do, here you have twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: So if I see a pattern, I just...

I: Anything that makes sense to you... [...]

S: I am trying to break it down into how to factor them, so if I see a trinomial I put them here [class 1], if I see a square root I put it here [class 2], if I see like a difference of squares I put it on the middle [class 3]. I am not sure where to put this one... [...] I'll keep them aside, an extra, these are the ones I don't know.

I: So what was again the rule? These are square roots.

S: If there's a square root I keep them here [class 2]. These seem all to have a difference of squares, I like those [class 3]. These are all trinomials [class 1]. These have a trinomial in the denominator but a difference of squares a kept it aside [expressions 5 and 9, members of class 3]. These are just to limits three and seven so I kept them aside [class 4]. The trigonometric I put them in this different category [class 5]... I wasn't sure where to put them.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 4, 8, 11, 15, 16	These are all trinomials.
2	3, 10, 13, 14, 18	There's a square root.
3	2, 5, 9, 12, 17	These have a difference of squares. I like those. These have a trinomial on the denominator but difference of squares I kept them aside.
4	6, 7	Limits three and seven.
5	19, 20	Trigonometric.

Figure A18 (Copy of Table 5.18a). Student S18's classification.

I: If you have let say... you are talking on the phone with a friend and you have to read this [card 8] to him, what would you say?

S: The limit as x approaches one of four x cube plus seven x minus nine.

I: Now, where you were telling me about the difference of squares, you said you like them, why?

S: It's a stupid reason but it's because they work out nice, like x square minus four is x minus two, x plus two.

I: They are kind of neat.

S: Yeah, they work out well. If you have to factor a trinomial you have to break it and then it's complicated. Different of squares is just two and two and it always works.

I: Ok, now I ask you to solve some limits, and if you kind of think aloud it'd be great for me, so I can keep track of what comes first to your mind and so on.

S: Ok, I am a bit [inaudible] with this. The first problem [2.1] x approaches one and we have x minus one so that doesn't work, we need to find a way to factor that out. The best thing I could do, let's see, if I could factor out the denominator, limit as x approaches one... we can take out an x and get x times x plus one and then we have x minus one over x plus one... I am really stuck here. Let's see if I started this right. This is as far as I can get now. How long do I have for each of them. Because if I could get this to be x minus one, this would factor well, and I'd just have limit of x. But right now, as far as I know, we have x approaches one and x minus one so we get zero divided by... divided by the denominator. And as far as I can see that could be zero. So limit as x approaches one, I don't like the fact that there's a zero on the numerator, but as far as I can remember all I can say is zero divided by one plus one, two, which is zero.

I: Ok. I'll ask you a few more and then I'll go back and ask you questions about each of them.

S: Ok. Oh, difference of squares [2.2], I like that. *X* approaches two, we keep the numerator, *x* plus three and *x* minus one. I factor out the denominator, *x* plus three, *x* minus three. We can get rid of the *x* plus three here. *X* approaches two, *x* minus one over *x* minus three, which would come out... negative one, as *x* approaches two.

I: Ok, what about this [2.3]?

S: You are giving me the ones I like, I see difference of squares. The first thing I'd do is to break it out. But I can already tell by the twenty five, that the five would pose a problem at the bottom. So *x* plus five, *x* minus five. The problem is that I need to get rid of the *x* minus five or else my denominator will end up being zero, which isn't good at all. What can I do to get rid of the denominator? How can I do to get rid of the *x* minus five? I am stuck.

I: What about this one [2.4]?

S: So it's not a difference of squares. How can I factor this? Well is the limit as *x* approaches one, so where would my problem be? There's no problem at all. Unless I could factor this... Say just to be nice I'd try... no never mind. The best thing I could do is say just not to factor and say the limit as *x* approaches one would be one plus four plus nine over three, which would equal fourteen thirds.

I: Ok. Say in this one. Do you think it was necessary to do the factoring?

S: Ah... I think it was necessary. I mean, if you do the limit as *x* approaches two without the factoring, I won't get something wrong like a zero on the denominator because two square is four minus nine... Was it necessary, no. But I was taught, if you could factor,

factor. The one I just did [2.4] I didn't factor, because I didn't get the time to check if it was factorable. But if it was, it would had been better to factor.

I: Better you mean... it would give you a different answer or...

S: I don't know. I just think is better to factor, to keep it safe. Let's say if I plug two in here [2.2] I get five times one, over... it still comes out to negative one. So it wouldn't make a difference not to factor it.

I: Do you think there could be a problem where there is a difference?

S: I guess in the sense.... better safe than sorry... I mean if you factor, you know that you factor well. I mean if there's something not factorable and you think you have to factor and you factor wrong is worst than not factoring it. But if you can factor, you kind of avoid mistakes, as long as you know how to factor...

I: And this was always the approach you had in cal one?

S: Basically I look at a problem and the first thing I see... and I always assume it is factorable, I mean, they never gave me a problem that wasn't factorable, so I wouldn't even ask whether it's factorable. I'd say, ok, where can I factor it. And I'd say ok let's look at the different categories. If I see a trinomial or a difference of squares and the method to factor them. And so long and so forth. But if it wasn't factorable... I never came across a problem that wasn't factorable.

I: And do you remember coming across with something like this [2.1]?

S: Yes, is one of the tricky ones. You have to think of a special... I don't know, I am not saying this would work, but you can multiply by negative one to inverse the signs of your equation and then you'd be able to simplify and it would work out.

I: And in this case?

S: I don't remember.

I: You don't remember what you should do?

S: I've been out of practice. The more I practice, the more I remember.

I: At this point you feel you forgot what would be the method for

S: In a situation like this, yes.

I: I have one more question about this, say about this one, that you've got that the limit equals negative one. What that means for you? What's the meaning you have in mind when you say the limit of this equals negative one?

S: I picture a function in a graph, I don't know how the function looks like, but I picture a function, and I see that as x approaches negative one, sorry as x goes to two, it would be negative one. So it would be closer and closer to negative one but it wouldn't necessarily reach it. So, well in some cases may be it reaches but as far as I know it doesn't reach. And let say you have an asymptote at negative one, and that would be it. From what I remember.

I: Ok, last one.

S: Oh... I am not a big fun of trigonometry... How does cos look again? I can't use a calculator to see the pattern, can I?

I: Yes, you can. [...]

S: Ok, this equals positive infinity... oh, no... Ok, I'd say positive infinity for now.

I: Which calculations you were doing?

S: Oh, trial and error. As x gets bigger what's y. Cos of ten, cos of a hundred. Then I went to e to the x and it gets bigger, I should have remembered. So basically you multiply and just approaches infinity, from what I remember. [...] Here [3.2] I do more trial and error with negative numbers. In this case, as x approaches negative infinity... we are getting closer and closer to zero. So I would say zero.

I: So exactly which calculations were you doing?

S: I was doing e to the negative ten, e to the negative a hundred, to see the pattern, and it gets smaller and smaller and smaller, approaching zero.

I: But you didn't try the multiplication.

S: No, because the way I see it is cosine, whether is a positive or a negative it keeps revolving around pi, pi over two, three quarters pi, around and around in the circle and it the number would never keep getting smaller or bigger, it just keeps revolving that circle. So basically what matters to me is e to the x.

I: I see.

S: And as x approaches zero, sine of zero is zero... so zero. That's all I can say.

I: Why would you say zero?

S: Well, sine of zero is zero, you plug in zero in the bottom and is zero. I can't see any way of factoring it, so I wouldn't... unless I use some sort of trigonometric identity, which I honestly can't stand...

I: But... could you use the calculator to try to check if that's zero or not?

S: Well, sine of zero is zero. So let's say you have sine of zero point one... sine of zero point zero zero one, so it approaches zero... Oh, does that mean that the limit doesn't exist? Yes, I don't think it exists. Because you can't divide by zero, so I'd say the limit doesn't exist.

I: I insist on the idea... do you think that you can use the calculator to check whether it doesn't exist or...

S: Well, I know that if I try... If you... because I am not using my calculator to find the limit, I am using the calculator to see the pattern, what the function is doing. I know if I put anything divided by zero I would get an error.

I: But do you think that this trial and error method to see the pattern...

S: Well, that's something I'll do if... basically, all the things I did are things that on a normal basis I would recognize without the calculator. But using the calculator for me, let's say if I blank out, ok, what happens as x gets bigger and bigger, what's e to the x, I'd put that in the calculator and I'd remember the pattern. Or let's say I forget is it cos of x that is zero or is it sine of x? So I'll use the calculator to remember. But not to determine the limits, I don't think I can use my calculator.

I: And do you think you can somehow get some help from the calculator to check if this infinity is really plus infinity?

S: I can't get help from the calculator to see if that's plus infinity. The calculator would just help me to remember what happens with e to the x, because I can't do e to a hundred in my head.

I: And what about the graphing features in your calculator, do you ever use them?

S: I haven't use them in a long time. I use them a lot in secondary school to graph the parabolas, in secondary five when we did conics. But since then I haven't use it.

I: Ok.

STUDENT S19

I: The first thing I ask you to do is... here you have twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: Ok. [...] Ok.

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	1, 2, 3, 15, 18, 19, 20	Zero over zero.
2	9, 10, 16, 17	A number over zero. Or zero over a number.
3	6, 7, 8,	A definite number.
4	4, 5, 11, 12, 13, 14,	Infinity over infinity. [Card 14] goes to infinity.

Figure A19 (Copy of Table 5.19a). Student S19's classification.

I: Can you tell me about the groups you did?

S: This was zero over zero [class 1]... This, zero over... No, infinity... no, a number over zero [class 2]. A definite number [class 3]. And either infinity over a number... or... a number over zero, again? No, that doesn't make sense. Something over infinity, infinity over infinity [class 4].

I: So it's infinity over infinity... or... This one is [class 1]?

S: Zero over zero.

I: And this one [card 14]?

S: It went to infinity.

I: This ones were definite numbers. Why this one would fall in this category?

S: Oh, no, it goes here.

I: Why there?

S: It's a number over zero.

I: Ok. And this ones... were zero over zero [class 1]?

S: Yes.

I: And this are a number over zero [class 2]?

S: Yes.

I: And this [expression 16]?

S: That's zero over a number.

I: Oh, so it's a number over zero or zero over a number?

S: Yes.

I: Ok. Let's say you are working with a friend, and you need to speak this aloud, say on the phone. What would you say?

S: Find the limit as x goes to one for four x cube plus seven x minus nine.

I: Ok. Next thing I ask you to solve some limits.

S: Ok. [2.1] [...] Can I use my calculator?

I: Yes. [...]

S: Ok.

I: So... here you chose this number instead of this one?

S: Yes.

I: Why?

S: Because it was closer to one.

I: And if you'd try a number even closer?

S: You would get even closer to what the number was... I did four nines... I got three zeros and a five and some other numbers.

I: Ok.

S: Then I'd say this goes to zero.

I: Do you remember another way of doing this, that's not a table of values?

S: No.

I: What about this one [2.2]? [...]

S: Oh, you can factor this one. And then you can cancel the x plus three and then you substitute the number... it gives you one over... one over minus one, minus one.

I: Why couldn't you do something similar in the previous one?

S: Because if you factor it... you get limit... you can't cancel two different...

I: What about this one [2.3]?

S: Ok. So... Factor them both. [...] I don't know what to do.

I: Why?

S: Because you can't cancel anything and the bottom equals zero.

I: Ok. If the bottom wasn't zero what would you do?

S: I would plug in five, in the equation.

I: What about this one [2.4]?

S: So here you just plug in the number. [...]

I: Ok. So I go back to the first one [2.1]. Why you didn't just plug in the number in here?

S: Because if you plug the number in the top you get zero over a number.

I: You get zero over...

S: Zero over two... which is zero. But it doesn't seem like a possible answer.

I: Why not?

S: Oh...

I: Ok. Do you remember when I gave you the problem, did you plug in the number?

S: Yes, and I saw it was zero over a number.

I: Ok. When you write here... let's say, here you wrote that this limit equals negative one... what do you have in mind. Does it have any meaning for you?

S: It's the slope of the tangent to that line... no. Sorry. That's the number when x equals two on the graph. So if this was plotted on a graph, at x equals two the number would be minus one.

I: One more set of problems. [...]

S: I don't remember how to do this.

I: Let's say you have to do them separately. Just the limit as x goes to infinity of e to the x and the limit as x goes to infinity of cosine of x. Could you do that?

S: No.

I: And if you do like you did for the first problem... a table of values?

S: Ok. Are you allow to do that with infinity though?

I: Yes...

S: Ok. [...] Ok.

I: Where did you get this numbers from? Which calculations you were doing?

S: Because...

I: Where were you pluging these values?

S: Here.

I: And here you got error? Do you know why you got error?

S: Because the number was too high.

I: And why this is infinity?

S: Because the number is going to be very very high, so it doesn't matter what you multiply it by, you would still get infinity.

I: And here is zero?

S: Yes.

I: Ok.

STUDENT S20

I: The first thing I'll ask you to do is to look at these twenty cards. I want you to classify them into groups, according to any rule that makes sense to you.

S: Any rule of my choice?

I: Yes.

S: How many groups? As much as I want?

I: Yes. [...]

I: Done?

S: Yes.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13, 14	With the infinity. I should take the biggest power up and down.
2	6, 7, 8, 9, 10	The answer is clear. These ones are solvable if you put in the number.
3	1, 2, 3, 15, 16, 17, 18, 19, 20	Indeterminate. If you substitute you get a number over zero.

Figure A20-1. Student S20's classification. In the conversation that follows, the student changed his classification to the one shown in table 20b. The difference consists in having a new class containing cards 19 and 20.

I: Can you explain me what was your rule?

S: The first one [class 1] is with the infinity, because I know I should take the biggest power up and down, oh, this one should be here [he removes card 6 from class 1 and places it in class 2]. Ok, I take the biggest number here [meaning the cards remaining in class 1], that's the rule I know.

I: How would that work in here [card 5]?

S: I take x square over x cube, so I take one over x. So the answer is zero.

I: And for this one [card 13], for example, how that will work?

S: This one… I forgot… based on… I should square it? Multiply by the conjugate…

I: And what about this one [card 14]?

S: This one is infinity minus infinity. I don't think it goes to infinity though... is positive... so the answer is zero.

I: But when you put them together in the same group, what were you thinking?

S: Just the infinity thing. I didn't look at one by one, I just thought I can take the biggest... like this one too [card 11], over x should be infinity, and here too [card 5], so this is one over x. [Even if he did realize that this technique does not work in all the cases in class 1, he does not change the class.]

I: Ok, and what about these [class 2]?

S: These ones, any numbers. But these ones are solvable if you put in the number, it gives an answer. If you substitute here [class 3] you get a number over zero, indeterminate.

I: Ok.

S: So this is when the answer is [inaudible] at the end.

I: The answer is...?

S: Clear.

I: Oh, in these ones. Straightforward.

S: Yes. You have to plug in first if it gives you an indeterminate you have to change the form, these ones need more work [for the ones in class 3]. This should be alone [for card 19], this is indeterminate again. This is equal to one but it's a special rule, like the cosine.

I: Do you remember what rule is that one?

S: Squeeze. I can do it with squeeze, but I know it's one… or zero… one, one. [He placed card 19 on a class of its own.]

I: So this will be separate now.

S: Yes. I just saw that. If there was the cosine one again, it'll be here too. [As he goes through the cards he placed in class 3 he stops at card 20.] This one, one over zero is infinity; sine of infinity is not infinity right? You can't find an answer. So this is an indeterminate? So you can't find a definite answer. I am learning about this, it's divergence [he's been studying divergent series in Calculus II which he is presently taking]. It diverges? I'll put it with this [he places it in the new class together with card 19]. [His talk is really rhetoric; he is not waiting for me to answer his questions.]

I: With this?

S: Or… Yes, same family, as an exception.

I: Ok.

S: So these two are like special exceptions.

S: These ones [the ones remaining in class 3] you should use rules, because if you plug the numbers in, like five minus five [card 17], will give you zero, so a number over zero, indeterminate, so you need to change the form. Either multiply by the conjugate or factor.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13, 14	With the infinity. I should take the biggest power up and down.
2	6, 7, 8, 9, 10	The answer is clear. These ones are solvable if you put in the number.
3	1, 2, 3, 15, 16, 17, 18	Indeterminate. If you substitute you get a number over zero.
4	19, 20	Exceptions.

Figure A20-2 (Copy of Table 5.20a). Student S20's final classification.

I: Ok. Next I'll ask you to solve four limits.

S: Solve them?

I: Yes, and as much as you can… when you are thinking, maybe think aloud.

S: [2.1] Ok, if I want to put one is gonna give me two, but zero over two is zero. So I need to either change this, factor this into x plus one… oh, no, zero. I just plug in numbers, and it doesn't give zero though, so zero over a number, two, which is zero.

S: [2.2] Ok, so I plug in the number first. Five over minus five, minus one.

S: [2.3] This I need to factor. Five squared minus twenty five will give me zero, so I need to change something… x squared minus twenty five… x plus two, x minus two… x minus five, x plus five, cancelling this will help me do nothing, even if I factor it will give me x minus two, x plus two, over x minus five, x plus five. So in both cases I'm gonna have a number over zero, which is, a number over zero is infinity, plus infinity… x squared is a very big number…

I: Ok, the last one.

S: [2.4] One plus four… ten over three. This one is easy.

I: And let say you need to read this [$\lim_{x \to -2} f(x) = 0$], how would you read it?

S: Limit of *f* of *x*... ok as *x* gets nearer to minus two, the function *f* of *x* is approaching zero.

I: Well. This is the last part. I ask you to find these three limits.

[...] [He wrote quickly the answer for the last limit, one, and writes a proof using squeeze theorem, that doesn't work.]

S: This one is tough [meaning the first two limits]... cosine... there's no finite number for it. This one is a very big big number. If you put cosine of a hundred is going to give you a number, if you put cosine of a bigger number is going to give you another number. So... did we see this in Cal I?

I: It depends on the teacher I guess... What about the exponential function.

S: Yes, this I know. *e* to the infinity is infinity. *e* to the minus infinity is one over *e* to the infinity, one over... zero. This I know. Multiplied also by a big number is going to give a big number.

I: Cosine of...

S: a big number...

I: will give you?

S: Will be between minus one and one. Pi by two and minus pi by two, no? Two different numbers... gives small numbers. Maybe not.

I: Could you use in some way the calculator to guess what these limits are?

S: If I don't know it, yeah, I can try bigger and bigger numbers to see what it approaches. [He tried in the calculator.] It gives different numbers. Let say cosine… gives a negative number.

I: Are you doing the calculation just for the cosine or for both…

S: No, just for the cosine. I know this is approaching infinity. [He tried more numbers.] Minus one. The answer is going to be between minus one, plus one. So whatever the x value is, even if it increases too, it is going to be oscillating. The value cannot be…

I: You mean just for…

S: Yes, just for the cosine. This is always infinity multiplied by the value [3.1]. This is zero multiplied by a number [3.2]. No, this is not infinity. I think this is zero. Should I give a definite answer now?

I: No. You said this is … why do you think is zero?

S: This is zero [for the exponential function], and this is approaching a small number [for the cosine].

I: Do you remember the graph of cosine?

S: Yes. It starts like this, right? [He started the graph at the point (0, 1) and graphs the cosine function to the right of the y-axis.]

I: And what about to the other side?

S: Same thing. Same thing as this. [He graphed to the left of the y-axis.] So even if it is a big number, the value is still gonna be between one and minus one. You can't…

I: So what the limit would be when x increases?

S: I am taking this in Cal II now, something about this. This diverges [the cosine function], doesn't give… Oh, that's the topic. Convergence and divergence. If it gives a definite answer, then it converges. If it doesn't, it diverges. This is a divergent limit [the limit of $cos(x)$ as x tends to infinity]. So there's no proper answer for it.

I: Ok. And what about when you multiply by the exponential function.

S: But this is unknown, I don't know what it is [3.1]. But this [3.2] is a number like say, between one a minus one, multiplying it by zero is gonna give zero, I think. This is my guess.

I: Ok. And what about this one? [3.1]

S: This is gonna give plus or minus infinity. If it's cosine of a negative number, maybe it's going to give you something negative. The number x chosen will give a negative number, multiplied by a big number gives minus infinity.

S: And this is a rule [3.3]. Squeeze.

STUDENT S21

I: The first thing I ask you to do is... here are twenty cards, I ask you to classify them according to any rule of your choice.

S: Like how easy they are?

I: Anyway you like, anything that makes sense to you. [...]

S: I put these together because the limit as x approaches infinity [class 1]. And this one I left it on its own [class 2] because it's simpler. This one [class 3] is also simpler but I put it with this one. I put all the ones with division in one pile [class 4]. I put the ones with sine function in one pile [class 5].

I: And this one [card 8] you left it alone because...?

S: This one looks easier because it's a trinomial, on its own. There's no division or square root signs.

I: And how do you read this [card 8]? If you have to read this aloud, what would you say?

S: I'd say the limit of four x cube of seven x minus nine as x approaches one.

I: Ok... These fractions [cards 11, 12 and 13] you didn't put them here [class 1] even if the limit was as x approaches infinity...

S: I didn't see that. I guess they should go with this pile [class 1].

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 11, 12, 13, 14	Limit as x approaches infinity.
2	8	Simpler. There's no division or square roots.
3	6, 7	Also simpler.
4	1, 2, 3, 9, 10, 15, 16, 17, 18	With division.
5	19, 20	With sine function.

Figure A21 (Copy of Table 5.21a). Student S21's classification.

I: The next thing, I ask you to solve some limits.

S: Do I write on the paper?

I: Yes, please.

S: The limit [2.1] is zero because if you put one where the xs are you get zero over two and so the limit is zero.

I: Ok. What about this one [2.2]?

S: This one [...] I think it would be negative one. If you put two where the xs are it would be two plus three is five times one over four minus one which is negative five. So negative one.

I: Ok. Just to know the way in which you are thinking the problem, when you looked at it, what was the first thing you did.

S: I thought that I might have to multiply out the top, but then I saw that it would be simpler maybe if I just put the five.

I: Ok. What about this one [2.3]? And if you can, while you are doing the problem, if you can think aloud so I can follow what you are thinking...

S: Ok. So at the top and bottom there's a square term, like x square and four, two square, and x square and five square. If you put in the five I don't think it would be an indeterminate form. Twenty five minus four nineteen over zero. I think you have to fix that somehow.

I: Why?

S: Because if it is over zero I think it could mean infinity. or it's not defined. So you have to... I think maybe L'Hôpital's rule... [...] At the top you could do also x plus two and x minus two. I think x plus two times x minus two over, it would not help though, x minus five times x plus five. It could be undefined? At five?

I: You mean the function or the limit?

S: Yeah, I think the function is undefined.

I: And how about the limit? Or let me ask you instead, for this problem here [2.1] that you wrote the answer to be zero, what is it that you have in mind when you say this is zero?

S: Well, because at the bottom is a number but at the top one minus one is zero so zero divided by anything is zero, so I don't have to go any further and I say is zero.

I: But the operation... When you say here the limit is zero or here is negative one [2.2], what's... does it have a meaning for you? [...]

I: No? What about this one then [2.4]?

S: Ok. I don't think there's any problem with this one because you could put in one. One to the power of anything is one, so it would be fourteen over three, and that would be the limit. So as the function... like on the x axis as it goes to one there's a limit of fourteen over three.

I: And what do you mean when you say there's a limit of fourteen over three?

S: It approaches the point fourteen over three on the y axis.

I: Ok, then what about this one. When x approaches five on the x axis...

S: Could it be that the limit is undefined? Because the function does something maybe like is not going on a straight line, it could be like a curve?

I: And what about this, let say you want to sketch on the y axis the idea of this, what would you sketch?

S: Well, there's the x axis and there's a point one on the x axis and [...] This is below seven [the 14/3]. Maybe close to five... whatever, it is fourteen over three. And I don't know how this looks like but maybe it could be, it could be something like this. No, no, that's an asymptote. So at fourteen over three, this is one. I think it could be something like that but I am really not sure.

S: [3.1] Well, there's an e, that would keep growing. Cosine goes between one and negative one. Well, the cosine you can't really find it, it keeps going up and down so the limit doesn't exist. For the e to the x it would be positive infinity. It might be that you have to do L'Hôpital's rule but I am not sure. Oh, no, because it's not a division. You have to make it a division. e x, because cosine is also one over secant. Then L'Hôpital's rule you can do the derivative of the one on top over the derivative of the one on bottom. The derivative of secant is secant x tan x, I think. I think the limit... Well, since cos is either one or negative one it would be positive infinity or negative infinity. But since it's to infinity, it can't really exist.

I: And what about this one [3.2]?

S: [...] If e is to something negative, it'd keep growing but negative. Cosine is going between one and negative one. And I think it would be kind of the same thing except since e would be negative for here the values are positive, for here they would be negative and vice versa.

I: Do you think that if you could use your calculator, you could check somehow your answers?

S: I don't think so, because the way it looks... it just looks it'd keep going up and down, and there wouldn't be a real limit.

I: And could you check somehow this phenomenon of going up and down? With the calculator?

S: No.

I: What about this one [3.3]?

S: Well, this is zero over zero, so you could do L'Hôpital's rule, I think. Because we learned this and we are being tested now on this. So it'd be cos x over... oh, no, that wouldn't work.

I: Why not?

S: Because the bottom it's just dx.

I: And...?

S: I don't think you can cancel out the dx. Or maybe you can and it would be cos of zero which is one.

I: What is it that you get here in the denominator when you cancel these out?

S: One.

I: Ok, do you remember any other way of doing this, that is not using L'Hôpital's rule.

S: I don't really remember. I'd say zero over zero, undefined.

STUDENT S22

S: Just, does it have to be a logical order?

I: Well, something that does makes sense to you.

S: O.k. [...] O.k.

I: That's it?

S: Yeah.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	2, 9, 17	Equal top and bottom, the powers equal to two. Can be factored and simplified.
2	6, 7	Constants.
3	3, 10, 13, 14, 18	With square roots. You have to rationalize. Common methods.
4	4	You can just simplify it and it will be nine over three.
5	19, 20	Trigonometric.
6	1, 5, 11, 12, 15, 16	Unequal top and bottom. Work out with rules.
7	8	Substitute the x. I just need to substitute the x.

Figure A22 (Copy of Table 5.22a). Student S22's classification.

I: O.k. Can you explain me the rule you chose?

S: O.k. Well, this one is just, is equal top and bottom, the powers equal to two [class 7]. Mmm, these are... these are just constants [class 3]. These are with square roots, so are by themselves [class 5]. These are unequal top and bottom, so I put them together [class 6]. And this has trigonom... eh, trigonometric [class 2]. And these ones can be factored and simplified [class 7].

I: O.k. so they go together?

S: Yeah.

I: What do we have here, lim? How do you read this? [Pointing at card 6.]

S: [Reading 6] Is the limit, is... well these are just constants, so the limit as x goes to infinity of seven so the answer is seven, 'cause these are just plain constants. Ah, this one is top and bottom are equal factors so it's nine over three, because you can just simplify it and it will be nine over three the limit [looking at 4]. Ah, these are square roots [class 5] so like there's gonna be like... if I have to take it... they are just square roots, so that's just a different... you just probably you have to rationalize and stuff like that, so there are like all these... all these common methods. These are like improper, I think that's what they're called, like top and bottom are not the same [class 6], so then I have to work that out with rules there. These are trigonometric [class 2] and this is just alone [showing 8] because... I think I just have to substitute x...

I: Can you read it?

S: Limit equals four x cube plus seven x minus nine... limit as x goes to one is four... of four x cube plus seven x minus nine... ? [Reading 8] And then these are just, these can be like factor and simplify to make it more easy to solve [class 7].

I: Next, I'll show you some problems like the ones you saw before but I'll ask you to solve them.

S: To solve it?

I: Yes. [...]

S: I don't know if they are right though... [...]

I: O.k. Let have a look at the problems. Can you tell me in which way you solve this, what was it that you did? [2.1]

S: Well, I just plugged in. [She first factored the denominator, but then erased her calculation.]

I: You just substituted?

S: Yes.

I: And for this one?

S: I simplified it... and after I simplified it, I substituted.

I: So here [2.1] you substituted but not here [2.2] right?

S: Yes, I simplified it because I thought it could be simplified but if I don't simplify it instead [tries the problem without simplifying, doing direct substitution] but I guess I could've done substitution, it doesn't matter... yeah, I could have right? Because it gives the same answer... ? Five divided by... four minus... yeah, it gives the same answer.

I: Do you know why in this case you don't need to simplify?

S: Yes, because nothing here equals zero so I didn't really have to simplify.

I: And what about this one? [2.3]

S: This one I don't know. I think... because I couldn't figure it out, 'cause the denominator equals to zero and I can't simplify to make it not equal to zero. Then I realize that it was irrational function I realize there's probably an asymptote at x equals five so then when x equals... approaches five aah, the limit as x approaches five is infinity, I guess, but I wasn't sure about it.

I: Did you try substitution first?

S: No, I just simplified it [she factored the denominator], but even, well even if... well I did it mentally and is zero at the bottom.

I: But do you remember when you did the problem, did you try substitution first?

S: No, I simplified it

I: And what about this one? [2.4]

S: In this one I tried long division and then realized it wasn't gonna work [she tried long division and then erased it], then I realize that's when you can do just substitution.

I: O.k. I will give you the last one [meaning the last set of problems, part 3].

S: I am bad? I'm not helping?

I: No, no, you are helping.

S: Oh my gosh [staring at the problems in part 3]. [...] Can I use cal two?

I: Yes, yes. [...]

S: O.k. These ones don't exist [3.1 and 3.2] and this one equal to zero [3.3].

I: How did you do this one? [3.3]

S: Well, this one I did it mentally I guess, in my head I pictured the graph of sine and then I figured when x approaches zero it's zero, and then zero, so it'll be zero divided by zero which is really indeterminate or like it doesn't exist also [she wrote "0 / doesn't exist"], that's what I think, or indeterminate. And this one it's just, I know, this is equal to ah... *e* to the infinity is infinity and then cos *x* is the same thing as this [pointing at a graph of cosine she did] so it has no limit as it approaches infinity, so just both don't exist.

I: O.k., as *x* goes to infinity, the cosine it's like this and you say the cosine of *x* doesn't have a limit. But you think it's the same for the product? When you multiply by e to the *x*?

S: Yeah, well, just doesn't exist.

I: And the same will be for this one [3.2] even if this changes [$-\infty$ instead of $+\infty$]?

S: Well, yeah. It's just going on the other direction, but like cos just goes on like this. So it just gonna give infinity in both ways... I guess... And everything times infinity should give infinity, so this just doesn't exist.

I: If you could use your calculator... could you test your guesses, somehow, with your calculator? Tell me before you punch anything what would you do with the calculator? Let say we are looking at this one [3.3].

S: Well, I'll put sine zero first. Then I'd realize is zero.

I: But sine *x* divided by *x*... So that wouldn't help... Let see what gives you when you put sine zero divided by zero.

S: Error.

I: Error. Yeah, so you can't do that.

S: So it doesn't exist.

I: Is there anything else you could do just to get a sense…?

S: Well, now I am thinking of cal two because we just did like indeterminate forms and stuff like that with L'Hôpital's rule, but then I don't know if I can apply to that because I don't remember if it has to be the derivative and I don't remember my exact rules.

I: What's the derivative of sine x?

S: The derivative? Cos x.

I: And the derivative of x?

S: Oh, yeah, but L'Hôpital's rule is integrate, you have to integrate… the integral of sine x, that's what I would do… now I mixed up my rules. But I don't know what I would do. I don't know I mixed up my rules.

I: Suppose using this calculator and instead of zero you put zero point one. [She just calculates sine of zero point one.] Yes, but the function is divided by x.

S: O.k., it approaches one. And then sine of point zero one divided by point zero one, one. So it approaches one. So as x approaches… point zero zero one divided by …

I: So you repeat with many more zeros… and what are you observing?

S: That it is approaching one.

I: You've got zero point nine nine nine...

S: Yes.

I: Is this convincing for you?

S: Yeah.

I: You would put equal one here?

S: Yeah.

I: So what equals one? Is it the limit? Or is it sine of x over x?

S: No, it's the limit as x approaches zero of sine x divided by x is equal to one.

I: You think you could use this idea to get the other two?

S: So I'll put really big numbers? But then cos it's just always oscillating. And then it's multiplication so it's just cos of... let say a big number [punching in the calculator], but it's just going like this [pointing at the graph she did]. O.k. times...

I: But for you it doesn't exist.

S: No... O.k., I'll do cos of five hundred first and then we'll do a bigger number after, times e to... error.

I: O.k. Cosine you say it's oscillating. Sometimes positive, sometimes negative.

S: [She did some calculations with smaller numbers.] So positive and minus infinity? So it doesn't exist.

I: You were saying before... that here doesn't make much difference [pointing 3.2] because the graph is the same. But what about for this one? [Pointing at the exponential function.]

S: Well, I think if I am not mistaken, the graph of e to the x is something like that [graphs the exponential function]... that I can check with the calculator... e to the zero is one, so it's going like that...

I: So what's the limit of e to the power of x as x goes to minus infinity?

S: It's zero.

I: And then you are multiplying it by cosine ...

S: Really big?

I: No, cosine is never really big or really small... it's between one and negative one.

S: So minus infinity times... if it's going to minus... if it's going to zero... Then it's zero because anything time zero... so it is equal to zero.

I: What it?

S: Negative... as e to the negative infinity the limit of x as x approaches... the limit of e to the x as x approaches negative infinity is zero.

I: And the limit of this whole function? [Meaning the product of the exponential with the cosine.]

S: Is zero... But you just convinced me two seconds ago that it was infinity.

I: No... why did I convince you?

S: Why? Because we were talking about this... now I don't remember.

I: No, I was just trying to follow if you were talking about e to the x only, or e to the x times cosine x, that the limit is equal to zero. But now you are convinced that limit of e to the x when x goes to minus infinity is zero.

S: Yes.

I: But what about this whole function?

S: I don't know. I don't know.

I: You don't know...

S: But then we can use like the rule [she starts writing the product rule for derivatives applied to this function]. But I don't even know if that's when you apply the rule... E to the x plus e to the x cos x... does not even it.

I: You mean using L'Hôpital's rule?

S: No, the quotient rule. Or the product rule. But when we use the product rule? Now I'm mixed up in my head.

STUDENT S23

I: These are twenty cards, I ask you to classify them according to any rule that makes sense to you.

S: So like...

I: You have to make groups... separate them in any way that has a sense to you. [...]

S: I am allowed to ask you what this limit is [card 20]?

I: Mmmm, why? In which way are you classifying?

S: Because this is one [card 19], I think… and this [card 20] is zero?

I: What's the rule you are using for the classification?

S:Well, these are complicated polynomials with cubes, these are ones that won't take me very long to solve, just factoring, stuff like that, this is most by the conjugate, this is on its own... [he is still doing the classification]. Can I make as many groups as I want?

I: Yes. [...]

S: Ok, I am done.

I: So what's the final classification? What are the groups?

S: Multiplying by the conjugate [class 1]. Cancelling out, very simple cancelling out [class 2 and 3]. More complicated, cubes [class 4 and 5]. No cubes and no multiplying by the conjugate [class 6]. This is a definite answer, polynomial [class 7].

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	3, 10, 18	Multiplying by the conjugate.
2	12	There's no xs. Simple cancelling out, $x \rightarrow \infty$.
3	2, 9, 15, 17	Simple cancelling out.
4	4, 5, 11	More complicate, cubes, $x \rightarrow \infty$.
5	1, 16	More complicate, cubes. They look very similar to me [...] you'd use the same technique to solve them.
6	13, 14	No cubes and no multiplying by the conjugate.
7	8	Definite answer.
8	6, 7	There's no xs.
9	19, 20	They are both sines. There are some similarities.

Figure A23 (Copy of Table 5.23a). Student S23's classification.

[In the following discussion he changed the classification so that the final result is what's shown in the table above.]

I: Why did you combine these two [cards 1 and 16 in class 5]?

S: They look very similar to me, like you'd use the same technique to solve them.

I: And this [class 8]?

S: Where there's no xs.

I: And this [class 7]?

S: Definite answers.

I: When you say definite answers, what do you have in mind?

S: Well, because is x to a number and it's all the same techniques. I guess I would separate divisions from the others.

I: And how do you think this is different from this one?

S: Mm, I guess is not. I guess this should go here, and so this, and that one. But not this one.

I: Why wouldn't you change that one?

S: Because of the cube, I remember the technique is not the same. I don't remember all the techniques.

I: And these two go together [class 9]?

S: Well, they are both sines, I know this is one [card 19], I am not sure about this one [card 20], but still there are some similarities.

I: Ok, so the next part, I ask you to solve some limits, and if you can sort of think aloud?

S: Ok, factoring the x, that's the first thing. Actually I don't know if that's going to help me or not. I think at that point I get... maybe I factor out the top to see what that gives. It'd be one minus one over x, in my head I replace it with one, so it's one times zero. So zero over two, the answer is zero.

I: Why your first step was to factor the x?

S: I don't know, I think... I just been trained to see factors.

[2.2. He factored, cancelled and substituted.]

S: [2.3.] I replace... so it's over zero, plus infinity? [2.4.] Factoring out the x square top and bottom I guess. Now I notice this is x to one. This cancels out... fourteen over three.

I: What were you saying about x to one?

S: After I did this steps... we use to that when it was only x to infinity. It doesn't change anything in here.

I: And if you had notice in the beginning that x goes to one, what would you do?

S: I would just put in the one, to see if it works out.

I: And if you look at the other problems now, do you think it was necessary to factor?

S: No, it wasn't.

I: And what could you have done instead?

S: Just replace. But I always factor, no matter what.

I: When you look at this expression [$\lim_{x \to -1} f(x) = 0$], how do you read it?

S: I think f of x equals zero.

I: And if I ask you to read it for me, what would you say?

S: Limit of the function f equals zero from x to negative two.

I: And does it have a meaning for you?

S: I don't know. I kind of forget the meaning of limits.

I: Ok, the last thing.

S: I know this is one [3.3]. I don't know why I remember this, but when I see sin x over x I remember is one.

I: Is sine of x over x that's is one or the limit of sine of x over x...

S: I think is the limit, but whatever you want...

I: Do you remember why this is one?

S: I remember looking at a graph and getting smaller, vaguely.

I: If you didn't know the answer was one. If you didn't remember, could you still figure it out?

S: Probably plugging numbers?

I: Ok, how would you do that?

S: Sine of five... and then closer to zero.

I: Which other numbers?

S: What do you mean?

I: Well, you said you'll plug in five first, what you'd do next?

S: Plug in one? First I plug in zero. Sine of zero is zero.

I: So what happens if you plug zero?

S: I guess it would be one [because of the zero over zero] or zero. Well, zero over zero is a... what's the word in...

I: Indetermination?

S: Yes, I don't know what I'd do.

I: And what about the other two.

S: I think there was a rule about this but I don't remember.

I: Do you think that you can use the calculator to figure out these limits?

S: If I was desperate, I'd probably try plugging in numbers.

I: And which numbers would you try?

S: I don't know, probably five, I like number five. I know that wouldn't work. Like I'd do every other question on the test and then come back to this one. Usually there are more problems with simple questions.

I: If you have to look at them independently [I write $\lim_{x \to \infty} e^x$ and $\lim_{x \to \infty} \cos(x)$] and in this case these limits [I write $\lim_{x \to -\infty} e^x$ and $\lim_{x \to -\infty} \cos(x)$]. Do you know what to do in this case?

S: e is just a number so to infinity it would be infinity. Cos is a different thing. Now I remembering the unit circle but I don't think that helps. Well, cos of any number gives you just another number. Infinity times any number is just infinity. I guess the answer is negative or positive infinity. Because with cos you can get positive or negative. And here [3.2] the same.

I: What happens with the function e to the x as x goes to negative infinity?

S: Oh, negative infinity, so zero. Thanks. So this one would be zero.

I: Why it would be zero?

S: e to a really small number would be zero. Oh, no, limit zero, thank you.

I: What do you mean by what you wrote there [next to 3.2 he wrote $\lim_{x \to 0}$]?

S: Is not zero, it's very close to zero.

I: So this you say zero [3.2]?

S: No, very close to zero.

I: And when multiplied by cosine... ?

S: Well, cosine of any number is just another number so...

STUDENT S24

I: So the first thing I'll ask you to do... here are twenty cards with different statements. I'll ask you to classify them into groups according to any rule that makes sense to you.

S: Ok, ok. [...]

I: Ok, can you explain me each of the groups?

Class	Members of class *(labels refer to Table 3.3)*.	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Just a number in them.
2	2, 5, 9, 12, 15, 16, 17	Squares. Squares at the top.
3	3, 10, 13, 14, 18	Roots. Roots at the top.
4	1, 4, 8, 11	Cubes. Cubes at the top.
5	19, 20	Sines

Figure A24 (Copy of Table 5.24a). Student S24's classification.

S: I kind of went through a pattern... These are with just a number in them [class 1]. These are with squares [class 2], these are roots [class 3], cubes [class 4] and sines [class 5]. Squares at the top, roots at the top, cubes at the top, and sines.

I: Ok, so that was the first thing that came to your mind when you started classifying these.

S: Yeah, just like that.

I: If you have to read this [card 8], let say you are talking with a friend on the phone, and you have to read this to your friend, what would you say?

S: The limit as x approaches one of four x cube plus seven x minus nine.

I: Ok. Next I'll ask you to solve some limits. And if you can sort of think aloud so I can have an idea of what you are thinking when you look at these...

S: Ok, so you can't put it in right away because it gives zero on the top. So... I don't write limit as I do it, I write it at the end. [She did not write limit at all.] The bottom is x times x plus one. [...] I think this is completely wrong. [She scratched her first step, which was factoring the denominator.] I can't remember...

I: You said before you can't...

S: You can't put it right away because it gives zero on top.

I: And what do you get below?

S: Two.

I: And what's zero divided by two?

S: It's just zero. You can just do that? Yes, so the limit equals zero.

I: What about this one? [2.2]

S: [...] Two plus three times two minus one over two square minus nine equals five times one over negative five equals negative one. [2.3] In this case I have zero on the bottom, is it? So *x* plus two times *x* minus two over *x* plus five times *x* minus five [...]. I don't know. [...] If it's zero on the bottom, it's just infinity. I don't know.

I: What did you just say about infinity?

S: If it's zero on the bottom it becomes infinity.

I: What is it that becomes infinity?

S: It's undefined. Is it possible it's just undefined? So twenty one over zero.

I: When you say undefined you mean twenty one over zero is undefined or the limit is undefined?

S: The limit is undefined.

I: Ok, what about this one? [2.4]

S: One cube plus four times one square plus nine over one square plus two. One plus four plus nine over one plus two. Fourteen over three.

I: Ok, let say maybe in this one... When you write, or when you see written the limit as *x* goes to two of this expression equals negative one... what's the meaning of that for you?

S: Like in a graph?

I: What does it bring to your mind? If the teacher is speaking or you read in a book the limit as x goes to two of this function equals negative one... [...]

I: What come to your mind? Nothing? A concept? An idea? A graph?

S: Well, a graph?

I: Ok, in terms of a graph... what?

S: Well, if x is approaching two, the graph is here at negative one [she draws a pair of axes and the point *(2, -1)*].

I: You mean that point?

S: Yes.

I: Ok, the last.

[She wrote wrong answers very quickly.]

I: Ok, why did you say this is infinity [3.1]?

S: Because *e* to the infinity is infinity.

I: And why did you put here minus infinity [3.2]?

S: I figured because it's negative infinity it would just be infinity.

I: So *e* to the negative infinity... do you think it's negative infinity? Why do you think that in both cases the multiplication, multiplying by the cosine, doesn't make a difference? [...] Because you say *e* to the infinity is infinity, right? But why when you multiply by the cosine you still get infinity?

S: Cosine of infinity, I don't know what that is. So I figured it would probably be a very big number.

I: Ok, if you could use somehow your calculator to check your answers... how would you do it?

S: I would try cosine of one and then cosine of a hundred and see if it gets to infinity...

I: Can you try?

S: [She tried some numbers.] Yes, it increases.

I: Which numbers did you try?

S: One and one hundred.

I: And what did you get.

S: That it increases.

I: Let say you were looking at this problem [3.2] and you couldn't guess the negative infinity. How could you use your calculator to try to find an answer?

S: Cosine of negative one... cosine of negative a hundred, so it gets bigger. E to the negative one... e to the negative a hundred... it's a much small number. It's a much smaller number. So probably that one will equals zero. E to the negative infinity just keeps going closer and closer to zero.

I: And what about the last one [3.3], can you try something similar with the calculator [as answer she wrote undefined and 0].

S: Yes, sine of zero is zero, so zero over zero... is that one? [She punches zero over zero on the calculator to check.] Or undefined? Undefined, so it's zero.

I: Why is it zero?

S: Because the top is zero. Sine x is zero so it'd be zero zero which is zero. Or is it undefined?

I: Well... it depends on the problem... I just want to be sure I understand what you are saying... You mean that sine of zero over zero is zero?

S: Zero over zero is zero.

I: Ok, so that's just for this, right? [I mean the expression $\frac{\sin(0)}{0}$ that she wrote before.]

S: Yes.

I: But what about the limit? Is there any difference between this expression here, the limit as x approaches zero of sine of x over x and this expression here [just $\frac{\sin(0)}{0}$]?

S: You get the limit when you put it in, when you make x equal whatever the number is here [in the expression $x \rightarrow c$].

I: Ok, but zero over zero is an undefined calculation, you cannot do it. So this is undefined.

S: So the limit is undefined.

I: But what is undefined is zero over zero. Still the limit might exist, depending on the problem. Do you think you have a way to check whether is the limit undefined or not?

S: There's probably a way...

I: Ok…

S: I don't remember.

I: In here [3.1 and 3.2] you chose one and one hundred and negative one and negative a hundred. Why did you choose those numbers?

S: Well, because negative infinity is a really big number, I mean really small number, so I figured negative one and negative a hundred, which is even a smaller number…

I: And do you think that this idea of picking numbers, like negative one, negative a hundred… Could you apply this idea to this problem [3.3]? Instead of replacing x by zero.

S: I guess so. I put zero because this [$x \to 0$] told me it should be zero. I guess you could, if it gives you a proper number.

I: Which kind of numbers could you use?

S: Maybe again the one and the hundred?

STUDENT S25

I: So the first thing is... these are twenty cards, I ask you to classify them according to any rule of your choice.

S: Like the answer?

I: Whatever you feel you say you want to classify these for yourself.

S: Ok. [...] Ok, I think that's it.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	4, 5, 6, 7, 8, 10, 17	Real answer.
2	2, 9, 11, 12, 14, 15, 16, 18, 20	Answer is infinity.
3	1, 3, 13, 19	Answer is zero over zero or infinity over infinity.

Figure A25 (Copy of Table 5.25a). Student S25's classification.

I: Can you tell me what was the rule for classification that you chose?

S: I chose if they have like a real answer [class 1], the ones where the answer was infinity [class 2], and when the answer was zero over zero or infinity over infinity [class 1]. I think so...

I: Ok. How do you read this [card 8]? If you have to read it aloud, what would you say?

S: How would I read the answer?

I: How do you read this sentence, this expression?

S: Oh, limit from... limit... limit going from... limit going x towards one of four x cube plus seven x minus nine. Limit of x going towards one, yeah.

I: Ok. Next I ask you to solve a few limits [2.1]. [...]

S: This I think is just zero, I think. Like if you just plug in you get zero over two.

I: So you just plug it in?

S: I don't remember if you are suppose to do the thing... to factor the highest degree from the top and the bottom, because I'd have one over one and one over one which is one.

I: Why would you have one over one?

S: Ah... because you have, because this would be... over one minus one over x... oh, no, so it'd still be zero, zero over two, so zero.

I: Ok. What about this one [2.2]?

S: [...] Negative five? You just put two where x is. [2.3] I don't remember if you are suppose to take the x square out... probably you get the same... just positive infinity.

I: Do you think it might be the case when you solve the problem just by substitution, like you did here, and when you do some algebra you get something different answers?

S: Well, I remember that sometimes yes. I think that is when you are going towards infinity, then you get a different answer when you do this.

I: And these are two different answers or one is the right one and the other not.

S: Yes, I think so, I think that if it's going towards infinity you have, I think I remember now, you have infinity over infinity, you are suppose to factor out the highest degree and then get a real number as your answer.

I: Ok, so this is your answer, and what about this?

S: I was saying, if x is going towards infinity...

I: But if they are different?

S: If they are different I think this is the right answer.

I: Ok.

S: [2.4] Fourteen over three.

I: Ok, the last thing I ask you to do.

S: Ok. [...] Well, I don't think cos of positive infinity exists. Doesn't just cos goes from negative one to one? So I'd write it doesn't exist. But I am sure there might be some sort of technique for these... or something.

I: So you are saying that the limit of the cosine function here doesn't exist. But what about this function, that is a multiplication.

S: Well, this would be positive infinity [for the exponential function], but it still wouldn't exist because cos of positive infinity doesn't exist.

I: And what about this [3.2]?

S: The same thing though this would be going towards zero [for the exponential function]. Yeah, that would be going towards zero, and that doesn't exist. So I don't remember if there was some sort of technique to solve this. I don't remember.

I: But if you had to... guess what the answer is...

S: I'd say they don't exist. And now I remember there was a technique for this too, I sort of remember the answer being one... equal one.

I: If you could use the calculator to try to answer these two problems [3.1 and 3.2], what would you do?

S: Well, it's going to say error if I put cos of two [he tries that]... I though cos could only be on one... but here I just get a value. Does cos go from negative one to one?

I: What do you mean by goes from negative one to one?

S: Like the values...

I: Do you remember the graph?

S: It's like that and it's... like that isn't it?

I: So what is it that goes from negative one to one? The x?

S: The y goes between negative one and one. So it's the arccos that doesn't go with more than one... So cos of a really big number gives me an error. Is it one, cos of infinity? Sorry, what did you ask me about the calculator?

I: If you could use to figure out the limits... would it help you in any way to use the calculator?

S: No, I don't think so, I don't think it would help me with the answer.

I: What's the meaning of the answer of this problem?

S: Well, at infinity, at wherever infinity is, this...

I: Or what about this one [3.3]... when you say that this equals one, what's the meaning of that answer?

S: Oh, I do not know, I thought that for some reason the answer is one, but if you plug in zero you get zero over zero.

I: But let's say it is one. What's the meaning of saying that the limit of this function equals one?

S: The function of sine of x divided by x gives you one.

I: The sine of x over x gives one? Or the limit?

S: Oh, right. The function of sine of x going to zero divided by x going to zero gives you one.

I: And when you say going to zero, what is that you are thinking?

S: Numbers very close to zero?

I: Numbers very close to zero for sine of x or for x?

S: Both?

I: So the operation here is sine of x divided by x when x is very close to zero... can you do that in the calculator?

S: Ah... Like a very very small number? [He does that.] I get one.

I: And using that idea, can you try something for these ones?

S: e to the one million, e to the ten million I get an error, for cos I get negative one, so it wouldn't exist if e cannot go to ten million. But I thought the function e was like this, so in my head I would have thought e to the x at positive infinity is just positive infinity.

I: Yes, the calculator is giving an error because the numbers are too big, so it doesn't fit in here.

S: So I'd get going to positive infinity... I am going to have cos of ten million...

STUDENT S26

I: These are twenty cards and I ask you to classify them into groups according to any rule that makes sense to you. [...] Ok, can you explain me what your rule was?

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	Constants.
2	2, 4, 9, 17	The thing on the top and the bottom.
3	1, 11, 12, 15	These the power is more in the top than the bottom.
4	5, 16	These the opposite.
5	3, 10, 13, 14, 18	Square roots.
6	8	
7	19, 20	Sines.

Figure A26 (Copy of Table 5.26a). Student S26's classification.

S: I did the constants [class 1]. And here I did the thing on the top and the bottom [classes 2, 3 and 4], these the power is more in the top than the bottom [class 3] , these ones the opposite [class 4] and these ones the roots [class 5]. And the sines [class 7].

I: Now I ask you to solve some limits and if you could think aloud...

S: [2.1] Ok, this is zero over two. So this is not an indetermination. [...] So, we just learn the L'Hôpital's rule... but that's for zero over zero. So I factor the bottom... But the limit is zero. Zero over two is zero. [2.2] Nine... I just put the numbers in... five times one over... [...] [2.3] Twenty five minus twenty five... I can't factor. [...] I want to get rid of the x minus five so I factor x minus five from the top. [...] The bottom approaches zero so it's going to be plus infinity of minus infinity, so I just plug in a positive number to see if it's positive or negative. The limit as x approaches five from positive... I put five point zero something, this is positive so plus infinity, and the other is minus infinity [he did not use the calculator for this]. [2.4] [He wrote the right result.]

I: Ok. I ask you a few questions about them... Why did you think you still factor here... In the beginning you did substitution, and you realized it was zero over two. Why did you still factor the bottom?

S: I tried maybe... because by factoring maybe it would cancel... could have been easier, but if it doesn't work I just plug it in.

I: Do you think you can get a different answer by doing the factoring than the answer you got in the beginning?

S: Probably it's not gonna work the factoring. You have x minus one and x plus one, so it won't work.

I: What about this one? You factor here and then...

S: I tried to factor an x minus five from the top but I still got an x minus five so it didn't work. Then it was approaching zero so it was infinity.

I: Do you know what was the rule for not to keep getting the x minus five?

S: The conjugate?

I: How should be this polynomial so that when you do long division you don't get a remainder?

S: This should be a factor of x plus five... a perfect factor. A four can't divide a five.

I: And then you went back here and you saw...

S: Well, something divided by something closer to zero is infinity. So I plugged a number bigger than five to see if was positive or negative.

I: So do you think all this calculation was needed or you could just avoid it?

S: You could just avoid it but... yeah, you could avoid it. If it's a number over zero you don't need this.

I: Just one more question... Why did you still do it? You didn't realize in the beginning that was a number over zero?

S: I realized somehow but whenever I see a fraction, I see the factoring first. If I look at this, I don't see x square minus four, I see x minus two, x plus two. By doing this I see it better, and then I saw the x minus five and that's zero and...

I: Ok. The last one. If you can, like before, let me know what you are thinking.

S: Yes. This is an indetermination of infinity times infinity... well cos... is infinity times a number. [...] The limit as x goes to infinity doesn't exist because this is a wave, so this limit [the product] does not exist.

I: What about the other one [3.2]?

S: This is the same graph [for cosine] and e to the x... same thing, it shouldn't exist because e to the x goes to infinity... because the graph of e to the x... let me see my calculator. e to the...

I: What did you try with the calculator?

S: I tried to put in some numbers. I tried a hundred and then a thousand, to see if it keeps growing or if it comes back.

I: And what did you get?

S: Putting a hundred I got negative, and then with a thousand I got positive, and then negative again. So this doesn't exist... both of them, because one is positive infinity and the other is negative infinity.

I: Do you want to try with the calculator [3.2]?

S: Cos of a negative angle times *e* to the... this is gonna come close to zero. So this is going to be *e* to the negative one hundred times cos... this is gonna get closer to zero. So the graph is like this... on this side doesn't exist but on this side it gets to zero.

I: What about this [3.3]?

S: This is one. It's zero over zero but if you do the L'Hôpital's rule is going to be cos over one, cos of zero over one, so this is one.

I: Do you remember any rule that is not using L'Hôpital's?

S: By doing the triangles in the circle, because the sine... if you draw the circle and a triangle like this, this is one, this is sine and this is cosine, so this is the same... what you do you look at it like this, the value here is bigger than this triangle and smaller than this triangle so when you do this smaller than this smaller than this and then you can cancel... do the limits and this limit is going to one and this limit is going to one so this has to go to one.

STUDENT S27

I: The first thing I ask you to do is... this are twenty cards, I ask you to classify them into groups according to any rule of your choice. [...]

S: Does it matter how many groups there are?

I: No. [...]

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	6, 7	The limit is to a number.
2	1, 2, 3, 8, 9, 10, 15, 16, 17, 18, 19, 20	The x is going to a number.
3	4, 5, 11, 12, 13, 14	Functions going to infinity.

Figure A27 (Copy of Table 5.27a). Student S27's classification.

I: Can you tell me what was the rule? How did you decide to put them in one pile or the other?

S: I figure it what was the limit was going to, like what kind of limits but I just wasn't sure...

I: Ok, so you put these two [class 1] together because...?

S: They both the limit is to a number.

I: And this pile?

S: They are all functions with the x going to a number [class 2]. And those are all functions going to infinity [class 3].

I: Ok, good. Then I ask you to solve some limits. And, as much as you can, when you are solving the problems, if you can think aloud so I can keep track of the ideas coming to your mind.

S: [2.1] Ok, I replace x with one, so it becomes one minus one over one plus one, which is zero over two... So the limit as it goes to the right and to the left... [...] [she wrote $\lim_{x\to 1^+}\frac{x-1}{x^2+x}=\frac{1}{2}$ and $\lim_{x\to 1^-}\frac{x-1}{x^2+x}=-\frac{1}{2}$] Then... it goes to... I don't remember what it happens when it does that.

I: Why do you say this is a half? What is the calculation that you did?

S: Because these numbers are a lot bigger than one and... oh, wait. Oh, no, it should be zero, shouldn't it? [She wanted to erase her work.]

I: No, please, leave it, just write on the side. If you want you can cross it. [She crossed it.]

S: Ok, so this should be zero [she wrote the result of the direct substitution 0/2 and then, this equals 0].

I: If you have to read this [2.1] aloud, so you have to read it to someone else, what would you say?

S: The limit of x minus one over x square plus one as x goes to one.

I: What about this one? [2.2]

S: Five times one over... minus five, minus one.

I: Why did you open this problem [2.1] into these two different problems?

S: Because the upper part of that one goes to zero as x approaches one, so you can't solve it.

I: Ok, next this [2.3]. [...]

S: This one you have to open it up also, I think [she means considering $x \to 5^-$ and $x \to 5^+$]. [...] Can I use my calculator?

I: Yes, of course.

[She tried some numbers in her calculator.]

S: So that approaches infinity... and this negative infinity.

I: What did you use the calculator for?

S: I just did twenty one divided by little little numbers close to zero.

I: Ok, last one [2.4]. [...]

S: Fourteen over three.

I: Ok, the last thing. [...] Why do you think that one [3.1] is plus infinity?

S: Because e to the power of very big numbers is very big numbers, and the cos of very big numbers I don't know. I thinks is bigger numbers, but I don't know. [...] This gets smaller, so it would be zero.

I: Do you think you could use the calculator to check if your guess is right?

S: What?

I: Because you say that here is zero... Why do you say here is zero?

S: This one I think of plus infinity but when I put big numbers it gets smaller, so that one goes towards zero and zero times anything is just zero.

I: Do you remember the graph of the cosine function?

S: Yeah.

I: Can you draw it?

S: I think it's [she gets a good sketch of the cosine function.]

I: Can you read on the graph what the limit would be?

S: No, not really.

I: And if you wanted to test... You were saying anything by zero gives you zero. Let's say you were not sure and you wanted to check with your calculator if this answer is right. Would you be able to test that? Would the calculator help you in any way?

S: I think so, because I could plug first one hundred and then a thousand and then...

I: Can you do it?

S: Yes. Cosine of a hundred is point eighty six, then cosine of a thousand is point fifty six, then cosine of ten thousands is negative point ninety five... So...

I: That would be to check for the cosine function, right? Could you check for the whole thing?

S: Yeah. E to the one hundred is two point three ten to the forty three, so a big number. Then time the cos which was point eighty six... it's a very big number. So it's a very big number. So maybe it does go to infinity?

I: What about this one [3.2]?

S: *e* of a power of negative a hundred is a very very negative big number so negative infinity... because the cosine of a negative number is the same as the cosine of a positive number, so that won't change.

I: Why negative infinity? Which of the two is going to negative infinity?

S: *e* to the *x*, I think.

I: What did you get here?

S: Oh, no no no, wait, zero. Sorry, yeah... times ten to the negative forty four.

I: What about the last one [3.3]?

S: I don't remember the trick for this one. What the trick was? Oh, my god. I believe it's zero, but I don't remember why.

I: Can you check with the calculator?

S: Well, sine of zero is zero. And zero... is divided by zero, so zero over zero. The limit... but I don't remember, I don't remember the proof for it.

I: Could you use the calculator to verify... to check if the thing you remember is right?

S: I don't remember what it was.

I: But I mean using the calculator as you did before.

S: Taking the sine of zero?

I: I mean doing some calculation that would help you to check...

S: Oh, I could take the limit as x approaches zero from the right of sine of x over x, for numbers bigger than zero, as point zero zero one [she does the calculation for point zero zero one], the limit approaches one. And the limit as x approaches zero from the left. [She does the calculation for negative point zero zero one.] It approaches one.

I: So what would you say about the limit?

S: That it approaches one?

I: I am curious now about the distinction you did here. Here you said, when you were calculating this part, you were saying this [$x \to 0^+$] is for numbers just above zero? What this [$x \to 0$] means for you then?

S: When x is exactly at zero.

STUDENT S28

I: These are twenty cards, with different statements... and I ask you to classify them into groups according to any rule that makes sense to you.

S: I start now?

I: Yes. [...]

S: There's so many possibilities.

I: In which sense.

S: Well, I could place them by order of polynomial or if the limit is infinity or definite number... Here I put all polynomials where the degree is higher in the numerator than in the denominator... This is a kind of tricky question...

I: But there is no right or wrong answer...

S: I know, but I am looking for an order that is... [inaudible] [...]

[He took paper and a pencil to quickly find the limits he could not figure out in his mind.]

[...]

S: Ok, I think am done.

Class	Members of class *(labels refer to Table 3.3).*	Phrases used by the student in response to the question "what was the rule of your choice?".
1	5, 7, 9, 14	The limit go to zero.
2	11, 12, 15, 16, 17	The limit is plus/minus infinity.
3	1, 2, 3, 4, 6, 8, 10, 13, 18, 19	Finite numbers that are not zero.
4	20	Divergent.

Figure A28 (Copy of Table 5.28a). Student S28's classification.

I: Ok, can you explain me what was the rule...

S: Well, I may have made a mistake in the calculation... Those are all the ones where the limit tends to one... ah, that go to zero [class 1]. Those are... this is plus infinity [class 2], this is... I am not sure, this is zero. Ok, these are zeros [class 1], these are plus infinity [class 2], these are finite numbers that are not zero [class 3], this is neither plus infinity nor minus infinity [class 20]... and this... Ok, [pointing at each of the classes] zero [class

1], plus infinity, minus infinity [class 2], divergent [class 4], and a number not zero [class 3].

I: Ok, say for this one [card 14], how do you know..., which calculation...

S: Well, when this tends to plus infinity this is going to tend to x, because this is going to be very big and once you do the square root of this the one will just vanish and you'll have minus x so it will be zero. It might not be some kind of theorem but I figure out is this way. And this one, is zero the numerator and the denominator is not zero therefore this goes to zero.

I: And in the beginning you said there were many ways to order these, to classify them. You were saying maybe looking at the polynomials or the limits... why did you choose this way of classifying?

S: Well, if I would have choose the limit [he means classifying according to $x \to c$] it would not have been specific enough... so I decided to put them in something that already tells you if you are going to have an answer finite, zero, divergent or plus infinity.

I: And why do you distinguish zero from finite? Why do you have a pile for zero and a pile for non zero answers? What's the difference in your mind?

S: Well... that's kind of strange... I don't know. For my mind zero... it's a number but is not as... it does not the same value, like in term of number, as another number. Because it's not actually a number, you don't have really anything. So this is why I put it as different from other finite numbers.

I: Ok, we move to the next part. I ask you to find these limits. And, if you can think aloud so I can track the ideas that come to your mind...

S: Yes, that's great. [In 2.1] I just evaluate the limit by doing the numerator which is zero, and the denominator is two, therefore the numerator approaches zero and the denominator approaches two, therefore it approaches zero.

S: [In 2.2] There's two ways of doing this, I just do the polynomial... extend the polynomial [he means performing the multiplication by distribution properties], so it's going to be four plus... four minus three... which will be five, and this is four minus nine which is minus five, so this is minus one. Or I could have done two plus three is equal to five, two minus one equal to minus one... I get two different answers, what's wrong... oh, ok, minus one.

I: What about this one? [2.3]

S: This was in the cards, it's zero, oh no, it's plus infinity, sorry. So is twenty five minus four over twenty five minus twenty five which is... over zero, which will tend to plus infinity... If I'm wrong you won't say it right?

I: Mmm, no.

S: Ok.

I: But I am more interested in the way you are thinking than in the final answer. Then, at the end we can talk about the answers if you want.

S: [In 2.4] I just replace it one plus four plus nine over one plus two, which is fourteen over three, and this is the answer.

I: Why in this problem... why do you think this was the first thing you did, instead of just doing this.

S: Because for some reason I find it easier to break in small polynomials... I don't know, I get confuse as you see [meaning the calculation mistake he did]... I don't see it, I see it better like this. It's easier for me this than this.

I: Ok. If I ask you to read this [$\lim_{x \to -2} f(x) = 0$] aloud, what would you say?

S: The limit when x tends to minus two of the function f of x is zero.

I: And what this means to you? When you read that, what's the meaning that comes to you?

S: That means that... well if you don't consider this [that *f(-2)* does not exist] I would say that at minus two is zero. But with this, it tells me that... I know that at this point it should be if it would have exist it should be at minus two but since it says it doesn't exist I will put an open dot.

I: So the last one [part 3].

S: There's nothing after?

I: No.

S: Oh, ok. Well, this goes to plus infinity [for the exponential function], this goes to... plus one, minus one [for the cosine function]. I could say... because this is plus or minus infinity. Well, my teacher would say that this limit won't exist, but I say it's plus infinity and minus infinity. And this is of course zero because this goes to zero [the exponential function] and this is plus one and minus one [the cosine function], this goes very rapidly to zero [the exponential], so you don't care if it's minus one of plus one. [He draws a very good sketch of the function $f(x) = e^x \cos(x)$.] And this [limit 3], well this is the one thing, because... and it's useful in proving the derivative of sine because, you'll use this that is one [he writes the definition of derivative of the sine function to show that you need this limit for the calculation]. And you also need this for the smaller gold approximation [inaudible] and when you make triangles very small it's going to be almost equal to the angle itself, so this one. [Inaudible.] And also when x is in radians, this function, the sine function... well, now I am talking about radians which is not... Well, the fact that it is in radians will give that the derivative is the cosine function. And it won't be the cosine if it is in degrees or other form of angles.

I: Do you know why this is one? Or do you have a justification for yourself of why this is one?

S: Well, you mean like a technical proof?

I: Well, something that you know... that you'd say I know it's one because...

S: Because you see that, if you close up... if you zoom very close this function, your sine is like this, and you have x like this, it's almost gonna be the same, therefore it's one. This is my proof [he laughs].

I: And do you remember any analytic proof?

S: Well, unless... I am not sure how to explain it, because they both have slope one so if you divide it, you'll get...

APPENDIX B

In this Appendix are shown the documents handed to the researcher in Mathematics Education participating in the triangulation process (see Chapter 3, Section 3.3) with whom I only had written communication.

Document 1: Instructions for assessing students' modes of thinking about limits in terms of Vygotsky's theory of development of scientific concepts

In his theory of concept development, Vygotsky distinguished three main stages a child goes through in developing conceptual (scientific) thinking: syncretic heaps, complexes and concepts (Vygotsky, 1987: 134-166), each with several phases. The most mature phase of complexes is pseudoconcepts. Each of these stages and phases is characterized by a *mode of thinking*. The general pattern of this development seems to be recapitulated each time a student embarks on the project of understanding or building a mathematical concept (Sierpinska, 1994).

Below is a description of the four modes of thinking – syncretic heaps, complexive thinking, pseudoconceptual thinking and conceptual thinking – with examples. In a separate document I am providing you with excerpts from transcripts of individual interviews with college students that have successfully passed a Calculus I course. The interviews were structured around 3 tasks the students were asked to engage with (they were "task-based interviews"). The excerpts are all drawn from the same part of the interviews, namely the part based on the task described below.

Students were given 20 cards. Each card contained a limit expression of the type $\lim_{x \to a} f(x)$ where *f(x)* was a constant, a polynomial, a rational function, division of functions involving radicals, or a trigonometric function, and a was either a constant or ∞. The students were asked to classify the 20 cards according to any rule of their choice.

The list of the 20 expressions on the cards is given at the end of these instructions. The expressions in this list have been numbered to facilitate communication of students' classifications in the thesis, but they were not numbered when presented to the students. Next to each expression in the list, two values have been written: the first one is the outcome of direct substitution, and the second is the value of the limit. Of course, this information was not given to the students, but it can be useful for the assessment since many students refer to these values in their classification..

What I am asking you to do is the following:

1. Read the descriptions of the Vygotskyan modes of thinking given below. (See Note 1.)

2. Read each interview excerpt carefully and analyze it to decide which Vygotskyan mode of thinking the student is using. (See Note 2.)

3. Write a short justification of your decision in each case.

Note 1: The description of the Vygotskyan stages of concept development given to the researcher were exactly those described in Chapter 2. He was also given the example that appears after that description.

Note 2: The researcher was given the excerpts of interviews with students S1, S6, S8, S14, and S28 corresponding to the first section of the interview, these can be found in pages 246-247, 277-278, 286-287, 324-327, and 409-411.

Document 2: The 20 expressions given to the students in part 1 of the interview.

Limit expression	Outcome of direct substitution	Value of the limit
1. $\lim_{x\to 3} \frac{x^3 - x^2 - 3x - 9}{2x^2 - 4x - 6}$	0/0	2
2. $\lim_{x\to 1} \frac{x^2 - 1}{(x-1)(x-3)}$	0/0	-1
3. $\lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x}$	0/0	1/[2sqrt(2)]
4. $\lim_{x\to \infty} \frac{9x^3 - x + 2}{3x^3 + 1}$	∞/∞	9/3
5. $\lim_{x\to \infty} \frac{x^2 - 25}{x^3 - 1}$	∞/∞	0
6. $\lim_{x\to \infty} 7$		7
7. $\lim_{x\to 5} 3$		3
8. $\lim_{x\to 1} 4x^3 + 7x - 9$	2	2
9. $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 3x - 4}$	0/-6	0
10. $\lim_{x\to 4} \frac{\sqrt{x} - 2}{x + 4}$	0/8	0

11. $\lim_{x\to\infty} \frac{x^3 + 2x + 1}{x^2 + 7x - 1}$	∞/∞	+∞
12. $\lim_{x\to\infty} \frac{x^2 - 1}{x - 1}$	∞/∞	+∞
13. $\lim_{x\to-\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$	∞/∞	-sqrt(2)/3
14. $\lim_{x\to\infty} \sqrt{x^2 + 1} - x$	∞ - ∞	0
15. $\lim_{x\to 1} \frac{x^2 - x - 2}{x - 1}$	-2/0	±∞
16. $\lim_{x\to 1} \frac{x^2 + 6x + 19}{x^3 - 3x + 2}$	26/0	±∞
17. $\lim_{x\to 5} \frac{x^2 - 4}{x^2 - 25}$	21/0	±∞
18. $\lim_{x\to 5} \frac{\sqrt{x + 20} - \sqrt{5}}{5 - x}$	[5-sqrt(5)]/0	±∞
19. $\lim_{x\to 0} \frac{\sin(x)}{x}$	0/0	1
20. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$		Doesn't exist.

Table B1. The first column corresponds to the expressions given to students for the classification task. The second column shows the outcome of direct substitution, and the third, the value of the limit.

Assessment on student's modes of thinking by a researcher in Mathematics education not involved in the process of this research

Student S1: Complex

a. "Difference of squares" vs. "Polynomials".

b. "[Trig] confuses me..."

While one of the classes was chosen affectively - "[Trig] confuses me so I put them together" - the classification seems to have been motivated primarily by the appearance of the limit expression. Student S1 has classified the limits without appealing to any unifying rule or hierarchy; furthermore, a unifying rule is impossible, since the classes are not mutually exclusive (e.g. Group 1 is a subset of Group 5). Therefore, the student is employing complexive thinking (with, perhaps, a syncretic motivation).

Student S6: Syncretic Image

a. "I spot it right away" vs. "Infinity over infinity"

b. #4 and #14 are both "Infinity over Infinity" but are classified as "I spot it right away".

Limits are classified by the ease with which the student can evaluate them. This is a purely affective, subjective classification, both impossible to formalize and independent of the limits themselves. Thus, the student is thinking syncretically. Student S6, explicitly used an affective relation as the classifying feature for Group 1.

Student S8: Complex

a. Mixture of descriptions of the expression ("Trig.") and methods of solution.

("Rationalization," "Factor," "Just plug in.").

b. "Just plug in" and "Rationalization" are not mutually exclusive (e.g. Limit 10).

This student is also employing complexive thinking. The classes are a mixture of descriptions of the expression itself ("Trig.") and the methods she would use to evaluate the limits ("Factor."). No universal rule is theoretically possible, since the classes overlap.

Student S14: Complex

a. "These ones I didn't know where to put them."

The student has classified the limits by the technique he would use to evaluate them. One of the student's groups is a catch-all class for limits which he is unable to evaluate using the techniques he knows. However, this is a reflection of the student's limited knowledge, and not an indication of affective classification. Since classification scheme results in ambiguous classes - limits for which you "divide by the highest degree" are also amenable to L'Hopital's rule - it is the result of complexive thinking.

Student S28: Concept

a. Precise categorization by the value of the limit

b. Idiosyncratic but unambiguous classification of numbers.

The student explicitly searches for a unifying key: "I am looking for an order that is...." The classification is unambiguous and the key is explicit. The student is at the level of concepts.

Discussion

The assessment matches exactly the modes of thinking that I inferred for these students at the technology level. Also, the particular behavior from which the researcher and myself inferred these modes is similar in each case. Thus, for student S1, the researcher wrote "student S1 has classified the limits without appealing to any unifying rule or hierarchy" and he noticed that group 1 ("difference of squares") is a subset of group 5 ("polynomials"). In my own analysis (see Chapter 5) I observed that the hierarchy of features changed from one class to another. Furthermore, we both observed that there was a syncretic motivation in the construction of the class containing the trigonometric objects.

Both, the researcher and myself, inferred that student S6 was thinking in syncretic images. The researcher wrote "limits are classified by the ease with which the student can evaluate them", I observed that his classification was based on a recognition of familiarity of the given expressions.

With respect to students S8 and S14, we both observed that they were employing complexive thinking. For example, in relation to student S8, I observed that another person would not be able to reconstruct her classification based on the phrases she provided, as for example, one would place expressions 9 and 10 in class 4 because both limits can be found by "just plugging in". The researcher wrote: "no universal rule is theoretically possible, since the classes overlap" based on the observation that "'Just plug in' and 'Rationalization' are not mutually exclusive (e.g. Limit 10)".

Finally, we both considered student S28 to be thinking in terms of concepts. The researcher wrote: "the student explicitly searches for a unifying key" [...] "The classification is unambiguous and the key is explicit. The student is at the level of concepts." While I observed that "the student was preoccupied for having a general conceptual key and he considered other criteria for classifying but discarded them for not being 'specific enough'. He made his classification key explicit: 'I decided to put them in something that already tells you if you are going to have an answer finite, zero, divergent or infinity'."