Symplectic Structures on Spaces of Polygons

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ABSTRACT

Symplectic Structures on Spaces of Polygons

Tuan Nguyen

A symplectic form on a smooth manifold is a differential 2-form on pairs of tangent vectors of the manifold which is closed and non-degenerate. A polygon in three-dimensional space is a closed polygonal line, or, more precisely, a polygon of m sides is a map ρ from the set of the first m integers into Euclidean vector space $\mathbb{R}^3$, such that the sum $\rho_1 + \rho_2 + \cdots + \rho_m$ equals to the zero vector. The vectors $\rho_1, \rho_2, \ldots, \rho_m$ are called the sides of the polygon ρ, and their lengths are called its side lengths. All polygons of fixed side lengths make up a space, and one may put symplectic structures on this space. In this text we shall describe two ways to do this; these ways, making use of a method called symplectic reduction, are due to Haussmann-Knutson [5] and independently to Kapovich-Millson [7], and have been shown to be equivalent by Hausmann-Knutson [5]. We begin in the first chapter with a compilation of the necessary definitions and results of group action and symplectic manifolds, including Hamiltonian action and symplectic reduction. In the second chapter we define precisely the space of polygons and describe the aforementioned symplectic forms on them.

There may be some group action on a symplectic manifold. If the group is a torus whose dimension is half of that of the manifold, and if the action is an effective Hamiltonian action, then the manifold corresponds to a figure in three-dimensional space called a Delzant polytope; in fact, Delzant polytopes completely classify such manifolds. By a result of Kapovich-Millllson [7], on the space of polygons of m sides of fixed side lengths, one may construct such a torus action if the

$m-3$ diagonals of the polygons, namely the lengths $|\rho_1+\rho_2|, \ldots, |\rho_1+\cdots+\rho_{m-2}|$, do no vanish. The space of polygons then corresponds to a Delzant polytope, and may then be identified with other symplectic manifolds corresponding to the same polytope. We shall describe in this text the case where polygons have 4 sides or 5 sides. Thus, in continuation of the second chapter, we start in the third chapter a compilation of necessary definitons and results on torus action and Delzant polytopes. Because in the case of polygons of 5 sides there appears an operation on the Delzant polytopes called blow-up, we describe briefly this concept in the same chapter. In the fourth chapter we describe torus action on the space of polygons due to Kapovich-Millson [7]; this action means geometrically rotations about the diagonals, and this idea is also described in the same section. We then apply the results to the case of polygons of 4 or 5 sides.

Finally, we compile in the appendix a more detailed list of certain definitions and results that appear in the text.

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Chapter 1

Symplectic structures

The main concepts in this chapter are those of a symplectic manifold and symplectic reduction. Example (1.8) is central to the result of section (2.2), in which a symplectic structure is defined for the space of polygons of fixed side lengths, as described in Hausmann-Knutson [5]. It is the culmination of examples (1.2), (1.5), (1.6), and (1.7).

1.1 Group action

In this section we give a concise compilation of the definitions of a Lie group and Lie algebra and of related concepts such as group action and the exponential map. The materials here is standard and is elaborated in relevant chapters in [4], [1], and [8].

A **Lie group** G is a smooth manifold in which elements satisfy the group axioms, namely for any elements g, h, and k of G, there exist a multiplication operation, denoted $(\cdot)$, such that:

1. $g \cdot h$ is an element of G;
2. $(g \cdot h) \cdot k = g \cdot (h \cdot k)$;
3. there exists an identity element, denoted 1, such that $1 \cdot g = g$;
4. for each g there exists an inverse element, denoted g^{-1}, such that $gg^{-1} = 1$;

moreover, the multiplication is assumed to be smooth in the sense that the map

$$\begin{array}{ccc} G \times G & \rightarrow & G \\ (g, h) & \mapsto & g \cdot h \end{array}$$

is a smooth map between manifolds.

The tangent space to the Lie group G at the identity is called its **Lie algebra** and is denoted by $\mathfrak{g}$. The name derives from the fact that a multiplication that satisfies a specific set of axioms may be defined for vectors in this tangent space.

We can pass from the Lie algebra $\mathfrak{g}$ to its Lie group G by the **exponential map**

$$\exp : \mathfrak{g} \to G, \tag{1.1}$$

which is characterized, for any element ξ of $\mathfrak{g}$, by the properties:

1. For any real numbers s and t, the usual power law holds, namely,

$$\exp(t\xi) \cdot \exp(s\xi) = \exp\big((t+s)\xi\big);$$

2. as the real number t varies, $\exp(t\xi)$ traces out a curve in G, and the tangent vector to this curve at $t = 0$ is ξ. In other words,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(t\xi) = \xi.$$

We shall work exclusively with matrix groups. For these, both the Lie groups and their Lie algbras consist of matrices, and the exponential map is the matrix series, that is,

$$\exp(\xi) = 1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \cdots . \tag{1.2}$$

Let G be a Lie group. G is said to be an **action** on a manifold M if to each of its elements g corresponds a diffeomorphism $\Phi(g)$ on M satisfying, for all elements g and h of G, the following conditions

1. $\Phi(gh) = \Phi(g)\Phi(h)$;

2. $\Phi(g^{-1}) = \big(\Phi(g)\big)^{-1}$;

3. $\Phi(1)$ is the identity diffeomorphism (where 1 is the unit element of the Lie group G).

The diffeomorphism $\Phi(g)$ that correponds to the element g of G will be written simply as Φ_g or g, and its value at a point x in M will be written $g \cdot x$.

The action of the Lie group G on the manifold M defines for each element in the Lie algebra $\mathfrak{g}$ of G a vector field on M. If ξ is an element of $\mathfrak{g}$, then as t

varies, $\exp(t\xi)$ traces out a curve in G, and if x is a point of M, then as t varies, $\exp(t\xi)\cdot x$ traces out a curve in M. The tangent vector to the latter curve at $t=0$ is associated to ξ. In this way, if T_xM denotes the tangent space to M at x, vector fields on M are obtained from elements of $\mathfrak{g}$ by the rule:

$$\begin{array}{lcl} \mathfrak{g} & \to & T_xM \\ \xi & \mapsto & \xi_M(x) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\big(\exp(t\xi)\cdot x\big), \end{array} \tag{1.3}$$

where ξ_M denotes the vector field associated to ξ.

1.2 Symplectic manifolds

The definitions of a symplectic form and a symplectic manifold given in this section can be found in [4] or [1]. These references also contain discussions of the standard theorem (1.1). Example (1.2) is mentioned in [7] and is worked out in detail in this section.

A smooth manifold M is called a **symplectic manifold** if on it there is a **symplectic form** ω, which by definition is a differential 2-form with the properties:

1. $\mathrm{d}\omega = 0$ (ω is closed);
2. if n denotes the dimension of the manifold M, then $\omega^n \neq 0$ at every point x of M (ω is non-degenerate everywhere).

Example 1.1. *The n-dimensional complex space $\mathbb{C}^n$ is a symplectic manifold with the symplectic form*

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

where the (x^i, y^i) are the coordinates of the copies of $\mathbb{C}$. We note that here $\mathrm{d}\omega_0$ equals 0, and moreover,

$$\omega_0^n = \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n$$

which shows that the form is non-degenerate everywhere.

The application of the symplectic form on two arbitrary tangent vectors of $\mathbb{C}^n$,

$$\xi_1 = \sum_{j=1}^{n}(x_1^j\frac{\partial}{\partial x^j} + y_1^j\frac{\partial}{\partial y^j}), \quad \xi_2 = \sum_{j=1}^{n}(x_2^j\frac{\partial}{\partial x^j} + y_2^j\frac{\partial}{\partial y^j}),$$

leads to the formula

$$\omega(\xi_1, \xi_2) = \sum_{k=1}^{n} (x_1^k y_2^k - x_2^k y_1^k).$$

The following example is mentioned in [5]. We present it in some detail in order to develop it further in later sections.

Example 1.2. *Let $M_{m\times 2}(\mathbb{C})$ denote the space of all complex m-by-2 matrices*

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix}.$$

Every tangent space to $M_{m\times 2}(\mathbb{C})$ may be identified with $M_{m\times 2}(\mathbb{C})$ itself. Let u and v be two matrices in the space, and denote by v^* the conjugate tranpose of v. The trace of the product uv^* is a complex number and so its imaginary part can be taken. The form

$$\omega(u, v) := \mathrm{Im}\big(\mathrm{tr}(uv^*)\big)$$

is a symplectic form on $M_{m\times 2}(\mathbb{C})$. For, by writing the columns of a matrix one after another in a row, every matrix corresponds to a vector in the complex space $\mathbb{C}^n$, with $n = 2m$. Let

$$\xi_1 = (\xi_1^1, \xi_1^2, \cdots, \xi_1^n), \quad \xi_2 = (\xi_2^1, \xi_2^2, \cdots, \xi_2^n)$$

be the complex vectors corresponding to the matrices u and v. The form

$$\langle u, v\rangle := \mathrm{tr}(uv^*)$$

is then equivalent to the form

$$\langle \xi_1, \xi_2\rangle := \sum_{k=1}^{n} \xi_1^k \bar{\xi_2}^k.$$

We now write down every complex component in terms of its real and imaginary part, namely,

$$\xi_\alpha^j = x_\alpha^j + \mathrm{i} y_\alpha^j, \quad j = 1, 2, \ldots, n; \alpha = 1, 2;$$

when this is put into the form above, we obtain

$$\langle \xi_1, \xi_2\rangle = \sum_{k=1}^{n} (x_1^k + \mathrm{i} y_1^k)(x_2^k - \mathrm{i} y_2^k) = \sum_{k=1}^{n} (x_1^k x_2^k + y_1^k y_2^k) + \mathrm{i} \sum_{k=1}^{n} (y_1^k x_2^k - x_1^k y_2^k).$$

From the end of the previous example the imaginary part of the form can now be seen to be a symplectic form.

Theorem 1.1 (Darboux). *There always exists a local choice of coordinate system such that locally any symplectic manifold transforms to the standard symplectic manifold $\mathbb{C}^n$ with the symplectic form $\omega_0 = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}y^i$.*

1.3 Hamiltonian action

The concepts of adjoint and co-adjoint actions, symplectomorphism, symplectic action, moment map and Hamiltonian action can be consulted in [4], [1], or [9]. Examples (1.5) and (1.6) are contained in [5] but certain aspects which will be needed in later sections are discussed in some detail in this section.

Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. Let us recall that $\mathfrak{g}$ is the tangent space of G at the identity.

Each element g of G may be made to correspond to a linear map on $\mathfrak{g}$. We take g and define a diffeomorphism on G by means of the formula

$$h \mapsto ghg^{-1}$$

for every h in G. The differential of this map at the identity 1 of G is then a linear map from $\mathfrak{g}$ onto itself which we denote by

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}. \tag{1.4}$$

The association of g with the linear map Ad_g is called the **adjoint action** of the Lie group on its Lie algebra.

As is known from linear algebra, any linear map of vector spaces produces a dual map of the dual vector spaces. If $A : V \to W$ denotes a linear map of vector spaces, its dual $A^* : W^* \to V^*$ is the map which is defined by composition, that is, the linear function α in W^* is brought to the linear function $\alpha \circ A$ in V^*.

$\mathfrak{g}$ is a vector space and so has a dual space which is usually denoted $\mathfrak{g}^*$. The adjoint action above therefore has a dual which is denoted by Ad_g^*; this is a linear map from the dual space $\mathfrak{g}^*$ into itself. We now associate to each element g of G the linear map

$$\mathrm{Ad}^*_{g^{-1}} : \mathfrak{g}^* \to \mathfrak{g}^* \tag{1.5}$$

and call this association the **co-adjoint action** of G on the dual space $\mathfrak{g}^*$.

Example 1.3. *We consider the Lie group of invertible complex matrices $GL(n, \mathbb{C})$ which has as its Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ all complex n-by-n matrices.*

The discussion below, however, applies to all matrix Lie groups. We therefore write G, $\mathfrak{g}$, and $\mathfrak{g}^*$ in places of $GL(n, \mathbb{C})$, $\mathfrak{gl}(n, \mathbb{C})$, and $\mathfrak{gl}^*(n, \mathbb{C})$.

Each matrix g in the Lie group G defines a diffeomorphism on G by means of matrix multiplication of the form

$$h \mapsto ghg^{-1}$$

for every matrix h of G. Taking the differential of this map at the identity we obtain the adjoint action

$$\begin{array}{rcl} \mathrm{Ad}_g : \mathfrak{g} & \to & \mathfrak{g} \\ \xi & \mapsto & g\xi g^{-1}. \end{array}$$

The coadjoint action can now be obtained from the formula (1.4), namely if α is an element of $\mathfrak{g}^*$, then $\mathrm{Ad}^*_{g^{-1}}$ is that map which brings it to the element $\alpha \circ \mathrm{Ad}_{g^{-1}}$ which also belongs to $\mathfrak{g}^*$.

Example 1.4. *The Lie group $SO(3)$ of all 3-by-3 matrices g with $\det g = 1$ and satisfy $g^{-1} = g^T$, where g^T denotes the transpose of g. The Lie algebra $\mathfrak{so}(3)$ consists of all matrices ξ such that $\xi^T = -\xi$.*

The adjoint and coadjoint actions may be obtained as in the example above. We shall, however, interpret these in terms of multiplication of matrices in $SO(3)$ by vectors in the Euclidean space $\mathbb{R}^3$. To do this, we shall use two facts. First, $\mathbb{R}^3$ has a Lie algebra structure which is equivalent to $\mathfrak{so}(3)$ by the correspondence

$$\begin{array}{rcl} c : \mathbb{R}^3 & \to & \mathfrak{so}(3) \\ (x, y, z) & \mapsto & \begin{pmatrix} 0 & -z & -y \\ z & 0 & -x \\ y & x & 0 \end{pmatrix}. \end{array}$$

Second, the Lie algebra $\mathbb{R}^3$ may be identified with its dual by the usual Euclidean scalar product, and that this corresponds perfectly with how $\mathfrak{so}(3)$ may be identified with its dual.

The adjoint action of $SO(3)$ on its Lie algebra $\mathfrak{so}(3)$ is the same as multiplication of matrices in $SO(3)$ by vectors in $\mathbb{R}^3$. For, if g is a matrix in $SO(3)$ and v a vector in $\mathbb{R}^3$, then the product gv corresponds to the element $gc(v)g^{-1}$ by the equation above.

The coadjoint action is also multiplication of matrices by vectors. For, if α is a vector in $\mathbb{R}^3$, it is brought by the coadjoint action to the element $\alpha \circ \mathrm{Ad}_{g^{-1}}$, where g is some matrix in $SO(3)$. Now, if v is a vector in space, we have, by the relation between the scalar product and orthogonal matrices,

$$\alpha \circ \mathrm{Ad}_{g^{-1}}(v) = \alpha \cdot g^{-1} v = \alpha \cdot g^T v = g\alpha \cdot v,$$

A **symplectomorphism** between two symplectic manifolds M_1 and M_2 having symplectic forms ω_1 and ω_2 is a diffeomorphism $\varphi : M_1 \to M_2$ that satisfies the condition $\varphi^* \omega_2 = \omega_1$.

Let M be a symplectic manifold and G a Lie group. Then G is said to be a **symplectic action** on M if it is an action on M, where every element in G is associated to a symplectomorphism on M.

Now suppose that G is a symplectic action on M. Again denote by $\mathfrak{g}$ the Lie algebra of G, and by $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Then a map

$$\phi : M \to \mathfrak{g}^* \tag{1.6}$$

is called a **moment map** if it satisfies, for every element ξ of $\mathfrak{g}$ and every point x of M, the conditions:

1. $\omega(\xi_M, -) = -\mathrm{d}\phi^\xi$, where the empty slot in the form is kept for tangent vectors, and where the real function ϕ^ξ on M is defined by the formula

   $$\phi^\xi(x) := \langle \phi(x), \xi \rangle;$$

   that is, the value of the function at the point x of M is obtained by applying the linear function $\phi(x)$ of $\mathfrak{g}^*$ on the element ξ of $\mathfrak{g}$. Moreover we recall that in the above formula ξ_M denotes the vector field obtained from the action of G on M [see definition at the end of section (1.1)].

2. $\phi \circ \Phi_g = \mathrm{Ad}^*_{g^{-1}} \circ \phi$, where Φ_g denotes the diffeomorphism corresponding to the action of the element g of G on M [see section (1.1)]. Therefore the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] & \mathfrak{g}^* \\ M \arrow[u, "\Phi_g"] \arrow[r, "\phi"] & \mathfrak{g}^* \arrow[u, "\mathrm{Ad}^*_{g^{-1}}"'] \end{tikzcd} \tag{1.7}$$

(This property is called the G-**equivariant** property.)

The action of G on M is said to be a **Hamiltonian action** whenever it has a moment map.

The following example is a development of example (1.2). It will be used later sections.

Example 1.5. *Let us recall that a square matrix α is called Hermitian if it equals to its conjugate transpose α^*. Let H_m denote the set of Hermitian matrices.*

Also, a matrix is called unitary if its inverse equals its conjugate transpose. Let U_m denote the set of unitary matrices. Moreover, we shall denote by $\mathfrak{u}_m$ the Lie algebra of U_m, and by $\mathfrak{u}_m^*$ the dual of $\mathfrak{u}_m$. It is known that $\mathfrak{u}_m^*$ may be identified with $\mathfrak{u}_m$ by the scalar product

$$\langle \xi, \eta \rangle = \frac{\mathrm{i}}{2}\mathrm{tr}(\xi^* \eta)$$

of any two unitary matrices; here tr *means the trace of a matrix.*

Each Hermitian matrix α may be made to correspond to an element of $\mathfrak{u}_m^*$ by multiplying it to any element ξ of $\mathfrak{u}_m$ and taking trace of the resulting product, namely,

$$\alpha(\xi) = \frac{\mathrm{i}}{2}\mathrm{tr}(\alpha\xi). \tag{1.8}$$

In this example we shall identify H_m with $\mathfrak{u}_m^*$.

Let $M_{m\times 2}(\mathbb{C})$ denote the space of all complex m-by-2 matrices

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix}.$$

We shall denote the elements of $M_{m\times 2}(\mathbb{C})$ by the letters $x, y, \ldots$ and so on. As discussed in example (1.2), the space $M_{m\times 2}(\mathbb{C})$ may be identified to every one of its tangent spaces, and moreover it is a symplectic manifold with the symplectic form being the imaginary part of the trace, more precisely,

$$\omega(x, y) := \mathrm{Im}\big(\mathrm{tr}(xy^*)\big).$$

The matrix product

$$xx^*$$

is equal to its conjugate transpose and so is an m-by-m Hermitian matrix. We shall verify below that the map

$$\begin{array}{rcl} \phi : M_{m\times 2}(\mathbb{C}) & \to & H_m \\ x & \mapsto & xx^* \end{array} \tag{1.9}$$

is a moment map for the action of U_m on $M_{m\times 2}(\mathbb{C})$, where the action is left matrix multiplication.

We shall use the symbols M, $\mathfrak{g}$, and $\mathfrak{g}^*$ in places of $M_{m\times 2}, \mathfrak{u}_m$, and $\mathfrak{u}_m^*$. Moreover, the diffeomorphism on M corresponding to the unitary matrix g will be denoted by Φ_g.

Thus, we verify the following two equations.

1. $\omega(\xi_M, -) = -\mathrm{d}\phi^\xi$. *If ξ is an element of $\mathfrak{g}$, the vector field ξ_M assigns to the point x of the manifold M the tangent vector [see equation (1.3)]*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\big(\exp(t\xi)\cdot x\big) = \xi x.$$

To evaluate the symplectic form ω on tangent vectors, we first note that for matrices x and y, the numbers $\mathrm{tr}(xy^*)$ and $\mathrm{tr}(yx^*)$ are complex conjugates, and so the imaginary part can be taken:

$$\omega(x,y) = \mathrm{Im}\big(\mathrm{tr}(xy^*)\big) = -\frac{\mathrm{i}}{2}\mathrm{tr}(xy^* - yx^*).$$

Now for tangent vectors ξx and y, we have

$$\begin{array}{rcl} \omega(\xi x, y) & = & -\frac{\mathrm{i}}{2}\mathrm{tr}(\xi xy^* - yx^*\xi^*) \\ & = & -\frac{\mathrm{i}}{2}\mathrm{tr}(\xi xy^* + yx^*\xi) \\ & = & -\frac{\mathrm{i}}{2}\mathrm{tr}(xy^*\xi + yx^*\xi) \\ & = & -\frac{\mathrm{i}}{2}\mathrm{tr}\big((xy^* + yx^*)\xi\big). \end{array}$$

On the other hand, the function

$$\phi^\xi : M \to \mathbb{R}$$

works according to the formula

$$\phi^\xi(x) = \frac{\mathrm{i}}{2}\mathrm{tr}(xx^*\xi).$$

Taking the differential of the function and applying it to the tangent vector y, we obtain

$$\mathrm{d}\phi^\xi_x(y) = \frac{\mathrm{i}}{2}\mathrm{tr}\big((yx^* + xy^*)\xi\big),$$

and so the equation claimed at the beginning verifies.

2. $\phi \circ \Phi_g = \mathrm{Ad}^*_{g^*} \circ \phi$ *(here* $g^{-1} = g^*$ *because the matrix is unitary).*

If x *is a point of the manifold* M*, it is brought by the composition* $\phi \circ \Phi_g$ *to the element* $(gx)(gx)^*$ *of* $\mathfrak{g}^*$*. If now* ξ *is an element of the Lie algebra* $\mathfrak{g}$*, then by equation (1.8),*

$$\big(gx(gx)^*\big)(\xi) = \mathrm{tr}\big(gx(gx)^*\xi\big).$$

On the other hand, the same point x *is brought by the composition* $\mathrm{Ad}^*_{g^*} \circ \phi$ *to the element* $xx^* \circ \mathrm{Ad}_{g^*}$ *of* $\mathfrak{g}^*$ *[see discussion of adjoint and coadjoint action for matrix group in example (1.3)], and this function applies on the element* ξ *of* $\mathfrak{g}$ *as follows:*

$$\begin{array}{rcl} xx^* \circ \mathrm{Ad}_{g^*}(\xi) & = & xx^*(g^*\xi g) \\ & = & \mathrm{tr}(xx^*g^*\xi g) \\ & = & (gxx^*g^*\xi) \\ & = & \mathrm{tr}\big(gx(gx)^*\xi\big). \end{array}$$

Hence the second equation verifies.

We conclude that the action of the unitary group on the space of complex m*-by-*2 *matrices is Hamiltonian.*

This example will similarly be used in later sections.

Example 1.6. *Using the notations of the above example, let* $M_{m\times 2}(\mathbb{C})$ *denote the symplectic manifold of all complex* m*-by-*2 *matrices*

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix},$$

the symplectic form being

$$\omega(x, y) := \mathrm{Im}\big(\mathrm{tr}(xy^*)\big).$$

Let H_2 *denote the space of* 2*-by-*2 *Hermitian matrices. Similarly to the example above, the map (below* 1 *denotes the identity matrix)*

$$\begin{array}{rcl} \psi : M_{m\times 2}(\mathbb{C}) & \to & H_2 \\ x & \mapsto & x^*x - 1 \end{array}$$

is a moment map for the action of the Lie group U_2*, where the Lie group action is right matrix multiplication.*

1.4 Symplectic reduction

Symplectic reduction will be presented in the form of a theorem, namely theorem (1.2). References [4], [1], and [2] contain the theorem and the justification of the various conditions used in the theorem. Examples (1.7) and (1.8) are due to Hausmann-Knutson [5].

Let a Lie group G be a Hamiltonian action on a symplectic manifold M with symplectic form ω, where the moment map for the action is denoted by the symbol

$$\phi : M \to \mathfrak{g}^*,$$

with $\mathfrak{g}^*$ being the dual space of the Lie algebra of the Lie group G.

Moreover, suppose that α is a **regular value** of ϕ, that is, α is an element of $\mathfrak{g}^*$ such that the differential $\mathrm{d}\phi$ is surjective at every point x of $\phi^{-1}(\alpha)$. With this assumption the space $\phi^{-1}(\alpha)$ is a submanifold of the manifold M.

Also, denote by G_α the group whose elements are those of G that fix α under the coadjoint action [see (1.5) for definition of coadjoint action], that is, G_α consists of all elements g of the Lie group G such that

$$\mathrm{Ad}^*_{g^{-1}}(\alpha) = \alpha.$$

With this condition, the space

$$M_\alpha := \phi^{-1}(\alpha)/G_\alpha$$

is well-defined.

Finally, suppose that G_α acts freely and properly on $\phi^{-1}(\alpha)$, where, by definition, an action is called **free** if the only diffeomorphism having any fixed point is the one corresponding to the identity element of the group, and it is called **proper** if the inverse image of the action of a compact set is also a compact set. Following this assmption, the space M_α defined above is a manifold.

Theorem 1.2 (Marsden-Weinstein). *The manifold*

$$M_\alpha = \phi^{-1}(\alpha)/G_\alpha$$

inherits a symplectic form ω_α from the symplectic from ω on the manifold M in the following way. Let

$$\pi : \phi^{-1}(\alpha) \to M_\alpha$$

be the projection map as shown in this diagram (here i *denotes the injection map)*

$$\begin{tikzcd} \phi^{-1}(\alpha) \arrow[r, "\mathrm{i}"] \arrow[d, "\pi"'] & M . \\ M_\alpha = \phi^{-1}(\alpha)/G_\alpha \end{tikzcd} \tag{1.10}$$

Then, a point p *of the manifold* M_α *can be the image under projection of any point* x *in the set* $\pi^{-1}(p)$*, and if* u *and* v *are tangent vectors to* M_α *at the point* p*, then they are images of some tangent vectors to the manifold* $\phi^{-1}(\alpha)$ *at* x*, say* u' *and* v'*. The value*

$$\omega(u', v')$$

is independent of the choice of the point x *in* $\pi^{-1}(p)$ *and also of the accompanying tangent vectors* u' *and* v'*, and therefore we can set*

$$\omega_\alpha(u, v) := \omega(u', v').$$

Looking at the diagram above, we may say that ω_α *is the symplectic form on* M_α *that satisfies the equation*

$$\pi_\alpha^* \omega_\alpha = \mathrm{i}_\alpha^* \omega.$$

The following lemma will be used in the example that follows it. Let $\mathbb{G}_2(\mathbb{C}^m)$ denote the Grassmannian space of complex planes in $\mathbb{C}^m$, in other words the space of all 2-dimensional complex linear subspaces in $\mathbb{C}^m$, and let $\mathbb{V}_2(\mathbb{C}^m)$ denote the Stiefel space of all orthogonal 2-frames in $\mathbb{C}^m$, in other words the set of all complex m-by-2 matrices

$$(\mathbf{a}, \mathbf{b}) := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \tag{1.11}$$

such that both column vectors have unit length, and moreover that they are orthogonal, that is,

$$\langle \mathbf{a}, \mathbf{b} \rangle := \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_m b_m = 0,$$

$$|\mathbf{a}| = \langle \mathbf{a}, \mathbf{a} \rangle^{1/2} = 1, \qquad |\mathbf{b}| = \langle \mathbf{b}, \mathbf{b} \rangle^{1/2} = 1.$$

Lemma 1.1. *Let* U_2 *denote the group of* 2*-by-*2 *unitary matrices (a unitary matrix* U *is a matrix such that*

$$U^* U = U U^* = 1 \tag{1.12}$$

where U^* denote the conjugate transpose of the matrix U and 1 the identity matrix). The Grassmannian space $\mathbb{G}_2(\mathbb{C}^m)$ is isomorphic to the space $\mathbb{V}_2/U_2$, where a unitary matrix acts on an frame in the Stiefel space by multiplication on the right.

Proof. By the Gram-Schmidt orthogonalization process, in every 2-dimensional linear subspace of $\mathbb{C}^m$, which is an element of $\mathbb{G}_2(\mathbb{C}^m)$, an orthonormal 2-frame, which is an element of $\mathbb{V}_2(\mathbb{C}^m)$, can be built from an arbitrary basis of the subspace. On the other hand, if Π is an element of $\mathbb{G}_2(\mathbb{C}^m)$ and $(\mathbf{a}, \mathbf{b})$ an orthonormal 2-frame in Π, the right action of any 2-by-2 unitary matrix U on the frame $(\mathbf{a}, \mathbf{b})$ is

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11}a_1 + u_{21}b_1 & u_{12}a_1 + u_{22}b_1 \\ u_{11}a_2 + u_{21}b_2 & u_{12}a_2 + u_{22}b_2 \\ \vdots & \vdots \\ u_{11}a_m + u_{21}b_m & u_{12}a_m + u_{22}b_m \end{pmatrix}$$

or, written more concisely,

$$(\mathbf{a}, \mathbf{b})U = (u_{11}\mathbf{a} + u_{21}\mathbf{b}, u_{11}\mathbf{a} + u_{22}\mathbf{b}). \tag{1.13}$$

This action leaves unchanged the subspace spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. Indeed, the vectors

$$u_{11}\mathbf{a} + u_{21}\mathbf{b}, \qquad u_{11}\mathbf{a} + u_{22}\mathbf{b}$$

are linearly independent, because any vector in Π has the form $\alpha\mathbf{a} + \beta\mathbf{b}$ for some complex numbers α and β, and then the equation

$$x(u_{11}\mathbf{a} + u_{21}\mathbf{b}) + y(u_{11}\mathbf{a} + u_{22}\mathbf{b}) = \alpha\mathbf{a} + \beta\mathbf{b}$$

always has a solution for x and y, as we can see by multiplying the conjugate transpose of the coefficient matrix on both sides,

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^* \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Therefore the space of orbits of the right action $\mathbb{V}_2(\mathbb{C}^m)/\mathrm{U}$ is the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$. ☐

The following example is a continuation of example (1.5) and (1.6). It will be used in later section.

Example 1.7. *We recall here the notations and results of example (1.6). There, the symbol $M_2(\mathbb{C})$ denotes the space of complex m-by-2 matrices*

$$(\mathbf{a}, \mathbf{b}) := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m. \end{pmatrix}$$

As discussed in that example, $M_{m\times 2}(\mathbb{C})$ is a symplectic manifold with symplectic form

$$\omega(x, y) := \mathrm{Im}\big(\mathrm{tr}(xy^*)\big),$$

and the action of the unitary group U_2 on it by right multiplication is a Hamiltonian action with the moment map

$$\begin{array}{rcl} \psi : M_{m\times 2}(\mathbb{C}) & \to & H_2 \\ (\mathbf{a}, \mathbf{b}) & \mapsto & (\mathbf{a}, \mathbf{b})^*(\mathbf{a}, \mathbf{b}) - 1, \end{array}$$

where 1 denotes the identity matrix, and H_2 the space of 2-by-2 Hermitian matrices (which can be identified with the dual of the Lie algebra of U_2).

Now, the inverse image $\psi^{-1}(0)$ of the zero matrix 0 consists of all matrices satisfying the equation

$$(\mathbf{a}, \mathbf{b})^*(\mathbf{a}, \mathbf{b}) = 1,$$

and therefore is the Stiefel space $\mathbb{V}_2(\mathbb{C}^m)$ [see equation (1.11) for the definition of the Stiefel space]. We conclude, from the theorem in this section, that the space $\mathbb{V}_2(\mathbb{C}^m)/U_2$, which is also the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$, inherits a symplectic form from that on $M_{m\times 2}(\mathbb{C})$, the way the symplectic form is obtained is described in the symplectic reduction theorem (1.2) above in this section.

Moreover, using the fact just obtained together with the result of example (1.5) [see equation (1.9) in the same example], we conclude that the moment map in that example (here H_m is the space of Hermitian matrices of size m),

$$\begin{array}{rcl} \phi : M_{m\times 2} & \to & H_m \\ (\mathbf{a}, \mathbf{b}) & \mapsto & (\mathbf{a}, \mathbf{b})(\mathbf{a}, \mathbf{b})^*, \end{array} \tag{1.14}$$

gives rise to a moment map for the action of the unitary group U_m on the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$, the action being left matrix multiplication. In other words, we have a moment map

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to H_m. \tag{1.15}$$

Let G be a Lie group and H a subgroup of G. The Lie algebra $\mathfrak{h}$ of H is a subspace of the Lie algebra $\mathfrak{g}$ of G, and the dual $\mathfrak{h}^*$ a subspace of $\mathfrak{g}^*$. The following lemma can be found in [1].

Lemma 1.2. *Let ϕ be the moment map for the Hamiltonian action of a Lie group G on a symplectic manifold M. Then, for any subgroup H of G, the composition of ϕ and the canonical projection*

$$M \to \mathfrak{g}^* \to \mathfrak{h}^*$$

is a moment map for the induced action.

Example 1.8. *In example (1.7) we obtained the moment map*

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to H_m$$

for the action of the unitary group U_m on the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$. The diagonal unitary matrices is a subgroup of U_m; the projection from the dual of its Lie algebra to $\mathfrak{u}_m^*$ is the map from H_m into $\mathbb{R}^m$ picking from each matrix its diagonal entries. By the lemma above, we obtain a moment map

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to \mathbb{R}^m \tag{1.16}$$

for the action of the diagonal unitary matrices on the Grassmannian.

Chapter 2

Symplectic structures on the space of polygons

In this chapter we define precisely the space of polygon of fixed side lengths. There are two ways to put a symplectic structure on this space, due to Haussmann-Knutson [5] and independently to Kapovich-Millson [7]; we give an exposition of these two methods in this chapter.

2.1 Polygons in space

The following general discussion about polygons is contained in [5] or [7].

A polygon of m sides in the three-dimensional Euclidean space $\mathbb{R}^3$ is a set of m vectors in the space that add up to zero; in other words, a polygon is a map

$$\rho : \{1, 2, \dots, m\} \to \mathbb{R}^3$$

such that

$$\rho_1 + \rho_2 + \cdots + \rho_m = 0.$$

The vectors ρ_1, ρ_2, $\dots$, ρ_m are called the sides of the polygon ρ. As these sides are vectors in the Euclidean space $\mathbb{R}^3$, we may compute their lengths by taking the Euclidean scalar product of each side with itself; the length of the side ρ_i will be denoted $|\rho_i|$.

We shall exclude in all of our discussion the polygon whose every side is zero.

By definition polygons are identified up to translations. Let us also note that vectors in $\mathbb{R}^3$ may be rotated, and there is a sense in each rotation, depending on whether the rotation matrix has determinant 1 or -1. We therefore may identify polygons up to orientation-preserving rotations, or just up to rotations, as we wish; orientation-preserving rotations will be called **proper rotations**.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an element of $\mathbb{R}^m$. We use the symbol ${}^m\mathcal{P}^3_+(\alpha)$ to denote the space of polygons with m sides in $\mathbb{R}^3$, all with side lengths $\alpha_1, \ldots, \alpha_m$; here we use the plus sign $(+)$ to identify polygons that are proper rotations of each other.

There are two apparently different ways to put a symplectic structure on the space ${}^m\mathcal{P}^3_+(\alpha)$. These two ways have been shown to be essentially equivalent by Haussmann and Knutson in the paper [5] . We shall only describe the two ways; their comparison is shown in the aforementioned paper.

2.2 The symplectic structure by unitary action

Lemmas (2.1) and (2.2) and further discussion about quarternions can be found in [8]. Otherwise, this section is an exposition of the results that are due to Hausmann-Knutson [5].

2.2.1 Preliminaries

Below are some additional facts about polygons that shall be used in the main theorems of this section. A polygon may be scaled by a positive factor, and by this we mean that every side of the polygon may be multiplied by the same positive scalar; the orientation of the polygon remains unchanged after such an operation. We have used the symbol ${}^m\mathcal{P}^3_+(\alpha)$ to denote the space of polygons of given side lengths, up to translations and proper rotations. We now drop the letter α in the symbol, and thereby introduce a new symbol, ${}^m\mathcal{P}^3_+$. This symbol will denote the space of polygons of m sides in the space $\mathbb{R}^3$, identified up to translations, proper rotations, and scaling.

Every polygon

$$\rho : \{1, 2, \ldots, m\} \to \mathbb{R}^3$$

has a perimeter which is defined as the sum of its side lengths:

$$|\rho| := |\rho_1| + |\rho_2| + \cdots + |\rho_m|.$$

We may multiply every side of a polygon by the same appropriate positive scalar to obtain a scaled polygon having perimeter equal to 2. As the space ${}^m\mathcal{P}^3_+$ consists of polygons up to scaling, every element of ${}^m\mathcal{P}^3_+$ has a representative of perimeter equal to 2. The **length map**

$$l : {}^m\mathcal{P}^3_+ \to \mathbb{R}^m \tag{2.1}$$

is defined as the map which assigns to each element of ${}^m\mathcal{P}^3_+$ the side lengths of the representative whose perimeter equals to 2.

We now describe a way to build a map, say

$$\Phi : \mathbb{G}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3_+, \tag{2.2}$$

from the Grassmannian into the space ${}^m\mathcal{P}^3_+$. The construction makes use of the results of the following proposition and lemmas.

Let the symbol ${}^m\mathcal{P}^3$ denote the space of polygons of m sides in the $\mathbb{R}^3$, identified up to translation, scaling, and rotations [and not proper rotations, that is why the plus sign $+$ has been dropped; for a discussion of rotations see the beginning of section (2.1), and for a discussion of scaling see the discussion at the beginning of this section].

Proposition 2.1. *There exists a surjective map, say*

$$\Phi_v : \mathbb{V}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3,$$

from the Stiefel space $\mathbb{V}_2(\mathbb{C}^m)$ *onto the space* ${}^m\mathcal{P}^3$.

We construct this map Φ_v by using an additional map, called the Hopf map and denoted $\phi : \mathbb{H} \to \mathbb{R}^3$, from the quarternions into the three-dimensional Euclidean space, which we now define. Let $\mathbb{H}$ be the algebra of quaternions, that is, the four-dimensional real vector space consisting of vectors of the form

$$q = a + bi + cj + dk$$

where a, b, c, and d are real numbers, and i, j, and k denote the basis vectors; this vector space moreover has an associative bilinear multiplication defined by the following rules: for real numbers ζ, η and vectors q_1, q_2, q_3,

1. $(\zeta q_1)(\eta q_2) = (\zeta\eta)q_1 q_2$;

2. $(q_1 q_2) q_3 = q_1 (q_2 q_3)$;

3. for the basis vectors,

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

$$i^2 = j^2 = k^2 = -1.$$

On the algbra of quarternions, the conjugation operation

$$q = a + bi + cj + dk \quad \mapsto \quad \bar{q} = a - bi - cj - dk$$

has the following properties

$$\overline{q_1 + q_2} = \bar{q}_1 + \bar{q}_2, \qquad \overline{q_1 \cdot q_2} = \bar{q}_2 \cdot \bar{q}_1$$

and leads to the definition of the norm of a quarternion,

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2.$$

Denote by $\mathbb{H}_0$ the space of imaginary quarternions, namely those of the form

$$\bar{q} = -q, \qquad q = xi + yj + zk.$$

In this space,

$$|q|^2 = q\bar{q} = x^2 + y^2 + z^2.$$

Therefore, $\mathbb{H}_0$ can be identified with the three-dimensional Euclidean vector space $\mathbb{R}^3$.

The **Hopf map**

$$\phi : \mathbb{H} \to \mathbb{H}_0 \simeq \mathbb{R}^3 \tag{2.3}$$

is defined by the formula

$$\phi : q \mapsto \bar{q} i q.$$

Let us obtain at this point a detailed formula for the Hopf map. If q is a quarternion, then we may write

$$q = a + bi + cj + dk = (a + bi) + (c + di)j,$$

and therefore any quarternion q may be written in the form

$$q = u + vj$$

for some complex numbers u and v. Using this form in the Hopf map we obtain the formula

$$\phi : u + vj \mapsto (\bar{u} - j\bar{v})i(u + vj) = i(\bar{u} + j\bar{v})(u + vj) = i(|u|^2 - |v|^2 + 2\bar{u}vj). \quad (2.4)$$

Proof of proposition (2.1). The map

$$\Phi_v : \mathbb{V}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3$$

is defined by making use of the Hopf map defined by formula (2.4) in the following way. We take an orthonormal frame and assign to it a set of vectors by the rule:

$$(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \mapsto (\phi(a_1 + b_1 j), \phi(a_2 + b_2 j), \ldots, \phi(a_m + b_m j)). \quad (2.5)$$

We shall verify below that the image of the frame $(\mathbf{a}, \mathbf{b})$ under the above map is a polygon of m sides having perimeter equal to 2. This polygon belongs to some class in the space ${}^m\mathcal{P}^3$; we take this class as the image of the frame $(\mathbf{a}, \mathbf{b})$ under Φ_v, thereby define Φ_v as a map from $\mathbb{V}_2(\mathbb{C}^m)$ into ${}^m\mathcal{P}^3$. The surjective property of Φ_v will be shown at the end of the proof.

The element

$$(\phi(a_1 + b_1 j), \phi(a_2 + b_2 j), \ldots, \phi(a_m + b_m j))$$

is a polygon with perimeter equal 2. Indeed, if $(\mathbf{a}, \mathbf{b})$ is a frame that belongs to $\mathbb{V}_2(\mathbb{C}^m)$, then

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{r=1}^{m} \bar{a}_r b_r = 0, \quad |\mathbf{a}| = \sum_{r=1}^{m} |a_r|^2 = 1, \quad |\mathbf{b}| = \sum_{r=1}^{m} |b_r|^2 = 1;$$

these properties, with the use of the formula (2.4), imply

$$\sum_{r=1}^{m} \phi(a_r + b_r j) = i(\sum_{r=1}^{m} |a_r|^2 - \sum_{r=1}^{m} |b_r|^2 + 2\sum_{r=1}^{m} \bar{a}_r b_r j) = 0,$$

$$|\phi(a_r + b_r j)|^2 = (|a_r|^2 + |b_r|^2)^2, \text{ or } \sum_{r=1}^{m} |\phi(a_r + b_r j)| = \sum_{r=1}^{m} |a_r|^2 + \sum_{r=1}^{m} |b_r|^2 = 2.$$

We now show that Φ_v is surjective or, specifically, given an arbitrary polygon of m sides whose the r-th side is the vector (x_r, y_r, z_r), there is a frame in $\mathbb{V}_2(\mathbb{C}^m)$

which will be mapped to it under the map (2.5). First, for a quarternion $a_r + b_r j$, the Hopf formula (2.4) shows that

$$\phi(a_r + b_r j) = i[(|a_r|^2 - |b_r|^2) + 2\bar{a}_r b_r j].$$

We shall write the complex numbers involved as $a_r = u_r + iv_r$ and $b_r = u'_r + iv'_r$. Then the formula above is equal to

$$\phi(a_r + b_r j) = [u_r^2 + v_r^2 - (u'_r)^2 - (v'_r)^2]i - 2(u_r v'_r - v_r u'_r)j + 2(u_r u'_r + v_r v'_r)k.$$

Moreover, we have the following useful identities:

$$|a_r|^2 = u_r^2 + v_r^2, \quad |b_r|^2 = (u'_r)^2 + (v'_r)^2, \qquad 2\bar{a}_r b_r j = 2(u_r u'_r + v_r v'_r)j + 2(u_r v'_r - v_r u'_r)k. \tag{2.6}$$

Let us look at the system of equations

$$\begin{array}{lcl} u_r^2 + v_r^2 - (u'_r)^2 - (v'_r)^2 & = & x_r \\ -2(u_r v'_r - v_r u'_r) & = & y_r \\ 2(u_r u'_r + v_r v'_r) & = & z_r. \end{array}$$

Since for the given polygon it is true that

$$\sum_{r=1}^m x_r = \sum_{r=1}^m y_r = \sum_{r=1}^m z_r = 0$$

we have

$$\sum_{r=1}^m u_r^2 + v_r^2 - (u'_r)^2 - (v'_r)^2 = \sum_{r=1}^m -2(u_r v'_r - v_r u'_r) = \sum_{r=1}^m 2(u_r u'_r + v_r v'_r) = 0.$$

It then follows from the last identity in (2.6) that $\sum_{r=1}^m 2\bar{a}_r b_r j = 0$, in other words the vectors

$$\mathbf{a} = (a_1, \ldots, a_m), \qquad \mathbf{b} = (b_1, \ldots, b_m)$$

are complex orthogonal. After normalization $(\mathbf{a}, \mathbf{b})$ becomes a frame in $\mathbb{V}_2(\mathbb{C}^m)$ and moreover the first equation in the above system is automatically satisfied. $\square$

Let us recall that a 2-by-2 unitary matrix U is one that satisfies $UU^* = 1$, where U^* denotes the conjugate transpose of U. Let U_2 denote the set of all 2-by-2 unitary matrices. In this set, the unitary relation $U^*U = 1$ implies that each matrix has determinant ± 1. Let SU_2 denote the matrices in U_2 with determinant equal 1.

Lemma 2.1. *SU_2 is isomorphic to the multiplicative group of unit quarternions.*

Proof. First, the unit quarternions forms a multiplicative group because the norm of a quarternion q,

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2,$$

satisfies the relation

$$|q_1 q_2| = |q_1||q_2|.$$

The algebra of quarternions $\mathbb{H}$ can be realized as 2-by-2 matrices in the following way

$$q = a + bi + cj + dk \mapsto A(q) = \begin{pmatrix} a - id & -bi - c \\ -bi + c & a + id \end{pmatrix} \tag{2.7}$$

It is then true that

$$A(q_1 q_2) = A(q_1)A(q_2), \qquad A(q_1 + q_2) = A(q_1) + A(q_2).$$

In particular, by realizing unit quarternions as 2-by-2 matrices, and by setting $x + yj$, where $x = a - id$ and $y = -c - ib$, one sees that

$$|q| = a^2 + b^2 + c^2 + d^2 = |x|^2 + |y|^2$$

and moreover that the matrix corresponding to q is

$$A(q) = \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix}.$$

Therefore $|q| = 1$ implies $A(q)^* A(q) = 1$ and $\det A(q) = 1$. $\square$

Let O_3 consist of all 3-by-3 matrices A with $A^T A = 1$, where A^T denotes the transpose of A. This relation implies that each matrix has determinant 1 or -1. Let SO_3 denote all matrices in O_3 whose determinants equal 1; it will be called the group of proper rotations in $\mathbb{R}^3$.

One recalls that the space of imaginary quarternions consists of the quarternions defined by the relation

$$\bar{q} = -q,$$

namely those having the form $q = bi + cj + dk$. Moreover, the formula

$$|q| = b^2 + c^2 + d^2$$

may be used to identify $\mathbb{H}_0$ with $\mathbb{R}^3$.

Lemma 2.2. *If q is a unit quarternion, then the transformation*

$$\alpha_q : x \mapsto \bar{q}xq, \qquad x \in \mathbb{H}_0 = \mathbb{R}^3 \tag{2.8}$$

is a proper rotation of the space $\mathbb{R}^3$.

Proof. Since $\bar{x} = -x$, the following formula is true:

$$\overline{\alpha_q(x)} = \overline{\bar{q}xq} = \bar{q}\bar{x}q = -\bar{q}xq,$$

which shows that $\bar{q}xq$ is an element of $\mathbb{H}_0$. The mapping α_q is linear and satisfies $\alpha_q(0) = 0$. Since it is true that the length of a vector remains unchanged after the transformation,

$$|\bar{q}xq| = |\bar{q}||x||q| = |x|,$$

the element α_q is determined by a matrix α_q in O_3. The function $f(q) = \det \alpha_q$ is a continuous function of q and takes only two values ± 1. But $f(1) = 1$ and $\mathrm{SU}_2 \approx S^3$ is a connected surface. Therefore, $\det \alpha_1 = 1$ always, and $\alpha_q \in SO_3$ for $|q| = 1$. $\square$

Now the map $\Phi : \mathbb{G}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3_+$ mentioned in formula (2.2) is defined by making use of the map $\Phi_v : \mathbb{V}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3$ defined in proposition (2.1). The latter satisfies the relation

$$\Phi_v\big((\mathbf{a}, \mathbf{b})U\big) = (\bar{U}\phi(a_1 + b_1 j)U, \bar{U}\phi(a_2 + b_2 j)U, \ldots, \bar{U}\phi(a_m + b_m j)U)$$

[see (1.13) for the formula of the action of a 2-by-2 unitary matrix on an orthonormal frame]. This relation is true because, after quarternions have been realized as 2-by-2 matrices [see (2.7)], the Hopf map itself satisfies the formula

$$\phi : qU \mapsto \overline{qU}iqU = \bar{U}\bar{q}iqU = \bar{U}\phi(q)U.$$

By lemmas (2.1) and (2.2), the map $\Phi_v : \mathbb{V}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3$ indeed induces the map $\Phi : \mathbb{G}_2(\mathbb{C}^m) \to {}^m\mathcal{P}^3_+$.

The result of example (1.8) is the deduction of the moment map

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to \mathbb{R}^m,$$

for the action of the diagonal unitary matrices, denoted by U_1^m, on the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$.

Lemma 2.3. *The moment map of example (1.8),*

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to \mathbb{R}^m,$$

is equal to the composition of the length map l in formula (2.1) and the map Φ in formula (2.2).

Proof. Let Π be an element of $\mathbb{G}_2(\mathbb{C}^m)$ with an orthonormal basis $(\mathbf{a}, \mathbf{b})$ [which is an element of $\mathbb{V}_2(\mathbb{C}^m)$; see formula (1.11) and the discussion preceding it for the definitions of these two spaces]. The diagonal of the matrix product

$$(\mathbf{a}, \mathbf{b})(\mathbf{a}, \mathbf{b})^* = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \begin{pmatrix} \bar{a}_1 & \bar{a}_2 \cdots \bar{a}_m \\ \bar{b}_1 & \bar{b}_2 \cdots \bar{b}_m \end{pmatrix}$$

shows that the image of Π under μ is

$$\mu(\Pi) = (|a_1|^2 + |b_1|^2, \cdots, |a_m|^2 + |b_m|^2)$$

[see formula (1.14), (1.15), and (1.16)]. On the other hand, from the formula (2.5),

$$(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_m & b_m \end{pmatrix} \mapsto (\phi(a_1 + b_1 j), \phi(a_2 + b_2 j), \ldots, \phi(a_m + b_m j)),$$

the length of each side can be computed and equals to

$$\begin{aligned} |\phi(a_i + b_i j)|^2 &= |\mathrm{i}(|a_i|^2 - |b_i|^2) + 2\bar{a}_i b_i k|^2 \\ &= |a_i|^4 + |b_i|^4 + 2|a_i|^2|b_i|^2 \\ &= (|a_i|^2 + |b_i|^2)^2. \end{aligned}$$

It is now seen that $\mu = l \circ \Phi_v$. □

2.2.2 The symplectic structure by unitary action

An element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of $\mathbb{R}^m$ is called **generic** when it satisfies the conditions:

1. Every component is non-negative: $\alpha_i \geq 0$ for $i = 1, \ldots, m$;
2. the sum of the components equals 2: $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 2$;
3. there is no element $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ of $\mathbb{R}^m$ such that

$$\sum_{i=1}^{m} \beta_i = 0 \quad \text{and} \quad |\beta_i| = \alpha_i, i = 1, \ldots, m.$$

(Thus a polygon whose side lengths are represented by a generic $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is always a non-lined polygon, since a lined-polygon of side lengths $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ always corresponds, after a direction on the line has been chosen and a sign given to every side length, to a set of numbers $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ with the sum $\sum_{i=1}^{m} \beta_i$ equal to 0; this set of number β is simply the sides of the lined-polygon.)

The element $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ of $\mathbb{R}^m$ is called **regular** if in addition to being generic as defined above, it also satisfies the condition $0 < \alpha_i < 1$ for all the component elements.

The result of example (1.8) is the deduction of the moment map

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to \mathbb{R}^m,$$

for the action of the diagonal unitary matrices, denoted by U_1^m, on the Grassmannian $\mathbb{G}_2(\mathbb{C}^m)$. It is shown by Hausmann-Knutson [5] that if α is a regular element of $\mathbb{R}^m$, then α is a regular value of the moment map [in the sense defined in section (1.4)]

$$\mu : \mathbb{G}_2(\mathbb{C}^m) \to \mathbb{R}^m$$

above.

Theorem 2.1. *If α is a regular element of $\mathbb{R}^m$, the polygon space ${}^m\mathcal{P}_+^3(\alpha)$ is a symplectic manifold diffeomorphic to the symplectic reduction $U_1^m \backslash \mu^{-1}(\alpha)$.*

Proof of theorem (2.1). The formula for the length map defined in (2.1) shows that for a given α,

$$l^{-1}(\alpha) = {}^m\mathcal{P}_+^3(\alpha).$$

Lemma (2.3) then shows that $l^{-1}(\alpha) = \mathrm{U}_1^m \backslash \mu^{-1}(\alpha)$. □

2.3 The symplectic structure by rotations

This section is an exposition of the materials contained in [7]. Specifically, except lemma (2.4) which can be found in [4], all results are discussed in [7].

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a set of positive real numbers. For a component α_i, let $S^2(\alpha_i)$ denote the sphere of radius α_i, and let

$$S_\alpha := S^2(\alpha_1) \times \cdots \times S^2(\alpha_m).$$

Each point on the sphere $S^2(\alpha_i)$ may be regarded as a vector in the Euclidean space $\mathbb{R}^3$. The group SO(3) acts on S_α by rotation on each component sphere.

Again denote by ${}^m\mathcal{P}^3_+(\alpha)$ the space of polygons with m sides in $\mathbb{R}^3$, all with side lengths $\alpha_1, \ldots, \alpha_m$, where the plus sign (+) identifies polygons that are proper rotations of each other. ${}^m\mathcal{P}^3_+(\alpha)$ may be regarded as a subset of S_α for which the sum of the component vectors add up to zero, up to rotations of the vectors. In other words, if we define a map $\mu : S_\alpha \to \mathbb{R}^3$ by

$$\mu(e) = e_1 + \cdots + e_m$$

then ${}^m\mathcal{P}^3_+(\alpha)$ equals to $\mathrm{SO}_3\backslash\mu^{-1}(0)$.

Theorem 2.2. *The map $\mu : S_\alpha \to \mathbb{R}^3$ defined by*

$$\mu(e) = e_1 + \cdots + e_m$$

is a moment map for action of SO_3 on S_α, and therefore, by the remark just before this theorem, the space of polygons ${}^m\mathcal{P}^3_+(\alpha)$ may be given a symplectic structure by the symplectic reduction $SO_3\backslash\mu^{-1}(0)$.

We describe first a symplectic form on S_α.

Lemma 2.4. *Let $S^2(r)$ denote the sphere of radius r. Let x be a point of $S^2(r)$, and let u, v be tangent vectors at x; these may be regarded as vectors in $\mathbb{R}^3$, with u and v orthogonal to x. Also, let the dot $(\cdot)$ denote the scalar product in $\mathbb{R}^3$, and let the cross $(\times)$ denote the vector product in $\mathbb{R}^3$. The form*

$$\nu_x(u, v) = x \cdot (u \times v)$$

is a symplectic form on $S^2(r)$.

Proof. The given form is a 2-form on the 2-dimensional manifold $S^2(r)$, and so it is closed.

Let u be a nonzero tangent vector. Non-degeneracy of the form means that there exists a vector v such that $\nu_x(u, v)$ is non-zero. We may take $v = u \times x$, because it is a vector in the tangent plane of $S^2(r)$ at x, and then $u \times (u \times x)$ is a vector parallel to x, and therefore

$$\nu_x(u, v) = x \cdot \big(u \times (u \times x)\big)$$

is non-zero. □

The symplectic form on S_α is the form

$$\omega := \sum_{j=1}^{n} \frac{1}{\alpha_j^2} \mathrm{p}_j^* \nu$$

where p_j denotes the projection map from S_α onto the component sphere $S^2(\alpha_j)$.

Proof of theorem (2.2). It suffices to verify the theorem for the one factor case. We shall write $S^2(r)$ in place of $S^2(\alpha_1)$. The Lie algebra $\mathfrak{so}(3)$ and its dual will be identified with $\mathbb{R}^3$ and its dual, and also $\mathbb{R}^3$ and its dual will be identified by the usual scalar product [see example (1.4)]. If ξ is an element in $\mathfrak{so}(3) = \mathbb{R}^3$, then the induced vector field is

$$\xi_M(x) = \xi \times x$$

for any point x in $S^2(r)$. Now for any tangent vector v at x,

$$\omega_x\big(\xi_M(x), v\big) = -\frac{1}{r^2} x \cdot [(\xi \times x) \times v] = -\frac{1}{r^2} x \cdot [(\xi \cdot v) x] = -\xi \cdot v.$$

On the other hand, $\mu(x) = x$, and therefore

$$\mathrm{d}\mu_x(\xi) = \xi.$$

Since the Lie algebra and its dual are identified by the scalar product, we have

$$\langle \xi, v \rangle = \xi \cdot v.$$

Thus $\omega(\xi_M, -) = -\mathrm{d}\mu$.

To verify the equivariant condition, let x be a point of $S^2(r)$. If g is an element of $\mathrm{SO}(3)$, the action of g on x is the matrix product $g \cdot x$, which under the moment map goes to $g \cdot x$. On the other hand, the same element x goes to x under the moment map, and the coadjoint action $\mathrm{Ad}^*_{g^{-1}}$, according to example (1.4), brings it to the element $g \cdot x$. $\square$

Chapter 3

Symplectic toric manifolds, Delzant polytopes, and Blow-up

In this chapter we present a brief discussion of a Delzant polytope and related concepts and results, all of which are necessary for the discussion in the last chapter. The materials below can be found in relevant chapters in [4], [1], or [2].

3.1 Symplectic toric manifolds

Let $\mathbb{T}^k$ denote the k-dimensional torus. The Lie algebra of $\mathbb{T}^k$ and its dual, $\mathfrak{t}$ and $\mathfrak{t}^*$, may be both identified with $\mathbb{R}^k$. As $\mathbb{T}^k$ is a commutative group, the adjoint action, and therefore the coadjoint action is the identity mapping. If $\mathbb{T}^k$ is also a symplectic action on a symplectic manifold M with symplectic form ω, the equivariant property in the definition of a moment map simplifies somewhat, so that a map

$$\phi : M \to \mathbb{R}^k$$

is called a moment map for the action of $\mathbb{T}^k$ on M if for every ξ of $\mathfrak{t}$ and every g of $\mathbb{T}^k$ [compare with definition (1.6)]:

1. $\omega(\xi_M, -) = -\mathrm{d}\phi^\xi$, where the empty slot in the form is kept for tangent vectors, and the function

   $$\phi^\xi : M \to \mathbb{R}$$

   is defined by the formula

   $$\phi^\xi(x) := \langle \phi(x), \xi \rangle;$$

   that is, the value of the function at the point x of M is obtained by applying the linear function $\phi(x)$ of $\mathfrak{t}^*$ on the element ξ of $\mathfrak{t}$. Moreover we recall that in the above formula ξ_M denotes the vector field obtained from the action of $\mathbb{T}^k$ on M [see definition at the end of section (1.1)].

2. $\phi \circ \Phi_g = \phi$, where Φ_g denotes the diffeomorphism corresponding to the action of the element g of $\mathbb{T}^k$ on M [see section (1.1)]; thus the map must be invariant under the action of the torus.

The action of $\mathbb{T}^k$ on M is called a **Hamiltonian action** whenever it has a moment map.

A **symplectic toric manifold** is a compact connected symplectic manifold M on which there is defined a Hamiltonian action of a torus $\mathbb{T}^k$ satisfying the requirements:

1. The action is **effective**, which by definition means that each element in $\mathbb{T}^k$ corresponds to only one symplectomorphism on M;
2. the dimension of $\mathbb{T}^k$ is half the dimension of M.

3.2 Delzant polytopes

Let M be a symplectic toric manifold [see definition in section (3.1)]. It is known that the image $\phi(M)$ of M under the moment map is a Delzant polytope, where, by definition, a **Delzant polytope** Δ in $\mathbb{R}^n$ is a convex polytope that satisfies the conditions:

1. There are n edges meeting at each vertex;
2. the i-th edge belonging to the vertex p may be written in the form $p + tu_i$ where $u_i \in \mathbb{Z}^n$ and $0 \leq t < \infty$;
3. the elements $u_i, i = 1, \ldots, n$ can be chosen to form a basis for $\mathbb{Z}^n$.

Now let M^T denote the set of fixed points of M under the action of the torus $\mathbb{T}^k$ on M. It is also known that the image $\phi(M^T)$ of the fixed points of the moment map are the vertices of the polytope $\phi(M)$.

Example 3.1. *The 2-sphere S^2 is a symplectic manifold with symplectic form $\omega = \mathrm{d}\theta \wedge \mathrm{d}h$. Let circle S^1 act on S^2 by rotation:*

$$\mathrm{e}^{\mathrm{i}t} \cdot (\theta, h) = (\theta + t, h).$$

The action is Hamiltonian with moment map

$$\phi = h,$$

where h *denotes the height function on the sphere. The Delzant polytope of the moment map is the closed interval* $[-1, 1]$.

The fixed points of the action are the North and South pole of the circle; their images under the moment map are -1 *and* 1 *which coincide with the vertices of the segment.*

Equivalently, the projective space $\mathbb{CP}^1$ *is a symplectic manifold with the Fubini-Study form*

$$\Omega = \frac{1}{4}\omega$$

where ω *is the symplectic form on the 2-sphere; the circle* S^1 *acts on* $\mathbb{CP}^1$ *by the action*

$$\mathrm{e}^{\mathrm{i}t} \cdot [w_0 : w_1] = [w_0 : \mathrm{e}^{\mathrm{i}t} w_1].$$

The action is Hamiltonian with moment map

$$\phi[w_0 : w_1] = -\frac{1}{2} \cdot \frac{|w_1|^2}{|w_0|^2 + |w_1|^2}.$$

The Delzant polytope in this case is the closed interval $[-1/2, 0]$. *The fixed points of the action are* $[1 : 0]$ *and* $[0 : 1]$*; their images under the moment map are* 0 *and* $-1/2$ *which coincide with the vertices of the segment.*

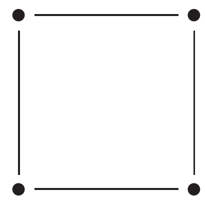
Example 3.2. *The* 2*-torus* $\mathbb{T}^2 = (S^1 \times S^1)$ *act on the symplectic manifold* $S^2 \times S^2$ *by the Hamiltonian action*

$$(\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t'}) \cdot \big((\theta, h), (\theta', h')\big) = \big((\theta + t, h), (\theta' + t', h')\big).$$

The moment map is

$$\phi\big((\theta, h), (\theta', h')\big) = (h, h').$$

The Delzant polytope of the moment map is the square $[-1,1]\times[-1,1]$.



The fixed points of the action are the cross product of the north and south pole of the component spheres; their images under the moment map are $(-1,1)$, $(1,1)$, $(1,-1)$, *and* $(-1,-1)$ *which coincide with the vertices of the square* $\phi(S^2\times S^2)$.

Alternatively, the torus $\mathbb{T}^2$ *acts on* $\mathbb{CP}^1\times\mathbb{CP}^1$ *by the Hamiltonian action*

$$(\mathrm{e}^{\mathrm{i}t},\mathrm{e}^{\mathrm{i}t'})\cdot\big([w_0:w_1],[w_0':w_1']\big)=\big([w_0:\mathrm{e}^{\mathrm{i}t}w_1],[w_0':\mathrm{e}^{\mathrm{i}t'}w_1']\big)$$

with moment map

$$\phi\big([w_0:w_1],[w_0':w_1']\big)=-\frac{1}{2}\left(\frac{|w_1|^2}{|w_0|^2+|w_1|^2},\frac{|w_1'|^2}{|w_0'|^2+|w_1'|^2}\right).$$

The Delzant polytope is the square $[-1/2,0]\times[-1/2,0]$. *The fixed points of the actions are*

$$\big([1:0],[1:0]\big),\quad\big([1:0],[0:1]\big),\quad\big([0:1],[1:0]\big),\quad\big([0:1],[0:1]\big)$$

whose images under the moment map are indeed the vertices of the square $\phi(\mathbb{CP}^1\times\mathbb{CP}^1)$.

Example 3.3. *The* 2*-dimensional projective space* $\mathbb{CP}^2$ *is a symplectic manifold with the symplectic forn*

$$\Omega:=\frac{\mathrm{i}}{2|z|^4}\sum_{j,k=1}^{n}\big(|z_j|^2\mathrm{d}z_k\wedge\mathrm{d}\bar{z}_k-\bar{z}_jz_k\mathrm{d}z_j\wedge\mathrm{d}\bar{z}_k\big).$$

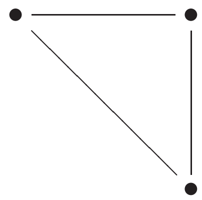
The 2*-dimensional torus* $\mathbb{T}^2$ *acts on* $\mathbb{CP}^2$ *by the Hamiltonian action*

$$(\mathrm{e}^{\mathrm{i}t},\mathrm{e}^{\mathrm{i}t'})\cdot[w_0:w_1:w_2]=[w_0:\mathrm{e}^{\mathrm{i}t}w_1:\mathrm{e}^{\mathrm{i}t}w_2]$$

with the moment map

$$\phi\big([w_0:w_1:w_2]\big)=-\frac{1}{2}\left(\frac{|w_1|^2}{|w_0|^2+|w_1|^2+|w_2|^2},\frac{|w_2|^2}{|w_0|^2+|w_1|^2+|w_2|^2}\right).$$

The Delzant polytope $\phi(\mathbb{CP}^2)$ is the triangle with vertices $(-1/2, 0)$, $(0, 0)$, and $(0, -1/2)$.



The fixed points of the actions are

$$[1:0:0], \quad [0:1:0], \quad [0:0:1]$$

whose images under the moment map coincide with the vertices of the triangle $\phi(\mathbb{CP}^2)$.

Theorem 3.1 (Delzant)**.** *There is a one-to-one correspondence between symplectic toric manifold and Delzant polytopes.*

3.3 Blow-up

3.3.1 Blow-up of a symplectic manifold

The concept of blow-up will be described very briefly in this section. A result on blow-up will stated at the end and will be used in a later section.

Let us describe first **symplectic blow-up at the origin of $\mathbb{C}^n$**, the standard symplectic space with the symplectic form ω_0 [see example (1.1)]; this is by definition the space

$$L = \Big\{ \big((z_1, \dots, z_n), [w_1 : \dots : w_n] \in \mathbb{C}^n \times \mathbb{CP}^{n-1} \big) \big| w_j z_k = w_k z_j \text{ for all } j, k \Big\}. \tag{3.1}$$

The space L may be seen as the space $\mathbb{C}^n$ where the origin has been replaced smoothly with the projective space $\mathbb{CP}^{n-1}$, for the following two reasons. First, the natural projection

$$\beta : L \to \mathbb{C}^n$$

is diffeomorphic when restricted to the subset $\beta^{-1}\big(\mathbb{C}^n - \{0\}\big)$; second, the preimage of the origin, $\beta^{-1}(0)$, is diffeomorphic to $\mathbb{CP}^{n-1}$.

$$\begin{tikzcd} L \arrow[r, "\beta"] \arrow[d, "\mathrm{p}"'] & \mathbb{C}^n \\ \mathbb{CP}^{n-1} & \end{tikzcd} \tag{3.2}$$

The blow-up L defined above is a complex submanifold of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$. A symplectic form on $\mathbb{C}^n \times \mathbb{CP}^{n-1}$ is $\omega_0 + \epsilon\Omega$, where

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i$$

is the standard symplectic form, ϵ is a positive real number, and

$$\Omega := \frac{\mathrm{i}}{2|z|^4} \sum_{j,k=1}^{n} \left(|z_j|^2 \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k - \bar{z}_j z_k \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k\right).$$

is called the Fubini-Study form on $\mathbb{CP}^{n-1}$.

The blow-up L, as a complex submanifold of $\mathbb{C}^n \times \mathbb{CP}^{n-1}$, inherits the symplectic form of the latter; in other words, the **symplectic form on the blow-up** L is $\beta^*\omega_0 + \epsilon \mathrm{p}^*\Omega$, where $\mathrm{p} : L \to \mathbb{CP}^{n-1}$ denotes the natural projection [see the diagram (3.2)].

Now let M be a symplectic manifold with symplectic form ω, and let x be an arbitrary point of M. By theorem (1.1), there exists a neighborhood of x which can be identified symplectically with $\mathbb{C}^n$, the n-dimensional complex space with the standard symplectic form ω_0. Therefore the symplectic blow-up at this point is done just as in the case of $\mathbb{C}^n$ above.

The following theorem will be used later.

Theorem 3.2. *The blow-up at one point of a symplectic manifold M is diffeomorphic to the connected sum $M\#\overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ denotes the n-dimensional complex projective space with reversed orientation.*

3.3.2 Blow-up of a symplectic toric manifold

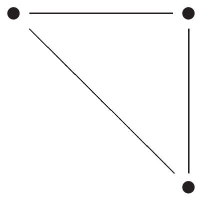
The following theorem will also be used in a later section.

Theorem 3.3. *Let M be a symplectic toric manifold with symplectic form ω, torus action $\mathbb{T}^k$, and moment map ϕ. If $x \in M$ is a fixed point under the action of $\mathbb{T}^k$, then the blow-up of M at x is a toric manifold; moreover, its Delzant polytope is obtained by replacing the vertex $\phi(x)$ of the polytope $\phi(M)$ by the n vertices*

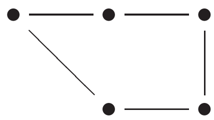
$$\phi(x) + \epsilon u_i, \quad i = 1, \ldots, n$$

where ϵ is a positive number of appropriate magnitude, and $u_1, \ldots, u_n$ the primitive inward-pointing edge vectors at $\phi(x)$, so that the rays $\phi(x) + tu_i$ for $t \geq 0$ form the edges of $\phi(M)$ at $\phi(x)$.

Example 3.4. *According to example (3.3), the projective space* $\mathbb{CP}^2$ *is a symplectic toric manifold whose the corresponding Delzant polytope is the triangle with vertices* $(-1/2, 0)$, $(0, 0)$, *and* $(0, -1/2)$.



As mentioned in that example, the fixed point of the torus action $[0 : 0 : 1]$ *is mapped to the vertex* $(0, -1/2)$ *of the polytope. A blow-up at the point* $[0 : 0 : 1]$ *then results in a new symplectic toric manifold whose Delzant polytope is obtained by cutting the triangle in the direction of the side opposite to the vertex* $(0, -1/2)$. *Therefore, the original polytope becomes*



According to theorem (3.2), the manifold after blow-up is diffeomorphic to the connected sum $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Chapter 4

Torus action on the space of polygons

We give in this chapter an exposition of the result that one may define a torus action on the space of polygons of non-vanishing diagonals. This result is due to Kapovich-Millson [7]. The examples in this chapter are elaborated in Haussmann-Knutson [5].

4.1 Torus action on the space of polygons of non-vanishing diagonals

In this section we give an exposition of the same materials found in Kapovich-Millson [7]. We shall use an alternative definition of a moment map given in [6]. Finally, except for the proof of the elementary lemma (4.1), all proofs are contained in [7].

We recall that ${}^m\mathcal{P}^3_+(\alpha)$ denotes the space of polygons in $\mathbb{R}^3$ of side lengths $\alpha = (\alpha_1, \ldots, \alpha_m)$, identified up to translations and proper rotations. Each polygon ρ in ${}^m\mathcal{P}^3_+(\alpha)$ has $m-3$ diagonals

$$\rho_1 + \rho_2, \ldots, \rho_1 + \cdots + \rho_{m-2},$$

some of which may be zero. Let M_α consist of the polygons in ${}^m\mathcal{P}^3_+(\alpha)$ none of the diagonals of which is zero. In M_α define the functions

$$\begin{array}{rcll} l_k : M_\alpha & \to & \mathbb{R}, & \quad k = 1, \ldots, m-2 \\ \rho & \mapsto & |\rho_1 + \cdots + \rho_{k+1}| & \end{array}$$

that measure the lengths of the diagonals. It will be shown below that corresponding to each l_k there is a flow ψ_k^t, where t denotes a real number, that rotates a part of ρ in M_α around the k-th diagonal, and that the period of $\psi_k^t(\rho)$ is 2π. In other words, we will have the following theorem.

Theorem 4.1. *The space M_α, consisting of polygons that have no zero diagonal, admits a free Hamiltonian action by a torus $\mathbb{T}^{m-3}$. The moment map is*

$$\begin{array}{rcl} \phi : M_\alpha & \to & \mathbb{R}^{m-3} \\ \rho & \mapsto & (l_1, \cdots, l_{m-3}). \end{array}$$

We shall use an alternative definition of a moment map, one that makes use of the concepts of Hamiltonian vector fields and Poisson brackets. Let f be a smooth function on a symplectic manifold M with symplectic form ω. Associated to f is the **Hamiltonian vector field** H_f defined by the relation

$$\omega(H_f, -) = -df.$$

The **Poisson bracket** of two smooth functions f and g on M is defined as

$$\{f, g\} := \omega(H_f, H_g),$$

where H_f and H_g are the Hamiltonian vector fields associated to f and g.

Now let a Lie group G be a symplectic action on M. Moreover, let $\mathfrak{g}$ denote the Lie algebra of G, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. The action is called a Hamiltonian action if there is a map

$$\phi : M \to \mathfrak{g}^* \tag{4.1}$$

such that for all ξ and η in $\mathfrak{g}$,

1. $H_{\phi^\xi} = \xi_M$, where here and below, the function $\phi^\xi : M \to \mathbb{R}$ is defined by the formula
$$\phi^\xi(x) := \langle \phi(x), \xi \rangle,$$
and ξ_M is the vector field on M defined by the formula [compare with formula (1.3)]
$$\begin{array}{rcl} \mathfrak{g} & \to & T_x M \\ \xi & \mapsto & \xi_M(x) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \big(\exp(t\xi) \cdot x \big), \end{array}$$
2. $\{\phi^\xi, \phi^\eta\} = -\phi_{[\xi,\eta]}$, where $[\xi, \eta]$ means the Lie bracket of the vectors.

The map $\phi : M \to \mathfrak{g}^*$ is called a moment map for the action.

The theorem above follows from the following results that will be developed for the space ${}^m\mathcal{P}^3_+(\alpha)$. In particular, the functions l_k will be derived from certain functions f_k on ${}^m\mathcal{P}^3_+(\alpha)$; moreover, associated to f_k are the flows φ^t_k which have the same geometric meaning just like the flows ψ^t_k associated to the l_k, and which imply that ψ^t_k can be found.

It was shown in section (2.3) that ${}^m\mathcal{P}^3_+(\alpha)$ inherits a symplectic form, by symplectic reduction, from the form

$$\omega := \sum_{j=1}^{n} \frac{1}{\alpha_j^2} \mathrm{p}_j^* \nu$$

on the space

$$S_\alpha = S^2(\alpha_1) \times \cdots \times S^2(\alpha_m),$$

where ν is the symplectic form on the sphere $S^2(\alpha_i)$ of radius α_i defined by

$$\nu_x(u, v) = x \cdot (u \times v),$$

and where the action is the action of SO(3) on the component sphere. We note that, generally, on a symplectic manifold M, the non-degeneracy of the form ω means that one may identify vector fields and 1-forms on M by the relation

$$X \mapsto \omega(X, -)$$

where X denotes a vector field on M. Thus, there always exists a Hamiltonian vector field for a given smooth function on M. On ${}^m\mathcal{P}^3_+(\alpha)$ define the functions

$$f_k : \rho = (\rho_1, \ldots, \rho_m) \mapsto \frac{1}{2}|\rho_1 + \cdots + \rho_{k+1}|^2, \qquad k = 1, \ldots, m-3$$

that correspond to the lengths squared of the k-diagonals of ρ in ${}^m\mathcal{P}^3_+(\alpha)$. It is shown in Kapovich-Millson [7] that the Hamiltonian vector field H_{f_k} associated to f_k is given by

$$H_{f_k}(\rho_1, \ldots, \rho_m) = (\mu_k \times \rho_1, \ldots, \mu_k \times \rho_{k+1}, 0, \ldots, 0)$$

where

$$\mu_k = \rho_1 + \cdots + \rho_{k+1}$$

is the k-th diagonal of ρ.

Proposition 4.1.

$$\{f_k, f_l\} = 0$$

for all k, l.

Proof. The Poisson bracket is anticommutative and therefore it may be assumed that $k < l$. Then

$$\begin{aligned}
\{f_k, f_l\} &= \omega(H_{f_k}, H_{f_l}) \\
&= \textstyle\sum_{i=1}^{k+1} \frac{\rho_i}{\alpha_i^2} \cdot \big((\mu_k \times \rho_i) \times (\mu_l \times \rho_i)\big) \\
&= \textstyle\sum_{i=1}^{k+1} [\rho_i \cdot (\mu_k \times \mu_l)] \\
&= \mu_k \cdot (\mu_k \times \mu_l) \\
&= 0,
\end{aligned}$$

where we have used these vector identities: If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors in $\mathbb{R}^3$, then

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) \\
\mathbf{a} \times \mathbf{b} \times \mathbf{c} &= (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{c} \cdot \mathbf{b})\mathbf{a}.
\end{aligned}$$

$\square$

The flows φ_k^t associated to f_k are the solutions of the following systems of ordinary differential equations

$$\begin{aligned}
\frac{\mathrm{d}\rho_i}{\mathrm{d}t} &= \mu_k \times \rho_i, \qquad 1 \leq i \leq k+1, \\
\frac{\mathrm{d}\rho_i}{\mathrm{d}t} &= 0, \qquad k+2 \leq i \leq m.
\end{aligned} \tag{4.2}$$

To solve the systems of equations, we identify the Lie algebra of SO(3) with $\mathbb{R}^3$ with the cross product $(\times)$ and, for each vector u in $\mathbb{R}^3$, define, from the operator

$$\begin{aligned}
\mathrm{ad}_u : \mathbb{R}^3 &\to \mathbb{R}^3 \\
v &\mapsto u \times v
\end{aligned}$$

the element $\exp(\mathrm{ad}_u)$ in SO(3) by the power series

$$\exp(\mathrm{ad}_u) := \sum_{n=0}^{\infty} \frac{(\mathrm{ad}_u)^n}{n!}.$$

The solutions of the systems of equations (4.2) is the result of the following proposition.

Proposition 4.2. *Suppose ρ is a polygon in ${}^m\mathcal{P}^3_+(\alpha)$ with sides $\rho_1, \ldots, \rho_m$. Then $\rho(t) := \varphi_k^t$ has sides $\rho_1(t), \ldots, \rho_m(t)$ given by*

$$\begin{aligned}
\rho_i(t) &= \exp(t\mathrm{ad}_{\mu_k})\rho_i, \qquad 1 \leq i \leq k+1 \\
\rho_i(t) &= \rho_i, \qquad k+2 \leq i \leq m.
\end{aligned}$$

Let us show first, by the following lemma, the interpretation of the flows φ_k^t as rotations of part of a polygon about the k-th diagonal.

Lemma 4.1. *Let Π denote the orientated plane in $\mathbb{R}^3$ orthogonal to the vector u. Then $\exp(\mathrm{ad}_u)$ is the rotation in Π through an angle of $|u|$ radians. In particular, the curve $\exp(t\mathrm{ad}_u)$ has period $2\pi/|u|$ and angular velocity $|u|$.*

We recall first some facts about rotations. An infinitesimal rotation R may be written as

$$R \simeq I + A$$

for some infinitesimal matrix A. Therefore,

$$R^T R \simeq I + A^T + A.$$

The orthogonal relation $R^T R = I$ required of a rotation implies that $A^T = -A$, or that $A = \theta \mathcal{J}$ for some real number θ and where $\mathcal{J}$ is the 2-by-2 antisymmetric matrix

$$\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, an infinitesimal rotation R may be written as

$$R \simeq I + \theta \mathcal{J}.$$

Rotation through a finite angle θ is obtained by accumulating infinitesimal rotations and has the form

$$R(\theta) = \lim_{N \to \infty} R\left(\frac{\theta}{N}\right)^N = \lim_{N \to \infty} \left(1 + \frac{\theta}{N}\mathcal{J}\right)^N = \mathrm{e}^{\theta \mathcal{J}}. \tag{4.3}$$

Proof of lemma (4.1). Choose u to be the z-axis, so that $u = (0, 0, |u|)$. If $v = (v_1, v_2, 0)$ is a vector in the plane Π orthogonal to u, then

$$u \times v = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & |u| \\ v_1 & v_2 & 0 \end{pmatrix} = |u|(-v_2)\hat{x} + |u| v_1 \hat{y}.$$

Thus ad_u corresponds to the operator

$$|u| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Comparing this form with the form in (4.3) it is seen that the lemma is verified. □

Proof of proposition (4.2). The last $n - k - 1$ sides remain constant throughout and so may be ignored. Define

$$\bar{\rho_1} = \rho_1 + \cdots + \rho_{k+1} = \mu_k, \qquad \bar{\rho_i} = \rho_i, \quad 2 \le i \le k+1.$$

We obtain then the following system of equations

$$\begin{aligned} \frac{\mathrm{d}\bar{\rho_1}}{\mathrm{d}t} &= \textstyle\sum_{i=1}^{k+1} \frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \sum_{i=1}^{k+1} \mu_k \times \rho_i = \mu_k \times \mu_k = 0 \\ \frac{\mathrm{d}\bar{\rho_i}}{\mathrm{d}t} &= \frac{\mathrm{d}\rho_i}{\mathrm{d}t} = \mu_k \times \rho_i = \bar{\rho_1} \times \bar{\rho_i}, \qquad 2 \le i \le k+1. \end{aligned}$$

Then $\bar{\rho}_1$, and therefore μ_k, is invariant under the flow. Integration yields

$$\rho_i(t) = \exp(t\mathrm{ad}_{\mu_k})\rho_i, \qquad 2 \le i \le k+1.$$

To find $\rho_1(t)$, we note that

$$\exp(t\mathrm{ad}_{\mu_k})\mu_k = \sum_{i=1}^{k+1} \exp(t\mathrm{ad}_{\mu_k})\rho_i = \sum_{i=1}^{k+1} \rho_i(t) = \mu_k$$

and therefore

$$\rho_1(t) = \mu_k(t) - \sum_{i=2}^{k+1} \rho_i(t) = \sum_{i=1}^{k+1} \rho_i(t) - \sum_{i=2}^{k+1} \rho_i(t) = \exp(t\mathrm{ad}_{\mu_k})\rho_1.$$

□

On account of the proposition just shown and of lemma (4.1), we have the following corollary.

Corollary 4.1. *The curve $\varphi_k^t(\rho)$ is periodic with period $2\pi/l_k$, where*

$$l_k = |\rho_1 + \cdots + \rho_{k+1}|$$

is the length of the k-diagonal μ_k of ρ.

The functions $l_1, \ldots, l_{m-3}$ are smooth on M_α and also satisfy the relation

$$\{l_k, l_l\} = 0. \tag{4.4}$$

Moreover, since $f_k = l_k^2/2$, we have

$$dl_k = \frac{df_k}{l_k}$$

and consequently the Hamiltonian vector fields associated to the l_k are

$$H_{l_k} = H_{f_k}/l_k.$$

Since the l_k are invariant under the flow φ_k^t, the solution procedure in the above proposition also works for H_{l_k}. Let ψ_k^t be the flows of H_{l_k}. We have the following proposition.

Proposition 4.3. *Suppose ρ is a polygon in M_α with sides $\rho_1, \ldots, \rho_m$. Then $\rho(t) := \psi_k^t$ has sides $\rho_1(t), \ldots, \rho_m(t)$ given by*

$$\begin{array}{rcl} \rho_i(t) & = & \exp(\mathrm{tad}_{\mu_k/l_k})\rho_i, \qquad 1 \leq i \leq k+1 \\ \rho_i(t) & = & \rho_i, \qquad k+2 \leq i \leq m. \end{array}$$

The interpretation of the flows is analogous: ψ_k^t rotates part of ρ about the k-th diagonal with constant angular velocity 1. Hence $\psi_k^t(\rho)$ has period 2π. We conclude that M_α admits a free action by a torus $\mathbb{T}^{m-3}$.

The Lie algebra of the torus may be identified with $\mathbb{R}^{m-3}$ in which, for any two vectors ξ and η one has $[\xi, \eta] = 0$. The basis vectors in $\mathbb{R}^{m-3}$ are associated with the functions l_k. On account of the relation (4.4) for the functions l_k, and on account of the fact that the flows ψ_k^t are solved directly in terms of the Hamiltonian vector fields associated to the l_k, it follows directly that the map

$$\begin{array}{rcl} \phi : M_\alpha & \to & \mathbb{R}^{m-3} \\ \rho & \mapsto & (l_1, \cdots, l_{m-3}). \end{array}$$

is a moment map for the action of the torus on M_α, according to the criteria in the definition of a moment map given at the beginning of this section. Theorem (4.1) is then established.

4.2 The space of polygons of 4 and of 5 sides

Here we apply the results above to some of the simplest examples. In reference [7] there are elaborations of these examples. The reference also contains more examples that require results not discussed in this text.

4.2.1 Polygons with 4 sides

The space of polygons with 4 sides in three-dimensional space is denoted, according to our notation, by the symbol ${}^4\mathcal{P}_+^3(\alpha)$, where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

is generic [see definition in section (2.2.2)]. For this space, the length of the diagonal, which according to theorem (4.1) is also the moment map,

$$\begin{array}{rcl} {}^4\mathcal{P}_+^3(\alpha) & \to & \mathbb{R} \\ \rho & \mapsto & |\rho(1) + \rho(2)| \end{array}$$

does not vanish when $\alpha_1 \neq \alpha_2$ or $\alpha_3 \neq \alpha_4$. This condition implies that the image of the moment map, which will be denoted by Δ, is the intersection of the two intervals

$$[|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2], \quad [|\alpha_4 - \alpha_3|, \alpha_4 + \alpha_3]$$

which is a closed interval. This coincides with the Delzant polytope in example (3.1). The space ${}^4\mathcal{P}^3_+(\alpha)$ is diffeomorphic to $\mathbb{CP}^1$.

4.2.2 Polygons with 5 sides

The space of polygons with 5 sides in three-dimensional space is denoted, according to our notation, by the symbol ${}^5\mathcal{P}^3_+(\alpha)$, where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

is generic [see definition in section (2.2.2)]. For this space, the lengths of the diagonals, which according to theorem (4.1) make up the moment map

$$\begin{array}{lll} {}^4\mathcal{P}^3_+(\alpha) & \to & \mathbb{R}^2 \\ \rho & \mapsto & \big(|\rho(1)+\rho(2)|, |\rho(1)+\rho(2)+\rho(3)|\big), \end{array}$$

do not vanish when $\alpha_1 \neq \alpha_2$ and $\alpha_4 \neq \alpha_5$. This condition implies that the image of the moment map, which will be denoted by Δ, is the intersection of the rectangular region

$$[|\alpha_1 - \alpha_2|, \alpha_1 + \alpha_2] \times [|\alpha_5 - \alpha_4|, \alpha_5 + \alpha_4]$$

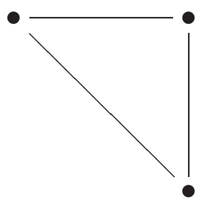
with the non-compact rectangular region that satisfies the following set of inequalities

$$x + y \geq \alpha_3, \quad y \geq x - \alpha_3, \quad y \leq x + \alpha_3, \quad x, y \geq 0.$$

The Delzant polytope Δ therefore has at most 7 sides.

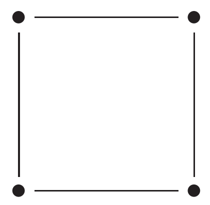
Figure 4.1: Δ as the intersection of two regions

We shall only look at the case when Δ has three and four sides. In the case Δ has three sides, it is the triangle

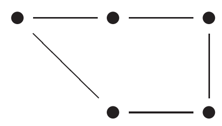


which coincides with the image of $\mathbb{CP}^2$ treated in example (3.3).

In the case Δ has four sides, it is either the square



which coincides with the image of $S^2 \times S^2$ treated in example (3.2), or it is the figure



which is obtained by blowing-up $\mathbb{CP}^2$ at a point as treated in example (3.4) (according to that example, the resulting manifold after the blow-up is diffeomorphic to $\mathbb{CP}^2 \times \overline{\mathbb{CP}^2}$).

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Appendix A

Differential forms and some related properties

In this appendix we give a brief discussion of differential forms and some related properties. More can be found in [3].

The algebraic rules for handling differential forms may be thought of as being suggested by the laws of transformations of integrals when the integration variables are changed. Let f be a function of the n variables $x^1, \ldots, x^n$. In the multiple integral

$$\int\int\cdots\int f(x^1, \ldots, x^n)\mathrm{d}x^1\mathrm{d}x^2\cdots\mathrm{d}x^n$$

we pick out the differentials $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$ and compose from them the entities (the indices run from 1 to n)

$$1, \mathrm{d}x^i, \mathrm{d}x^i \wedge \mathrm{d}x^j, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k, \ldots, \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n, \qquad i < j < k, \ldots \tag{A.1}$$

which we shall treat as being independent from one another. On these entities we define an algebra according to the rule

$$\mathrm{d}x^i \wedge \mathrm{d}x^j = -\mathrm{d}x^j \wedge \mathrm{d}x^i. \tag{A.2}$$

For example, let

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i.$$

It then follows that the product of ω_0 with itself n times is, using the rule above,

$$\omega_0^n = \sum_{i_1,\ldots,i_n=1}^{n} \mathrm{d}x^{i_1} \wedge \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n} \wedge \mathrm{d}y^{i_n} = \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n.$$

A **differential q-form** is either an expression of the form

$$f_{i_1\cdots i_q}\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}$$

or more generally a sum of such expressions.

To obtain a $q+1$-form from a q-form we define the **differential operation** d according to the following rule. If f is a function of n variables, we take the differential of f according to the formula of calculus:

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i. \tag{A.3}$$

For a single expression

$$\omega = f_{i_1 \cdots i_q} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$$

we define

$$\mathrm{d}\omega = \mathrm{d}f_{i_1 \cdots i_q} \wedge \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_q}.$$

Finally, for a general q-form which is a sum of single expressions as above, we take the differential of each term in the sum and add the results. For example, if there are only two variables which we shall denote x and y, the differential of the form

$$\omega = x\mathrm{d}y$$

is the form

$$\mathrm{d}\omega = \left(\frac{\partial x}{\partial x} \mathrm{d}x + \frac{\partial x}{\partial y} \mathrm{d}y \right) \wedge \mathrm{d}y = \mathrm{d}x \wedge \mathrm{d}y.$$

The following identities are useful for our text. By (A.2) an expression

$$f_{i_1 \cdots i_q} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_n}$$

is necessarily zero if any two indices coincide. Hence, if there are only n independent variables $x^1, \ldots, x^n$, the differential of any n-form is necessarily zero. For example, if

$$\omega = f \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$$

the differential of ω is

$$\mathrm{d}\omega = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}x^1 \cdots \mathrm{d}x^n$$

which must be zero, since every term in the sum contains a repeated index and so vanishes.

Moreover, according to (A.3), $\mathrm{d}f$ vanishes if f is a constant function. It follows that the differential of any combination of the independent entities (A.1) vanishes. For example, if

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

then $\mathrm{d}\omega_0$ vanishes.

Differential forms may be treated as objects that operate on tangent vectors. Let us recall here only the essential points; for further discussion please see [3]. Every point of an n-dimensional manifold may be given a set of local coordinates, say $x^1, \ldots, x^n$, associated with a local orthogonal coordinate system. If f is a function given in terms of the local variables, the rates of change of f, in the directions of the coordinate axes of the local coordinate system, at a point are given by the partial derivatives

$$\frac{\partial f}{\partial x^i}, \qquad i = 1, \ldots, n,$$

all evaluated at the point. These partial derivatives are obtained by evaluating the partial derivative operators

$$\frac{\partial}{\partial x^i}, \qquad i = 1, \ldots, n$$

on f at the given point. We treat these operators as independent entities and form from them the various sums

$$\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$$

(here a^i denote arbitrary real numbers) which we shall call **tangent vectors** at the given point of the manifold. Using the rule of calculus,

$$\frac{\partial x^i}{\partial x^j} = \delta_{ij},$$

we set up the following rule: at each point of the manifold, the differentials $\mathrm{d}x^i$ are to be treated as a dual basis to the operators $\partial/\partial x^j$. Then 1-forms belong to the dual of the tangent space at the point, and they operate linearly on the tangent vectors there. For example, given the total change of a function f at a point, namely the 1-form

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \mathrm{d}x^i,$$

we obtain the rate of change in a given direction by computing (at the point in question)

$$\frac{\partial}{\partial x^j}(\mathrm{d}f) = \frac{\partial f}{\partial x^j}.$$

There is a generalization of the above that treats n-forms as objects that operate on n tangent vectors according to some algebraic rules. Let us mention the following useful formula. If v_1 and v_2 are two tangent vectors at a point of a manifold and, say $\mathrm{d}x^1 \wedge \mathrm{d}x^2$, is a differential 2-form, then one may apply the form to the tangent vectors; the value is the determinant of the matrix $\big(\mathrm{d}x^i(v_j)\big)$. For example, for the 2-form

$$\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

and two arbitrary tangent vectors of $\mathbb{C}^n$,

$$\xi_1 = \sum_{j=1}^{n}(x_1^j \frac{\partial}{\partial x^j} + y_1^j \frac{\partial}{\partial y^j}), \quad \xi_2 = \sum_{j=1}^{n}(x_2^j \frac{\partial}{\partial x^j} + y_2^j \frac{\partial}{\partial y^j}),$$

the application of the form on the tangent vectors is the sum of the determinants of the matrices

$$\begin{pmatrix} \mathrm{d}x^i(\xi_1) & \mathrm{d}x^i(\xi_2) \\ \mathrm{d}y^i(\xi_1) & \mathrm{d}y^i(\xi_2) \end{pmatrix},$$

which is equal to

$$\omega(\xi_1, \xi_2) = \sum_{k=1}^{n} \big(\mathrm{d}x^k(\xi_1)\mathrm{d}y^k(\xi_2) - \mathrm{d}x^k(\xi_2)\mathrm{d}y^k(\xi_1)\big) = \sum_{k=1}^{n}(x_1^k y_2^k - x_2^k y_1^k).$$