Teaching fractions through a Measurement Approach to prospective elementary teachers:
A design experiment in a Math Methods course

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Abstract

Teaching fractions through a Measurement Approach to prospective elementary teachers: A design experiment in a Math Methods course

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In this study we give an account of a teaching experiment on fractions to prospective elementary teachers, which took place in winter 2014 in a Teaching Mathematics course in an Elementary Education undergraduate program at a North-American university. The experiment was an adaptation for teacher education of the "Measurement Approach" to teaching fractions developed by the psychologist V.V. Davydov for the elementary mathematics curriculum (Davydov & Tsvetkovich, 1991).

The research had the characteristics of a design experiment, with a *phase of reflection* on the sources of meaning of fractions appropriate for the elementary school, as well as *preliminary trials* with one year before (winter 2013) preceding the *implementation of the experiment* in a "mature form." We had two overarching goals in the design conception: fostering future teachers' quantitative reasoning and cultivating a positioning relative to the course institution that is more conducive to accepting the approach – that of university students acquiring theoretical knowledge. In the description and the retrospective analysis of the teaching intervention we follow the realization of these goals at three levels: the overall organization of the material and tasks in *the course* by the instructor, *the classroom* interactions between the instructor and the students in lectures, and *individual* reasoning without mediation by the instructor.

We found that the Measurement Approach encouraged a culture of systemic justification in the classroom with some students adopting flexibly and creatively the proposed models of reasoning within a given theory. However, the risk of students' imitating only certain aspects of

these models – such as words, sentence structures, or procedures – ran high, with many students using the theory only as “decoration”, without adequate understanding. Furthermore, although spontaneous engagement with quantitative reasoning for establishing validity of statements about fractions or for explaining realistic problems was rare, it was present in several students, in encouraging forms. Very few students adopted such reasoning, but those who did, exhibited sophisticated and varied strategies for solving problems, which demonstrated robust understanding of the fraction of quantity theory.

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1 INTRODUCTION AND RATIONALE

This thesis gives an account of a teaching experiment on fractions to prospective elementary teachers. The experiment was an adaptation for teacher education of the "Measurement Approach" to teaching fractions developed by the psychologist V.V. Davydov for the elementary mathematics curriculum (Davydov & Tsvetkovich, 1991). In a "mature form" – an eight-week series of lectures and labs – the experiment was run in winter 2014 in a Teaching Mathematics course – what is typically called a *math methods* course – in an Elementary Education undergraduate program at a North-American university.

Our research had the characteristics of a "design experiment" in the sense of Cobb et al. (Cobb, diSessa, Lehrer, & Schauble, 2003), bearing some resemblance with "didactic engineering" in the French tradition of mathematics education research, or "didactique des mathématiques" (Artigue, 1989; Godino, et al., 2013).

The experiment was preceded by a phase of reflection on the sources of the meaning of fractions appropriate for elementary school mathematics and how it could be taught to prospective teachers. This reflection led us to Davydov & Tsvetkovich's epistemological analyses of the notion of fraction (ibid.), and that is where our idea of teaching fractions in the context of measurement was born. In short, the argument for this idea goes as follows. Fractions originate in human activities of measuring as a way of quantifying certain aspects of things: a fraction expresses a relationship between two measured quantities. By contrast, the practice of breaking things corresponds rather to the modern notion of rational number as a result of division: it may make sense to divide things to teach division. Moreover, r*ational number as division* indeed beautifully matches the behavior of its realistic counterpart, *fraction-relation between measured quantities*: taking together two quantities that are fractions of the same quantity and asking what fraction the sum is of that quantity "gives the same result" as adding two rational numbers. But the conceptualization of rational number *as division* of whole numbers was spurred by internal theoretical concerns having to do with the foundations of mathematics as a domain of knowledge grounded in axiomatic theories. For all its theoretical power, this concept loses the connection to "measured things" – rather, rational number as division owes its very attractiveness for abstract algebra to this detachment from reality. Finally, this stands to argue that rational number as division is perhaps not the route to go for teaching children mathematics: we would have to teach a sophisticated concept – rational number – with tools too rudimentary to capture its essence – cutting cakes and pizzas into pieces. Cutting cakes could work, but only if we retained an interest in the measurements of the resulting pieces with respect to one or another quantifiable aspect that interests us: cardinality of sets, surface area, weight, etc. Otherwise we may think – and children's thinking this way produces systematic errors – that a fraction is *a* piece of cake, when it very much isn't (pun intended).

We tried a few teaching situations with future teachers based on the idea of fraction as a relationship between measured quantities in a pilot study in the winter of 2013. Based on this experience – marked by a strong resistance from future teachers to some of the teaching situations we proposed – we revised the design of the situations, but did not give up the main idea of the approach: fractions arising from measurement activities. We decided to re-try our introduction to fractions with a new group of future teachers, in the winter of 2014. This time, however, we developed a comprehensive program for introducing future teachers to the Measurement Approach to fractions and supplied the students with "Course Notes" – in guise of a textbook for the course – containing the essential ideas of the approach and many exercises to understand and practice them. We also attended more carefully to the fact that we were to teach this approach to undergraduate students enrolled in a math methods course – this context turned out to explain some of the opposition to the ideas we had attempted to put forward. Thus our approach gained more solid mathematical and didactical foundations, or, more generally, the features of a clearly defined design, ready to be tested in the classroom.

In implementing the design we focused on two overarching goals we had set for it: fostering future teachers' quantitative reasoning and cultivating a positioning relative to the course institution that is more conducive to accepting the approach. The description and analysis of the teaching intervention, in the present study, follows the realization of these goals at three levels: the overall organization of the material and tasks in *the course* by the instructor, *the* *classroom* interactions between the instructor and the students in lectures, and *individual* reasoning without mediation by the instructor.

An a-posteriori analysis serves as the basis for discussion about the validity of the design in relation to the a-priori analysis – this is the internal validation process characteristic of the didactic engineering methodology (i.e., as opposed to comparison with a control group performed in other types of instructional interventions).

1.1 Goals of the design

Through the a-priori analysis – rooted in available knowledge in the literature and our preliminary trials – we have set *two objectives for the didactic engineering*:

- To develop future teachers' quantitative reasoning
- To influence future teachers' positions in the institution of mathematics methods courses

The first is the one that captures the spark that started this study: our view that elementary instruction in mathematics should aim at developing children's quantitative reasoning. The second objective is subsumed to the first in the sense of encouraging the university students' attitudes that may favor learning of quantitative reasoning, but it is also, more generally, related to how we envision learning at university for prospective teachers, especially within the particular framework of mathematics methods courses.

I will justify the emergence of these goals in detail in chapters 3 and 4, as I speak about the reflection and conception phases of the instructional design, and examine their realization in chapter 5, which details and analyzes the actual implementation. In this introductory chapter, however, before casting the work in the conceptual structure amenable to close study, I give a general characterization of our goals in the form of a general vision and a few examples.

1.1.1 Quantitative reasoning in elementary mathematics

In elementary school, children do not learn about the rational numbers of modern mathematics. If they did, then, for example, to justify that $\frac{2}{5}$ is less than $\frac{3}{7}$, it would be enough to say that 2×7 is less than 3×5. But children encounter fractions in problems that involve some real-life situation or objects, and their teachers should not get away with such a justification alone. To talk about any piece of mathematics meaningfully at this level, there must be some connection with the context suggested in the problem. That those numbers have to be somehow attached to concrete things is an idea that few stakeholders in elementary mathematics education – be it parents, teachers, policy makers, or researchers – would disagree with. But how to effectively mediate this attachment of numbers – in particular, fractions – to the concrete, how to maintain it when performing operations in problem situations, or how to organize the take-off to abstraction are just some of the questions that we have to confront if we want to even keep the topic of fractions in the elementary mathematics curriculum.

Neither the interest in playing with real pizzas, manipulatives, and electronic applications ("apps") for fractions, nor the aesthetics of developing the theory of rational numbers, *alone,* justifies the need to learn fractions. The former remains just that, an indulged interest if it is not "realized through its direction", says John Dewey (1900/1990, p. 40), the philosopher of education whose view of learning was nothing if not of an experience-based, child-centered process. The latter runs the risk of "deforming the taste and creating snobs" if it is in and of itself the goal of teaching, says Henri Lebesgue (1932, p. 198), the mathematician who initiated the creation of the modern theory of integration based on the notion of measure. The progressive philosopher of education is concerned with keeping the mathematics in the experience, the enlightened mathematician – with keeping the experience in mathematics. Inspired by both, we believe that doing meaningful mathematics at elementary level requires thinking about quantities – modeling the reality in terms of those aspects of objects or of the relations between them that can be measured (e.g., the surface area, the number of calories, the weight of a pizza). Before tracing more precisely the inspiration and the conceptualization of the notion of fraction of quantity – central to our development of the Measurement Approach – we illustrate, in broad strokes, what we mean by quantitative reasoning, through three examples. The first is extracted from a research paper about fractions and multiplicative reasoning, the second is an episode from a workshop on fractions in realistic situations conducted by a future teacher, and the third is a reflection on a personal experience of modeling using mathematics with a child. They all feature a proportionality problem: the first example highlights the difference between a solution based on equations and a solution based on quantitative reasoning, the second touches on adults' preference for algorithm at the expense of mathematical insight in connection with real life, and the third acknowledges the difficulty of revealing, for a child, such insight.

1.1.1.1 Example 1: Quantitative reasoning versus cross-multiplying in a proportionality problem

Thompson and Saldanha (2003) have long advocated for attending with greater clarity to quantities, as attributes of objects that can be measured with units, and to operations on quantities, as acts of imagining what one does when performing them. They discuss the poor level of fractions knowledge and teaching in North-America, and propose to address it by enhancing multiplicative reasoning. This is different, they show, from being able to perform a multiplication. A problem that they quote, given by Post et al. (1993) to a group of elementary teachers brings home this important distinction: ""Melissa bought 0.46 of a pound of wheat flour for which she paid $0.86. How many pounds of flour could she buy for one dollar?" The problem could be solved by setting up the equation $\frac{0.46}{0.83} = \frac{x}{1}$ and "cross-multiplying" to then solve for x. The teachers from the sample, although able to do this, were far from having a coherent understanding what it meant and justifying why this "rule" should work to produce a true answer. Thompson and Saldanha (ibid.) propose how such understanding could look if quantitative reasoning was used:

> *If 0.46 lb costs $0.83, then $0.01 (being 1/83rd of $0.83) will purchase 1/83rd of $0.46 lb. Thus $1.00 will purchase 100/83 of 0.46 lb. A more sophisticated expression of the same reasoning would be: $1.00 is 100/83 as large as $0.83, so you can buy 100/83 of 0.46 lb for $1. (Thompson & Saldanha, 2003, pp. 14-15).*

The two authors reflect on the kind of teaching that would encourage such thinking:

> *What conceptual development might lead to such reasoning? A variety of sources suggest it is through the development of a web of meanings that entails conceptualizations of measurement, multiplication, division, and fractions. We*

emphasize conceptualizations of measurement, multiplication, division, and fractions. This is not the same as measuring, multiplying and dividing. The latter are activities. The former are images of what one makes through doing them. (p.15)

1.1.1.2 Example 2: Problem situations from real-life do not naturally lead to explanations involving quantities

We first saw the phenomenon of heavily favoring the algorithm in the first instantiation of our teaching of fractions using measurement, in the Winter of 2013. As we were trying to stimulate future teachers' taste for using fractions to make sense of their worlds, we gave them the task of preparing a group activity for their peers, where they would teach a lesson on fractions inspired by a real-life experience. Somewhat naively, we had thought that this sort of authentic experiences (theirs rather than ours) would engage them naturally in meaningful, quantitative explanations. One of the future teachers was passionate about baking, and proposed a problem-solving situation involving a recipe she had actually tried. She prepared a lesson on conversions between the metric and imperial systems of measurement. She tried this lesson with 8 classmates, acting as her students.

As a preamble, she outlined some features of the two systems for her audience (playing the roles of pupils) and circulated various objects in the group: measuring cups, spoons, a scale, etc. Then, handing out a worksheet, she asked them to "practice conversions" using the following conversion equations as givens (the second and the third contain redundant information):

[1] $1\ cup = 240\ ml = 8\ fl\ oz = 1/2\ \ pint\ (liq) = 16\ tbsp = 48\ tsp$

[2] $15\ ml = 1\ tbsp$

[3] $2\ tbsp = 1\ oz$

The recipe called for quantities such as $480\ ml$, $360\ ml$, $240\ ml$, $180\ ml$, $60\ ml$, and $5\ ml$ to be measured in cups, tablespoons, teaspoons, and ounces. The members of the group were given some time to reflect and solve. After this, the teacher-student asked some of her pupil-classmates, to present their solutions; she praised them for their work, without commenting on the content of their proposals. She then introduced, as a recipe in its own right, what she called "the method of cross product": setting up a proportion, with one unknown, x, and solving for it algebraically to find the desired imperial measures. She did this despite the fact that all of the quantities in the recipe were multiples of $60\ ml$ ($480\ ml$, $360\ ml$, $240\ ml$, $180\ ml$, $60\ ml$) or a divisor of it ($5\ ml$) thus suitable for more intuitive strategies for conversion, and without reflecting on the meaning of the proportion in light of the obvious multiplicative relations involved. Moreover, she ignored some participants' ideas that alluded to such quantitative relations (e.g., $360\ ml$ is $1\frac{1}{2}\ cups$ because it is made of of $240\ ml$ and $120\ ml$, which are $1\ cup$ and $\frac{1}{2}\ cup$, respectively). She also mentioned, in passing, that if decimals are obtained when solving the equation, they *should* be converted to fractions. More surprising was, however, the participants' critique of her lesson: she was praised for the fun and "hands-on" activity (she had passed measuring utensils around), that would be, in their view, very appealing to children. One person from the group did notice that the method may be too "algebraic" for children, and proposed that the x in the proportion be replaced by a question mark. They seemed oblivious to the missing link between the proposed activity and the ensuing calculation: a sensible explanation involving the quantities at hand.

1.1.1.3 Example 3: Reasoning quantitatively to model a realistic situation is difficult

I was recently reminded of the difficulty to explain things quantitatively when the same proportionality problem came up in a real, rather than realistic, situation. My 8-year old daughter came back from her gymnastics class one Saturday visibly upset: her coach didn't believe she had woken up at 8.30 a.m., yet managed to be in the downtown gym by 9 a.m. The coach knew she lived on the South Shore and thought it was impossible to achieve such a feat. Now, I do math with my children quite often, without planning it but using spontaneously arising opportunities: most of the time as playful discussions triggered by an actual quantifiable event, sometimes as pure number games we invent. But this was not the usual ad-hoc game or challenge that we delight in at the dinner table or in the car. Maya asked for my help, because she felt embarrassed in front of her team, and powerless in redeeming her credibility (even if her coach had apparently just joked lightly about it). She had a solid understanding of multiplication, and in relation to it, some grasp of division; she had just learned the metric units of length. I had never talked to her deliberately about speed. However, our Civic has a digital speedometer showing numbers that – the children seem to understand – are related to how fast we are going (they can feel *that*). So I decided to help her solve a mathematical problem to prove she was telling the truth; I had to do it on the spot, and rather fast, so as not to lose the authenticity of the situation and work with her fired up motivation. And from setting up the problem, to explaining, and finally writing it for communication, I found myself navigating a web of decisions and obstacles that underlined the non-triviality of even the most basic modeling for the purpose of teaching. First, I had to use numbers that my daughter could work with: 10.7 km – the actual distance from our home to the gym – could not work because she didn't know decimals; I set it at 12 km, rather than at 10, because it would fit better, I thought, multiplicatively, with the 60 min measure of time needed to conceptualize the speed. Then, I decided quickly that the speed must be 80 km/h, even if 60km/h would have worked to solve her practical question of arriving in time; this decision was rather didactical: 60 km/h would have been too trivial, even for a novel situation such as this one was for her. So the problem for my daughter to solve was: "How long does it takes to get from home to the gym at a speed of 80 km/h, if the distance between the two is 12 km?" Then came the part of explaining what does a reading of 80 on the speedometer mean (Maya seemed to easily accept that 80 is a good representative of the many readings of the speedometer – I chose not to use the word average for now). I told her that it is the distance traveled in 1h, or 60 minutes, and she seemed to get it but when I reformulated the problem again as a proportionality situation she was stuck: "If it takes 60 minutes to travel 80 km, how long does it take to travel 12 km?" I had to improvise a few more problems with the same structure, but more relatable, involving steps and seconds, which she actually played out by walking and counting before arriving at the following critical insight: "I need to find how much time is needed for 1 km, then multiply it by 12 to find out how long the drive was." It was also her idea that dividing 60 by 80 is the same as dividing 6 by 8: at this point I wasn't sure if she had she lost – or rather abandoned – the quantitative connection. Also, I was uncertain about whether she understood this relation multiplicatively, or rather just thought of ignoring a zero (would she have thought, for example that, 42 divided by 56 is the same as 6 divided by 8?), but I let it be in order to maintain the flow in the reasoning. Interestingly, she didn't convert to seconds, which would have made for a division in whole numbers, but drew six circles, which, she said, represent clocks, and broke them into "quarters" to distribute over 8, getting three quarters as a result, which she wrote in words: "trois quarts." I asked "How long is that?" instead of the more school-like "Of what?" or "What is the unit?", and she answered "75 seconds". I thought this came from thinking about money, and said it: "This is not money", to which she answered "Oh, yeah! [laughing] It's 45 minutes…No, wait! This is a minute clock [pointing to the circle], so it's…45 seconds". Multiplying by 12 and getting the final result of 9 minutes was straightforward, and, in hindsight, I thought that 11 km – the number closer to the real distance – would have worked quite easily also. Finally, writing down the solution so that she could convincingly present it to her coach was a lesson in and of itself: she had to include her other assumptions (such as that there are just two lights with a stop time of 30 seconds, or that there was no traffic on a Saturday morning) and a timeline, with the wake-up time, and the time taken for breakfast and getting dressed specified. I encouraged writing the solution schematically, but intelligibly, and using mathematical symbols, in particular the equal sign, grammatically (e.g., she initially wrote 60 km = 80 minutes).

Maya was very satisfied with the result, and was happy to be able to, in her words, "prove I'm right." I was left wondering about the unpredictability of such modeling of reality using mathematics with children, about the constraints posed by the child's cognitive abilities and present state of mathematical knowledge, about walking the thin line between letting her own the explanation and telling too much, and about letting slide some learning opportunities for the sake of the flow of the explanation. I also questioned whether I stayed true to the quantitative approach to problem solving we were professing for future teachers.

The design presented in the thesis is an attempt at equipping prospective teachers with the knowledge base for working with children in such situations involving realistic contexts and concrete objects. The main problem we see, and ultimately hope to address, is the separation that inevitably tends to occur – at the outset, in elementary school – between such contextual problems and numerical calculations, which raises the question: why use word problems with realistic situations at all in school? We think they should be used, but we should develop better ways to reason about them.

1.1.2 The future teacher as a university student enrolled in a mathematics methods course

In Canada, the organization of teacher education is regulated at the provincial level. In the province of Quebec, teacher education is set up as undergraduate programs of Education at universities. These programs take, usually, four years to and include between two and four one-term courses aimed at preparing the prospective elementary school teachers for *teaching mathematics* – they are referred to, usually, as the *math methods courses*. It is hard to even discuss, outside the scholarly domain of mathematics education research, what these courses are about: what is there to learn about the teaching of mathematics that is so simple. Everybody supposedly knows mathematics at this level, and then there are the ubiquitous calculators for the practical day to day needs of doing arithmetic. Besides, prospective teachers learn child psychology or other relevant domains for this level of education in separate designated courses in their program. However, as I tried to show in the above examples, more profound knowledge for teaching children mathematics cannot be assumed in adults who have been through formal schooling. In addition, prospective teachers, and perhaps, most people, don't even perceive a need to acquire such knowledge at university.

Yet there is, for example, cutting edge research of children's intuitive thinking of mathematics as they enter schools, e.g. (Carpenter, Fennema, Franke, Levi, & Empson, 1999), or of children's learning of particular elementary level topics, such as fractions, e.g. (Brousseau, Brousseau, & Warfield, 2014)). There is also the meta-knowledge of mathematics resulting from epistemological and cognitive analyses of various concepts, e.g. (Davydov & Tsvetkovich, 1991). It is the latter that inspired our goal of developing future teachers' quantitative reasoning, but, in addition, we argue that prospective teachers should learn such knowledge as university students. More specifically, we propose a body of knowledge that is rather theoretical, aimed at fostering future teachers' habits of thinking analytically, within a theory that is built, in the course, using definitions and precise language, and justifications using inductive and deductive reasoning. We argue that such an approach not only contributes to the rather formidable task of laying an appropriate intellectual foundation for our students through university education, but, more practically, it distances them from their own learning experiences in elementary school, which has a substantial influence or their attitudes towards learning in math methods courses.

1.2 Outline of the Thesis

Chapter 2 contains the methodology and the conceptual framework. In Chapter 3, I present a preliminary study of the concept of fraction. Chapter 4 is dedicated to the ideas we derived from the preliminary trials in the winter of 2013 and the design conception for the experiment to be carried out in the winter of 2014. Chapter 5 is the description and the retrospective analysis of the experiment, organized according to a framework suggested by the Theory of Didactical Situations (Brousseau, 1997). In the first section we look at the long term organization of the course; in the second, we turn to the analysis of classroom interactions; in the third, we analyze individual students' application of knowledge without mediation by the instructor. Chapter 6 contains a discussion of the results by examining the realization of the two outlined goals.

2 INSTRUCTIONAL DESIGN METHODOLOGY AND SUPPORTING CONCEPTUAL FRAMEWORK

In this chapter I explain in a systematic way how I carried out my research: what one usually includes under *theoretical framework* and *methodology*. What these two categories refer to, in research about teaching according to a *design*, deserves some clarification.

I will start by discussing an ambiguity in the meaning of *instructional design*, on one hand, as a product of practicing research in mathematics education as a "design science", and, on the other hand, as a research methodology that specifies how to conceive and evaluate educational interventions aimed at developing a particular product, such as a curriculum, a sequence of lessons, or a piece of software. With both meanings, as product and as method, research-based design for improving the teaching of mathematics is known as *Ingénierie Didactique* (which we translate as Didactic Engineering, DE) in the French didactique, and as Design-Based Research (DBR) in English language mathematics education research. I discuss how the two paradigms are different, especially as specifications for carrying out research, and explain in what sense my methodology is closer to the French approach to instructional design. I give special attention to the role of theory in the design and formulate, in this context, how I choose to use, instead of a purely theoretical framework, a combination of theoretical perspectives and practitioner knowledge to articulate a *conceptual framework* (rather than a "theoretical framework") as my skeletal structure of justification, as suggested by (Eisenhart, 1991).

Having clarified the categories of *conceptual framework* and *methodology*, I instantiate them for the present study in the second section. I will present the phases of our instructional design as supported by various components of the conceptual framework. In this section I also include the procedures for data collection and analysis.

2.1 INSTRUCTIONAL DESIGN: AS PRODUCT OF AND METHODOLOGY IN MATHEMATICS EDUCATION

A few authors have been promoting the idea, recently, that one of the most fruitful ways of carrying out research in mathematics education is to practice it as a "design science". Wittman

(1995) articulates such a proposal in a convincing manner by answering the foundational question “What is mathematics education research?” (MER), a question which, at the time his paper was published, had emerged in the research community more and more often. The times were ripe, he argued, to define an identity for our domain that reflects its specific status and relative autonomy with respect to related or relevant disciplines such as sociology, anthropology, pedagogy, or mathematics, linguistics, computer science, and logic. This identity can be derived from orienting it towards improving practice. Keeping an eye on practical issues delineates a set of specific tasks, which he labeled as *the core* of MER, distinguishing it from other related areas of research. The components of the core include, in particular:

- *analysis of mathematical activity and of mathematical ways of thinking,*
- *development of local theories (for example, on mathematizing, problem solving, proof and practising skills),*
- *exploration of possible contents that focus on making them accessible to learners*
- *critical examination and justification of contents in view of the general goals of mathematics teaching,*
- *research into the pre-requisites of learning and into the teaching/learning processes,*
- *development and evaluation of substantial teaching units, classes of teaching units and curricula,*
- *development of methods for planning, teaching, observing and analysing lessons, and*
- *inclusion of the history of mathematics education. (Wittman, 1995, p. 357)*

Relations with other disciplines are crucial – warranting an interdisciplinary approach – as they bring about an exchange of ideas and ways of investigating the core systematically, but theories and methods are only relevant insofar as they receive specific meaning by being linked to the core. Only advances in the core count, according to the author, as the measuring stick by which progress in the whole domain should be assessed. In this, mathematics education research has a kinship with other domains, such as music, engineering and medicine:

> *For example, the composition and performance of music must take precedence over the history, critique and theory of music; in mechanical engineering the construction and development of machines is paramount to mechanics, thermodynamics and research of new materials; and in medicine the cure of patients is of central importance when compared to medical sociology, history of medicine or cellular research. (Wittman, 1995, p. 357)*

Moreover, it is the creative process leading to innovations oriented to improving teaching, of both theoretical and practical importance, that provides MER with a robust self-concept, versus appendance to neighboring disciplines. Work within the core, ultimately aimed at producing “artificial objects”, such as teaching units, or sequences of teaching units, to be tested in various educational “ecologies”, defines MER as a design science.

Our research belongs to this kind of work covering almost all the components outlined by Wittman. The product is an approach for teaching fractions to elementary teachers. In developing it, epistemological analysis of the mathematical topic, in particular, of the origin of fractions in human activities versus their status in mathematics as a research domain, was crucial. We also carefully examined and justified ways of delivering the content to prospective teachers and the possible reverberations for teaching it to children by taking into account cognitive aspects – e.g., that teachers must re-learn in a different way what they have seen before, or that encapsulating fraction as a relationship between quantities is a cognitively demanding task. We held under scrutiny the content and the ways of delivering it in regards to the general goals of mathematics teaching by asking questions such as: What should children learn in school? What should prospective teachers learn at university?

Essentially, as Wittman proposed, we practiced mathematics education as a creative endeavor, but also exposed the work to systematic testing:

> *Indeed the quality of these constructions depends on the theory-based constructive fantasy, the "ingenium ", of the designers, and on systematic evaluation, both typical for design sciences. (Wittman, 1995, p. 363)*

The approach was far from being mechanistic – this would obviously raise concerns or, at the least, be naïve, in the domain of education. Wittman quotes Malik (1986) who advances a new "systemic-evolutionary" paradigm for design sciences, which takes into account complexity and self-organization of living systems. This view, in particular, allows, in the domain of mathematics education, a shift from a transmission-of-knowledge model of learning – from the teacher to the passive student – to knowledge arising from a system consisting of teacher, students, resources, constraints, etc. By adopting a "home-grown" theory in mathematics education – the theory of didactical situations by Brousseau (1997) – we followed Wittman's recommendations of using achievements from the core already available: this theory, indeed, served us well for modeling teaching situations in a systemic fashion.

Cobb (2007), in his paper about "putting philosophy to work" for carrying out research, maintains that it is this feature of a theory – its usefulness for understanding learning processes – that should serve as one of the main criteria for selecting one or another theoretical perspective, even more so when mathematics education is practiced as design science. He and his collaborators are some of the pioneer practitioners of design-based research in North-America, while Artigue and several other French researchers have been advancing the mathematics education domain in this way, even before their American counterparts, in the form of *didactic engineering*. Both traditions of research have the overarching goal of providing robust evidence for instructional approaches, thus very much retaining the applied status of the domain.

In time, this led to the specification of a research methodology associated with instructional design, which, on one hand, responds to the need to base the design on some fundamental theoretical principles, and, on the other hand, provides the setup for empirical testing and systematic validation. One then talks about didactical engineering (DE) or design-based research (DBR) also in the understanding of methodologies for theory-driven educational interventions, usually involving some innovative teaching sequence. The similarities as well as the substantial differences between DBR and DE, as outlined by their proponents and active users, make an interesting point of discussion for my research. As part of the more general discussion about carrying out instructional design research, I outline some features of my own methodology, by reference to this comparison, before articulating it concretely, in the next section. This discussion also has important consequences for the role I attribute to theoretical framing in my research – I carry it here as a matter of coping with multiple theoretical perspectives, before spelling out, in the next section, the theoretical constructs used in the present study.

In a recent paper, Godino and his collaborators (2013), highlight an important difference between DBR – a term designating a family of methodological approaches prevalent in Anglo-Saxon literature – and DE – the French-born methodology. Namely, while DE is closely connected to a particular theory – the theory of didactical situations (Brousseau, 1997) – DBR can be supported by a multitude of theoretical perspectives. DE relies on the theory of didactical situations (TDS) both in the conception and the analysis of the design. I will not present TDS here but it is important to mention, at this point, that one of its important assumptions is that, to each piece of mathematical knowledge there corresponds a *fundamental situation* which captures its epistemological essence, in the sense that the target mathematical knowledge will be the optimal solution to the question is raises. This implies that in the teaching-learning situations that we design, there should be an intrinsic, epistemological, motivation to construct this particular piece of mathematical knowledge. Adopting TDS results in the central role of epistemological questions in DE. This justifies the inclusion, as the first phase of DE methodology, of a preliminary analysis of the mathematical knowledge to be taught. The phases of DE methodology (e.g., according to (Artigue, 1989)) are:

1. Preliminary studies
2. A-priori analysis and design
3. Implementation
4. A posteriori analysis and validation

By, contrast, Godino notes, DBR methods focus on the planning, experimentation, and evaluation of educational interventions in naturalistic contexts (corresponding to phases 2, 3, and 4 of DE) while the study of epistemological issues is notably absent. We find the following three phases for design experiments in a paper written by active researchers in this domain in the US, as they are examining their own methods (Cobb, diSessa, Lehrer, & Schauble, 2003):

- Planning the experiment
- Conducting the experiment
- Conducting retrospective analysis

Our methodology is closer to the French tradition of DE in that we, too, cultivate an epistemological emphasis: for example, we undertake a preliminary study of the sources of meaning for fractions. However, we do not consider this kinship as excluding implementation in naturalistic contexts, as Godino seems to frame the difference between the French and Anglo-Saxon paradigms. Moreover, and related to this, we also use the Theory of Didactical Situations, but not, as Godino characterizes DE, to orient the methodology towards testing and developing the theory itself. Of particular interest – the issue at stake here – seems to be one of the aims of teaching postulated by TDS, playing an important role in its early applications to design instructional interventions: achieving a high-degree of *a-didacticity*, i.e., creating situations where students learn through feedback from their learning environment, without direct interaction with the teacher (the teacher's actions are directed only at creating such an ideal *milieu*). In DE's early days, the preliminary analyses aimed at the construction of a fundamental situation for the target mathematical knowledge, on which the design of teaching situations with such a potential could be based. We, too, search for good problems, but acknowledge, based partially on theoretical considerations, but also on initial trials that this is hardly achievable, especially at university. Yet later developments in the French paradigm of research do accommodate more naturalistic observations (Artigue, 2009; 2014) where TDS takes into account also the importance of student-teacher interaction in learning. We use TDS primarily as a way of studying learning in a systemic manner – i.e., as an analytical framework rather than as design program. We focus on the relations among three systems – the student, the teacher, and the content – rather than on one of them alone, or just on the relation between two of them. Viewing the process of learning like this accommodates other theoretical perspectives, such as Vygotsky's sociocultural theory of learning through acculturation, or sociological theoretical perspectives that account for the strong imprint of particular institutions. I combine multiple theoretical perspectives as well as practitioner knowledge (that is, our knowledge of mathematics and our experience of teaching it) to produce what Eisenhart called, in her often quoted paper about the use of frameworks in one's research projects, a *conceptual framework* "as a skeletal structure of justification" (Eisenhart, 1991, p. 209).

In terms of the evaluation of the design, both DBR and DE methodologies differ from classical design experiment in that they are mainly qualitative and do not measure pre and post characteristics of experimental and control groups. But the two differ in the following notable aspect: in DE one deals with hypotheses created before the experimentation phase, which are sought to be validated explicitly, versus micro-adjustments throughout the experimentation in DBR [e.g. (Artigue, 1989), (Cobb, McClain, & Gravemeijer, 2003)]. This stance, Godino argues, also relates to the role of theory in the research:

> *Although both approaches are mainly qualitative (or mixed), an important difference is the establishment in DE of a priori hypotheses (before the experience design), while DBR tends to a qualitative posture, assuming that theories emerge from the data. Although the methodology stages are very similar, there is a greater influence of the theory in DE that seeks explicitly validation (even if internal) of the previous hypotheses, and the previous analyses of the design are detailed and complete. (Godino, et al., 2013)*

Our research is closer to DE in that we establish overarching goals at the beginning and use them as foci for observation in the experimentation phase, but do not treat them as hypotheses that are either confirmed or infirmed. In particular, we do not predict expected behaviors of the students, but rather adopt an interpretivist view to make "empirically grounded claims about the conditions for the possibility of all students' learning" (Cobb, McClain, & Gravemeijer, 2003, p. 4). Some components of our conceptual framework emerged, as predicted by Eisenhart, while carrying out the intervention or its analysis.

2.2 PRESENT STUDY: PHASES OF THE RESEARCH AND SUPPORTING CONCEPTUAL FRAMEWORK

I describe here the *instructional design methodology* undertaken in this study (I use neither the terms DE or DBR) by detailing the phases of the research, each supported through an argumentation structure with elements of my conceptual framework.

Drawing on Eisenhart (1991), I build a structuring framework for the research by combining theoretical and practical elements insofar as I find them useful for my research in instructional design. I quote from this author:

> *Crucially, a conceptual framework is an argument that the concepts chosen for investigation, and any anticipated relationships among them, will be appropriate and useful given the research problem under investigation. Like theoretical frameworks, conceptual frameworks are based on previous research and literature, but conceptual frameworks are built from an array of current and possibly far-ranging sources. The framework used may be based on different theories* <u>*and*</u> *various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem. (Eisenhart, 1991, p. 209)*

Cobb (2007) considers the "usefulness" criterion particularly relevant when mathematics education is viewed as design science, and frames it in terms of the work that different theoretical positions can do to contribute to our understanding and realization of learning processes. The second criterion, he posits, should be how the individual is conceptualized.

The Theory of Didactical Situations (TDS) is an essential component of my analytical framework, perhaps the overarching one. In particular, it inspired the epistemological analysis of the concept of fraction that we carried out in the preliminary phase of the research, where we asked the question: what are the sources of meanings for fractions? Secondly, through the concept of *fundamental situation* we were able to look at what kind of problem, or set of problems, are good problems to incite the desired knowledge of fractions. But the most attractive part, for us, in this theory, is its view of learning as a process that balances *acculturation*, i.e., student's interaction with the teacher, and *adaptation*, i.e., student's interaction with the problems set up for her. From here we zoom in to conceptualize each of the two processes: the Vygotskian perspective for teacher-mediated learning of culturally constructed concepts, and a cognitive model of a learner interacting with a mathematical problem (Balacheff, 2013). I also employed the distinction between theoretical and practical thinking, developed by Sierpinska and collaborators (Sierpinska, Nnadozie, & Oktaç, 2002; Sierpinska, 2005; Sierpinska, Bobos, & Pruncut, 2011) in the context of teaching linear algebra to high-achieving students at university. The categories proposed in Sierpinska's model were useful, on one hand, in giving a finer description of the mental processes required or encouraged by certain tasks, and, on the other hand, in justifying the content we proposed in view of the goals of mathematics education at different levels. An essential component of my conceptual framework resulted from zooming out (i.e, beyond the didactical situation) to consider the mathematics methods course as an institution whose specific characteristics imprint on the learning of mathematics that takes place.

TDS keeps in center stage the piece of mathematics onto which teaching and learning occurs; this sensitivity to content also called, extensively, on my own knowledge of mathematics (sometimes logic an, as well as my analytical stance on teaching mathematics at college or university – these components represent the practical framework in my research.

In the following sections, I describe all these theoretical and practical perspectives in connection to the different stages of our research.

2.2.1.1 Phase 1: Preliminary studies of the concept of fraction

The first phase of the research took place sometime before the first time the course was taught in the Winter semester of 2013. I report on it in Chapter 3.

The course in question was a Teaching Mathematics course – colloquially called a "math methods" course – in an Elementary Education undergraduate program at a North-American university. Two large areas of content had to be covered in a 13-week long course: fractions and geometry, with fractions taking up most of the time, 8 weeks. The research team consisted of myself, the instructor of the course (hereinafter called the Instructor), and another graduate student, interested in the geometry part of the course.

In this phase of the research we aligned with the Theory of Didactical Situations in the preoccupation with issues of epistemology of the piece of mathematics concerned, in our case – fractions. The preliminary study thus took the form of a literature review, with the following angle: how does the concept of *fraction* in school mathematics compare to that of *rational number* in scholarly mathematics? This initial search was partly prompted by our background in more advanced mathematics: I had a graduate degree in pure mathematics, with a specialization in mathematical analysis, while the Instructor had been a research mathematician for a few years in the domain of abstract algebra. We both had taught exclusively in mathematics departments, at college and university, and, as mathematics education researchers, our interests were in the teaching and learning of university or college level mathematics, such as linear algebra, calculus, or, more recently, the algebra topic of absolute value inequalities. In the textbooks we had at our disposal – for teaching fractions to future teachers – the notion of rational number seemed to have emerged anew, we began to think, as a result of an enormous amount of research on the learning and teaching of fractions with somewhat converging results that made their way into teacher training materials, reform documents, and even textbooks for children. Fraction seemed to inhabit a totally different world here, one that we needed to explore in more depth: "slices of pizza" and "pairs pf numbers" were certainly not just the two endpoints on a path of increasing complexity and abstraction.

That mathematics as a school subject is practiced quite differently than mathematics as a science is quite visible, but the appearance is that school mathematics evolves, as a body of knowledge, somewhat by accumulation – of content and methods – into scholarly mathematics. This "illusion" is created, on one hand, by the organization of the mathematics curriculum across the grades, culminating with the study of academic mathematics at university in science programs, pure or applied. On the other hand, even for non-science academic paths, we hold this idea of children emulating aspects of the mathematicians' work: solving situational problems, reasoning and reflecting, and using mathematical language (Quebec Ministry of Education, 2001), or even having a productive disposition towards mathematics (Kilpatrick, Sawfford, & Bradford, 2001).

The purpose of the preliminary study was to examine the relations between knowledge of fractions in scholarly mathematics and knowledge of fractions in school, as portrayed in research papers, textbooks for future teachers, or textbooks for children.

Our distinction between kinds of knowledge on fractions can be explained in terms of Chevallard's (1985a) anthropological theory of didactics (ATD). According to ATD, it is impossible, or uninteresting, from the point of view of the mathematics educator, to describe how one knows a particular domain of mathematics if the institutional context in which the knowledge is formed is not taken into account. In an elementary mathematics textbook, for example, proper and improper fractions are treated separately, the latter perhaps even after some arithmetic operations on fractions are introduced. From the point of view of the educator, this organization may be justified by a perceived cognitive difficulty associated to "taking more parts than the whole" (i.e., given $\frac{5}{4}$ how can you take 5 parts out of 4 parts?). On the other hand, in academic mathematics – say, in an abstract algebra book – fractions are defined as equivalence classes of pairs of integers or ratios of integers, *any* integers (with some restrictions), and, for the mathematician, the distinction proper/improper fraction is irrelevant. Both pieces of knowledge may be labeled, generally, as mathematics, but it is only when the institutions in which they "live" are considered, that the logic behind these different organizations of knowledge becomes comprehensible. For Chevallard, *institution* is a primitive term (he does not define it); he provides a framework for studying mathematics knowledge as a human activity among others, as it is practiced in various institutions, such as theoretical or applied mathematics, engineering mathematics, school mathematics, etc.

Within ATD, school mathematics and scholarly mathematics are different institutions because they entail different types of "know-how", organized and legitimized by different theoretical assumptions. In the preliminary study phase of our research we follow Chevallard's macro-level approach perspective to viewing institutions, by distinguishing between research mathematics and school mathematics. (We take a micro-level perspective only in the next phase – the design conception, by looking at *the institution of math methods courses at university*). We look at mathematics or history of mathematics books, for the first, and at mathematics education research papers and teacher training materials, for the second.

The reasoning behind this has to do with the purpose of studying mathematics as institutionalized practice, following Chevallard's program: going beyond the analysis of what is designated as knowledge of fractions in different contexts, getting at the rationale for designating it as such, and thus unpacking the "science" behind it.

This approach allowed us to identify two types of knowledge of fractions in scholarly mathematics – abstract algebraic and analytic – and two instructional approaches for fractions promoted for school mathematics, the partitioning and the measurement approach. To understand the relations between them, inspired by Chevallard (1985a), we isolated the theoretical justifications that support each of these types of knowledge. This allowed us to relate scholarly mathematics to school mathematics knowledge of fractions, not only with respect to the content, but also concerning more general epistemological aspects, such as ways of knowing or goals of learning.

This discussion, based on scholarly materials corresponding to the standards of practice – from mathematics and mathematics education research – enabled us to edge on answering the following more fundamental question, which bears also on our self-perceived identities as practitioners of mathematics and as mathematics educators: what kind of mathematics should there be in school mathematics (especially as concerns fractions)? In Chevallard's analytical terms: what kind of *didactic transposition*, if any, is envisioned, from research mathematics to school mathematics?

We report on this preliminary studies in Chapter 3.

2.2.1.2 Phase 2: Preliminary trials and design conception

In the second phase of the research, we conducted preliminary trials of teaching fractions in measurement contexts. This was in the Winter of 2013, in the above mentioned mathematics methods course. We reflected on this experience and developed the Measurement Approach in a mature form with more solid mathematical and didactical foundations, or, more generally, the features of a clearly defined design, ready to be tested in the classroom. I report on this phase in Chapter 4, but clarify here the theoretical perspectives that guided the reflection and ensuing design conception.

In the preliminary studies we looked at what research (and mathematics) tells us about teaching fractions in elementary school. Our epistemological analysis inspired the idea of teaching fractions to prospective teachers using measurement activities, more generally, of forwarding a concept of number as measure, as conceived in mathematical analysis. We assumed that future teachers' learning would occur in a way predicted by Brousseau's idea of learning by *adaptation*: namely, the student, by being confronted with a good problem, acting by her own motivation, produces the desired piece of mathematical knowledge as the optimal solution to that problem. It is the teacher's role to pose problems that have the potential to result in the desired knowledge. The teacher, ideally, searches for this epistemological essence to inform the teaching process and then constructs corresponding problems to be solved by students. The essence of fractions was measurement, and we had this idea of teaching fractions through situations of problem posing on measurement, where we would help students look at their everyday life and environments to come up with interesting problems on measurement. The Instructor modeled this behavior in class: she described real-life situations where the notion of fraction serves for purposes of measurement, and solved various mathematical questions involving fractions resulting from these situations. Alas, the students not only did not engage in such behavior but even resisted the ideas promoted in the course, very strongly, sometimes even displaying an emotional attachment to certain ways of thinking.

This led to the restructuration and enlargement of our conceptual framework. We saw the need, in particular, to zoom in at a more micro-level institutional analysis, by considering the Teaching Mathematics courses as an institution in and of itself, the same way as we had considered the institution of Prerequisite Mathematics Courses in our previous research (Sierpinska, Bobos, & Knipping, 2008). Our students in the Teaching Mathematics course were not behaving at all like the students in our mathematics courses. Most notably, discussions about a mathematical item were hijacked by students' expressed views about what children should do in class, for example, engage in "hands-on activities", or about how children learn, for example, “visually”, or about what “works in schools.” We were able to identify, empirically, several positions taken by prospective teachers in the course, seen as an institution, and the tasks that we were giving them influenced these positions. We describe the positions of Former Pupil, Teacher, Popular Educator, and University Student resulting from the preliminary trials in Chapter 4.

In the design, we focused on how we wanted to shape the University Student position in the institution of Teaching Mathematics courses, and this, in relation to the goal of teaching fractions through measurement as a way of developing quantitative reasoning. This multilayered structure of goals speaks of the complexity of training teachers: the goal of developing a concept of number in the context of quantities is embedded in the larger goal of shaping future teachers’ as theoretical thinkers, able to think systemically, reason analytically, and engage in reflection.

First, we had to reconsider the concept of *fundamental situation for teachers*. For children, who have never seen fractions before, constructing fractions in context of measurement out of an *epistemological need* may work: when one wants to measure a quantity and the existing units of measurement do not represent it precisely enough, a change of unit into finer ones may be necessary.

But it is not clear what would make future teachers *want to know fractions as measurement*. This brings to light a larger theoretical debate about what counts as “occupations” for them in the sense of Dewey (1900/1990): so as to be motivating through one’s own experience but, at the same time, to embody social and scientific values?

The answer is not obvious.

A large-scale study by Frykholm (1999) shows that beginning teachers do crave practical advice about classroom teaching. The teachers in this study felt that, despite believing in the ideals of the reform, they couldn’t implement them in class, because they were not made to enact them in methods courses. One student, for example, remarked:

We need more practical applications to go along with the theory. We always talk about providing real-life experiences and connections in math classes. Shouldn't the same thing be done for us in math-ed classes? (Frykholm J. , 1999, p. 94)

But their view of what counts as practice may be at odds with their instructors'. The following excerpt from a conversation with an instructor of mathematics methods courses illustrates this point:

Halfway through the course, I […] said, "Is there anything we haven't talked about? […] They were saying, "Well, when are going to talk about lesson planning?" I thought we had been talking about lesson planning the whole time. We've been talking about how you'd teach this, there was an implicit lesson structure to that, but it hadn't had lesson planning in big letters at the top of it, so we hadn't done lesson planning. (Anonymous, 2009)

Moreover, their expectations may be embedded within a widespread view, in popular culture, of the role of teacher education programs. Paul Bennett, an experienced educator and the lead consultant at Schoolhouse Consulting Inc, spoke on the CBC show *The Current*, to decry Ontario's Ministry of Education reform of teacher education programs in 2013, in particular, the plan to extend the teacher training programs – previously one-year long – to two years. He argued that teacher training should emphasize practice, rather than theory, and insisted on certain outcomes of university teachers' education as crucial, while theory is actually a danger to avoid. We can glean at what he means by practice from the following excerpt:

[Future teachers] need to learn essentially how to work with parents, how to build partnerships, they need to know more about classroom management, they need to know how to motivate students and they need […] more practical experience, less theory. So, there's a danger here that in making [the program] two years they'll get more theory, not less, and less practical experience. (Bennett & Johnson, 2013)

This popular view, in fact, clashes with Dewey's progressive view of what should be done in methods courses. Even over a hundred years ago, when the idea of teacher education belonging in the research university was in its infancy, at a time when elementary teachers were mostly trained in normal schools, with a curriculum focusing on the technical issues of effectively managing large groups of children, John Dewey maintained that the most fundamental aspect of teacher education in colleges should be theory. In his essay *The relation of theory to practice in education*, Dewey (1907) inquired, in his characteristic dialectical approach, if emphasis in teacher education in college, especially in the part he refers to as "practice work in teacher education" (perhaps the equivalent of today's methods courses), should be put on practical, how-to-issues, or on theoretical aspects, that build on subject matter knowledge as well as the fundamental disciplines that inform education, such as psychology and philosophy. His answer was that both are important, but academic training of teachers should be an ultimately intellectual affair:

> *On one hand, we may carry on the practical work with the object of giving teachers in training working command of the necessary tools of their profession; control of the technique of class instruction and management; skill and proficiency in the work of teaching. With this aim in view, practice work is, as far as it goes, of the nature of apprenticeship. On the other hand, we may propose to use practice work as an instrument in making real and vital theoretical instruction: the knowledge of subject-matter and of principles of education. This is the laboratory point of view. The contrast between the two points of view is obvious […] From one point of view, the aim is to form and equip the actual teacher; the aim is immediately as well as ultimately practical. From the other point of view, the immediate aim, the way of getting at the ultimate aim, is to supply the intellectual method and material of good workmanship, instead of making on the spot, as it were, an efficient workman. Practice work thus considered is administered primarily with reference to the intellectual reactions it incites, giving the student a better hold upon the educational significance of the subject-matter he is acquiring, and of the science, philosophy, and history of education. (Dewey, 1904, pp. 9-10)*

In the design conception, we sided with Dewey, in seeing the role of university education in methods courses as that of training autonomous, intellectually fit individuals, who would be able to justify their claims in the classroom, and further grow with practice. This view is also promoted as one of our province's reform ideals (Martinet, Raymond, & Gauthier, 2001). We saw this as learning out of a *cultural* need.

But we also preserved Dewey's "occupations" concept (employed by him only in the context of children's learning), as well as Brousseau's idea of learning out of *epistemological* need, by imagining *a fundamental situation for teachers* as the problem of explaining a piece of mathematics involving fractions to children by means of *realistic problems*. We define, in turn, realistic problems, not as some real-life, complex "situational problems", but as problems usually encountered in elementary mathematics, for example:

> There are 8 people in a lifeboat. They have only 5 litres of drinking water and they must share it equally.
>
> (a) To what fraction of the whole $5\ L$ will each person be entitled?
>
> (b) To what fraction of $1\ L$ will each person be entitled?

The concept of fraction of quantity would be the optimal solution to the instructional problem of solving this elementary mathematics problem with children.

In the next section we talk about the concepts that framed the Instructor's actual teaching in the design implementation.

2.2.1.3 Phase 3 Implementation and Phase 4 Retrospective analysis

I use concepts of Brousseau's Theory of Didactical Situations (Brousseau, 1997) to describe and analyze the implementation of the design through the Experiment. This phase is reported in Chapter 5.

Several researchers, working in the French didactique paradigm, have recognized that it can be very difficult to create a-didactical situations, i.e., to cultivate a "breeding ground" for the epistemological need to acquire a piece of mathematical knowledge, especially when it comes to the more abstract nature of knowledge taught at university (Perrin-Glorian, 2008; González-

Martín, Bloch, Durand-Guerrier, & Maschietto, 2014). We have experienced this difficulty first hand, in the Pilot-Study, and discussed the issues that contributed to it, that one encounters in the reality of teaching mathematics to future teachers at university. This fact does not make TDS any less interesting: one need not interpret it as an ideal model of practice to pursue. When it is understood as a framework that helps understand practice, in particular aspects of it that are crucially linked to the mathematical knowledge at stake (Herbst & Kilpatrick, 1999), it can be very productive. TDS can be used for the analysis of a mathematical teaching practice, even when the researcher does not intervene in the design of the lessons (Hersant & Perrin-Glorian, 2005).

Artigue (2009), whose work with the methodology of didactical engineering (DE) has been consistently based on TDS, emphasized the increasing importance of naturalistic observations in the classroom (versus the previous, much more heavily-scripted interventions) as one of the features of the TDS-inspired methodology today. Reacting to Gonzales's paper about the use of TDS at university, in particular to their claim that "the milieu at these levels usually comprises abstract mathematical signs, which are not yet seen by the students with their entire mathematical meaning" (González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014, p. 120), Artigue (2014) stressed the effectiveness of TDS in the presence of three invariants – all, indeed, guiding my research. She stated:

> *However, as a researcher familiar with TDS, I consider it to be a theory that combines three important ingredients for research in mathematics education in an original and powerful way:*
>
> - *a systemic approach, and a vision of learning balancing adaptation and acculturation processes;*
> - *an epistemological sensitivity whose value for research at university level cannot be denied; and,*
> - *through the idea of didactical engineering, a powerful technique for elucidating didactical phenomena and testing research hypotheses.*

What it offers to research at university level finds its source in these fundamental characteristics. (Artigue, 2014, p. 138)

More importantly for us, this understanding of TDS accommodates the Vygotskian perspective, according to which scientific knowledge has to be mediated by a teacher (Vygotsky, 1986). The theory of fraction of quantity which we developed in the measurement approach, in particular, could not be learned adaptively, without teacher mediation. But this does not preclude a certain autonomy of the learning individual. Here is how. In our approach, the concept of fraction of quantity was introduced explicitly via a definition using precise language, and further conceptualizations of operations and relations between fractions of quantity were achieved within a theory, which we called “the fraction of quantity theory” (FoQ theory). Once the definition of fraction of a quantity has been accepted by the student as an element of a mathematical theory, he or she could argue about the validity of statements in this theory without regard to the instructor’s authority as the teacher in the course or a mathematician, but only by reasoning, as being consistent or not consistent with the accepted definition. The truth of the logical and mathematical consequences of a definition does not depend upon the authority of the utterer, but only on the soundness of the reasoning. The students had the same tools at their disposal as the teacher and in this sense were equal to the teacher, and therefore autonomous in their reasoning. This reasoning is, however, cognitively quite demanding. In the specification of the Measurement Approach we describe the level of cognitive sophistication required to comprehend these concepts using Sierpinska et al.’s model of theoretical thinking (Sierpinska, Nnadozie, & Oktaç, 2002; Sierpinska, 2005; Sierpinska, Bobos, & Pruncut, 2011).

TDS, as a systemic approach, also serves as an organizer for the retrospective analysis of the teaching experiment. I will use the following four-pole diagram proposed by Brousseau, quoted in (Perrin-Glorian, 2008), to describe teaching as an activity balancing the processes of acculturation and independent adaptation (Figure 2.1). Knowledge emerges as a result of three didactic relations.

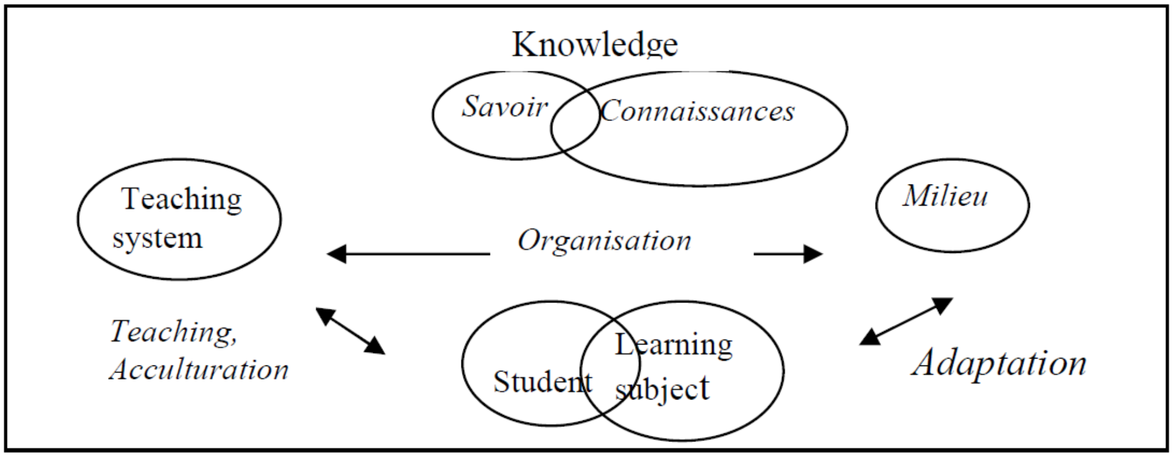

Figure 2.1. A system for describing and analyzing the Experiment, inspired by (Perrin-Glorian, 2008)

I will separate the analysis in three levels, each corresponding to one of the arrows in the diagram, i.e., to one of the processes of interaction:

1. **Organization** refers the teacher's influence on the milieu, understood largely as all with which the student interacts: tasks, instructional resources, peers. More precisely, I use the term *milieu*, in the way suggested by (Hersant & Perrin-Glorian, 2005) as the "objective reference for interactions" (p. 144).

At level 1, I look at the overall organization of the teaching sequence – the unit of analysis is the whole course. In chapter 4, I describe the *savoir*, or the target mathematical knowledge, thus already touch on aspects in the organization of the milieu by the Instructor, especially with regards to the underlying theoretical structure (the Measurement Approach). In the retrospective analysis I look at other more concrete aspects of the organization of the resources in the course, in particular with respect to the actions required in tasks.

2. **Acculturation** is the direct interaction between teacher and student. The *Student-Learning subject* distinction is appropriate here, in that it points to the influence of the course institution on the learning that takes place in such interactions: one may act out of an interest to pass an upcoming test on a given topic, versus out of a need to understand it.

At level 2, I take the classroom community as the unit of analysis, and look at how the participants – Instructor and FTs – interacted against a given mathematical backdrop.

3. **Adaptation** refers to the interaction between student and milieu. In TDS this point of interaction has been, traditionally, the most interesting to study. This is because the students' *connaissances*, individual knowledge developed by action on the milieu, concretized in specific behaviors, helps reveal the functioning of the "black box" milieu.

At level 3, I zoom in at an even finer grain level, on the cognizing individual, by looking in great detail at the students' solutions to a given problem, without direct mediation from the Instructor. I use a structuring framework for the analysis here inspired by Balacheff's (2013) model of learners' conceptions, which portrays *the connaissances* of the learning subject rather than *the savoir*, targeted by the Instructor. We examine students' behaviors in problem solving by asking the following questions:

- What <u>problem</u> is the learner, de facto, solving?
- What <u>strategies</u> is she using and how does she establish the <u>validity</u> of her solutions?
- What <u>language</u> does she use to solve it?

This approach is meant to cover the whole sphere of practice in students' responses to the mathematical problem proposed, and has the merit of modeling the rationality of students' solutions (i.e., versus, for example, dismissing a solution as incorrect from the point of view of *savoir*).

The separation along the three didactical relations in the retrospective analysis is inevitably somewhat artificial: for example, the student-milieu interaction can be observed also at the second level of analysis as we look at the interactions between participants in the course. But this organization of the analysis serves our purposes well, not only because it preserves, at all levels, in line with the TDS inspiration, the sensitivity to the mathematics involved, but also because, more generally, it makes possible the ambitious goal of the overall evaluation of an instructional design which is, in principle, different from, say, a unilateral measurement of students' learning of a particular topic or an evaluation of the available instructional resources. Validation of the design should be internal – between the a-priori and a-posteriori analyses, as emphasized by Artigue (2009) – but, I maintain that it does not concern solely FTs learning in the course. At all three levels of analysis I can examine in a systemic way the two foci of interest for my research: the quantitative reasoning and the shaping of the position of University Student. At the first level, for instance, even if the student is not featured, we could look at the possibility to create tasks in the MA that destabilize the Former Pupil position we have seen the students taking so often taking in the Pilot-Study. The second and the third levels, however, give greater insight into the more practical goal of appraising students' learning.

I added an essential methodological tool in hindsight – after having analyzed most of the data and realizing I had used it extensively – which was not inspired by my theory-driven conceptual framework but by my background as a mathematics instructor at college level. It is what the literature on research methods calls the practical framework (Eisenhart, 1991). I used mathematics and logic, for example, to describe the structure of mathematical statements and examine students' solutions. I relied on my own quantitative reasoning in discussing modeling situations. In my analysis of the lectures I resorted to the same analytical framework that I employ extensively in my organization of lectures at university or college: according to it I think of each problem is an "episode" aimed explicitly at making a particular point about the mathematics at hand.

2.2.1.3.1 Data sources and methods of analysis

The research team consisted of myself and the Instructor for the design experiment on fractions of the present study. We conceived the design together in the first two phases of the design experiment through the *Preliminary studies of the concept of fraction* and reflections that ensued after the *Preliminary trials*. I did not actively participate in the Implementation of the Experiment: I sat in the classroom, as a field observer, throughout all the lectures and the labs in both the Winter 2013 and Winter 2014 instantiations of the course (the Pilot-Study, and the Experiment, respectively), but I did not teach or act as a teaching assistant in the course. However, on occasion, I talked to the students informally, sometimes at their request to explain something, other times at their initiative of sharing their experiences in the course. I collected the data and performed the Retrospective Analysis.

The following data was generated in the two runs of the course:

(1) Two sets of course artefacts (Winter 2013, Winter 2014): course outlines, Power Point lectures, Instructor's and teaching assistant's notes from meetings or e-mails with students;
(2) Two sets of copies of the students' written work for the course (Winter 2013, Winter 2014);
(3) One set of detailed classroom field notes from lectures and labs along with pictures of the board and easel for the Winter of 2014;
(4) One set of notes of the research team meetings for the Winter of 2014.

The data was extensive, and the project of examining the design within the boundaries of a doctoral dissertation rather ambitious. As mentioned, we used the TDS conceptual framework to deal with this complexity in order to give an overall portrayal of what went on.

We followed the interpretative research paradigm where "key incidents" are used to illustrate assertions about the design. The inspiration comes from a paper by (Knipping, Straehler-Pohl, & Reid, 2012), where one episode in a mathematics classroom is used to illustrate how the teacher's discourse in the span of only one problem-solving session reinforces the stratification of the class into those who do and those who do not have access to the vertical discourse (Bernstein, 1999). The "key incident" is thus "key" in that "it represents concrete instances of the working abstract principles of social organization" [(Wilcox, 1980, p.9) quoted in (Knipping, Straehler-Pohl, & Reid, 2012)]. The data analysis, in our case, follows entirely the method of "key incident analysis" – we maintain that design experiments cannot be evaluated quantitatively, using more than simple descriptive statistics, although studies of particular variables amenable to such research can be generated from the study. As observed by the researchers from The Design-Based Research Collective (2003), however, generating randomized trials is not the end goal of design based research:

> *However, randomized trials are not necessarily the appropriate end goal of our research approach; we do not understand issues of context well enough yet to guarantee that randomized trials are the best means to answer the questions we*

> *care about. The use of randomized trials may hinder innovation studies by prematurely judging the efficacy of an intervention. Additionally, randomized trials may systematically fail to account for phenomena that violate this method's basic assumptions—that is, phenomena that are contextually dependent or those that result from the interaction of dozens, if not hundreds, of factors. Indeed, such phenomena are precisely what educational research most needs to account for in order to have application to educational practice. We would suggest, however, that design-based research can generate plausible causal accounts because of its focus on linking processes to outcomes in particular settings, and can productively be linked with controlled laboratory experiments or randomized clinical trials (cf. Brown, 1992) by assisting in the identification of relevant contextual factors, aiding in identification of mechanisms (not just relationships), and enriching our understanding of the nature of the intervention itself. (The Design-Based Research Collective, 2003, p. 6)*

To give an example, "key incidents" included a student's question during the Instructor's office hours: *How can I explain the equivalence of fractions*? The Instructor's efforts to help the student triggered a way of thinking that led to the creation of the definition of fraction of quantity. Another example of a key incident was in a lecture, where the Instructor, in reaction to students' response, had to change her planned mode of reasoning about quantities. I also classify, however, as a "key incident" students' written productions in response to a question given at the end of the course, despite the fact that I report it in a summary form of description with percentages of students giving one or another interpretation to a given problem. Based on such simple, descriptive, statistics, I make no quantitative claim.

Any general statements made in the report, were, however, not the results of analyzing single episodes, but rather of systematic reading of the entire data corpus and using Glaser and Strauss's method of producing categories grounded in the data through an iterative process of conjecturing, refining, or refuting emergent rubrics (Glaser & Strauss, 1967), using data triangulation. In doing this, the goal is to describe and explain the classroom realizations in relation to the a-priori analysis, but as Cobb, McClain and Gravemeijer (2003) contend, rather than implying homogeneity of all students' learning, we make "empirically grounded claims about the conditions for the possibility of all students' learning" (Cobb, McClain, & Gravemeijer, 2003, p. 4). Bednarz and Proulx (Proulx, 2005; Bednarz & Proulx, 2005) suggest, in fact, that the impact of teacher education is hardly controllable: case studies of five pre-service teachers' practices revealed that even from the same university methods course, teachers appropriated very different elements. While three of them grasped the general principles and themes brought forth in the program, one saw the course as a source of inspiration for teaching resources, such as specific activities or problems to be used in his teaching, and one took what he learned for granted, guaranteed to be optimal, and replicated it in his teaching without actually knowing why.

We maintain that it would be naïve to claim that the goal of a teaching experiment is to establish *what works*. The goal of design experiments in educational research, following (Cobb, diSessa, Lehrer, & Schauble, 2003), is rather to develop theories that allow us to better understand a "learning ecology."

Both the data collection and the empirical analysis were deliberate and theory-driven in the sense that I did not enter into the collection or the analysis of the data with a *tabula rasa* mind. As Erickson (1986) pointed out in his seminal paper on interpretative research, such romantic vision of the fieldworker is not warranted by contemporary philosophy of science or cognitive psychology. On the contrary, my participation in the course setting and subsequent analysis were guided by the assumptions about the sources of meaning of fractions appropriate for elementary school and the conjectures about how they could be taught to prospective teachers provisioned in the design. I agree with Erickson (1986) that such a conception of fieldwork and analysis, as *deliberate inquiry into a setting*, "does not place shackles on intuition and serendipity", but rather highlights the main concern of method as that of "[bringing] research questions and data collection into a consistent relationship, albeit an evolving one" (Erickson, 1986, p. 140).

As already mentioned, in the description and analysis of the Experiment, I had two, interrelated, foci of interest, inspired by the a-priori analysis and design: the reasoning about fractions quantitatively (of all participants), and the positions of future teachers in the Teaching

Mathematics institution, in particular the shaping of the University Student position. The retrospective analysis takes the shape of an in-depth discussion of the realization of the two goals set for the design.

3 Preliminary studies: the concept of fraction in school mathematics and in research mathematics

In this chapter, I examine the relations between knowledge of fractions in research mathematics and knowledge of fractions in elementary mathematics education, particularly such knowledge of fractions as research papers, textbooks for future teachers, or reform documents claim teachers should possess.

3.1 Fractions in mathematics

In academic mathematics, the term "fractions" is used as an informal name for elements of the set of "rational numbers" as a subset of real numbers in mathematical analysis, or in reference to a generalization of rational numbers in abstract algebra, "the field of fractions of an integral domain." Accordingly, there are two types of knowledge of fractions in mathematics: analytic and algebraic. We will show that the two conceptualizations rely on different theories to "produce numbers" – *the theory of measure and the principle of permanence* – which result in different epistemologies of the numerical domain. We start by presenting the algebraic conceptualization of fraction.

3.1.1 Fractions in abstract algebra

In abstract algebra, a rational number is defined formally as an element of the "quotient set $\mathbb{Z}\times \mathbb{Z}^*/\sim$, where $(a,b)\sim(c,d)$, if and only if $a\times d=b\times c$." (MacLane & Birkhoff, 1967) More informally, this means that a rational number is a pair of integers (a,b), where $b\neq 0$, and it is assumed that two such pairs (a,b) and (c,d) represent the same rational number if $a\times d=b\times c$. The set of all pairs equivalent to a particular pair (a,b) is then denoted by the symbol $\frac{a}{b}$[1]. So the rational number represented by the symbol $\frac{a}{b}$ is, in fact, a whole set, namely the set of all pairs of integers (c,d) such that $a\times c=b\times d$. The theory then defines operations of addition, subtraction, multiplication and division on the rational numbers in this

[1] In formal abstract algebra, the symbol $[a,b]$ is also sometimes used.

sense, and proves that these operations are well defined (i.e., do not depend on the choice of the representative pair of each rational number[2]), and that addition and multiplication have "good properties": are commutative and associative. These definitions and proofs, and some other details, then allow to claim that the system of all rational numbers with the four operations constituted a "field", an algebraic structure defined axiomatically by a certain number of properties (such as commutativity, associativity, and invertibility of addition and multiplication). A field allows more operations than a "ring" (e.g., integers form a ring, but not a field); it allows division of any element by any other element that is not zero.

This notion of rational number is based on a constructive approach to number systems whereby new sets of numbers are constructed from old ones by pairing: the integers are constructed as pairs of natural numbers so that addition is invertible, while the rational numbers are constructed as pairs of integers so as to make also multiplication to be invertible.

The role of theory for this development of numbers in general, and rational numbers in particular, is played by "the principle of permanence", which states that number systems should be extended from a "nucleus" (i.e., natural numbers) so that the observed regularities in the arithmetic of the original elements should be preserved in the extended systems. This principle can be found in the first theoretical treatment of rational numbers in the works of Bolzano and Ohm in the 19th century (Mainzer, 1991). In 1851, Bolzano developed a theory of rational numbers, as sets of numbers that are closed with respect to the four elementary arithmetic operations, while Ohm, in 1834, in a similar fashion, was interested in defining rational numbers "solely through the basic truths relating to addition, subtraction, multiplication, and division" [quoted in (Mainzer, 1991, p. 22)]. In 1867, Hankel pinned down explicitly the idea behind the algebraic construction of number systems: "The laws of these operations determine the system of conditions […] which are necessary and sufficient to define the operation formally." [quoted in (Mainzer, 1991, p. 22)].

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[2] This means that, for example, $\frac{1}{2}+\frac{3}{4}=\frac{3}{6}+\frac{6}{8}$.

Thus, the formal construction of rational numbers as pairs of integers presupposes an abstract idea of number, as an element of a number system, detached from concrete meanings, an idea of relatively recent date in the history of mathematics – in fact, Hankel formulated the principle in the context of his work on complex numbers, which was where giving up the intuitive understanding of numbers was necessary.

The construction of real numbers does not follow naturally from this approach – real numbers are not sets of equivalent pairs of rational numbers – and, in fact, it is not even interesting from an abstract algebraic perspective. Regularities, problems of abstraction, problems of representation, and abstract differences come at the forefront in the algebraic epistemology of rational numbers. The field of rational numbers is the smallest field of characteristic zero (no sum of 1's is equal to zero); it is a prime field (no subset of it is a field itsef); any prime field of characteristic zero is isomorphic to the field rational numbers (in other words, any field of characteristic zero contains a copy of the rational numbers field). The construction of a quotient structure by means of an equivalence relation of an underlying set is a fundamental technique in the study of algebra; more generally, with respect to a given relation, the same mathematical object can have many instantiations. For example, with respect to congruence modulo 3, 4 and 7 are instantiations of the same equivalence class $[1]$. The same construction applies to the ring of polynomials. Rational numbers share properties with algebraic numbers. In particular, a field of algebraic numbers has a ring of algebraic integers, which turn out to be the roots of monic polynomials with integer coefficients. They behave like ordinary integers, in particular, they form unique factorization domains. Just as the rational numbers are "the natural habitat" of integers, algebraic number fields are the natural habitat of algebraic integers.

3.1.2 Fractions in analysis

In analysis, a rational number is defined as the ratio of two integers. The conceptualization of number as measure serves as the theoretical support for this view of rational number. The real number line provides a model for this organization of the numerical domain, where segment length is measured. There is a bijective correspondence between the points on a straight line and real numbers: any point of the line corresponds to a number, designating the length of the segment from a fixed origin, given a unit of length for the measurement of segments. In particular, a point corresponds to a rational number, if the length of the segment from the defined origin to that point is not a multiple of the chosen unit of measure, but it is a multiple of subunit of it, obtained by dividing it in equal parts. In other words, both the length of the segment to be measured and length of original unit of measure are integer multiples of another length. The number $\frac{1}{b}$ stands for the rational number that measures the subunit obtained by dividing the original unit into b equal parts, and $\frac{a}{b}$ is the rational number that measures a quantity that contains a of these subunits. The existence of incommensurable quantities, on one hand, and the continuity of the straight line, on the other hand, leads to the construction of real numbers.

It is probably the analytic conception of rational number, and in general, of number, that is the closest to the actual one held by mathematicians for working purposes. As Lebesgue (1932) remarked in one of his pedagogical papers, the measure of quantities is primordial in applications of mathematics, but it also furnishes the notion of number, the object of study in analysis. The number line model provides both geometric intuition, and room for formalization when it is endowed with arithmetic, order and completeness properties through axioms. In particular, the completeness property requires non-algebraic concepts (Dedekind cuts, Cauchy sequences, or decimal expansions). Wheeler (1974) calls it the geometric viewpoint of number systems, and notes that, in this view, each set of numbers is seen as a subset of another set: rational numbers represent a subset of the real numbers set, integers a subset of the rational numbers set, and natural numbers a subset of the integers. Here, $+2$ is the same number as 2. The rational number $\frac{3}{1}$ represents the measure of the same quantity as the number 3. In the algebraic construction, by contrast, the old numbers are the "material of construction" of the new ones. Subsets of the extended sets "behave like" (i.e., they are isomorphic to) the sets they result from: the subring $R^{'} = \{[r,1]/r \in \mathbb{Z}\}$ of the field of rational numbers is isomorphic to $\mathbb{Z}$, by an isomorphism that maps r to $[r,1]$. In other words, the set of rational numbers of the form $\frac{r}{1}$, with the operations of addition, subtraction and multiplication has the same structure and properties as the ring of integers." (i.e., is "isomorphic" with it).

3.2 FRACTIONS IN MATHEMATICS EDUCATION

Mathematics educators agree that the formal concept of fraction (analytic or algebraic) cannot be taught as is, because it fails to convey meaning to children. But this is a truism, and we are left with the question: how exactly should we organize the take-off to abstraction in schools?

In this section we review research about the teaching of fractions to children and distinguish two approaches for deriving fractions knowledge from concrete contexts.

We start by presenting the partitioning approach, as portrayed in research, textbooks for future teachers, and some textbooks for children. Here fractions originate primarily in activities of partitioning either a discrete set of objects or a continuous quantity. We remark here the tendency to use the concrete (e.g., division of things, figures, etc.) to model "up to" the abstract algebraic notion of fraction (Behr, Harel, Post, & Lesh, 1993).

We then present the measurement approach which inspired our design, along with the hypothesis advanced by its author , that the partitioning approach – or at least or more narrow version of it that ends up being prevalent in schools – is ultimately associated with the abstract algebraic notion of rational number. By contrast, the measurement approach is more akin to the analytic concept of number, as argued by Lebesgue. This exposition is organized already in the form of an argument for our preference of the practice of measurement as the source of fractions in our design.

3.2.1 Partitioning activities as the source of fractions

Fractions originate in partitioning activities: a discrete set of objects or a continuous quantity is divided into equal size pieces, and an expression called "fraction" is introduced to quantify the relation that can be established between a part of the original whole and the whole. The expression is a pair of integers, on which one denotes the number of the equal size pieces in the part, and the other – the number of the pieces in the whole.

Instructional approaches where partitioning is the main element of teaching fractions feature prominently in research on fractions. In their review of research into the area of initial fraction concepts, Pitkethly and Hunting (1996), identify it as one of the basic mechanisms for the construction of rational number knowledge, quoting the work of Streefland (1991), and Behr, Lesh, Post and Silver (1983) to support this claim. For Streefland (1991) the equal sharing or partitioning approach is “realistic” because it calls on students’ own intuitions from real life as well as their experiences with whole numbers. In his study, five clusters of activities were designed to help students move gradually from the concrete contexts of equal sharing to the formal rules for operations with fractions. The results were positive: for the fourth-grade students participating in the experiment the equal sharing approach served, on one hand, as the context for forming the initial idea of fraction, and on the other hand, as the source of concrete models for solving various fraction problems. Behr, Lesh, Post and Silver (1983) found that if children are given ample experiences in partitioning through various manipulative aids, they understand better the compensatory relationship between the size and the number of parts into which a whole is partitioned.

Thus, partitioning is set up as the cognitive precursor to knowledge of fractions in mainstream research about fractions in North-America (the Rational Number Project, e.g. (Behr, Harel, Post, & Lesh, Rational number, ratio, and proportion, 1983)), but also in the significant work of Streefland within the Dutch tradition of Realistic Mathematics Education (Streefland, 1991; 1993). Within the Rational Number Project, however, we see a conscious preoccupation with emphasizing other meanings of fractions, besides the “part-whole” one resulting from partitioning. The authors’ work is inspired by the seminal work of Kieren, which provides one of the most largely accepted models of *good understanding of fractions* in mathematics education research. Kieren (1993) considers “rational numbers as humanly knowable such knowledge in terms of human activity” (p. 51). He answers the question “What would a person know – be able to do – if he or she knew fractional or rational number?” (p. 57) by identifying five subconstructs[3] of the rational number construct: part-whole relation, quotient, measure, operator, and ratio. Yet, Kieren assigns the part-whole subconstruct the most important role in understanding all the other interpretations and in generating language about fractions, and in

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[3] Kieren initially identified four subconstructs, all having the part-whole relation as the basis.

relation to this, the ability to partition a continuous quantity or a discrete set of objects into equal-sized parts is considered to be the cognitive precursor to knowledge of fractions.

Other studies inspired by such seminal research take on more in-depth analyses that focus on one, or another aspect of these theories, and in various contexts. For example, a considerable number of studies within the Rational Number Project examine the gradual differentiation, as well as the synthesis of the various subconstructs (Behr, Lesh, Post, & Silver, 1983). Others look at how children perform on tasks involving one particular subconstruct. For example, Noelting (1980) observed the stages in children's responses to the task of comparing the concentration of orange juice mixtures, ranging from a basic comparison of the terms in the proportion, to a final stage where ordered pairs were conceptualized as members of an equivalence class. Another line of inquiry may go towards examining teachers' knowledge of fractions: for example Charalambous and Pitta-Pantazi (2005) found that pre-service teachers' procedural knowledge of fractions (e.g., algorithms for addition, multiplication, etc.) may obstruct their conceptual understanding of fractions, defined as the mastery of the five subconstructs.

In North-American textbooks for future teachers, the treatment of fractions is inspired by some of the mainstream ideas from research. Two such ideas – both inspired from the mainstream research quoted above – stand out and appear to have important consequences for the treatment of the material.

One of them is partitioning as the unifying theme for learning fractions among a web of other meanings of fractions.

Another idea is that the main problem affecting the teaching and learning of fractions is that symbolic manipulations are taught at the expense of conceptual understanding.

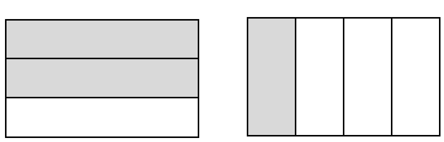
First, the introduction of fractions, for example, in both (Reys, Lindquist, Lambdin, & Smith, 2006) and (Sowder, Sowder, & Nickerson, 2010) emphasizes the multiple meanings of fractions in applications, with partitioning as a unifying theme for learning fractions. We do find here traces of the subconstructs theory: in (Reys, Lindquist, Lambdin, & Smith, 2006), fractions are presented with three meanings: part-whole, quotient, and ratio; the measure and the operator view are absent. In (Sowder, Sowder, & Nickerson, 2010) fractions are conceived of as part-whole and division (this encompasses both the quotient interpretation and the measure). In (Reys, Lindquist, Lambdin, & Smith, 2006) the *part-whole* meaning of fraction refers to the partitioning of a whole into equal parts, of which a certain number is considered. For example, under this meaning, the fraction $\frac{3}{5}$ refers to a whole that has been partitioned into five parts of which three are considered. The *quotient* meaning is based on a "sharing" model: $\frac{3}{5}$ is how much each of 5 persons gets if 3 cookies are equally shared. Lastly, the fraction $\frac{3}{5}$ may also refer to a *ratio*: for example, in a class with 6 boys and 10 girls, there is a ratio of 3 boys for every 5 girls (so the number of boys is the fraction $\frac{3}{5}$ of the number of girls). This treatment of fractions is certainly inspired by the work of Behr, Cramer, Harel, Lesh and Post in *The Rational Number Project*. These researchers talk about five meanings of rational number: the three we have already mentioned (part-whole, quotient, and ratio), as well as the interpretation of rational number as *measure* (e.g., $\frac{3}{5}$ from the beginning of a segment to the end), and as *operator* (e.g.: $\frac{3}{5}$ of 15). They argue that, from a pedagogical perspective, the main task is to provide a sufficiently detailed description of the "multiple personalities" of rational number. This assures, they claim, an appropriate theoretical foundation for the organization of students' learning experiences in this domain (Behr, Harel, Post, & Lesh, Rational number, ratio, and proportion, 1983). The authors of textbooks for future teachers seems to follow these recommendations. Furthermore, in the textbooks for future teachers the preoccupation with partitioning gains a practical layer compared to research from which it is inspired. We learn, for example, in (Reys, Lindquist, Lambdin, & Smith, 2006), that regions, lengths, or sets can all serve as models for the part-whole meaning of fractions, and advantages and disadvantages of each are discussed: for example a circle is easy to see as a whole, but hard to partition, while a rectangle is considered easy to partition, but difficult to notice as a whole.

Secondly, as an answer to the gap between symbolic manipulations and conceptual understanding, the textbooks for prospective teachers encourage multiple representations: Reys et al. (2006) quote research by (Cramer, Post, & DelMas, 2002) to justify pictorial, physical, verbal, real-world, and symbolic representations. Therefore, a significant part of the chapter is dedicated to concrete and pictorial models that enhance, in the authors' view, students' conceptual understanding of fractions and operations with fractions, beyond mere algorithms. The use of words along with counting as children's skills that teachers might call on in the teaching of fractions (e.g., counting thirds, fourths, etc.), especially as a way to introduce addition and subtraction of fractions with like denominators, is also emphasized. With regards to symbolic representations, much emphasis is put on the necessity to gain insight into the meaning of fractions before moving to symbols for fractions. For example, before introducing the symbol $\frac{2}{3}$, the authors suggest that children first make the connection between a pictorial model of the fraction and the words "two thirds." Also, before solving an exercise such as finding the value of a in $\frac{2}{3}=\frac{a}{12}$, the children should first be familiar with pictorial models for equivalent fractions which would naturally lead to the generalization that multiplying/dividing the numerator and the denominator by the same number results in an equivalent fraction.

The preoccupation with finding words or pictures to provide such conceptual understanding results, indeed, in some meaningful representations. But there was an ad-hoc and ingenious quality to them that I found intimidating even as a practitioner of much more advanced mathematics. There is one idea that seems to guide the exposition: children will derive the rules naturally if given the opportunity. For example, for the problem: "Cyrilla ate $\frac{2}{3}$ of a cake, and Carey ate $\frac{1}{4}$ of the cake. How much of the cake did they eat altogether?" the authors have the expectation:

> *Children may come up with various ways to approach the problem. If they have drawn pictures to compare fractions, they may draw pictures like these:*

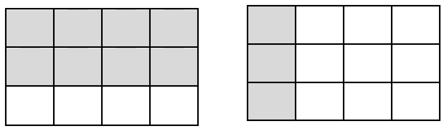
Math sentence: $\frac{2}{3}+\frac{1}{4}=?$ *Math computation*



$$\frac{2}{3}$$
$$+\frac{1}{4}$$

Cyrilla $\frac{2}{3}$ *Carrey* $\frac{1}{4}$

Show both portions on each cake:



$$\begin{array}{rl} \frac{2}{3} & \rightarrow \frac{8}{12} \\ +\frac{1}{4} & \rightarrow \frac{3}{12} \end{array}$$

Cyrilla $\frac{8}{12}$ *Carrey* $\frac{3}{12}$ $= \frac{11}{12}$

(Reys, Lindquist, Lambdin, & Smith, 2006, p. 301)

Moving on to textbooks for children, we also find the intention to foster the various meanings of fractions, and partitioning as the main source of fractions. An analysis by Lemoyne and Barallobres (2006) of Quebecois school textbooks illustrates, however, that the choice for one or another meaning is primarily based on two factors: maintaining the continuity of operations with natural numbers, and introducing the algorithm as fast as possible. In one of the textbooks they describe, the former leads to introducing, as an intermediate, the multiplication of whole numbers by fractions as repeated addition using the part-whole meaning of fractions; when two fractions are multiplied, the area model of multiplication (Sowder, Sowder, & Nickerson, 2010) is used in which fractions feature as operators. Furthermore, the algorithms are introduced right after the observation of various pictorial and geometrical models, in which fractions feature as part-whole relation or measure. The models and activities that accompany them don't play the role of illustrating the necessity of defining multiplication in a manner that matches the actions involved; instead, children are supposed to "verify" the rule by observing that the results in each particular case coincide with the result given by the algorithm. The algorithm appears thus as an observed "regularity" rather than a generalization of the partitioning actions involved:

> *Il nous faut un algorithme de multiplication de fractions qui respecte les résultats précédents (les 4 exemples présentés). L'algorithme suivant satisfait ces exigences […] Pour multiplier deux fractions, il suffit de multiplier les numérateurs ensemble et les dénominateurs ensemble. (Breton, 1993, p. 228)*

In their analysis of the presentation of fractions in textbooks belonging to different didactical traditions, (Freiman & Volkov, 2004) notice that Canadian textbooks appear to follow the trend dictated by research where children are to be given ample experiences with "real-life situations" and multiple perspectives in order to build appropriate fraction concepts. The authors notice, however, the unrealistic expectation from children, but also the burden that is placed on the teacher:

> *We can see that in many aspects the textbook closely follows recommendations of Bezuk and Cramer. Each piece of knowledge is introduced exclusively by the means of a "real life" situation. The situation leads to a mathematical problem that, in turn, requires a construction of concept of objects or of operations. Being placed in this learning situation, the student is asked to explore the problem, to create appropriate models (often using pictorial form or real manipulatives), to try to understand the data and to build relationships. The students are formally no more obliged to follow one model, one calculation rule; they are encouraged to make a sensible choice of an appropriate strategy. But how will they know which choice is sensible? The textbook does not help to understand it. It seems that a lot depends on the teacher. (Freiman & Volkov, 2004, p. 3)*

The list of strategies that the textbook for children proposes for adding fractions, for example, is as follows (ibid., p.3)

- *Draw pictures (stripes);*
- *Use real objects (eggs);*
- *Find a common denominator;*
- *Use a calculator.*

It is indeed startling to see that the teacher is left to their own devices in linking the "drawing of stripes" to the "use of the calculator."

3.2.2 Measurement activities as the source of fractions

Davydov proposes that fractions originate in measurement activities. But he asks the question about the appropriate context purposefully: where do fractions originate in reality, or, more precisely, in human practices? In his words: what are the "object sources" of the concept of fraction? (Davydov & Tsvetkovich, 1991)

Historically, fractions appeared as devices for measurement long before the formal construction of the system of rational numbers in theoretical mathematics (Courant & Robbins, 1953). However, to emphasize measurement as the source of fractions in order to simply replicate their original development would be naïve; in fact, as Courant & Robbins (1953) argue, it is precisely clinging to the reality of measurement that acted as an *epistemological obstacle* (Bachelard, 1938/83; Brousseau, 1997; Sierpinska, 1994) for conceiving of an arithmetic for fractions; neither the Greeks, nor the Egyptians achieved it, despite having a sophisticated understanding of ratio and proportion – the Egyptians even had a symbol for fractions (Caveing, 1992; Fowler, 1992):

> *As we have seen, this extension of the number concept was made possible by the creation of new numbers in the form of abstract symbols like, 0, -2, and ¾. Today, when we deal with such numbers as a matter of course, it is hard to believe that as late as the seventeenth century they were not generally credited with the same legitimacy as the positive integers, and that they were used, when necessary, with a certain amount of doubt, and trepidation. The inherent human tendency to cling to the "concrete", as exemplified by the natural numbers, was responsible for this slowness in taking an inevitable step. Only in the realm of the abstract can a satisfactory system of arithmetic be created. (Courant & Robbins, 1953, p. 56)*

But the operationalization of fractions into an arithmetic which fulfilled a theoretical need within mathematics came with its own set of epistemological obstacles – in particular the obstacle of excessive formalism. Davydov's curriculum (Davydov & Tsvetkovich, 1991), which inspired the MA, derives fractions from measurement but also considers the inherent abstraction in the concept of fraction, thus overcoming both epistemological obstacles.

On the one hand, fractions function as a measuring "tool": if we want to measure some quantity, the existing units of measurement may not represent it precisely enough, and we have to change those units into some finer ones. For example, consider the problem of measuring a given distance, say between the desk and the board in a classroom, using somebody's foot as unit. We may find that it measures, say, $10\ feet$ and a bit. A more precise measurement then requires a smaller unit of measure – say the same person's thumb. But instead of measuring the whole distance using the thumb, it is more economical to first find the relationship between the foot and the thumb, and *convert* to thumbs using multiplication rather than measure the whole distance with thumbs. Thus, if, for example, $1 foot = 9\ thumbs$, and the remaining bit measures $2\ thumbs$, then, altogether, the distance between the desk and the board measures $10 \times 9\ thumbs + 2\ thumbs = 92\ thumbs$. A fraction then arises to express the sought out measure in *feet*: it will be $\frac{92}{9}$ of a foot, where the numerator is the measure of the given distance, and the denominator is the measure of the foot, both expressed in thumbs.

Let us note here that this operation of change of unit shows how closely fractions are connected to the operation of multiplication. The practice of change of unit is the object source of both concepts (Davydov V. V., 1991). Reducing the concept of multiplication to repeated addition, common in elementary school education, is misleading even if one of the factors is a whole number. It ignores the basic precondition of the operation: repeated addition requires the construction of those equal groups or quantities that can be "repeatedly added." Each such group is, in fact, a new unit in which the whole quantity is being measured. The "multiplication problems" in elementary school usually present the equal groups or quantities as already formed; so what remains to be done is only the repeated addition. A typical example, quoted by Davydov is: A factory produces 9 cars each day; what is its weekly output if the factory operates 6 days a week? (Davydov V. V., 1991, p. 18) The problem of the weekly output of the plant is already half solved by deciding to use the daily production as a unit and finding out information about it. Just as the reduction of multiplication to repeated addition conceals the underlying operation of change of unit, reduction of fractions to breaking an object in equal parts and taking some of them hides the fact that these parts function, in fact, as units of measure – in measuring the whole object and the set of its chosen parts.

On the other hand, the resulting fraction is explicitly conceived of as a relationship between the given amount and the given unit: $\frac{92}{9}$ is a relationship between two quantities, rather than a quantity in and of itself. This is a significant abstraction, and the measurement practice, by actually considering *the measures of two quantities*, affords such detachment from the objects at hand. However, Davydov argues (following Vygotsky (1986)), a leap to such higher order thinking about quantities is not spontaneous: fraction, as an abstraction, is not born naturally from the concrete situations in which children interact; rather, it exists in the teacher's mind to begin with and it is reconstructed by the children with the help of significant scaffolding on the part of the teacher.

In Davydov's teaching experiments on multiplication, for example, the first instructional situation requires children to measure the amount of water in a large container using a small glass (Davydov V. V., 1991, p. 34). The teacher lets the children experience the difficulty of the task, and in the second instructional situation, she suggests the use of another, larger measuring unit – a mug – and goes on to reveal "the *secret* for a new method of working" (Davydov & Tsvetkovich, 1991, p. 37):

> *It is not useless for us to have measured the water with the mug – although it won't help us right away. We can still find the number of glassfuls. How? Here is how. Watch and listen carefully. We used a mug instead of a little glass – fine, this made our work easier, but we did not find how many glassfuls there are in one of this mugfuls. Can we find that out now? How? We have to pour water into the mug and measure it with the glass.*

The teacher then performs the measurement, and asks the children leading questions, guiding them to the desired conclusion and synthesis of the measurement process: "We took five glassfuls six times" (p. 40), which in turn is written down – on the board and in children's notebooks – as the multiplication "five multiplied six times" (p. 40).

Davydov and Tsvetkovich (1991) criticize the prevailing method of acquainting students with fractions that derive the concept of fraction from the practice of dividing objects (we will call this approach the Division Approach[4], DA):

> *This whole process of acquainting students with the fractional number has the following basic characteristics:*
>
> 1. *The students are shown that some objects ("units") can be divided and subdivided into equal parts ("fractions"), or that the students themselves can distinguish equal parts in these objects. Each, or all, or some aggregate of the fractions can be the object of direct observation.*
> 2. *The students designate some number of fractions ("three orange sections") with a whole number, but at the same time they indicate the whole number of parts from which the first aggregate ("three orange sections out of ten") was isolated.*
> 3. *Certain verbal expressions are commonly used for similar, simultaneous demonstration of a pair of whole numbers: "one out of seven", or "one-seventh;" "three out of ten", or "three-tenths." These pairs of numbers are fractions. The children learn these expressions by observing the appropriate situations or by being required by their teacher to "take" some number of parts from a group. (Davydov & Tsvetkovich, 1991, p. 94)*

Further on the same page, the authors summarize "the goal of the preliminary instruction" of fractions in the Division Approach so: "the elementary school student [shall] clearly conceive of

[4] We did not want encompass all the research where partitioning activities are the main sources of fractions knowledge (as presented in section 3.2.1) under the same term of "Division Approach"; we refer more strictly here to the approach for teaching fractions depicted by Davydov, and likely the prevalent one in today's elementary schools (although this is a claim to be confirmed empirically).

the fraction as one or several equal parts of a whole circle, square, paper strip, or unit." This understanding of fractions is called the "visual conception of fraction." (Davydov & Tsvetkovich, 1991, p. 94)

In this approach (feature 1 in the last quote), "fraction" refers to an object as a part of another object.

The authors maintain that this approach is based on the conception of fraction as a consequence of the will to broaden the set of integers in order to allow unconstrained division. The instructional objective of this approach is thus the abstract notion of rational number. Having children experience the partitioning of concrete objects, then drawings of geometric figures, and finally abstract numbers, reflects an attempt to teach fractions as solutions to equations of the type: given number times what number is equal to a given number? (e.g., $7 \times WHAT\ NUMBER = 8$). Fractions as the result of division is thus basically an abstract algebraic idea, where fraction is a pair of whole numbers - the preliminary activities of physical division of things are only meant to alleviate the difficulties inherent in grasping this idea.

3.2.3 Relationships to mathematics and real life

We will compare the two approaches for teaching fractions by looking at the relations of school knowledge of fractions, first, with mathematical knowledge, and then, with real life. This separation is somewhat artificial as it underscores a dichotomy between mathematics and real life; however, we found that these two kinds of relations tend to recur as the major themes in both theoretical and ideological discussions of school epistemology.

3.2.3.1 Relationship to mathematics

Kieren (1993) discusses the relations of the subconstruct theory to the larger domain of mathematical ideas, arguing that, as a basis for a fractions curriculum, it provides opportunities to experience many important mathematical ideas, covering the diverse domains of number, space and geometry, logic and algebra, and infinity. For example, the partitioning of quantities as a basis for the quotient subconstruct can be related to the idea of infinitesimals in calculus, but also to algebraic ideas of equivalence, when the same quantity is considered, partitioned in different ways. The measure construct brings forth geometrical aspects of fractional numbers: they can be thought of as a set of linear measures providing experience with order and the infinitely large. The operator understanding provides a concrete model for the composition of functions, but it also emphasizes the algebraic notions of multiplicative inverse and identity elements. The notion of operator is also close to the modern understanding of number as a scalar, in vector space theory, for example, making a quantity larger or smaller, keeping its direction, or changing it. Finally, one of the interesting domains for the application of fractions is probability, when the ratio understanding is promoted.

Kieren's argument is that the subconstructs theory provides school students, relatively early on, with a natural window on the entire domain of mathematics. The fundamental assumption in his theory reflects a constructivist conception: such mathematical understandings build on children's personal experiences and interest in applications. There are important consequences in terms of the mathematical knowledge for teaching (Ball, Thames, & Phelps, Content Knowledge for Teaching: What Makes It Special?, 2008) that teachers should have of fractions, assuming this theory justifies their practice. Teachers should have knowledge of the concrete situations, as well as an arsenal of pictorial models or manipulatives, that cover the full range of fractions subconstructs, and understand how they are linked to various mathematical domains. Moreover, notions of equivalence of fractions as well as algorithms for operations must be derived as generalizations of partitioning activities. The latter, in particular, require some very imaginative ideas that cannot be picked up ad-hoc, without special training. Teachers should also be able to evaluate the textbook problems critically, interpret and organize children's intuitive strategies and their errors in light of the subconstructs framework, in particular with respect to the opportunities for conveying important mathematical ideas. This is especially true since the textbooks that teachers have at their disposal may hinder proper understanding, as shown by the analysis of textbooks performed by Lemoyne and Barallobres (2006) or (Freiman & Volkov, 2004). Failure to acquire and articulate these elements of knowledge can lead to the situation, deplored by many diagnostic studies in the field, where fractions are taught with a focus on symbolic manipulations, mimicking only superficial aspects of the subconstruct theory (such as the use of pictures or manipulatives), or, more generally, of constructivist ideas.

Teacher training materials such as (Reys, Lindquist, Lambdin, & Smith, 2006) or (Sowder, Sowder, & Nickerson, 2010) provide some *examples* of “good fractions teaching”, but lack an explicit articulation of the theory with the practice of teaching. It becomes the task of elementary mathematics methods courses to provide future teachers with the missing link in the form of analytical tools to perform the tasks of teaching (Osana, Sierpinska, Bobos, & Rayner, 2010).

Davydov’s (Davydov & Tsvetkovich, 1991) introduction of fractions using the measurement of quantities materializes certain general psychological and pedagogical premises, but also relies heavily on mathematical analyses of the concept of fraction. On the one hand, he argues, measurement of quantities retains a historical significance as the origin of fractions, in the same line as Courant and Robbins (1953). On the other hand, Davydov finds inspiration in the work of mathematicians such as Klein, Lebesgue, or Kolmogoroff, to show that introducing rational numbers in the context of measurement of quantities is preferable to introducing them as pairs of whole numbers. He quotes Klein to argue that, while the modern construction of fractions as a pair of integers is more rigorous and purer, the new entities have no application for the activity of “measurement of quantities with which we have to work” (p. 101). Moreover, they confine the learner to the realm of whole numbers, without providing him or her with the intuitions that are unique to fractions. Lebesgue (1932) went further, to contend that the measurement of quantities is no less scientific, proposing an approach that breaks the whole number limitation in an even more radical way. He postulates that all numbers in school be based on measurement, proceeding immediately from natural numbers to real numbers. In his original exposition, he employs decimal notation, by subdividing units successively in tenths. He enthusiastically advocates the use of the decimal numeration system, as one of the most important inventions in the history of science – that the Ancient Greeks didn’t have at their disposal – one that, moreover “young minds reason more easily on” (Lebesgue, 1932, p. 181). In elementary school, children could be acquainted with the operation of measurement that produces finite decimal expansions, with the problem of precision in measurement making it necessary to construct a general concept of real number. As we know, even Lebesgue’s abstract integration theory has its germ in this basic geometric intuition: his abstract measure is a generalization of length on the basis of countable additivity. Lebesgue's way of constructing the integral of a function analytically by partitioning the range of the function provides its power, but this approach is possible because of a very intuitive, geometric characterization of the integral in terms of measures of sets. The integral on interval $[a, b]$ of a function f is the difference between the 2-dimensional measures of the planar sets bounded by the graph of f and lying above and below the x-axis, respectively. Then the procedure of integration by partitioning the range works for highly discontinuous functions (where partitioning the domain doesn't do the job) because it starts with the goal to be attained: to collect values of f differing by very small amounts. Thus, measuring, as concrete as it may be at the source, is a thread that runs through the fabric of all mathematical analysis.

Davydov and his collaborators' approach (Davydov & Tsvetkovich, 1991) corresponds to the measure theoretic conception of number, characteristic of mathematical analysis. Kieren (1993) maintains that an approach based on applicational meanings of fractions provides opportunities for experiencing the whole domain of mathematics. However, fractions treatments that bear similarities with the subconstructs theory (whether explicitly inspired by it or not), in particular with respect to the focus on partitioning activities and on using various models to introduce operations, are closer to the abstract algebraic conception of number, notwithstanding their apparent emphasis on concreteness. Two important features of these treatments are essentially algebraic, going beyond Davydov's hypothesis that division of numbers is modeled as division of things: the consistent bias towards whole numbers and the focus on operations with fractions.

It can be argued that *the principle of permanence* could provide a powerful heuristic for teaching numbers in school: extending number systems from a "nucleus" – the natural numbers – based on observed regularities in their arithmetic *justifies* the arithmetic operations in a coherent fashion for different kinds of numbers. The extension from natural numbers to integers is possible by removing "the embargo" on subtraction, while the rational numbers would be the collection where unrestricted division becomes possible. The focus would be on maintaining the properties of the fundamental operations observed in the natural numbers. Kilpatrick's treatment of numbers in *Adding it up* (2001) is quite convincing: he uses concrete and pictorial representations for numbers to provide an intuitive basis for fractions, and derives commutativity, associativity, and distributivity properties "empirically." But the more primitive work of connecting the fraction to the "real-life situations" encountered in school by looking at quantities is missing.

3.2.3.2 Relationship to real life and ways of knowing

From an ideological standpoint, the relationship of mathematics with real life is often at the center of both defining the goals of mathematics education and prescribing ways of learning mathematics in schools, that is, concerning both the "why" and the "how" of knowing mathematics. For example, one of the "core learnings" in elementary mathematics defined by the Quebec Ministry of Education in its reform documents is to "View [mathematics] knowledge as a tool that can be used in real-life" (2001, p. 138). At the same time, mathematics should be anchored in reality because children learn better if they can relate it to their concrete experiences, and perceive it as authentic. In the Quebec Education Program, for example, the tool to achieve this is the situational problem.

In approaches where fractions originate in partitioning activities, the connection with children's everyday experiences is crucial. Behr, Lesh, Post and Silver (1983) found that if children are given ample experiences in partitioning through various manipulative aids, they understand better the compensatory relationship between the size and the number of parts into which a whole is partitioned. For Streefland [(1991) (1993)], the activities of equal sharing and splitting up groups of sharers into subgroups are "realistic" because the tasks involved call on students' intuitions of fair sharing from real life as well as their experiences with whole numbers. Students move gradually from the concrete contexts of equal sharing to the formal rules for operations with fractions. Streefland and his collaborators write extensively about the need to build resistance to the tendency to associate what one is learning about fractions with the natural numbers. N-distractors resistance – as they call it – is also built by means of concrete situations: for example, a child performs a distribution to arrive at $\frac{1}{2}+\frac{1}{2}=1$, which contradicts his initial calculation $\frac{1}{2}+\frac{1}{2}=\frac{2}{4}$. The children in Streefland's research (1993) move from accepting both results as correct, thus not experiencing the envisaged cognitive conflict, to more or less spontaneous refutations of errors associated with N-distractors. However, for virtually the entire group in Streefland's research, the presence of N-distractors proved to be quite stubborn, and even after a period when the children showed some stability in overcoming them, relapses occurred especially in the context of calculating with fractions, when children seemed to have lost contact with the insights acquired from concrete contexts.

Children in Streefland's research (Streefland, 1991) also didn't do very well on tasks requiring them to place a fraction on a number line: for example, they would establish order on it, but be rather careless in determining precise differences. They would prefer to calculate using prices rather than the straight line model. However, this way of thinking about fractions, which echoes the analytic conception of number – was not the main objective in Streefland's approach. In line with Freudenthal, he aimed rather at teaching the children the mathematics of practitioners. The following excerpt illustrates the kind of activities found in realistic mathematics education:

> One day when father didn't come home for lunch, Els Fractured made an omelet for herself and the three children. There was plenty going on here. (In the first place, the recipe drew attention, which we don't deal with here.) Our attention is focused on how the omelet is divided. Els and the children each took two slices of bread and covered them with a share of the omelet. The assignment was how to distribute the omelet over the eight slices of bread. *(Streefland, 1991)*

In mathematics education research, Freudenthal was the pioneer of "realistic mathematics education." His view of the relationship of mathematics with real-life was quite sophisticated: the "realism", for Freudenthal, is to be found in the process of "mathematization", as an activity of structuring reality. This means, literally, "making more mathematical" by employing, in the activity of solving problems, tools and strategies that reflect characteristics of mathematics such as generality, certainty, exactness, or brevity (Gravemeijer & Terwel, 2000). Importantly, the subject matter of the problems refers to both real-life and mathematical content, depending on the problem at hand. In the recent conceptualization of realistic mathematics education the tenet that mathematics should be anchored in reality means that the starting point of a teaching sequence is *experientially real* for students, i.e., not necessarily involving realistic situations. For young children, however, the priority should be to mathematize everyday reality, since mathematical content cannot yet be experientially real. Besides, since most of them will not become mathematicians, it is the practitioners' mathematics that they should be acquainted with. Vergnaud (1994) shares this point of view and believes that school mathematics and school learning shouldn't be disconnected from real-life mathematical competencies and real-life learning:

> *This opposition is misleading in the sense that no mathematical procedure observable in real-life situations cannot potentially be found in the classroom, provided students are offered a variety of situations to deal with, rather than stereotyped algorithms. (Vergnaud, 1994, p. 45)*

Chevallard had a different point of view. In one of his papers (1989/2007), he questions the very purpose of teaching mathematics to children in school, disputing the view according to which school mathematics should find inspiration in everyday life, which, in turn is permeated by mathematics in all of its aspects. He makes a distinction between the explicit, or visibly used, mathematics (in science, engineering, business, etc.) and the implicit mathematics, which is "embodied", or crystallized in objects of all kinds, and argues that the mathematics that is omnipresent in people's lives is of the implicit kind, while the explicit kind is concealed to the layman. Mathematicians work, paradoxically, to demathematize social practices, while, at the same time, mathematizing (implicitly) the objects and the techniques available. The teaching of mathematics has become a way to make mathematics culturally visible, and embraced a somewhat apologetical discourse, by attempting to justify mathematics in terms of its utility, which, in turn, was wrongly understood in terms of the individual's needs, rather than the society's. Lebesgue (1932) has a similar view when he says that one may have to try hard to find practical uses of fractions: the mechanic who has to employ fractions when tapping a screw is more likely to understand fractions on the spot, in the context, rather than to see the connection between his job and the school knowledge. Educators, says Chevallard (1989/2007), need not worry about the *need* to learn "the niceties of fractions addition" (p. 5), since anybody can use a pocket calculator to perform arithmetic. The attempt, by some educators, to find aspects of the real world that can be modeled through mathematics, is futile, as the visible mathematics in most people's lives is banal. Chevallard proposes, instead, that mathematics be studied, like other subjects (such as history, or literature), as cultural initiation:

> *Such an initiation should result in an awareness of society as a complex whole made up of many deeply-interrelated components, most of them hardly visible and understandable from the outside. It should avoid some major pitfalls, address elemental, not necessarily elementary, questions, and beware of unrealistic realism. If it could cast off the sanctified fallacies that I earlier criticized, the "mathematics at work" movement might in this respect show us the way. (Chevallard Y. , 1989/2007, p. 8)*

Chevallard's criticism, however, is mainly directed at the secondary level school mathematics, which is the weak link, he says, among primary and tertiary mathematics, in terms of "naïve" realism. Although fractions are usually part of elementary level mathematics, they are always somewhat on the cusp of higher level mathematics, one foot into the learning of algebra. Moreover, the choice of one or another approach for teaching fractions supports different epistemologies of mathematics, in particular of the numerical domain, which, as we have seen, may have long lasting effects on children's learning of mathematics in higher grades. Davydov's curriculum (Davydov & Tsvetkovich, 1991) fits the goal of teaching mathematics as "cultural initiation" better. The measurement approach is set, from the start, on an abstract notion of number as measure or relation between quantities, and proceeds to ascend from the formal to the concrete. Davydov's approach, inspired by Vygotsky, assumes that theoretical knowledge is acquired through instruction within conceptual systems, rather than spontaneously, abstracted from reality. In one of his columns for the Mathematical Association of America on-line Keith Devlin (2008) observes that this is the way professional mathematicians primarily know: by learning the rules of a system – which may have no relation to reality, nor any meaning to begin with – , following them through conscious effort, and after enough experience, moving intuitively, as if "playing" within the system. In fact, many mathematicians speak enthusiastically about how mathematical formalism not only does not exclude intuitive thinking, but is also the source of new understandings and intuitions. In his book on the history of numbers, Dantzig (1954) offers such a perspective even with regards to the highly abstract algebraic construction of rational numbers: he argues that the extension of the number systems according to the principle of permanence is not at all an artificial and arbitrary affair if natural numbers are understood as symbols that are subject to a system of rules; it is rather “an inevitable process of natural evolution” (Dantzig, 1954, p. 98).

We saw that the relations of school knowledge of fractions with mathematicians’ knowledge of fractions – invisible at first – are quite explicit in the theoretical justifications that support the two approaches. Such relations concern not only the more “technical” distinction between the algebraic and the analytic conception of fractions, but also the different conceptualizations of the numerical, or, more generally, the ways of knowing in mathematics. When fractions originate in partitioning activities, it is possible to accede to the whole domain of mathematics when under the subconstructs theory: the operator, measure, quotient, ratio subconstructs, all supported by the part-whole relation, can foster both algebraic and analytical understandings. The partitioning approach, however, as we have shown,– in textbooks for future teachers and children – ends up taking an algebraic path with its focus on operations painstakingly and inconsistently modeled with pictures and diagrams, as well as confinement within the whole numbers domain. The measurement approach, by contrast, is, from the beginning, set squarely on an analytic conception of fraction as measure, which is consistent throughout the organization of the instruction on numbers. This is, in fact, the main feature that distinguishes it from the preferred approach in North-American curricula: a definite conceptualization of the notion of number as a relation between quantities. The cognitive price may be bigger for the measurement, compared to the equal sharing approach, but it provides an important gain: access to the notion of real number. As Streefland’s research (1993) showed, it takes much effort to overcome the whole number obstacle, and its presence may prove stubborn as far way up as college, when students still haven’t gotten an intuition of the density property of the rational numbers system. The obstacle is particularly powerful in the context of operations, where performing arithmetic on fractions requires that two numbers be “encapsulated” as one entity that can be operated on.

The two instructional approaches also reflect different conceptions of ways of knowing in mathematics. Freudenthal (1983) and his followers make a solid point for learning based on real experiences, and the equal sharing approach is based on such a view of mathematics knowing in school. Given that the notion of fraction is, in a way, more connected to algebra than to arithmetic, and that it is very demanding in terms of theoretical thinking, it is perhaps worth exploring, if at this point in elementary mathematics education, different ways of knowing, that nurture theoretical thinking, should not be fostered. This would correspond more closely to the teaching of mathematics as “cultural initiation”, as advocated by Chevallard (1989/2007). In Davydov’s approach, this was achieved in an essentially Vygotskian view of learning, through a more direct teaching method, which included guidance, hints, suggesting answers, staging mistakes or pointing to conflicts.

4 PRELIMINARY TRIALS AND DESIGN CONCEPTION

In the first part of this chapter I present two categories of issues identified in the first run of the experiment of teaching fractions to future teachers in the context of measurement: future teachers' lack of a coherent knowledge base for fractions and their attitudes as influenced by the institutional setting of the course.

In the second part I present the design for the second instantiation of the course as a two-pronged response to these issues. Firstly, I describe our adaptation of the Measurement Approach for math methods courses and further justify, on *epistemological* and *cognitive* grounds, our decision to introduce it as the fractions knowledge base for future teachers. Secondly, I turn more directly to the *didactical* dimension of the design, and discuss how we planned to influence the students' positioning in the course, thus taking into account the specific institutional setting considered – a mathematic methods course for undergraduate students studying at university to become elementary school teachers.

4.1 ISSUES IDENTIFIED IN THE PILOT-STUDY

In the first run of the experiment (Winter 2013) – I call it the Pilot-Study – we had the deliberate goal of developing future teachers' competence in *mathematical problem posing* in the context of quantities encountered in real life. Problem posing would encompass teacher's tasks such as: asking questions that enhance students' learning; modifying existing problems; writing original problems to reach given instructional objectives; judging the difficulty of a problem and modifying it to make it more or less challenging for students. This goal corresponds to at least one of the core competencies for the teaching profession outlined in the official MEQ guide for teacher training: "To develop teaching/learning situations that are appropriate to the students concerned and the subject content with a view to developing the competencies targeted in the programs of study" (Competency #3, in (Martinet, Raymond, & Gauthier, 2001, p. 69)).

The content was going to be *fractions* and *geometry*, but we didn't set out *to teach* the two topics according to a defined sequencing or separation. Rather, we had assumed that future teachers ("FTs") had common content knowledge (Ball, Thames, & Phelps, Content Knowledge for Teaching: What Makes It Special?, 2008) of both, and we focused on engaging them in creating problems by using such knowledge to model realistic situations (i.e., situations involving some physical reality). Given an origami construction, or a measurement activity, for example, we would encourage FTs to ask questions about the concrete aspects of the situation, and then help them see the mathematical nature of questions and formulate them in a more precise and decontextualized form. Ideally, we thought, FTs would be stimulated to look for such situations in their surroundings and everyday life, and come up with problems themselves, designed so as to satisfy certain instructional objectives. We did envision meaningful learning of topics such fractions, decimals, percentages, or geometric relationships, but we thought it would occur in the process of mathematizing real life situations. In particular, the concept of fraction, inspired by Davydov's approach to teaching fractions in elementary school (Davydov & Tsvetkovich, 1991), was derived in class during such a realistic activity, in the context of measurement: a fraction was defined as the ratio of an amount to the unit in which the amount is measured (this is a conception of number, in general; the number is a fraction only if the amount and the unit are commensurable). This concept of fraction, as relation between measured amounts or quantities was consistently used by the Instructor thereafter, yet continuously ignored by the FTs, some of whom even exhibited an emotional resistance to it. FTs appeared strongly attached to the idea of fraction derived from division of things into equal parts, where the fraction is not an abstract relationship, but "a thing", i.e., one or more of these parts taken together.

More generally, we found, firstly, that FTs knowledge of the mathematics they will be expected to teach was insufficient. With respect to fractions, in particular, FTs lacked a coherent knowledge base as the necessary condition for posing mathematically interesting questions as teachers.

Secondly, we discovered that our vision of the university student as an individual who is there to learn specific content in their discipline and develop, in the process, certain habits of rationality, was continuously undermined – not by the students' unwillingness or lack of opportunity, but by the dynamics encouraged by the Teaching Mathematics (TM) course institution as it was organized at the time. The tasks typical for this institution, in particular, prompted FTs to act like teachers; for example, they required designing a teaching activity or reflecting on children's errors. FTs appeared to take upon this role swiftly, but were both unprepared from a disciplinary perspective (i.e., did not know the math), and acted based on their own, strongly ingrained, ideas of what constitutes good teaching. The theoretical tools that the Instructor introduced were irrelevant for such teachers' actions, insofar as FTs consistently relied, and with relative confidence, on their previous experience in schooling and unexamined popular wisdom as bases for their teaching decisions. Furthermore, we observed that the setup of the course as a school institution where students need to obtain a certain grade in order to reach their career goals, inevitably raised the issue of assessment, which affected, in turn, the FTs and the Instructor's sense of agency in the course.

Thus, FTs learning of fractions turned out to be greatly influenced not only by their previous knowledge of fractions (or the poor quality thereof), but also by their rapport with the Teaching Mathematics course institution. I review these issues below using the institutional theoretical perspective inspired by Chevallard (1985a), and Ostrom (2005).

4.1.1 Issue 1: FTs' knowledge of fractions was insufficient for the Teacher position

In the first instantiation of the course it became clear to us the FTs common content knowledge of fractions had serious gaps. This meant more than what we had expected gleaning from research in mathematics teacher education before the course started: it wasn't just their pedagogical content knowledge of fractions that had to be attended to, but also, simply, their knowledge of fractions as a topic in mathematics. A crucial component in the Pilot-Study, in the Winter of 2013, had been engaging FTs in *problem posing* activities, thus assigning them to the Teacher position: they were asked to write original problems to serve certain instructional objectives, to ask questions that foster understanding, to judge the difficulty of a problem and modify it to make it more or less difficult for the hypothetical child, to address certain misconceptions, etc. Despite appreciating the relevance of this type of task for their future profession – indeed evaluating and creating problems is an important component of teachers' professional lives – FTs struggled with the mathematics at hand in several ways.

First, many the problems posed by FTs were too easy, imprecise, or nonsensical, from a mathematical point of view, despite the possibilities offered by the situations they were supposed to be derived from. For example, when given the following picture (Figure 4.1) and asked to write a few related questions about “Teddy’s flushable wipes”, we saw questions like “Three moms bought a crate with 60 boxes of wipes from Costco to share. How many boxes will each mom get?”, “Mia needs 540 wipes. If there are 50 wipes per package are there any *left over*? “, or “There are 98 wipes but 50 babies’ bottoms to wipe. How many wipes would you need to wipe each baby?”

Figure 4.1. A realistic picture given to FTs in a problem posing task

The Instructor also included less open-ended tasks where FTs had to create problems of a given structure; for example:

> Consider the following problem: “Five sevenths of a bag of apples is 12 apples. How many apples are in the bag?” Explain why this is not a very good problem. Reformulate the problem to remove the flaw. There are several possibilities; offer two.

Many students did not grasp the structure of the problem, thus did not solve it correctly. We reproduce below one student’s answer containing the typical incorrect solution (Figure 4.2), but also pointing to other habits we consistently observed, which throw any significant mathematics out the window in posing problems: the unnecessary conversion to decimals (e.g., converting $\frac{1}{4}$ to 0.25), the use of “benchmark fractions” ($\frac{1}{4}$, $\frac{1}{2}$), the lack of clarity in formulating the question (“$\frac{1}{4}$ of a bag of 12 apples. How many apples?”):

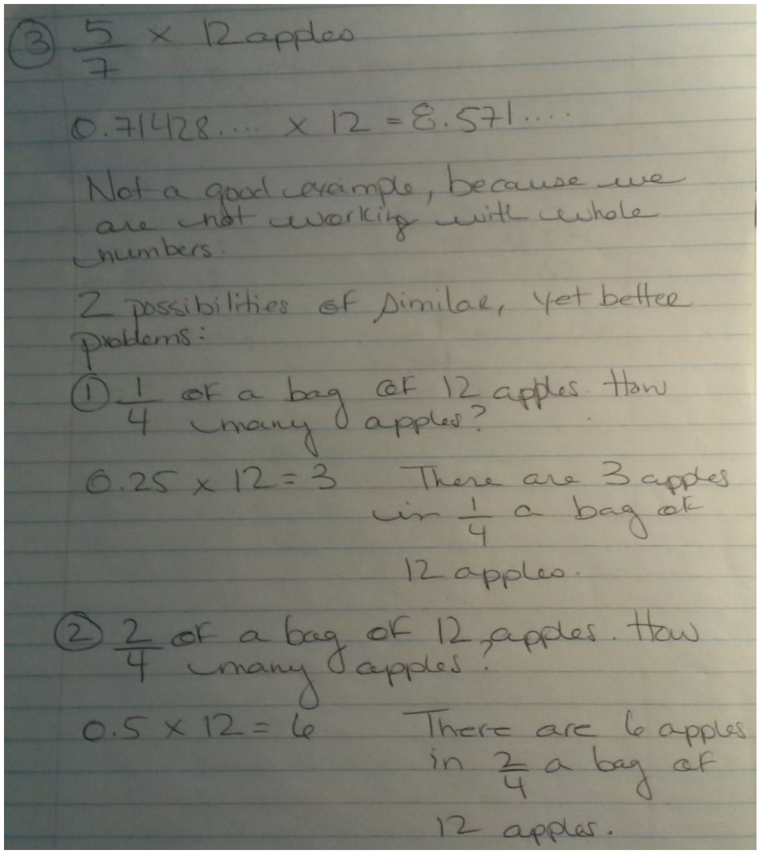


Figure 4.2. An FT's solution to a problem along with similar problems she invented

Secondly, although it was rare that the FTs didn’t know how to solve their own posed problems, there were many mistakes in their solutions, mostly related to terminology and notation, but also, more generally, to the grammar of sentences involving mathematics. It was not uncommon to see sentences such as: “The fractions are proportionate in numerator and denominator”, “$\frac{1}{4}$ of a bag of 12 apples. How many apples?”, “The ratio is uneven”, or “The area is $235.41\ cm$”. Clearly, we were not concerned about respecting some prescriptive grammar rules, but rather about the sense and clarity that such phrases or sentences lacked.

Finally, when slightly more difficult problems were posed in class by the Instructor, FTs encountered great difficulties. In a questionnaire she gave at the beginning of the Pilot-Study, only 4 of the 35 students answered the following question by calculating the areas of the pizzas and finding the fractions of these areas.

> Mary ate $\frac{3}{4}$ of an 8-inch pizza. Jane ate $\frac{1}{2}$ of a 10-inch pizza with the same filling. Which one had less pizza? Justify your answer.

Most of them, instead, divided 8 by 4 and multiplied the result by 3 to find the amount of pizza that Mary had, and divided 10 by 2 and multiplied the result by 1 to find the amount of pizza that Jane had, while others compared the abstract fractions $\frac{1}{2}$ and $\frac{3}{4}$ by "cross-multiplication", erroneously concluding that Jane had less pizza.

The problem was certainly more difficult because of the non-linear dependence of the area on the diameter, but the massive failure to observe this detail was just one example among many other instances where FTS struggled with problems that were either more complex or had a twist requiring more than obvious numerical computations. Others such instances include proposing teaching activities for certain mathematical topics that entirely missed the essence of said topic: e.g, designing an activity for teaching Pythagoras's theorem where the essential assumption of having a $90°$ angle is not even mentioned.

We thought that to even begin thinking about an easier question that would guide a child to understanding a division problem, for example, perhaps by means of an "inverse" multiplication problem, or by analogy with whole numbers, the FTs must first be able to solve correctly problems that require division – one must recognize the "how many times a quantity goes into another quantity" structure that calls for division in a problem like "A beekeeper collected $5\frac{1}{4}\ l$ of honey and wants to store it in $\frac{3}{4}\ l$ cans. How many cans will he fill?" Or, to create an interesting comparison of a sequence of fraction problems, by modifying the quantities that the fractions are taken of, one must interpret correctly such a piece of information in solving the problem. In other words, to see such quantities as values of a didactic variable, one must be able to first recognize them as relevant variables in solving. Overall, our experience throughout the Pilot-Study led us to believe that FTs needed to learn more mathematics in order to be able to teach anything meaningful.

The Measurement Approach was designed to fulfil this role – of the fractions knowledge base for future teachers. As I will show, we develop its specialized nature – as knowledge base *for teachers* – in its theoretical premises, rather in the more visible demands to "act as a teacher" posed to students. We hypothesized that this approach would help them reason about fractions in the context of quantities.

4.1.2 Issue 2: FTs acted as Former Pupils and Popular Educators to carry out Teacher's actions, and as University Students interested in their grades

We identified four positions that FTs adopted in the Teaching Mathematics (TM) institution during the Pilot-Study: Teacher, Former Pupil, Popular Educator and University Student. Like in (Sierpinska, Bobos, & Knipping, 2008) these labels resulted – and will be described – as a result of applying the "grounded theory" approach, i.e., based on empirical observation rather than on postulating some ideal versions of these positions. DeBlois and Squalli (2002) also observed these categories (with the exception of Popular Educator) as epistemological positions that prospective teachers adopt in relation to the knowledge acquired during their in-service training.

The Instructor's attempt to have students use analytical tools – such as definitions or precise methods in problem posing tasks – which we would classify as belonging to the Teacher position – was met with considerable resistance. Clearly the genuine epistemological and cognitive obstacles related to the notion of fraction may explain why FTs would not spontaneously use, for example, the measurement technique demonstrated in the classroom to design an activity for children involving non-standard units. But what we found most striking and felt unprepared for was the difficulty to *enlist* the candidate teachers to learn anything new. This barrier didn't lay in some sort of apathy, or just general lack of interest, on the part of the students, such as we sometimes encountered in students from the same demographic taking mathematics in college or university.

It was rather specific to the TM institution: FTs adopted the positions of Former Pupil or of Popular Educator when asked, in tasks, to perform Teacher's actions. That is, we did expect that it would be a challenge to present any new way of looking at elementary level mathematics to someone who has already been through formal schooling, but the attachment to certain algorithms or methods in teaching contexts was also accompanied by a strong belief (not always justified) that "they work", derived from one's personal experience with learning the topic at hand. Furthermore, FTs appeared to have a certain resourcefulness in "educationese", a form of progressive education rhetoric that they used to support their views: FTs spoke like Popular Educators using phrases such as "hands-on learning", "meaningful learning experiences", "child-centered lesson", "the visual learner", and so on, in posing problems and solving them as part of a proposed activity for children. But these words were empty categories – of no consequence or relation to the mathematics involved in an activity.

From the position of Former Pupils, FTs often complained that the Instructor was making something that they believed to be straightforward and easy, "unnecessarily complicated". Not only would FTs fail to produce activities that would use a given definition or method, but they would also actively dispute the relevance of certain ideas, when "simple" algorithms (such as "cross-multiplication") or diagrams can be used in such activities. In one of the homework assignments, we included the following two tasks – one of solving a problem, the other of inventing a similar one and writing a plan of a classroom activity around it:

Solve the following problem

1. In class, you have seen a measurement technique that consisted in counting how many times a given unit fits into the line measured and finding the remainder, then counting how many times that remainder fits into the unit and finding perhaps yet another remainder, then counting how many times the new remainder fits into the old and perhaps finding a third remainder, and so on until no remainder is left or is so small that we decide to ignore it. Then, you find how many times the smallest remainder you found fits into both the unit and the line you want to measure with it. This way, the smallest remainder works as a common "sub-unit" (like inches are a subunit of a foot) that fits a whole number of times

into both the line measured and the unit we started with. The ratio of these whole numbers gives you the measure of the line measured in terms of the initial unit.
Use this technique to measure the longer side of the rectangle below using the shorter side as your unit. Be patient and find at least two remainders.

Invent a problem

2. Invent a measurement problem similar to Problem 1 above, and write the plan of a classroom activity in which this problem would be incorporated. Your plan should contain at least the following elements:

- The target age group of students
- The main mathematical ideas or techniques necessary to solve the problem
- Materials
- Expected solutions and their validity
- Implementation: how the problem would be posed to children
- Questions to ask children in the context of the problem, to extend the problem, test children's understanding, etc.

The responses took on what we thought were irrelevant aspects of the given problem such as the use of rectangles or of one object to measure another object. In most cases FTs missed the essential features of the measurement technique (which was, in fact, the Euclidean algorithm of finding a common unit to measure two lengths), such as measuring with a non-standard unit, or presenting the result of the measurement in the form of a fraction. For example, one of the proposed activities would require children to measure the large side of an eraser using the small side, but the measures of both lengths would be given beforehand in cm: $5\ cm$ and $3\ cm$. Furthermore the *Expected solutions and their validity* part read as follows:

> There are 6 halves (0.5) in 3 and there are 10 halves (0.5) in 5. 6 halves go into 10 halves, like 3 over 5:
>
> $$\frac{3}{5} = \frac{6}{10}$$
>
> Proof: cross-multiply
>
> $$(3 \times 10) \div 5$$
>
> $$30 \div 5 = 6$$

The cross-multiplying can be done in "two seconds" the student replied to the attempt to convince her to use the learned technique, and "it's always worked for me", so "why should I use the other method?"; furthermore, when told her that this would be an activity *to introduce fractions* – hence neither fractions nor decimal numbers should be assumed as known – she answered "but decimals are much easier." In other words, the decisions for posing or solving problems, were rooted in FTs previous experiences in learning the topic at hand. They would adopt the position of Former Pupil to support decisions and resist change with surprisingly fierce conviction.

When teaching mathematics, by contrast, we find less emotion and always a few students that simply flourish after adopting a certain method, or idea, even it if appears more cumbersome: students appear more comfortable in the University Student position, even in remedial courses where they are not there to learn mathematics by choice, but rather to fulfill a prerequisite. For example, in one of the remedial courses I teach at college I propose that an equation of the type $\frac{2}{7}x = \frac{3}{5}$ be solved using the more general method of dividing both sides by the number that x is multiplied with, that is $\frac{2}{7}$, rather than "getting rid of the denominators" by multiplying both sides with their lest common multiple, in order to work with integers only. There are always students who take the Former Pupil position and bring up the latter method, and argue it's easier ("That's how I did it, and it works!"), but they can also rationally appreciate viewing the equation in a structural way as $ax = b$ and thus treating $\frac{2}{7}$ and $\frac{3}{5}$ the same way as the whole numbers in the equation $5x = 15$; this view, in turn, gives them much better control in more complicated equations, where the method of "getting rid of the denominators" is more likely to lead to mistakes. But in the TM course it was as if there was really nothing to learn: it is true that the mathematical knowledge for teaching is of considerable depth, but FTs would not even engage in scratching the surface when prompted to perform Teacher's actions.

Future teachers' belief that "there is nothing to learn" in mathematics teaching methods courses, may, in fact, be a common belief among the general public, and even among mathematicians. Deborah Ball's work to explicate the domain of "mathematical knowledge for teaching" seems to be a response to this belief (Ball, Thames, & Phelps, Content Knowledge for Teaching: What Makes It Special?, 2008). In his book, "The trouble with Ed schools", Labaree (2004) carries an in-depth discussion of the public perceptions about the failure of education schools in the US, and he provides some explanations for such a poor reputation. There is a gap, he amply argues, between the difficulty of the teaching job and the perception of learning to teach as very easy; this contrast, he argues, is the core problem in the way of the professionalization of teaching. He looks at the roots of this gap, by examining why teaching is such a demanding form of professional practice on one hand, and why it appears so easy to become a teacher, on the other hand.

"The problem of client cooperation" is one of the characteristics that make teaching enormously difficult. He observes how medical schools are successful at preparing physicians to treat many diseases by direct intervention without the patient making an effort to cooperate;

they are much less successful at treating disorders such as obesity, for example, and, as a result, they tend to relegate these "low-yield" therapies to counselors, naturopaths, and other human improvement practitioners. But the teacher cannot even attempt to be effective without the student's complete involvement. The rate of success is probably low in such a profession, yet, education schools do not have the luxury to opt out of preparing professionals for whom "changing people is the whole job" (Labaree D. , 2004, p. 41). He discusses several other problems, peculiar to the teaching profession: "The problem of compulsory clientele", "The problem of emotion management", "The problem of structural isolation", and "The problem of chronic uncertainty about the effectiveness of teaching."

The work of the teacher is thus Sisyphean, yet it looks so easy: it seems that teacher educators and their FT students sit on the two sides of this disconnect. One of the explanations is to be found in what Labaree terms "The Apprenticeship of Observation": having been a long time in school before taking university level courses in education, future teachers think that they know enough already to succeed as teachers. By comparison to other professions, for which one needs to possess a very specific set of skills, mostly concealed from the public eye, the teaching profession seems to hold no mystery for future teachers by the time they enter university. Labaree's analysis draws sharp differences between the teaching profession and other professions, painting, in the process, a very distinctive type of professional practice, which explains both the perceptions of the public about the failure of education and prospective teachers' epistemological positions in relation to knowledge needed for teaching.

Deblois and Squalli's study (2002) perform a finer grain analysis of FTs reticence to learn in mathematics methods course, especially when it comes to resorting to theoretical knowledge in mathematics methods courses. They assert, inspired by (Britzman, 1991), that the conceptions one has regarding teaching and learning act like filters to give priority to certain facts versus others. Moreover, these conceptions are particularly powerful at the moment of planning a learning situation (Raymond, 1997), that is when Teacher's actions are called upon (Sierpinska & Osana, 2012). By contrast, during classroom interventions, it is what happens at that moment that influences the most the decision-making process. Furthermore, Deblois and Squalli approach the issue of influencing FTs' conceptions about teaching and learning by describing three epistemological positions that they take with respect to using the knowledge acquired during their in-service training: *the former pupil, the university student,* and *the teacher*. The authors were interested, in particular, in how FTs react to children's errors. As a *former pupil*, the future teacher has certain conceptions with regards to teaching and learning, which are strongly influenced by the way he or she has been taught. As it is often the case, such previous experiences include working on problems for which the solutions are predictable and the answers held by the teacher. Such experiences create the expectation, in FTs, to learn about "methods of teaching" corresponding to the "right" teaching model (Deblois & Squalli, 2002, p. 215). As *university students*, however they are supposed to function in a different setup, where aspects such as the nature of mathematical concepts, the development of children's thinking, or ideas about teaching are discussed in depth, without providing ready-made answers, formulas or recipes for solving math problems or designing classroom activities. In particular, when this stance is adopted, children's errors are conceived of as a component of their learning, which elucidates, for the teacher, ways in which the child may be guided. More importantly, as observed by (Borasi, 1987), it is only when the emphasis is put on the level of abstraction in which the error occurs, that the future teacher can better grasp both the child's learning process and the nature of the mathematical concept at hand. Finally, from the position of *the teacher* – and here the authors, unlike us, refer only to the *stage* component of the program – FTs are expected to propose learning situations but also to intervene, on the spot, to adapt their interventions to children's errors. Moreover, by affecting certain variables of the problem situation, they may even seek to provoke errors, thus bringing on discussions that would allow the child to understand better.

The most interesting result of Deblois and Squalli (2002), for our purpose, was that teachers predominantly took the position of *former pupil* when analyzing children's errors. This, in turn, led them to punctual interventions to rectify them, which although having made reference to specific conceptual elements, failed to connect such concepts in a more global fashion. The researchers' suggestion is that FTs may be led to hold more complex conceptions about teaching and learning only by destabilizing the central role of the *former pupil* stance that they undertake when analyzing teaching/learning situations, in particular with respect to children's errors. The main friction, the authors suggest, is present between the *university student* and the *former pupil* position, while the latter may exist in relative congruence with the *teacher* position (and, I would add, thus perpetuating certain conceptions about teaching and learning in schools). This is exactly what we saw in the course: FTs embrace problem posing tasks – thus willingly assuming the role of the Teacher (we understand the position more broadly to include not only actual teaching but also teaching preparation activities) from the perspective of a Former Pupil, resisting the use of analytical tools entailed by the University Student position.

Some future teachers' resistance to embracing the theory in mathematics methods courses can also be explained, more generally, by the fact that they *feel* what the right way to teach is based on their previous schooling experiences or their personal interactions with children. This reliance on an "inner voice" or subjective thoughts, feelings, and experiences is characteristic for the epistemological position of "subjective knowers'', as described by (Belenky, Clinchy, Golberger, & J.M., 1997), and is accompanied by the view that analysis almost ruins knowledge, as well as by the need to be listened to without being criticized. The feedback received from peers when FTs carried out an actual lesson in the Pilot-Study seemed to respect this need: it was always praise with only modest suggestions made in relation to superficial aspects such as the kind of manipulatives used. In a whole class discussion following a teaching activity, where some FTs acted as teachers and the others– as their pupils, when the Instructor started to point out some shortcomings of the simulated lessons, from the point of view of the mathematical concepts used in them, one FT got up and accused the Instructor of being "rude" to those whose lessons suffered of these shortcomings, even if she did not mention any names. This FT then left the room and two other FTs followed.

FTs' use of jargon in designing teaching activities – what I termed as the position of Popular Educator – was more puzzling and appeared less investigated in the teacher education literature. It is likely that future teachers have had ample exposure to constructivist ideas, or more generally, to progressive education principles, as promoted in reform documents, through educational theory courses at university, or even by school administrators or media. That is where they might have picked up expressions such as, "hands-on and collaborative learning" or "child-centered lesson." FTs used them when describing the implementation of teaching activities, when discussing a workshop, but also in interactions among them or with the Instructor. But, as already mentioned, these were empty categories, of no consequence for the unraveling of the planned mathematics lesson. Moreover such – rather technical – terms would accompany fairly trivial problems. One FT, for example, proposed the following problem for a classroom activity aimed at grades 3-5:

> There were 1200 packages of Teddy's flushable wipes delivered to Willow elementary school, yesterday. Each classroom receives $\frac{1}{6}$ of the total. How many packages of wipes does one classroom receive?

The implementation was then conceived as follows:

> I would love to find a similar type of product to count and use for the problem. I would adapt the equation's total, to the total the group would count but still ask what $\frac{1}{6}$ of that would be. I suppose asking the whole class to participate would be too much, so I would likely split them in two and give two activities simultaneously. I would try to give them *a realistic situation* and *hands-on meaningful learning experiences.* Relating the question to them or real situations and asking them to solve the problem helps *engage* them. It often has a lot to do with how you pose the questions; tone of voice, etc.

FTs were, however, very concerned with assessment – this was one point of interest that would revert them to the student role. In the Pilot-Study, as soon as they started solving Homework Assignments, FTs were very preoccupied with their grades. They would constantly ask the Instructor and the teaching assistant about the grading scheme applied to their work. We thought that their inquiries were legitimate – this is a course they needed to pass to reach a desired career goal – but the conversations were difficult because of the nature of the tasks given in the Pilot-Study. Most of the tasks – and of the grade weight – required FTs to invent problems and plan activities for children (we give more details about these tasks in a subsequent section). The Instructor had grading schemes that took into account the mathematical correctness, the use of certain required ideas or techniques, the matching to a given structure, but there were many other aspects of the activity that were open or just difficult to grade. For example, when grading an activity that FTs planned based on an invented problem, the Instructor didn't include a rubric for the complexity of the problem – it was too impractical to have a scale of complexity, so she couldn't penalize the students for posing too easy a problem. Or, when the Instructor asked FTs to pose a series of related questions, she had in mind that in answering a question one should use the answer to the previous question; some of the students interpreted "related" to mean just that all the four questions would be about wipes, or about the same daycare. The Instructor wanted to have such open-ended questions and grading schemes as a way to preserve the richness of experiences that FTs bring to class, and to encourage independent thinking, but still, "within the box" – by reference to the concepts learned in class. Yet, rather than modeling real situations with mathematics, judging the difficulty of problems and modifying them to make them more or less difficult, or to address certain misconceptions, FTs discussed superficial aspects of the situation – how to organize the seating, how to use one's voice, or use manipulatives to "make it fun. Discussions about how children think or act would also inevitably come up when students proposed ideas for the implementation of an activity. All these aspects weren't built into the assessment scheme, which resulted in a sense of loss of agency for both the Instructor and the students as participants in the TM institution, despite the legitimacy of the concerns on both sides.

As a response to the issues related to FTs positioning in the course, the design entailed shaping a position of University Student for FTs in the TM course institution, and a detachment from the more naturally adopted ones of Teacher, Former Pupil or Popular Educator. We hypothesized that the course organization may reinforce or destabilize one or another of these positions, and, in turn, shape the more desired one of University Student preparing to become an elementary school teacher.

4.2 DESIGN OF THE EXPERIMENT

4.2.1 The Measurement Approach: a knowledge base for future teachers' quantitative reasoning

In this section, I present the Measurement Approach (MA) we developed as a knowledge base for future teachers, as it was planned before the Experiment. Firstly, I include an analytic narrative vignette of an encounter and subsequent exchange between the Instructor and a future teacher, Edith (not the real name), in the Pilot-Study. These events were crucial in the Instructor's development of the definition of fraction of quantity – the central concept in MA – while also pointing to what could be identified as a fundamental situation for future teachers (Brousseau, 1997). Secondly, I highlight the assumption, underlying Davydov's (1991) curriculum that the origin of fractions is in measurement practices and propose the need to adapt it for the mathematics education of future teachers in the light of the vignette. In the third section, I present the decision to practice an explicit distinction between quantities and pure numbers as an important part of this adaptation. In the fourth section, I justify the decision to teach fractions to future teachers in the form of a theory – which we call the theory of fractions of quantities (abbreviated FoQ theory) and present two examples of how the theory has been constructed in the course notes that accompanied the course. Finally, in the fifth section I illustrate the difference between the standard approach of teaching fractions through division (the Division Approach, or DA) and the Measurement approach on an example of solving an elementary level problem of fractions multiplication.

4.2.1.1 A fundamental situation for teachers and the concept of fraction of quantity as a solution

I start with the reconstruction of events that inspire the Instructor to formulate the definition of a fraction of quantity, central in the Measurement Approach, the way it eventually appeared in the "Class Notes" for the TM course.

<u>ANALYTIC NARRATIVE VIGNETTE: THE BIRTH OF THE DEFINITION OF FRACTION OF QUANTITY</u>

- Episode 1: Edith consults the instructor before designing her teaching simulation workshop

Edith wants to do her workshop on fractions equivalence. She asks for the Instructor's advice during office hours, posing the question: *How can I explain equivalence of fractions to children?*

The Instructor says that equivalence of fractions is related to change of unit of measurement measurement. She draws some pictures to represent how the same amount can be represented as two different-looking fractions of another amount: a round pizza (circle) divided into 6 slices (6 equal circular sectors), two of which are shaded to mean that they have been eaten. She says: if I take one slice as a unit of measurement, then I can say that $\frac{2}{6}$ of the pizza was eaten; if I take 2 slices as a unit, then I can say that $\frac{1}{3}$ of the pizza was eaten; so $\frac{2}{6}$ of the pizza and $\frac{1}{3}$ of the pizza are two equivalent fractions of the pizza because they represent the same amount.

- Episode 2: The instructor organizes her explanation in a short text and sends it to Edith

The explanation in the face-to-face meeting with Edith was not as smooth as presented here; the Instructor hesitated in choosing the shape of the pizza and then in choosing the right words. She was generalizing the idea at the same time as she was giving the example and responding to Edith's questions for clarification. She was starting to see how this notion of equivalence of fractions calls for identifying the notion of fraction of a quantity as a notion in its own right, deserving its own special definition. She also came to realize that the notion of abstract fraction as a special type of abstract number, defined, at the beginning of the course as a ratio of an amount to a unit used to measure it, is too complex and too abstract to explain to children or teachers like Edith. Its understanding requires an understanding of more basic concepts, such as what it means that an amount is measured in some units and when we can say that this amount is such and such fraction of another amount. So, after the student left, the Instructor organized her explanations in a neat text with precise diagrams (using rectangles rather than circles, because those diagrams were easier to generate with the geometry software – Cabri (Cabrilog, 2009)), and sent it to the student by email. The text is reproduced below. In this text, the Instructor tried (not very successfully) to make a link between the concept of equivalent fractions and the abstract concept of number introduced in the course, as ratio between an amount and the unit used to measure it.

<u>Text sent in an email to Edith as an attachment right after the meeting</u>

A number is a ratio of an amount (call it A) to some other amount (call it u) used to measure A. The number tells us how many times u fits into A.

A fraction (in abstraction) is a number that can be represented as a ratio of two whole numbers:

$$\frac{a}{b}$$

So an abstract fraction is, in fact, a ratio of ratios.

In elementary school, we speak mainly of fractions of some given amounts.

But what do we mean when we speak of a **fraction of an amount**?

What do the numbers a and b represent in the context of amounts?

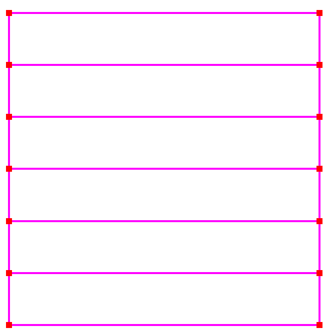
I'll explain it on an example.

Suppose I say that I ate $\frac{3}{6}$ of a rectangular pizza.

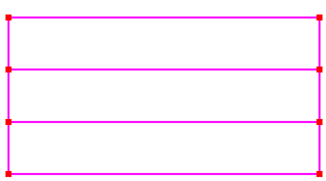
Suppose the picture below represents this rectangular pizza:

This means that if I use the amount of pizza represented below as a measuring unit:

then the rectangular pizza measures 6 such units

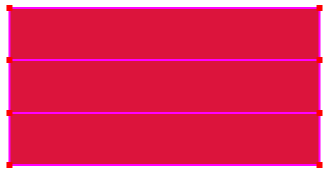


and the amount I ate measures 3 such units:



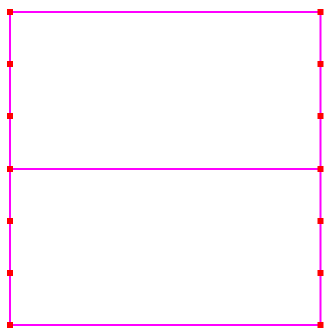
So I can say that I ate $\frac{3}{6}$ of the pizza.

If I now use a bigger unit to measure the amounts of pizza, say, the amount represented below:



then this new unit measures 3 old units.

One whole rectangular pizza measures 2 new units



and the amount I ate measures 1 new unit.

So I can say that I ate $\frac{1}{2}$ of the rectangular pizza.

But the amount I ate is the same, whatever the units I measure it with.

So saying that I ate $\frac{3}{6}$ of the rectangular pizza, and saying that I ate $\frac{1}{2}$ of that same amount of pizza mean the same thing.

- Episode 3. Presenting this idea of fraction of a quantity and of equivalent fraction in a lecture

In one of the labs after the encounter (towards the end of the course, in the Pilot-Study) the ideas developed above were communicated to students in the form of an instructional objective for a teaching activity, along with means for its realization. This presentation was meant to serve as a model for FTs Problem Book that students had to write as their final assignment for the course. The definition of fraction of quantity as a relationship between two measured quantities became more crystalized. Here are the relevant slides from this Lab:

<u>Slide 1</u>

INSTRUCTIONAL OBJECTIVE:
STUDENTS WILL UNDERSTAND THE CONCEPT OF FRACTION IN THE CONTEXT OF QUANTITIES

The means to achieve this objective are many and varied:

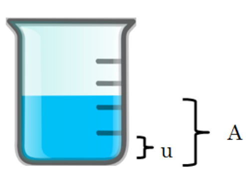
- Brief and concise definition of the word "fraction"
- Explanation of the meaning of the expression "This quantity is $\frac{a}{b}$ of that other quantity"
  - in general terms
  - and on examples of use covering a range of possibilities (fractions less than one, and greater than one; fractions of the same quantity and fractions of different quantities, etc.)
- Graphical representations
- Physical representations

<u>Slide 2</u>

INSTRUCTIONAL OBJECTIVE:
STUDENTS WILL UNDERSTAND THE CONCEPT OF FRACTION IN THE CONTEXT OF QUANTITIES

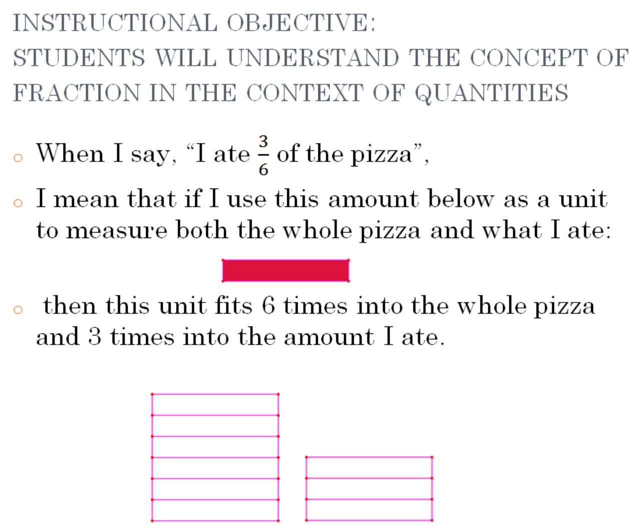
Explanation of the meaning of the expression "This quantity is $\frac{a}{b}$ of that other quantity"

- a and b are whole numbers.
- So they are numbers.
- And numbers tell us how many times a certain unit fits into another amount which is measured with this unit.

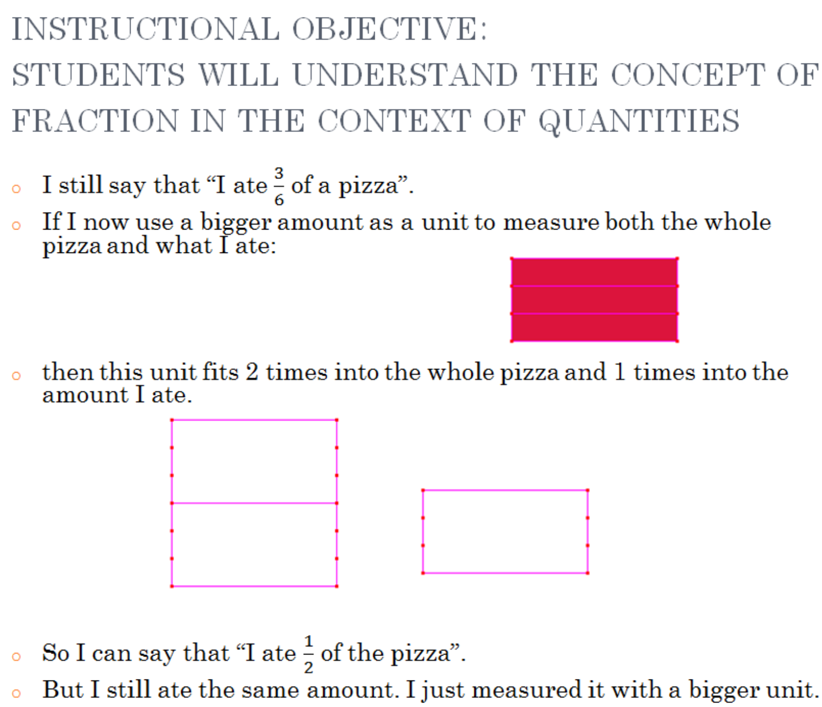


$$\frac{A}{u} = 2 + \frac{1}{2}$$

Slide 3



Slide 4



In the meeting between the Instructor and the future teacher we called Edith we can identify a *fundamental situation* for future teachers (Brousseau, 1997): *How to explain equivalence of fractions?* The FT would not be concerned with solving a particular problem on fractions equivalence, for example, the equivalence of $\frac{1}{2}$ and $\frac{3}{6}$ in a problem about comparing how much pizza two children had or with teaching fractions equivalence to a particular audience (e.g., to children), but rather with finding a meaningful explanation in a realistic context, i.e., problems that one encounters in elementary mathematics. To say that $\frac{1}{2}$ is equivalent (or equal to) $\frac{3}{6}$ because $1 \times 6 = 2 \times 3$ is a legitimate justification in the context of the structural theory of rational numbers (section 3.1.1). In this theory, rational numbers are defined as a system of ordered pairs of integers subject to exactly this relation of equivalence: two ordered pairs, (a, b) and (c, d), where b and d are not 0, refer to the same rational number if and only if $a \times d = b \times c$. This explanation would not make sense in elementary school, where meanings of mathematical concepts must be drawn from their sources in human practices familiar to students. But it would not be reasonable, either, to construct a kind of "ad hoc" realistic explanation of equivalence of fractions that would have no bearing on or would be unrelated with realistic explanations of other important concepts related to fractions which also have to be introduced in elementary school, particularly the arithmetic operations. To make sense, mathematical concepts must be related, connected within a conceptual system. Thus, an optimal solution to the problem of *explaining equivalence of fractions* seems to be a concept that one can use consistently to explain not only the equivalence relation of fractions but also other operations on fractions as "images of what one makes through doing them" (Thompson & Saldanha, 2003, p. 15). In MA, this central concept is the concept of "fraction of a quantity", where fraction as a number is not yet abstracted from the context of measurable quantities. We propose a definition of the concept of fraction of quantity (we elaborate on it below), and consistently build a whole theory of fractions of quantities on it, as a system of conceptualizations of various relations and operations on fractions of quantities, all the while maintaining the commitment to anchor the explanations in realistic situations of measuring and comparing quantities. In this theory, equivalence of fractions corresponds to measuring quantities in different units. Addition of two fractions of a given common reference quantity corresponds to taking the two fractions together and comparing this new quantity to the reference quantity; this comparison may require measuring one or both of the quantities in different units. The operation of multiplication appears in several particular operations on fractions of quantities: taking a fraction of a quantity is, in fact, multiplication of an abstract fraction by this quantity; taking a fraction of a fraction of a quantity is an iteration of this form of multiplication, etc. Division is involved in two types of problems. One type is: The given quantity is the given fraction of what quantity? The solution is obtained by means of the operation of *division of a quantity by an abstract fraction*, which consists in finding the unit in which the given quantity was measured – by dividing the quantity by the numerator – and then multiplying this unit by the denominator, which gives the unknown reference quantity. Another type is: The given quantity is what fraction of another given quantity? Here, the operation is *division of a quantity by a quantity*. It requires finding, if possible, a *common unit* to measure both quantities so that each measures a whole number of such units. This last idea is fundamental in our definition of fraction of a quantity. We explain it in the next paragraph.

We define fraction of quantity as a relationship between *a quantity* and *another* measured with *a common unit*, itself a quantity: a quantity $\boldsymbol{Q1}$ is a fraction $\frac{a}{b}$ of another quantity $\boldsymbol{Q2}$, if there is a common unit $\boldsymbol{u}$, such that $\boldsymbol{Q1}$ measures a such units $\boldsymbol{u}$, and $\boldsymbol{Q2}$ measures b such units $\boldsymbol{u}$ (where a and b are whole numbers, and $b \neq 0$). We can then explain fractions equivalence easily as a change in unit: by either breaking down units of measurement into smaller ones, or reversely, grouping them into larger units, and producing different measurements of the two considered quantities, different representations of the same multiplicative relationship between the two quantities are obtained. These representations are “equivalent fractions.” For example, we can say that $100\ g$ is both $\frac{2}{3}$ and $\frac{4}{6}$ of $150\ g$, because, if we use $\boldsymbol{u} = 50\ g$ as a unit of measure, then $100\ g$ measures 2 such units $\boldsymbol{u}$ and $150\ g$ measures 3 such units $\boldsymbol{u}$, while if we use a unit twice as small, $\boldsymbol{w} = 25\ g$, then $100\ g$ measures 4 such units $\boldsymbol{w}$ and $150\ g$ measures 6 such units $\boldsymbol{w}$.

MA, as a design, has a few essential features that I will present in detail below. The crucial role of measuring quantities is inspired by Davydov’s approach to teaching fractions in elementary school. As an adaptation for future teachers we propose that an explicit distinction be introduced between quantities and abstract numbers. Another such adaptation is the organization into a theory: the fraction of quantity theory, or FoQ, theory. Essentially, these ideas were subsumed to the goal of giving future teachers the tools to explain a piece of mathematics involving fractions. Issues of how to deliver fractions in elementary school to children were beyond the purpose of the study – for example, we didn't address the question of how to construct a lesson plan for the topic of fractions equivalence; we were concerned squarely with building future teachers' a mathematical knowledge base that would serve them across various curricular and institutional constraints. However, in conceiving the MA, we did examine critically assumptions about the goals of elementary mathematics education and the role of university-based teacher education, and we considered children's readiness and future teachers' backgrounds in relation to various cognitive demands.

4.2.1.2 MA: an approach for teaching future teachers

No matter how attractive Lebesgue's idea of using the problem of precision in measurement to construct all numbers, as we have shown in Chapter 3, the idea of fraction in the context of division of things is rather the standard in today's elementary teaching and even in textbooks for future teachers. We find the abstract-algebraic justification for fractions construction even in NCTM materials for teaching fractions (in the form of "Big Idea 1"):

> *Big Idea 1*. Extending from whole numbers to rational numbers creates a more powerful and complicated number system.
>
> Have you ever wondered why we even have fractions? (…)
>
> *Reflect 1.1* Six times a number is 42. What is the number? Suppose that you change the first statement in the problem to 'Six times a number is 32'. Now what is the number?
>
> Reflect 1.1 suggests one need for fractions.
>
> (The National Council of Teachers of Mathematics, 2010, p. 10)

The standard Division Approach is a reality that we cannot ignore if we want to take into account the "real-life situations" of our students, future teachers in North-American schools.

The Measurement Approach is inspired by Davydov's approach, but takes the reality of teachers into account in several ways.

First, from the onset, we practice a clear distinction between quantities and pure numbers; we discuss this choice in the next section. While not introducing fractions as a necessary consequence of a practice of measuring in hands-on activities, we retain, from Davydov's approach, the crucial role of measurement by making quantities the focus of study. When we consider situations of equal sharing and partitioning, typical in the DA (e.g., sharing cookies, cutting pizzas, etc.), we make the point of specifying the quantities that are to be compared (weight, top surface area, calories, amount of fat, etc.). It is typical of the DA to leave the quantities unspecified and refer to them as if they were physical objects. In MA, we aim at making students aware that fractions refer to abstract relations between measurable aspects of objects, not between objects as such.

Our definition of fraction of quantity reflects this less radical stance: a fraction is a relationship between a quantity $\boldsymbol{Q1}$ and a quantity $\boldsymbol{Q2}$, rather than between $Amount$ and $unit$, where "unit" refers to a – standard or non-standard – unit of measure, as in Davydov's approach to teaching fractions to children (Davydov & Tsvetkovich, 1991). The reference quantity $\boldsymbol{Q2}$ in our definition does not have to represent a unit of measure. Our definition applies not only to phrases such as "the length of this string is $\frac{7}{8}\,ft$" (where the length of the string is $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ is the standard unit $1\,ft$ of length), but to any pair of quantities. For example – to statements such as "the weight of this piece of cake is $\frac{3}{4}$ of the weight of the whole cake", which, modulo the specification of the quantity (weight), belongs to the kind of statements common in the DA. This allows a more natural adaptation to the context of partitioning objects, where a quantity that corresponds to "some part" is a fraction of another quantity that corresponds to "the whole" or to another "part." The definition only emphasizes aspects left implicit in the DA: the need to measure both quantities with a common unit and the operations of change of unit and unit conversion. Assumption of the "common unit" replaces the condition of dividing the whole into "equal parts" made in DA. In fact, one of the first planned activities accompanying the introduction of the definition was to cut an apple in two parts, somewhere in the "middle" of the apple. In DA, the parts would be called "halves" and each assigned the symbol $\frac{1}{2}$. In MA, after cutting the apple in two parts, each and both are first weighed on a kitchen scale, and then their volumes are found, and the results recorded in a table. If, in the case of weight, the whole apple is found to weigh 198 g, and the parts 98 g and 100 g, then the teacher declares that the part weighing $98\ g$ is the fraction $\frac{98}{198}$ of the weight of the whole apple, when a common unit $\boldsymbol{u} = 1\ g$ is used; a larger common unit, $\boldsymbol{u}' = 2\ g$, produces the equivalent fraction $\frac{49}{99}$.

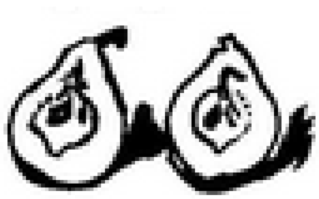
Another feature of the design, meant as an adaptation for teachers of Davydov's approach, is to organize the material for the course into a theory, the FoQ theory, which was to be taught as such, to the future teachers. The idea was to build a solid concept of fraction of quantity, which can produce quantitative explanations for relations and operations on fractions. We did this by assigning stable meanings to concepts and techniques, based on definitions and mathematical reasoning applied to quantities, using informal deduction. To formulate solutions and results we used precise language and paid attention to the technical meanings we had assigned to various terms. In solving or posing problems we analyzed their structure and classified them into types, thus reflecting the nature of teacher's work. Finally, the theory of fractions of quantities is clearly consistent with the theory of rational numbers, but the two are distinct as theories; they are developed to answer different concerns and fulfill different epistemological functions. The concept of rational number should eventually be abstracted from that of fraction of quantity, but we did not consider explanations rooted in abstract algebra as satisfactory at the level of elementary mathematics education.

4.2.1.3 MA: distinguishing between quantities and abstract numbers

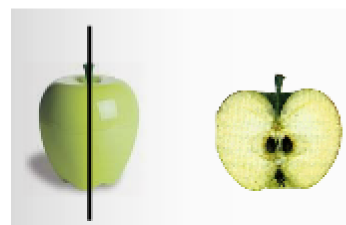
In MA, we study fractions of quantity first, and then, separately, rational numbers. More generally, MA distinguishes between denominate numbers (e.g. $900\ calories$, $375\ g$, $3\ slices$) and abstract numbers (e.g., 900, 375, 3). Denominate numbers are represented by numerals accompanied by names of units or names of things, while abstract numbers – by numerals only. Denominate numbers designate quantities, defined as measurable aspects of objects. Some object – say the pizza that Jack ate – can be quantified in terms of energy value as

$900\ calories$, of weight as $375\ g$, or of cardinality as $3\ slices$. We say how much of *what* we refer to. By contrast, abstract numbers can say how many times one quantity is bigger than another quantity: in the phrase, $1\ foot$ is 12 times larger than $1\ inch$, $1\ foot$ and $1\ inch$ are quantities, while 12 is an abstract number. But abstract numbers can also be considered only in reference to other abstract numbers, without any reference to quantities. For example, 4 can be defined as $2 + 2$, and $\frac{3}{8}$ can be defined as the number which, multiplied by 8 gives 3.

In DA, representations and their description suggest that fractions refer to parts of things (Figure 4.3).



"When anything is divided into two equal parts, one of the parts is called one half of the thing." (Colburn, 1884, p. 99)



"Une demi-pomme." (Guimond, 2014)

Figure 4.3. Representations of fractions common in the Division Approach (Sierpinska & Bobos, 2014, p. 8)

We avoid using such mental shortcuts in the MA. Fraction symbols explicitly represent relationships between quantities that are identified and named. For example, $\frac{3}{8}$ refers not to *the* pizza that Jack ate, but to the relationship between the energy value of the pizza he ate ($900\ calories$) and the energy value of the whole pizza ($2400\ calories$), or between the weight of the pizza he ate ($375\ g$) and the weight of the whole pizza ($1000\ g$), or between the number of pieces he ate ($3\ pieces$) and the total number of pieces ($8\ pieces$). The common units used to measure each pair of quantities are, respectively, $300\ calories$, $125\ g$, and $1\ piece$. Different units would give different, equivalent fractions. For example when the unit of weight is changed from $125\ g$ to $25\ g$, we obtain the equivalent fraction $\frac{15}{40}$. When we pay attention to quantities in this way, we not only reveal an assumption left implicit in modeling the real situation of cutting a large pizza into 8 pieces – that the cut distributes the weight and the caloric value evenly among the pieces – but, most importantly, we distinguish between *the thing* and *the number*. $\frac{3}{8}$ is not the pizza that Jack ate; it is the relationship between two quantities, be it energy value, weight, or number of pieces.

Tracing a sharp, explicit distinction between abstract numbers and quantities is worth considering in the educational domain, especially in the preparation of future teachers, for a few reasons. On the one hand, the teaching of mathematics to children in elementary school does rely greatly on interactions with the real world where numerals, or names of numbers, designate quantities. This connection to the real world is not only desirable, but also, from a cognitive developmental point of view, likely impossible to bypass. On the other hand, the crystalized notion of pure number, detached from concrete meanings so painfully in the history of mathematics, already inhabits school mathematics, and becomes unavoidable as soon as operations with fractions are involved. But the domain of abstract numbers in elementary mathematics, even considering all variability across different curricula, does not map isomorphically to this familiar reality. In other words, abstract numbers and quantities are all mixed up in elementary mathematics: teachers must explain to children the mathematics of practitioners, where fractions come up in quantitative contexts, while dealing with abstract fractions – a handy, albeit sophisticated, mathematical tool produced by mathematicians not so long ago, in the 19th century.

In theoretical mathematics, abstract fractions are constructed as equivalence classes of a relation among ordered pairs of integers (a, b), with $b \neq 0$ (as described in section 3.1.1) This construction, which has a certain aesthetic appeal, is, today, sometimes introduced to mathematics students in an Abstract Algebra class at the university. But, as Chevallard's (1985b) historical analysis of the passage from arithmetic to algebra in French textbooks shows, in the 1970s, this construction was taught in secondary schools in France (Figure 4.4). This approach to fractions was characteristic of the focus on algebraic structures underlying the New Math reforms (called "Mathématique Moderne" in France) in the 60s and 70s. The reforms were later abandoned, but Chevallard's study shows that traces of the formal approach to the exposition of fractions stubbornly pervade mathematics textbooks even in redesigned curricula. The same traces were detected in NCTM's postulate of "*Extending from whole numbers to rational numbers creates a more powerful and complicated number system*" as a rationale for the existence of fractions.

6 Ensemble $\mathbb{Q}$

Nombres rationnels

RAPPELONS :

Dans l'ensemble $\mathcal{F}$ des fractions, nous avons défini la relation $\mathcal{R}$ suivante :

Quelles que soient les fractions $\frac{a}{b}$ et $\frac{c}{d}$, on a : $\left(\frac{a}{b} \mathcal{R} \frac{c}{d}\right) \longleftrightarrow (ad = bc)$.

Nous avons établi (p. 48) que $\mathcal{R}$ est une relation d'équivalence dans $\mathcal{F}$.

Pour traduire que deux fractions $\frac{a}{b}$ et $\frac{c}{d}$ sont liées par $\mathcal{R}$, nous avons écrit : $\frac{a}{b} = \frac{c}{d}$,

et nous avons dit que les fractions $\frac{a}{b}$ et $\frac{c}{d}$ sont équivalentes.

DÉFINITION :

Nous donnons la définition suivante :

$\boxed{\textbf{Chaque classe d'équivalence selon } \mathcal{R} \textbf{ est appelée un nombre rationnel.}}$

Nous désignons par $\mathbb{Q}$ l'ensemble des nombres rationnels.

REPRÉSENTANT D'UN RATIONNEL.

Soient x un élément de $\mathbb{Q}$ et $\frac{a}{b}$ une fraction, élément de $\mathcal{F}$.

Nous disons que x **est un nombre rationnel dont un représentant est** $\frac{a}{b}$ **si et seulement si** x **est la classe d'équivalence selon** $\mathcal{R}$ **de la fraction** $\frac{a}{b}$.

Le nombre rationnel x est donc l'ensemble des fractions $\frac{c}{d}$ équivalentes à $\frac{a}{b}$; nous avons : $x = \left\{\frac{c}{d} \middle| c \in \mathbb{Z},\ d \in \mathbb{Z}^*,\ \frac{c}{d} = \frac{a}{b}\right\}$.

Remarquons que toute fraction équivalente à $\frac{a}{b}$ est un représentant du nombre rationnel x.

CHAPITRE 6. ENSEMBLE $\mathbb{Q}$ 67

Figure 4.4. A page from a 1978 French textbook for grade 9 (Mathématiques 4e, Monge), as reproduced in (Chevallard, 1985b, p. 67).

One can ask: What could be the rationale for continuing to teach fractions at school, especially elementary school? Haven't they lost their utility in common calculations since the invention of decimals, and even more since the ubiquitous use of calculators? The famous mathematician

Henri Lebesgue (1932), one of the forefathers of the modern theory of integration, was asking himself this question because he doubted not only the pragmatic but also the mathematical value of fractions: he was able to construct real numbers based entirely on the notion of measure and decimal representations of numbers. He speculated on the reasons behind the apparent educators' fascination with the domain of fractions and a possible resistance to eliminate them altogether from the elementary curriculum: could it be by the fact that fractions provide an endless source of exercises for young children? Is it because the structural theory of fractions (outlined above) offers a construction that captures the beauty rather than the utility of mathematics?

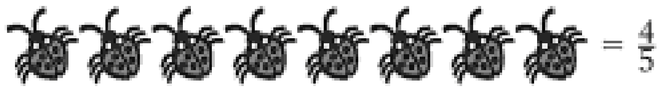
In France, the criticism of the excessively theoretical orientation of the reform, led to the design of a new curriculum, in 1978-79, where fractions acquired a new status, as the central topic in the study of numbers. The focus on fractions came at the expense of a thorough treatment of real numbers, but the taste for rigor and purism still accompanied the redesigned curriculum with regard to fractions. Whether the purely algebraic approach – where rational numbers are constructed as pairs of integers, under an equivalence relation – or a more realistic conception is adopted, where the real numbers are given and fractions are real numbers that can be represented as quotients of integers, the preoccupation with the abstraction of rational numbers as equivalence classes persisted, more or less explicitly. For example, in one of the textbooks presented by Chevallard, the notation $\frac{a}{b}$ stands, according to the authors, for two different things: the pair (a, b), and the exact quotient of a by b. This leads to an odd usage of the equal sign: "the rational number $\frac{14}{4}$ is equal to the rational number $\frac{7}{2}$; the fraction $\frac{14}{4}$ is not equal to the fraction $\frac{7}{2}$ (because the couples $(14,4)$ and $(7,2)$ are distinct)" (Chevallard Y. , 1985b, p. 68). Chevallard comments on the teacher-student interaction with respect to fractions in this case: the teacher acts as the "architect" of a rich, rigorous, and subtle construction that is hardly accessible to the student, while the student 's role is to solve computational exercises of increasing complexity. Just as it is the case with the use of algebraic language, communication is poor as long as the teacher demonstrates a general equality using variables, while the student's role is reduced to applying it in particular cases, substituting concrete numbers for the letters.

Ma (2013) makes a similar observation about the organization of elementary mathematics in the US as envisioned in the NCTM Standards. She argues that the current "Strands" structure proposed in the NCTM Standards is the result of an early 60s reform of elementary mathematics education, when, in the wake of the Sputnik launch, it was decided that more advanced mathematics could and should be taught at elementary level. The New Math movement was full of interesting ideas such as pursuing advanced concepts of set theory and functions, or higher order cognitive abilities such as problem-solving and "thinking like a mathematician"; these ideas were organized as strands that would unify the entire curriculum: Strand 1 - Number and Operations; Strand 2 – Geometry; Strand 3 – Measurement; Strand 4 - Applications of Mathematics; Strand 5 - Sets; Strand 6 - Functions and Graphs; Strand 7 - The Mathematical Sentence; and Strand 8 - Logic. In Ma's view, despite subsequent reforms, including the "back to basics" framework adopted in the 80s, still several decades after, the "strands" structure continues to undermine its very purpose: to unify elementary mathematics curriculum. Throughout the years, these "strands" are being removed and modified – or given the different name of "standards" – without justification concerning their role as unifying structure. We can add that the result of preserving the "Strands" structure resulted in pushing a greatly refined mathematical idea into a context that does not justify its emergence. This is the case for the rational number concept envisioned by NCTM materials: the *Big Idea*. *Extending from whole numbers to rational numbers creates a more powerful and complicated number system* is supposed to emerge in the context of pizza cutting.

Ma (ibid.) contrasts this approach to that in the Chinese curricula which focus on arithmetic in the context of quantities as the core that unifies all of school mathematics. The resulting theory of school arithmetic has its inspiration in two sources: the commercial tradition where operations on fractions and whole numbers are demonstrated in worked out examples of practical situations, and the Euclidean tradition of Greek mathematics where a definition system is used to organize such operations. It is precisely this power of arithmetic of quantities to develop abstract thinking step by step that is overlooked in elementary mathematics education; according to Ma, this is a “blind spot” in today’s US elementary education (Ma, 2013, p. 23):

> *One reason that U.S. elementary mathematics pursues advanced ideas is that the potential of school arithmetic to unify elementary mathematics is not sufficiently known. This is a blind spot for current U.S. elementary mathematics. One popular, but oversimplified, version of this trend is to consider arithmetic to be solely “basic computational skills” and consider these basic computational skills as equivalent to an inferior cognitive activity such as rote learning. Thus, for many people arithmetic has become an ugly duckling, although in the eyes of mathematicians it is often a swan.*

The juxtaposition of the advanced idea of rational number with concrete contexts involving division of physical objects is fraught with obstacles, many of which are well documented in the research on the teaching and learning of fractions by children. Such mixing up explains, perhaps, not only children’s, but also future teachers overreliance on an understanding of fraction as “a part of a whole” (Kerslake, 1986), and why a measure of something is confused with the thing itself. Furthermore, known students’ difficulties with fractions such as the limited informal understanding derived from experience in relation to fractions compared to whole numbers, as well as a focus on syntax rather than semantics (Kilpatrick, Sawfford, & Bradford, 2001) are not at all surprising when the rational number is taught. After all, that is exactly what one deliberately does in abstract algebra – ignoring the concrete and focusing on form. Finally, a constant reminder to “mind the units” in solving fractions problems by mathematics educators in textbooks for future teachers is, paradoxically, a symptom of not taking units seriously. One is supposed to decide what “the unit” is in each situation: it seems to be the abstract number 1 but it gets associated to whatever the “whole” is in the situation at hand. In the example below, extracted from such a textbook, a quantity ($8\ bugs$) is equated to a pure number ($\frac{4}{5}$).



or:

$$8\ bugs = \frac{4}{5}$$

(Lamon, 2012, p. 102)

Besides the inappropriate usage of the equal sign, which itself is the source of many a difficulty in doing mathematics later in algebraic contexts (Knuth, Stephens, McNeil, & Alibali, 2006), the reader is left to their own devices to figure out that the "unit" is the number 1, which in turn *is equal to* $10\ bugs$, while the number $\frac{4}{5}$ *is equal to* $8\ bugs$. Who is to say that the number $\frac{4}{5}$ is not actually *equal to* $16\ antennae$ or $32\ spots$? We believe that future teachers' fractions knowledge derived from both their own schooling and their immersion in such materials for teaching fractions warrants the explicit separation we propose between abstract numbers and quantities.

4.2.1.4 MA: developing a theory of fractions of quantities

Through this design, we set out to develop an approach where the quantities at hand are essential in any problem situation and serve as the very fabric of explanation. Moreover, we did this in a systematic manner: in conceptualizing all relations and operations on fractions we started from a definition of fraction of quantity and built all the other meanings on it.

Thus we developed the course with the deliberate intention to teach a concept of fraction of quantity, rather than that of rational number. We gave it a strong theoretical flavor as a way to provide robust tools for explaining mathematics, particularly fractions, as they arise in elementary mathematics in quantitative, rather than abstract, contexts. Davydov advocates the theorization of working with quantities as a way to develop children's concept of number early on in their education:

> *Although primary school students work a great deal with concrete numbers, they still do not obtain actual concept of them. The reason for this […] is the empiricism and pragmatism of the teaching manual interpretation of the essence of the concept of quantity and the operation of measurement. Only by overcoming these deficiencies and increasing the emphasis given in teaching to the theoretical explanation of the meaning of this concept and operation we will be able to give students an understanding of their real mathematical content as early as the primary grades. (Davydov V. V., 1991, p. 4)*

On another level, with regards to the kind of knowledge acquired by future teachers – undergraduate students in education, the decision to teach fractions of quantities in the form of a theory could be well justified by a view of universities as institutions where one acquires intellectual fitness, manifested in habits of thinking systemically, reasoning analytically and practicing deep reflection, i.e., developing theoretical thinking (Sierpinska, Nnadozie, & Oktaç, 2002; Sierpinska, 2005; Sierpinska, Bobos, & Pruncut, 2011).

As mentioned before (in section 2.2.1.2), Dewey (1907) settled the theory vs. practice in teacher education debate in favor of theory even at a time where teacher education had only just started residing in research universities. Yet there were educators after him that still pursued the avenue of designing courses for teachers based on lists of skills and techniques put together by observing what teachers actually do in class (Zimpher, 1986). More recently, students are known to long for "real-life applications" that embody reform ideals (Frykholm J. , 1999) but we have seen also more experienced educators vehemently decrying a focus on theory as a danger to be avoided – this was, for example, an experienced educational consultant's view on a popular radio-show as he was discussing the idea of lengthening teacher education programs in Ontario (Bennett & Johnson, 2013). This view, however, is no longer the norm encouraged today by researchers in mathematics education, or by reform documents (Martinet, Raymond, & Gauthier, 2001; Proulx, 2005). On the contrary, teachers are envisioned as autonomous individuals, who are able to justify their actions in the classrooms, to understand their roles within larger systems of knowledge and culture, and to modify their practices upon thoughtful reflection. Modern psychology recognizes that such habits of theoretical thinking don't come naturally to people and, as such, they must be fostered through conscious instructional effort.

The Measurement Approach makes such an effort primarily by developing a theory of fractions of quantities –the FoQ theory. The definition of fraction of a quantity (FoQ) is the foundation of this theory:

> *Let $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ be quantities of the same kind. Let a and b be two whole numbers, with $b \neq 0$. We say that the quantity $\boldsymbol{Q1}$ is the fraction $\boldsymbol{\frac{a}{b}}$ of the quantity $\boldsymbol{Q2}$ if there exists a common unit $\boldsymbol{u}$ such that $\boldsymbol{Q1}$ measures a units $\boldsymbol{u}$ and $\boldsymbol{Q2}$ measures b units $\boldsymbol{u}$.*
>
> *The number a is called the numerator and the number b is called the denominator of the fraction $\frac{a}{b}$. (Sierpinska, 2013, p. 13)*

We say, for example, that $15\ l$ is $\frac{5}{7}$ of $21\ l$, because there is a common unit $\boldsymbol{u} = 3\ l$ such that the first quantity measures $\boldsymbol{5u}$ and the second quantity measures $\boldsymbol{7u}$.

Systemic thinking, one of the features of theoretical thinking, is targeted in MA by a focus on definitions, on the hypothetical character of statements, and on proof (Sierpinska, Nnadozie, & Oktaç, 2002; Sierpinska, 2005; Sierpinska, Bobos, & Pruncut, 2011). A whole arithmetic of fractions is proposed through precise definitions of what it means to perform various quantitative operations. Statements about quantities are deemed hypothetical, and awareness of one's assumptions is strongly encouraged. In particular, when a new concept or operation is introduced in the theory, one must not use concepts or operations that have not yet been introduced. Decisions about truth are to be made by means of proofs, by reference to accepted definitions, and logical relations within a system and not (or not only) on images evoked by terms, common beliefs or feelings. *Analytic thinking*, another feature of theoretical thinking, is fostered by encouraging attention to the technical meaning of terms (e.g., the term *quantity* is defined precisely as the measurable aspect of an object), and by sharpening sensitivity to formal symbolic notations and to the structure and logic of language. Finally, MA cultivates *reflective thinking* by constant upkeep of an inquisitive attitude: one is expected not only to solve a problem on fractions of quantities, for example, but also to reflect on it in order to identify its structure and synthetize a class of problems of the same type.

We illustrate, by means of two examples extracted from the latest edition of the textbook for the course (Sierpinska & Bobos, 2014), how relations and operations on fractions of quantities are studied in the MA in ways that foster theoretical thinking. The commitment is to explain relations and operations in terms of quantitative practices. In a larger sense, we approach the very intuitive practice of measurement with disciplinary insight – i.e., mathematical sophistication – to develop a powerful concept of fraction of quantity.

ADDITION OF FRACTIONS

Just as the question of "equivalent fractions" corresponds to the practice of *measuring quantities with different units*, the problem of "adding fractions" corresponds to the practice of *taking quantities together.* When such quantities are fractions of the same quantity, one may look for an expression that represents the sum of two fractions of a quantity by a single fraction of this quantity. The problem of addition of fractions of a quantity is formulated as follows:

> *If A is a fraction $\frac{a}{b}$ of some quantity Q, and B is a fraction $\frac{c}{d}$ of the same quantity $\boldsymbol{Q}$, what fraction of the quantity $\boldsymbol{Q}$ are the quantities A and B taken together?*
> *(Sierpinska, 2013, p. 38)*

To arrive at the answer we use carefully chosen examples which we solve using the definition of FoQ and then generalize.

Example 4.1

Jack and Jill were sharing a $240\ g$ box of chocolate fudge. Jack took $\frac{1}{6}$ of the fudge and Jill took $\frac{3}{4}$ of the fudge. What fraction of the fudge did they take together?

Example 4.2

Jack and Jill were sharing a certain quantity of chocolate fudge. Jack took $\frac{1}{6}$ of the fudge and Jill took $\frac{3}{4}$ of the fudge. What fraction of the fudge did they take together?

I include the solution to Example 4.2 to illustrate the reasoning:

If the whole fudge measures 6 units $\boldsymbol{u}$, then Jack took $1\boldsymbol{u}$.

If the whole fudge measures 4 units $\boldsymbol{w}$, then Jill took $3\boldsymbol{w}$.

Thus together, they took 1 unit $\boldsymbol{u}$ and 3 units $\boldsymbol{w}$: to express these quantities taken together as a fraction of the total quantity of fudge $\boldsymbol{Q}$, both the sum quantity and $\boldsymbol{Q}$ must be measured with the same unit. The new unit $\boldsymbol{v}$ must fit a whole number of times into $\boldsymbol{u}$, $\boldsymbol{w}$ and $\boldsymbol{Q}$; in other words, we must find whole numbers p, q, and M, such that:

$$\boldsymbol{u} = p\boldsymbol{v}$$

$$\boldsymbol{w} = q\boldsymbol{v}$$

$$\boldsymbol{Q} = M\boldsymbol{v}$$

But $\boldsymbol{Q}$ is at once $6\boldsymbol{u}$ and $4\boldsymbol{w}$, which means that:

$$M\boldsymbol{v} = 6 \times p\boldsymbol{v} = 4 \times q\boldsymbol{v}$$

The problem is reduced to finding whole numbers p, q and M, such that:

$$M = 6 \times p = 4 \times q$$

Any common multiple of 6 and 4 is a solution for M, while p and q are factors that multiply by 6 and 4, respectively, to give M. The smallest solution is when M is the *least* common multiple, in this case 12, with $p = 2$ and $q = 3$. The following conversion equations ensue:

$$\boldsymbol{u} = 2\boldsymbol{v}$$

$$\boldsymbol{w} = 3\boldsymbol{v}$$

$$\boldsymbol{Q} = 12\boldsymbol{v}$$

Thus Jack took $1\boldsymbol{u} = 1 \times 2\boldsymbol{v}$ and Jill took $3\boldsymbol{w} = 3 \times 3\boldsymbol{v}$. Altogether, they took $11\boldsymbol{v}$, and since the total quantity of fudge was $12\boldsymbol{v}$, it means that the sum of Jack and Jill's fractions of the fudge is $\frac{11}{12}$ of the total quantity of fudge. We represent this operation as follows:

$$\frac{1}{6}+\frac{3}{4}=\frac{1\times 2}{6\times 2}+\frac{3\times 3}{4\times 3}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$$

which suggests the generalization: the sum of the two fractions of the same quantity $\boldsymbol{Q}$ is the fraction $\frac{a\times p+c\times q}{M}$ of the quantity $\boldsymbol{Q}$, where M is a common multiple of the denominators b and d, and p and q are whole numbers, such that $M = b \times p = d \times q$.

MULTIPLICATION OF FRACTIONS

One of the problems involving "multiplying fractions" corresponds to the practice of *taking a fraction of a fraction of a quantity* (abbreviated as FoFoQ). The problem of finding a fraction of a fraction of a quantity can be formulated as follows:

> *If a quantity $\boldsymbol{Q1}$ is a fraction $\frac{a}{b}$ of a quantity $\boldsymbol{Q2}$, which itself is a fraction $\frac{c}{d}$ of a fraction $\boldsymbol{Q3}$, what fraction is $\boldsymbol{Q1}$ of $\boldsymbol{Q3}$?*
> *(Sierpinska, 2013, p. 38)*

Using the definition of fraction of quantity, the solution is: $\boldsymbol{Q1}\ is\ \frac{a\times c}{b\times d}\ of\ \boldsymbol{Q3}$. The expression $\frac{a\times c}{b\times d}$ is thus arrived at as a technique proven to work for solving the practical problem of *taking a fraction of a fraction of a quantity.*

In the case of FoFoQ, the generalization is organized in steps, modeled by concrete examples:

Example 4.3

If a quantity $\boldsymbol{Q1}$ is $\frac{1}{3}$ of a quantity $\boldsymbol{Q2}$, which itself is $\frac{1}{2}$ of some other quantity $\boldsymbol{Q3}$, then $\boldsymbol{Q1}$ is $\frac{1}{6}$ of $\boldsymbol{Q3}$, which generalizes as $\frac{1}{b}$ of $\frac{1}{d}$ of $\boldsymbol{Q}$ is $\frac{1}{bd}$ of $\boldsymbol{Q}$

Example 4.4

If a quantity $\boldsymbol{Q1}$ is 4 of a quantity $\boldsymbol{Q2}$, which itself is $\frac{1}{3}$ of some other quantity $\boldsymbol{Q3}$, then $\boldsymbol{Q1}$ is $\frac{4}{3}$ of $\boldsymbol{Q3}$, which generalizes as a of $\frac{1}{b}$ of $\boldsymbol{Q}$ is $\frac{a}{b}$ of $\boldsymbol{Q}$

I include the solution to the last example to illustrate the solution process using the definition:

If $\boldsymbol{Q1}$ is 4 of $\boldsymbol{Q2}$, then for some unit $\boldsymbol{u}$, $\boldsymbol{Q1}$ measures $1\ \boldsymbol{u}$ and $\boldsymbol{Q2}$ measures $4\boldsymbol{u}$.

If $\boldsymbol{Q2}$ is $\frac{1}{3}$ of $\boldsymbol{Q3}$, then for some unit $\boldsymbol{w}$, $\boldsymbol{Q3}$ measures $3\boldsymbol{w}$ and $\boldsymbol{Q2}$ measures $1\boldsymbol{w}$.

This implies that $\boldsymbol{Q2}$ measures, simultaneously, $1\ \boldsymbol{u}$ and $1\ \boldsymbol{w}$.

So $1\ \boldsymbol{u} = 1\ \boldsymbol{w}$: the units are the same.

Therefore, $\boldsymbol{Q3}$ measures $3\ \boldsymbol{w} = 3\ \boldsymbol{u}$.

Since $\boldsymbol{Q1}$ measures $4\boldsymbol{u}$ and $\boldsymbol{Q3}$ measures $3\boldsymbol{u}$, then $\boldsymbol{Q1}$ is $\frac{4}{3}$ of $\boldsymbol{Q3}$.

Example 4.5

If a quantity $\boldsymbol{Q1}$ is the fraction $\frac{2}{3}$ of a quantity $\boldsymbol{Q2}$, which itself is $\frac{1}{7}$ of some other quantity $\boldsymbol{Q3}$, then $\boldsymbol{Q1}$ is $\frac{2}{21}$ of $\boldsymbol{Q3}$, which generalizes as $\frac{a}{b}$ of $\frac{1}{d}$ of $\boldsymbol{Q}$ is $\frac{a}{b \times d}$ of $\boldsymbol{Q}$

Finally, the generalization, based on an example of general type:

Example 4.6

If a quantity $\boldsymbol{Q1}$ is the fraction $\frac{2}{3}$ of a quantity $\boldsymbol{Q2}$, which itself is $\frac{4}{7}$ of some other quantity $\boldsymbol{Q3}$, then $\boldsymbol{Q1}$ is $\frac{8}{21}$ of $\boldsymbol{Q3}$, which generalizes as $\frac{a}{b}$ of $\frac{c}{d}$ of $\boldsymbol{Q}$ is $\frac{a \times c}{b \times d}$ of $\boldsymbol{Q}$.

The work is done here within the confines of a system built from scratch, i.e. assuming only the definition of fraction of quantity and knowledge of whole numbers in the context of quantities. In Example 4.1 we avoid the need to represent quantities by numbers with non-zero decimal parts (e.g., if instead of $240\ g$ we had used $210\ g$ in the first example) or even decimal approximations (e.g., if instead of $240\ g$ we had used $220\ g$): this is because the introduction to addition of fractions does not assume previous knowledge of decimals. These choices are made explicit in the Course Notes (Sierpinska, 2013; Sierpinska & Bobos, 2014).Furthermore, reflection on the first two examples of addition (Example 4.1 and Example 4.2), leads to the insight that the measure of the whole quantity of which the fractions are to be added is not relevant to finding *what fraction* of that quantity do the fractions constitute together. In particular, the quantity $\boldsymbol{Q}$ can be either given as a concrete number of units, allowing thus to find A and B and reduce the problem to that of adding whole numbers of some unit, or it may not be given, thus requiring a different approach to finding the sum, as shown in the above solution. Finally, analytic thinking is in high demand for mastering both the logic and structure of the argument and the mathematical notation at hand, in particular the algebra needed to formulate successive generalizations based on observed patterns – this is the case for finding the fraction of a fraction of a quantity (Example 4.3 to Example 4.6).

As it can be gleaned from these examples, the MA requires quite the taste for abstraction and rigorous reasoning. There is more behind this deliberate choice than the lofty ideal of teaching such a disposition at university. An important reason is that the concept of fraction is intrinsically heavy from a cognitive point of view; it is a transitional domain between arithmetic and algebra. MA recognizes the need for higher order abstractions when it comes to fractions compared to whole numbers and tackles it explicitly, rather than giving the illusion of a seamless transition from whole numbers to fractions, or of treating them as disconnected topics. Another reason why we propose that future teachers undertake such a dramatic departure from the usual ways of dealing with elementary mathematics is because it may avoid the confusion with what they already know about fractions. In our first attempt to teach fractions arising from measurement, we were struck by the stubbornness of the obstacles created by their own knowledge of fractions coming in the course. Building on such previous knowledge and personal experiences was not satisfactory. McClain (2003) avoids exactly this reliance of previous knowledge to develop preservice teachers' understanding of multidigit arithmetic and the concept of place value, by modifying the "Candy Factory" instructional sequence, designed initially for elementary students (Cobb, Yackel, & Wood, 1992; Bowers, 1996): while in the children's version of the task 10 candies are packed in a roll, and 10 rolls in a box, in the grown-ups' version rolls and boxes are made of 8 subunits. Finally, an elaborate development of a theory of fractions of quantities was aimed at developing future teachers

“Specialized Content Knowledge” (Ball, Thames, & Phelps, Content Knowledge for Teaching: What Makes It Special?, 2008) for the teaching of fractions at elementary level. In other words, it’s teachers’ knowledge, and not knowledge that should be imparted, as is, to children. In fact, most likely, as teachers, they will operate within heavy institutional constraints, and with very different developments of the topic of fractions, limited by the curricula and textbooks at hand. The flexibility afforded by a profound understanding of fractions within such constraints is one of the desirable outcomes of the MA.

4.2.1.5 MA vs. DA, an example: calculating a fraction of a fraction of a quantity

I will illustrate the difference between MA and DA by reasoning, from the perspective of a teacher, on a realistic problem – this is the term I use for problems typically encountered in elementary mathematics, involving some elements of a physical reality. I will use a problem which involves finding a fraction of another fraction of some quantity.

This example is included in the latest edition of the textbook for the course (Sierpinska & Bobos, 2014).

4.2.1.5.1 A fraction of a fraction problem in the Division Approach

Consider the following problem:

> Aileen went to a party and took $\frac{3}{4}$ of a cake back home with her. The next day, she noticed that her roommate had eaten half of what she brought back. How much of the whole cake was eaten? [Adapted from (Osana & Royea, 2011)]

To begin with, the question “how much of the whole cake was eaten” would be somewhat confusing if some concrete quantity of the cake was specified (e.g., the cake weighs $1\ kg$): would it require to find a quantity in grams or the fraction it represents of the whole cake? But since a quantity – e.g., weight or surface – is not given in relation to imagined cake, one gets accustomed to understanding that it requires finding a “fraction of the whole cake”.

Secondly, speaking about a fraction of an object (a cake) and not a fraction of a quantity (e.g., the weight of the cake, or its volume, or bottom surface area, or number of pieces, etc.) is characteristic of DA. From the point of view of the goal of this approach (to teach rational numbers and operations on them), what is meant, really, is that no matter what kind of quantity is considered, if the whole cake measures 4 units $\boldsymbol{u}$ of that quantity, Aileen took back home an amount measuring 3 such units $\boldsymbol{u}$, and her roommate ate an amount measuring half of the $3\,u$, then the measure of the amount eaten is always the same fraction of the $4\,\boldsymbol{u}$, no matter what the value of that unit $\boldsymbol{u}$ is assumed to be. The teacher may have in mind the rational number $\frac{3}{4}$ as the relationship between some 3 units $\boldsymbol{u}$ and 4 units $\boldsymbol{u}$, no matter what kind of quantity is considered, but the children are justified to interpret $\frac{3}{4}$ literally as a material piece of cake. The problem has a well-defined solution: the measure of the amount eaten is $\frac{5}{8}$ (or some equivalent fraction) of the measure of the whole cake.

So, in fact, children are expected to solve not one problem but *a whole class of problems* at the same time.

Younger children are expected to give a general solution to this class of problems by representing the problem situation using drawings: dividing a schematic picture of the cake into 4 equal parts; shading one of the parts to represent the part of the cake eaten at the party; dividing the rest into two equal parts; shading one of them as the part eaten by the roommate (Figure 4.5).

So far, the solution only "simulates" the actions of the protagonists in the situation. But now comes the decisive and most difficult step: to divide the whole cake and the part eaten into parts of the same size, so that the numerator and the denominator of the fraction that represents the relationship between them can be found.

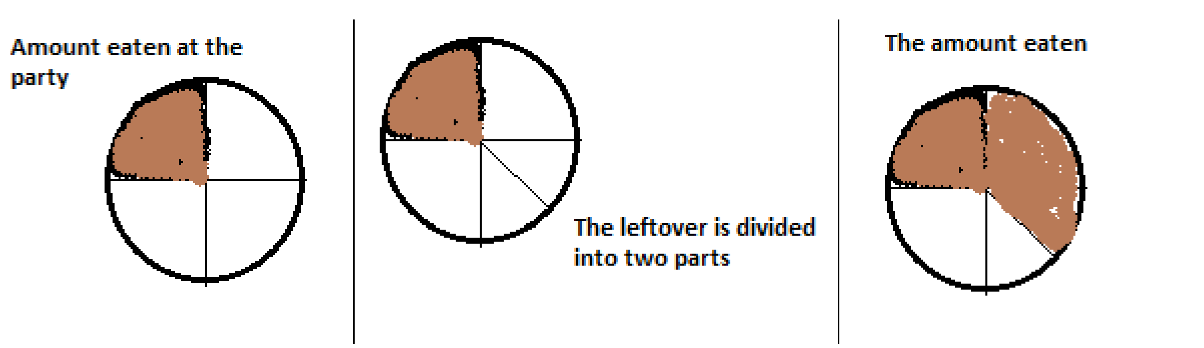


Figure 4.5. The first steps of solving the "cake problem."

Children are expected to “see” that the line used to divide the leftover into two parts divides the “quarter” of the cake into two smaller *equal* parts. If all the quarters are divided in two, then the amount eaten is made of 5 such small parts while the whole cake is made of 8 such parts (Figure 4.6). So the amount eaten is $\frac{5}{8}$ of the whole cake.

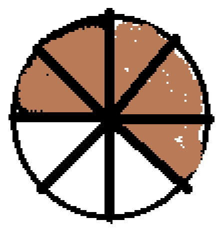


Figure 4.6. Dividing the cake and the amount eaten into smaller pieces.

This way the problem is solved, although, for the children, it may remain mysterious what the cutting of the quarters of the cake in two represents, in terms of the measures of the quantities of the cake. The key to seeing this solution as a solution to a whole class of problems is to understand that a **conversion of units** has taken place: in order to produce a fraction as the quantitative relationship between the measures of the amount eaten and the whole cake, the two quantities have to be measured with *the same unit*. If children fail to see that, they can only see it as a smart graphical solution to a particular problem about a round cake.

Older students would solve the problem using formal numerical operations only, ignoring the quantities in their calculations altogether or perhaps only mentioning “the cake” in the final answer. For example,

$$\frac{1}{2}\times\frac{3}{4}=\frac{3}{8} \qquad \left(1-\frac{3}{4}\right)+\frac{3}{8}=\frac{1}{4}+\frac{3}{8}=\frac{2}{8}+\frac{3}{8}=\frac{5}{8}$$

So $\frac{5}{8}$ of the cake was eaten.

In this solution, the quantitative operation of unit conversion is completely absent and the numerical operation of “expansion of a fraction” ($\frac{1}{4}=\frac{2}{8}$) has been substituted for the quantitative operation of change of unit.

In the two solutions, the meaning of the fractions operations, in particular of multiplication, remains covert: it is, at best, an empirical observation in the first case, and a procedure, in the second, often a succession of the two, where the operations are supposed to be abstracted from the physical model. The difficult step of unit conversion remains implicit.

4.2.1.5.2 A fraction of a fraction of a quantity problem in the Measurement Approach

In MA, before presenting a problem such as quoted above, students would be asked to solve a sequence of problems where the quantity of the whole cake would be specified. By making explicit the quantity of the cake in concrete units, this approach both reduces the cognitive cost of the solution and encourages students to see the essential structure underlying different problems. For example,

> Aileen brought an 800 grams cake to a party. After the party, $\frac{3}{4}$ of the cake was left over and she took it back home with her. The next day, she noticed that her roommate had eaten half of what she brought home.
>
> a. What fraction of the weight of the whole cake was eaten altogether?
>
> b. What would be the answer if the whole cake weighed 1 kg instead of 800 grams?
>
> c. What if the cake weighed 1500 grams?

Students can answer those questions by sequentially calculating fractions of concrete quantities. Question (a) can be solved by first calculating $\frac{3}{4}$ of $800\ g$, finding that it is $600\ g$ and concluding that the amount eaten at the party was $200\ g$. The amount eaten at home is $\frac{1}{2}$ of $600\ g$, $300\ g$. The total amount eaten is therefore $500\ g$ which is $\frac{5}{8}$ of the weight of the cake (if $100\ g$ is used as the common unit of measure).

Solving problems (b) and (c) would lead to the same or equivalent fractions, an observation that students would be expected to reflect upon and offer explanations. If a general solution independent of the concrete weight of the cake is not spontaneously proposed by some students, the teacher would encourage them to solve a more general problem, like the one usually given children without preliminaries in the Division Approach, and guide them in solving it.

> Aileen brought a cake to a party. After the party, $\frac{3}{4}$ of the cake was left over so she took it back home with her. The next day, she noticed that her roommate had eaten half of what she brought back. What fraction of the whole cake was eaten?

In helping students to solve this problem the teacher would stress the moment of change of unit and unit conversion in the reasoning, and would repeatedly ask them to be explicit about the quantities: Three-quarters of what quantity? One-half of what quantity? What is your unit of measurement here?

4.2.2 Shaping the University Student position in Teaching Mathematics courses: fostering learning by acculturation in addition to adaptation

The design entailed, in particular, shaping a position of University Student for FTs in the TM course institution, and a detachment from the more naturally adopted ones of Teacher, Former Pupil or Popular Educator.

In the Pilot-Study the students seemed to have learned little by independent adaptation to the *milieu* alone, i.e., by solving the given tasks, interacting with peers, or using available instructional resources. The problems at hand, in particular, did not make knowledge about fractions *epistemologically necessary*, i.e., they did not function as *a-didactical situations* where the use of a certain piece of knowledge is justified entirely by the inner logic of the situations, and not by the instructions from the teacher or by the didactical contract. A high degree of a-didacticity, where the Instructor merely provides the necessary tools and sets up good problems, without explicitly telling students how to use them, is naturally more difficult to achieve at university level, especially for mathematics courses, because of the level of abstraction already reached at this point (González-Martín, Bloch, Durand-Guerrier, & Maschietto, 2014).

In TM courses the difficulty is compounded. If *knowledge of fractions* is targeted, situations such as the measurement of the thicknesses of sheets of paper proposed by Brousseau and his collaborators (Brousseau, Brousseau, & Warfield, 2014), could not work to create epistemological necessity for FTs, because they already know fractions and decimals. If, on the other hand, *knowledge of fractions for teaching* is aimed at, we did imagine a fundamental situation where FTs would have to design a lesson or explain a problem on fractions (in section 4.2.1.1) but, with no feed-back from the milieu – i.e., no children to give it to – FTs could not be expected to construct such knowledge spontaneously. Even if an audience did exist, on-the-spot certainty about the effectiveness of teaching is hard to impossible to achieve (Labaree D. , 2004).

In the reality of teaching future teachers methods course, we decided to settle for less in terms of FTs autonomy as a prerequisite for learning – that is, more in terms of the Instructor's explicit guidance. When used to analyze classroom situations in a systemic manner (Perrin-Glorian, 2008) the theory of didactic situations (TDS) postulates teaching as an activity that balances <u>acculturation</u>, where learning occurs through direct interaction between the *student* and the *teacher* and <u>adaptation</u> where learning results from the interaction of the *learning subject* with the *milieu* (which is, however, carefully designed by the teacher).

In the Experiment, one of the substantial changes was to foster learning by acculturation rather than by adaptation. This decision belonged to the Instructor, and a-priori, consisted of planning to teach in a way consistent with her identity of a professor in a mathematics department. The most visible consequence reflecting this conscious decision would be to have more direct instruction, or *telling*. This is not the opposite of constructivist teaching, although there is indeed a genuine tension between embracing students' independent thinking and telling students, for example, to use a certain method to solve a problem.

Cobb and other researchers he quotes (Cobb, 2007) point to the dangers of embracing constructivist ideas in a simplistic manner, in particular with respect to downplaying the practice of telling, and the passive role that is attributed to the teacher in this context:

> *The most prominent case in which attempts have been made to derive instructional implications directly from a background theory is that of the development of the general pedagogical approach known as constructivist teaching. This pedagogy claims to translate the theoretical contention that learning is a constructive activity directly into instructional recommendations. As Noddings (1990) and Ball and Chazan (1994) observe, it is closely associated with the dubious assertion that "telling is bad" because it deprives students of the opportunity to construct knowledge for themselves. For his part, J.P. Smith (1996) clarifies that adherents of the pedagogy tend to frame teachers' proactive efforts to support their students' learning as interfering with students' attempts to construct meaning for themselves. As Smith demonstrates, in emphasizing what the teacher does not do compared to traditional instructional practices, the teacher's role is cast in relatively passive terms, thereby resulting in a sense of loss of efficacy. (Cobb, 2007, p. 5)*

Labato, Clarke, & Burns Ellis (2005) not only tackle directly the telling/non-telling tension but also propose a reformulation of telling as "initiating", through a shifting of the foci of telling in three areas: (a) from the form to the function of the acts of communication; (b) from the procedural to the conceptual content of the new information; (c) from the isolated action to the relationships to other actions (Labato, Clarke, & Burns Ellis, 2005, p. 102). Their theoretical considerations and empirical data go a long way towards restoring the credibility of initiating and eliciting students' actions, and certainly demonstrate an approach that is in better alignment with the progressive education ideas as advocated by Dewey (1900/1990). We find a similar type of direct instruction in the teaching episodes documented in (Davydov & Tsvetkovich, 1991): the children are guided, for example, towards a very particular understanding of the notion of multiplication through a series of carefully sequenced eliciting actions from the part of the teacher. In both the above mentioned studies a striking feature was the admirable skill of the instructor, who clearly satisfied the three conditions on telling proposed by Labato et al. (2005).

The instructors of mathematics methods courses typically fit this profile of highly-knowledgeable teachers, so this type of more structured interaction with their future teachers students appears be an interesting option. For the Instructor in the present study, the initiation was going to be in doing mathematics as a cultural practice – it reflected her background as a mathematician and her affiliation in a mathematics department. But such structured initiation can also take place with regard to mathematics education theory. In one of the TM courses profiled in Sierpinska & Osana (2012), the instructor required of her students, quite precise application of theoretical concepts, as in the following example by asking them to apply Ginsburg's (1989) "psychological principles of counting"; this approach reflected her background as a cognitive psychologist.

> Statement: "Children must grasp the one-to-one principle before they can effectively apply the stable order principle".
>
> True or false? Justify.
>
> (Sierpinska & Osana, 2012, p. 122)

In the Experiment, the class was going to look like a typical mathematics classroom at university, where the body of knowledge at hand – in this case the Measurement Approach – would be taught systemically, with meanings established by definitions and techniques justified by reasoning. Importantly, FTs would have a textbook at their disposal with theory, examples, solved and unsolved problems. This organization was also expected to afford a more objective discussion with respect to assessment. The challenge, for the Instructor, reflecting the need to reconcile acculturation and independent adaptation, would be to do just enough scaffolding so as to not allow the production of "good solutions" by mere imitation, without adequate knowledge.

5 Implementation of the Experiment: Description and Analysis

In this chapter, I present *the Experiment*, as it unfolded in the Winter 2014 semester. I organize this presentation in three sections. In the first, I present the long-term organization of the course – especially with regards to the content and tasks. In the second, I describe the teacher-student interactions, taking the classroom community as unit of analysis. In the third, I bring the analysis at a finer grain level – the individual – to look at FTs responses to a question they solved after the end of the course.

5.1 Organization: teacher-milieu

I analyze here how the Instructor influenced the tasks and other features of the content of Measurement Approach for teaching fractions to the particular group I considered – the students of the Teaching Mathematics course offered as part of a university-based education program. As discussed in the description of our conceptual framework *an institutional perspective* (Chevallard Y. , 1999; Barbé, Bosch, Espinoza, & Gascón, 2005; Ostrom, 2005) is particularly useful for presenting the perspective of someone planning or evaluating an entire course. In this section, in particular, it gives me the tools to discuss characteristics of the tasks and content that influence the positions taken by the students in the course. In particular I will first show that the Instructor emphasized the University Student position in the content by teaching topics systematically and comprehensively, without assuming that FTs already knew some topics usually learned in elementary school. Secondly, she replaced the problem posing tasks, typical for the Teacher position, with tasks that were closely embedded in the taught material: to give a fuller grasp of this feature of the long-term organization I contrast it with the organization of the Pilot-Study.

5.1.1 Comprehensive coverage of topics, including prerequisites

The University Student is a *learner* of new material: in the Experiment the instructor assumed FTs knew fractions as children, but not as educated adults. Thus she covered all the usual topics in the fractions domain not just as a way to foster systemic thinking, but also to address potential difficulties and misconceptions, to clarify terminology, and even teach anew certain concepts. I reproduce below the overall organization of the content, based on my field notes from the 8-week long Experiment. I formulate the organization in terms of eight "topics", the first six of which were devoted to the FoQ theory, the seventh – to the abstract theory of fractions, and the eighth – to the connection between the two theory.

1. Number
    a. Abstract and denominate numbers
    b. Quantities and their kinds
2. The concept of fraction of quantity
    a. Definition
    b. Mixed numbers
    c. Decimal fractions
    d. Not all pairs of quantities of the same kind can be related by a fraction
3. The concept of fraction of fraction of quantity
4. Addition and subtraction of fractions of quantities
5. Multiplication and division of fractions in the context of fractions of quantities.
    a. The context of buying and selling. Proportional quantities
    b. The context of areas
6. Comparison of fractions of quantities
7. Systems of numbers. Theory of abstract fractions
8. Connecting the concept of fraction of quantity with the concept of abstract fraction in the contexts of:
    a. Ratio
    b. Percents

The level of detail in content pursued by the Instructor in the lectures can be gleaned from the treatment of the sequence of topics written in bold typeface in the list above. I zoom in on this sequence to show the Instructor's influence on the milieu, refraining, however, from the discussion of the teacher-student interactions against this backdrop.

As already quoted in section 4.2.1, the definition of FoQ, introduced after the explicit distinction between abstract numbers and quantities, was formulated as follows:

> *Let $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ be quantities of the same kind. Let a and b be two whole numbers, with $b \neq 0$. We say that the quantity $\boldsymbol{Q1}$ is the fraction $\frac{a}{b}$ of the quantity $\boldsymbol{Q2}$ if there exists a common unit $\boldsymbol{u}$ such that $\boldsymbol{Q1}$ measures a units $\boldsymbol{u}$ and $\boldsymbol{Q2}$ measures b units $\boldsymbol{u}$.*
>
> *The number a is called the numerator and the number b is called the denominator of the fraction $\frac{a}{b}$. (Sierpinska & Bobos, 2014)*

Mixed numbers are derived fairly easily from the definition, and with a quantitative interpretation: when, in the fraction $\frac{a}{b}$, the numerator is greater than the denominator ($a > b$), the quantity $\boldsymbol{Q1}$ contains the quantity $\boldsymbol{Q2}$ a whole number of times, and a remainder which is itself a quantity that is a fraction of $\boldsymbol{Q2}$. More precisely, if $a = q \times b + r$ where $0 \leq r < b$, then $\boldsymbol{Q1}$ measures $(q \times b)\boldsymbol{u} + r\boldsymbol{u} = q\ of(b\boldsymbol{u}) + r\boldsymbol{u} = q\ of\ \boldsymbol{Q2} + \frac{r}{b}\ of\ \boldsymbol{Q2}.$ This last expression is then traditionally written as $q\frac{r}{b}$ of $\boldsymbol{Q2}$, and $q\frac{r}{b}$ is called a $mixed\ number$. For example, suppose a quantity is $\boldsymbol{Q1}$ is $\frac{17}{5}$ of a quantity $\boldsymbol{Q2}$. This means that for some unit , $\boldsymbol{Q2}$ measures $5\boldsymbol{u}$ and $\boldsymbol{Q1}$ measures $17u$. Since $17 = 3 \times 5 + 2$, then $17\boldsymbol{u} = (3 \times 5)\boldsymbol{u} + 2\boldsymbol{u} = 3\ of(5\boldsymbol{u}) + 2\boldsymbol{u} = 3\ of\ \boldsymbol{Q2} + 2\boldsymbol{u}$. This means that the quantity $\boldsymbol{Q1}$ contains the quantity $\boldsymbol{Q2}$ three times and there is a remainder of 2 units $\boldsymbol{u}$, which is a quantity that is $\frac{2}{5}$ of $\boldsymbol{Q2}$. Thus $\boldsymbol{Q1}$ is said to be $3\frac{2}{5}$ of $\boldsymbol{Q2}$, and $3\frac{2}{5}$ is what we call a *mixed number*. FTs could *learn*, in a connected manner, the notion of mixed number of a quantity in relation to that of a fraction of a quantity. However, as preparation for mixed numbers, before drawing such conceptual connections, the Instructor treated certain pre-requisites in some depth: Factors/Divisors of whole numbers, Prime numbers, Greatest Common Divisor (GCD), and Division with a remainder. The explanations were supposed to be intuitive (i.e., with concrete examples and drawings):

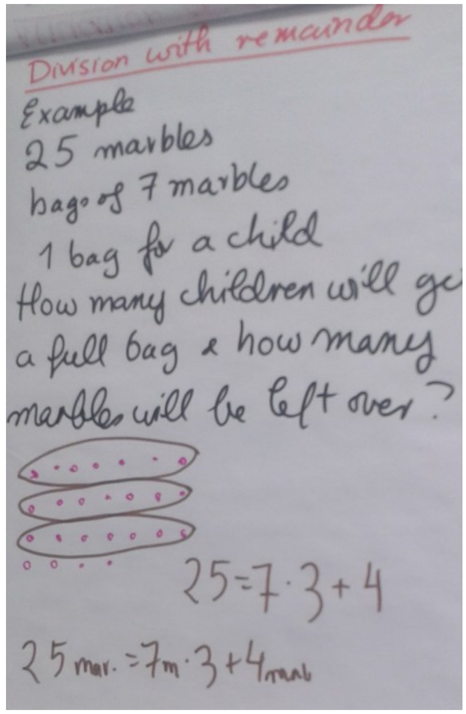


Figure 5.1. Division with remainder - prelude to Mixed numbers

But there was also explicit attention paid to:

- the representation that respects the distinction, practiced in the course, between abstract numbers and quantities: the Instructor writes two separate sentence (the last two lines in Figure 5.1)
- mathematical terminology: the Instructor says that 25 is the dividend, 7 is the divisor (mentions here the ambiguity of the term, hence the choice to use the term “factor” to distinguish), 3 is the quotient, and 4 is the remainder.
- the assumption that the remainder is less than the divisor: the Instructor questions it by proposing the alternate statement: $25 = 7 \cdot 2 + 11$.

The next topic in the sequence, *decimal fractions*, that is fractions with denominators $10, 100, 1000, 10000, etc.$, benefitted from the same detailed account, aimed at bridging gaps in FTs understanding of the more general idea of decimal representation. We had witnessed, in the Pilot-Study, how FTs were quick to use calculators to find decimal representations of fractions, but fell short not only of making sense of the quantitative operations involved, but also in understanding the connection between two ways of representing numbers, in particular with respect to positional or place-value notation.

The Instructor started the sequence on decimal fractions with fractions whose denominators are powers of 10, which can be written “horizontally” in one line, as decimal fractions. Examples with metric units were used to make the point:

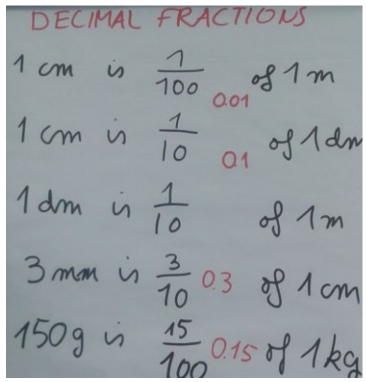


Figure 5.2. Decimal fractions: fractions whose denominator is a power of 10

Next, she continued with fractions that are equivalent to decimal fractions:

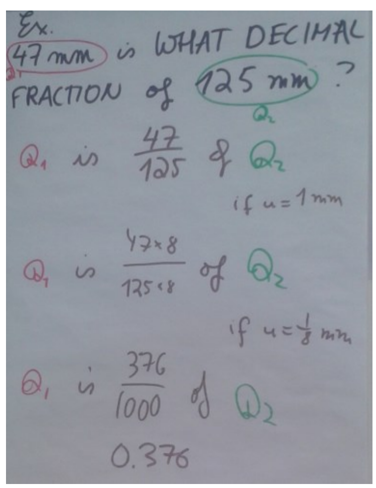


Figure 5.3. Fractions equivalent to decimal fractions: denominator is a factor of a power of 10

Finally, she discussed the case of fractions that cannot be represented as fractions with denominators that are powers of 10:

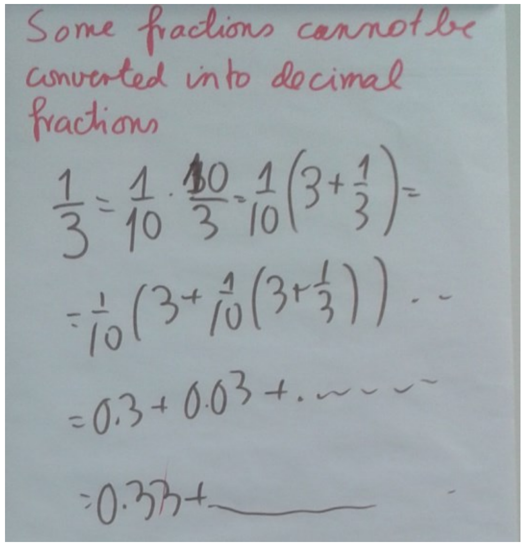


Figure 5.4. Fractions that cannot be represented as decimal fractions

Throughout the three cases, the instructor emphasized the meaning of the decimal expansion – whether finite or infinite – of a given fraction. The following example served exactly this purpose by stressing the difference between 0.6 and $0.666\ldots$ (infinitely many $6's$) in quantitative terms:

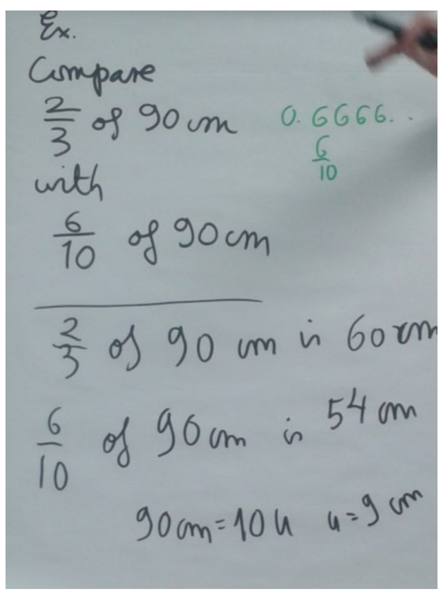


Figure 5.5. Difference between 0.6 and 0.666...: a quantitative explanation

The last topic in the sequence – *Pairs of quantities of the same kind which cannot be related by a fraction* – presents the idea of incommensurability of pairs of quantities, thus hinting at the abstraction “irrational number”, in the way suggested by Lebesgue (1932). In particular, one can only say that the length of the circumference of a circle is *approximately* a certain fraction of the length of its diameter, with a desired degree of precision (e.g., $\frac{31}{10}$, $\frac{314}{100}$, $\frac{3141592}{1000000}$, and so on), but without ever giving a perfectly precise value (i.e., the value of π). It was not done in class at the beginning of the course, but assigned as reading in the textbook and discussed later in the context of ratios, as a generalization of the concept of fraction. This idea of irrational number is within reach in the domain of elementary mathematics precisely through the concept or ratio of quantities (as Lebesgue suggested). A general understanding of number systems (including a concept of real number as a convergent series of rational numbers) – covered in the last part of the Experiment – was considered a necessary part of an elementary teacher’s content knowledge, not only for mere esthetic appreciation, but also for a more profound and connected understanding of number.

5.1.2 Favoring University Student’s actions over Teacher’s actions in tasks

Another measure aimed directly at improving FTs’ mathematical knowledge directly was to consciously change the tasks that FTs had to solve in the course, in particular with regards to the *actions* required to complete them (Sierpinska & Osana, 2012).

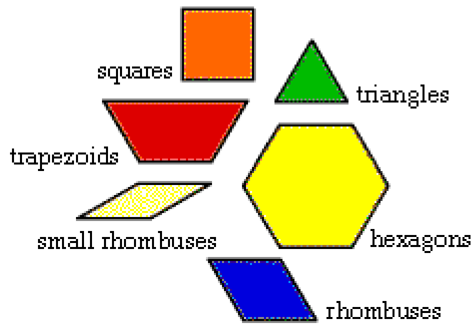
In the Pilot-Study, the homework always included tasks that required *actions* simulating those of a *practicing teacher*, ranging from <u>inventing problems</u> to <u>preparing teaching activities</u> and <u>reflecting on them</u>.

I include below a typical weekly homework assignment from the Pilot-Study: it always contained three parts: “Solve problems”, “Invent problems”, and “Plan an Activity”. The second and the third components clearly simulated Teacher’s actions as he or she is planning a lesson or envisioning the unfolding of one; moreover, they also accounted for more than three quarters of the grade weight (The Plan an Activity task was usually assigned 12 marks, and the Invented Problems – between 1 and 3 marks, out of a total of 16 marks for the homework).

Typical Homework Assignment in the Pilot-Study (Winter 2013)

Solve the following problems

1. Find whole numbers a and b such that $\frac{8}{13} = \frac{a}{26} + \frac{39}{b} + \frac{15}{65}$
2. "Pattern blocks" refers to the flat plastic blocks in the shapes of typical geometric figures:



Source: http://mathforum.org/sum95/suzanne/active.html

Answer the following questions referring to the pattern blocks:

a. How much of the hexagon is the triangle?
b. How much of the triangle is the hexagon?
c. How much of the hexagon is the blue rhombus?
d. How much of the blue rhombus is the trapezoid?
e. Assuming that the full angle $(360°)$ is the unit, what are the measures of the angles in the vertices of
    i. the hexagon
    ii. the trapezoid
    iii. the smaller rhombus
    iv. the larger rhombus?

Invent some problems

3. Consider the following problem: "Five sevenths of a bag of apples is 12 apples. How many apples are in the bag?" Explain why this is not a very good problem. Reformulate the problem to remove the flaw. There are several possibilities; offer two.
4. The problem we discussed in class, "How do $\frac{1}{2}$ of a 10-inch pizza and $\frac{3}{4}$ of an 8-inch pizza compare in terms of their areas?" illustrated the fact that the order relation between fractions as abstract numbers does not always hold in the context of amounts of things. Invent another problem that would illustrate the same point. Use different fractions and do not use amounts of pizza.

Activity

5. Describe a classroom activity organized around one of the problems you have invented above. Your plan should contain at least the following elements:
    - The target age group of students
    - The main mathematical ideas or techniques necessary to solve the problem

- Materials
- Expected solutions and their validity
- Implementation: how the problem would be posed to children
- Questions to ask children in the context of the problem, to extend the problem, test children's understanding, etc.

Beside the weekly homework assignments such as above, by the middle of the course, each FT had to organize a 40 minutes workshop where they would simulate teaching a lesson with their peers in the role of pupils. They could enact one of the activities produced in the weekly homework assignments or invent something new. As a final assignment in the course, FTs had to write a "Problem Book" with 12 problems and activities for elementary school on the topics of fractions, ratio and proportion and geometry. The book was to be addressed to teachers, not directly to children. In the Problem Book, FTs could include revised versions of the problems and activities invented and described in their homework assignments or tried in the workshops, but they could also invent new ones.

In the Experiment, by contrast, most of the tasks required actions that a university student would perform in the course of learning mathematics. According to Sierpinska and Osana's (2012) taxonomy of tasks in pre-service elementary education courses, these actions belong to the Student's, rather than the Teacher's repertoire. Moreover, at a finer level, the Instructor emphasized the Student's *Mathematical* actions (doing mathematics) versus the *Behavior* (mainly course participation) or the *Didactic-Theoretical* actions (actions on theoretical constructs from mathematics education theories).

In the following example of a weekly homework assignment from the Experiment FTs are expected to <u>deduce an answer from the definition</u> and <u>to calculate</u> and <u>compare results</u>. They are required <u>to explain</u> the results of the comparison: this action could also be thought as belonging to the Teacher, but the formulation of the problem – and the overall culture of the course (which I will discuss in more detail in the next section) – encourages, more generally, the kind of explanation that somebody doing mathematics would provide based on the data at hand – in this case information about quantities – and, potentially, some theory.

Typical Homework Assignment in the Experiment (Winter 2014)

1. $15\,mL$ of olive oil is $\frac{3}{7}$ of what volume of oil (in mL)? Deduce the answer from the definition of a fraction of a quantity, in writing. Justify your reasoning.
2. A pie has been baked in a square tray, 5 inches by 5 inches. The pie's weight was half a kilo.
   The pie has been cut into 25 square pieces of equal size.
   Mary ate 4 pieces and Joe ate 6 pieces.
   a. The weight of the amount of pie Mary and Joe ate together is what fraction of the weight of the whole pie?
   b. The top surface area of the amount of pie they ate together is what fraction of the top surface of the whole pie?
   c. Compare the results of your calculations in parts a, and b. What do you notice and how can you explain it?

The two homework assignments reproduced in this section are representative of all the assignments given in the two courses and reflect the significant change in terms of what the Instructor required the FTs to do in the course. In the Winter of 2013 (Pilot Study) the assessment components and their respective weight in the final grade were: Participation and professionalism (10%), Homework Assignments (24%), Workshop animation (15%), Report from Workshop (11%) and Problem Book (40%). In the Experiment, the grading scheme was much more similar to a typical university science course, with the significant addition of two timed tests: Quizzes (20%), Homework Assignments (30%), Mid-term test (20%), and Final (30%). Thus, overall, the grade weight of tasks that required Teachers' actions significantly decreased: from 84% in Pilot-Study (i.e., all the components except Participation and professionalism and 75% of the grade for Homework Assignments – corresponding to the Activity plan part) to only 10% in the Experiment, corresponding to one of the Homework Assignments. This is much less even compared to the two TM courses analyzed in (Sierpinska & Osana, 2012): 48% and 67%, respectively.

In the single homework assignment that required Teacher's actions, FTs had to invent a problem (part 2.) with a structure similar to one that they had previously solved (part 1.):

Problem posing in the Experiment (Winter 2014)

1. Consider the following three problems:

Problem 1. If $7/8$ of a box of paper clips contains 273 paper clips, then how many paper clips are there in $5/13$ of the box?

Problem 2. If $7/8$ of a box of matches contains 272 matches, then how many matches are there in $5/13$ of the box?

Problem 3. $7/8$ of a bag of marbles contains 273 marbles, then how many marbles are there in $7/15$ of the bag?

a. Which of these problems are solvable?
b. Write solutions for those that are solvable.
c. Which of these problems are not solvable, and why?

2.

a. Invent a problem of the type you have seen in PART 1. of the assignment. That is - a problem of the type:

If the given fraction a/b of a certain quantity Q (kind of quantity mentioned but exact measure in units not given) is the given number A of [some objects that cannot be cut into pieces], how many [of these objects] are there in another given fraction c/d of Q?

b. Solve the problem.

3.

a. Ask a partner to solve the problem you invented in PART 2. In return, your partner will ask you to solve the problem he or she invented.
b. Explain to the partner how you solved his or her problem. The partner will explain to you how he or she solved your problem. Keep the partner's solution and take notes of his or her explanations.
c. Together with the partner, find out what must be the relations between the numbers a, b, c, d, and A for the problem of the type described in PART 2 to be solvable.
d. Write, individually, a report of your work in tasks (a)-(c):
   i. How did your partner solve the problem you gave him or her?
   ii. Was this solution as you expected or different? What was different? Was it correct?
   iii. Describe and explain the solution you and your partner have arrived at in task (c).

The first part of the assignment already puts the solver in a meta-position with regards to a mathematical problem. The second calls for a generalization of the observations, scaffolding the solver's work, however, by identifying the relevant variables. By the imperative verb phrase "Invent a problem" in the second part, the task could be assigned to the Teacher's actions category, but it rather brings into focus the mental operations of identifying the structure of the problem and its variables, operations that *a rational individual* would perform in order *to solve* a problem, and not necessarily *to teach* it. The category Teacher's actions is perhaps too coarse to give a precise description of this kind of action, essential in teachers' practices: good teachers of mathematics always solve *and* share their own insight into the process of problem solving.

5.2 ACCULTURATION: TEACHER-STUDENTS

As mentioned in Chapter 2, I analyzed the teacher-students interactions with the lenses I use when doing my work as a teacher of mathematics. I give an illustration below, before undertaking the analysis of a lecture in the Experiment

When I teach, for example, limits at infinity, I have a plan made of episodes I envision in my lecture – each linked to a precise idea about the mathematics at hand that I plan to get students to learn through that particular lesson. The coarse structure may look like this:

- Give an **intuition of limits at infinity as "what happens after a long time"**: a concrete problem (e.g., box-office sales of a movie in time)
- Give **the definition of limit at infinity** and discuss the following ideas using the above example: **"x sufficiently large"**, **"$f(x)$ made as close as we want"**, the difference of meaning in **the equal sign in $f(x) = \cdots$ versus $\lim_{x \to \infty} f(x) = \cdots$**
- Give several examples of functions (constant, polynomial, rational, trigonometric) and have students plug numbers on their calculators so that they know **how to find limits numerically;** demand **answers of the form "Limit is [some number or infinity] because I can make f [as close as I want to it or as large as I want], by taking x large enough"** by asking concrete questions, e.g., "How do I get [given function] within, say, 0.00001 of 3?"
- Give examples where plugging in large numbers allowed by calculator gives wrong answers so that **taking x as large one wants is imagined rather than performed**
- Give graphs and show concretely **how one looks at graphs to find limits** coordinating gesture with words
- Give **techniques for finding limits** organized according to types of functions

Within this broad structure there are the problems or examples that I use: they, too, are categories that I hold critical for students' full grasp of the concept of limit at infinity. They even become relatively stable over time. They are sequenced to give the students as comprehensive a grasp of each of the goals I had set in the larger structure, but each makes a particular point that contributes to one facet of the larger goal: e.g., “rational function with leading term written last, not first, in the numerator”, “exponential function that doesn’t work with algebraic techniques”, “piece-wise graph with a constant y for all x larger than a given value” etc.

I analyze here a typical lecture in the Experiment, which I present, as above, via a succession of *mathematical-didactic episodes*, or chunks of lectures each distinctly aimed at teaching a particular aspect of the mathematics at hand. There are a few reasons why I do this. First, it puts the abstract ideas of the MA against a realistic backdrop: the reader can be brought closer to “seeing” what happened in the classroom. For example, we did make provision, in the MA design, for an explicit distinction between abstract numbers and quantities, but I present here how it was actually done, including relevant details – which turned out to be relevant for FTs’ learning – such as the written representation on the board – using colors, diagrams, etc. – as a didactical aid. Secondly, I share with the Instructor, and with many of my colleagues, this practice of structuring the lecture, in terms of the mathematics at hand and the points we want to get through to students, be it an actual definition, a typology for problems, a common mistake in writing, a way of looking at graphs, an instance where intuition is contradicted, or the opposite, when it is more useful than a popular method, and so on. It is then worth mentioning as a powerful analytical tool of the mathematician who approaches a piece of mathematics for teaching. Finally, on this analytical structure I present the interactions during the lecture between the Instructor and the students, FTs, in full detail, by including relevant quotes from field notes, and analyze them with regards to the areas of interest defined a priori: *reasoning about quantities* and *institutional positions* of participants in the course. Furthermore, I will link the assertions I make based on this analysis to other instances in the data that confirm, refine, or, even refute them, thus allowing a complex characterization of the MA design.

5.2.1 Mathematical-didactic episodes in a typical lesson in the Experiment

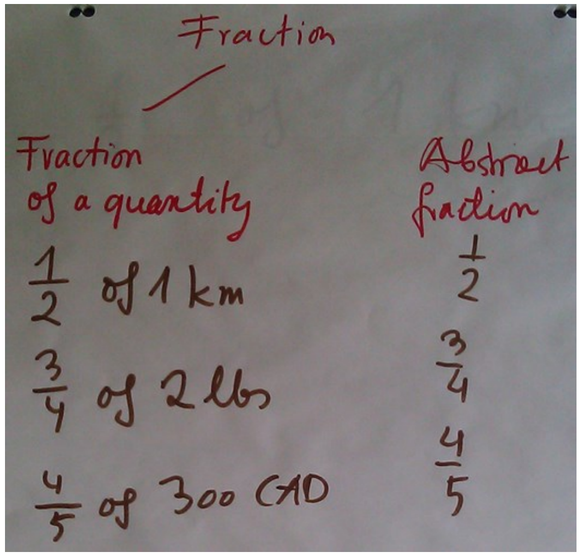
The lesson took place in one of the lectures at the beginning of the course: it contained the introduction to the notion of fraction of quantity. The style of interaction, as I will describe it below, was remarkably consistent in the lectures throughout the course – compared to the first run of the experiment, where there was much higher variability in participants' interactions even in the lectures (e.g., teacher-led whole class interactions, student-led lesson simulations, prolonged activities in small groups, etc.) .The class was rather typical for lecture-style university mathematics classes, as we know them through our practice: the Instructor did most of the talking, she wrote on the board (or easel) extensively while reasoning out loud. She asked questions to the general audience, where typically either one or a few students engaged in the dialogue, or a detectable reaction by the majority was present (e.g., many students answering at the same time, or looking puzzled at a question and nobody answering it). I describe the mathematical-didactical episodes below, each in one sub-section. The format of presentation is as follows: the title, in bold typeface, captures the main idea of the episode, and it reflects my above-mentioned analytical stance as a mathematics college instructor; in the body of the subsection, I include quotes from the field notes and descriptive details, followed by an analysis guided by my conceptual framework, in particular, the focus on working with quantities and institutional positions.

5.2.1.1 Prequel: An explicit distinction between abstract fractions and fractions of quantity, and the appropriateness of the two notions for different levels of school mathematics

The Instructor started the lesson on fraction of quantity by following up on the definition of quantity and the distinction in general between abstract and denominate numbers with a distinction between abstract fractions and fractions of quantity. I include the relevant quote from the field notes:

> [01] Instructor: Just as we distinguish, in general, between denominate numbers and abstract numbers, we distinguish between abstract fractions, which are abstract numbers, and fractions of quantities which are denominate numbers. For example (writing on the easel while speaking): $\frac{1}{2}\ of\ 1\ km$; $\frac{3}{4}\ of\ 2\ lbs$; $\frac{4}{5}\ of\ 300\ CAD$ are fractions of quantities; the expressions refer to denominate numbers, and therefore to quantities; $\frac{1}{2}$, $\frac{3}{4}$, $\frac{4}{5}$ are abstract fractions.

[02]



[03] Instructor: Abstract fractions are elements of a number system, rational numbers, with operations of addition, multiplication, etc. Abstract fractions and fractions of a quantity are defined differently, and operations on them – addition, multiplication, etc. – are defined differently. But definitions of operations on abstract fractions have been inspired by operations on fractions of quantities, which were invented first, in practical situations of measurement. In practice we use mostly fractions of quantities and in elementary school we teach mostly fractions of quantities. Abstract fractions belong more to high school mathematics. I will teach fractions of quantities first; I will talk about abstract fractions later in the course.

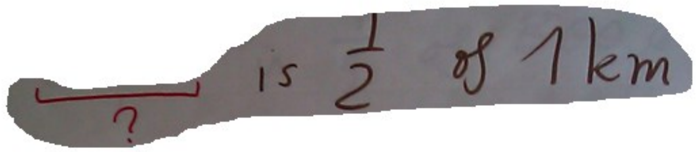
Quote from field notes 1. Fraction of quantity lesson: Prequel

The distinction between fractions of quantities and abstract fractions is given formally and emphasized in writing. The Instructor justifies it, putting FTs in the Teacher position – the position of the reflective practitioner (Dewey, 1933) rather than of the teacher concerned with the immediate task of teaching something to children.

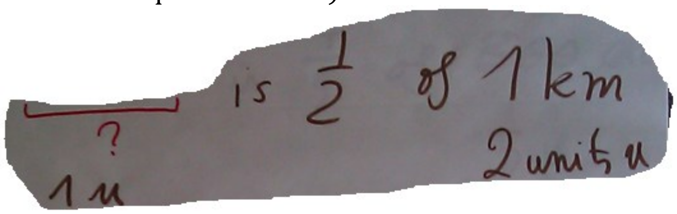
5.2.1.2 Episode 1: Definition of FoQ by generalization from an example of the type "[What quantity] is [given fraction] of [given quantity]?"

The second episode included three worked out examples of the same type that led up to the definition of fraction of quantity:

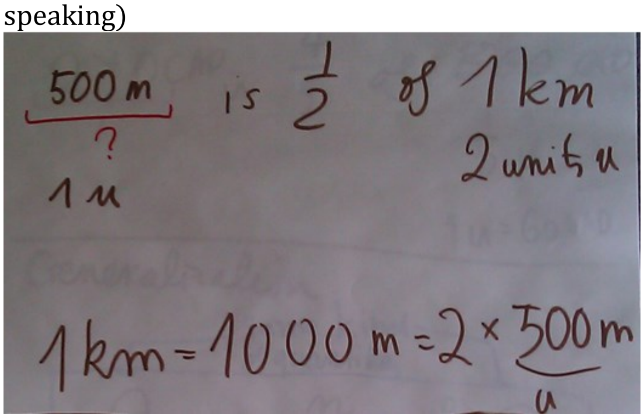
[04] Instructor: Let's go look at our examples. What concrete denominate numbers do these expressions in the example refer to? (pointing to space over the question mark)

[05] 

[06] Instructor: You probably know what is one-half of 1 *kilometer*…
[07] FTs: 500 *meters*
[08] Instructor:… here I'll explain how we get this number (adds 2 *units* u under 1 km and $1u$ under the question mark)

[09] 

[10] Instructor: So what is this unit u? We can find it by converting 1 *kilometer* to *meters*; it is 1000 *meters* and 1000 *meters* is two times 500 *meters*, so the unit u is 500 *meters*. So one-half of 1 *kilometer* is 500 *m*eters (completes the solution on the board while speaking)

[11] 

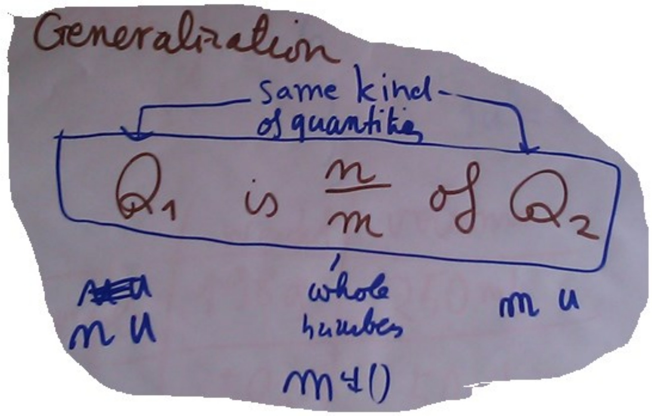
Quote from field notes 2. Fraction of quantity lesson: episode 1

This was followed by two more, identical in structure, examples: *What quantity is* $\frac{3}{4}$ *of* $2\ lb$? *What quantity is* $\frac{4}{5}$ *of* $300\ CAD$*?* The Instructor worked out one of them in the same way as above, and asked the students to solve the last one on their own in their notes, without requesting that somebody solves it at the board (but writing the solution herself with input from some students in less than a minute after asking).

These examples were to lead to the generalization in the form of the definition of fraction of quantity:

[12] Instructor: Now, generalization. The format of the statements we have been working on in the example, was: some quantity is a fraction of another quantity. One quantity, call it

$\boldsymbol{Q1}$, “$\boldsymbol{Q}$” for quantity, is a fraction, call it $n\ over\ m$, of another quantity, call it $\boldsymbol{Q2}$. These quantities must be of the same kind; it doesn’t make sense to say that a distance, for example, is a fraction of a weight. The numbers n and m are whole numbers (says this while writing on the board)



[13]
[14] FT: Whole numbers, you mean integers?
[15] Instructor: Are you asking if n or m can be negative?
[16] Same FT: (surprised; didn’t quite expect this kind of question; not clear what she means by ‘integers’) Mmmm, integers… yes, negative…
[17] Instructor: In abstract fractions, n and m can be any integers, but then we are talking about rational numbers. In the context of fractions of quantities, n and m are positive integers, because they represent measures; n can be zero, but m cannot be zero, because it doesn’t make sense to take a fraction of zero units of a quantity.

Quote from field notes 3. Fraction of quantity lesson: episode 1, continued

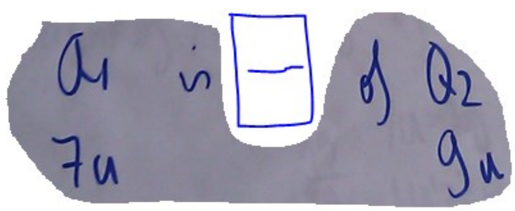
The instructor gives in an initial intuition for the idea of fraction of quantity, connecting it to some common knowledge about reality – “you probably know what is one half of one kilometer” (line 06) – but she writes the problem on the board in a sentence format that sets up the example for generalization to the definition of fraction of quantity (FoQ). This initial intuition is the source of the *definiendum* part of the FoQ, i.e., the term being defined, while the *definiens* is the proposed correspondence in terms of measurements of quantities. The generalization is based on three examples with the same sentence structure, yet the subtle hypothetical character of the definition statement – $\boldsymbol{Q1}$ is the fraction $\frac{a}{b}$ of $\boldsymbol{Q2}$ <u>if</u> [*condition is satisfied*] remains somewhat concealed, once it is uttered and produced as text (lines 12 and 13). We see a few didactical aids at play here: reinforcement through similar structures, schematic representations using color coding, the “worked-out” example. FT are cast in their University Student role, learning new material, which is acknowledged as cognitively demanding.

In an FT's question (lines 14 and 16) we see that some of the assumed knowledge about numbers is not necessarily owned, the student did not know the difference between whole numbers and integers. In the Instructor's answer we see a spontaneous quantitative explanation, to a question that was explicitly related to the FoQ theory: "m cannot be zero, because it doesn't make sense to take a fraction of zero units of a quantity".

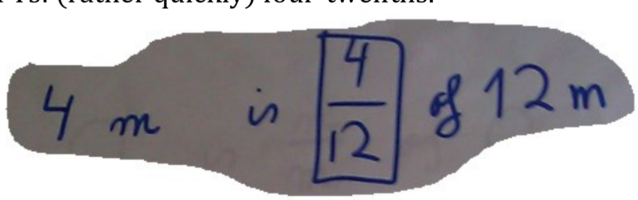
5.2.1.3 Episode 2: Direct application of FoQ definition on examples with a different surface structure: [Given quantity] is [what fraction] of [given quantity]?"

In this part, the Instructor followed up on the definition with exercises that were also interrogative statements, but with the word "what" moved to a different position, thus requesting a different unknown:

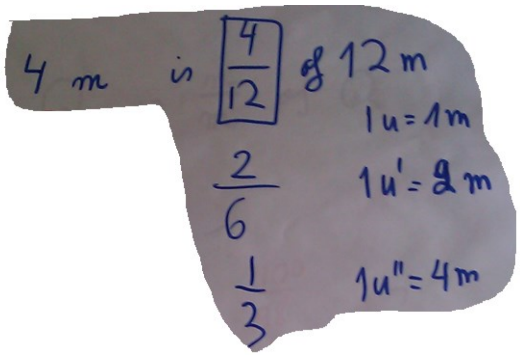
[18] Instructor: Example: Suppose $\boldsymbol{Q1}$ measures 7 units $\boldsymbol{u}$, and $\boldsymbol{Q2}$ measures 9 units $\boldsymbol{u}$. $\boldsymbol{Q1}$ is what fraction of $\boldsymbol{Q2}$?



[19]
[20] FTs: (hesitation, nobody answers)
[21] Instructor: Suppose $\boldsymbol{Q1}$ is 7 meters and $\boldsymbol{Q2}$ is 9 meters.
[22] FTs: seven-ninths.
[23] Instructor: Another example. 4 meters is what fraction of 12 meters?
[24] FTs: (rather quickly) four-twelfths.



[25]
[26] FT: Shouldn't we always reduce the fraction to the simplest form?
[27] Instructor: We can simplify it if we take a different unit. With the $\frac{4}{12}$ the unit we used was 1 meter. But if we measure the quantities with a bigger unit, for example, 2 meters, the fraction will be…?
[28] FTs: $\frac{2}{6}$
[29] Teacher: and if the unit is 4 meters…
[30] FTs: $\frac{1}{3}$



[31]
[32] Instructor: In the context of fractions of quantities, fraction reduction means change of unit to measure the quantities. The bigger the unit, the simpler the fraction.

Quote from field notes 4. Fraction of quantity lesson: episode 2

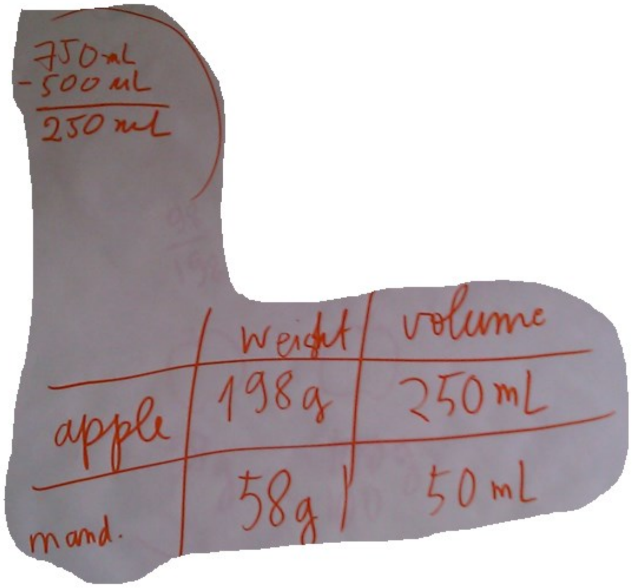
The students have difficulty working with quantities when non-standard units are involved – the Instructor improvises, reverting to standard units, which quickly helps the students. We have the "worked-out" example again, reinforced through two similar exercises of identical sentence structure. This, along with didactical aids such as writing the rectangle in the middle or the writing the corresponding units for each fraction, makes for a highly structured environment for reasoning about quantities. But this is countered by a spontaneous quantitative explanation again, involving conversion of unit, which works to answer FT's question from the Former Pupil position: "Shouldn't we always reduce the fraction to the simplest form?"

5.2.1.4 Episode 3: A fraction is a relationship between quantities of the same kind; it is not an object that is a part of another object. Case 1: Object 1 <u>is not</u> a fraction in the everyday sense of the word of Object 2

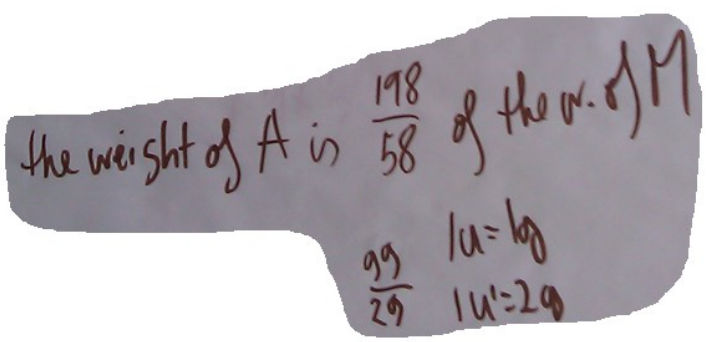
In this part of the lesson the Instructor proposed a measuring activity, followed by writing statements about the fraction relationships between the resulting measurements.

[33] Instructor: Okay, so this expression $\frac{n}{m}$ is a relationship between two quantities of the same kind. Quantities – not objects. Take this apple, for example (takes a big green apple). An apple is a physical object, not a quantity in itself. But we can associate some quantities with it: its weight, its volume, the quantities of sugar, fiber, or calories it contains.
[34] Instructor: (Holds the apple and a tangerine up in the air) Is this mandarin orange a fraction of this apple? In the sense of this definition, not in the everyday sense of the word "fraction".
[35] FTs (collectively): NOOOO!

[36] Instructor: Why? Because it is not a part of it? What if I ask, is the weight of this mandarin a fraction of the weight of this apple?
[37] FTs: (unsure, but some say) Yes
[38] Instructor: Let's weigh them on the kitchen scale. (Produces the scale, which causes general merriment among the students. Gives the apple, the tangerine and the scale to a student. Then the volumes of the two fruits are measured (by another student) using Archimedes' principle and a graded jar filled up to 500 mililiters with water. The results of the measurement are written on the board)



[39]
[40] Instructor: So the weight of the apple is what fraction of the weight of the mandarin?
[41] FTs: $\frac{198}{58}$.
[42] FTs (a couple of students): Can we simplify?
[43] Instructor: Okay. This fraction $\frac{198}{58}$ was obtained by measuring the two weights with 1 gram. What bigger unit can we use?
[44] FT: 2 grams.
The Instructor then writes on the board:



[45]

Quote from field notes 5. Fraction of quantity lesson: episode 3

The Instructor directly addresses the focus on quantities in mathematics with an activity aimed at distinguishing between the meaning of the word in everyday language and its technical meaning in mathematics. In this case the first object is not a fraction of the second object in the everyday sense of the word, but if one considers quantities associated with the two objects then one quantity regarding the first object is a fraction of one quantity (of the same type) regarding the second object. Also, in the process she addresses the exclusive part-whole conception of fractions by rerouting the reasoning to the FoQ definition – which allows for fractions relationships other than part-whole. The students enjoy the active part of the lecture; two students applauded this aspect of the lecture in an informal conversation after the third lecture (where no such activity had been programmed) saying that "we should have more hands-on activities like when we used the scale, that's how children learn." Such instances, of reference to some vague ideas about good teaching, which we categorized as students' positioning in the Popular Educator role, were, however, much sparser in the Experiment, compared to the Pilot-Study.

Again, one student asks if she can simplify, and the Instructor responds with the same quantitative justification, and consistent written representation.

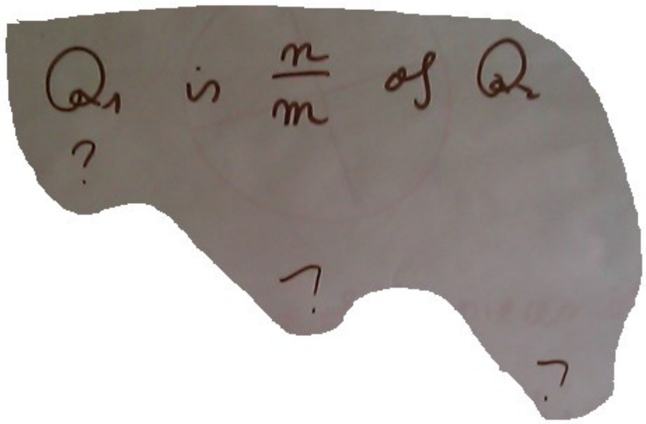
5.2.1.5 Episode 4: Types of problems about fractions of quantities with the same structure as the definition

The Instructor next generalized the solved exercises as types of problems that could be created by letting one of the variables in the definition be unknown, while the other two are given:

[46] Instructor: This definition of fraction of a quantity already suggests three broad types of problems about fractions of quantities, depending on which element of the proposition:

[47] $Q1$ is $\boldsymbol{\frac{n}{m}}$ of $Q2$

[48] is unknown: Type I: $Q1$ is unknown; Type II: $Q2$ is unknown; Type III: $\frac{n}{m}$ is unknown (writing on the easel)



[49]
[50] Instructor: You can also be asked, given $Q1$, n/m and $Q2$, with what unit you have measured the two quantities?
[51] FT: So in fact you have 4 types of problems.
[52] Instructor: Yes, if I write $Q1$ is $\frac{n}{m}$ of $Q2$ measured with unit u. Then u can also be an unknown. So these are broad types of problems, and you will have all these types in your homework.

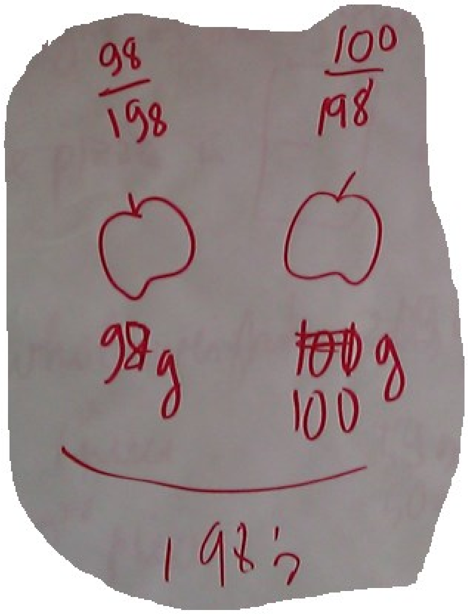
Quote from field notes 6. Fraction of quantity lesson: episode 4

Building on the two previously presented categories of exercises, with the sentence structure similar to the definition, the Instructor generalizes to types of problems about fractions of quantities. To a student's remark – a sensible linguistic observation – the Instructor responds with adding another clause "when…" that allows the u to be unknown, as well, thus making for 4 types of problems associated to the definition of fraction of quantity.

5.2.1.6 Episode 5: A fraction is a relationship between quantities of the same kind; it is not an object that is a part of another object. Case 2: Object 1 <u>is</u> a fraction in the everyday sense of the word of Object 2

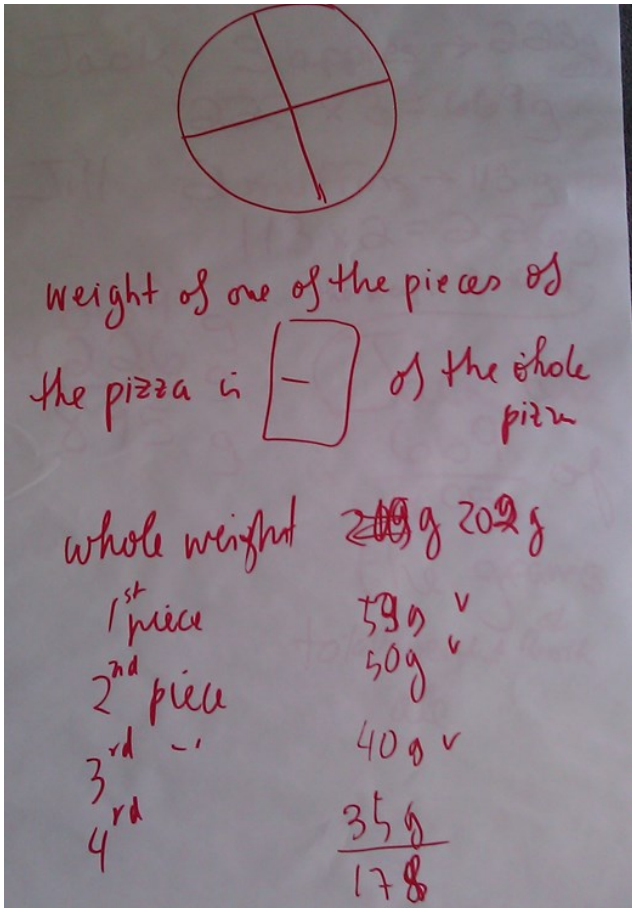
In this episode the teacher revisits Episode 3, reinforcing the meaning of the word fraction in mathematics, versus its meaning in everyday life. This time an object (a piece of apple) is indeed a fraction – in the everyday meaning of the word – of another object (an apple).

[53] Instructor: Cuts the apple in two pieces, one slightly bigger than the other. Is this piece of the apple a fraction of the whole apple?
[54] FTs: (many say YES; a few, NO)
[55] Instructor: An object is not a fraction of another object. But if I ask, is the weight of this piece of apple a fraction of the weight of the whole apple then the answer is Yes. (Pieces of apple are weighed, fractions are written)



[56]

[57] Instructor: (produces a pizza, cut into 4 very unequal slices.) In the traditional teaching of fractions, a slice of a pizza cut into 4 pieces would be said to be one-quarter of the whole pizza. But if we take the quantities seriously, then different results may obtain. (The whole pizza and the pieces are weighed by two FTs, but there is some problem because the weight of the pieces does not add up to the weight of the whole pizza. Specification of the fractions is abandoned) All right. Reality is messy.



Quote from field notes 7. Fraction of quantity lesson: episode 5

The Instructor exemplifies again the way one must reason about two quantities if the word fraction is to be understood in its technical, mathematical sense. She emphasized working with quantities by resorting to the definition, and is consistent in using the rectangle in the middle as a cue to use it. The Instructor puts the students in the Teacher's position, again cultivating their reflective attitude by comparing approaches to thinking about objects when teaching in elementary school (DA vs MA). This allocation to the Teacher's position is extended into the Instructor's remark that "Reality is messy."

5.2.1.7 Episode 6: Writing statements about fractions of quantities

In the episode the Instructor gave a problem for the whole class: it required writing statements about quantities based on some given data. Two students were prompted to come to the board to solve it.

[58] The Instructor displays the problem on a slide and reads the first question:

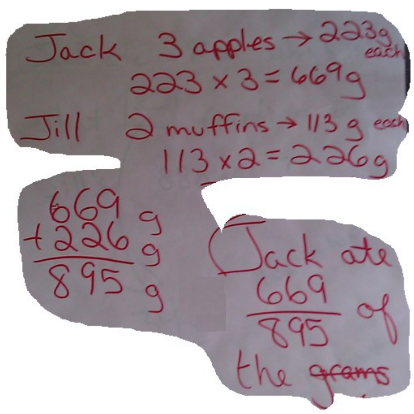
Practice exercise 1. For lunch at school, Jack ate 3 apples and Jill ate 2 muffins.

a. Can we say that the quantity that Jack ate is $\frac{3}{2}$ of the quantity that Jill ate? Why yes or why not?

b. Suppose you know that Jack ate 3 large apples weighing about 223 grams each and that there are about 116 calories in each such apple. Suppose you know that the 2 muffins Jill ate were both medium blueberry muffins weighing about 113 grams each and that each such muffin carries about 444 calories. Make four sensible fraction statements about the quantities that Jack and Jill ate for lunch.

[59] FTs (answering to the first question, immediately): NO.

[60] Instructor lets students work on Question b. She then asks a student to write her solution on the board (see below, the word grams was not crossed out in the student's solution).

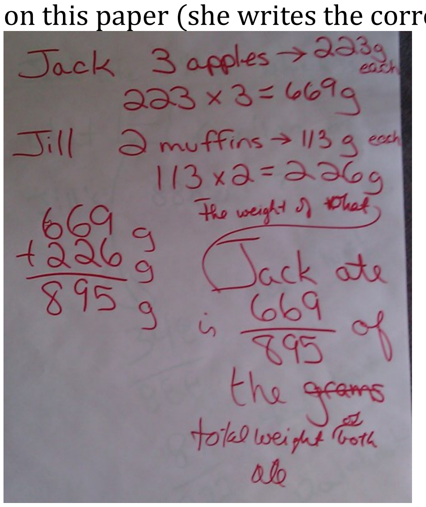


[61]

[62] Instructor: That's shorthand, right? This sounds a bit awkward.

[63] FT1 (who produced the solution): The weight of what Jack ate is $\frac{669}{895}$ of the total weight they both ate.

[64] Instructor: Write the whole sentence in your solutions. Here, there is not enough space on this paper (she writes the correction herself).



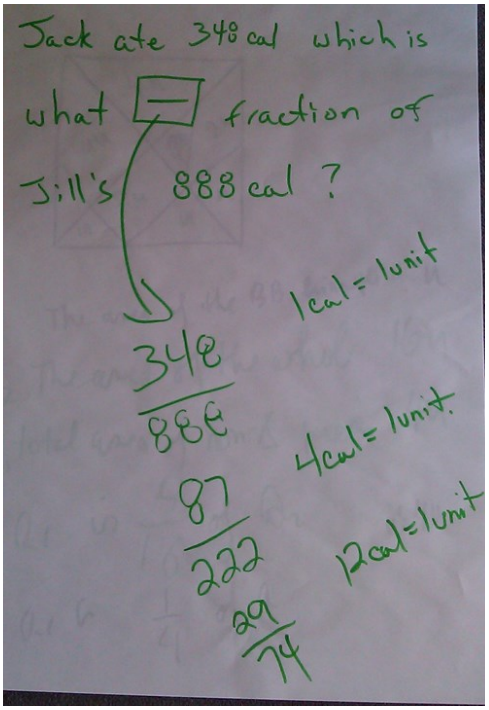
[65]

[66] Instructor (to another student): Do you want to present your statement?

[67] Teaching Assistant (sitting near that student): They wrote problems.

[68] Instructor: Ok, write your problem.

[69] FT2 writes:



[70]

[71] FT3: Shouldn't he be writing u, u prime and u double prime, because these are not the same units?

[72] Instructor: Yes, if he used letter symbols, but he just wrote words; he could have written "another unit" in words, perhaps.

Quote from field notes 8. Fraction of quantity lesson: episode 6

The problem now contains more context, thus giving the impression of a realistic problem – one that somebody would encounter in schools, as a teacher. But the Instructor puts the FTs in the University Student role by requiring them use the freshly developed "school knowledge" about quantities: first to answer a question related to two previous episodes (3 and 5) which focuses attention on the technical meaning of fractions, and then to produce "fraction statements", again based on the practiced sentence structure. She doesn't seek the explanation for the answer to the first, i.e., to the "Why yes or why not?", appearing satisfied with the collective correct answer – although this would have been an interesting pursuit for gauging students' resort to the FoQ theory for justification.

The two students' solutions to the second question give us a glimpse into how the same teaching – even highly structured – produces diverse outcomes in terms of students' learning. The two students were exposed to both formal knowledge about quantities and spontaneous explanations. One of them understood "fraction statements" in a literal way, producing a sentence with a fraction in it, complete with the word "grams", thus following the instruction of "representing quantities with unit" (although she doesn't do it when performing calculations before producing the sentence). This "imitation game" seems to sacrifice common sense about grammatical English sentences – this is what the Instructor appeals to in order to help her correct the sentence: "This sounds a bit awkward". The other student internalized the Instructor's spontaneous explanation about unit conversion, picked up the "trick" of the rectangle in the middle as a placeholder and the corresponding units for each fraction, and even went beyond that and took the Teacher's position, by writing interrogative sentences about fractions, closely following the syntactic structure of the demonstrated examples. These sentences are somewhat "stiff" from a natural language point of view, but they are nonetheless clear.

Also interesting is the instance of linguistic sensitivity – a student asks about the appropriate notation for units when her peer doesn't reflect the distinct units in writing; the Instructor, in turn, answered with a remark about algebraic notation versus natural English language.

5.2.1.8 Episode 7: Use of the fraction word in everyday life (approximations)

In this episode the Instructor talks about the approximations made in fraction statements, when the word fraction is used in its everyday meaning. Again she gives the problem to the whole class, time to solve it, then talks to a student who offered his solution:

[73] The Instructor displays the problem on a slide and reads it:

> Practice exercise 2. The driving distance from Montreal to Toronto is 544 km. On your way, you pass by Kingston. From Montreal to Kingston, the distance is 293 km. As you pass by Kingston when traveling to Toronto, what fraction of your journey will you have completed, approximately? [a map of the road is displayed]

[74] FT: I'd say that it's approximately one-half.
[75] Instructor: If it were one-half, then 293 km should fit twice into 544 km. Does it? How much is two times 293 km?
[76] FTs: 586 km.
[77] Instructor: So that would be a rough approximation. In everyday usage, we make such rough approximations. We say that one piece of the apple is one-half of the whole apple, and we don't mention whether we mean weight or volume, or… we just say, vaguely, an "apple."

Quote from field notes 9. Fraction of quantity lesson: episode 7

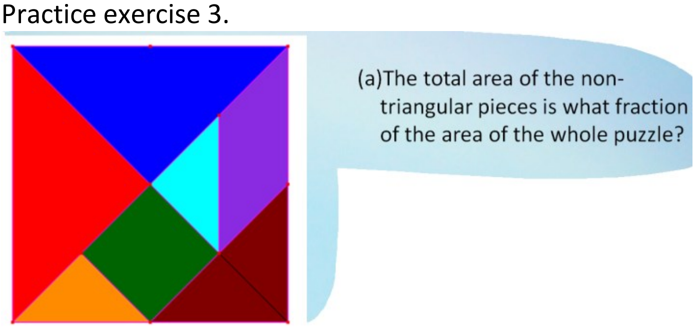
The problem has the appearance of a realistic problem, but the question, and the interaction that takes place about it, emphasizes the difference between reasoning about quantities via a precise definition (within a theory), and reasoning about quantities in everyday usage. This calls, indirectly, upon the episodes 3 and 5.

5.2.1.9 *Episode 8: Use of the fraction word in mathematics (precise specification)*

In the third practice exercise the Instructor formulated the question to hint at the technical meaning of the word fraction in mathematics.

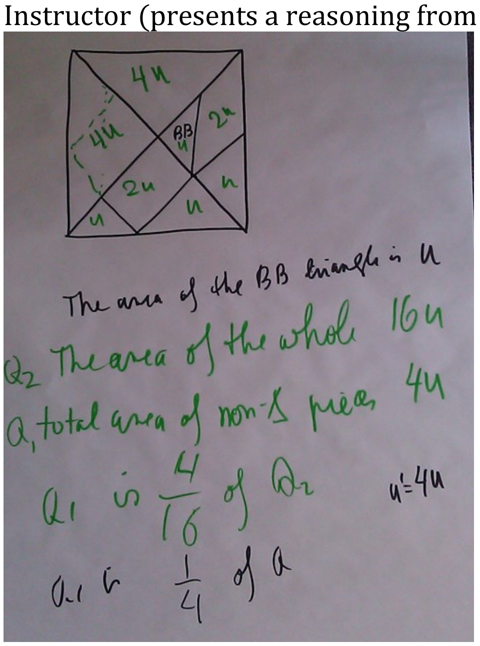
[78] The Instructor displays the problem on a slide and reads it:

Practice exercise 3.



*

[79] FT1: We could measure the area with the yellow triangle.
[80] FT2: Can I measure the area with the baby blue triangle? Then the whole area is 16 and the triangle fits twice into the square and twice into the parallelogram, so it's one-fourth.
[81] FT3: This way there is no math in it.
[82] Instructor (presents a reasoning from the definition on the board):



[83]

Quote from field notes 10. Fraction of quantity lesson: episode 8

This problem, too, reads like a realistic problem, but subtle linguistic cues suggest solving it using FoQ theory: the quantity – area – is specified instead of asking about an object as a fraction of another object. The question more typical for the division approach – "The non-triangular pieces are what fraction of the whole puzzle?" – could be ambiguous otherwise (e.g., one could talk about number of pieces and compare cardinality of two sets) but likely not for the students in the audience – the possible confusion would have been interesting to make explicit to stimulate their reflective attitude from the Teacher position.

One of the students proposes a sensible solution involving reasoning about quantities in which she measures one of the one quantities using another. With respect to the FoQ theory, the only visible error is that the student says "the area is 16" without mentioning the unit. It is telling to see that another student dismisses the reasoning as "not mathematical." The Instructor may appear to the students as writing a different solution, when, in fact, she is writing the suggested solution using the elements of the theory – for her, it's a way of representing the justification with the representational tools offered by a given theory; for the students, the algebra signals the legitimacy of a solution different from their own. This is a difficult problem in mathematics education at tertiary level in particular: sometimes good reasoning is sacrificed by the students themselves only because it does not appear to satisfy certain perceived criteria about how mathematics should *look*.

5.2.2 Discussion: quantitative reasoning in classroom interactions

I could identify two modes of reasoning about numbers, in particular fractions, quantitatively, initiated by the Instructor in class interactions – I present them by reference to the mathematical-didactical episodes recounted above, and link them to other key events throughout the Experiment.

One of the two modes was **planned quantitative reasoning,** that is, reasoning about quantities as a way of constructing the FoQ theory, as planned before the Experiment. The second one was **spontaneous quantitative reasoning,** where FoQ theory is applied to questions or problems that are realistic in the sense that they resemble problems encountered in elementary school mathematics. Each of the two modes reveals different aspects about the interaction between teacher and students, or about the process of acculturation. Observing the first – where problem solution scenarios are scripted beforehand – brings into focus the perennial dilemma of the educator: would it not create imitating behaviors without the desired knowledge? Would FTs *really* be reasoning about quantities or just copying the Instructor's steps? Looking at the second – when the teacher poses realistic problems or questions without requiring a precise method or tool for solving – may provide answers to whether it is at all possible for students to reason in terms of quantities spontaneously and what may be the obstacles for engaging them in this kind of reasoning. But there is another, very interesting aspect of both the Instructor and the FTs' reasoning about quantities in problems or questions that don't require explicitly the use of FoQ theory: the realization of such reasoning provides validation of FoQ theory as actually practical for teaching fractions in school. If one is able to solve problems that do not resemble "school material" – in this case learnt in a university course – by producing sensible quantitative explanations rather than empty algorithms, then the MA has indeed the potential to serve as a flexible knowledge base for teachers' practices.

5.2.2.1 Planned quantitative reasoning: theory building

I will first talk about planned quantitative reasoning in classroom interactions. I use this term to encompass all reasoning about quantities that leads to building the FoQ theory. Entering the course, the Instructor had most of the theoretical "infrastructure" in place for teaching fractions with a focus on measurable quantities. In the delivery of the course, the organization of the material where all the operations and relations on fractions of quantities were to be derived from the central concept of fraction of quantity was indeed very visible – thus making for a rather faithful following of the a-priori design. Said operations on fractions of quantities were stated (and written on the board) as titles or subtitles in the lectures (or labs) and they also largely corresponded to the contents of the textbook (Sierpinska, 2013). Thus reasoning about quantities was not to be abstracted by students from problem situations a-didactically: the Instructor taught such reasoning formally by initiating the students in theory building.

In the above lesson the distinction between abstract fractions and fractions of quantity was taught through a dedicated episode – the Prequel – and so was the definition of fraction of quantity in Episode 1. We see that an abstract concept, in this case *fraction of quantity*, is linked to a physical reality: the symbol $\frac{1}{2}$, i.e., two whole numbers placed on top of each other and separated by a line, is supposed to capture something about *what exists in the real world*, which is suggested by the symbols $500\ m$ and $1\ km$, i.e., whole numbers attached to units of measurement. Observation of real world starts with focusing on quantities, as measurable aspects of physical objects. The presence of physical reality in problems is a key condition for teaching mathematical content at elementary level, in particular fractions: as we have shown, the theory of abstract fractions developed just recently in mathematics is not cognitively appropriate for children so young. But a connection with reality already exists in the Division Approach and its typical didactical "aids" (models of sliced pizzas, pictures, diagrams, apps, etc.). In the Measurement Approach, however, the Instructor makes the connection of the physical reality with an abstract concept the very *object of theorizing*, rather than a device for alleviating difficulties for an existing abstract concept.

In Episode 1 the theorizing to produce the definition of fraction of quantity takes place in three steps

- an example from physical reality formatted to fit the structure of the definition, that the students would intuitively know: it is assumed common knowledge that the symbol $\frac{1}{2}$ is read "a half" and a half of one kilometer is 500 meters (line [06] in the Quote from field notes 2).
- a precise meaning for the fraction symbol proposed by associating the top number with a multiple of a unit and the bottom number with another multiple of the same unit using a schematic representation: 1 unit under the first quantity corresponding to the top number of the fraction and 2 units under the second quantity corresponding to the second quantity (line [09] in the Quote from field notes 2)
- the formulation of the definition: the *definiendum*, based on the sentence structure of the given example, the *definiens* based on the proposed correspondence between the fraction symbol and the two quantities.

The concept of fraction of quantity thus captures what exists in reality but in a precise, stable, way, which appeals heavily to FTs' theoretical thinking. The "fraction" word acquires a technical meaning after being used more loosely only in the initial example (systemic-definitional thinking). Attention is paid to the structure of the statements to produce generalization (analytical-linguistic thinking). The definition has an "if – then" structure that highlights the hypothetical character of mathematical statements (systemic-hypothetical thinking). Furthermore, one is called to reflect on the need to develop a theory of fractions of quantities for the elementary level of school mathematics, on the difference between the technical and the everyday meaning of the word fraction, or on the possibility of creating different types of problems based on the missing variable (reflective thinking). The cognitive load remains – it is the tradeoff for the efficiency of the developed theory in solving and posing problems. The Instructor uses worked out examples, linguistic patterns, color coding, and schematic representation not only to alleviate the difficulties, but also to be consistent within a system of notation.

Thus the Instructor created a culture where the University Student position is intended as that of a scientist in the making who builds models from which consequences that agree with observations can be derived. Vertical discourse, a form of "coherent, explicit, and systematically principled structure, hierarchically organized" (Bernstein, 1999, p. 159) – reflected also in the written representation – is practiced by the Instructor and encouraged, even demanded, via institutional tools – such as in-class or homework exercises from the students. This way, the Instructor did not act as the sole architect of an elegant construction, who relegates to the student the mastering of procedural matter, as Chevallard (1985b) views the teacher-student interaction in conventional teaching of the abstract notion of rational number.

Students participate by working on theoretical exercises usually by writing, analyzing or justifying statements similar to the ones demonstrated by the teacher as part of building the theory. This was the case for the examples in Episode 1 and 2 – in each case 2 examples were demonstrated by the Instructor, the third was left to the students, and also for the exercise of writing "fraction statements" in Episode 6. One of the responses to the latter, by an FT who created interrogative sentences that matched the structure of the *definiendum*, along with the intervention of another FT who was interested in details about notation used (lines 66-72), provide examples of students' engaging with the analytical thinking envisaged by the Instructor. So does another FT's remark about the possibility of having 4, instead of 3 types of problems – she had noticed that there are 4 variables in the sentence.

This type of actions, from the students, where they successfully demonstrated solutions that closely followed the Instructor's approach were common throughout the course, and generally resulted in pointed discussions in the classroom, always related to the content at hand. Compared to the Pilot-Study, the Instructor reported having regained a sense of agency from being able to carry discussions in problem solving by reasoning about quantities within a theory, and by focusing on the correctness and consistency of the solution at hand, rather than, for example, on the simplicity of the method used according to personal experience, or on its imagined reception by children according to presumed educational principles such as, "it must be visual for children to understand it." Such remarks were still present, although much less frequently, but the risk of confusion between social and mathematical agency – a significant problem in mathematics education (Balacheff, 1991) – appeared highly diminished compared to the Pilot-Study.

The lecture also provides evidence – which we could link to similar incidents in the Experiment – of the issues associated with students' reasoning about quantities in such highly scripted exercise-solution scenarios.

One of them is the risk of imitation, by FTs, without adequate understanding. It was present, in the lecture, in the first FT's response in Episode 6 to the task of "writing fraction statements" – where the student only copied some elements of Instructor's discourse (e.g., writing the units, grams, in the statement she produced), but failed to see the underlying structure of the statement to be produced. The Instructor's reaction is a didactic tool whose power, perhaps, we tend to overlook: the prompt to create sentences that "sound good in English" – i.e., using one's internal grammar to produce correct mathematics. Another type of imitation I observed, as the course advanced, – pertaining more to mathematical grammar – was the assimilation, by FTs, of words from the Measurement Approach "vocabulary" with incorrect usage:

“common measurement” instead of common denominator, “quantity” to refer to fraction and the other way around, or “unit” to refer to factor, are just a few such examples. These cases, when contrasted with the correct reproduction, by some FTs, of Instructor’s techniques and vocabulary point to the perennial paradox in teaching captured by Brousseau’s concept of “Topaze effect” (1997, p. 25) : the teacher must alleviate the cognitive load of a tasks through didactic coding and hints but at the same time she must devolve to the students the responsibility so solve a novel problem and learn in the process. The first student writing fraction statements in Episode 6 seemed to have rather imitated the given format, without identifying the critical variables correctly – in particular the quantities and their measures; the second student was able to use the knowledge much more flexibly.

Another issue was FTs’ difficulties with algebra in the context of theory building. I saw it first in the general confusion when the Instructor uses, in an example in Episode 2, an arbitrary unit, rather than some standard unit, such as grams or meters, and names it “$\boldsymbol{u}$.” It was rather surprising how, just changing from an arbitrary $\boldsymbol{u}$ to meters – a mere letter in the written representation – quickly brought students to produce the desired result. Also, in Episode 8, one FT confesses not knowing “how to write [her solution] mathematically.” This comes as no surprise, if we read research by Thompson demonstrating that quantitative and algebraic reasoning are closely linked (Thompson, 1989): his idea is that doing algebra, especially in problem solving, involves reasoning about quantities and relationships between them without knowing their values (understood as the numerical results of measuring them). As the theorizing became more complex, students’ difficulties with handling “unknown” units, in particular performing operations on them – which effectively amounted to doing algebra – became increasingly difficult for the students. This was the case in the lesson where fraction multiplication was modeled as the quantitative operation of taking a fraction of a fraction of a quantity. I explain below.

The Instructor performed, in class, successive generalizations to find the solution to the problem of fraction of fraction of quantity (as described in Section 4.2.1.4), thus practicing vertical discourse in class. The example that led to the generalization $\boldsymbol{\frac{1}{b}}$ **of** $\boldsymbol{\frac{1}{d}}$ **of** $\boldsymbol{Q}$ **is** $\boldsymbol{\frac{1}{bd}}$ **of** $\boldsymbol{Q}$, was schematized as follows:

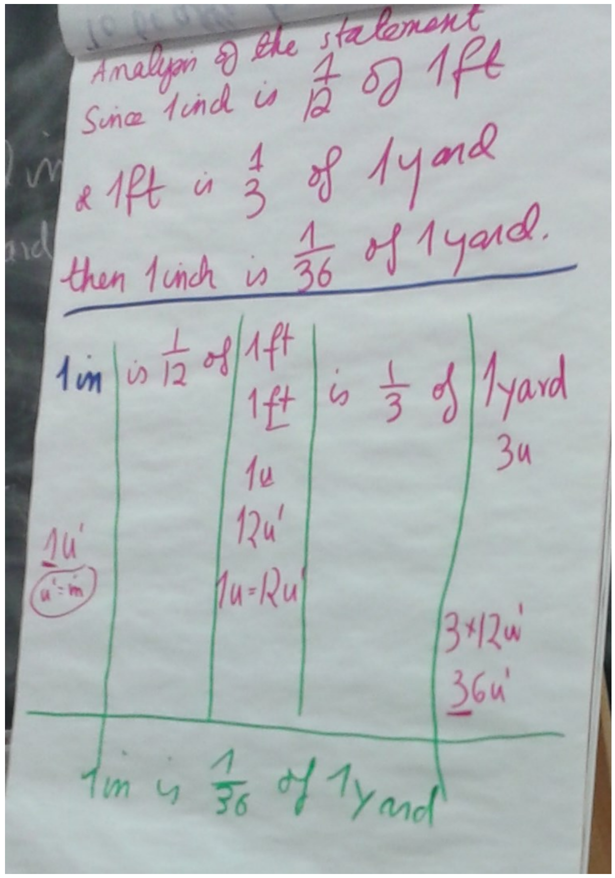


Figure 5.6. Picture of Instructor's writing on the easel: finding a fraction of a fraction of a quantity using the FoQ definition, step 1

There are two fraction statements, involving three quantities: $1\ in$, $1ft$, and $1\ yard$. The relation between the first and the second, and the second and the third, are known, the fractions: $\frac{1}{12}$ and $\frac{1}{36}$, respectively. The second statement is written under the first, such that the second quantity, which appears in both is in the same column. This way of writing was supposed to organize the conversion of units for the purpose of relating the first quantity ($1\ in$) with the third ($1\ yard$): the second quantity, after the definition is applied to both statements, is both $1\boldsymbol{u}$ and $12\ \boldsymbol{u}$**:** this gives the conversion equation $1\boldsymbol{u} = 12\boldsymbol{u}'$, which serves to express the third quantity, initially $3\boldsymbol{u}$, as $3 \times 12\boldsymbol{u}'$, which gives $36\ \boldsymbol{u}'$**.** Finally, that allows to relate the first quantity, which measures $1\boldsymbol{u}'$ with the third, which measures now $36\ \boldsymbol{u}'$, to conclude, based on the definition of fraction of quantity that $1\ in$ is $\frac{1}{36}$ of $1\ yard$.

The conversion is not straightforward when the fractions are not unitary: the conversion equation between $\boldsymbol{u}$ and $\boldsymbol{u}'$ that does not allow for expressing the first and the third quantity directly with the same unit in order to relate them through a fraction; in this case it is necessary to find a third unit to measure the second quantity, such that it fits a whole number of times into $\boldsymbol{u}$ and $\boldsymbol{u}'$. The Instructor explained it in class using the same schematic representation:

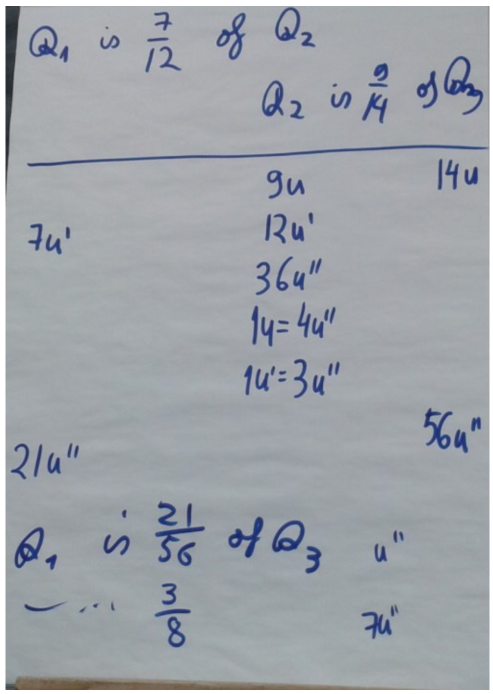


Figure 5.7. Picture of Instructor's writing on the easel: finding a fraction of a fraction of a quantity using the FoQ definition, step 2

The unit $\boldsymbol{u}'$ to measure $\boldsymbol{Q2}$ is found using the least common multiple of 9 and 12. Finally, this allows expressing both $\boldsymbol{Q1}$ and $\boldsymbol{Q3}$ with the same unit $\boldsymbol{u}''$**,** which means that $\boldsymbol{Q1}$ is $\frac{21}{56}$ of $\boldsymbol{Q3}$ or, if a unit 7 times as large as $\boldsymbol{u}''$ is used, $\boldsymbol{Q1}$ is $\frac{3}{8}$ of $\boldsymbol{Q3}$.

Most FTs appeared to have something to say about this development of the FoFoQ problem: it was clearly something they perceived as one of the most difficult topics in the course. The main obstacle – having to do with reasoning about quantities with algebra – could be identified in the prequel to this lecture, where the step of unit conversion was isolated and introduced separately:

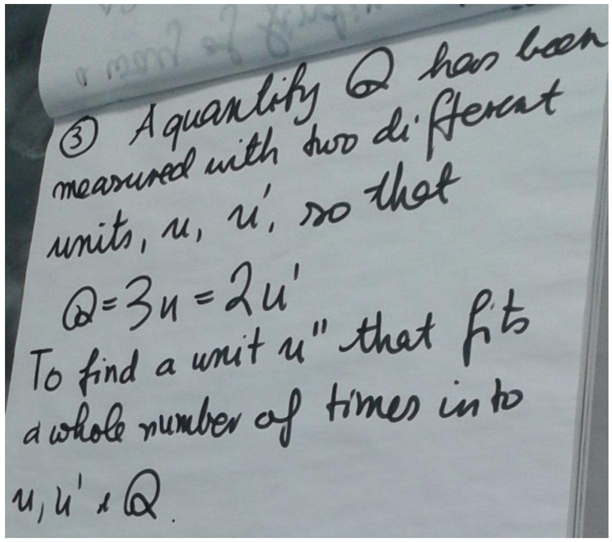


Figure 5.8. Picture of Instructor's writing on the easel: a problem involving conversion of units in preparation for the finding of a fraction of a fraction of quantity

The students, having been acquainted with the technique of finding the least common multiple answered 6, but when asked why, could not produce a quantitative explanation, and declared being confused by the "$\boldsymbol{u}$ thing". The Instructor repeated the explanation on a line segment diagram and wrote the solution on the easel.

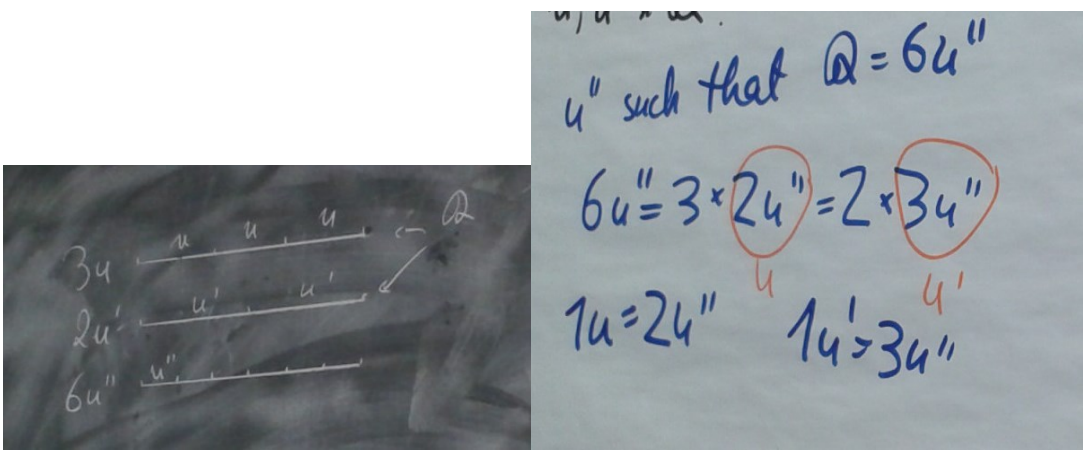


Figure 5.9. Picture of Instructor's writing on the board and on the easel: solution to a problem involving conversion of units

But in the following, almost identical example – finding $\boldsymbol{u}''$, given that $\boldsymbol{Q} = 9\boldsymbol{u} = 12\boldsymbol{u}'$ – problem persisted to some extent: some students said 36, but, when asked, didn't know why. They had difficulties *reading* the expression of the quantitative idea of unit conversion in algebraic terms: "I solved it using bar diagrams, but couldn't explain it mathematically", said one FT. When the Instructor wrote the following equation (first and second line in Figure 5.10) a student asked "Why isn't there a $\boldsymbol{u}$ near the 9?",

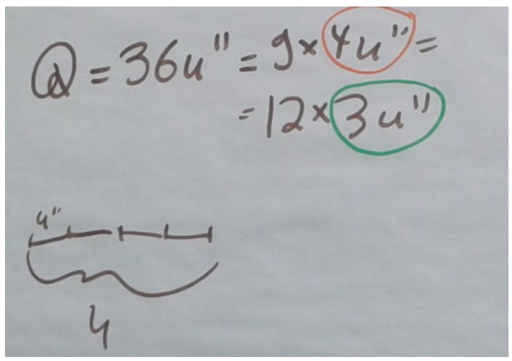


Figure 5.10. Picture of Instructor's writing on the easel: relations between units expressed algebraically

The Instructor replied "It's replaced by $4\boldsymbol{u}''$. I discovered a conversion.", and added a segment diagram (third line in Figure 5.10 above) to express the conversion equation for $\boldsymbol{u}$ and $\boldsymbol{u}''$, and another segment diagram on the board for the equality expressed in the equation .

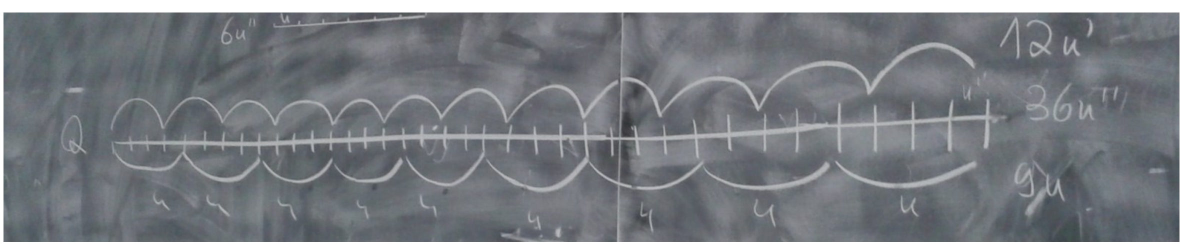


Figure 5.11. Picture of the board: relations between units expressed in a segment diagram

These aids – color coding, line segments, repeated examples - seemed to work: although struggling with the FoFoQ problem, some students appeared to succeed after intensive practice on similar exercises in class and in the lab; they declared that the line segments diagrams, in particular, help a lot in grasping the quantitative relations (i.e., conversions of units) and expressing them algebraically. Others, when met with this genuine obstacle, disputed the very relevance of learning this in a methods class. One student, in particular, declared, during a lab problem-solving session that "using the theory to justify is just a mechanical exercise in doing algebra" and that "you can't do this in school." Another student questioned the approach by saying that "I don't get why we have to go through the 'unit thing' when I can solve this in 2 seconds in my head." These concerns raise a deeper problem rooted in FTs' perception of the learned material from a meta perspective. These students did not see themselves, as University Students, in the position of experts-to-be, participating in the process of building the theory, which explains why things work the way they do. In particular, as future teachers, this means knowing why the problem of taking a fraction of a fraction of quantity is solved by multiplying the numerators, and the denominators, respectively, of the two fractions. This perception is a great obstacle in the way of the professionalization of elementary mathematics teaching, and should be perhaps addressed explicitly. One way was, in the course, as trivial as simply reinforcing the student's responsibility for this kind of knowledge through assessment. In almost every lecture FTs had questions related to what is required of them and considered acceptable on a test: "Is a picture ok?", "should I write the whole $\boldsymbol{u}$ process?", or "do I have to do the *since…then…* thing?" These were legitimate questions, and appeared to be mostly tools for students to learn the accepted practices, rather than solely preoccupation with grades. The Instructor used them to establish students' responsibility for learning the theory.

5.2.2.2 Spontaneous quantitative reasoning: solving realistic problems

The second mode was **spontaneous quantitative reasoning**: throughout the course the Instructor also consistently *reasoned quantitatively* in relation to problems or *ad hoc* students' questions that were not specifically linked to the taught theory. One such instance was in the presented lecture, in Episode 1 when two different students asked if they should simplify a fraction that was not irreducible. The Instructor answered with a quantitative interpretation of fraction simplification as "taking a bigger unit." Although in the Measurement Approach conversion of units was an important topic to be discussed, it got introduced spontaneously, in class, as the result of a question that was not specifically derived from within the FoQ theory.

Similarly, in one of the later lectures, when one student, concerned with what is "the right way to write on a test" asked if she should write " $\frac{5}{5}$ or $\frac{1}{1}$ ", the Instructor produced an *ad hoc* quantitative explanation "5 fifths of one cup of flour is one cup", which provides a rather indirect answer to the FT's question. Another context for working with quantities spontaneously was through realistic problems: problems that looked like the ones one would encounter in the elementary school. These cases would not qualify as spontaneous on the part of the Instructor: many such problems were included in the course notes written before the semester began – even with solutions relying on FoQ theory – and those proposed in class were also part of planned lessons. But their solutions were not scripted by the Instructor beforehand through worked out examples: the students were usually not provided with models for solving them, but rather were expected to use their understanding about fractions of quantities to solve them and provide sensible explanations. This was the case for the problems in Episodes 7 and 8, where only subtle cues would suggest the use of the learned knowledge to solve them. Such problems were present throughout the Experiment, in class, homework assignments, and timed tests. The following is a realistic problem solved in one of the labs, which although rather easy, can be quite confusing when one relies on the Division Approach to solve it:

> <u>Realistic problem solved in lab</u> There are 8 people in a lifeboat. They have only 5 litres of drinking water and they must share it equally.
> (a) To what fraction of the whole $5\ L$ will each person be entitled?
> (b) To what fraction of $1\ L$ will each person be entitled?

The expected solution involved reasoning about fractions of quantities in the context of proportional quantities; the Instructor wrote the following solution on the board:

<u>Solution to (a)</u>

Number of people	Number of 5 L (5 L = u)
8 people	1 u
1 person	X u

Since 1 person is 1/8 of 8 people, then X u is 1/8 of 1 u.

So each person will be entitled to 1/8 of 5 litres.

Solution to (b)

Number of people	Number of L (1 L = u)
8 people	5 L
1 person	Y L

Since 1 person is 1/8 of 8 people, then Y L is 1/8 of 5 L.

Since 1/8 of 5 L is 5/8 L, then each person will be entitled to 5/8 L.

This mode of reasoning brought students closer to practice – it appeared as more relevant in terms of the application of the Measurement Approach for teaching. Moreover, when such problems arose, either as *ad hoc* questions or as realistic problems, the Instructor observably put the FTs in the Teacher position by discussing, as a reflective practitioner on the pitfalls of ignoring quantities (typical for the Division Approach), or the disadvantages of relying on algorithms to solve problems (e.g., cross-multiplication, bringing to the same denominator when unnecessary, etc.). Another such context when the Instructor expected FTs to engage spontaneously with quantitative reasoning, was by framing such thinking in terms of quantities as the solution to a fundamental situation for teachers "How would you explain this to a child?" Such context revealed a different view of "explaining"; a demonstration, in practice, of how, in the absence of a robust understanding of quantities, the child who asks "why" is answered with the "how". This was the case when a student FT produced the following solution to the problem of finding the volume of water in a beaker containing $\frac{3}{4}\ L$ to which $\frac{2}{5}\ L$ is added (he added the units and the equal signs only at Instructor’s prompt):

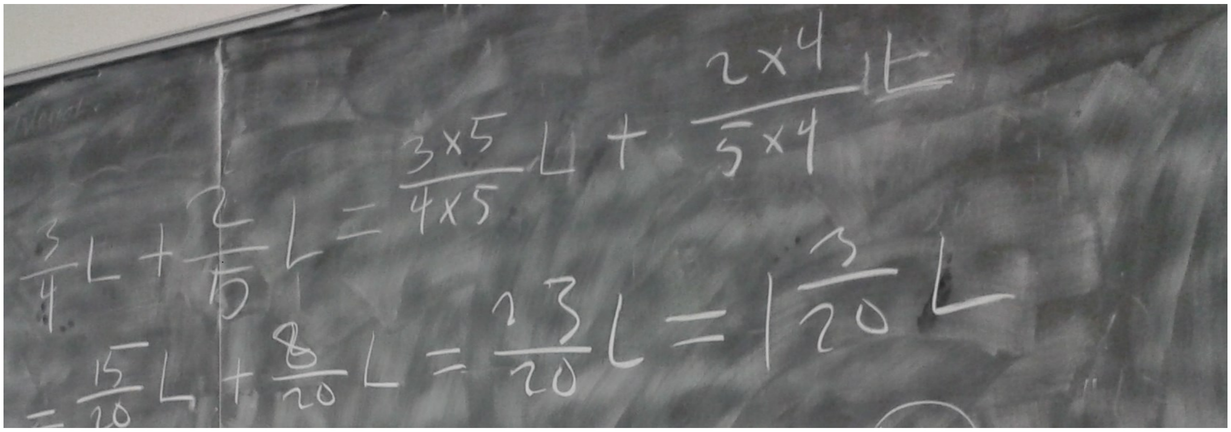


Figure 5.12. Picture of the board: A student's solution to a problem of addition of fractions of quantitties

The solution was correct, but when the Instructor asked the student to “explain to a child who doesn’t know the formula” what he did, the student described, in words, the procedure of adding the fractions by bringing to the same denominator. To justify why he multiplied by 4, and 5, respectively, the student responded: “it is the least common denominator”. At Instructor’s repeated request “Why would you think of doing it”, the student replied: “to keep it consistent, because if we multiply the bottom by 4 we should multiply the top as well”. The FT thus explains exclusively in terms of numerical calculations, showing how to perform a procedure on pure numbers, and the connection to the problem about total volume of water in the beaker is absent.

It was rare, in general, to see students engage in reasoning about quantities in realistic problems or to explain a procedure in terms of “images of what one makes through doing them” (Thompson & Saldanha, 2003, p. 15) when the problem was not pre-formatted to match the format of a previously solved example. It was, however, not unseen. One instance is when a student tried to solve the following problem about percentages using the definition of FoQ:

> *The regular price of a bookcase was* 283.50 \$*. It is now being sold at a reduced price,* 250 \$*. What percent reduction was it?*

She had tried to solve it on her own using in two ways: first, by setting up $\boldsymbol{Q1} = 283.50$ \$ and $\boldsymbol{Q2} = 250$ \$, and then reversely, letting $\boldsymbol{Q1} = 250$ \$ and $\boldsymbol{Q2} = 283.50$ \$, and was puzzled by the inconsistency between the results obtained. It was not only her spontaneous use of the theory that was remarkable – showing systemic thinking, but also her initiative to write an e-mail to the whole class community – including the Instructor – to ask for input. This signaled an important shift when what is up for criticism and debate is not a person, but a mathematical production – indeed a behavior signaling success in cultivating a reflective attitude. I reproduce below the text of her e-mail and the attachment with her solution.

> Text of e-mail sent by a student on FirstClass to ask for peers' input
>
> Hello,
>
> I am very confused as to how we can differentiate between $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ when solving percentage problems. Also why is it that the percentage aren't interchangeable as seen in the attached photo.
>
> Any explanation would be greatly appreciated.
>
> Attachment to e-mail

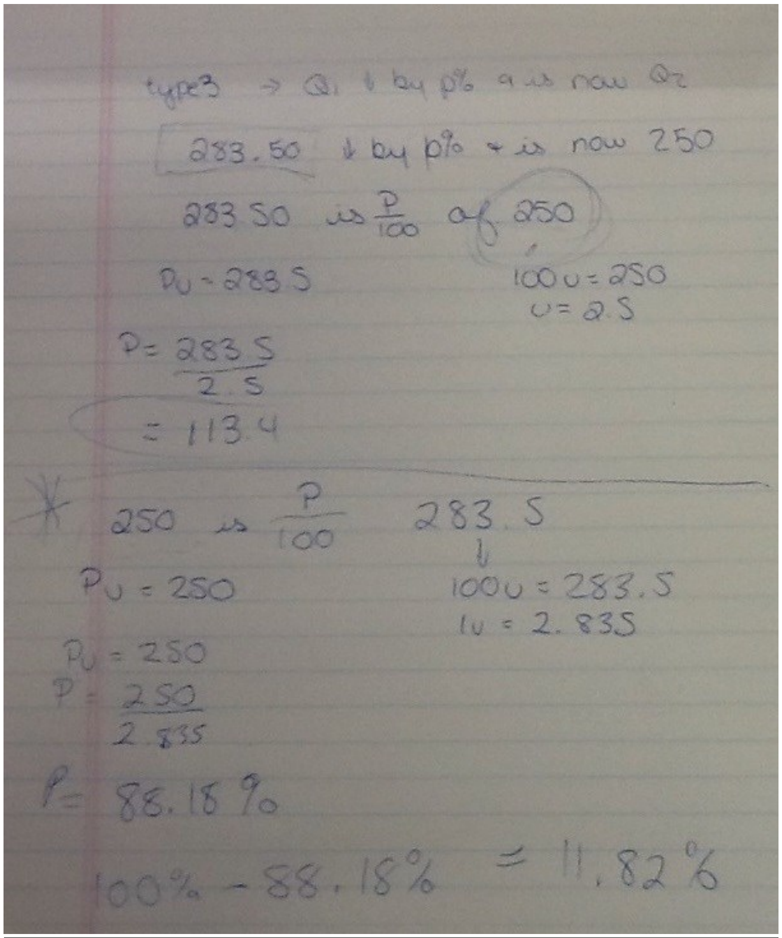


Figure 5.13. A student's solution containing the spontaneous use of the definition of FoQ, e-mailed to the classroom community for consultation

5.3 ADAPTATION: STUDENT-MILIEU

I described, in the previous section, the interactions between the participants when dealing with fractions of quantities, by means of some key incidents that tended to recur throughout the Experiment. The Instructor did mathematics – quantitative reasoning – in front of the students as initiation into the practice of theory building, and into solving realistic problems by applying the theory. We saw, already in FTs reactions, some of the obstacles that could be expected with this approach, but also some evidence of success with cultivating a University Student position where one develops theoretical thinking and experience a shift from purely numerical calculations to quantitative operations.

The focus was, however, more on the Instructor's realization in practice – that of teaching in a mathematics methods course – of the Measurement Approach designed before the Experiment. I turn now more precisely to students' learning, by picking up on some of the observation threads about it in the previous section and analyzing them in depth here. I look at FTs reasoning in terms of quantities, without the Instructor's mediation, and in particular at whether they imitate familiar solutions – and how well they do it – or engage more spontaneously with quantitative concepts. For presentation in this thesis, I chose my analysis of students' behaviors when faced with the question:

> Use the definition of a fraction of a quantity to justify the statement:
> $1\frac{1}{8}\ lb$ is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}\ lb$.

and had to respond to it without any teacher intervention or access to course materials. Such questions were dealt with in the course before, so students could be tempted to imitate previously seen solutions. But, as I will show, it was impossible to respond correctly to the question without a thorough understanding of its structure and relevance of the definition to justify it. The difference between this question and the similar ones tackled before in the course was that the former were true, and the given one is false, although nothing in the formulation of the question suggests this possibility – it is not formulated as a "True or False?" question. This "entrapment" produced a rich range of students' behaviors and thus became a good source of information about students' understanding of the FoQ theory. Issues of representation, particularly in writing – which appeared important in the teaching – were also revealed in their responses to the question.

This marks the shift from the *savoir* targeted by the Instructor to the *connaissances* actually developed by the students.

5.3.1 The Question and initial discussion

Five weeks after the end of the part of the course on fractions, students were asked to respond, in writing, to the question already mentioned above:

Use the definition of a fraction of a quantity to justify the statement:
$1\frac{1}{8}\ lb$ is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}\ lb$

This question was given in the final exam (so they had prepared for it in advance), which contained altogether nine questions, all to be solved within a 3-hour timeframe. The students were not allowed to consult their notes (i.e., this was a closed-book exam).

Such questions – to justify that or verify if a given quantity is the given fraction(s) of another given quantity – were familiar to the students. They were discussed in class and given on assignments and tests. They appeared in several forms. For example, students could be asked two questions instead of one: (a) justify that [some quantity $\boldsymbol{Q1}$] is the fraction [e.g., $\frac{3}{7}$] of [another quantity $\boldsymbol{Q2}$], and (b) show that [$\boldsymbol{Q1}$] is also the fraction [e.g., $\frac{24}{56}$] of [$\boldsymbol{Q2}$].

In a handout given to students in one of the labs, as preparation for a class test, the Instructor included a section on *Equivalent fractions*, containing a set of practice exercises along with explanations. An exercise, similar to the one I considered, was given:

Use the definition of a fraction of a quantity to justify the statement:
$3\frac{1}{2}\ ft$ is both $\frac{7}{10}$ and $\frac{42}{60}$ of $5\ ft$.

The notes, however, did not contain the particular solution to this exercise. Instead, a whole class of problems was discussed, by not specifying the two quantities: it was demonstrated that if one quantity is $\frac{7}{10}$ of another quantity, then it must also be $\frac{42}{60}$ of that quantity, and vice-versa:

In more general situations, where the quantities are not specified, the justification why a certain quantity Q1 can be both, say $\frac{30}{36}$ and $\frac{5}{6}$, of some quantity Q2, consists in showing that, if Q1 is $\frac{30}{36}$ of Q2 then it must necessarily be also $\frac{5}{6}$ of Q2, and vice versa. The argument goes as follows:

If Q1 is $\frac{30}{36}$ of Q2 then, by definition, Q1 = 30 u and Q2 = 36 u for some unit u. If we take a unit u' that is 6 times larger than u (1 u' = 6 u), then Q1 = 5 u' and Q2 = 6 u'. This implies that Q1 is $\frac{5}{6}$ of Q2.

Conversely, if Q1 is $\frac{5}{6}$ of Q2, then, for some unit w, Q1 = 5 w and Q2 = 6 w. If we take a unit w' 6 times smaller than w (so that 1 w = 6 w') then Q1 = 30 w' and Q2 = 36 w'. This implies that Q1 is $\frac{30}{36}$ of Q2.

Therefore Q1 is both $\frac{30}{36}$ and $\frac{5}{6}$ of Q2.

The concept captured in the worked out example as well as in the question we chose for analysis (hereinafter called the Question) is that of *fraction equivalence*. In the FoQ theory, the equivalence of fractions is the equivalence of ways to express the multiplicative relationship between two quantities, depending on the choice of the common unit used to measure the quantities. For example, we can say that $750\ g$ is both $\frac{3}{4}$ and $\frac{15}{20}$ of $1\ kg$, because, if we use $250\ g$ as a unit of measure, then $750\ g$ is $3 \times 250g$ and $1\ kg$ is $4 \times 250\ g$, and if we use a unit 5 times smaller ($50\ g$), then $750\ g$ is $15 \times 50\ g$ and $1\ kg$ is 20×50 g.

In the Question, the equivalence means that, if $1\frac{1}{8}lb$ is $\frac{3}{7}$ of $2\frac{3}{4}lb$ using one unit, then it is also $\frac{24}{56}$ of $2\frac{3}{4}\ lb$ if we use a unit that is 8 times smaller. But there is a trap in this question, because $1\frac{1}{8}lb$ is not $\frac{3}{7}$ of $2\frac{3}{4}lb$. There is no unit that would fit 3 times into $1\frac{1}{8}\ lb$ and 7 times into $2\frac{3}{4}lb$. Would the students have developed enough "quantitative sense" in the course to even verify if 1 1/8 lb is indeed 3/7 of 2 ¾ lbs without prompting? (Asking to "verify" rather than to "justify" the statement could be such a prompt; it was avoided by asking students to "justify" the statement.) If they did not verify whether $1\frac{1}{8}\ lbs$ is $\frac{3}{7}$ or $\frac{24}{56}$ of $2\frac{3}{4}\ lbs$, did their reasoning still carry some features of quantitative reasoning, or were their arguments restricted to the numerical operation of reduction of fraction $\frac{24}{56}$ by dividing the numerator and the denominator of the fraction by 8? These were the questions we were asking ourselves. The definition of a fraction of a quantity to which the question refers was the following:

> *Let $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ be quantities of the same kind. Let a and b be two whole numbers, with $b \neq 0$.*
>
> *We say that the quantity $\boldsymbol{Q1}$ is the fraction $\frac{\boldsymbol{a}}{\boldsymbol{b}}$ of the quantity $\boldsymbol{Q2}$ if there exists a common unit $\boldsymbol{u}$ such that $\boldsymbol{Q1}$ measures a units $\boldsymbol{u}$ and $\boldsymbol{Q2}$ measures b units $\boldsymbol{u}$.*
>
> *The number a is called the numerator and the number b is called the denominator of the fraction $\frac{a}{b}$.*

This definition was given in this form in the Course Notes and quoted several times during the course in class.

Thus, to solve the problem correctly, according to this definition, means to exhibit a unit, say $\boldsymbol{u}$, such that $1\frac{1}{8}\ lb$ measures 3 such units $\boldsymbol{u}$, and $2\frac{3}{4}\ lb$ measures 7 such units $\boldsymbol{u}$. Finding such a unit and the respective measures of the two quantities in terms of $\boldsymbol{u}$, amounts to finding *one way* of representing the quantitative relationship between $1\frac{1}{8}\ lb$ and $2\frac{3}{4}\, lb$, namely, as the fraction $\frac{3}{7}$. In an identical fashion, one has to find another unit, say $\boldsymbol{u}'$, such that the first quantity measures $24\boldsymbol{u}'$ and the second quantity measures $56\boldsymbol{u}'$; this would prove that $\frac{24}{56}$ is another way to represent the relationship between the two quantities, using a unit that is 8 times smaller.

At first sight, the problem contains the easiest type of task to be solved (twice) using the FoQ theory: applying the definition directly to check if a given quantity is a given fraction of another given quantity. One does not have to produce this fraction; it is already given – as opposed, for example, to the task of finding what fraction one quantity is of another, which is tantamount to dividing a quantity by a quantity, a notoriously difficult operation for students. But the problem turned out to be difficult for several reasons, outlined below.

Difficulty 1. The two quantities are not given as whole numbers of units

One reason is that the two given quantities are themselves fractions of quantities: $1\frac{1}{8}$ of 1 pound, and $2\frac{3}{4}$ of 1 pound. We did not expect students to use the definition of a fraction of a quantity to justify their conversions of the mixed numbers into improper fractions of pounds, or into ounces and in explaining, e.g., why $\frac{9}{8}$ of $16\ oz$ is $18\ oz$. But they could, in theory, perceive the question as requiring such justifications, thus making the requirement of using the definition much more demanding.

Moreover, the fact that the two given quantities are fractions of quantities makes the finding of a common unit less than trivial. Compare, for example the two structurally identical sentences below:

$35\ ml$ is $\frac{5}{11}$ of $77\ ml$

$1\frac{1}{8}\ lb$ is $\frac{3}{7}$ of $2\frac{3}{4} lb$

In this first sentence, to find the unit, one operates with whole numbers, and divisions/multiplications can be easily done as mental calculations: the unit is obviously $7\ ml$. In the second, one needs to do divisions/multiplications of fractions, which are not only introduced much later than the definition, but also, are built upon it. In other words, one has to use the most elementary piece of knowledge to prove the sentence – the definition of a fraction of quantity, but has to operate with tools that are built upon the definition – the multiplication and division of fractions of quantities, in this case, by a whole number. There is also the additional – although not necessary – step of converting mixed numbers to common fractions.

The students have the option of converting pounds to ounces, thus having the given quantities measured in whole number of units; having denominators factors of 16 makes this option more visible: the given statement could be modified as follows:

$$18\ oz\ \ is\ \ \frac{3}{7}\ \ of\ \ 44\ oz.$$

But changing from pounds to ounces is still an extra step to take, which adds to the cognitive load of the problem.

Difficulty 2. The statement is a conjunction of two statements

Although the statement could be broken into two structurally identical ones, both directly verifiable by the definition, it is not formulated as such. To break the given statement:

$1\frac{1}{8}\ lb$ is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}\ lb$

into a conjunction of two statements:

$1\frac{1}{8}\ lb$ is $\frac{3}{7}$ of $2\frac{3}{4}\ lb$

AND

$1\frac{1}{8}\ lb$ is $\frac{24}{56}$ of $2\frac{3}{4}\ lb$

is a mental operation that is fraught with logical subtleties, thus requiring a cognitive leap that is not as direct as it may seem when AND is to be understood as a logical operator.

Difficulty 3. The statement is false and the student is not warned about this possibility

The first given quantity is neither the fraction $\frac{3}{7}$ nor $\frac{24}{56}$ of the second one. The statement would be true only if one of its three variables had a different value:

- *The first given quantity is rather* $\boldsymbol{\frac{9}{22}}$ *of the second given quantity;*
- *The first given quantity is the given fraction of* $\boldsymbol{2\frac{5}{8}\,lb}$
- $\boldsymbol{1\frac{5}{28}\,lb}$ *is the given fraction of the second given quantity.*

On the other hand, $\frac{3}{7}$ and $\frac{24}{56}$ are equal as abstract numbers. Thus the statement can be disproved in various ways, while also being likely to give the illusion that it is true in more than one way. Moreover, even its formulation as affirmative rather than interrogative (e.g., Is $1\frac{1}{8}\ lb$ both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}\ lb$?) could certainly make students less inclined to spot the contradiction.

5.3.2 A-priori analysis of the structure of the question

In this section, we analyze the logical structure of the Question, in particular with respect to its correspondence to the definition, and we outline several possible paths for solving it.

5.3.2.1 Step 1: Matching the statement to the definiendum of the definition of a fraction of a quantity

In the definition of a fraction of a quantity the *expression to be defined* – also known as the *definiendum* – is:

$$\mathbf{Q1}\ is\ \frac{a}{b}\ of\ \mathbf{Q2} \quad (S)$$

It is thus a whole sentence[5] (and not just a word or a phrase, as in many definitions). This definition does not say, "A fraction is…"; it defines what it means for one quantity to be a given fraction of another quantity, which is a complex structure in itself. Let us call this structure S.

To solve the problem correctly using the definition, in particular to discover the contradiction, one has to reconstruct, from the given string of words, a *conjunction of two sentences* with the same structure, say $S1\ AND\ S2$:

$$\underbrace{\left[1\frac{1}{8}\mathrm{l}b\ \ is\ \ \frac{3}{7}\ \ of\ \ 2\frac{3}{4}lb\right]}_{S1}\ AND\ \underbrace{\left[1\frac{1}{8}lb\ \ is\ \ \frac{24}{56}\ \ of\ \ 2\frac{3}{4}lb\right]}_{S2}$$

[5] We use the word *statement* as declarative sentence that can be true or false, thus as a logical entity; we use the word *sentence* in its usual sense, as a grammatical entity.

This parsing of the given statement is necessary in order to "see" in it the structure of the *definiendum* in its entirety, and thus refer to all the four variables that appear in the definition of the fraction of a quantity ($\boldsymbol{Q1}, \boldsymbol{Q2}, a, b$) when justifying it.

Thus, the first operation in order to prove the given statement is to create $S1$ and $S2$, which are instantiations of the structure S, with given values $1\frac{1}{8}lb$, $2\frac{3}{4}lb$, 3, 7 and $1\frac{1}{8}lb$, $2\frac{3}{4}lb$, 24, 56, substituted successively for $\boldsymbol{Q1}, \boldsymbol{Q2}, a, b$.

The word AND has the meaning of the truth-functional operator (conjunction): both sentences should be understood as statements with a truth value. If one of the statements is false, their conjunction – the given statement – is false. At least one of the two statements has to be validated using the definition straightforwardly (Step 2). The other one can be proved independently in a similar fashion, or it can be deduced from the first statement (Step 3).

5.3.2.2 Step 2: Using the definition to prove one of the two statements

The *definiens* (that which defines) of the fraction of a quantity poses several conditions. All are essential, but, for practical purposes, in particular in the context of the given statement, it would be acceptable not to check all of them. We distinguish between *main definitional conditions*:

(1) $\boldsymbol{Q1}$ measures a units $\boldsymbol{u}$, which can be written as: $\boldsymbol{Q1} = a\boldsymbol{u}$

(2) $\boldsymbol{Q2}$ measures b units $\boldsymbol{u}$, which can be written $\boldsymbol{Q2} = b\boldsymbol{u}$

and *secondary definitional conditions*:

(3) $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ are quantities of the same kind

(4) a and b are whole numbers

(5) b is not zero

The first operation in using the *definiens* to produce an acceptable proof of, say, $S1$, is to substitute all the given values in the main definitional conditions:

$$1\frac{1}{8}lb = 3\boldsymbol{u}$$

$$2\frac{3}{4}lb = 7\boldsymbol{u}$$

This might remain a mere syntactic operation if one does not understand the statements as having a truth value to be verified: if both hold for some common unit $\boldsymbol{u}$ then $S1$ is true.

Only such hypothetical stance can lead one to perform the next couple of operations: producing the unit $\boldsymbol{u}$, say by division of $1\frac{1}{8}lb$ by 3 to verify condition (1), and then multiplying $\boldsymbol{u}$ by 7 to see if $2\frac{3}{4}lb$ obtains, thus verifying condition (2). An equivalent approach would be to divide the first quantity by 3 and the second by 7 and see if the same value for the unit obtains.

It is, of course, crucial that both conditions (1) and (2) hold for the same unit $\boldsymbol{u}$ in order to prove the statement $S1$ (or $S2$) using the definition. Using just one of them to produce a unit is of no value for the validation of S1.

5.3.2.3 Step 3: Proving the other statement

If Step 2 is performed correctly for, say, S1, and the meaning of AND as a logical operator is correctly understood, then checking the validity of $S2$ by definition is not necessary: $S1$ would be established false and the conjunction $S1$ and $S2$ is automatically false (it is true only when both $S1$ and $S2$ are true).

But if Step 2 is not correctly performed (i.e., it is not established that $S1$ is false) or, more generally, if $S1$ were to be true, then there are at least two possible strategies to prove (or disprove) $S2$. One of them is to proceed exactly as for $S1$, by checking if the main definitional conditions are satisfied. Another approach is to notice, or deduce, a relationship between the units corresponding to $S1$ and $S2$, say $\boldsymbol{u}$ and $\boldsymbol{u}'$. In particular, assuming that S1 is true, i.e., $1\frac{1}{8}lb = 3\boldsymbol{u}$ and $2\frac{3}{4}lb = 7\boldsymbol{u}$, if one takes a unit that is 8 times smaller (this is possible), i.e., $1\boldsymbol{u} = 8\boldsymbol{u}'$, the definitional conditions for $S2$ follow easily: $1\frac{1}{8}lb = 3\boldsymbol{u} = 3\ \times 8\boldsymbol{u}' = 24\boldsymbol{u}'$ and

$2\frac{3}{4}lb = 7\boldsymbol{u} = 7 \times 8\boldsymbol{u}' = 56\boldsymbol{u}'$. The units $\boldsymbol{u}$ and $\boldsymbol{u}'$ need not be calculated concretely, as quantities (numbers of pounds or ounces in this case).

5.3.3 Analysis of students' responses

In this section I analyze students' responses to the Question. The 38 students whose answers we considered produced solutions that were very different from one another. Only 5 students actually disproved the statement as false. However, all the remaining students produced pieces of argumentation corresponding to their own interpretations of the given statement. There was, of course, one correct mathematical interpretation – which we discussed above, and which 5 of the students produced – but we were looking for the rationality in each student's solution by reconstructing their reasoning.

To do this we used a structuring framework inspired by Balacheff's (2013) model of learners' conceptions, which portrays *the connaissances* of the learning subject rather than *the savoir*, targeted by the Instructor. We examined students' behaviors in problem solving by asking the following questions:

- What problem is the learner, de facto, solving?
- What strategies is she using and how does she establish the validity of her solutions?
- What language does she use to solve it?

This approach is meant to cover the whole sphere of practice in students' responses to the mathematical problem proposed, and has the merit of modeling the rationality of students' solutions (i.e., versus, for example, dismissing a solution as incorrect from the point of view of *savoir*).

We grouped the students in the following 5 clusters of interpretations of the given statement, based on *the problem* or *the problems* they appeared to be tackling in their answers. We labeled them as follows:

- **Interpretation 1: A statement about equivalence of fractions of quantities**
- **Interpretation 2: A relation between pairs of abstract numbers**

- **Interpretation 3: Two sentences about fractions**
- **Interpretation 4: A conjunction of statements about equivalence of fractions of quantities**
- **Other interpretations**

We present the interpretations below, in descending order of each cluster's cardinality: 10 students adopted Interpretation 1, 9 students – Interpretation 2, 6 students – Interpretation 3, and 5 students – Interpretation 4. The remaining 8 students each wrote solutions that were almost unique among their peers. In particular, their interpretations of the given statement led them to solve each a different problem, or even several different problems within the same solution.

The main characteristics of students' solutions from the **first** cluster was that they ignored the concrete values of the quantities $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ in establishing the truth value of either $S1$ or $S2$. Essentially, however, they considered $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ either as variables, or with their concrete values in order to show a relationship between units and thus argue that the two fractions are equivalent as fractions of quantities. The problem was, for them, *to justify that a quantity can be both* $\frac{3}{7}$ *and* $\frac{24}{56}$ *of another quantity*. In logical terms, the solutions of this group contained the proof of the conjunction of $S1$ and $S2$, *assuming* that $S1$ and $S2$ are true – a condition they failed to check.

The focal point for the students in the **second** group consisted only of the numbers $3, 7, 24$, and 56: for them the task was *to show that* $\frac{3}{7}$ *" is the same as"* $\frac{24}{56}$. There were no quantities involved; the other parts of the sentence were not essential in the justifications of this group. We didn't write "$\frac{3}{7} = \frac{24}{56}$" above because, in some of these students' solutions the focus is on showing a relation among four whole numbers, rather than two numbers which are fractions: it is not clear that they conceive of $\frac{3}{7}$ (or $\frac{24}{56}$) as *one number*.

The students in the **third** group did parse the given statement as two separate sentences. We use the word "sentence", rather than "statement" here, to emphasize the fact that students in this group did not question the truth of these sentences. The problem, for them, was: *to show* *that* $1\frac{1}{8}\ lb\ \ is\ \ \frac{3}{7}\ \ of\ 2\frac{3}{4}\ lb$ *and that* $1\frac{1}{8}\ lb\ \ is\ \ \frac{24}{56}\ \ of\ 2\frac{3}{4}\ lb$ by using the definition as a procedure with certain steps, rather than as a control structure. Consequently, the word “and” did not act as a logical operator and when contradictions arose at various points in this procedure, the students did not reflect, or act upon them. Furthermore, the students in this group appeared to view the fractions in the Question as abstract numbers, rather than as fractions of quantities.

Finally, the students in the **fourth** group, like the previous group, separated the given statement into the same two sentences, but regarded them as hypothetical statements, whose truth value is to be established: *Is* $1\frac{1}{8}\ lb\ \ \ \frac{3}{7}\ \ of\ 2\frac{3}{4}\ lb$ *? and Is* $1\frac{1}{8}\ lb\ \ \ \frac{24}{56}\ \ of\ 2\frac{3}{4}\ lb$ *?* They used the definition to establish the truth and reflected in various ways about the ensuing contradictions: by proposing a change of one of the given quantities, or the given fractions, or simply stating that the statement is false.

I present all the interpretations below by detailing not only the problems each student was solving, but also the strategies they used to establish validity, in particular whether they used elements of the FoQ theory meaningfully in their solutions to produce quantitative arguments versus purely numerical calculations. I also comment on the way in which they employed language to express ideas, not only in terms of correctness of mathematical terminology and grammar, but also in terms of the overall coherence of the written representation. The 38 students’ names are coded from FT1 to FT38.

5.3.3.1 Interpretation 1: A statement about equivalence of fractions of quantities

Ten of the 38 students who responded to the question interpreted the given statement this way: they more or less ignored the concrete values of the quantities and focused on the equivalence of fractions $\frac{3}{7}$ and $\frac{24}{56}$ of a quantity. They explained this equivalence by a relationship between the units of measure corresponding to the fractions. Some expressed this relationship qualitatively (unit corresponding to the latter fraction is smaller than the unit corresponding to the former); others expressed it quantitatively (unit corresponding to the latter fraction is 8 times smaller than the unit corresponding to the former). In writing about the units, some students used the letter “u” as an abbreviation of “unit” and others – algebraically, as a variable. In the latter case, algebraic expressions of the relationships between the units corresponding to fractions appeared in some students’ responses; e.g., $u = 8\,u'$. Eight of these ten students’ arguments were well structured and written using correct mathematical notation and deductive discourse.

Two of the students, FT29 and FT38, ignored the values of the quantities explicitly, by calling them Q1 and Q2. FT29 used the letters u and u’ as variables; FT38 – as an abbreviation. FT29 expressed the relationship between the units algebraically. We quote:

> *“A quantity is $\frac{24}{56}$ of a quantity Q2 when measured with a common unit u. When both quantities are measured with a unit u’ such that $u' = 8\,u$ then Q1 which was $24\,u$ becomes $8\,u'$ and Q2 which was $56\,u$ becomes $7\,u'$. Q1 is $\frac{3}{7}$ of Q2.” (FT29)*
>
> *“If Q1 is $\frac{3}{7}$ of Q2, Q1 measures $3\,u$ and Q2 measures $7\,u$. If we take a unit 8 times smaller then Q1 would measure $24\,u$ and Q2 would measure $56\,u$. This means that Q1 is both $\frac{3}{7}$ and $\frac{24}{56}$ of Q2.” (FT38)*

Other students referred to the quantities by their numerical values but these values were not logically essential in their arguments; these values acted like “dummy variables.” They were not essential logically, but some students may have believed that mentioning the values was essential to obtain a good grade: that it was a part of the “didactic contract” in this exercise.

This is the case, in particular, of FT17. This student first converted the mixed numbers to common fractions $\frac{9}{8}\,lb$ and $\frac{11}{4}\,lb$ and then replaced $\frac{11}{4}\,lb$ by $\frac{22}{8}\,lb$ justifying it by ‘taking a unit that is twice as small’. But the student did not use all these conversions in the remaining, essential part of his response:

"Now $\frac{9}{8}\,lb$ *is both* $\frac{3}{7}$ *and* $\frac{24}{56}$ *of* $2\frac{3}{4}\,lb$ *because* $\frac{3}{7}$ *is the same amount as* $\frac{24}{56}$ *if you utilize a unit of measure that is 8 times smaller.* $3\ \times 8 = 24\,u \quad 7\ \times 8 = 56\,u$*." (FT19)*

FT27 also started by taking into account the concrete values of the quantities: she calculated the [candidates for] the units corresponding to $\frac{3}{7}$ and $\frac{24}{56}$ by dividing $1\,\frac{1}{8}\,lb$ by 3 (getting $\frac{9}{24}\,lb$), and then by 24 (getting $\frac{9}{192}\,lb$). But she did not verify if $\frac{9}{24}\,lb$ multiplied by 7 or $\frac{9}{192}\,lb$ multiplied by 56 gives the second quantity. She "concluded" from this that the unit corresponding to $\frac{3}{7}$ is "simply greater" than the unit corresponding to $\frac{24}{56}$, but the fractions "maintain the same value." Her explanation was qualitative: she did not state how many times greater the unit corresponding to $\frac{3}{7}$ would have to be. She used the letter "u" as an abbreviation, not as a variable.

We quote her response in full below. Note that her response started with "matching the statement to the *definiendum* of the definition of a fraction of a quantity" (Step 1 in section 5.3.2) and verifying the secondary definitional conditions: $Q1$ and $Q2$ are quantities are of the same kind: they are pounds; $3, 7, 24$ and 56 are whole numbers and 7 and 56 are not zero. Many students started their solutions this way. In the second and third paragraphs of her solution, FT27 appears to start verifying the primary definitional conditions: she finds candidates for the value of the units from the first quantity but she does not verify if their multiples corresponding to the denominators equal the second quantity. We surmise, therefore, that she treats these two paragraphs as a continuation of the "matching" Step 1: she exhibits the units corresponding to the two fractions. She starts her third paragraph – the essential element of her argument – by "therefore", although this argument holds independently of the concrete values of the units she has calculated. She claims that her main argument follows "from the definition", as required.

"Let Q1 and Q2 be quantities of the same kind (lb), where a and b are whole numbers (3 & 7 or 24 & 56) and $b \neq 0$. Q2 is a/b of Q2 if such a unit, u, exists so that Q2 is a x u and Q2 is b x u.

1 1/8 lb is 3/7 of 2 ¾ lb where u is equal to 9/24 lb , or 3/8 lb.

1 1/8 lb is 24/56 of 2 ¾ lb where u is equal to 9/192 lb , or 3/64 lb.

Therefore the unit size where Q1 is 3/7 of Q2 is simply greater than the unit size where Q1 is 24/56 of Q2, but the fractions maintain the same value." (FT27)

In their solutions, students in this group did capture the essential aspect of the problem, i.e., they understood that the equivalence of fractions of quantities consists in a change of the unit in which the quantities are measured. The key understanding, held by all 10 students in this group, was that the relationship between two quantities could be expressed by means of different pairs of whole numbers, corresponding to different units. They all showed that a quantity could be both $\frac{3}{7}$ and $\frac{24}{56}$ of another quantity, essentially by exhibiting the relationship between two different units, either as a qualitative relation ($\boldsymbol{u} > \boldsymbol{u}$'), or as a quantified one ($1\boldsymbol{u} = 8\boldsymbol{u}$'). But no one in this group of students attempted to show that the given quantity, $1\frac{1}{8}\ lb$, was indeed $\frac{3}{7}$ or $\frac{24}{56}$ of the other given quantity, $2\frac{3}{4}\ lb$. The given values of the variables $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ were not seen as influencing the truth-value of the given statement.

In logical terms, the flaw of all the arguments in this group was that they thought that it sufficed to prove only that $S1$ implies $S2$ (or that $S2$ implies $S1$) without validating any of the statements S1 or S2. If the students proved, e.g., that $S1$ implies $S2$ and also that $S1$ is true, then the law of *modus ponens* would allow them to claim that $S2$ is true as well. Without proving that $S1$ is true, the validity of $S2$ cannot be claimed.

5.3.3.1.1 Problem: To justify that a quantity can be both $\mathbf{\frac{3}{7}}$ and $\mathbf{\frac{24}{56}}$ of another quantity

The problem that students in this group were solving was to justify one or both of the following statements:

- If a quantity is $\frac{3}{7}$ of another quantity then it is also $\frac{24}{56}$ of that quantity (P1.1[6])
- If a quantity is $\frac{24}{56}$ of another quantity then it is also $\frac{3}{7}$ of that quantity (P1.2)[7]

Each of the above statements is the converse of the other. Some students believed they have to show both, imitating the argument the teacher demonstrated in the preparation for the midterm, in the context of abstract quantities, $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$, and abstract units, $\boldsymbol{u}$ and $\boldsymbol{u}$'.

This was a problem about fractions of quantities, not abstract fractions. Students were not proving the equivalence of the fractions $\frac{3}{7}$ and $\frac{24}{56}$ in abstraction from the quantities of which they were fractions. They all interpreted the relation between the pairs of numbers $3, 7$ and $24, 56$ in terms of a relationship between the units used to measure two quantities.

But only two of the students in the group (FT29 and FT38, quoted above) explicitly abstracted from the given values of the quantities ($1\frac{1}{8}\ lb$ and $2\frac{3}{4}\ lb$) using the symbols $Q1$ and $Q2$ in reference to them. They were literally proving *exactly* one or the other of the above two implications: FT29 correctly proved P1.1 and FT38 correctly proved P1.2.

The rest of this group, however, did not show such awareness. The definitional conditions $\boldsymbol{Q1} = a\boldsymbol{u}$ and $\boldsymbol{Q2} = b\boldsymbol{u}$, were typically instantiated with the particular values of the variables given in the statement, in sentences such as: $1\frac{1}{8}\ lb = 3u$ (or $1\frac{1}{8}\ lb$ measures $3u$), $2\frac{3}{4}\ lb = 7u$, $1\frac{1}{8}\ lb = 24u$', or $2\frac{3}{4}\ lb = 56\ u$'. The solutions in Figure 5.14 (FT35) and in the transcribed solution below (FT32) are examples of such behavior, and they include also the proof of the converse.

[6] Problem 1 corresponding to Intepretation 1.
[7] Problem 2 corresponding to Interpretation 1.

The greater notational consistency of FT35's response indicates that F35 is more in control of the meaning of the discourse she is producing than F32.

> *"If 1 1/8 lb is 3/7 of 2 ¾ lb then 1 1/8 lb is equal to 3u and 2 ¾ is equal to 7u for some unit u. If we take a unit u' that is 8 times smaller, then 1 1/8 = 24u' and 2 ¾ = 56 u'*[8]*. This implies that 1 1/8 lb is 24/56 of 2 ¾ lb. Conversely, if 1 1/8 lb is 24/56 of 2 ¾ lb then 1 1/8 lb is equal to 24w and 2 ¾ is equal to 56w, for some unit w. If we take a unit w' that is 8 times larger than (sic!) 1 1/8 = 3u*[9] *and 2 ¾ = 7 u. This implies that 1 1/8 lb is 3/7 of 2 ¾ lb." (FT32)*

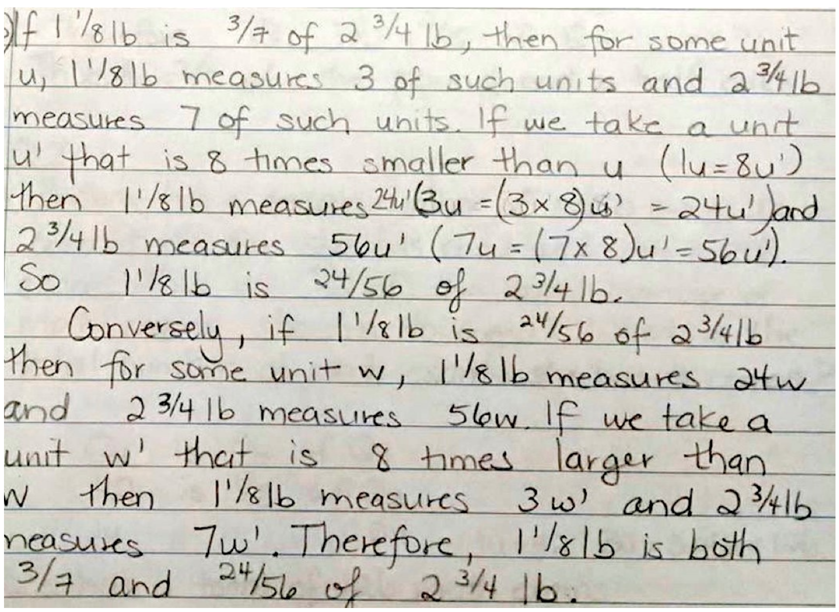

If 1 1/8 lb is 3/7 of 2 3/4 lb, then for some unit u, 1 1/8 lb measures 3 of such units and 2 3/4 lb measures 7 of such units. If we take a unit u' that is 8 times smaller than u (1u = 8u') then 1 1/8 lb measures 24u' (3u = (3 × 8)u' = 24u') and 2 3/4 lb measures 56u' (7u = (7 × 8)u' = 56u'). So 1 1/8 lb is 24/56 of 2 3/4 lb.
Conversely, if 1 1/8 lb is 24/56 of 2 3/4 lb then for some unit w, 1 1/8 lb measures 24w and 2 3/4 lb measures 56w. If we take a unit w' that is 8 times larger than w then 1 1/8 lb measures 3w' and 2 3/4 lb measures 7w'. Therefore, 1 1/8 lb is both 3/7 and 24/56 of 2 3/4 lb.

Figure 5.14. FT35 proved that a quantity is 3/7 of another quantity if and only if it is 24/56 of that quantity.

[8] Note the inconsistency in these expressions: an abstract number, 1 1/8 (2 ¾) is equated to a quantity: 24u' (56u') is a number of pounds.
[9] There is a notational lapsus here which may indicate that the student is not quite in control of the meaning of the learned discourse she is producing.

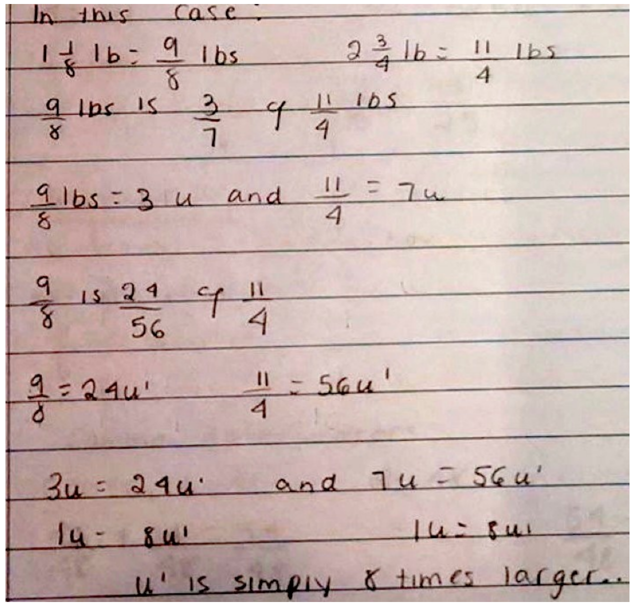
5.3.3.1.2 Strategies and validity

The ten students in this group had two ideas on how to solve the problem. Seven students used an essentially *algebraic* approach: the conversion equation $1\boldsymbol{u} = 8\boldsymbol{u}'$ was either deduced from the two sets of definitional conditions for $S1$ and $S2$, or assumed true and used to deduce $S2$ from $S1$ or $S1$ from $S2$. The three other students *found two concrete units*, fractions of pounds or ounces, in order to explain that one unit is larger than the other. In both cases, the equivalence of fractions of quantities was validated using FoQ theory.

We give examples of students' strategies below.

One of the algebraic approaches was to assume that the main definitional conditions hold for $S1$ (i.e., that $1\frac{1}{8}\ lb = 3\ \boldsymbol{u}$ and $2\frac{3}{4}\ lb = 7\ \boldsymbol{u}$) and *claim that* " $1u = 8u'$ " proves $S2$ (i.e., that $1\frac{1}{8}\mathrm{l}b\ \ is\ \ \frac{24}{56}\ \ of\ \ 2\frac{3}{4}lb$). This was the case for the solutions of FT32 and FT35. The essential flaw of these solutions was precisely this assumption that the equalities $1\frac{1}{8}\ lb = 3\boldsymbol{u}$ and $2\frac{3}{4}\ lb = 7\boldsymbol{u}$ in $S1$, and $1\frac{1}{8}\ lb = 24\boldsymbol{u}$ and $2\frac{3}{4}\ lb = 56\boldsymbol{u}$ in $S2$ are true, or, rather, do not need justification. These equalities were not understood as needing justification: as couples of conditions to be fulfilled in order to justify why $1\frac{1}{8}\mathrm{l}b\ \ is\ \ \frac{3}{7}\ (\frac{24}{56})\ \ of\ \ 2\frac{3}{4}lb$. The students performed the syntactic operation of matching the given statement to the definition, in particular to the two main definitional conditions, but did not see these conditions as hypothetical. They did not grasp the meaning of the definition, focusing instead on the form of the equations.

Another approach, exemplified in Figure 5.15, was *to deduce* $1\boldsymbol{u} = 8\boldsymbol{u}'$ from both $S1$ and $S2$, based on the transitivity of equality: if $1\frac{1}{8}\ lb = 3\boldsymbol{u}$ and $1\frac{1}{8}lb = 24\boldsymbol{u}'$, then $3\boldsymbol{u} = 24\boldsymbol{u}'$, whence $1\boldsymbol{u} = 8\boldsymbol{u}'$. In a similar fashion, the same conversion equation ($1\boldsymbol{u} = 8\boldsymbol{u}'$) results from $2\frac{3}{4}\ lb = 7\boldsymbol{u}$ and $2\frac{3}{4}\ lb = 56\boldsymbol{u}'$, which confirms its validity. Again, the particular values of the variables, $1\frac{1}{8}\ lb$ and $2\frac{3}{4}lb$ had no bearing on the truth of the either $S1$ and $S2$; they played the role of "dummy variables."



In this case:

$1\frac{1}{8}$ lb = $\frac{9}{8}$ lbs $\quad$ $2\frac{3}{4}$ lb = $\frac{11}{4}$ lbs

$\frac{9}{8}$ lbs is $\frac{3}{7}$ of $\frac{11}{4}$ lbs

$\frac{9}{8}$ lbs = 3 u and $\frac{11}{4}$ = 7u

$\frac{9}{8}$ is $\frac{24}{56}$ of $\frac{11}{4}$

$\frac{9}{8}$ = 24u' $\quad$ $\frac{11}{4}$ = 56u'

3u = 24u' and 7u = 56u'

1u = 8u' $\quad$ 1u = 8u'

u' is simply 8 times larger..

Figure 5.15. FT25 deduced the conversion equation 1u = 8u'
from the main definitional conditions using transitivity of equality.

The other strategy was *to calculate the two units concretely* and compare them; two students did this. One of them was FT27, already quoted at the beginning of section 5.3.3.1. In her solution, the student appeared to be using only two of the conditions, $1\frac{1}{8}\ lb = 3\boldsymbol{u}$ and $1\frac{1}{8}\ lb = 24\boldsymbol{u}'$, to calculate two concrete units by division: $9/24\ lb$ and $9/192\ lb$ are the results of $1\frac{1}{8}\ lb$ divided by 3, and 24, respectively. With these two units, the student then made the observation that one unit is larger than the other (a qualitative observation about quantities). The conclusion stated in terms of the two sentences, "Q1 is 3/7 of Q2" and "Q1 is 24/56 of Q2", hints at the student's tautological interpretation of these sentences as "formulas" that are always true, regardless of the values "plugged" into them. But more striking is perhaps the asymmetry of the argument, compared to the previously quoted solution of FT25: only the given values of $\boldsymbol{Q1}$, a, and c are relevant – those of $\boldsymbol{Q2}$, b, and d are left out. According to the definition, to justify that two quantities are in a $\frac{3}{7}$ relationship, for example, one has to exhibit a unit $\boldsymbol{u}$ such that $1\frac{1}{8}\ lb = 3\boldsymbol{u}$ and $2\frac{3}{4}\ lb = 7\boldsymbol{u}$. In the theory, verifying the definitional conditions thus amounts to an existence proof – exhibiting a unit that verifies the two main conditions. Formally, the two conditions have equal status, while in practice, there is, indeed, an apparent asymmetry: the solver would typically assume one condition to hold in order to produce the unit $\boldsymbol{u}$, for example by dividing $1\frac{1}{8}\ lb$ by 3, and then checking the other condition by multiplication.

Yet, even in this group, where we found some sound arguments for equivalence of fractions of quantities, this asymmetry of treatment of $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ appeared more than once. In the solution of FT37, presented in Figure 5.16, the student first writes the main definitional conditions for $S1$, "matching" $18\ oz$ with 3 units, and $46\ oz$ (a conversion mistake: it should be $44\ oz$) with 7 units. But even with the added convenience of whole numbers of units, the student uses the given values of $\boldsymbol{Q1}$ only, $18\ oz$, to calculate a unit of $6\ oz$ ($18\ oz\ \div 3\ = 6\ oz$). The other unit is produced by dividing the first unit by 8 ($6\ oz\ \div 8$), based on the observation that the unit corresponding to $\frac{24}{56}$ is 8 times smaller. Like in the previous solution, of FT27, the given value of $\boldsymbol{Q2}$ is of no consequence. Unlike in the previous solution, however, the values of b and d are featured to link the numerical relationship between two abstract fractions with the quantitative relationship between units. The student shifts between numerical and quantitative operations, but appears to hold a stable link between the two: in the last paragraph of the solution she writes the equality of two fractions and explains it by a change in unit.

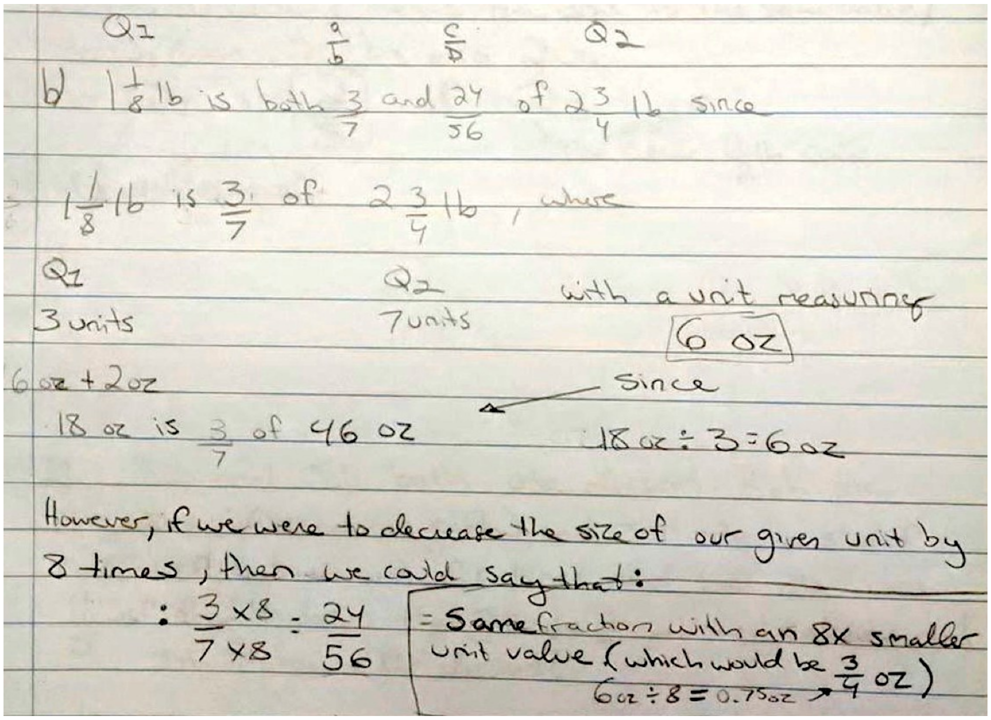
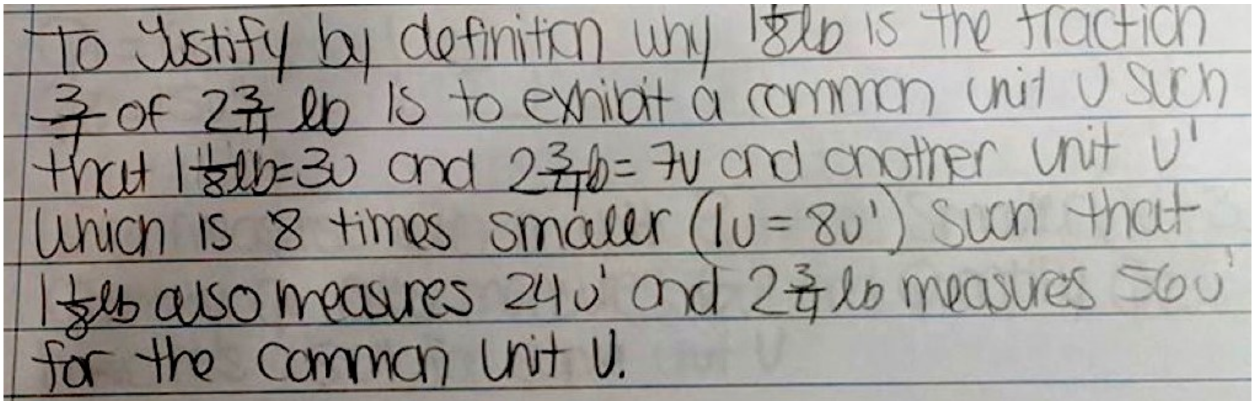


Figure 5.16. FT37's solution: finding a candidate for the unit from Q1 but not checking if it measures Q2 the required number of times.

5.3.3.1.3 Language

All students in the group used algebra correctly when describing quantitative relations such as "a unit that is 8 times smaller": $1\boldsymbol{u} = 8\boldsymbol{u}'$. The students' use of the logical connectives specific to mathematical deductive discourse was also correct: "since", "however", "if", "then", "conversely" do create meaningful coherence relations. Writing in two columns, using arrows, framing results, and other features of page organization reveal generally sound trains of thought. The remarkable common feature of these solutions was this coherence in language. But it is precisely this purely formal manipulation of structures that accounts for the shortcomings of the solutions in this group. The students start sentences with "If" but don't grasp its meaning as a condition introducer. An even more flagrant instance of semantic error appeared in the solution of FT21 (Figure 5.17).



To Justify by definition why $1\frac{1}{8}$ lb is the fraction $\frac{3}{7}$ of $2\frac{3}{4}$ lb is to exhibit a common unit u such that $1\frac{1}{8}$ lb $= 3u$ and $2\frac{3}{4}$ lb $= 7u$ and another unit u' which is 8 times smaller $(1u = 8u')$ such that $1\frac{1}{8}$ lb also measures $24u'$ and $2\frac{3}{4}$ lb measures $56u'$ for the common unit u.

Figure 5.17. FT21 explained what it means to justify a statement using the definition, mimicking the teacher's explanation.

FT21's response reads as an unfinished solution. She says what she is expected to do but does not do it: "To justify […] is to exhibit […]" – one expects the "exhibition" to ensue. The expectation is to see concrete units satisfying the relation – or at least an attempt to produce them, but this meaning eludes the student: perhaps, for her, the units *are* exhibited, they are u and u'.

There were a few minor inconsistencies in notation, but they appeared to be related to linguistic sensitivity rather than to "bugs" in reasoning: the use of the equals sign to signify what is being done, the use of the same letter u for two different units, awkward formulations such as "the unit size where Q1 is 3/7 of Q2". Grammaticality with respect to the FoQ theory was also mostly satisfied, with only some instances where the standard units (lb, oz) have been omitted.

The only exception was a student, FT17 (Figure 5.18), already quoted partially at the beginning of section 5.3.3.1, who claimed that "$\frac{3}{7}$ is the same amount as $\frac{24}{56}$", a statement which is ungrammatical in the context of FoQ theory because the quantities of which $\frac{3}{7}$ and $\frac{24}{56}$ are fractions are not mentioned. Perhaps for FT17 the fractions *are* the quantities, rather than a relationship between them. His other language "glitches", such as making claims that do not follow from the premises, using deductive jargon ("Since", "Now") and incorrect terminology (GCD instead of LCM), as well as failing to capture algebraically the relationship expressed in words ("a unit that is 8 times smaller" as "$3 \times 8 = 24u \quad 7 \times 8 = 56u$") point also to a shaky reasoning trajectory with the help of the definition, despite his understanding of the essential fact that equivalent fractions arise with the change of units.

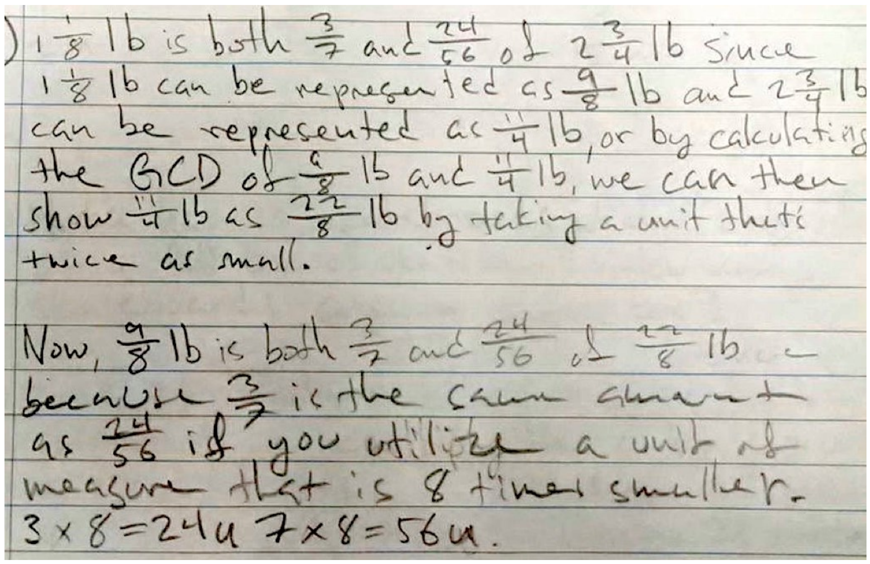


Figure 5.18. FT17's solution contains several inconsistencies in language with respect to both general mathematical terminology and FoQ theory

5.3.3.2 Interpretation 2: A relation between pairs of abstract numbers

Nine students interpreted the given statement as the task of demonstrating a relation between two pairs of abstract numbers. As we will show below, this interpretation does not necessarily mean the equality of the fractions $\frac{3}{7}$ and $\frac{24}{56}$, as we would think of it in mathematics; it is not clear that whether students in this group conceived of these expressions as representing *one number*. They were indeed preoccupied with numerical rather than quantitative operations and relations, but sometimes their understanding of abstract numbers was not correct, even from the perspective of common content knowledge (Ball, Thames, & Phelps, Content Knowledge for Teaching: What Makes It Special?, 2008). We saw, for example, the common error, associated with the whole numbers obstacle signaled in the literature about children's understanding of fractions (Streefland, 1993), to think of $\frac{3}{7}$ as "smaller" than $\frac{24}{56}$ because each of the numbers in the first pair is smaller than the numbers in the second pair. Students did perform the typical procedures for simplifying fractions, as if to demonstrate the equality $\frac{3}{7} = \frac{24}{56}$ but some solutions didn't even include this equality but rather equalities involving whole numbers such as $3 \times 8 = 24$ and $7 \times 8 = 56$.

The theory of fractions of quantities was not used to solve the problem by any of the students in this group. Although the definition appeared frequently in some form in their solutions, it seemed to play the role of a ritual that had to be performed rather than a means of control of validity. The language was less coherent, and ungrammaticalities – both with respect to natural language and mathematical accepted usage – were much more frequent than in the previous group.

5.3.3.2.1 Problem: 3/7 is the same as 24/56

The problem, for this group, was to show that $\frac{3}{7}$ is "the same" as $\frac{24}{56}$. "The same" does mean "equals" here, not only because some students omitted the equals sign, but also because there is evidence that they didn't think of the two fractions as two numbers that are equal, but rather as four numbers satisfying some relationships. In the example below (Figure 5.19), belonging to FT3, the equality of the two fractions, as we understand it in mathematics, although declared, is clearly not grasped: $\frac{3}{7}$ is 8 times smaller than $\frac{24}{56}$.

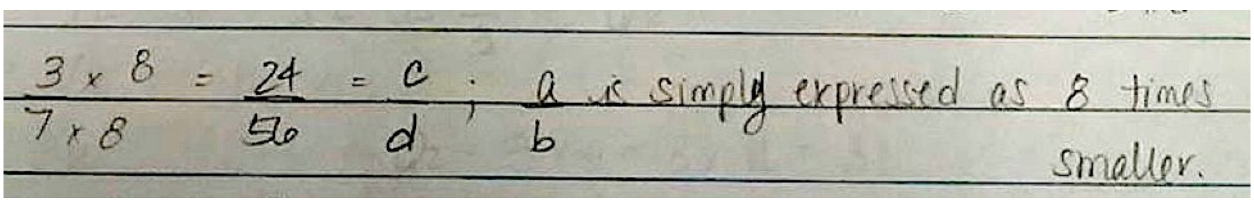


Figure 5.19. FT3 writes the equal sign between the two fractions, then writes that one is 8 times smaller than the other.

In the next example (Figure 5.20), the student mentions the common factor of 8, as if simplifying fractions, but again, it is not clear if she sees them as equal numbers – she does not write the equal sign between the two fractions.

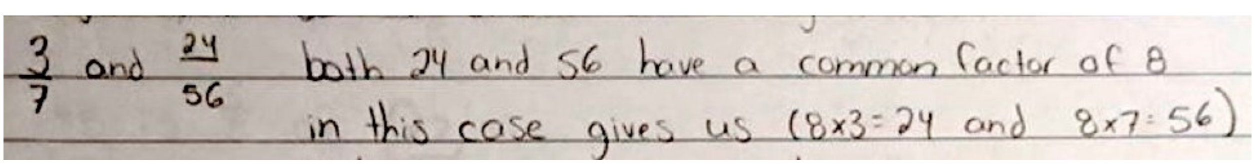


Figure 5.20. FT12 multiplies both numerator and denominator by 8 as if to justify the fractions equivalence numerically, but does not write the equals sign between the two fractions.

5.3.3.2.2 Strategies and validity

To solve the problem most students (8) used variants of a fraction simplification procedure. They noticed a common factor of numerator and denominator as in the examples above or they performed successive simplifications, such as shown in Figure 5.21.

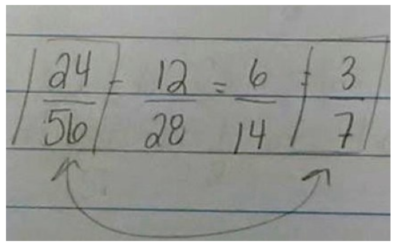


Figure 5.21. FT13 justifies fractions equivalence numerically using successive simplifications

One student suggested a cross-multiplication (Figure 5.22)

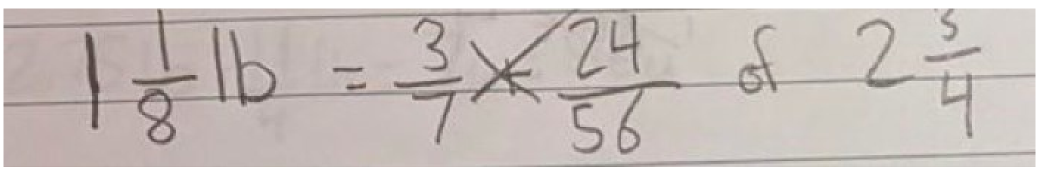
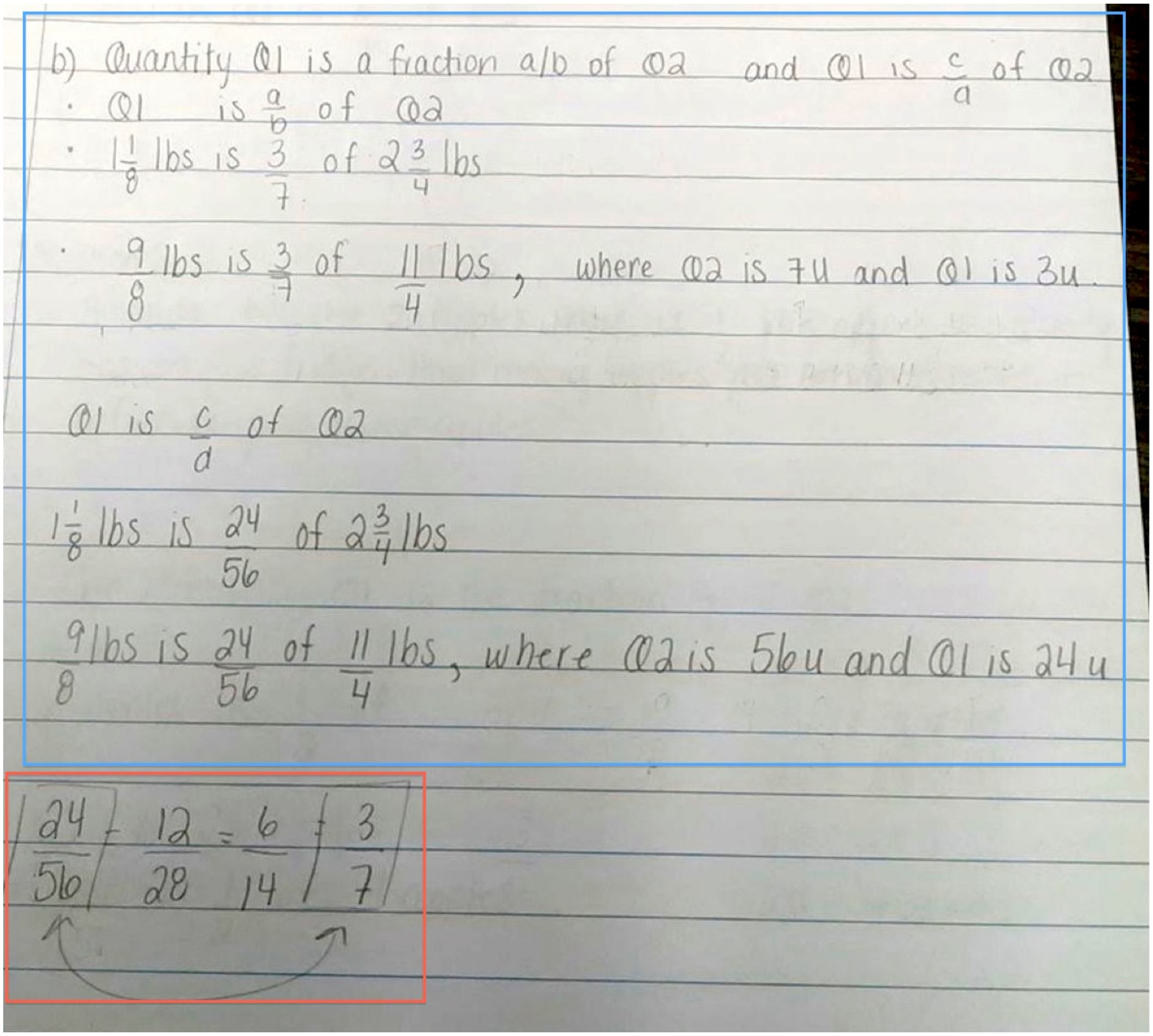


Figure 5.22. FT13 justifies fractions equivalence numerically by suggesting "cross-multiplication."

With one exception, however, these procedures were amply accompanied by bits and pieces of the FoQ theory. This was an important characteristic of all the justifications in this group: a numerical justification embellished with a piece of theory. In some cases, the disconnect between the two was striking, like in the case of FT13's solution, quoted in full below (Figure 5.23). The student started by breaking the given statement into $S1$ and $S2$, and then substituting the given values of a, b, c, and d in the main *definiens* conditions. However, she did not perform the complete substitution of the given values: she left the variables $Q1$ and $Q2$ when writing these conditions, instead of replacing them with the given values $1\frac{1}{8}\ lb$ and $2\frac{3}{4}\ lb$. The passage to a numerical statement of equality of fractions undergoing successive simplifications is abrupt (the quantitative justification in the upper frame , and the numerical one in the bottom frame); there is no attempt to link the two.



b) Quantity Q1 is a fraction a/b of Q2 and Q1 is $\frac{c}{d}$ of Q2

- Q1 is $\frac{a}{b}$ of Q2
- $1\frac{1}{8}$ lbs is $\frac{3}{7}$ of $2\frac{3}{4}$ lbs
- $\frac{9}{8}$ lbs is $\frac{3}{7}$ of $\frac{11}{4}$ lbs, where Q2 is 7u and Q1 is 3u.

Q1 is $\frac{c}{d}$ of Q2

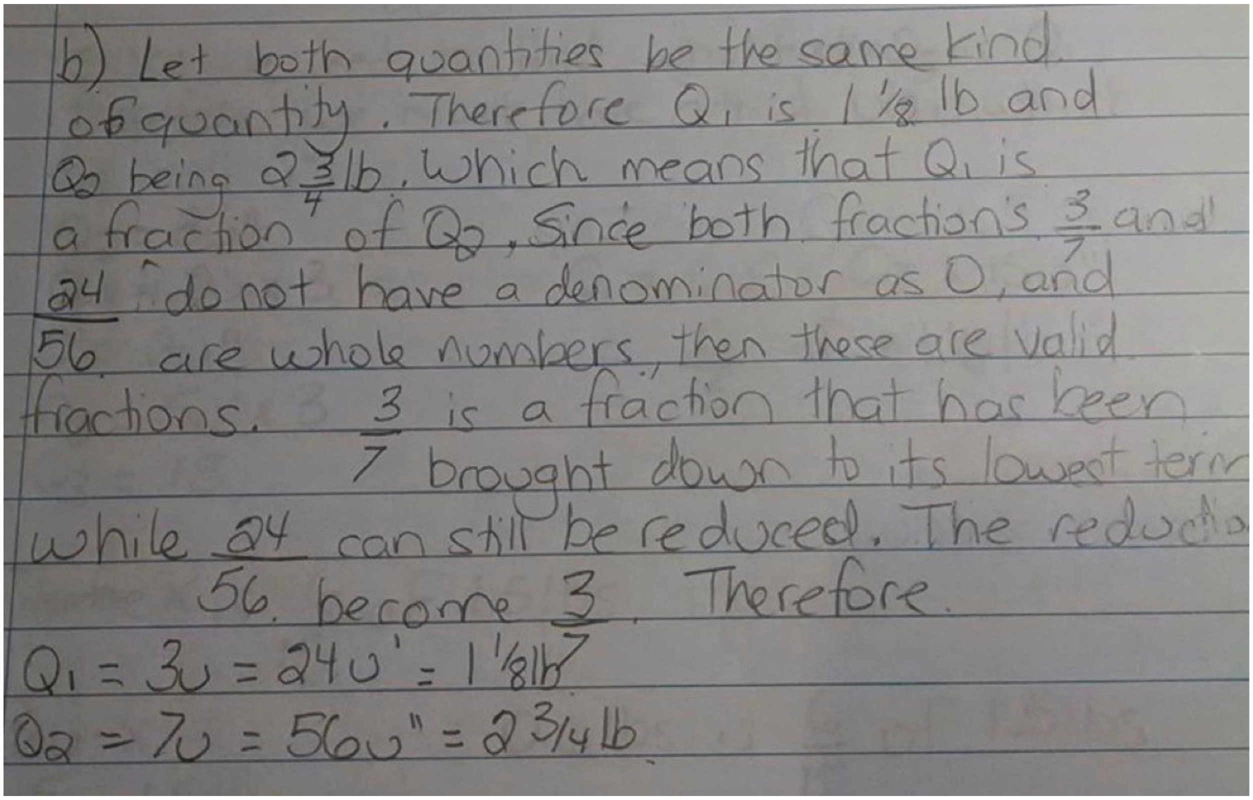
$1\frac{1}{8}$ lbs is $\frac{24}{56}$ of $2\frac{3}{4}$ lbs

$\frac{9}{8}$ lbs is $\frac{24}{56}$ of $\frac{11}{4}$ lbs, where Q2 is 56u and Q1 is 24u

$\frac{24}{56} = \frac{12}{28} = \frac{6}{14} = \frac{3}{7}$

Figure 5.23. FT13's solution contains elements of FoQ theory (in the part framed in blue) as "decoration" to the numerical justification of fractions equivalence (in the part framed in red)

An interesting behavior for ascertaining validity, portrayed in FT8's solution included below (Figure 5.24) was to invoke the secondary definitional conditions, such as the requirement to have quantities of the same kind or to have whole numbers as numerator and denominator; the given fractions are therefore "valid" for the student whose solution we include below. The numerical justification is then interposed: $\frac{3}{7}$ is in a "reduced" form, contrary to $\frac{24}{56}$. This, in turn, serves to ascertain another pair of equalities stemming from the definition.


b) Let both quantities be the same kind of quantity. Therefore Q_1 is $1\frac{1}{8}$ lb and Q_2 being $2\frac{3}{4}$ lb. Which means that Q_1 is a fraction of Q_2, Since both fraction's $\frac{3}{7}$ and $\frac{24}{56}$ do not have a denominator as 0, and are whole numbers, then these are valid fractions. $\frac{3}{7}$ is a fraction that has been brought down to its lowest term while $\frac{24}{56}$ can still be reduced. The reductio $\frac{24}{56}$ become $\frac{3}{7}$. Therefore.

$Q_1 = 3u = 24u' = 1\frac{1}{8}$ lb

$Q_2 = 7u = 56u'' = 2\frac{3}{4}$ lb

Figure 5.24. FT8's solution: validation of secondary definitional conditions

While it is possible that FT8 saw the connection between the procedure of fraction reduction and the measurements of the given quantities by means of two different units as expressed in the last two lines of the solution, this insight is not captured in writing.

Almost all the students in the group employed theory as a form of prose that had to embellish the numerical justification of fractions equivalence: they either simply recited the definition, *plugged* in some of the given values, or even calculated the units, but didn't draw any conclusions from such purely syntactic operations.

One of the students in the group, however, merged the theory and the numerical justification in a less obvious way, giving us some hints on how some students understand the elements of the fraction of quantity definition. Her solution is reproduced below, in full (Figure 5.25). She argued that "the fractions are the same amount" – a mistake we have seen before of conceiving of the fraction as a quantity, rather than a relationship between quantities. The explanation that follows sheds some light on her definition-based conception of fraction of quantity: the numerator of the fraction is a quantity, and the denominator is the other quantity. While the student does make the important remark that the different units of measurement are at stake, the letters a and b don't stand for the numbers of units, but for actual quantities: the fraction $\frac{a}{b}$ is $\frac{3u}{7u}$ "and" $\frac{24u_1}{56u_2}$ (the units probably "cancel out" – an algebraic operation). The equivalence is then explained in a rather numerical way: both numerator and denominator are multiplied by 8.

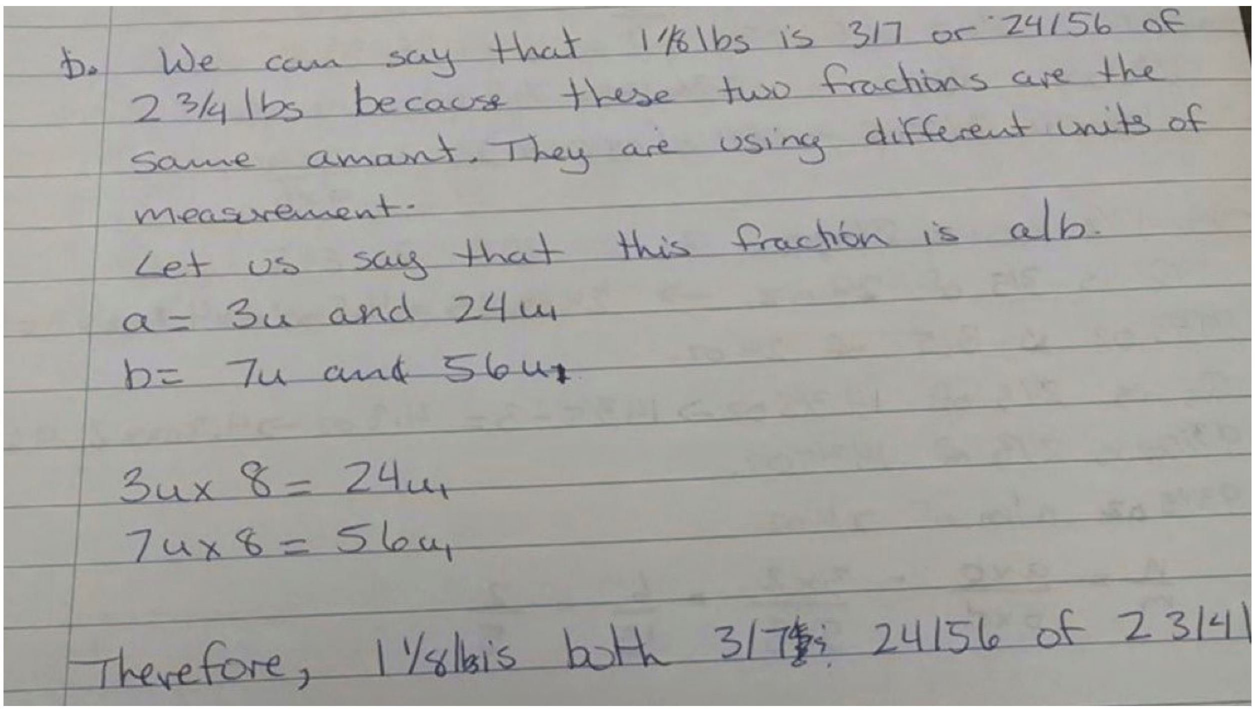
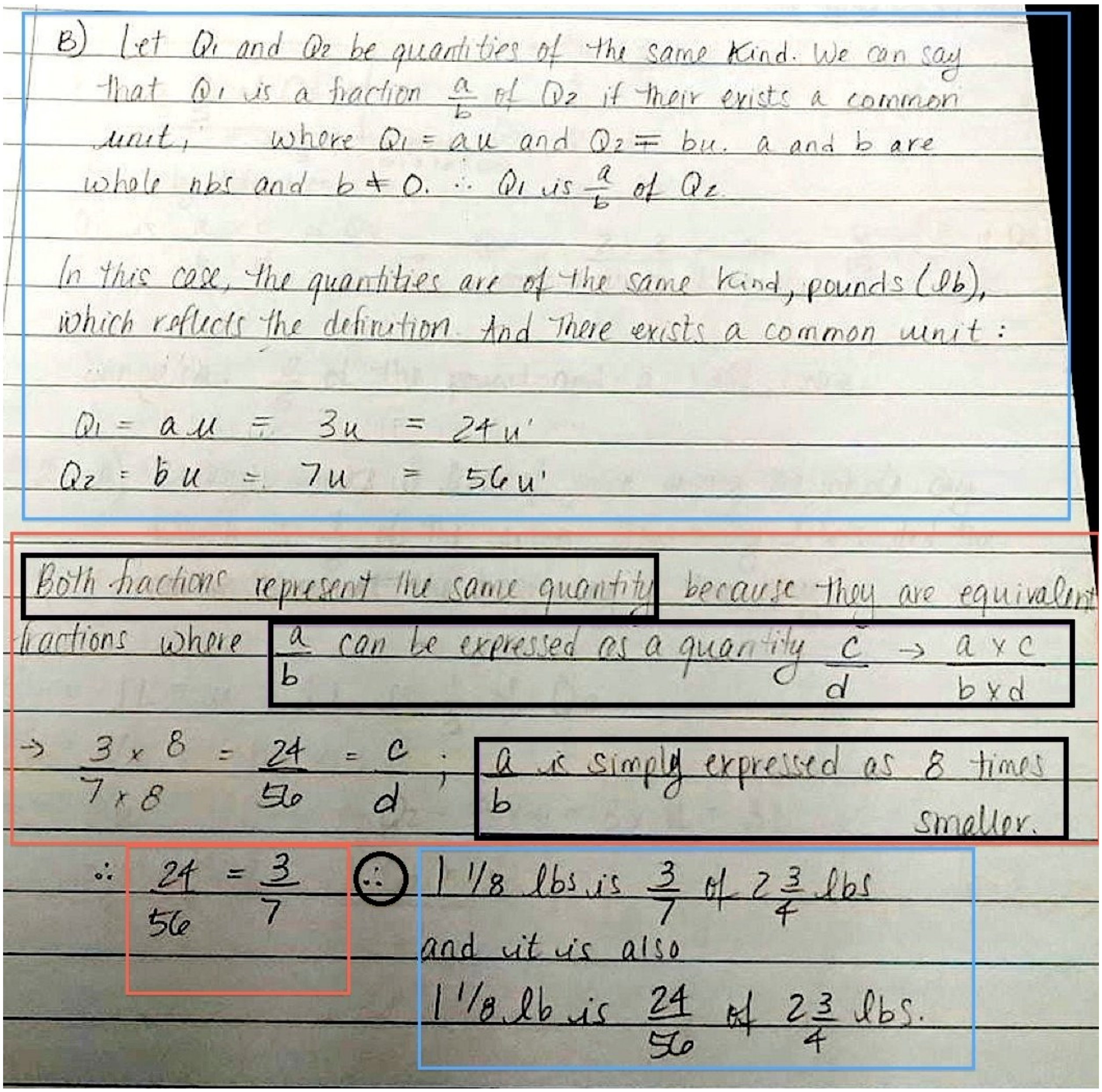


Figure 5.25. FT24's solution

5.3.3.2.3 Language

There were many more language inconsistencies in this group compared the group that produced Interpretation 1. The most salient one reflected the discontinuity in reasoning between the recitation of some piece of theory and a numerical justification. A typical such chain of reasoning was: matching some variables from the definition with the given values, simplifying the fractions numerically, stating that either S or $S1$ and $S2$ are true (sometimes this conclusion was not explicitly stated). These distinct pieces of the argument are linked by connectives such as "therefore", "since", "which means that" or symbols, such as "→" or ∴, as if they follow from one another, although that is never the case. We annotated the solution in

Figure 5.26 to symbolize this succession. We framed in blue the quantitative statements: the first rectangle is matching the statement to the definition, the second is the reformulation of S as $S1$ and $S2$. The red frame contains the numerical justification. We framed in black inconsistencies in language.



B) Let Q_1 and Q_2 be quantities of the same Kind. We can say that Q_1 is a fraction $\frac{a}{b}$ of Q_2 if their exists a common unit, where $Q_1 = au$ and $Q_2 = bu$. a and b are whole nbs and $b \neq 0$. $\therefore$ Q_1 is $\frac{a}{b}$ of Q_2.

In this case, the quantities are of the same Kind, pounds (lb), which reflects the definition. And there exists a common unit:

$Q_1 = au = 3u = 24u'$

$Q_2 = bu = 7u = 56u'$

Both fractions represent the same quantity because they are equivalent fractions where $\frac{a}{b}$ can be expressed as a quantity $\frac{c}{d} \rightarrow \frac{a \times c}{b \times d}$

$\rightarrow \frac{3 \times 8}{7 \times 8} = \frac{24}{56} = \frac{c}{d}$; $\frac{a}{b}$ is simply expressed as 8 times smaller.

$\therefore \frac{24}{56} = \frac{3}{7}$ $\therefore$ $1\frac{1}{8}$ lbs is $\frac{3}{7}$ of $2\frac{3}{4}$ lbs
and it is also
$1\frac{1}{8}$ lb is $\frac{24}{56}$ of $2\frac{3}{4}$ lbs.

Figure 5.26. FT3's solution contains elements of FoQ theory (in the part framed in blue) as "decoration" to the numerical justification of fractions equivalence (in the part framed in red); language inconsistencies are framed in black

The student starts by reciting the definition and then by validating a definitional condition that, although not unessential, is not very relevant in the context of proving the given statement. She replaces only the given values of a, b, c, and d in the main definitional conditions, leaving those of $Q1$ and $Q2$ out. "There exists" means, for her, the writing of u and u', rather than the production of two concrete units. *The definition has been used*, as per the requirement of the Question! Now she moves to the numerical register, not without linguistic mistakes (framed in black): with respect to FoQ theory ("fractions represent the same quantity"), algebraic notation (that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent fractions is symbolized by $\frac{a\times c}{b\times d}$), and even numeracy ("$\frac{a}{b}$ is simply expressed as 8 times smaller" – the fraction is not conceived as one abstract number, but as two numbers, indeed smaller). Finally she *deduces* – see the circled symbol ∴ – that the given quantitative statement, rephrased as $S1$ and $S2$, is true.

Such incoherence in language, as expected, represents more than "bugs" that can be fixed by using the right kind of connective, a different arrangement of phrases, or more rigorous mathematical notation. This is not clumsy writing, but rather the outward representation of the student's concepts and the logical relations that govern them.

Figure 5.27 and Figure 5.28 present an example of a linguistic inaccuracy that is not just careless writing. When reading a student's solution (when marking, for example) we would be quite forgiving of her use of the word "and" instead of the equal sign, as in FT24's solution previously quoted (Figure 5.27).

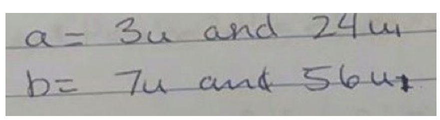


Figure 5.27. FT24 uses "and" instead of "="

It is ungrammatical (with respect to mathematical language), but readers in teaching and learning contexts would probably not be too concerned about it. Yet, the equations that follow (Figure 5.28), signal that the misuse of the equal sign runs deeper than a mistake in representation; they contradict the previous statements ($24u_1$ cannot be *equal to* $3u$ and $3u \times 8$):

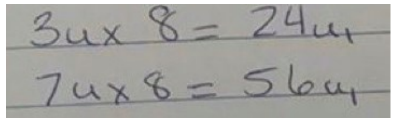


Figure 5.28. FT24: next sentences in contradiction with the previous one

As we hypothesized in the previous quotation of this solution (Figure 5.25), this way of writing, rather than being just a slip in notation points to student's understanding of the $\frac{a}{b}$ in the definition as an algebraic fraction $\frac{3u}{7u}$, which "becomes" $\frac{24u_1}{56u_1}$ when both numerator and denominator are multiplied by 8; the units u and $u1$ are variables that cancel out. We get a hint of this understanding of the unit as a common factor of the numerator and denominator in FT4's solution (Figure 5.29), when she writes that 8 is the common unit (framed in black):

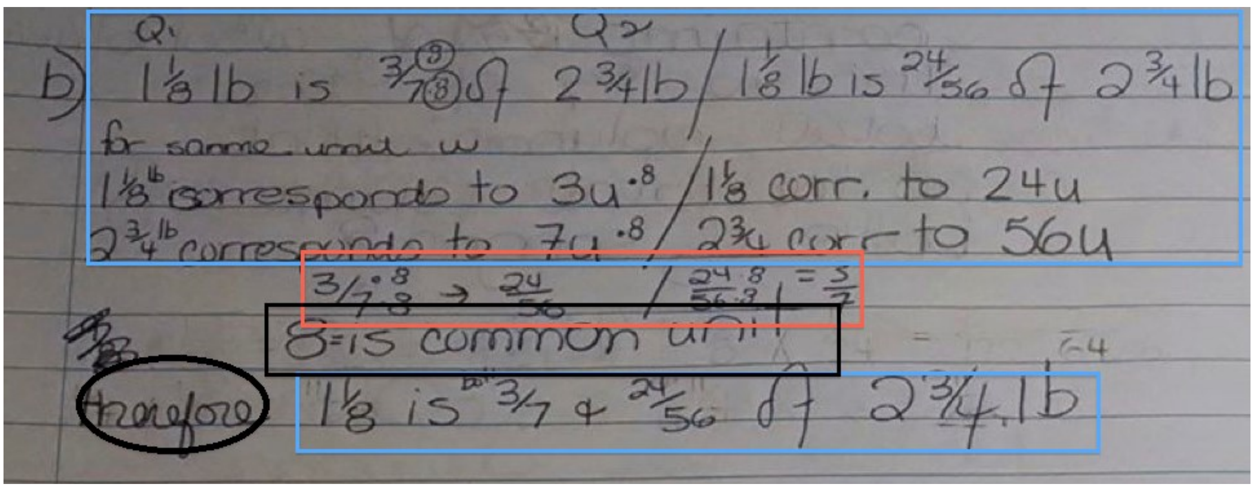


Figure 5.29. FT4's solution

Just as in the previously analyzed solution the student starts by matching the given statement to the definition – this time distinguishing, from the beginning, between $S1$ and $S2$ and plugging in also the given values of $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ – but swiftly moves to the numerical justification: multiplies then divides the numerators and the denominators by 8. The latter is what ultimately serves to justify S.

5.3.3.3 Interpretation 3: Two sentences about fractions

The six students to which we attributed this interpretation of the given statement approached $S1$ and $S2$ separately in their solutions. The solutions read as if the students were checking the truth value of the two statements separately. However, none of the students in this group found a contradiction in either $S1$ or $S2$, and thus they all proceeded to "check" the other sentence, in an identical fashion. It is thus not clear whether the students in this group even doubted the truth of these statements, and thus their checking amounted more to performing some of the operations expected when applying the definition of FoQ. When a contradiction arose, the students in this group did not appear to reflect on it – at least in writing, and mostly ignored it. Moreover, there is no indication that they saw the relationship between the two statements, i.e., that the given quantities are the same in both and that their relationship can be represented differently by two fractions – through the two sentences – by choosing a larger or smaller unit. Thus, their interpretation is not of a conjunction of statements, but of two completely distinct sentences (i.e., for them, the given values of $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ may well have been different between $S1$ and $S2$). Finally, despite resorting to the typical procedures for verifying the statement according to the FoQ definition – e.g., dividing $\boldsymbol{Q1}$ by a to get $\boldsymbol{u}$, multiplying the $\boldsymbol{u}$ by b to get $\boldsymbol{Q2}$ – there is evidence to suggest that the operations they performed belonged to the numerical rather than the quantitative domain. The statements were about fractions as abstract numbers, rather than about fractions of quantities.

5.3.3.3.1 Problem: 1 1/8 lb is 3/7 of 2 ¾ lb. 1 1/8 lb is 24/56 of 2 ¾ lb.

All six students in this group parsed the given statement as two separate sentences – we called them $S1$ and $S2$ – each of them having the structure $\boldsymbol{Q1}\ is\ \frac{a}{b}\ of\ \boldsymbol{Q2}$. This would be "task zero" when the definition of fraction of quantity is introduced: showing that a quantity is a fraction of another quantity. We recognized the students' intention to solve two such problems in that they wrote separate arguments, usually in two columns, as if the two statements didn't share the given values of $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$. Also, with one exception, all the students approached both $S1$ and $S2$ with the typical procedure learned in class for solving a task of this type: finding the unit by dividing one of the given quantities by the numerator of the fraction, then multiplying the result by the denominator to get the other quantity. The exception was a student who set out to "solve" $S1$, but was unsuccessful in even finding a unit.

5.3.3.3.2 Strategies and validity

In theory, proving $S1$ and $S2$ separately is a valid approach for establishing that their conjunction is also true. Formally, one needs not find the relationship between the units corresponding to $S1$ and $S2$ in order to validate both of them, and consequently, their conjunction. But the students in this group did not seek to determine whether these statements are true or false (no question mark in the ***Problem*** heading above), but rather, in their words, *to solve* them in the sense of: to do with them actions perceived as the expected response to sentences with this particular structure: $\boldsymbol{Q1}\ is\ \frac{a}{b}\ of\ \boldsymbol{Q2}$.

FT28, whose solution we reproduce below (Figure 5.30) even writes this sentence as a model at the beginning of her work (framed in black). Then, the equals sign in the main definitional conditions does not carry hypothetical meaning (circled in red). The student divides the given value of $\boldsymbol{Q2}$ by that of b to find the unit, and then multiplies the result by the value of a to get the expected $\boldsymbol{Q1}$. This would be a correct approach, as long as the second operation were carried for verification. But it turns out that the interpretation of the equation $\boldsymbol{Q1} = a\boldsymbol{u}$ as the interrogative statement *Is $\boldsymbol{Q1}$ equal to $a\boldsymbol{u}$?* is more subtle and elusive for students than we had expected (especially since the other definitional condition is implicitly validated by the division). The student writes $Q1 = 3u = 3 \times \frac{11}{28} = 1\frac{1}{8}$, but does not notice that $3 \times \frac{11}{28} \neq 1\frac{1}{8}$. The procedure is carried identically for $S2$, and with the same fundamental flaw: the equality $24 \times \frac{11}{224} = 1\frac{1}{8}$ is not questioned.

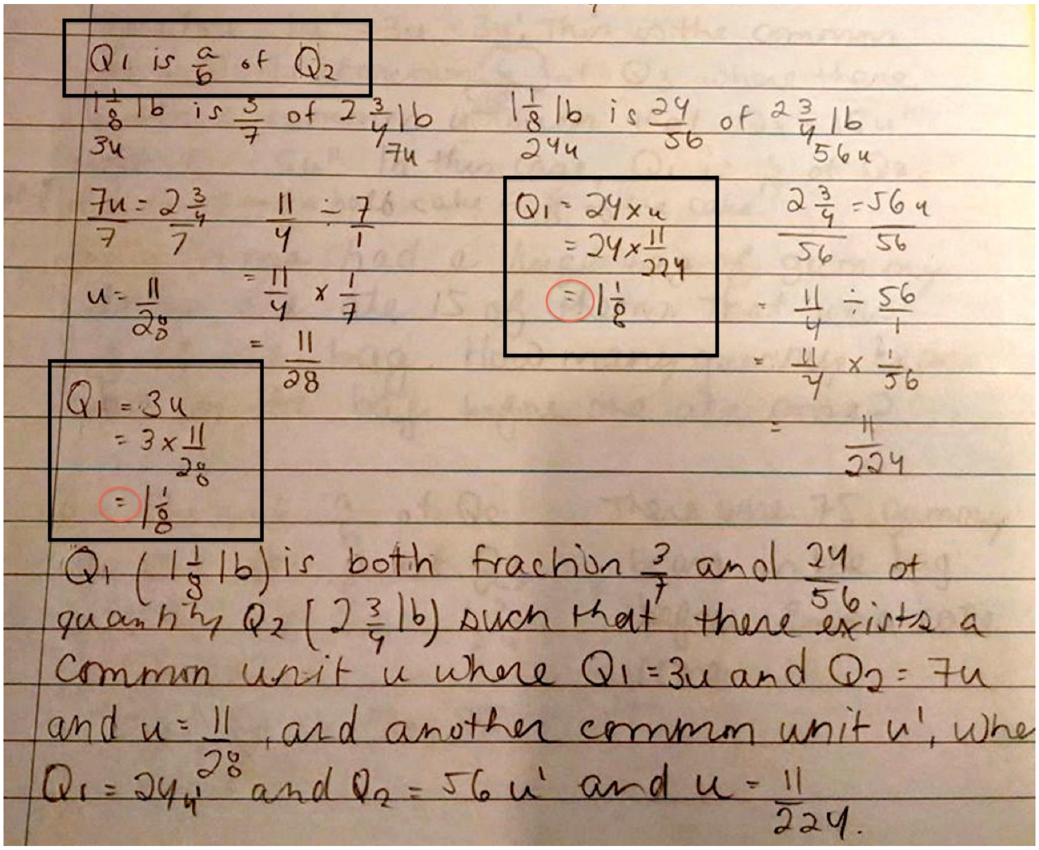


Figure 5.30. FT28's solution

Another student (FT11, in Figure 5.31) had an almost identical strategy, with a slight variation: she first converts the given quantities from pounds to kilograms ($0.51\ kg\ and\ 1.25\ kg$, respectively), but then, like FT28, she calculates the unit from one definitional equation by division: $1u = 0.51\ kg \div 3 = 0.17\ kg$, and appears to multiply it in the form of the second definitional equation, $7 \times u = 1.25\ kg$, without actually questioning the truth of the latter.

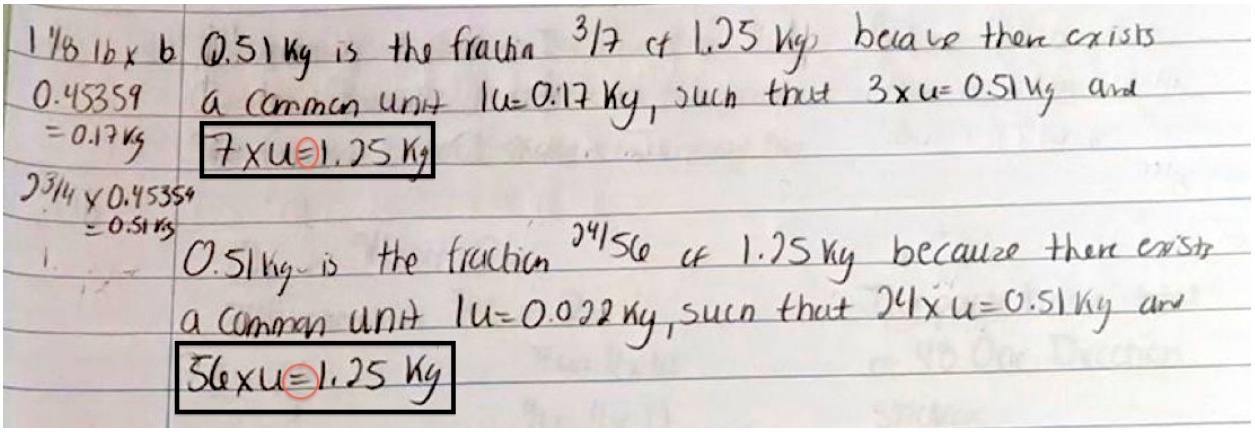


Figure 5.31. FT11's solution.

Like FT28, whose solution we quoted in Figure 5.30 just above, FT11 didn't actually perform the multiplication she has written; had she done that, she would have noticed that $7 \times 0.17 \neq 1.25$. It is somewhat troublesome that these solutions could pass as correct if the given statement were true. They speak not only of the need to address such lack of hypothetical thinking in students, with regard to mathematical statements, but also of the immense challenge, for educators, of producing tasks that reveal this kind of deep misunderstandings of theoretical concepts.

The three remaining students had a similar strategy in that they calculated the unit by division from one definitional equation, but then actually did multiply it *as if* to check the other definiens equation. None of them, however, disproved the given statement, or either of $S1$ and $S2$, based on such a calculation which didn't yield the expected quantity. For them, too, the task was to carry out the procedure associated with this type of statement, rather than to check the truth of the statement. I quote below a solution where this approach is very visible, with operations on abstract fractions being carried out almost blindly as procedures with no control structure (FT26, in Figure 5.32). First the student calculates a unit $\boldsymbol{u}$, by dividing $\boldsymbol{Q1}$ by a, i.e., $\frac{9}{8} \div 3$. Rather than directly noticing that the result is $\frac{3}{8}$ (nine eighths divided by three), she does the more mechanical multiplication with the inverse, $\frac{1}{3}$, to get $u = \frac{9}{24}$. When multiplying this u by the given b, she doesn't get the given $Q2$, i.e., $2\frac{3}{4}$, but $2\frac{3}{8}$. This inconsistency is of no consequence in the solution. She moves to applying the same procedures to $S2$, but it's exactly this empty manipulation of numbers that produces not only the same contradiction as above, but also a series of other confusions (all framed in the excerpt below). She first calculates the unit to be $\frac{9}{192}$ or $\frac{3}{64}$ by means of the same "inverse and multiply" technique. But then she also writes $u = \frac{9}{216} = \frac{1}{24}$ and $u = \frac{9}{8}$, so, in all, three different values of u (not to count the unit for S1, where the same letter u is used, as a tag). She multiplies the first such value $u = \frac{3}{64}$ by 3, to get $Q1$, as if forgetting that this is how she got it in the first place (by dividing $Q1$). The last result, now $u = \frac{9}{8}$, is multiplied by 7, to get $Q2$, but the resulting inconsistency goes unnoticed again. Throughout the solution, the standard unit, in this case pounds, is absent, giving the impression of an entirely numerical approach, with operations performed on abstract numbers rather than on fractions of quantities. Yet, a closer look reveals also a not so strong sense of number, in the way the operations are carried out as repetitious procedures, with no intuition for shortcuts or memory of previous results (notice, for example, how the student cancels the 1's when multiplying fractions).

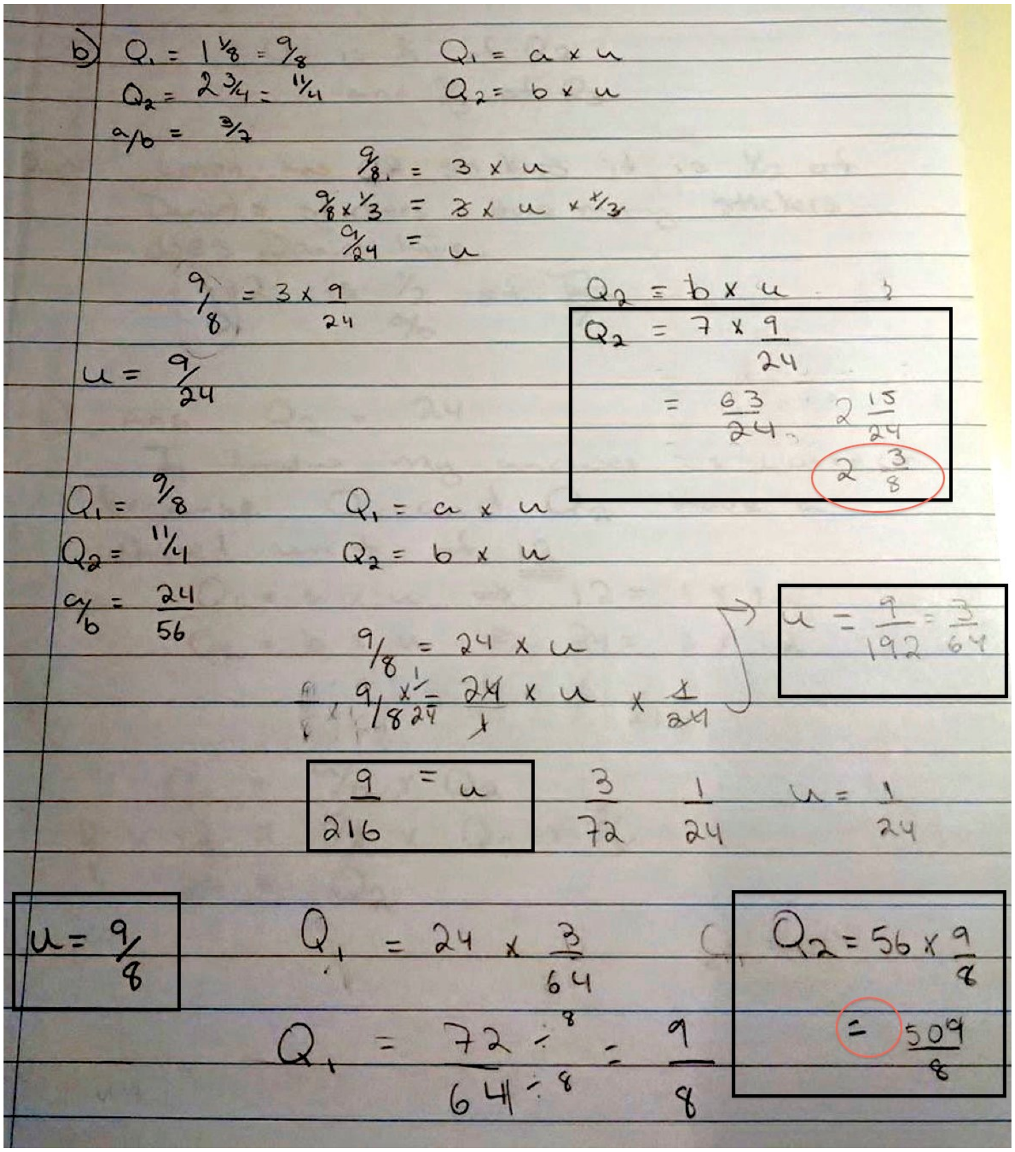


Figure 5.32. FT26's solution

The other two students in this group also found a unit by division from one of the main definitional conditions, but they neither wrote the second equation as if it was true without multiplying (like the FT28 and FT11), nor completely ignored the result of the multiplication (like FT26). Instead they got results that were close to the expected quantities, and explicitly accepted them as “good enough.” For example, FT19, whose first part of the solution is reproduced below (Figure 5.33) is comfortable writing that the unit is *about* $6.2\ oz$ – the quantity she obtained by dividing $44\ oz$ by 7, in the case of $S1$. It makes sense that when checking whether $3 \times u = 18\ oz$ she is willing to accept $18.6\ oz$ as an approximate result. The same rough results are accepted in the case of $S2$.

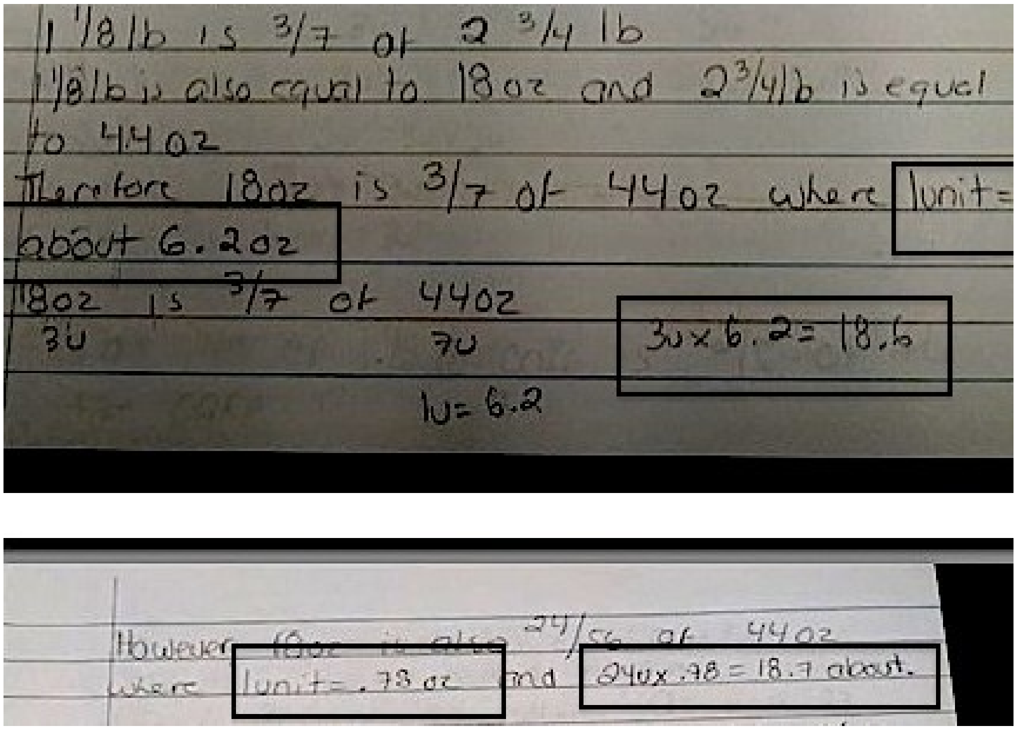


Figure 5.33. FT19's solution – first part.

To validate the statement within relatively small margins of error, especially while acknowledging them as this student did, is more desirable than just writing the required operations without reflecting on their results, or perhaps not performing them altogether. But this student also seeks validation in her common knowledge of fractions in the second part of the solution (Figure 5.34), and she does it by abruptly moving to a numerical justification that is not connected to the quantitative work she had just done. The only remnant is the use of the word “unit”, now assigned a different meaning, that of common factor of numerator and denominator.

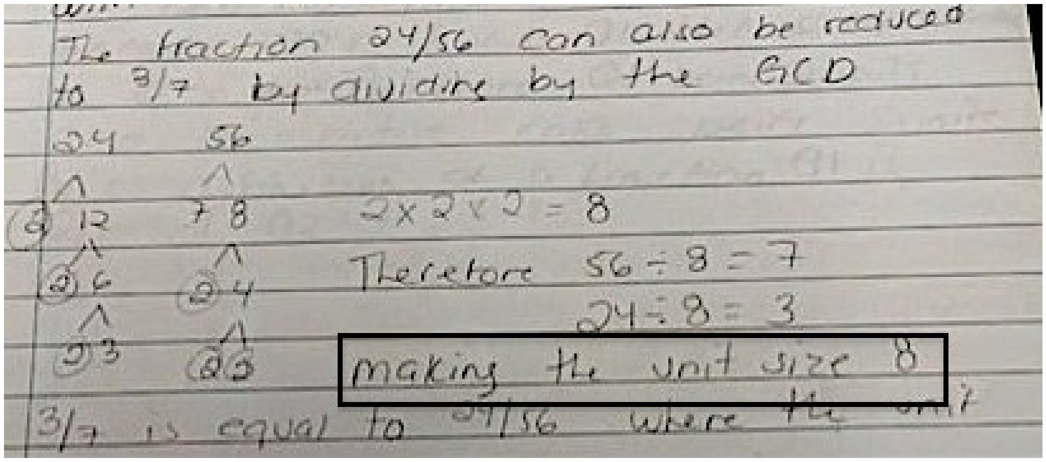


Figure 5.34. FT19's solution - second part.

The lack of number sense is apparent in FT19's solution above, as well: the so-called factorization trees would not be necessary for relatively small numbers such as 24 and 56; the factors can be directly derived from the multiplication table up to 10, expected to be learned by all children at school. Moreover, the numbers 3 and 7 are strong hints for the sought greatest common divisor. However, it may also be that the student uses this "prop" not so much because she needs it in order *to find* the greatest common factor, but because she thinks this is the legitimate procedure to demonstrate in her proof of the statement – a habit she may have acquired in school.

The other student in this group who also accepted approximate results, FT15 (Figure 5.35), maintained a quantitative stance for the entire solution, although the use of decimals, like in the above solution, is not warranted by the FoQ theory without further justification (i.e., one "does not know" what $6.2\ oz$ is at this point). After changing the given quantities from mixed numbers to improper fractions of a pound and bringing them to the same denominator, the student eliminates the common denominator and rewrites the statement preserving only the numerators of the given fractions of a pound. This is a procedure that many students are quite familiar with from the practice of solving equations – usually linear, in one variable – that contain fractions. They were taught to bring to the same denominator and then to get "rid" of it – operating with fractions is to be avoided. Few of them, however, understand this last step as the operation of multiplying both sides by the common denominator. In this context, the procedure of “getting rid of the denominator” is actually valid: if both quantities are increased by the same factor, then the multiplicative relationship between them, expressed as a fraction, is the same. The method is even quite effective: it is easier to deal with whole numbers of pounds instead of fractions of pounds. But when the justification of the procedure is not given – and likely not internalized either – it remains just that: a procedure with no anchoring in real understanding of multiplicative relationships between quantities (or numbers, for that matter).

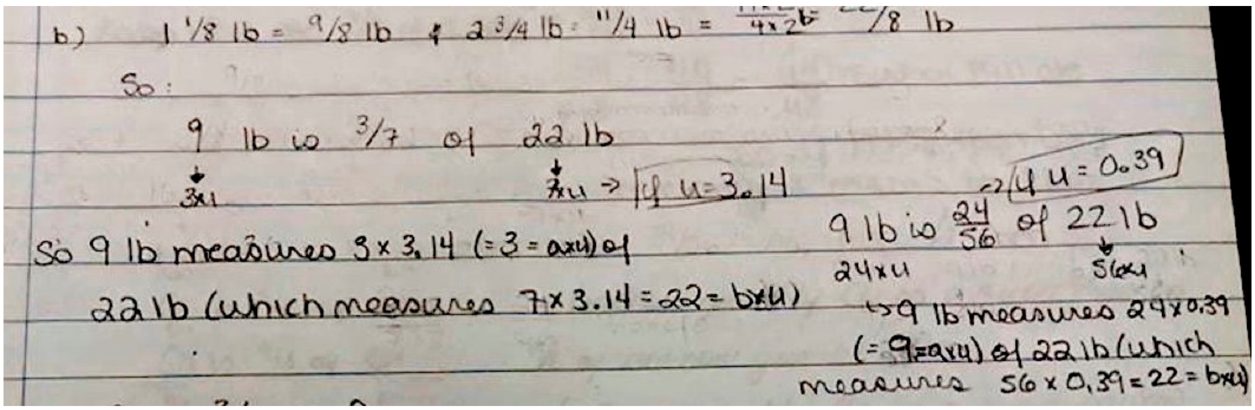
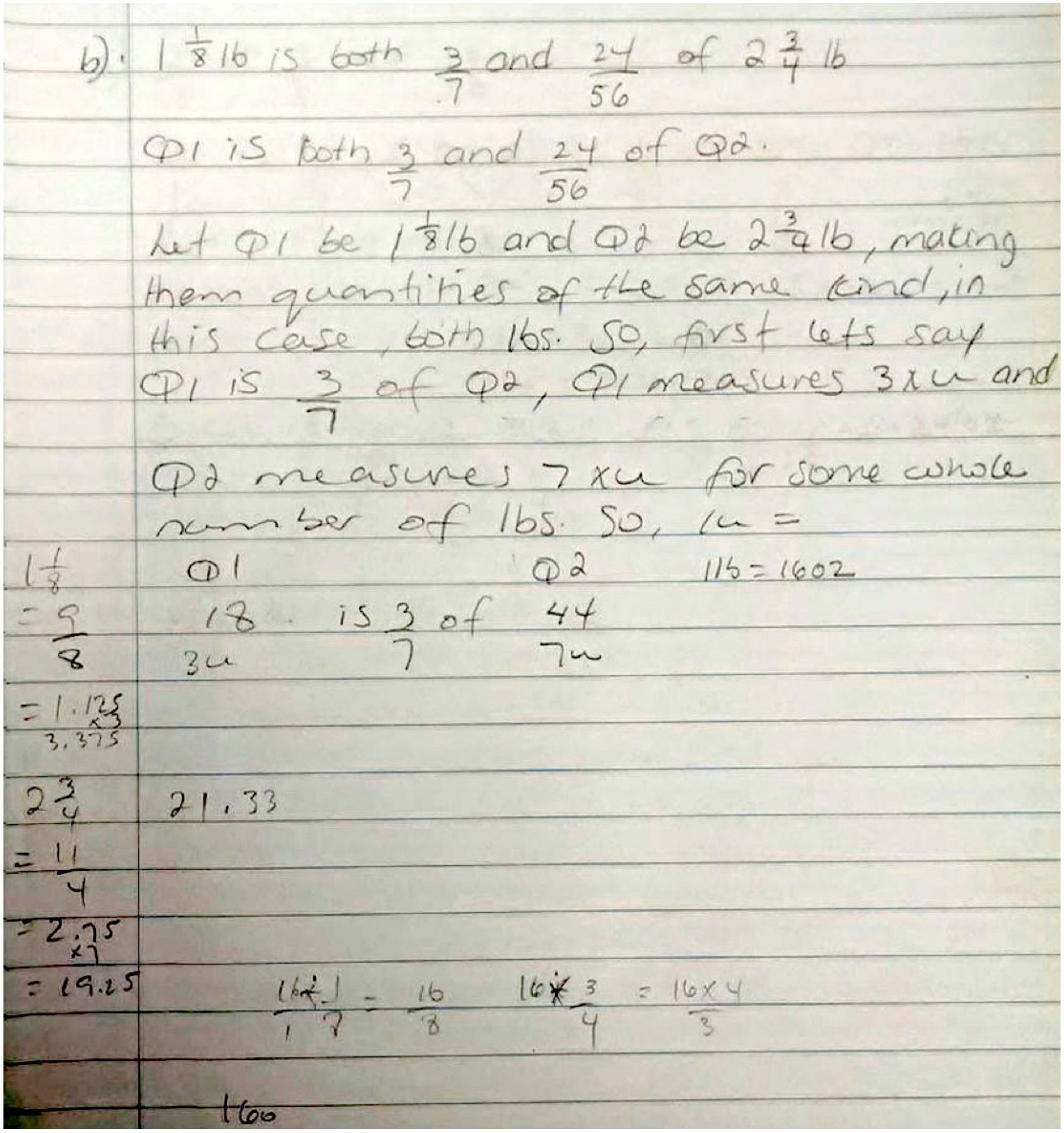


Figure 5.35. FT15's solution.

Finally, an outlier in this group, FT9 (Figure 5.36), although processing the given statement the same way as his peers – two separate sentences to be “solved” – was unsuccessful in finding the unit for the sentence he first tackled (he does have the intention to solve both: “So, *first* let’s say…). The effort to use the definition really stands out in his solution, but so is the barrier he faces when it comes to make it operational.

The student first rewrites the given statement, replacing the particular quantities $1\frac{1}{8}\ lb$ and $2\frac{3}{4}\ lb$, with the letters $Q1$ and $Q2$, as if to match it to the known model (he further writes: “Let $Q1$ be $1\frac{1}{8}lb$, and $Q2$ be $2\frac{3}{4}lb$”). $Q1$ and $Q2$ are both measured in lbs; therefore, he concludes, they are quantities of the same kind (incorrectly identifies the “kind” as lbs, instead of weight) .

He then focuses on the statement that $Q1\ is\ \frac{3}{7}\ of\ Q2$, and proceeds to writing $3u$ under $Q1$ and $7u$ under $Q2$ correctly aiming at finding $1u$: “$1u\ = \cdots$”. It is not clear whether he grasps the meaning of this schematic organization of the information commonly used in class: does he understand that $3\boldsymbol{u}$ written under $\boldsymbol{Q1}$ means that $\boldsymbol{Q1}$ is equal to 3 times $\boldsymbol{u}$? We ask this question because he appears to get stuck here, although the conversion to ounces should have made the unit quite visible if he had correctly grasped this representation. He does several other things, but does not succeed in finding u. On the side, he changes the given quantities to improper fractions and then to decimal numbers by division of numerator with denominator: he thinks the results of the division are the sought-after $\boldsymbol{u}$, as evidenced by the fact that he multiplies the results by 3 and 7, respectively, which does not seem to validate anything for him. Finally, he performs a multiplication and division of abstract fractions, it's not clear why: $\frac{16}{1} \times \frac{1}{8} = \frac{16}{8}$ and $16 \div \frac{3}{4} = 16 \times \frac{4}{3}$. His numerical calculations on the side don't seem to alleviate the obstacle he encountered in operating with the definition, even in terms of obtaining some numerical consistency to serve as validation – as such calculations did for other students. He left the problem unsolved.



b). $1\frac{1}{8}$ lb is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}$ lb

Q1 is both $\frac{3}{7}$ and $\frac{24}{56}$ of Q2.

Let Q1 be $1\frac{1}{8}$lb and Q2 be $2\frac{3}{4}$lb, making them quantities of the same kind, in this case, both lbs. So, first lets say Q1 is $\frac{3}{7}$ of Q2, Q1 measures 3 x u and Q2 measures 7 xu for some whole number of lbs. So, 1u =

$1\frac{1}{8}$ $=\frac{9}{8}$ $=1.125$ $\times 3$ 3.375

$2\frac{3}{4}$ $=\frac{11}{4}$ $=2.75$ $\times 7$ $=19.25$

Q1 | Q2 | 1lb = 16oz
18 is $\frac{3}{7}$ of 44
3u | 7u

21.33

$\frac{16 \times 1}{1\ \ 8} = \frac{16}{8}$ $16 \times \frac{3}{4} = \frac{16 \times 4}{3}$

160

Figure 5.36. FT9's solution.

5.3.3.3.3 Language

Students in this group generally wrote structured, somewhat coherent proofs (with the notable exception of the student who left the problem unfinished, FT9). A significant feature of their style was the organization into two almost identical argumentation threads, corresponding to $S1$ and $S2$. As before, we can hypothesize about their reasoning about the given statement from such outward manifestation: they likely processed it as two sentences (i.e., not statements whose truth value is to be established) of the type $\boldsymbol{Q1}\ is\ \frac{a}{b}\ of\ \boldsymbol{Q2}$, to be proved separately. It is not impossible that they did grasp the meaning of “and” as a logical conjunction, i.e., they understood that both $S1$ and $S2$ had to be validated, but, significantly, they didn't make anything of the fact that $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ from the structure S shared their given values in $S1$ and $S2$. While all of them calculated the units corresponding to both, none noticed their relationship, either. The students in this group operated on the propositional variables as categories (e.g., the category $\boldsymbol{Q1}$) without reflecting on their particular instantiation. We could say that they performed operations at the syntactic, rather than at semantic level. This is also evident in their reactions to the contradictions resulting from the multiplication that ensued the finding of the unit by division. The meaning attached to the sentence containing this multiplication (or lack thereof), in particular to its predicate – the equals sign – was pivotal in this groups' failure to discover that $S1$ and $S2$ are false:

- two students (FT28 and FT11) wrote the equals sign followed by the expected result – the value of $\boldsymbol{Q2}$ – after a multiplication that they didn't actually perform: $3 \times \frac{11}{28}$ is not equal to $1\frac{1}{8}$, nor is 7×0.17 equal to 1.25. (Figure 5.37) The equality sentence is a formula, or even a tautology: true regardless of the values of the propositional variables.

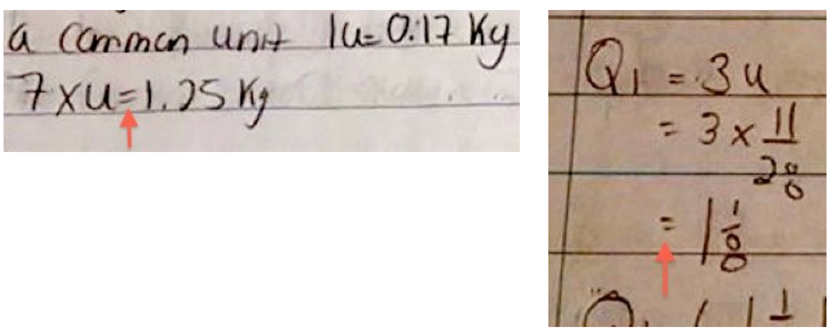


Figure 5.37. FT28's and FT11's use of the equal sign

- one student (FT26) wrote the equals sign followed by the actual result of multiplication (she didn't write the equals sign throughout the subsequent fraction simplifications). She did not compare it to the expected result, the value of $\boldsymbol{Q2}$ in this case ($2\frac{3}{4}\,lb$). The equality sentence is a procedure to be performed; the result is neither anticipated nor reflected upon. (Figure 5.38)

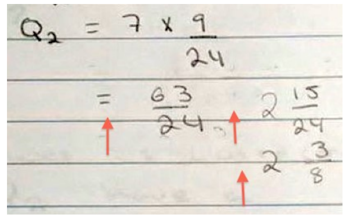


Figure 5.38. FT26's use of the equal sign

- two students (FT19 and FT15) wrote the equals sign followed by the result of multiplication which was an approximation of the expected result, and was accepted as good enough. An equality does not have to be precise. (Figure 5.39)

Figure 5.39. FT19's and FT15's use of the equal sign

- one student (FT9) didn't conceptualize the correspondence between the quantities and the multiples of units as equality. This student likely didn't understand his matching of $3u$ with $18\ [oz]$ as the equality $18\ oz = 3u$. (Figure 5.40)

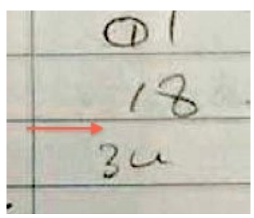


Figure 5.40. FT9's use of the equal sign

Finally, all the students in this group often forgot to write the standard units (pounds or ounces) when performing operations. This could be just a slip in writing or lack of rigor when performing rough work – this is a common phenomenon. We could attribute it to the habit of working with abstract numbers, when one forgets to pay attention to the *writing* of the units (although one is thinking of them). The definition of a fraction of quantity is, in fact, interesting and makes for a valid proving tool even when the quantities are replaced by abstract numbers. But in this particular group we have also seen some awkward calculations even when the appearance is of mastering abstract numbers (e.g. dividing $\frac{9}{8}$ by 3 by multiplying with the inverse, simplifying the $1's$ when multiplying fractions, using factorization trees for fairly "easy" numbers, 24 and 56). Thus the problems may run deeper than carelessness in writing.

5.3.3.4 Interpretation 4: A conjunction of statements about equivalence of fractions of quantities

Only five students questioned the validity of the given statement, by asking if, individually, either $S1$ or $S2$ are true. Their strategies for answering this question as well as for dealing with the resulting contradiction varied considerably. All the students in the group clearly understood both the conditional nature of the definitional conditions and the meaning of AND as a logical operator requiring that the two sentences be true. They *used* the definition to prove that the sentence is false; three of the five students in the group, however, felt the need to give, in addition, a numerical argument about the equivalence of the two fractions as abstract numbers. Of the five students, also three (not the same subset) proposed a resolution for the contradiction – by either changing the given fractions or one of the two given quantities. Their written arguments had minor glitches but the language was coherent overall, in particular with respect to the succession of ideas to produce a global argument.

5.3.3.4.1 Problem: Is 1 1/8 lb 3/7 of 2 ¾ lb? Is 1 1/8 lb 24/56 of 2 ¾ lb?

The problem perceived by the students in this group in the given statement was *to verify* whether $S1$ and $S2$ hold. I put the question mark in the title of this section to signify this group's hypothetical attitude with respect to the truth of the two statements about fractions of quantities. All the given values were relevant in studying the validity of such statements, not just those of the fractions. Clearly, this broader interpretation of the problem is the result of an understanding of fractions as relations between quantities, rather than as quantities themselves, or as pairs of numbers.

5.3.3.4.2 Strategies and validity

Two of the five students in this group, FT2 and FT5, examined the definitional conditions applied to either one or both of $S1$ and $S2$, and stated that they cannot be true: no such unit exists. These two students didn't produce the unit by division of one of the given quantities by 3, 7, 24, or 56, but probably performed some mental calculations, which were not made explicit in writing. In reading the solution below, of FT2 (Figure 5.41) one is convinced that the problem is actually quite easy: it's enough to convert to ounces *to see* that this can't work. The definition is fully operational in establishing validity. *If only one asks the question*, that is. It's not even necessary to check both statements: it's enough to take a hypothetical stance (circled in black in the figure) and say that the given statement would indeed be true under this assumption. The ensuing justification of the fractions equivalence is arguably one about abstract numbers (framed in blue), while the definition of fraction of quantity is no longer used, although it is alluded to when the common factor for the fraction simplification is referred to as the unit.

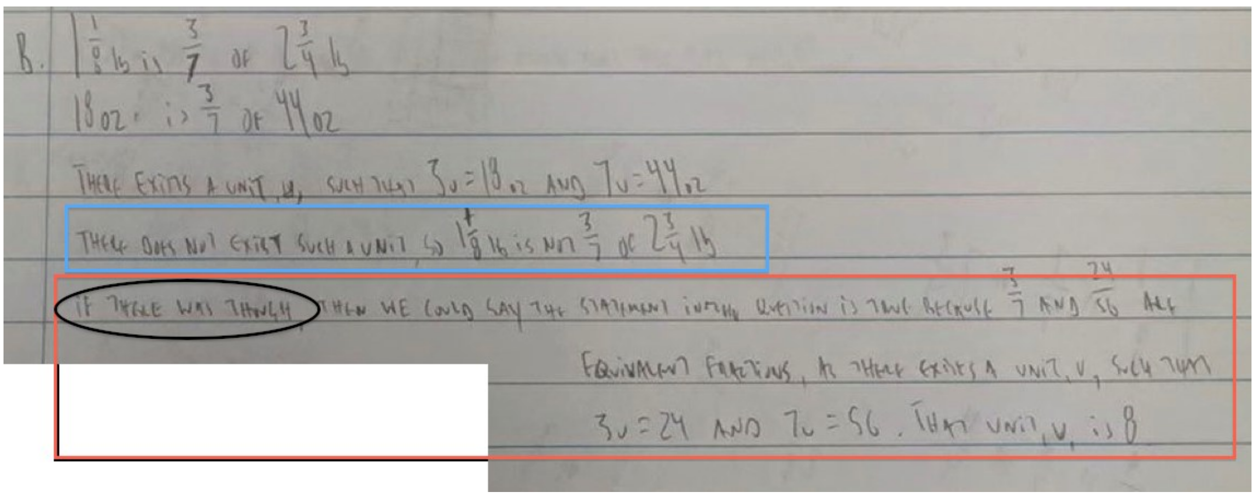


Figure 5.41. FT2's solution.

The next student we quote, FT5 (Figure 5.42), also required that the definitional conditions be met – this time for both S1 and S2 – but it is not clear that the conclusion at the bottom (the letter F, standing for false, circled in black) is validated by checking these conditions for some hypothetical unit, or by the numerical calculation on the side (framed in red).

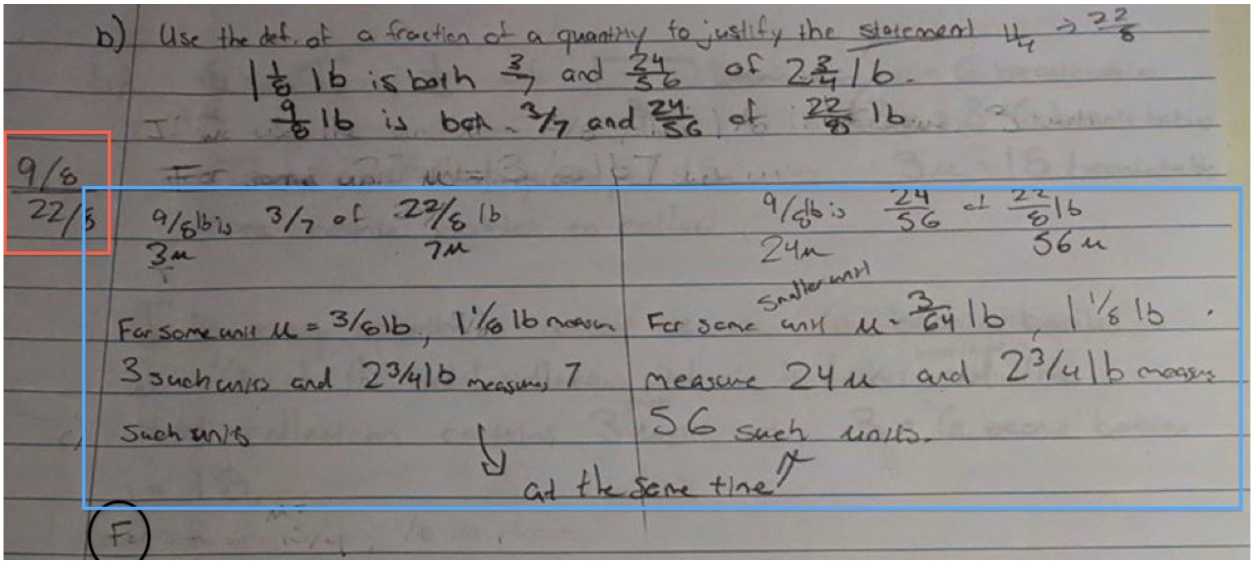


Figure 5.42. FT5's solution.

The remaining three students (FT6, FT34, and FT14) were more explicit in their arguments, with respect to why one of the sentences is not true. All of them resorted to calculating concrete units, and noticed the contradiction as a result. FT6, for example, first calculated the unit by dividing the given $\boldsymbol{Q1}$ by 7 (framed in black at the bottom of the Figure 5.43); she then multiplied it by 7 and noticed that the result is not $1\frac{1}{8}\ lb$, as expected (framed in black nearer the top of the figure).

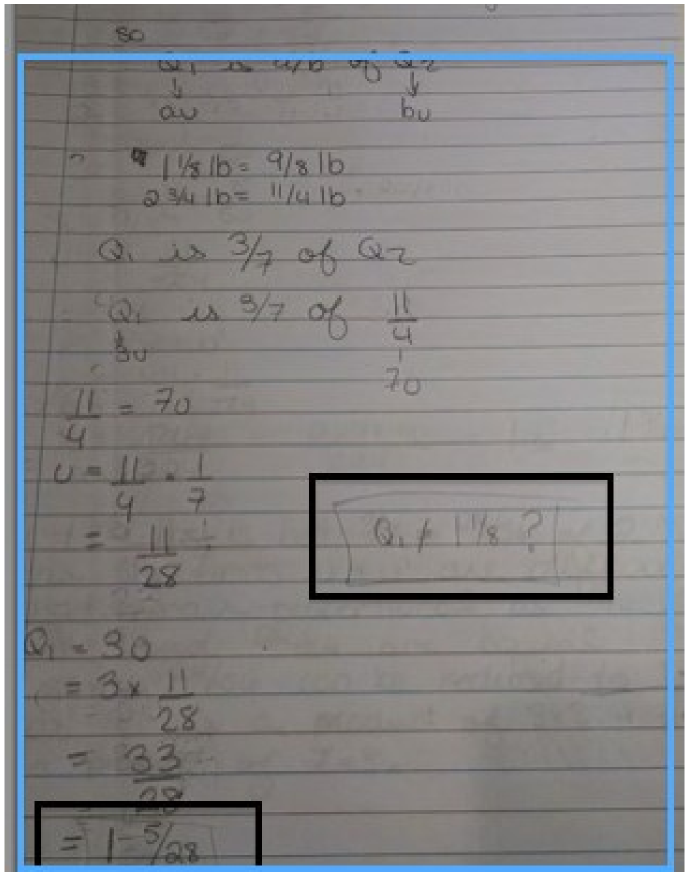


Figure 5.43. FT6's solution - first part.

She appeared to question the validity of $S1$ by noticing that $\boldsymbol{Q1}$ is not the expected value (as suggested by the question mark), and went on to apply the same reasoning onto $S2$, which confirms that indeed the value of $\boldsymbol{Q1}$, with all else unchanged, is $1\frac{5}{28}$ $[lb]$. She then rewrote the given statement to reflect the contradiction she discovered (Figure 5.44).

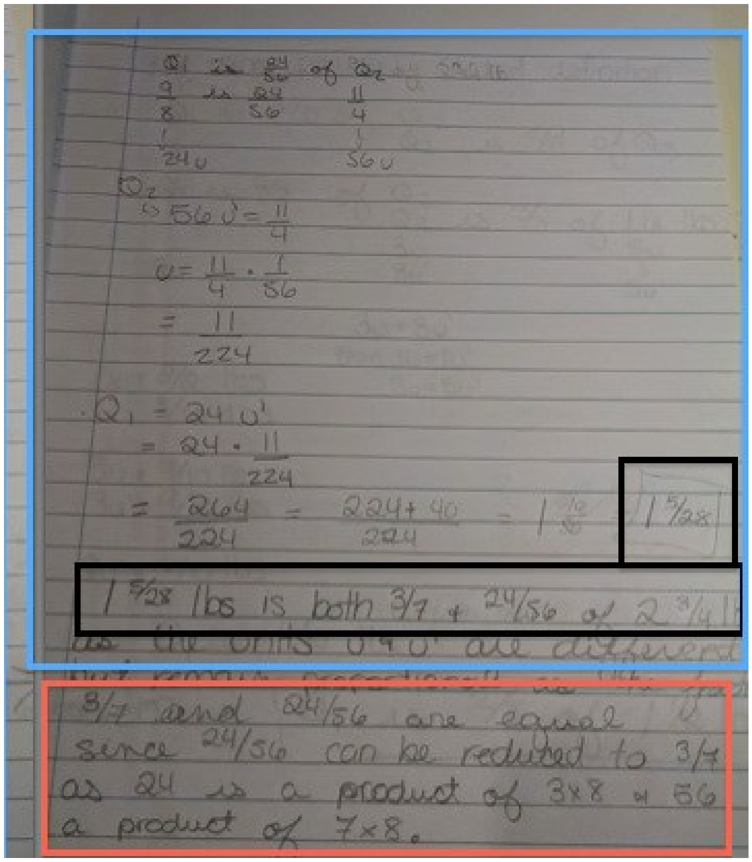


Figure 5.44. FT6's solution - second part.

But, like FT5 and FT2, FT6 felt the need to reconcile the quantitative argument (framed in blue in Figure 5.44) with a numerical one (framed in red), and perhaps the falsity of the statement with the part of it that looked true: the two fractions $\frac{3}{7}$ and $\frac{24}{56}$ are equal numbers; the reduction is left implicit but suggested when she writes that $24\ is\ the\ product\ 3 \times 8$, and $56\ is\ the\ product\ 7 \times 8$.

Only two students (FT 34 and FT14) produced entirely quantitative arguments. Both of them, like FT6, resolved the contradiction by modifying the value of one of the three variables in the given statement, in their case the value of $\frac{a}{b}$, but also, significantly, *stated that the given statement is false*. One of them, for example, started by checking the validity using the definition, on her rough copy, and even appeared to gauge the mistake by comparing her result with the given quantity (Figure 5.45).

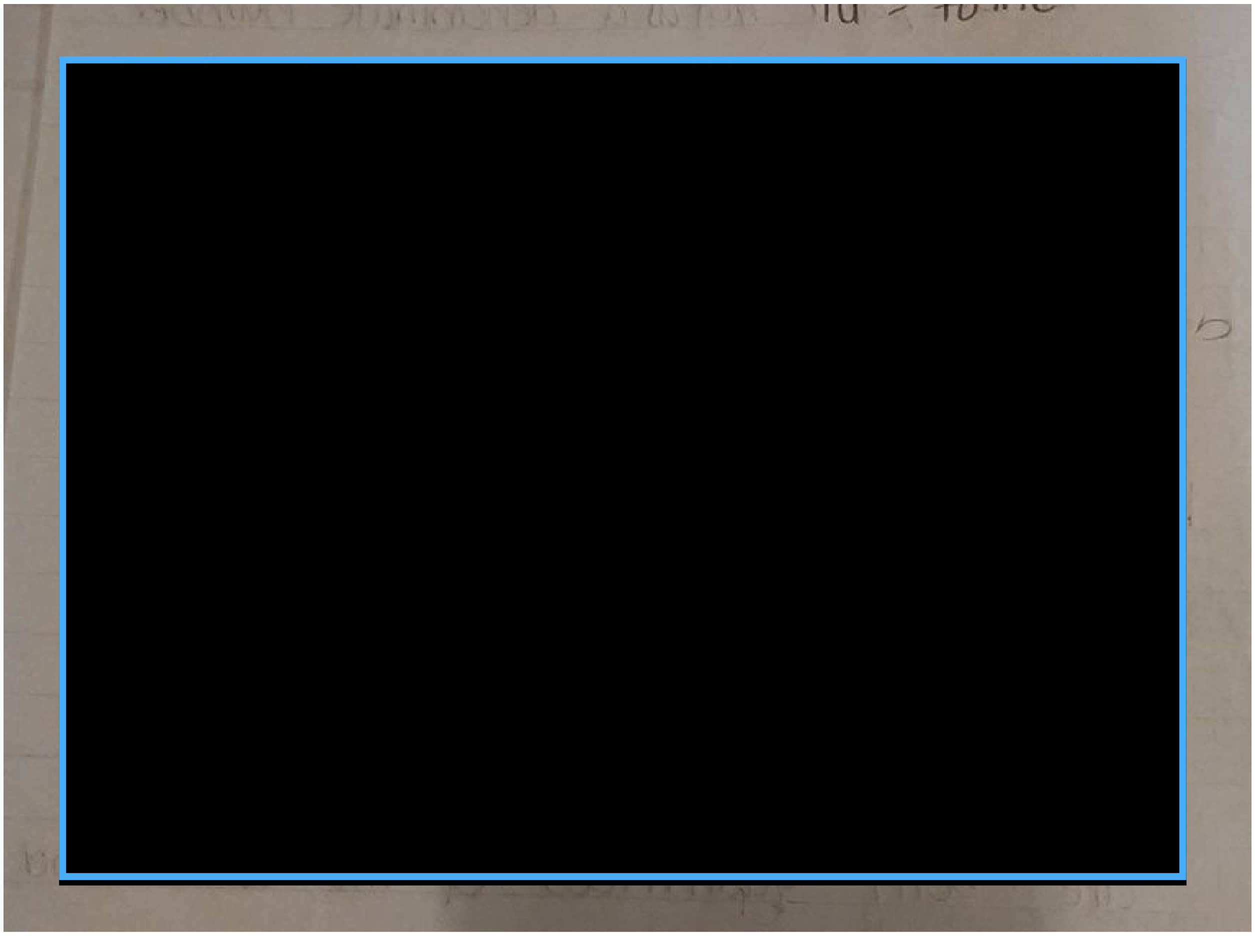
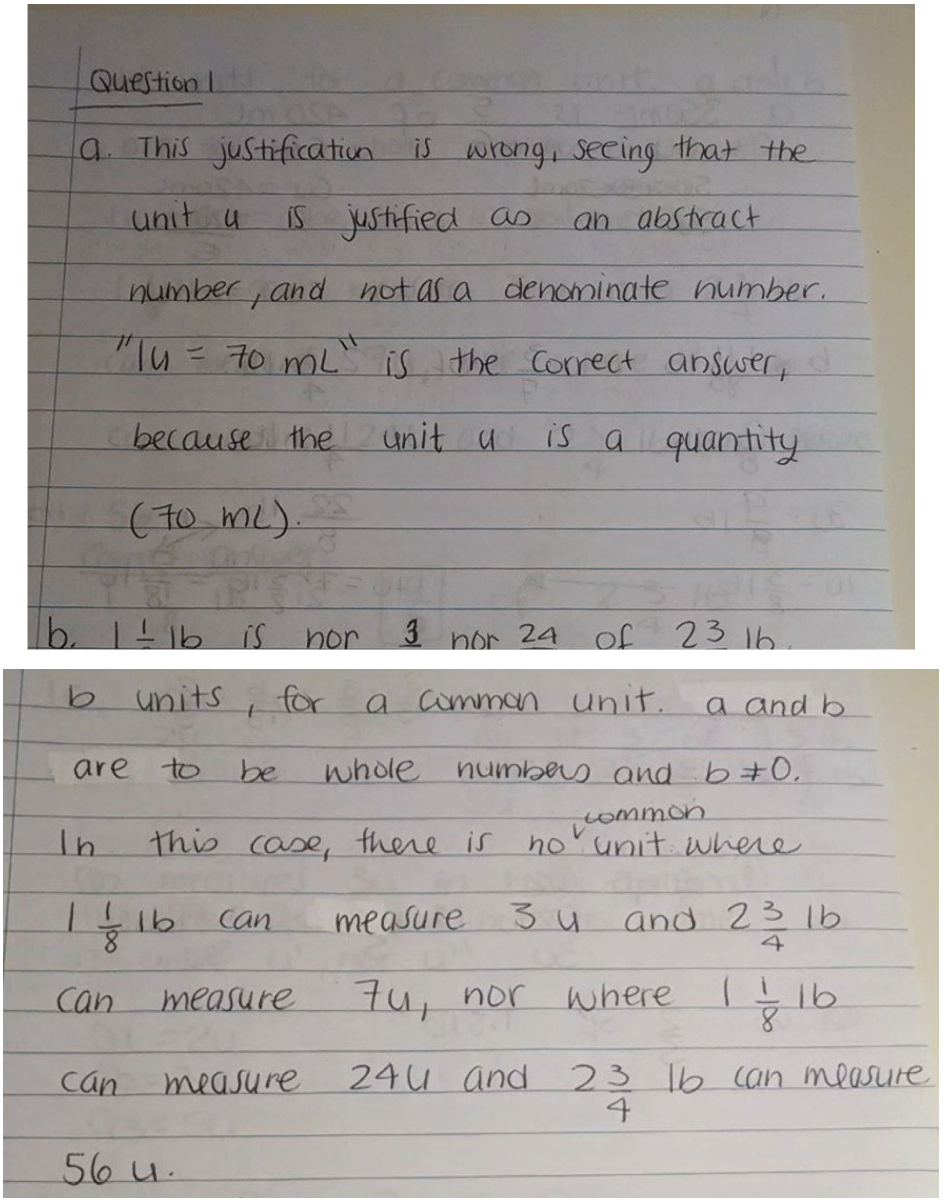


Figure 5.45. FT34's rough calculations before the written answer.

Her solution is a solid combination of theoretical *and* practical thinking. The definition – all of its variables – are indeed fully relevant for establishing the truth of the given statement: in the clean copy of her solution, she goes systematically about checking all the conditions of the definition, and although some of them hold, the fraction $\frac{3}{7}$ fails to fulfill the main definitional conditions (Figure 5.46).



Question 1

a. This justification is wrong, seeing that the unit u is justified as an abstract number, and not as a denominate number. "1u = 70 mL" is the correct answer, because the unit u is a quantity (70 mL).

b. $1\frac{1}{8}$ lb is nor 3 nor 24 of $2\frac{3}{4}$ lb.

b units, for a common unit. a and b are to be whole numbers and $b \neq 0$.

In this case, there is no common unit where $1\frac{1}{8}$ lb can measure 3 u and $2\frac{3}{4}$ lb can measure 7u, nor where $1\frac{1}{8}$ lb can measure 24u and $2\frac{3}{4}$ lb can measure 56 u.

Figure 5.46. FT34's solution: discovering the contradiction.

She then set out explicitly to resolve the contradiction, to find "the correct answer", which, for her is the fraction that represents the relationship between the two given quantities (Figure 5.47). She used the theory again to find the unknown $\frac{a}{b}$, but her practical flair is also evident when she brings the two quantities, fractions of pounds, to the same denominator. Finding the common unit $\frac{1}{8}lb$ is then a mere observation (clearly supported by a good intuition the multiplicative relation at hand between unitary and non-unitary fractions).

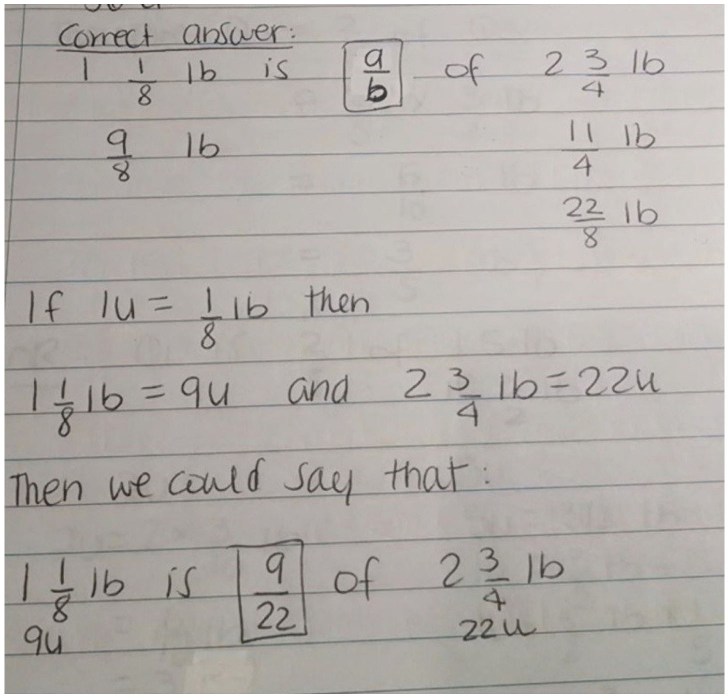
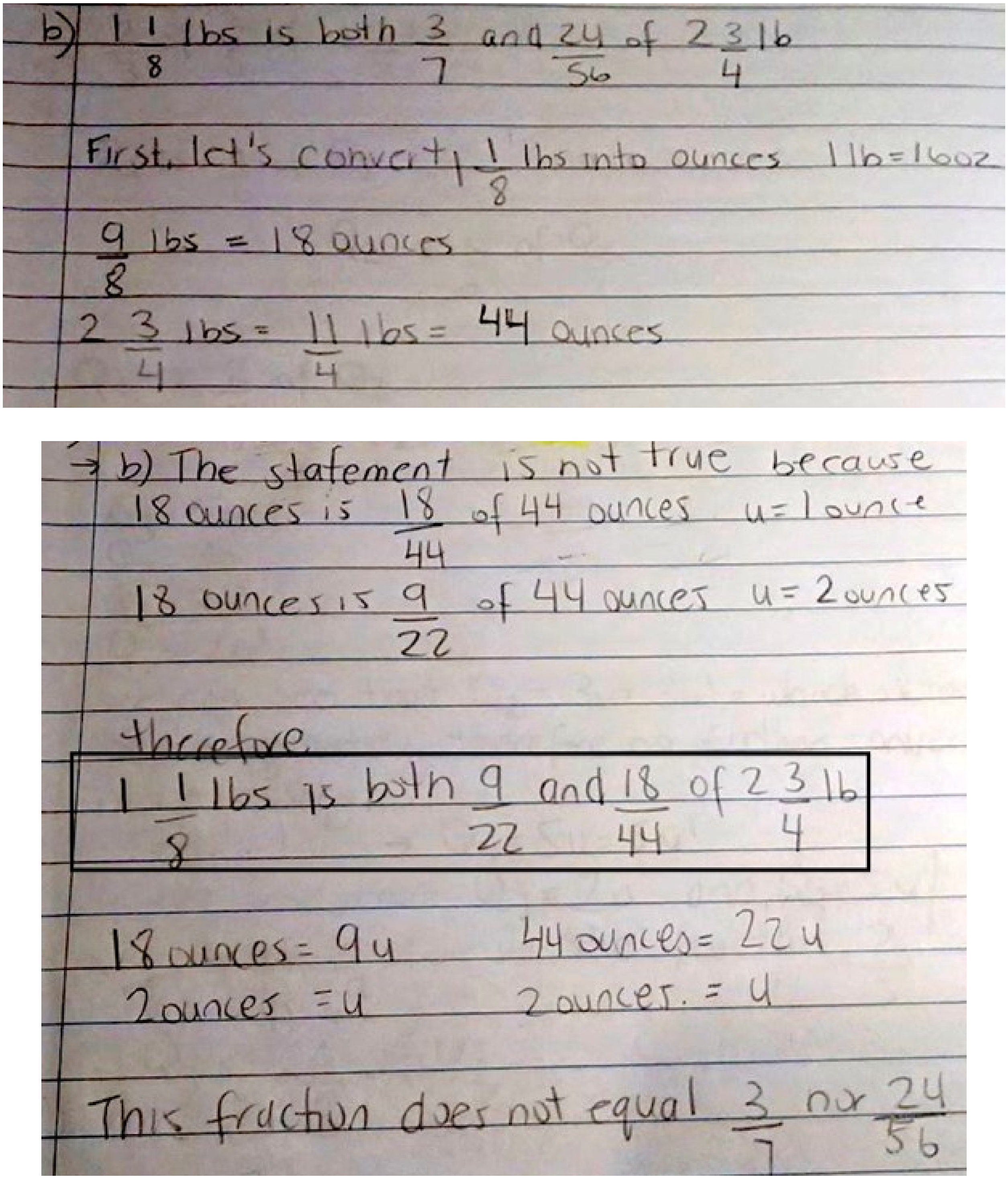


Figure 5.47. FT34's solution: resolving the contradiction.

Also very practical is FT14's idea to convert the given quantities from pounds to ounces, thus easily producing the fraction between them by setting the unit at 1 ounce (Figure 5.48). This demonstrates her flexible understanding of the definition. Interestingly, this student not only "fixed the mistake", but also *created* a statement of the same type as the given one, in finding not one, but two equivalent fractions that represent the relationship between the two given quantities. While other students, like the ones in the first group we analyzed, had the abstract idea of the notion of fraction equivalence as change of unit, this student took complete ownership of it and applied it with ease to produce a new problem. This is really the kind of behavior we would wish to see in our future teachers, and where we would want the utility of the theory to lie.



b) $1\frac{1}{8}$ lbs is both $\frac{3}{7}$ and $\frac{24}{56}$ of $2\frac{3}{4}$ lb

First, let's convert $1\frac{1}{8}$ lbs into ounces   1 lb = 16 oz

$\frac{9}{8}$ lbs = 18 ounces

$2\frac{3}{4}$ lbs = $\frac{11}{4}$ lbs = 44 ounces

→ b) The statement is not true because

18 ounces is $\frac{18}{44}$ of 44 ounces   u = 1 ounce

18 ounces is $\frac{9}{22}$ of 44 ounces   u = 2 ounces

therefore

$\boxed{1\frac{1}{8} \text{ lbs is both } \frac{9}{22} \text{ and } \frac{18}{44} \text{ of } 2\frac{3}{4} \text{ lb}}$

18 ounces = 9u   44 ounces = 22u

2 ounces = u   2 ounces = u

This fraction does not equal $\frac{3}{7}$ nor $\frac{24}{56}$

Figure 5.48. FT14's solution.

5.3.3.4.3 Language

All the five students in the group wrote sophisticated arguments that reflected their inquisitive attitude towards the problem. Four of them clearly signaled the contradiction in linguistic terms, either using interrogative statements, even just symbolically, as FT6 did: “$Q1 \neq 1\frac{1}{8}$” or negating the given statement in a full coherent sentence, demonstrating sharp mastery of logic and meticulous attention to detail (notice the “common” added in hindsight), as FT34 did (Figure 5.49).

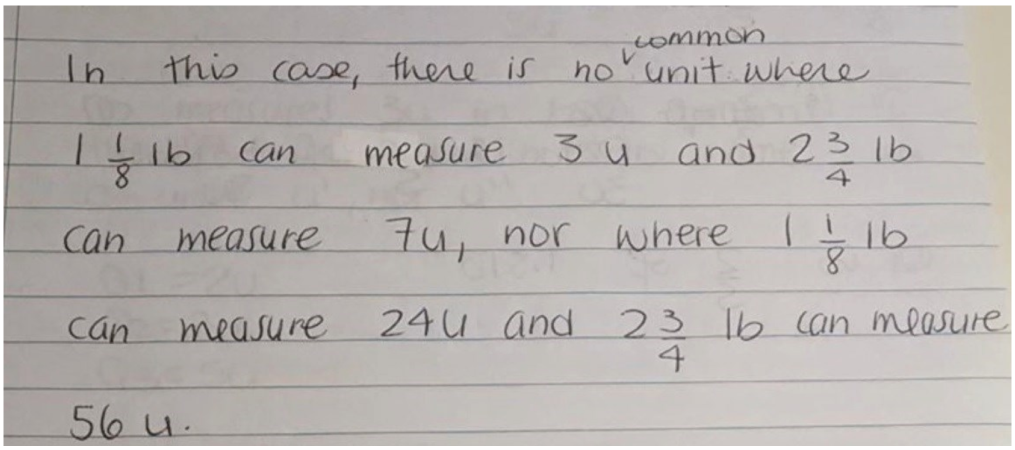

In this case, there is no common unit where $1\frac{1}{8}$ lb can measure 3 u and $2\frac{3}{4}$ lb can measure 7u, nor where $1\frac{1}{8}$ lb can measure 24u and $2\frac{3}{4}$ lb can measure 56 u.

Figure 5.49. A part of FT34’s solution

Only one of the students (FT5) more timidly wrote just the symbol “F”, for false, without clearly stating what the contradiction is.

More generally, what stood out for this group was that they were “telling a story” in their arguments: every single sentence was part of a global argument – nothing was superfluous, or merely decorative. A question mark led to further inquiry that resolved it; a negative statement about the definitional conditions was followed by a conditional statement, demonstrating quite the hypothetical disposition: “There does not exist such a unit…”; “If there was though….” Their writing styles were quite different, which is what one expects even within a group of expert mathematicians: one writes a six-line proof with delightful simplicity that makes you think “of course, this is so obvious”, while another takes two pages and uses a whole linguistic arsenal – sentence structure, connectives, brackets, arrows, underlining, etc. – to produce a sophisticated argument that leaves no threads untied.

Finally, while there are linguistic “glitches”, such as not writing the units (lbs) or using the same letter to designate two variables (u for both units), they go readily unnoticed, as long as the coherence doesn’t suffer because of them. Where this “forgiveness” line is drawn remains, for most students – future teachers a mystery. They often think that they have to “write the unit”, or “use algebra”, or, more generally “write mathematically”, but don’t understand the underlying motivation of such prompts from the teacher (e.g., being aware of the units for the purpose of establishing the accurate relationship), and often apply such advice idiosyncratically. By contrast, and perhaps to the students’ surprise, experts’ standards are less prescriptive about rigor in language as long as the ideas are there.

5.3.3.5 Other interpretations

Eight students (out of 38) wrote solutions that were almost unique among their peers’. In particular, their interpretations of the given statement, led them to solve each a different problem, or even several different problems within the same solution. Only 3 of the 8 had a somewhat similar approach. There were, however, certain commonalities among these 8 solutions. One of them was the incorrect use of mathematical notation, and, in general, the ungrammatical use of both sentence structure (syntax) and meaning (semantics). The second shared feature was they were somewhat missing the point of the problem, insofar as the question was really about *fractions equivalence*. All the other groups saw this theme, whether as equivalence of fractions of quantities, of fractions as pairs of numbers, or even of statements about fractions; all these meanings – however partial with respect to the instructor’s intended problem – eluded the students in this last group. The consequence (or perhaps the cause?) was a focus on very punctual, particular parts of the statement, and the attempt to match them with either some aspect of the theory or some known techniques. This included, for example, using the definition to convert the given quantities from pounds to ounces, or bringing all the fractions in the problem to the same denominator. Others misclassified the given task completely, and attempted procedures corresponding to other types of tasks. One of the solutions, for example, corresponded to the problem of finding the fraction of a fraction of a quantity, another hined at the procedure of changing mixed numbers to improper fractions.

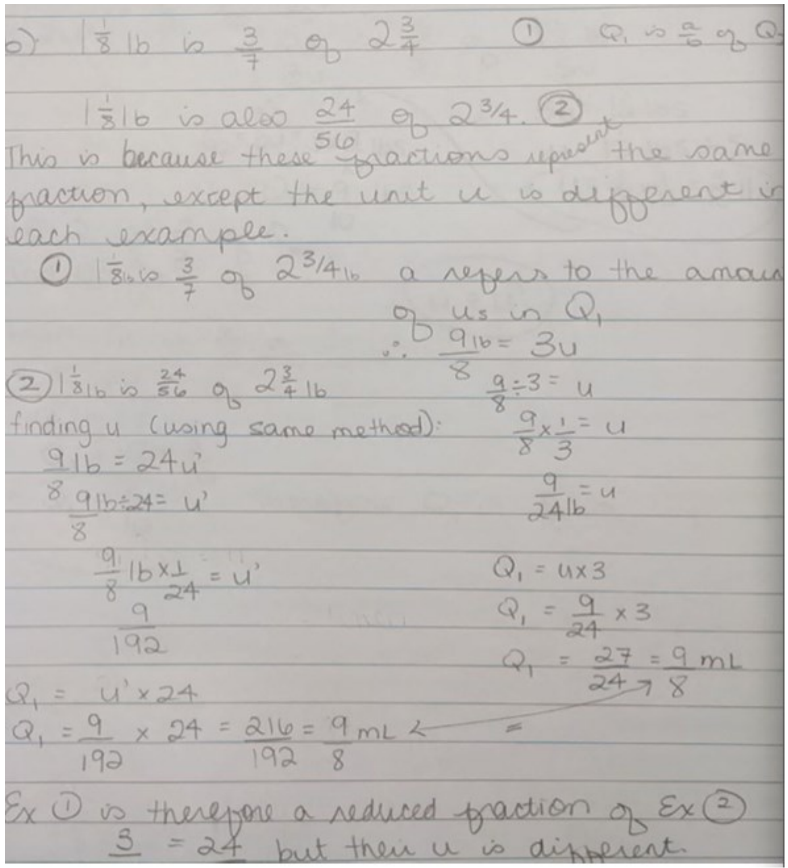
The use of theory, the definition of fraction of a quantity in particular, as decoration, which we have seen before, is, in this group, full-blown. It is as if the students used "discourse markers" to show the teacher that they have studied the theory. They recited the definition, plugged in given numerical values in the *definienum* ($\boldsymbol{Q1}\ is\ \frac{a}{b}\ of\ \boldsymbol{Q2}$), wrote the letter u or of the word unit, and used elements of deductive jargon such as "therefore."

I will go over some of the problems that these eight students tackled in their solutions. Producing these categories was a difficult task, mainly because the students didn't write a coherent narrative, at least when checked against the intended purpose of the given problem. To put it simply: it was hard to decipher what these students were doing. The theoretical flourishes, as well as the use of some legitimate procedures for the problem at hand, complicated our analysis even further, but also provided us with some insights about the students' understanding of the definition of FoQ, in particular about its role in problem-solving.

5.3.3.5.1 Student FT7

5.3.3.5.1.1 Problem: to measure $1\frac{1}{8}\ lb$ *with different units*

The student measures the first quantity, $1\frac{1}{8}\ lb$, with two units: she appears to perceive each such measurement as a separate problem; she even labels them "1" and "2." She finds a unit u corresponding to the fraction $\frac{3}{7}$ by dividing the quantity $1\frac{1}{8}\ lb$ by 3 (she obtains $\boldsymbol{u} = \frac{9}{24}\ lb$). She then checks, by multiplication, if this same quantity $1\frac{1}{8}\ lb$ indeed measures 3 units $\boldsymbol{u}$. This is nothing but a verification of her division, but she appears to believe that she is making a step towards verifying the truth of the given statement (or responding correctly to the given examination question). Next, she does the same for the fraction $\frac{24}{56}$: she finds a unit u' by dividing the quantity $1\frac{1}{8}\ lb$ by 24 (obtaining $u' = \frac{9}{192}$ – she forgets to write "lb"), and then checks if $1\frac{1}{8}\ lb$ measures 24 units u', by multiplying $\frac{9}{192}$ by 24. (Figure 5.50)



a) $1\frac{1}{8}$ lb is $\frac{3}{7}$ of $2\frac{3}{4}$ ① Q_1 is $\frac{a}{b}$ of Q_2

$1\frac{1}{8}$ lb is also $\frac{24}{56}$ of $2\,3/4$. ②

This is because these fractions represent the same fraction, except the unit u is different in each example.

① $1\frac{1}{8}$lb is $\frac{3}{7}$ of $2\,3/4$lb a refers to the amou of us in Q_1

∴ $\frac{9}{8}$lb = 3u

$\frac{9}{8} \div 3 = u$

$\frac{9}{8} \times \frac{1}{3} = u$

$\frac{9}{24\text{lb}} = u$

$Q_1 = u \times 3$

$Q_1 = \frac{9}{24} \times 3$

$Q_1 = \frac{27}{24} = \frac{9}{8}$ mL

② $1\frac{1}{8}$lb is $\frac{24}{56}$ of $2\frac{3}{4}$ lb

finding u (using same method):

$\frac{9}{8}$lb = 24u'

$\frac{9}{8}$lb ÷ 24 = u'

$\frac{9}{8}$lb × $\frac{1}{24}$ = u'

$\frac{9}{192}$

Q_1 = u' × 24

$Q_1 = \frac{9}{192} \times 24 = \frac{216}{192} = \frac{9}{8}$ mL

Ex ① is therefore a reduced fraction of Ex ②

3 = 24 but their u is different.

Figure 5.50. FT7's solution.

5.3.3.5.1.2 Strategies and validity

FT7 breaks the given statement into $S1$ and $S2$, modeling after the *definiendum*, written in the upper right corner (this familiar structure is one of her controls for solving the problem: $S1$ and $S2$ are "examples" of it). She does make the pertinent observation that a different unit is at stake in each. There is, however, no follow up on this observation in terms of showing the connection between the unit sizes and the given fractions despite calculating both units. The fruitful follow-up here would have been to notice the relationship between the two unit candidates: $\boldsymbol{u} = 8\boldsymbol{u}'$.

Separately for $S1$ and $S2$, she divides $1\frac{1}{8}\ lb$, changed to an improper fraction $\frac{9}{8}\ lb$, by 3 and 24, respectively, by multiplying by the inverse. She obtains units $\frac{9}{4}\ lb$ and $\frac{9}{192}$, respectively (one is a quantity, the other – an abstract number). In each case she multiplies these units back by 3 and 24, respectively, to obtain "$\frac{9}{8}\ ml$" (sic!) in both cases, seemingly unaware that she is actually performing a check of her division, rather than a verification of the other definitional condition. She draws an arrow connecting the two identical results with the equals sign underneath: this validates the problem of measuring a quantity with different units, rather than the intended problem of checking whether a quantity is a fraction of another quantity. She concludes that "Ex1" (which refers to the whole statement $S1$) is a "reduced fraction" of "Ex2", and writes the equality of the two fractions, stressing that "u is different." There is an effort to represent the problem in the quantitative terms, but only in hindsight, and the operations are ultimately numerical: the unit "lb" appears at the beginning of the solution; it is not present throughout, and at the end a weight of $\frac{9}{8}$ pounds becomes the volume of $\frac{9}{8}ml.$ The phrase "reduced fraction" is also typical for the numerical approach.

5.3.3.5.1.3 Language

To say that "Ex 1 is a reduced fraction of Ex 2" is a grammatical mistake: a sentence cannot *be* a fraction. Another mistake is "these fractions represent the same fraction", which is an expression from a different theory: the theory of abstract fractions. It points, we surmise, to the student's confinement within her previous knowledge of fractions and unsuccessful attempts to employ the theory of fractions of quantities.

5.3.3.5.2 Students FT1, FT30 and FT31

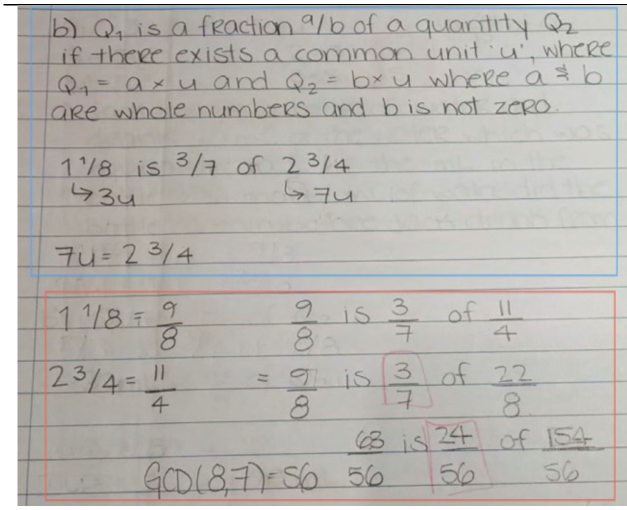
5.3.3.5.2.1 Problem: Bring the fractions $\frac{9}{8},\frac{11}{4},\frac{3}{7},\frac{24}{56}$ to the same denominator

Students FT1, FT30 and FT31 perceived the problem as one of finding a common denominator for several fractions. They essentially brought the fractions $\frac{9}{8}$, $\frac{11}{4}$, and $\frac{3}{7}$ to the same denominator, 56, thus matching that of the remaining variable, $\frac{24}{56}$.

5.3.3.5.2.2 Strategies and validity

In one of the solutions (FT31) there is a clear disconnect between satisfying the "use the definition" requirement (lines [1]-[4] in Figure 5.51) and what she sees, in the given statement, as the problem at hand (lines [5]-[9]). She first reproduces the text of the definition and appears to reference the main definitional conditions using arrows, and writing just one of them, to then move abruptly to the numerical procedure of bringing all the fractions to the same denominator.

Original solution of FT31 **Transcript**



[01] $Q1$ is a fraction $^{a}/_{b}$ of a quantity $Q2$ if there exists a common unit 'u', where $Q1 = a \times u$ and $Q2 = b \times u$ and a & b are whole numbers and b is not zero.

[02] $1\,^{1}/_{8}$ is $^{3}/_{7}$ of $2\,^{3}/_{4}$

[03] ↳ 3u ↳ 7u

[04] 7u = $2\,^{3}/_{4}$

[05] $1\,^{1}/_{8} = \frac{9}{8}$

[06] $2\,^{3}/_{4} = \frac{11}{4}$

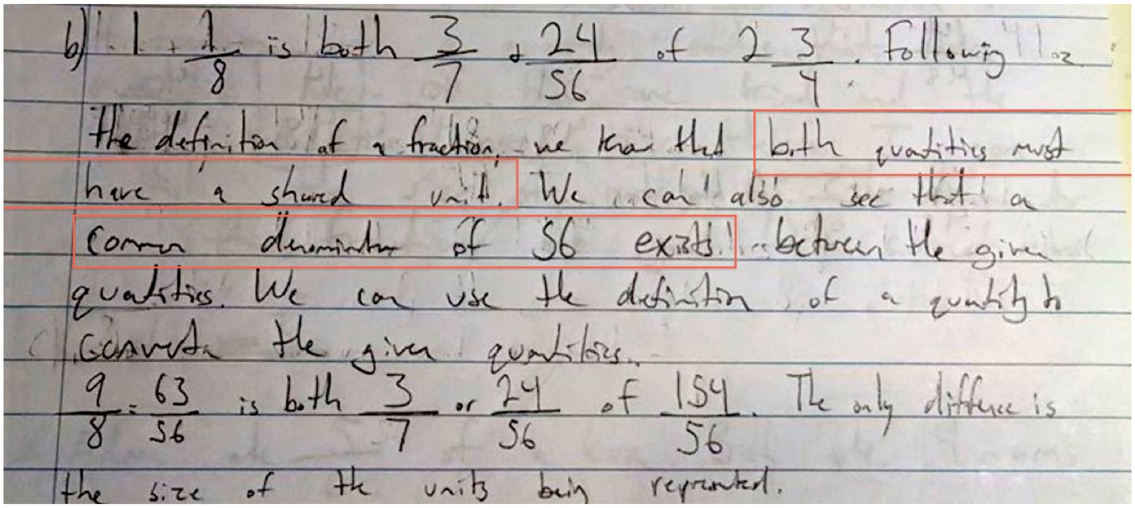
[07] $\frac{9}{8}$ is $\frac{3}{7}$ of $\frac{11}{4}$ = $\frac{9}{8}$ is $\frac{3}{7}$ of $\frac{22}{8}$

[08] GCD(8, 7) = 56

[09] $\frac{63}{56}$ is $\frac{24}{56}$ of $\frac{154}{56}$

Figure 5.51. FT31's solution and its transcript

For the other two students, on the other hand, the theory, as they understand it, is intertwined with the numerical argument, and forms an argument that is difficult to interpret. It is possible that the main source of the confusion is thinking of the *common unit* as a *common denominator* or a *common factor*. Thus, instead of measuring two quantities with a common unit these students bring two fractions to a common denominator, using a common factor.

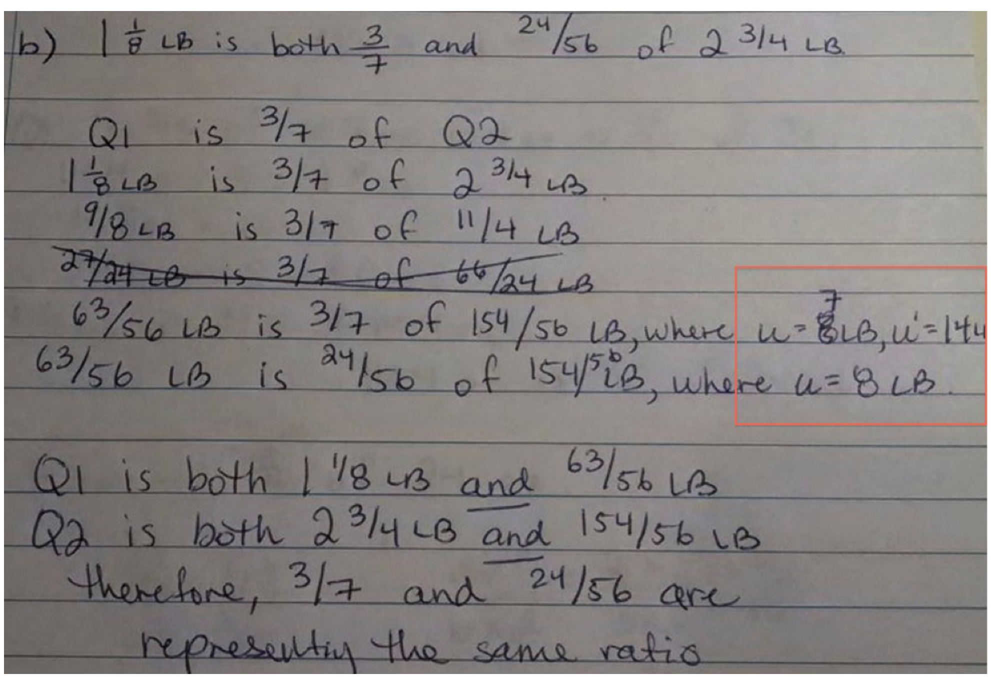


b) $1\frac{1}{8}$ is both $\frac{3}{7}$ & $\frac{24}{56}$ of $2\frac{3}{4}$. Following the definition of a fraction, we know that both quantities must have a shared unit. We can also see that a common denominator of 56 exists between the given quantities. We can use the definition of a quantity to convert the given quantities.

$\frac{9}{8} = \frac{63}{56}$ is both $\frac{3}{7}$ or $\frac{24}{56}$ of $\frac{154}{56}$. The only difference is the size of the units being represented.

Figure 5.52. FT30's solution.

In FT30's solution (Figure 5.52) the association could be related to the similarity between the meanings of the words "shared" and "common." In FT1's solution (Figure 5.53) the student clearly identifies the units with the common factors for amplifying the numerators and the denominators of the fractions in the problem. The more or less conscious idea behind this approach appears to be that since the numerical values of the given $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ can be represented by fractions with the same denominator, 56, so can their ratio.



b) $1\frac{1}{8}$ LB is both $\frac{3}{7}$ and $24/56$ of $2\,3/4$ LB

Q1 is $3/7$ of Q2

$1\frac{1}{8}$ LB is $3/7$ of $2\,3/4$ LB

$9/8$ LB is $3/7$ of $11/4$ LB

~~27/24 LB is 3/7 of 66/24 LB~~

$63/56$ LB is $3/7$ of $154/56$ LB, where $u = \cancel{8}\,7$ LB, $u' = 144$

$63/56$ LB is $24/56$ of $154/56$ LB, where $u = 8$ LB.

Q1 is both $1\,1/8$ LB and $63/56$ LB

Q2 is both $2\,3/4$ LB and $154/56$ LB

therefore, $3/7$ and $24/56$ are representing the same ratio

Figure 5.53. FT1's solution.

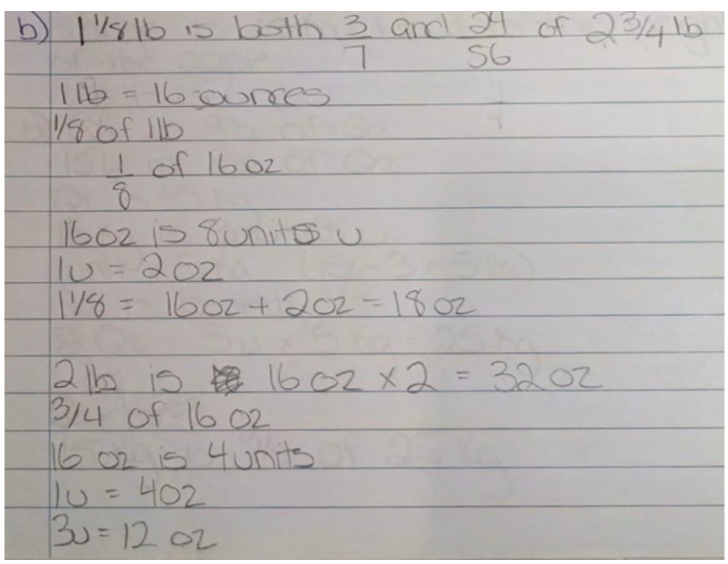
5.3.3.5.2.3 Language

In FT31's solution, there is no flagrant incoherence, other than the misnomer GCD for what is actually the least common multiple – we have seen this before in other groups. In the other two solutions we see, however, a significant common feature: the use of deductive language with unsubstantiated claims. FT1, in particular, in the last lines of her argument writes that "3/7 and $24/56$ are representing the same ratio" *follows from* the following two statements: $Q1$ is both $1\frac{1}{8}\ lb$ and $63/56\ lb$; $Q1$ is both $2\frac{3}{4}\ lb$ and $154/56\ lb$. This, again, as we have seen before, is more than just a linguistic mistake that could be corrected by a reorganization of paragraph or sentence structure, or by replacing the incorrect connectives with the right ones. It is rather a symptom of associations, in students' reasoning, between words or tags rather than between concepts that are part of a system.

5.3.3.5.3 Student FT33

5.3.3.5.3.1 Problem: Convert the given quantities, $1\frac{1}{8}\,lb$ and $2\frac{3}{4}\,lb$, from pounds to ounces

FT33 engaged in a detailed justification, using the definition, of the conversion of the measures of the given quantities from pounds to ounces (Figure 5.54).


b) 1 1/8 lb is both $\frac{3}{7}$ and $\frac{24}{56}$ of 2 3/4 lb
1lb = 16 ounces
1/8 of 1lb
$\frac{1}{8}$ of 16oz
16oz is 8units u
1u = 2oz
1 1/8 = 16oz + 2oz = 18 oz
2lb is 16oz x 2 = 32oz
3/4 of 16 oz
16 oz is 4units
1u = 4oz
3u = 12 oz

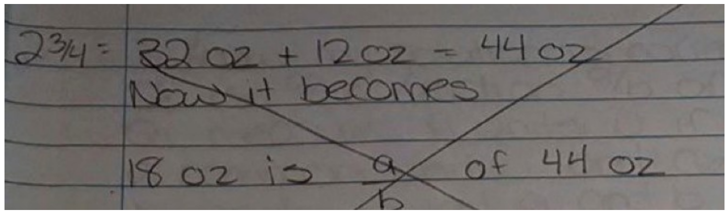
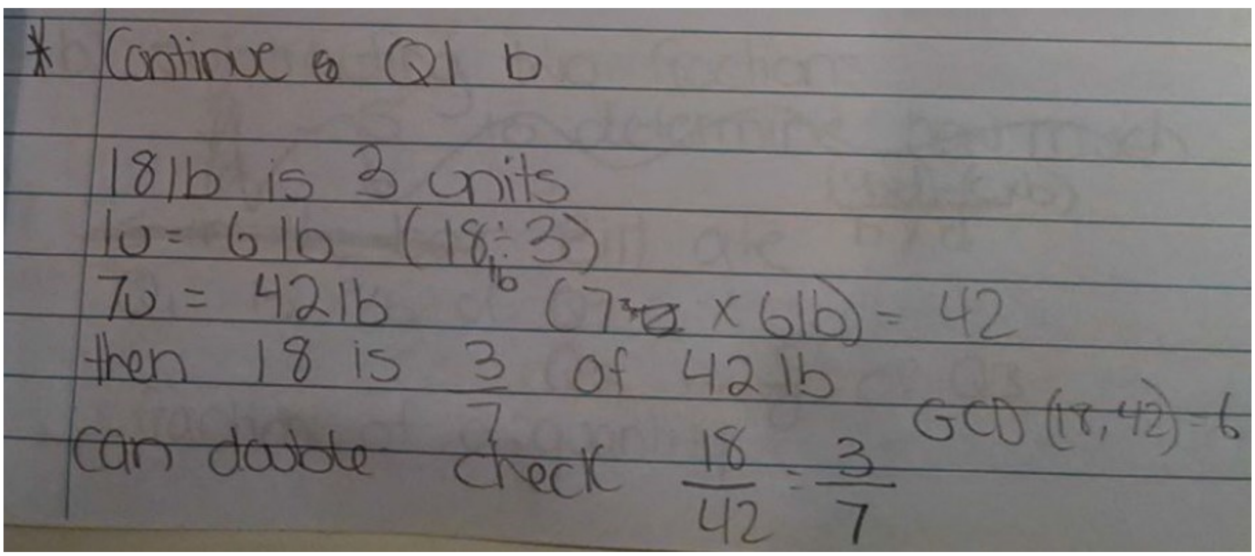
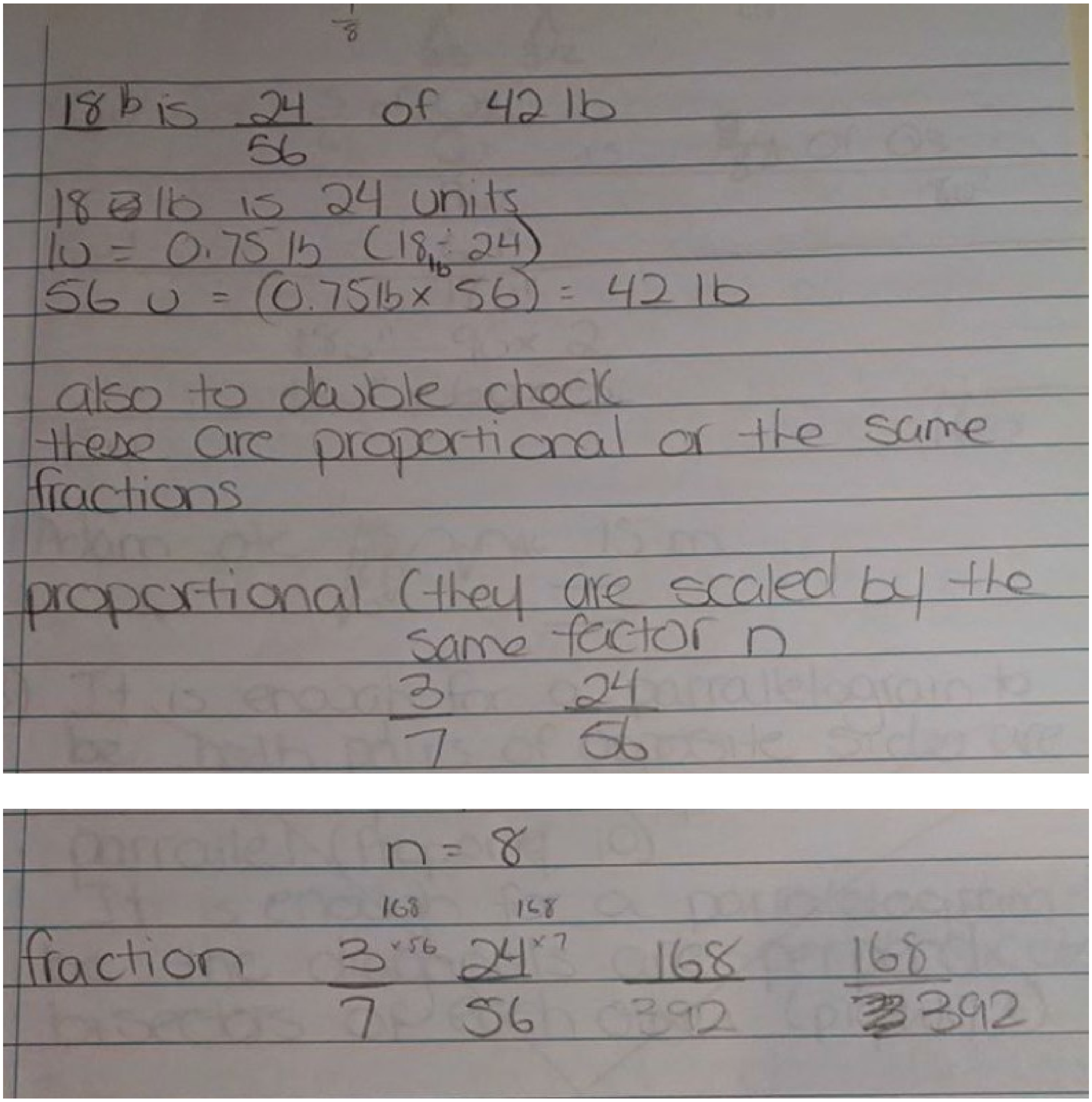


Figure 5.54. FT33's solution – part 1.

The last part appears to be crossed out because she discovered that there is a mistake in the problem. In the subsequent part of her solution, she corrected the 44 to 42. I include a reproduction of that rest of her solution below (Figure 5.55). She switched from oz to lb, but this is a minor mistake.




18 lb is $\frac{24}{56}$ of 42 lb

18 lb is 24 units
1u = 0.75 lb (18 lb ÷ 24)
56 u = (0.75lb × 56) = 42 lb

also to double check
these are proportional or the same
fractions

proportional (they are scaled by the
same factor n
$\frac{3}{7}$ $\frac{24}{56}$

n = 8

fraction $\frac{3^{\times 56}}{7}$ $\frac{24^{\times 7}}{56}$ $\frac{168}{392}$ $\frac{168}{392}$

Figure 5.55. FT33's solution – part 2.

5.3.3.5.3.2 Strategies and validity

The student correctly converts $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ from pounds to ounces, based on the conversion equation $1lb = 16\ oz$. She does that by converting the fractional parts of the given quantities using the definition of a fraction: to convert $\frac{1}{8}lb$ she looks for the quantity that is $\frac{1}{8}$ of the equivalent $16\ oz$. Since, by definition, $16\ oz$ is $8\boldsymbol{u}$, she finds the unit – also the missing quantity – to be $2\ oz$. The mixed fraction of a pound $1\frac{1}{8} lb$ is then the sum of $16\ oz$ and $2\ oz$, i.e. $18\ oz$. The same approach yields the measurement in ounces of the second quantity – $44\ oz$. To solve this problem she uses the definition of fraction of quantity correctly. Finally, she rewrites the statement to be proved with the given values of $\boldsymbol{Q1}$ and $\boldsymbol{Q2}$ now in ounces and $\frac{a}{b}$ not replaced by the given particular values, i.e., with either of the fractions $\frac{3}{7}$ and $\frac{24}{56}$. It is not clear why she then crosses out this part. We can only presuppose that she thought of the fraction between the two quantities as an unknown that needs to be found (she found the other two), instead of questioning whether the fraction between the two quantities is $\frac{3}{7}$ and $\frac{24}{56}$. This is not an unfruitful thread of thinking to follow (one of the students in the group that disproved the statement did exactly this: found the fraction between the two quantities and compared it to the given $\frac{3}{7}$ and $\frac{24}{56}$), but the student seemed overwhelmed by the task at this point.

5.3.3.5.3.3 Language

The student's solution reads approximately well, but the main problem seems to be the breaks in coherence that take place at least twice in the solution, without much explanation. She also forgets to write the standard units lb and oz a couple of times.

5.3.3.5.4 Student FT36

5.3.3.5.4.1 Problem: To show that if $\boldsymbol{Q1}$ is $\frac{3}{7}$ of $\boldsymbol{Q2}$ and $\boldsymbol{Q2}$ is $\frac{24}{56}$ of $\boldsymbol{Q3}$ then $\boldsymbol{Q1}$ is $\frac{3\times24}{7\times56}$ of $\boldsymbol{Q2}$

FT36 misinterpreted the problem as a problem of representing a fraction of a fraction of a quantity by a single fraction of this quantity (Figure 5.56).

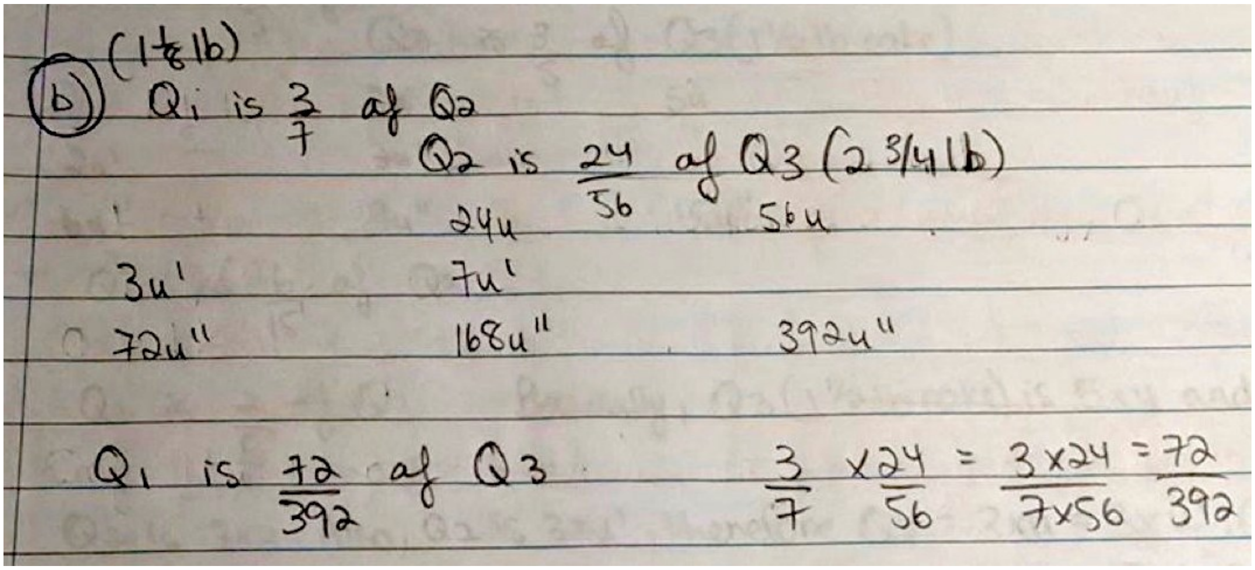


Figure 5.56. FT36's solution.

5.3.3.5.4.2 Strategies and validity

The student derived the answer to her problem by reasoning from the definition of the fraction of a quantity and showed that the same fraction can be obtained using the numerical operation of multiplication of the two abstract fractions.

5.3.3.5.4.3 Language

Instead of interpreting the given statement as: $\boldsymbol{Q1}\ is\,\frac{a}{b}\ AND\ \frac{c}{d}\ of\ \boldsymbol{Q2}$, she reads it as

$\boldsymbol{Q1}\ is\,\frac{a}{b}\ OF\ \frac{c}{d}\ of\ \boldsymbol{Q2}$.

5.3.3.5.5 Student FT23

5.3.3.5.5.1 Problem: $1\frac{1}{8}lb\ \ is\ \ \frac{9}{8}lb$ *and* $2\frac{3}{4}lb\ \ is\ \ \frac{22}{8}lb$

The student's solution is presented in Figure 5.57. For this student, the problem was one of conversion, using FoQ theory, of mixed numbers to improper fractions.

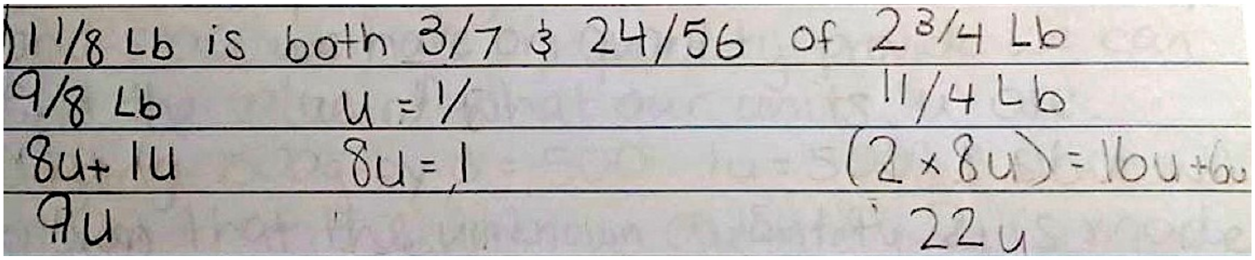


Figure 5.57. FT23's solution.

5.3.3.5.5.2 Strategies and validity

The student changes the given quantities from mixed numbers to improper fractions using the learned procedure, by letting a unit u be $\frac{1}{8}lb$ (she does it schematically – it is not clear that she understands the meaning of the procedure). The procedure is meaningless, however, when applied to the fractions $\frac{3}{7}$ and $\frac{24}{56}$. It is not clear if or how she validates the results.

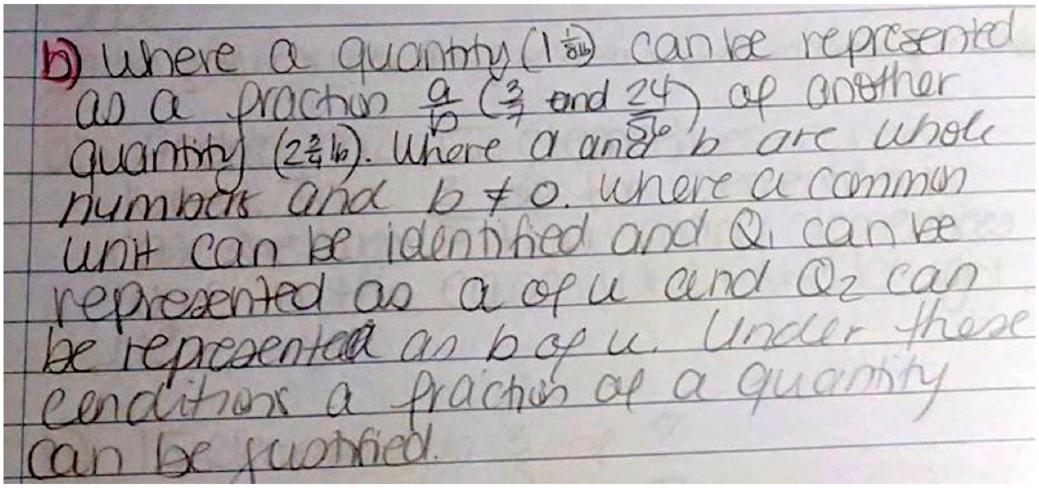
5.3.3.5.5.3 Language

Her expression is laconic, without connectives or explanations, in either words or algebraic notation. She uses the letter u for the unit, conceived of as an abstract number, namely $\frac{1}{8}$. She erased the denominator 8 in the line where she expressed what u is equal to (not clear why). The equals sign is used incorrectly $2 \times 8u = 16u + 6u$.

5.3.3.5.6 Student FT20

5.3.3.5.6.1 Problem: List the definitional conditions

We reproduce the student's solution in Figure 5.58.



b) Where a quantity ($1\frac{1}{8}$ lb) can be represented as a fraction $\frac{a}{b}$ ($\frac{3}{7}$ and $\frac{24}{56}$) of another quantity ($2\frac{3}{4}$ lb). Where a and b are whole numbers and $b \neq 0$. where a common unit can be identified and Q_1 can be represented as a of u and Q_2 can be represented as b of u. Under these conditions a fraction of a quantity can be justified.

Figure 5.58. FT20's solution

The solution is transcribed Figure 5.59. In an awkward fashion, FT20 states what needs to be proved to justify the given statement, but does not do it (lines [2] and [3]).

[10] Where a quantity ($1\frac{1}{8\,lb}$) can be represented as a fraction $\frac{a}{b}$ ($\frac{3}{7}$ and $\frac{24}{56}$) of another quantity ($2\frac{3}{4}$ lb).
[11] Where a and b are whole numbers and $b \neq 0$.
[12] Where a common unit can be identified and Q1 can be represented as a of u and Q2 can be represented as b of u.
[13] Under these conditions a fraction of a quantity can be justified.

Figure 5.59. FT20's solution

This hypothesis about the problem that FT20 was solving follows from the following interpretation of her response:

The quantity $\boldsymbol{Q1}$ = $1\frac{1}{8}$ lb can be represented as a fraction $\frac{a}{b}$ ($\frac{3}{7}$ and $\frac{24}{56}$) of the quantity $\boldsymbol{Q2}$ = $2\frac{3}{4}$ lb if a and b are whole numbers and $b \neq 0$ and a common unit $\boldsymbol{u}$ can be identified such that $\boldsymbol{Q1}$ can be represented as a of $\boldsymbol{u}$ and $\boldsymbol{Q2}$ can be represented as b of $\boldsymbol{u}$. Under these conditions the statement that the first quantity is a fraction of the second quantity can be justified.

5.3.3.5.6.2 Strategies and validity

The student's answer contains a recitation of the definition with the given values appearing in brackets next to the three variables. She doesn't check any of them, but appears to believe that listing them validates “a fraction of a quantity” (it is not clear that she understands what object this expression (FoQ) refers to.

5.3.3.5.6.3 Language

The definitional conditions are listed in ungrammatical sentences starting with the adverb “Where”, which perhaps she associates with deductive discourse one uses in mathematics. She also speaks of justifying a noun phrase (fraction of a quantity), instead of *justifying* a sentence (a quantity is a fraction of another quantity).

5.3.4 Discussion: quantitative reasoning in an “old” question with a twist

The first observation is that the question produced a high variety of answers. It is significant that we could identify four categories of solutions and, in addition, eight (out of 38, or about 21%) singularities, for a question that was not open ended and of a type that has been discussed in class on various occasions. We believe that the fact that the question was never solved in the exact same way and that it had a “twist” (that is was a false statement) may have greatly contributed to this variability in students’ answers.

I organize the discussion of this question by first reviewing the characteristics of FT’s solutions in each interpretation that stood out as relevant for our design goals, or more generally, as a portrayal of students’ *adaptation* to the Measurement Approach. While I do mention some percentages as a way of giving a summary description of the results, I make no claim of generalizability based on such simple statistics; as I mentioned before our interpretative approaches affords generalization based on different principles (Eisenhart, 2009).

Ten students, out of 38 (26%), belonged to the **Interpretation 1** group. This was the largest group and they all produced sensible arguments, capturing the essence of the problem: the equivalence of fractions of quantities. They proved the equivalence by demonstrating a quantitative relation between two units used to measure the two given quantities. In doing this they resorted to the theoretical constructs provided in the course and made them operational for producing justifications within the given system. Their productions read mostly well, as coherent written speech with few inconsistencies, and mathematical notation used adequately.

We could say that this group fulfilled the expectation of engaging in the kind of behavior we envisioned they would, as university students doing mathematics, in particular with respect to focusing on quantities. All of the students in this group failed to establish the statement as false because they didn't engage in hypothetical thinking so as to ask if the relationship between the given quantities is indeed $\frac{3}{7}$ (or $\frac{24}{56}$) before proceeding to show it is both one and the other of these fractions. This flaw in their arguments could be attributed to a logical miss: proving only that $S1$ implies $S2$ (or that $S2$ implies $S1$) without validating any of these statements. Another explanation could be that they imitated the solution given by the Instructor in one of the labs to a problem with a similar syntactic structure, but where the quantities were variables rather than concrete values. We have expressed this concern before, in relation to worked out examples of a given structure, but, in this group, the variety of students' strategies – mostly departing from the one employed by the Instructor – is a sign of stable learning rather than shaky imitative behavior.

Nine students, out of 38 (24%) belonged to the **Interpretation 2** group. The students in second largest group focused only on the pure numbers in the given statement, ignoring, in a way, the whole structure of the sentence – at least as being relevant for their justifications. More significantly, the given quantities played no role in the solution. All the students in this group did produce a justification that would be considered valid in the theory of rational numbers for the equivalence of $\frac{3}{7}$ and $\frac{24}{56}$. As we have shown, however, there is not even a guarantee that they conceived them as fractional numbers – something we might expect from research which documents children's difficulties with fractions stemming from "the whole number obstacle" (Pitkethly & Hunting, 1996), especially after instruction based on partitioning activities (Streefland, 1991). Although the students in this group did not engage in quantitative reasoning, the FoQ theory was not absent from most of their solutions. The given values of the fractions and of the quantities were "plugged in" the definition of fraction of quantity but with no operational power. The theory was mostly "decorative." This resulted in writing that either abruptly changed from quantitative discourse to numerical operations, or awkwardly combined the two in imprecise use of notation and terminology and overall lack of clarity.

Six students, out of 38 (16%) belonged to **Interpretation 3** group. In this group the students considered the whole structure of the statement and actually broke it into the two sentences that formed the conjunction. They appeared to actively use the theory of fractions of quantities to justify the statement, by using the definition to verify if the two statements are each, individually, true. The students in this group did more than plugging values into the definition: they performed the operations of division and multiplication to find the units corresponding to the two sentences. However, these operations were undertaken as a procedure – one that was perhaps perceived as typical for this type of problems – which had, again, no consequence on establishing whether these statements were true or false. The telling feature was the use of the equals sign: equalities were not perceived as hypothetical, as they should have been as definitional conditions. A fundamental way of employing a definition in mathematics – to check whether a given object is *something* – was not grasped by the students in this group.

Five students, out of 38 (13%) belonged to the **Interpretation 4** group. These five students saw, in the given statement, the two identically structured sentences, but also questioned them as statements, i.e., as an information-bearing declarative sentence. Significantly, they considered the concrete values of the quantities to establish first the relation between them and check it against the given fractions – before discussing the equivalence issue. The definition, for this group acquired full operational power, leading them to disprove the statement when one of the main definitional conditions was not satisfied. Each solution can be discussed as an interesting case study with regards to the issue of transfer of knowledge disseminated in class: despite the fact that the question was far from being open ended it spurred quite different problem solving behaviors among the five students in this group. One student, very practically, converted the given quantities to ounces to easily observe that they are not in the given relationship; his solution was only a few lines. At the other end, another student, meticulously checked all the conditions of the definition – even the non-essential ones – to produce a rigorous proof that the statement is false; her solution covered a couple of pages. We have talked often about ungrammaticality in students' written expression; our analysis of this group's solutions shows that our concern is not with prescriptive formal rules, but rather with clarity. Students, in general, may interpret the teacher's prompt to write correctly as the requirement to write a lot, according to a certain form of discourse, in rigorous algebraic notation, when, in fact, in writing mathematics, one is concerned with sense and meaning. As mentioned, the students in this group, *made sense* because they "told a story" in the form of a global argument, with occasional lack of rigor or glitches in notation.

Finally, something has to be said about the answers we grouped under **Other Interpretations**. Eight students (out of 38, so 21%) wrote solutions that we grouped together only because they didn't produce arguments for fractions equivalence, either quantitative or numerical. We had some hypotheses about their interpretations of the given question, but, overall, it was difficult to explain some of the problem-solving behaviors exhibited in this group. We observed, as a common feature, an effort to "match" fragments of the discourse specific to the theory of fraction of quantities (such as "unit" or "amount") or, more generally, of mathematical deductive discourse ("where", "therefore"). Most often these words were not used according to their accepted meaning: for example, "unit" stood for factor, and "therefore" was not a logical connector. In general, only words, rather than full sentences were imitated. The definition for this group was also not operational; in some cases it was merely recited, with even substitution of the given values ("plugging in" the definition) not completed. Numerical operations such as bringing all the fractions in the problem to the same denominator, or removing all the denominators, perhaps reflected a habit of performing certain procedures on specific cues without considering the underlying justification (for example, being used to "getting rid of denominators").

A polarized distribution of students' learning as a result of the Measurement Approach emerges from their solutions to this particular question. Of the 38 students, 15 (those belonging to Interpretations 1 and 4), or about 40%, engaged in quantitative reasoning, and more generally, in justifications using the theoretical tools provided in the course. It is encouraging to see, in these two groups, a degree of creativity, particularly high in the fourth group, as reflected by the great variety of strategies used. This demonstrates that those students took ownership of the knowledge communicated in the course and applied it according to their own, internalized, understanding. Yet more than a half of the students remained confined within the numerical domain of their previous knowledge of fractions, in a procedural fashion, appropriating only superficial aspects of the taught theory of fraction of quantities. Such distribution of highly positive and highly negative results highlights a dilemma: should we renounce teaching something that is too difficult for some students or should we teach it nonetheless for those who stand to greatly benefit from it?

6 CONCLUSIONS

From the mathematics teacher education literature we came to the math methods course expecting that prospective teachers' knowledge of fractions may be lacking, especially, in what goes beyond the mastery of certain rules or standard algorithms, usually for performing operations In fact, in terms of conceptual understanding – in-depth knowledge of fundamental ideas and relations between them (e.g., according to (Rittle-Johnson & Siegler, 1998)) – preservice teachers have the same lacunae, producing systematic errors, as the ones documented in children (Newton, 2008).

In the first instantiation of the math methods course described in the thesis (the Pilot Study), we saw, in more nuanced ways, how such gaps were manifested in a domain of tasks that resembled the practice of a real teacher quite closely: problem posing in the context of quantities encountered in real life. The mathematical problems created by the students as part of the course work were indeed inspired by real-life situations, but these problems were missing the key concepts they were supposed to embody, unsolvable because of missing data or unclear formulation, or using incorrect notation or terminology. But mostly, despite the rich potential afforded by the real-life inspiration, *the problems were trivial.* The solutions and accompanying activities for the invented problems were only sometimes incorrect; most often they didn't fulfil the requirement to serve as explanations based on the quantities featured in the problem. Prospective teachers, instead, resorted to purely numerical algorithms as solutions to problems that could have been easily solved by simple reasoning about the quantities involved. The effort to reach the child was channeled not on linking context to operation, but on making the lessons "fun", "child-centered", and "hands-on", by measures such as having children work in groups, changing one's tone of voice, or bringing objects to class. Thus, when asked to take the Teacher position, rather than using the taught mathematical concepts and techniques, or even everyday common sense about quantities, the students invoked their own school experiences to argue in favor of "simple" algorithms that "work" (Former Pupil position) as well popular discourse about what constitutes progressive, reform-oriented, education (Popular Educator position). This is not surprising if we follow

Labaree's argument in "The trouble with Ed Schools" (Labaree D. , 2004): by the time one sits in education schools one sees the profession in action for many years and probably has deep-rooted ideas and models about what good teaching is. Moreover, student-teachers genuinely embrace reform ideals, if only on the surface, as we learn from Frykholm's study (1999).

The issue of assessment became contentious in this context. The open-ended nature of the tasks that the students routinely solved in class and as homework – usually of the form "Invent an activity" – meant that they did not have to punctiliously follow a procedure or apply a concept. The students had to actually produce immediate transfer to the rather practical situation of activity planning for the classroom, and for this, they appeared to naturally fall back on what they knew, rather than use, spontaneously, what we had thought would be useful analytical tools. Yet, all the participants – Instructor, students, teaching assistant – were still caught within the institutional confines of a university course, where the students had to produce work that would be graded, in order to pass the course and further pursue their career goal of becoming elementary teachers.

We designed the course for its next run, the Experiment reported in this thesis, by considering all these issues, while still holding on to the idea of teaching fraction as relation between measured quantities, which we thought justified on cognitive and epistemological grounds. Our approach to the design of the course would not merely aim at reducing the frustration and uncertainty about assessment. Making the students happy and our marking job easier was clearly not the objective, but we did acknowledge the legitimacy of students' concerns and did endeavor to be sensitive to their preoccupation with knowing what is required of them to pass the course.

Students appeared to genuinely hold, on the one hand, the belief that they already know most about what works in school and, on the other, the confidence that they will learn the rest on the job. The math methods course was but a gatekeeper. Were they right perhaps? We had to consider in all seriousness questions about the goals of learning mathematics in elementary schools but also the goals of learning "teaching of mathematics" at university. Could we actually contribute, in the math methods course, to their becoming professional teachers? The course design embodied our answers inspired by Dewey (a philosopher), Lebesgue (a mathematician), and Davydov (a psychologist), to mention but a few of the scholars who influenced our work. But this view was influenced from outside the Ivory Tower as well. When, in prospective teachers' proposed activities and demo classes, teaching mathematics consisted in a detailed specification of procedures for performing operations, I felt the kind of an unease similar to Lockhart's lament (2009), at a time when one of my children just entered first grade. I would want a teacher for her who could *see* the mathematics in children's thinking, even if it is a sentence, not a mathematical equation, who could gently nudge their informal strategies to more formalized practices, who could see the big picture of interrelationships between concepts so as to intuit where a child's idea comes from or where it's headed, and who could, on-the-spot, seize the opportunity afforded by an error rather than downright dismiss it. I would like her teacher to express herself clearly in writing and in speech, to reason logically, and persuade with argumentation. I would like her to model a behavior where it's ok to stop and think, and to not know the answer to a question. Her teacher should not "tell the child to save her wondering" (Knipping, Straehler-Pohl, & Reid, 2012), nor should she bring manipulatives to class just for fun or as a reward for appropriate behavior (Moyer, 2001). I'd rather my child did no math at all, than be told: "yours is not to reason why, just invert and multiply": if the school subject "mathematics" does not fulfil the goal of nurturing children's intellectual activity, we should perhaps replace it altogether with more innovative pursuits – such as game design or linguistics – as a way to develop scientific and social insight.

In Dewey's words again, in order to achieve the goal of cultivating children's genuine thinking "in deed, not in mere word", the teaching of any subject matter to future teachers in an academic setting should focus on that subject's embodiment of thought processes:

> *It is the business of normal schools and collegiate schools of education to present subject-matter in science, in language, in literature and the arts, in such a way that the student both sees and feels that these studies are significant embodiments of mental operations. He should be led to realize that they are not products of technical methods, which have been developed for the sake of the specialized branches of knowledge in which they are used, but represent fundamental mental attitudes and operations--that, indeed, particular scientific*

methods and classifications simply express and illustrate in their most concrete form that of which simple and common modes of thought-activity are capable when they work under satisfactory conditions. In a word, it is the business of the "academic" instruction of future teachers to carry back subject-matter to its common psychical roots. (Dewey, 1907)

On our part, we thought that reasoning about quantifiable aspects of objects and relations between them captures a way of inquiry into reality that mathematics, as a human practice, has always been advancing. It is also more intimately related to children's thinking than purely numerical operations performed according to algorithms. Thus a focus on quantities – and conceiving of fractions explicitly as relations between them – was paramount in our design. Furthermore, mathematics can serve as one, amongst many, domains of developing future teachers' habits of thinking analytically within a given system as well as a reflective mindset, as a part of the larger pursuit of a university education. What we formulated as the goal of shaping the University Student position in the math methods course as an institution, did not entail forcing the students into an epistemological submission to the authority of the university teachers. A priori, the explicit theoretical nature of the Measurement Approach was meant to empower future teachers to equal-to-equal mathematical discussions with the university teacher. The major shift in our approach was that autonomy was not the prerequisite, but the expected result, of learning in the course.

To what extent have these expectations been fulfilled? I will review the findings contained in the previous chapter's discussion sections.

Assessment remained important for the students, but was no longer a contentious issue

In our first experience in the course (the Pilot-Study) we were inclined to interpret students' expressed frustration about the grading of their assignments as an exclusive preoccupation with passing the course or obtaining a good grade. And we thought it was detrimental to their learning. But modifying the tasks – problems given in class and homework – to no longer include Teacher's actions (such as planning an activity for children) had the result of completely diffusing the emotion in discussions between the Instructor and the students. In the Experiment, students' inquiries about assessment persisted, but, by destabilizing their positions of Former Pupil and Popular Educator, the tasks had the result of focusing discussions about assessment on the content at hand. This, in turn, led us to interpret students' preoccupation with grades rather as their way of grasping what is important to learn, or simply as a manner of asking questions they may have been habituated into. Given the institutional constraints we faced, where behaviorist principles are built into the grading systems, this is not surprising, but, we conclude, students' preoccupation with grades can be reframed as a fruitful source of interaction about the taught content.

A culture of systemic justification was established

We described in section 5.2 how the Instructor and the students typically interacted. We showed that one of the modes of interaction involved the Instructor building the fraction of quantity theory according to highly scripted structures. There was certainly an asymmetry in student-teacher relations: the students' voices are not very present in our description. The asymmetry is, however, intellectual and not social, taking the form of a communication, from teacher to student, of a base of knowledge that the teacher possesses and the student does not. Evidence that the Instructor is fully aware of this asymmetry can be gleaned from her use of various didactical aids to help the students to level up in the proposed domain of knowledge. She used a variety of tools to reduce the intrinsic cognitive load of the material: worked out examples, color coding, schematic organization. But, essentially, she practiced vertical discourse, by justifying all the steps in the building of the theory, even when it was very hard, such as in the successive generalization of the operation of taking a fraction of a fraction of a quantity. When solving problems she did not encourage procedures based on superficial aspects of the problem, but reflected on the deep structure and even generalized across various structures to invite students to reflect on the types of problems proposed.

The 15 students belonging to the Interpretation 1 and Interpretation 4 groups appeared to have adapted to the culture created in the classroom of justification within the system of the theory of fractions of quantities, despite the fact that 10 of them (Interpretation 1) did not succeed in disproving the given statement. But the rest (23 students) continued to rely on procedural knowledge of fractions and operated only on abstract numbers: some of them to successfully justify the equivalence of two rational numbers (9 students – Interpretation 2), while others to simply carry out known procedures or fragments of them without the need for justification (6 students – Interpretation 3 and 8 students – Other Interpretations).

Risk of imitating behavior without adequate knowledge is high

We did not encounter this risk in the first instantiation of the course (the Pilot Study). But in the Experiment it appeared to run high. In the lecture presented in the section 5.2 we saw its realization in full swing, when a student, when asked to write "fraction statements", after having seen the Instructor write several with slightly modified deep structure, produced a statement that only superficially followed this structure. She did not even check her fraction statement against her natural grammar, thus producing a sentence that sounded awkward in English. Her colleague, by contrast, responding to the same question, like the five students belonging to Interpretation 4, appropriated the Instructor's shared concepts and methods much more flexibly, and engaged in creative problem posing, and, respectively, problem solving, behavior. More strikingly, however, more than half of the students answering a question that was not entirely novel, only "pasted", in some form, pieces of the fraction of a quantity theory, into their solution, without operationalizing it to produce arguments: some of them "plugged-in" values in the definition, demonstrating a more structural understanding, while others only used certain words or phrases from the theory, almost as a decoration for their solutions. The disconnect between the FoQ theory and the numerical operations *de facto* used aligns with findings from the literature that speak of preservice teachers' difficulty with linking conceptual and procedural knowledge with regards to fractions operations (Osana & Royea, 2011).

One possible explanation for students' difficulties in successfully using the "language" of the Instructor may lie in their difficulties with algebra. During the lectures the students had a hard time following the explanations when the level of the theory required reasoning using variables more heavily. An instance of this difficulty, which we described in section 5.2, was when the students complained about not being able to follow the Instructor's theorization of the operation of taking a fraction of a fraction of a quantity. Later on, some of the students confessed that they were able to follow the procedure mechanically but were could not connect it to the quantities at hand meaningfully. An eventual detachment is not all bad: in fact herein lies the power of algebra, and its rationale. However, the difficulty with reasoning using variables (such as $\boldsymbol{u}$ or $\boldsymbol{Q}$) – as exemplified by some students' use of the letter $\boldsymbol{u}$ as a name for unit – points to the deeper, genuine obstacle to reasoning about quantities abstractly, without knowing (or ignoring) their concrete values in some units of measurement, which is what algebra is all about.

Spontaneous engagement with quantitative reasoning is non-zero but rare

Only five of the students who answered the Question engaged in more spontaneous quantitative reasoning by considering the concrete quantities given in the statement. In the lectures, we have seen also that, when solving realistic problems, which did not take one of the formats specific to the newly learned FoQ theory, the students did not, when prompted, explain their solutions based on the quantities at hand. An instance of this behavior was when the Instructor asked a student to explain what he did when he added two fractions: the student described the procedure of adding fractions by bringing them to the same denominator. When asked further to explain why, he justified by saying that "the denominators must match." We have seen, in this, a different view – between the Instructor and the student – of what explaining means: for the Instructor it means connecting the operations to the actions on quantities, while for the student it equates to saying how to manipulate the numbers. The rationale of the Measurement Approach to provide tools, for future teachers, to give robust explanations involving quantities (not necessarily to children, rather to themselves) was perhaps missed by some students. Their view of it as an epistemologically equivalent alternative to numerical procedures can be gleaned from a student's remark during one of the lectures: "I don't get why we have to go through the 'unit thing' when I can solve this in 2 seconds in my head." Besides reinforcing the students' responsibility for learning the theory, perhaps some meta discussion should be carried explicitly in the course, of the kind we carried at places in this study, about the importance of attending to quantities when teaching children mathematics. As we have seen, to their detriment, a lack of such awareness may even lead to future teachers dismissing perfectly valid intuitive approaches that do not resemble learned algorithms, because they "don't look mathematical." Realistically perhaps only a sensitization to this issue can be achieved without the reality of teaching to children being close enough. Stephens (2003) argues that the most educators can hope for is to provide prospective teachers in methods courses with “sound conceptual knowledge [of the mathematics they will teach]” and some pedagogical content knowledge which must then be developed in the field through both teaching and professional development activities:

> *Content knowledge is invariably refined as a result of initial teaching experience. Pedagogical content knowledge is in a different category. Those completing initial teacher preparation can, at best, be viewed as novice holders of such knowledge. Such knowledge typically develops as a result of professional practice and opportunities to reflect upon that practice. It is sometimes called situated or case-specific knowledge. It cannot be taught or imparted in the same way as conceptual knowledge. The more seriously one takes this latter kind of knowledge as fundamental to the profession of teaching, the more seriously one has to view regulating entry to the profession as a process that commences with initial training but necessarily must continue well beyond that stage.* (Stephens, 2003, p. 785)

In the Measurement Approach, the conceptual content knowledge of mathematics was designed, a-priori, to take into account epistemological, cognitive and didactical dimensions of children’s and teachers’ learning; it is not, thus, a separate category from the knowledge for teaching, as suggested in the above quote. However, we agree that a full grasp of its relevance for teaching, as well as concrete ways of applying it, can only be acquired through practice. Ideally, such practice should take the form of an apprenticeship, starting with the shadowing of an experienced teacher. One such avenue is proposed in a recent New York Times opinion piece by Jal Mehta (Mehta, 2015), who proposes that teacher education be treated as a medical residency, as a way to restore teaching as a highly qualified professional activity:

> *We need an education system that can do for teachers what medical residencies do for doctors. These institutions would be modeled after teaching hospitals, designed to provide vertically integrated training over the first three years,*

eliminating the current split between teacher preparation institutions responsible for year one and the school districts responsible for years two and three.

Teachers would learn to teach in these specialized residency schools, and they would gradually take on more responsibility as they showed more competence. These "teaching" schools would attract master teachers who want to educate new entrants to the profession and could also function as places to learn the latest in cutting-edge research and practice.

Such a system would assure the public that tenured teachers had met significant standards and signal to prospective teachers that teaching is a demanding endeavor. It would move teacher training into the "field" and support new teachers in their aims to meet raised standards. (Mehta, 2015)

We add, that, like medical residents, future teachers would enter "the field" with sound theoretical knowledge rather than just mastery of the social skills needed to deal with parents and administration or of the jargon about reform ideals, while still replaying their own school experiences in the classroom. This is the goal that we have emphasized in our conception of the design of a Teaching Mathematics course for pre-service elementary teachers. How initial preparation of teachers is coordinated with subsequent professional development remains an important issue to be discussed, one that deserves further research.

Establishing or measuring the causal links between features of the design and the desired outcomes was not an objective of the study. The high variability of the results in students' learning we have displayed is a symptom of the irreducible complexity of teaching and learning situations when viewed from the point of view of a *design science*, and is related to what Labaree (2004) labeled as "the problem of chronic uncertainty about the effectiveness of teaching." The main contribution of this study is the argument that hard knowledge can be taught in elementary mathematics methods courses: in our case, it was a theory of fraction of quantity, inspired by a fundamental way of thinking about quantifiable aspects of objects as a fundamental way of doing mathematics. We demonstrated ways in which students can be empowered to adopt creative ways of thinking “within the box” in situations where previous procedural knowledge or mastering a “genre” of discourse is not enough for succeeding.

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