INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI®

NOTE TO USERS

This reproduction is the best copy available.

UMI

Invariant Measures for Inner Functions

Wael Bahsoun

A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University.
Montreal. Quebec. Canada.

March 2000

Canada

Abstract

Invariant Measures for Inner Functions

Wael Bahsoun

This thesis describes the chaotic behavior of inner functions in the unit disk and in the upper half plane. Absolutely continuous invariant measures for inner functions. ergodicity and exactness will be discussed. Moreover. under some conditions. we prove that the restriction of an inner function to $\mathbb{R}$ is ergodic if and only if the Julia set is the real line.

Acknowledgments

I would like to thank professor Pawel Góra who introduced me to the field of dynamical systems. Moreover. I am grateful to him for his help and support during my Master's and for his wise supervision throughout this thesis.

Contents

Introduction

The areas of dynamical systems and ergodic theory are rich in connections with subjects such as mathematical physics. statistics. number theory. geometry and biology. Although this field is relatively new in mathematics. it has led to interesting applications in other branches of science. One of the basic ideas in the field of dynamical systems is the theory of invariant measures. Absolutely continuous invariant measures are believed to be of physical importance.

In the late 1970's. attention had been called to transformations of the unit disk. and the upper half plane. that preserve Lebesgue measure. Inner functions are bounded analytic functions in the unit disk that have a limit a.e. on the unit circle. The study of ergodic properties of inner functions was begun by Aaronson [Aa] and Neuwirith [Ne].

In this thesis. we are going to present results on inner functions and their invariant

measures through proofs often more detailed than those in the original papers. Then, we will prove a new result for a class of meromorphic functions which are inner functions in the upper half plane.

In Chapter 1, we are going to present some preliminaries. In Chapter 2, we are going to discuss the chaotic behavior of inner functions in the unit disk starting with absolutely continuous invariant measures. After that, we will focus our attention on the case of a finite Blaschke product. In Chapter 3, we will move the results of Chapter 2 from the unit disk to the upper half plane. In addition, we discuss some strong results due to Aaronson [Aa] such as proving that ergodic inner functions are exact. Finally, we prove new results for a class of meromorphic functions which are inner functions in the upper half plane.

Chapter 1

Background Information

In this chapter we are going to present basic information from complex analysis, complex dynamics, hyperbolic geometry, measure theory and ergodic theory. For more information in complex analysis and measure theory we refer the reader to [Ru], and in ergodic theory to [BG].

1.1 Complex Analysis

In this section we assume familiarity with the definition of an analytic function and some of its properties such as the maximum modulus principle. We will present the notion of an inner function with some of its properties in the unit disk and in the upper half plane. The theorems of this section can be found in [Ru].

Definition 1.1 *The open unit disk is $D = \{z : |z| < 1\}$, the closed unit disk is $\overline{D} = \{z : |z| \leq 1\}$ and the unit circle is $S^1 = \{z : |z| = 1\}$.*

3

Definition 1.2 *An inner function in D is a function $F(z)$ such that $|F(z)| \leq 1$ $\forall z \in D$ and $|F(e^{it})| = 1$. a.e. on S^1.*

Theorem 1.1 *If $\{a_n\}$ is a sequence in D such that $a_n \neq 0$ and $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. if k is a nonnegative integer. and if $B(z) = z^k \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}$. $z \in D$. then $B(z)$ is an inner function and has no zeros except at the points a_n and the origin if $k > 0$.*

Proof. Consider $\sum_{n=1}^{\infty} |1 - \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}|$. The nth term is $|\frac{a_n + |a_n| z}{(1 - \bar{a}_n z) a_n}|(1 - |a_n|) \leq \frac{1 + r}{1 - r}(1 - |a_n|)$ if $|z| \leq r$. If $\sum_{n=1}^{\infty}(1 - |a_n|) < \infty$. then $\sum_{n=1}^{\infty} |1 - \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}|$ converges uniformly by the Weierstrass M-test in any compact subset of D. Therefore. $B(z) = z^k \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}$ converges uniformly in any compact subset of D. Consider $f(z) = \frac{a_n - z}{1 - \bar{a}_n z}$. Then.

$$|f(z)|^2 = \frac{-a_n \bar{z} + |a_n|^2 - z \bar{a}_n + |z|^2}{1 - a_n \bar{z} - \bar{a}_n z + |a_n|^2 |z|^2}.$$

Observe that $|f(z)| = 1$ for $|z| = 1$. Therefore. if $z \in D$ then $|f(z)| < 1$ by the maximum principle. Since each factor of $B(z)$ is less than 1 in D. it follows that $|B(z)| < 1$ in D. To complete the proof. we must show that $|B(e^{it})| = 1$ a.e.. on S^1. Let $B_n(z) = \prod_{j=1}^{n} \frac{a_n - z}{1 - \bar{a}_n z} \frac{|a_n|}{a_n}$ for each $n \geq 1$. Since $B(z)/B_n(z)$ is holomorphic in D. $|B(z)/B_n(z)|$ is subharmonic in D. Recall that the means of a subharmonic function are decreasing when the radius is increasing. Then. for $0 < r < r' < 1$ we get

$$\int_{-\pi}^{\pi} |B(re^{it})/B_n(re^{it})| dt \leq \int_{-\pi}^{\pi} |B(r'e^{it})/B_n(r'e^{it})| dt.$$

4

Fix r and let $r' \to 1$. Since $|B_n(e^{it})| = 1$ on S^1 we get

$$\int_{-\pi}^{\pi} |B(re^{it})/B_n(re^{it})| \, dt \le \int_{-\pi}^{\pi} |B(e^{it})| \, dt \text{ a.e..}$$

Let $n \to \infty$ and use Fatou's lemma to obtain

$$\int_{-\pi}^{\pi} dt \le \int_{-\pi}^{\pi} |B(e^{it})| \, dt.$$

Since $|B(e^{it})| \le 1$. we get $|B(e^{it})| = 1$. a.e. on S^1. ∎

Definition 1.3 *The product $B(z)$ produced in the above theorem is called a Blaschke product.*

Theorem 1.2 *Suppose c is a constant. $|c| = 1$. $B(z)$ is a Blaschke product. μ is a positive Borel measure which is singular with respect to Lebesgue measure. Let*

$$F(z) = cB(z) \exp\{ -\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \}.$$

$z \in D$. Then $F(z)$ is an inner function in D and every inner function in D is of this form.

Proof. Let $g = \frac{F}{B}$. Then $\mathrm{Log}|g|$ is the Poisson integral of $-d\mu$. Therefore. $\mathrm{Log}|g| \le 0$. i.e.. $|g| \le 1$. Since $F = Bg$. then $|F(z)| = |B(z)||g(z)| \le 1$ for any $z \in D$. Since μ is singular with respect to Lebesgue measure then $D_\mu = 0$. a.e. on S^1. where D_μ denotes the Radon-Nikodym derivative of the measure. By theorem 11.10 in [RU]. the radial limits of a harmonic function equal to D_μ. a.e. on S^1. Therefore. $\lim_{r \to 1} \mathrm{Log}|g(re^{it})| = D_\mu(t)$. a.e. on S^1. Then $\mathrm{Log}|g| = 0$. a.e. on S^1. Hence. $|g| = 1$.

5

a.e. on S^1. It follows that $|F| = |B||g| = 1$. a.e. on S^1. Therefore. F is an inner function.

Now. consider F to be any inner function. Let $B(z)$ be the Blaschke product formed with the zeros of F. Consider $g = \frac{F}{B}$. Then g is zero free in D. It follows that $\text{Log}|g|$ is harmonic. Since $g = \frac{F}{B}$. then $|g| = 1$ a.e. on S^1. By the maximum principle $|g(z)| \leq 1$ in $\overline{D}$. Then. $\text{Log}|g| \leq 0$. $\text{Log}|g|$ is harmonic. then it can be written as the Poisson integral of its boundary values

$$\text{Log}|g(z)| = -\int_{-\pi}^{\pi} \frac{1 - |z|}{|e^{it} - z|^2} d\mu(t).$$

Since. $\text{Log}|g| = 0$. a.e. on S^1. then $D_\mu = 0$. a.e. on S^1. It follows that μ is singular. Now. observe that $\text{Log}|g|$ is the real part of

$$h(z) = -\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t).$$

This implies that $g = c \exp(h)$ for some constant c with $|c| = 1$. Thus.

$$F = cB(z) \exp\{-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\}$$

is the form of an inner function in D ∎

Definition 1.4 *A singular inner function is an inner function of the form*

$$F(z) = c \exp\{-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)\}.$$

$z \in D$. where μ is a positive Borel measure which is singular with respect to Lebesgue measure and c is a constant with $|c| = 1$.

6

Definition 1.5 *The upper half plane is defined by* $\mathbb{R}^{2+} = \{z : Imz > 0\}$.

Definition 1.6 *An inner function in* $\mathbb{R}^{2+}$ *is an analytic function* $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ *such that for a.e.* $x \in \mathbb{R}$. $F(x + iy) \to T(x) \in \mathbb{R}$ *as* $y \to 0$. *The measurable function* $T : \mathbb{R} \to \mathbb{R}$ *is the restriction of* F *to* $\mathbb{R}$.

1.2   Complex Dynamics

The subject of complex dynamics is large and rapidly growing. Fatou and Julia started this theory when they independently discovered the dichotomy of the Riemann sphere into sets now bearing their names. We will present some of the information that we need for the remaining chapters. For more details we refer the reader to [Mi].

Definition 1.7 *The Riemann sphere is defined by* $\hat{C} = C \cup \{\infty\}$ *where* C *is the complex plane.*

Definition 1.8 *A family of analytic functions having a common domain of definition is called normal if every sequence in this family contains a locally uniformly convergent subsequence.*

Definition 1.9 *Let* $f : \hat{C} \to \hat{C}$. *The Fatou set is defined by* $F(f) = \{z : \exists\ a$ *neighborhood* U_z *s.t.* $\{f^n\}_{n=1}^{\infty}$ *is a normal family in* $U_z\}$. *The Julia set is defined by* $J(f) = \hat{C} \backslash F(f)$.

Definition 1.10 *Let* $f : \hat{C} \to \hat{C}$ *and* $f(z_0) = z_0$. *i.e., let* z_0 *be a fixed point of* f. *Then. we say that:*

7

1. z_0 is an attracting fixed point if $|f'(z_0)| < 1$.

2. z_0 is a neutral fixed point if $|f'(z_0)| = 1$.

3. z_0 is a repelling fixed point if $|f'(z_0)| > 1$.

Definition 1.11 *Let $f(z) = \frac{az+b}{cz+d}$, $ad - bc \neq 0$. where a, b, c, d are complex constants. A function of this form is called a Möbius function.*

Lemma 1.1 (Schwarz) *If $f : D \to D$ is a holomorphic map with $f(0) = 0$. then the derivative at the origin satisfies $|f'(0)| \leq 1$. If equality holds. then f is a rotation about the origin. that is $f(z) = \lambda z$ for some constant $\lambda = f'(0)$ on the unit circle. If $|f'(0)| < 1$. then $|f(z)| < z$ for all $z \neq 0$.*

Proof. Consider the function $g(z) = \frac{f(z)}{z}$ for $z \neq 0$ and $g(0) = f'(0)$. Observe that $g(z)$ is well defined and holomorphic throughout D. Since $|g(z)| < \frac{1}{r}$ when $|z| = r < 1$. it follows by the maximum modulus principle that $|g(z)| < \frac{1}{r}$ for all z in the disk $|z| \leq r$. Taking the limit as $r \to 1$. we see that $|g(z)| \leq 1$ for all $z \in D$. Again by the maximum modulus principle. we see that the case $|g(z)| = 1$ with $z \in D$. can occur only if the function $g(z)$ is constant. If we exclude this case $\frac{f(z)}{z} = \lambda$. then it follows that $|g(z)| = \frac{f(z)}{z} < 1$ for all $z \neq 0$. and similarly that $|g(0)| = |f'(0)| < 1$.

∎

1.3 Hyperbolic Geometry

In this section we are going to present the Poincaré metric or the hyperbolic metric not only for the geometry itself, but for the application of this geometry to function theory. This metric is considered to be a useful tool in proving some theorems in dynamical systems. Let us consider the fractional linear transformation $S(z) = \frac{az-b}{bz-\overline{a}}$ with $|a|^2 - |b|^2 = 1$. Observe that $S(z)$ maps D conformaly into itself.

Definition 1.12 *Consider* $z_1, z_2 \in D$ *and set* $w_1 = S(z_1), w_2 = S(z_2)$. *Then .*

$$w_1 - w_2 = \frac{az_1 + b}{\overline{b}z_1 + \overline{a}} - \frac{az_2 + b}{\overline{b}z_2 + \overline{a}}$$

$$= \frac{z_1 - z_2}{(\overline{b}z_1 + \overline{a})(\overline{b}z_2 + \overline{a})}$$

and

$$1 - \overline{w_1}w_2 = 1 - \frac{\overline{a}\overline{z}_1 + \overline{b}}{\overline{b}\overline{z}_1 + a}\frac{az_2 + b}{\overline{b}z_2 + \overline{a}}$$

$$= \frac{1 - \overline{z}_1 z_2}{(\overline{b}\overline{z}_1 + a)(\overline{b}z_2 + \overline{a})}.$$

Hence,

$$\left|\frac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right| = \left|\frac{w_1 - w_2}{1 - \overline{w}_1 w_2}\right|.$$

Define $\delta(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \overline{z}_1 z_2}\right|$ *to be the Poincare distance between two points in* D. *Observe that* δ *is a metric in* D. *Another fact can be also read from the identity*

$$1 - \delta(z_1, z_2)^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \overline{z}_1 z_2|^2}.$$

9

If $z_1 \rightarrow z_2$ we get

$$\frac{|dz|}{1 - |z|^2} = \frac{|dw|}{1 - |w|^2}.$$

The shortest arc from zero to any other point is along the radius. Hence the geodesics are circles orthogonal to S^1. They can be considered straight lines in a geometry, the hyperbolic or the Poincare geometry of the disk. The hyperbolic distance from 0 to $r > 0$ is

$$\int_0^r \frac{2dr}{1 - r^2} = Log \frac{1 + r}{1 - r}.$$

From the above definition we can notice that the hyperbolic geometry does not satisfy the parallel straight line axiom in the Euclidean geometry.

Lemma 1.2 *(Schwarz-Pick) Let $f : D \rightarrow D$ be an analytic map. Then either f is linear, or else for each $z_0 \in D$ the inequalities*

$$\frac{|f(z) - f(z_0)|}{|1 - \overline{f(z_0)}f(z)|} < \frac{|z - z_0|}{|1 - \overline{z_0}z|}, \quad z \neq z_0$$

and

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} < \frac{1}{1 - |z_0|^2}$$

hold.

Proof. Let $f(z_0) = w_0$. Let T_1, T_2 be transformations taking D onto D with $T_1(z_0) = T_2(w_0) = 0$. So, $T_1(z) = \frac{z - z_0}{1 - \overline{z_0}z}$ and $T_2(w) = \frac{w - w_0}{1 - \overline{w_0}w}$. Define an analytic

function o on D by $o(z) = T_2 f(T^{-1}z)$. Then

$$
\begin{aligned}
o(0) &= T_2 f(T^{-1}(0)) \\
&= T_2 f(z_0) \\
&= T_2(w_0) \\
&= 0
\end{aligned}
$$

and $|o(z)| \leq 1$ if $|z| < 1$. By Schwarz's lemma either $|o(z)| < |z|$, $z \neq 0$ or else $o(z) = cz$ where $|c| = 1$. In the later case let $z = T_1\varsigma$, $\varsigma \in D$. We have

$$
\begin{aligned}
f(\varsigma) &= T_2^{-1} o(T_1\varsigma) \\
&= T_2^{-1} c T_1(\varsigma)
\end{aligned}
$$

In this case f is a linear transformation of the unit disk into itself. In the former case where f is not linear we find that $|T_2 f(\varsigma)| < |T_1(\varsigma)|$ for all $\varsigma \in D$, $\varsigma \neq z_0$. Therefore,

$$
\frac{|f(\varsigma) - f(z_0)|}{|1 - \overline{f(z_0)}f(\varsigma)|} < \frac{|\varsigma - z_0|}{|1 - \overline{z_0}\varsigma|}, \quad z \neq z_0.
$$

As a conclusion, we can say that an analytic mapping of the unit disk into itself which is not linear decreases the noneuclidean distance between any two points. Now, we are left to prove that if f is not linear, then

$$
\frac{|f'(z_0)|}{1 - |f(z_0)|^2} < \frac{1}{1 - |z_0|^2}.
$$

For fixed z_0 define g by

$$
\begin{aligned}
&\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)} \frac{1 - \overline{z_0}z}{z - z_0}, \quad z \neq z_0 \\
&f'(z_0) \frac{1 - |z_0|^2}{1 - |f(z_0)|^2}, \quad z = z_0
\end{aligned}
$$

11

. The function $g(z)$ is analytic. and by the first part of the lemma $|g(z)| < 1$ for $z \neq z_0$. From the maximum principle we conclude that $|g(z_0)| < 1$. which is equivalent to

$$\frac{|f'(z_0)|}{1 - |f(z_0)|^2} < \frac{1}{1 - |z_0|^2}.$$

Now. we can conclude that an analytic mapping of the unit disk into itself decreases the noneuclidean length of an arc. ■

1.4 Measure and Ergodic Theory

In this section our aim is to present basic definitions and theorems in ergodic theory. The reader who would like to know more about ergodic theory will find much more in [BG].

Definition 1.13 *Let X be a space. Let β be a σ-algebra of Borel sets of X and μ be a measure defined on β. Then $(X.\beta, \mu)$ is called a measure space.*

Definition 1.14 *If $(X.\beta, \mu)$ and $(Y.\pounds.\nu)$ are measure spaces. and f is a mapping of X into Y. then f is said to be measurable provided that $f^{-1}(B)$ is a measurable set in X for every measurable set B in Y.*

Definition 1.15 *Let $(X.\beta, \mu)$ be a measure space and let $T : X \longrightarrow X$ be a measurable function. Then μ is said to be T invariant if for any set $B \in \beta$ we have $\mu(B) = \mu(T^{-1}(B))$.*

12

Lemma 1.3 *Let (X, β, μ) be a measure space and let $T : X \rightarrow X$ be a measurable function. Then μ is T invariant if and only if $\int f d\mu = \int f \circ T d\mu \; \forall f \in L^1(X, \beta, \mu)$.*

Lemma 1.4 *Let (X, β, μ) be a measure space and let $T : X \rightarrow X$ be a measurable function. Then μ is T invariant if and only if $\int f d\mu = \int f \circ T d\mu \; \forall f \in C^o(X)$.*

Proof. One way follows from the previous lemma. The other way follows from the Riesz representation and the Hahn-Banach theorems which are stated and proved in [Ru]. ∎

Definition 1.16 *Let $T : X \rightarrow X$ be any transformation. The nth iterate of T is denoted by T^n. i.e.*

$$T^n(x) = T \circ T \circ \ldots\ldots\ldots \circ T(x)$$

n times. $\{T^n(x)\}_{n \geq 0}$ is called the orbit of x.

Theorem 1.3 *(Poincaré Recurrence Theorem) Let (X, β, μ) be a probability measure space . Let $T : X \rightarrow X$ be a measurable transformation which preserves μ. Let $E \in \beta$ such that $\mu(E) > 0$. Then almost all points of E return infinitely often to E under iterations of T.*

Proof. Let E be a measurable set with $\mu(E) > 0$, and let us define the set B of points that never return to E. i.e.. $B = \{x \in E : T^k(x) \notin E, k = 1, 2, \ldots\}$. We will prove that

$$T^{-i}(B) \cap T^{-j}(B) = \emptyset.$$

13

for $i > j \geq 0$. If $x \in T^{-i}(B) \cap T^{-j}(B)$, then $T^j(x) \in B$ and $T^{i-j}(T^j(x)) = T^i(x) \in B$, which contradicts the definition of B. Hence, we have

$$\sum_{i=0}^{\infty} \mu\left(T^{-i}(B)\right) = \mu\left(\cup_{i=0}^{\infty} T^{-i}(B)\right) \leq \mu(X) = 1.$$

Since μ is T-invariant, this implies that $\sum_{i=0}^{\infty} \mu(B) \leq 1$. Therefore, $\mu(B) = 0$. ∎

Definition 1.17 *Let $T : X \to X$ be a transformation. A point $x \in X$ is called T-recurrent if and only if $\exists$ a strictly increasing sequence of positive integers $(n_i)_{i=0}^{\infty}$ such that*

$$x = \lim_{i \to +\infty} T^{n_i}(x).$$

Definition 1.18 *Let (X, β, μ) be a probability measure space. Let $T : X \to X$ be a measurable transformation. T is called ergodic if for every set $B \in \beta$ with $T^{-1}B = B$ we have that $\mu(B) = 0$ or $\mu(B) = 1$.*

A more general definition of ergodicity is the following

Definition 1.19 *Let (X, β, μ) be a measure space. Let $T : X \to X$ be a measurable transformation. T is called ergodic if for every set $B \in \beta$ with $T^{-1}B = B$ we have that $\mu(B) = 0$ or $\mu(B^c) = 0$.*

Lemma 1.5 *T is ergodic with respect to μ if and only if, whenever $f \in L^1(X, \beta, \mu)$ satisfies $f = f \circ T$, then f is a constant function.*

14

Definition 1.20 *We call a measure preserving transformation $T : X \to X$ on a probability measure space (X, β, μ) an exact endomorphism if $\bigcap_{n=0}^{\infty} T^{-n}\beta = \{X, \emptyset\}$ up to sets of zero measure (i.e., if $B \in T^{-n}\beta$. for every $n \geq 0$. then $\mu(B) = 0$ or $\mu(B) = 1$).*

From the above definition one can observe that if T is exact then it is ergodic.

Proposition 1.1 $T : X \to X$ *is exact if for any positive measure set A with $T^n A \in \beta$. $n \geq 0$. $\mu(T^n A) \to 1$ as $n \to \infty$.*

Proof. First we remark that T is exact if every measurable set A satisfying for arbitrary n the relationship $A = T^{-n}(T^n A)$ is either of measure zero or measure one. For such a set A. it is clear that $\mu(A) = 1$ if $\mu(A) > 0$. as $\mu(T^n A) = \mu(A)$ and so $\lim_{n \to \infty} \mu(T^n A) = \mu(A) = 1$ if $\mu(A) > 0$. ∎

Definition 1.21 *Let (X, β, μ) be a normalized measure space. Then $T : X \to X$ is said to be non-singular if and only if $T_*\mu \ll \mu$. i.e., if for any $A \in \beta$ such that $\mu(A) = 0$. we have $T_*\mu(a) = \mu(T^{-1}A) = 0$.*

Proposition 1.2 *Let T be a non-singular transformation of (X, β, μ). If T is exact. then*

$$\|T^{*n} f\|_1 \to 0 \text{ as } n \to \infty \ \forall f \in L^1. \ \int_X f d\mu = 0$$

Proof. Let $f \in L^1$. $\int_X f d\mu = 0$. There are functions $g_n \in L^\infty$. such that $\|g_n\|_\infty = 1$. and $\int_X f g_n \circ T^n d\mu = \|T^{*n}\|_1$. Any weak-*-limit of $\{g_n \circ T^n\}$ is measurable

15

with respect to $\cap_{n \geq 1} T^{-n} \mathfrak{B}$, and hence is constant by exactness. Whence.

$$\lim_{n \to \infty} ||T^{*n} f||_1 = \lim_{n \to \infty} \int_X f g_n \circ T^n d\mu = 0.$$

and the proposition follows. ∎

Chapter 2

Invariant Measures For Inner

Functions In The Unit Disk

2.1 Absolutely Continuous Invariant Measure

Proposition 2.1 *Let $F : D \to D$ be an inner function that has a fixed point in D. $F(0) = 0$. Then the restriction of F to S^1 $.o : S^1 \to S^1$, preserves Lebesgue measure.*

Proof. Let v be a continuous real valued function on S^1. Since F is analytic in D then $v \circ F$ is harmonic in D and the Poisson integrals of $v \circ F$ and $v \circ o$ are equal because both have the same boundary values; therefore, we can write

$$\int_{S^1} v(e^{it}) \frac{1 - |F(z)|^2}{|e^{it} - F(z)|^2} dt = \int_{S^1} v \circ o \left(e^{it}\right) \frac{1 - |z|^2}{|e^{it} - z|^2} dt.$$

for any $z \in D$. Since $F(0) = 0$, we get

$$\int_{S^1} v(e^{it})dt = \int_{S^1} v \circ o(e^{it})dt.$$

By Lemma 1.4, we conclude that if $F(0) = 0$, the restriction of F to the unit circle preserves Lebesgue measure. ∎

Corollary 2.1 *Let* $F : D \longrightarrow D$ *be an inner function that has a fixed point in* D. $F(z_0) = z_0$. *Then the restriction of* F *to* S^1, $o : S^1 \longrightarrow S^1$, *preserves the Poisson measure* $\frac{1-|z_0|^2}{|e^{it}-z_0|^2}dt$, *where dt is Lebesgue measure.*

Proof. Let v be a continuous real valued function on S^1. Since F is analytic in D then $v \circ F$ is harmonic in D and the Poisson integrals of $v \circ F$ and $v \circ o$ are equal because both have the same boundary values; therefore, we can write

$$\int_{S^1} v(e^{it})\frac{1-|F(z)|^2}{|e^{it}-F(z)|^2}dt = \int_{S^1} v \circ o(e^{it})\frac{1-|z|^2}{|e^{it}-z|^2}dt.$$

for any $z \in D$. Since $F(z_0) = z_0$ we get

$$\int_{S^1} v(e^{it})\frac{1-|z_0|^2}{|e^{it}-z_0|^2}dt = \int_{S^1} v \circ o(e^{it})\frac{1-|z_0|^2}{|e^{it}-z_0|^2}dt.$$

By Lemma 1.4, we conclude that the restriction of every inner function F with $F(z_0) = z_0$, $z_0 \in D$, to the unit circle preserves the Poisson measure. ∎

Proposition 2.2 *Let* $F : D \longrightarrow D$ *be an inner function. Let* o *be the restriction of* F *to* S^1 *and* μ *be* o *invariant, where* $d\mu = \frac{e^{it}-|z_0|^2}{|e^{it}-z_0|^2}dt$. dt *is Lebesgue measure. Then* F *has a fixed point in* D.

18

Proof. Let v be a continuous real function on S^1 then $v \circ F$ is harmonic in D and the Poisson integrals of $v \circ F$ and $v \circ o$ are equal

$$\int_{S^1} v \circ \varphi(e^{it}) \frac{1 - |z_0|^2}{|e^{it} - z_0|^2} dt = \int_{S^1} v(e^{it}) \frac{1 - |F(z_0)|^2}{|e^{it} - F(z_0)|^2} dt.$$

Since o preserves μ then by Lemma 1.4 we get

$$\int_{S^1} v \circ o\left(e^{it}\right) \frac{1 - |z_0|^2}{|e^{it} - z_0|^2} dt = \int_{S^1} v\left(e^{it}\right) \frac{1 - |z_0|^2}{|e^{it} - z_0|^2} dt.$$

Hence.

$$\int_{S^1} v\left(e^{it}\right) \frac{1 - |F(z_0)|^2}{|e^{it} - F(z_0)|^2} dt = \int_{S^1} v(e^{it}) \frac{1 - |z_0|^2}{|e^{it} - z_0|^2} dt.$$

Therefore.

$$v(F(z_0)) = v(z_0).$$

Since this is true for any continuous function. it follows that $F(z_0) = z_0$. i.e.. F has a fixed point in D. ∎

2.2 Denjoy-Wolff Theorem

Lemma 2.1 *Suppose U is hyperbolic disk. $F : U \longrightarrow U$ is analytic and F is not an isometry with respect to the hyperbolic metric. Then either $F^n(z) \longrightarrow \partial U$ for all $z \in U$. or else there is an attracting fixed point for F in U to which all orbits converge.*

Proof. Since F is not an isometry then $\delta(F(z). F(w)) < \delta(z. w)$ for all $z. w \in U$ by Schwarz-Pick lemma. In particular for any compact set $K \subset U$. there is a constant

19

$k = k(K)$ such that $\delta(F(z), F(w)) \leq k\delta(z, w)$. $z, w \in K$. Suppose there is a $z_0 \in U$ whose iterates visit some compact subset L of U infinitely often. Take K to be a compact neighborhood of $L \cup F(L)$ then $\delta(z_{m+2}, z_{m+1}) \leq k\delta(z_{m+1}, z_m)$ whenever $z_m \in L$. This occurs infinitely often. so $\delta(z_{n+1}, z_n) \to 0$. Thus by continuity any limit point $\xi \in L$ of the sequence $\{z_n\}$ is fixed by F. which is attracting because $\delta(F(z), \xi) \leq \delta(z, \xi)$ in some neighborhood of ξ. Since the iterates of F form a normal family they converge on U to ξ. ∎

Theorem 2.1 (**Denjoy-Wolff**) *Suppose that* $F : D \to D$ *is analytic and not Möbius. Then, there is a unique point* $\alpha \in \overline{D}$ *such that* $F^n(z) \to \alpha$ *as* $n \to \infty \forall z \in D$.

Proof. If the orbit of zero visits any compact set infinitely often. the previous lemma provides a fixed point in D. Thus. we may assume that the orbit of zero accumulates on ∂D. The remaining part is to show that this accumulation point is unique. Define $F_\epsilon = (1 - \epsilon)F(z)$ which is analytic and maps to a compact subset of D. Let z_ϵ be the fixed point of F_ϵ. and let D_ϵ be the hyperbolic disk centered at z_ϵ with radius $\delta(0, z_\epsilon)$. Since F_ϵ is contracting then $F_\epsilon(D_\epsilon) \subset D_\epsilon$. Now consider D_ϵ in the Euclidean metric. Zero is on the boundary of D_ϵ. Let D_0 be any of the limits of the circles D_ϵ's as $\epsilon \to 0$. see Figure 1. D_0 is a Euclidean disk with zero on its boundary and $F(D_0) \subset D_0$. Hence. the orbit of zero never leaves D_0. Thus. the point of tangency of D_0 and ∂D is a unique limit point of the orbit of zero. ∎

Definition 2.1 *(**Denjoy-Wolff point**) Since the point* α *produced by the Denjoy-Wolff theorem is unique, it is called the Denjoy-Wolff point.*
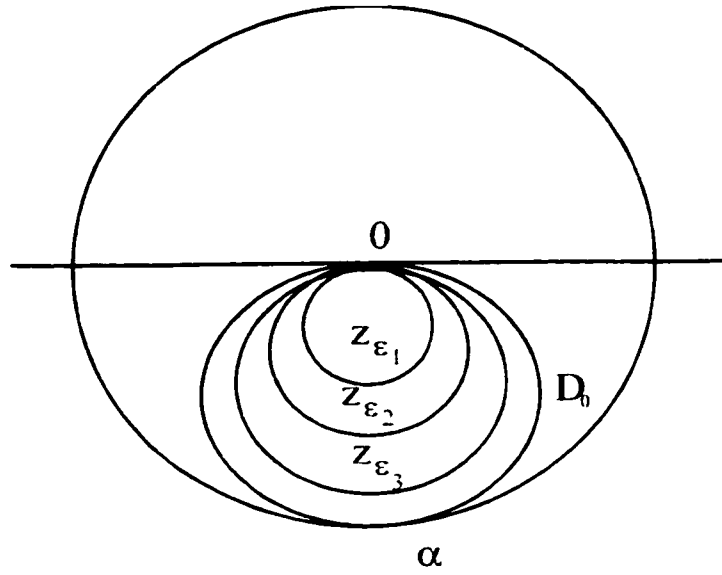
Figure 1

Now. we are going to present a special definition for recurrence.

Definition 2.2 *A measurable map* $T : X \rightarrow X$ *of a measure space* $(X. \text{ß}. \mu)$ *is recurrent if the set*

$$3^{\bullet} = \bigcap_{N \geq 0} \bigcup_{n \geq N} T^{-n}(3)$$

of all points $x \in X$ *such that* $T^n(x) \in 3$ *for infintely many values of* $n \geq 0$ *satisfies* $\mu(3 - 3^{\bullet}) = 0$ *for every* $3 \in \text{ß}$.

Theorem 2.2 *If* $F : D \rightarrow D$ *is an inner function and there exists a point* $z \in D$ *such that*

$$\sum_{n \geq 0} (1 - |F^n(z)|) < \infty.$$

then the Denjoy-Wolff point α *of* F *belongs to* S^1 *and* $\lim_{n \to \infty} F^n(w) = \alpha$ *for a.e.* $w \in S^1$.

21

Proof. Assume $z \in D$ is such that $\sum_{n \geq 0} (1 - |F^n(z)|) < \infty$: this implies that the Denjoy-Wolff point α of F belongs to S^1. Given a Borel set $S \subseteq S^1$, let

$$S^* = \bigcup_{N \geq 0} \bigcap_{n \geq N} F^{-n}(S)$$

denote the set of all points $w \in S$ for all $n \geq N$. It is enough to prove that $m(S^*) = 1$ for every open neighborhood S of α, because taking a decreasing sequence of open neighborhoods S_k of α such that $\bigcap_{k \geq 0} S_k = \{\alpha\}$, we obtain

$$m\left(\bigcap_{k \geq 0} S_k^*\right) = 1:$$

all that remains to do is to observe that $w \in \bigcap_{k \geq 0} S_k^*$ implies that for each neighborhood S_k of $\alpha \exists N_k \geq 0$ such that $F^n(w) \in S_k \forall n \geq N_k$, which means that $\lim_{n \to \infty} F^n(w) = \alpha$. Since $m\left(\bigcap_{k \geq 0} S^*\right) = 1$, this will prove the theorem.

To estimate $m(S^*)$, we recall that the Poisson measure, $d\mu_{z_0} = \frac{1 - |z_0|^2}{|e^{it} - z_0|^2} dm$, is equivalent to the Lebesgue probability measure and consider the characteristic function v of S^c. Then $v \circ F^n$ is the characteristic function of $F^{-n}(S^c)$. It follows that

$$\mu(F^{-n}(S^c)) = \int (v \circ F^n d\mu_z)) = \int v d\mu_{F^n(z)} = v^*(F^n(z)).$$

Since

$$S^{*c} = \bigcup_{N \geq 0} \bigcap_{n \geq N} F^{-n}(S^c).$$

we obtain

$$\mu_z(S^{*c}) \leq \inf_{N \geq 0} \sum_{n \geq N} \upsilon^*(F^n(z)).$$

We also have

$$\frac{\upsilon^*(F^n(z))}{1 - |F^n(z)|} = \frac{1}{1 - |F^n(z)|} \int_{S^c} \frac{1 - |F^n(z)|^2}{|w - F^n(z)|^2} d\mu_w = (1 + |F^n(z)|) \int_{S^c} \frac{1}{|w - F^n(z)|^2} d\mu_w$$

Let us now assume that S is an open neighborhood of α. Then

$$\lim_{n \to \infty} \frac{\upsilon^*(F^n(z))}{1 - |F^n(z)|} = 2 \int_{S^c} \frac{1}{|w - \alpha|^2} d\mu_w.$$

Hence there exists $C > 0$ such that $\upsilon^*(F^n(z)) \leq C(1 - |F^n(z)|)$ for large $n \geq 0$ and

therefore

$$\sum_{n \geq N} \upsilon^*(F^n(z)) \leq C \sum_{n \geq N} (1 - |F^n(z)|)$$

for large $N \geq 0$. But our hypothesis is $\sum_{n \geq 0} (1 - |F^n(z)|) < \infty$. from which it

follows that $\inf_{N \geq 0} \sum_{n \geq N} \upsilon^*(F^n(z)) = 0$. Therefore. $\mu_z(S^{*c}) = 0$. implying that

$m(S^{*c}) = 0$. i.e.. $m(S^*) = 1$ ∎

2.3 Ergodicity and Exactness

Proposition 2.3 [Ne] *Let F be an inner function with $F(0) = 0$. Suppose that F is not invertible. Then, the restriction of F to S^1 is exact.*

Proof. Let $E \in \bigcap F^{-n}(\beta)$. Then for every n we can find a set $E_n \in \beta$ so that $E = F^{-n}(E_n)$. Since Lebesgue measure is preserved by F (Proposition 2.1). we have $m(E) = m(E_n)$ for all n. Letting χ_E and χ_{E_n} be the characteristic functions of E and E_n. respectively. we find that

$$\int \chi_E(t) \frac{1 - |z|^2}{|e^{it} - z|^2} dt = \int \chi_{E_n}(F^n(t)) \frac{1 - |z|^2}{|e^{it} - z|^2} dt = \int \chi_{E_n}(t) \frac{1 - |F^n(z)|^2}{|e^{it} - F^n(z)|^2} dt$$

for all $z \in D$ and all n. Since $z = 0$ is an attracting fixed point. then $\lim_{n \to \infty} F^n(z) = 0$. Therefore. $\lim_{n \to \infty} \frac{1 - |F^n(z)|^2}{|e^{it} - F^n(z)|} = 1$ uniformly on S^1 for all $z \in D$. Thus.

$$\int \chi_E(t) \frac{1 - |z|^2}{|e^{it} - z|^2} dt = m(E) + \lim \int \chi_{E_n}(t) [\frac{1 - |z|^2}{|e^{it} - z|^2} - 1] dt$$

$$= m(E)$$

Therefore. $\chi_E(t)$ is constant a.e. on S^1. Hence. $m(E) = 0$ or $m(E) = 1$. It follows that the restriction of F to S^1 is exact. ∎

Now. we are going to present a general definition for ergodicity. This definition is mainly used when we don't have a measure preserving transformation.

Definition 2.3 *Let* $T : S^1 \to S^1$ *be any transformation.* T *is said to be ergodic if whenever* $T^{-1}(B) = B$, *then* $m(B) = 0$ *or* $m(S^1 \backslash B) = 0$.

Proposition 2.4 [Ne] *Let* F *be an inner function with Denjoy-Wolff point* $\alpha \in S^1$. *Assume that* $F'(\alpha) = \gamma$, $|\gamma| < 1$, *and* F *has an analytic extension around* α. *Then the restriction of* F *to* S^1 *is not ergodic.*

Proof. Since F is analytic in a neighborhood of α, there is an $\epsilon > 0$ and $\eta < 1$ such that $F'(z) \neq 0$ and $|F(z) - \alpha| \leq \eta |z - \alpha|$ for $|z - \alpha| \leq \epsilon$. Let $G = \{z : |z - \alpha| \leq \epsilon\}$: then $\forall z \in G$

$$\frac{\alpha - F^n(z)}{\gamma^n} \to A(z) \text{ for } |z - \alpha| < \epsilon:$$

where $A(z) = \{t \in S^1 : F^n(t) \to \alpha\}$. Clearly $A(z)$ is F invariant and $\{|z - \alpha| < \epsilon\} \cap S^1\} \subseteq A(z)$, and $m(\{|z - \alpha| < \epsilon\} \cap S^1\}) > 0$. Therefore, $m(A(z)) > 0$. If F were ergodic then $A(z) = S^1$ a.e. and this completes the proof. ∎

Lemma 2.2

$$\int_{S^1} |p_z - p_w| dm = \frac{4}{\pi} \sin^{-1} \delta(z, w).$$

where δ *denotes the Poincaré distance in* D *and* p_z *denotes the Poisson kernel.*

Proof. Set $d(z, w) = \int_{S^1} |p_z - p_w| dm$. Suppose that $\gamma : D \to D$ is a Möbius transformation. Its S^1-restriction g is an invertible non-singular transformation of (S^1, β, m). Write

$$d(z, w) = \sup_{o \in L^\infty(m)} \int_{S^1} (p_z - p_w) o\, dm.$$

Observe that $p_z \varnothing = p_z \varnothing \circ g$. Hence.

$$d(z, w) = \sup_{\varnothing \in L^\infty(m)} \int_{S^1} (p_{\gamma(z)} - p_{\gamma(w)}) \varnothing dm = d(\gamma(z), \gamma(w)).$$

It follows that

$$d(z, w) = d(0, \left| \frac{z - w}{1 - \overline{w}z} \right|).$$

for $r \in [0, 1)$ we get

$$d(0, r) = \frac{4}{\pi} \sin^{-1} r.$$

Hence, the lemma. ∎

Theorem 2.3 [Aa] *Let $F : D \to D$ be an inner function. Then F is exact if and only if $\delta(F^n(z_1), F^n(z_2)) \to 0$ as $n \to \infty$. where δ denotes the Poincaré distance in D.*

Proof. First, suppose that F is exact. By Proposition 1.2

$$\|F^{*n}g\|_1 \to 0 \ \forall \ g \in L^1, \ \int g = 0.$$

In particular, by Lemma 2.2. for $z_1, z_2 \in D$.

$$\delta(F^n(z_1), F^n(z_2)) = \sin\left(\frac{\pi}{4} \|F^{*n}(p_{z_1} - p_{z_2})\|_1 \right) \to 0.$$

Conversely, suppose that $\lim_{n \to \infty} \delta(F^n(z_1), F^n(z_2)) = 0$. Let ß denote the Borel σ-algebra of S^1. Let $A \in \bigcap_{n \geq 0} F^{-n}(\text{ß})$. then $A = F^{-n}(A_n)$ with $A_n \in \text{ß}$. $n \geq 0$. Consider the characteristic functions χ and χ_n of A and A_n respectively. Since $A =$

$F^{-n}(A_n)$. then $\chi_n \circ F^n = \chi$. Therefore. the boundary values of $\chi_n \circ F^n$ and χ are equal. Thus. for $z_1, z_2 \in D$ we get

$$\int_{S^1} \chi(t) \frac{1 - |z_1|^2}{|e^{it} - z_1|^2} dt - \int_{S^1} \chi(t) \frac{1 - |z_2|^2}{|e^{it} - z_2|^2} dt$$
$$= \int_{S^1} \chi_n(t) \frac{1 - |F^n(z_1)|^2}{|e^{it} - F^n(z_1)|^2} dt - \int_{S^1} \chi_n(t) \frac{1 - |F^n(z_2)|^2}{|e^{it} - F^n(z_2)|^2} dt$$

for each $n \geq 0$. Now. choose a Möbius map $T_n : D \longrightarrow D$ with $T_n(0) = F^n(z_1)$. Then.

$$\int_{S^1} \chi_n(t) \frac{1 - |F^n(z_1)|^2}{|e^{it} - F^n(z_1)|^2} dt - \int_{S^1} \chi_n(t) \frac{1 - |F^n(z_2)|^2}{|e^{it} - F^n(z_2)|^2} dt$$
$$= \int_{S^1} \chi_n(T_n(t)) dt - \int_{S^1} \chi_n(T_n(t)) \frac{1 - |T_n^{-1}(F^n(z_2))|}{|e^{it} - T_n^{-1}(F^n(z_2))|} dt$$
$$= \int_{S^1} \chi_n(T_n(t)) [1 - \frac{1 - |T_n^{-1}(F^n(z_2))|}{|e^{it} - T_n^{-1}(F^n(z_2))|}] dt.$$

But $\lim_{n \to \infty} T_n^{-1}(F^n(z_2)) = 0$. since $\delta(T_n^{-1}(F^n(z_2)).0) = \delta(F^n(z_2).T_n(0)) = \delta(F^n(z_2).F^n(z_1)) \to 0$ as $n \to \infty$. Hence.

$$\lim_{n \to \infty} \frac{1 - |T_n^{-1}(F^n(z_2))|}{|e^{it} - T_n^{-1}(F^n(z_2))|} = 1.$$

Therefore.

$$\int_{S^1} \chi_n(t) \frac{1 - |F^n(z_1)|^2}{|e^{it} - F^n(z_1)|^2} dt = \int_{S^1} \chi_n(t) \frac{1 - |F^n(z_2)|^2}{|e^{it} - F^n(z_2)|^2} dt.$$

Therefore.

$$\int_{S^1} \chi(t) \frac{1 - |z_1|^2}{|e^{it} - z_1|^2} dt = \int_{S^1} \chi(t) \frac{1 - |z_2|^2}{|e^{it} - z_2|^2} dt \; \forall \; z_1. z_2 \in D.$$

Hence. χ is a constant function and $m(A) = 0$ or $m(A) = 1$. It follows that F is exact. ∎

The proofs of the following theorem and corollary are found in [DM]. Since we are going to use them later in the text, we repeat the proof.

Theorem 2.4 *If* $F : D \to D$ *is an inner function and there exists a point* $z \in D$ *such that* $\sum_{n \geq 0} (1 - |F^n(z)|) = \infty$, *then the restriction of* F *to* S^1 *is recurrent.*

Proof. Assume $z \in D$ is such that $\sum_{n \geq 0} (1 - |F^n(z)|) = \infty$. We shall prove that $\mu_{p_z} (B - F^{-1}(B)) = 0$, where μ_{p_z} is the Poisson measure, for every Borel set $B \subseteq \partial D$ such that $F^{-1}(B) \subseteq B$, which ensures recurrence. Given a Borel set $B \subseteq \partial D$ and $F^{-1}(B) \subseteq B$, assume $A \subseteq B - F^{-1}(B)$ is a Borel subset and υ is a characteristic function of A. Then, $\upsilon \circ F^n$ is the characteristic function of $F^{-n}(A)$. Then, we get

$$\mu\left(F^{-n}(A)\right) = \int (\upsilon \circ F^n)\, d\mu_{p_z} = \int \upsilon\, d\mu_{p_{F^n(z)}} = \upsilon^*\left(F^n(z)\right)$$

where υ^* denotes the Poisson integral of υ. Also,

$$\frac{\upsilon^*\left(F^n(z)\right)}{1 - |F^n(z)|} = (1 + |F^n(z)|) \int_A \frac{1}{|w - F^n(z)|^2} d\mu_{p_w}.$$

Let $\xi = \lim_{n \to \infty} F^n(z)$ be the Denjoy-Wolff point of F; again $\xi \notin \overline{A}$ implies

$$\lim_{n \to \infty} \frac{\upsilon^*\left(F^n(z)\right)}{1 - |F^n(z)|} = (1 + |\xi|) \int_A \frac{1}{|w - \xi|^2} d\mu_{p_w}.$$

Now, $A \subseteq B - F^{-1}(B)$ and $F^{-1}(B) \subseteq B$. Therefore,

$$F^{-j}(A) \cap F^{-i}(A) = \emptyset,$$

28

for all $0 \leq j < i$ and we have

$$1 \geq \mu_{p_z} \left(\bigcup_{n \geq 0} F^{-n}(A) \right) = \sum_{n \geq 0} \mu_{p_z} \left(F^{-n}(A) \right) = \sum_{n \geq 0} v^* \left(F^n(z) \right).$$

Suppose that $\mu_p(A) > 0$. Then

$$\lim_{n \to \infty} \frac{v^* \left(F^n(z) \right)}{1 - |F^n(z)|} > 0.$$

which guarantees the existence of $K > 0$ and $N \geq 0$ such that $v^* \left(F^n(z) \right) \geq K \left(1 - |F^n(z)| \right)$ for all $n \geq N$. Since

$$1 \geq \sum_{n \geq 0} v^* \left(F^n(z) \right)$$

and by our hypothesis, we get

$$1 \geq \sum_{n \geq N} v^* \left(F^n(z) \right) \geq K \sum_{n \geq N} \left(1 - |F^n(z)| \right) = \infty.$$

This contradiction proves that $\mu_p(A) = 0$ for every Borel set $A \subseteq B - F^{-1}(B)$ satisfying $\xi \notin \overline{A}$. Since $B - F^{-1}(B)$ is a measurable subset of S^1. it is a countable union of such sets A. plus "maybe" the point ξ itself. it follows that $\mu_p(B - F^{-1}(B)) = 0$.

■

Corollary 2.2 *If $F : D \to D$ is an inner function and the restriction of F to S^1 is recurrent, then it is exact.*

Proof. Let $F : D \to D$ be an inner function with a recurrent boundary map and let us prove that it is exact. Later in the text, Corollary 3.1 will guarantee that ergodic inner functions are exact. Therefore, to prove this corollary, it suffices to show that F is ergodic. Suppose $A \subseteq S^1$ is a Borel set such that $F^{-1}(A) = A$ and let us consider the characteristic function v. Then, $v \circ F = v$ and since they are harmonic functions, each can be represented as the Poisson integral of its boundary value; hence, $v^* \circ F = v^*$, where v^* denotes the Poisson integral of v at a certain point. Therefore, if $g : D \to C$ is any holomorphic function with $\operatorname{Re} g = v^*$, then

$$\operatorname{Re}(g \circ F) = \operatorname{Re}(g) \circ F = v^* \circ F = v^*.$$

and one may choose $b \in \Re$ such that

$$(g \circ F) = g + ib.$$

Let us assume that $b \neq 0$. Defining $G : D \to C$ by $G = \exp(-2\pi|b|^{-1}g)$, we obtain

$$|G(z)| = \exp\left(-\frac{2\pi}{|b|}\operatorname{Re} g(z)\right) = \exp\left(-\frac{2\pi}{|b|}v^*(z)\right) \leq 1$$

and $(G \circ F)(z) = \exp\left(-\frac{2\pi}{|b|}g((F(z)))\right) = \exp\left(-\frac{2\pi}{|b|}g(z) \pm 2\pi i\right) = \exp\left(-\frac{2\pi}{|b|}g(z)\right)$

30

$= G(z)$. that is. G is bounded and F-invariant. It follows that $G \circ F^n = G$ for all

$n \geq 0$. which means that $G - G(a)$ is zero at every $F^n(a)$. for all $n \geq 0$ and every

$a \in D$. Given $a \in D$. then by Theorem 1.1 we get

$$G - G(a) \equiv 0$$

or else its sequence of zeros $a_0, a_1, \dots$ satisfies

$$\sum_{n \geq 0} (1 - |a_n|) < \infty.$$

If the later follows. then we would obtain

$$\sum_{n \geq 0} (1 - F^n(z)) < \infty.$$

and then by Theorem 2.2. the Denjoy-Wolff point ξ of F would belong to S^1 and the

$\lim_{n \to \infty} F^n(z) = \xi$ for a.e. $z \in S^1$. Since. our hypothesis is the recurrence of F. this

cannot happen and we are left with $G - G(a) \equiv 0$. But this implies that G. and then

g and Re$g = v^*$. are all constants. Hence. v is constant because its Poisson integral

is constant. Therefore. $v = 0$ a.e. or $v = 1$ a.e. proving that $m(A) = 0$ or $m(A) = 1$.

The other case. $b = 0$ is proved by applying the same argument to $G = \exp(-g)$.

which is again bounded and F-invariant. ∎

2.4   The Chaotic Behavior of a Finite Blaschke

Product and The Shape of Its Julia Set

In this section we are going to present the chaotic behavior of a finite Blaschke product. This is a special case of the general theory, presented for illustration and as an introduction to Chapter 3.

Case $B^*(0) = 0$

First, let us consider the following Blaschke product $B^*(z) = e^{i\alpha} z \Pi_{j=2}^m \frac{z - a_j}{1 - \bar{a}_j z}$, $|a_j| < 1$.

Obviously, zero is a fixed point of $B^*(z)$. We will prove that $J(B^*(z)) = S^1$. Let

$g(z) = \Pi_{j=2}^m \frac{z - a_j}{1 - \bar{a}_j z}$ and observe that $|g(0)| < 1$. Then, $(B^*(z))' = e^{i\alpha}(g(z) + z g'(z))$.

Therefore, $|(B^*(z))'(0)| = |g(0)| < 1$. It follows that $z = 0$ is an attracting fixed point

of $B^*(z)$. Let $f(z) = \frac{z - a_j}{1 - \bar{a}_j z}$. For $|z| = 1$ we have $|f(z)| = 1$. Hence, $f(S^1) = S^1$ and

by the maximum principle $f(D) = D$. Then, $\sup |f(z)| \leq \rho < 1 \ \forall \ z \in D$. Therefore,

$|B^*(z)| \leq \rho |z| < 1$ and $|(B^*)^n(z)| \leq \rho^n |z| \to 0$ as $n \to \infty$. Then every point in D is

attracted to zero. Hence, $D \subset F(B^*)$. Also, $B^*(\infty) = \infty$: implies ∞ is a fixed point.

Consider the conjugation $\varphi(z) = \frac{1}{z} = w$. Let $G(w) = \varphi \circ B^* \circ \varphi^{-1}(w)$. Therefore

$$G(w) = \varphi \circ B^* \left(\frac{1}{w} \right)$$

$$= \varphi \left(e^{i\alpha} \frac{1}{w} \Pi_{j=2}^{m} \frac{\frac{1}{w} - a_j}{1 - \overline{a}_j \frac{1}{w}} \right)$$

$$= \varphi \left(e^{i\alpha} \frac{1}{w} \Pi_{j=2}^{m} \frac{1 - a_j w}{w - \overline{a}_j} \right)$$

$$= e^{-i\alpha} w \Pi_{j=2}^{m} \frac{w - \overline{a}_j}{1 - a_j w}.$$

Then $G(0) = 0$. Let $h(w) = \Pi_{j=2}^{m} \frac{w - \overline{a}_j}{1 - w a_j}$ and observe that $|h(0)| < 1$. Then $|G'(w)| = (h(w) - wh'(w))$. Therefore, $|G'(0)| = |h(0)| < 1$. Hence, 0 is an attracting fixed point of $G(w)$, i.e., ∞ is an attracting fixed point of $B^*(z)$. Now, we will show that for $z \in U$, $U = \{z : |z| > 1\}$, z is attracted to ∞. Let $f(z) = \frac{z - a_j}{1 - \overline{a}_j z}$. For $|z| \geq R > 1$ we have $\inf |f(z)| \geq R > 1$. Therefore, for $z \in U$ we have $|B^*(z)| \geq R|z|$. Then $|(B^*)^n(z)| \geq R^n |z| \to \infty$ as $n \to \infty$. Hence every point in U is attracted to ∞. Then $U \subset F(B^*)$ and it follows that D is completely invariant in $F(B^*)$. The following theorem can be found in [Be].

Theorem 2.5 *If Ω is the union of components of F which is completely invariant in F, then $J = \partial \Omega$.*

Since D is invariant in $F(B^*)$ by the above theorem the Julia set of $B^*(z) = e^{i\alpha} z \Pi_{j=2}^{\infty} \frac{z - a_j}{1 - \overline{a}_j z}$ is S^1.

General Case

Now consider the general case $B(z) = e^{i\alpha}\Pi_{j=1}^{m}\frac{z-a_j}{1-\overline{a_j}z}$. By the Denjoy-Wolff theorem $B(z)$ has a unique fixed point in $\overline{D}$. Suppose $B(z_0) = z_0$, $z_0 \in D$, and consider $\phi(z) = \frac{z-z_0}{1-\overline{z_0}z}$. Observe that $\phi(z_0) = 0$ and that $\varphi(z)$ preserves D. Using $\phi(z)$ we can conjugate $B(z)$ into $B^*(z) = e^{i\alpha}z\ \Pi_{j=2}^{m}\frac{z-a_j}{1-\overline{a_j}z}$. We proved that $J(B^*) = S^1$. Therefore, if $B(z)$ has a fixed point in D, then $J(B) = S^1$.

Now we are going to look for the case when the fixed point of $B(z)$ belongs to S^1. Here, we present examples with $J(B) = S^1$ and $J(B)$ as a Cantor like set of S^1. Consider $B(z) = \frac{z-a}{1-\overline{a}z}\frac{z-\overline{a}}{1-az}$. Observe that $B(1) = 1$, i.e., $z = 1$ is a fixed point.

$$B'(z) = \frac{(1-\overline{a}z)(1-az)(2z-a-\overline{a})-(z-a)(z-\overline{a})(-a-\overline{a}+2|a|^2z)}{(1-\overline{a}z)^2(1-az)^2}$$

Then $|B'(1)| = \frac{|2(1-|a|^2)|}{|1-a|^2}$.

We will check some cases where $z = 1$ is an attracting fixed point. Consider $a = -\sqrt{\frac{2}{3}}$, then $|B'(1)| = \frac{2}{1+2\sqrt{6}} < 1$. Then $z = 1$ is an attracting fixed point. Other cases where $z = 1$ is an attracting fixed point can be checked by taking $a = -\frac{1}{2}$, $a = \frac{-1-i}{2}$ or $a = \frac{-1+i}{2}$.

For discussing the shape of Julia set in this case consider the symbolic space $\sum$

34

$= \sum_d = \{1.2.\ldots\ldots\ldots d\}$ which endowed with the metric

$$\delta(\sigma, \tau) = \sum_{k=1}^{\infty} \frac{|\sigma_k - \tau_k|}{d^k}$$

is compact and totally disconnected metric space. The follwing theorems are found in [Mi].

Theorem 2.6 *For each hyperbolic rational function f there exists a continuous map $o : \sum \longrightarrow J$ such that*

$$o \circ S = f \circ o$$

where S is the shift operator $S : \sum \longrightarrow \sum$. $(\sigma_1, \sigma_2.\ldots\ldots\ldots) \longrightarrow (\sigma_2, \sigma_3.\ldots\ldots\ldots)$.

Theorem 2.7 *Suppose that f is a hyperbolic map and F is connected. Then J is totally disconnected and the map $o : \sum \longrightarrow J$ is a homeomorphism.*

Proposition 2.5 *If $B(z)$ has an attracting fixed point on S^1. then $J(B)$ is a Cantor like set of S^1.*

Proof. When the attracting fixed point of $B(z)$ is on S^1. we conclude that $F(B)$ is connected because the attracting fixed point is in $F(B)$. Therefore. from the above theorems. we get that $J(B)$ is totally disconnected in the unit circle. The above theorems are with respect to Poincaré metric. Then. $J(B)$ is totally disconnected in Poincaré metric is equivalent to saying that $J(B)$ is a Cantor like set of S^1 according to Euclidean metric. ∎

35

Now. we are going to check a case when z =1 is a neutral fixed point of $B(z)$. Consider $a = \frac{-1}{3}$. Then $B'(1) = \frac{2(1-\frac{1}{3})}{(1+\frac{1}{3})^2} = 1$. Therefore. $|B'(1)| = 1$. i.e.. $z = 1$ is a neutral fixed point of $B(z)$.

The following lemma and theorem are proved in [Be] for rational functions: we are going to repeat the proof to make sure that it does not depend on rationality.

Lemma 2.3 *Let* $R : \hat{C} \longrightarrow \hat{C}$ *be any map.Suppose that* F_0 *is a forward invariant component of* $F(R)$. *If there exists a constant limit function with value* ξ. *then* ξ *is a fixed point.*

Proof. A function o is a limit function on a component F_0 of $F(R)$ if there is some subsequence of $\{R^n\}$ which converges locally uniformly to o on F_0. The class of limit functions on F_0 is denoted by $\Phi(F_0)$. Since $\{R^n : n \geq 1\}$ is normal in F_0. then $\Phi(F_0)$ is non empty and each map is analytic in F_0. If F_0 is forward invariant. and if o is a limit function in F_0. then $o(F_0)$ lies in the closure of F_0: in particular. if o is constant with value ξ. then $\xi \in F_0 \cup \partial F_0$. Moreover. if z is in F_0. then so is $R(z)$. Therefore. on some sequence of integers n tending to infinity. we get

$$R(\xi) = R\left(\lim_{n \to \infty} R^n(z)\right) = \lim_{n \to \infty} R^n(R(z)) = o(R(z)) = \xi.$$

Therefore. ξ is a fixed point. ∎

36

Theorem 2.8 *Let* $R : \hat{C} \to \hat{C}$. *Suppose that* F_0 *is a forward invariant component of* $F(R)$. *and that* $R^n \to \partial F_0$ *as* $n \to \infty$. *Then there is a rationally indifferent fixed point* ξ *of* R *in* ∂F_0 *such that* $R^n \to \xi$ *locally uniformly on* F_0 *as* $n \to \infty$. *and* $R'(\xi) = 1$.

Proof. The Lemma preceding the theorem proves that there is a fixed point ξ in ∂F_0 such that $R^n \to \xi$ locally uniformly on F_0 as $n \to \infty$. Thus ξ is uniquely determined by the action of R on F_0. So. it remains to prove that $R'(\xi) = 1$. Obviously. $|R'(\xi)| \geq 1$. otherwise ξ would be attracting fixed point of R so would lie in $F(R)$. It is also true that $|R'(\xi)| \leq 1$. otherwise ξ would be repelling fixed point of R; however. $R^n(z)$ can converge to a repelling fixed point ξ if $R^n(z) = \xi$ for some n and as z belongs to Fatou and ξ belongs to Julia. this cannot happen. It follows that $|R'(\xi)| = 1$. By conjugation. we may assume that $\xi = 0$ and $\infty \in \partial F_0$. Then put $\lambda = R'(0)$ and note that as $|\lambda| = 1$. R is injective in some neighborhood U of the origin. We construct a forward invariant subdomain W of $F_0 \cap U$. Now. take any point ξ_0 in W. and for $n \geq 1$ define the functions o_n on W by $o_n(z) = \frac{R^n(z)}{R^n(\xi_0)}$. Now we want to show that $\{o_n\}$ is normal in W. First. o_n does not take 0 and ∞ in W because R^n does not. As $W \subset U$. R is injective on W. As $o_n(\xi_0) = 1$ non of the o_n takes the value $0. 1. \infty$ in $W - \{\xi_0\}$. It remains to prove that it is normal near ξ_0. By normality. we can find a sequence $\{o_{n_j}\}$ which converges locally uniformly to some constant function o on $W - \{\xi_0\}$. Now. choose a disk centered at ξ_0. and lying in F_0 and B be its interior and ∂B be its boundary. using the Cauchy integral formula and

letting $n_j \to \infty$. we get

$$\varphi_n(z) = \frac{1}{2\pi i} \int_{\partial B} \frac{\varphi_n(w)}{w-z} dw \to \frac{1}{2\pi i} \int_{\partial B} \frac{\varphi(w)}{w-z} dw$$

and the convergence is uniform near ξ_0. Notice that $\frac{1}{2\pi i} \int_{\partial B} \frac{\varphi(w)}{w-z} dw$ is holomorphic in B with value 1 at ξ_0 and it is $\varphi(z)$ when $z \neq \xi_0$. It follows that $\{\varphi_{n_j}\}$ converges uniformly through out W where $\varphi(z_0) = 1$. Therefore. $\varphi_{n_j} \to \varphi$ locally uniformly on W. Now. for all n.

$$\varphi_n(R(z)) = \varphi_n(z) \left(\frac{R^n(R(z))}{R^n(z)} \right) = \varphi_n(z) \left(\frac{R(R^n(z)) - 0}{R^n(z) - 0} \right).$$

Letting $n \to \infty$ through the sequence n_j. we obtain $\varphi(R(z)) = R'(0)\varphi(z) = \lambda\varphi(z)$. Still we need to prove that $\lambda = 1$. First. a non constant locally uniformly limit function of injective analytic maps is injective by the Hurwitz's Theorem. Thus. either φ is constant in W or it is injective in W. If φ is constant. then its value at ξ_0 is one. and since $\varphi(R(z)) = \lambda\varphi(z)$ we deduce that $\lambda = 1$. If φ is not constant on W. then it has an inverse φ^{-1} which maps $\varphi(W)$ onto W. However. since $\varphi(R(z)) = \lambda\varphi(z)$ we obtain that $\varphi(R^n(\xi_0)) = \lambda^n\varphi(\xi_0) = \lambda^n$. Since $|\lambda| = 1$. so there is an increasing sequence of integers n_j such that $\lambda^{n_j} \to 1$. It follows that $\varphi(R^{n_j}(\xi_0)) \to 1$ and as the open set $\varphi(W)$ contains $1 = \varphi(\xi_0)$ we see that $\varphi(R^{n_j}(\xi_0)) \in \varphi(W)$ for sufficiently large j. For this j. we have.

$$R^{n_j}(\xi_0) = \varphi^{-1}(\lambda^{n_j}) \to \varphi^{-1}(1) = \xi_0$$

and this is false because $R^n \to 0$ on W. recall that we conjugated ξ to 0. Therefore. we deduce that φ is necessarily constant and that $\lambda = 1$. ∎

Proposition 2.6 *If $B(z)$ has a neutral fixed point, then it is rationally indifferent.*

Proof. By Theorem 2.8 we get that if $B(z)$ has a neutral fixed point then it is rationally indifferent. ∎

Theorem 2.9 *Let f be any function. If $deg(f) \geq 2$ then every rationally indifferent cycle of f lies in $J(F)$.*

Definition 2.4 *Let $f : \hat{C} \to \hat{C}$. U is called an attracting petal of f at 0 if and only if $f(\overline{U}) \subset U \cup \{0\}$ and $\bigcap_{k \geq 0} f^k(\overline{U}) = \{0\}$. V is a repelling petal if and only if it is an attracting petal of f^{-1}.*

Theorem 2.10 *Suppose f is any function where $f(z) = z(1 - az^n + \ldots\ldots)$. $a \neq 0$. $n \geq 1$. near 0. Then f has exactly n Leau domains. L_1. L_2....... L_n corresponding to the boundary point 0, and $\{f^n\}$ converges to 0 locally uniformly in L_k. Each L_k contains an invariant attracting petal.*

Proposition 2.7 *If $B(z)$ has a neutral fixed point petals will appear. Moreover, we may have one of the following cases:*

1. The repelling petals are part of the unit circle. In this case $J(B) = S^1$.

2. The attracting petals are part of the unit circle. In this case $J(B)$ is a Cantor like set of the unit circle.

3. The neutral point is repelling from on side and attracting from the other side. In this case $J(B)$ is a Cantor like set of the unit circle.

39

Proof. Since the neutral point of $B(z)$ is rationally indifferent. then by Theorem 2.10 petals will appear. We will discuss the three possible cases.

For the first case. the only possible attractor on S^1 is the neutral fixed point. which is repelling in this case. Moreover. $B(S^1) = (S^1)$. then no arc of the unit circle can be attracted to a point in D. Therefore. no arc in S^1 is in $F(B)$. Hence. $J(B) = S^1$

For the second case. the attracting petals are subsets of components of $F(B)$ and the boundary of there basin is in $J(B)$.Therefore. $F(B)$ is connected. This implies $J(B)$ is totally disconnected. by Theorem 2.7. i.e.. $J(B)$ is a Cantor like set of S^1. The third case is similar to the second case. ∎

Definition 2.5 *If f is an analytic map of a domain U onto a domain G and if γ is a curve in G. then it is. in general not true that there is some curve Γ such that $f \circ \Gamma = \gamma$. If Γ exists. then it is called a lift of γ.*

The following theorem and corollary are proved in [DM] for rational functions: we are going to repeat the proof to make sure that it does not depend on rationality.

Theorem 2.11 *Let U be a fixed parabolic basin of a map $f : \hat{C} \to \hat{C}$ and let $o : D \to D$ be a lifting of $f : U \to U$ via a uniformization $\upsilon : D \to U$ such that $\upsilon \circ o = f \circ \upsilon$. Then. for every $z \in D$ and all $\alpha > \frac{1}{2}$. the inequality*

$$1 - |\varphi^n(z)| \geq \frac{1}{n^\alpha}$$

holds for all sufficiently large n.

Proof. We shall only prove that there exists a point $z_0 \in D$ such that, for all $\alpha > \frac{1}{2}$.

$$1 - |\varphi^n(z_0)| \geq \frac{1}{n^\alpha}$$

holds for all sufficiently large n. From this, the same relation holds for all $z \in D$ because the boundary map of f is recurrent by Theorem 2.4. It follows that it is exact by Corollary 2.2. Therefore, by Theorem 2.3 $\lim_{n \to \infty} \delta(\varphi^n(z), \varphi^n(z_0)) = 0$, where δ is the Poincare distance between two points in D. Hence,

$$\lim_{n \to \infty} \frac{|\varphi^n(z) - \varphi^n(z_0)|}{1 - |\varphi^n(z_0)|} = 0$$

for all $z \in D$. Hence, given $z \in D$, we have that

$$1 - |\varphi^n(z)| \geq 1 - |\varphi^n(z_0)| - |\varphi^n(z) - \varphi^n(z_0)|$$
$$\geq (1 - \epsilon)(1 - |\varphi^n(z_0)|)$$

holds for all $\epsilon > 0$ and sufficiently large n. This implies that z also satisfies

$$1 - |\varphi^n(z)| \geq \frac{1}{n^\alpha}.$$

Now we need to prove that

$$1 - |\varphi^n(z_0)| \geq \frac{1}{n^\alpha}$$

holds for all sufficiently large n. So, let p be the parabolic fixed point of f; then $p \in \partial U \subseteq \hat{C}$. $f'(p)$ is a root of unity and $\lim_{n \to \infty} f^n(z) = p$ for all $z \in U$. Without

41

loss of generality we can assume $p = 0$, $f'(p) = 1$ and the Taylor series of f at $z = 0$ is

$$f(z) = z - z^k + \sum_{n > k} a_n z^n$$

for some $k \geq 2$. The basic theory of parabolic fixed points (see [Mi]) implies that there exists an $\eta > 0$ such that U contains the sector

$$S = \{z \in C : |z| < \eta, |\arg(z) - \frac{\pi m}{k-1}| < \frac{\pi(1-\epsilon)}{k-1}\}$$

and, moreover, $f^n(S) \subseteq S$ for all large n, say $n \geq N$ and $\lim_{n \to \infty} f^n(z) = 0$ and $\lim_{n \to \infty} |\arg(f^n(z) - \frac{\pi m}{k-1})| = 0$ for all $z \in S$.

Without loss of generality, we may assume that $m = 0$. From the Taylor expansion of f at $z = 0$ it follows that for each $z \in S$ there exists $C > 0$ such that

$$|f^n(z)| \geq C \left(\frac{1}{n}\right)^{\frac{1}{k-1}}$$

for all n. Let $v : D \to U$ be the uniformization of U; write S^0 for the connected component of $v^{-1}(S)$ such that $o^n(S^0) \subseteq S^0$ for all $n \geq N$. Set $\eta = \frac{1-\epsilon}{k-1}$ and choose a branch $g : S \to C$ of $z^{\frac{1}{2\eta}}$ which leaves the positive half line $\{t \in \Re : t \geq 0\}$ invariant. Then $u(z) = \operatorname{Re} g(z)$ is a harmonic function of S that vanishes on the sides of S. Consider the harmonic function $u \circ v : S^0 \to \Re$ and observe that there exists $K > 0$ such that

$$-K \log|z| \geq (u \circ v)(z)$$

42

for all $z \in S^0$. In fact, to check this inequality, it suffices to verify it at the boundary of S^0. It holds on the part of ∂S^0 mapped by υ onto the sides of the sector S, since there $u \circ \upsilon$ vanishes. Taking K large enough we can make it also true on the part ∂S^0 that υ maps onto $\{z \in C : |z| = \eta, |\arg(z)| < \frac{\pi(1-\epsilon)}{k-1}\}$, since $u \circ \upsilon$ is bounded there. Now, for $z_0 \in S^0$ we have $\varnothing^n(z_0) \in S^0$ for $n \geq N$ and

$$
\begin{aligned}
-K \log|\varnothing^n(z_0)| &\geq u \circ \upsilon \varnothing^n(z_0) = u(f^n(\upsilon(z_0))) \\
&= Reg(f^n(\upsilon(z_0))).
\end{aligned}
$$

Since $\lim_{n \to \infty} |\arg(f^n(\upsilon(z_0)))| = 0$ it follows that

$$
\lim_{n \to \infty} |\arg g((f^n(\upsilon(z_0))))| = 0
$$

and therefore

$$
\begin{aligned}
Reg(f^n(\upsilon(z_0))) &\geq \frac{1}{2}|g(f^n(\upsilon(z_0)))| \\
&= \frac{1}{2}|f^n(\upsilon(z_0))|^{\frac{1}{2\eta}}
\end{aligned}
$$

for large n. Combining this result with

$$
|f^n(z)| \geq C\left(\frac{1}{n}\right)^{\frac{1}{k-1}}
$$

we obtain, for large n,

$$
\begin{aligned}
-K \log|\varnothing^n(z_0)| &\geq \frac{1}{2}C^{\frac{1}{2\eta}}\left(\frac{1}{n}\right)^{\frac{1}{2\eta(k-1)}} \\
&= \frac{1}{2}C^{\frac{1}{2\eta}}\left(\frac{1}{n}\right)^{\frac{1}{2\eta(\epsilon-1)}}.
\end{aligned}
$$

Since ϵ is arbitrary, we get the result. ∎

43

Corollary 2.3 *If U is a fixed parabolic basin of a map $f : \hat{C} \to \hat{C}$. then the restriction of f to ∂U is exact and recurrent with respect to the harmonic class of ∂U.*

Proof. the harmonic class of ∂U is the class of measures on the Boreal σ-algebra of ∂U which are equivalent to a harmonic measure μ_p for some $p \in U$. Let $\upsilon : D \to U$ be a uniformization mapping. i.e.. a holomorphic covering map of U: it always exists if U^c contains at least three points. as in our case. since U is regular. the regularity of U also implies that the logarithmic capacity of U^c is positive. Then. the radial limit of $\upsilon^* \left(e^{i\theta}\right) = \upsilon\left(re^{i\theta}\right)$ a.e.. on ∂U. Moreover. υ^* transforms a harmonic measure μ_p on ∂U in the harmonic measure μ_q on ∂D. where $q \in \upsilon^{-1}(\{p\})$. More precisely. $\mu_q\left((\upsilon^*)^{-1}(A)\right) = \mu_p(A)$ for every Borel set $A \subseteq \partial U$. To see this. take a continuous function $g : \partial U \to \Re$ and let $g^* : \overline{U} \to \Re$ be its harmonic extention. Then

$$\lim_{r \to 1} (g^* \circ \upsilon)\left(re^{i\theta}\right) = g\left(\upsilon^*\left(e^{i\theta}\right)\right)$$

a.e. on ∂D and by a theorem of Fatou.

$$\int (g^* \circ \upsilon)\,d\mu_q = g^* \circ \upsilon(q) = g^*(p) = \int g\,d\mu_p.$$

From this. $\mu_q\left((\upsilon^*)^{-1}(A)\right) = \mu_p(A)$ follows by applying standard approximation methods.

Now. lift $f : U \to D$. we obtain an inner function $o : D \to D$ such that $\upsilon \circ o = f \circ \upsilon$. with Denjoy-Wolff point $p^* \in \partial D$ since $\upsilon^* : \partial D \to \partial U$ satisfies $\mu_q\left((\upsilon^*)^{-1}(A)\right) = \mu_p(A)$. for each Borel set $A \subseteq \partial U$ and every point $p \in U$. $\upsilon(q) = p$. it is clear that the ergodic properties of the boundary map of o. thus the recurrence and the exactness

44

of $f/\partial U$ with respect to the harmonic class of ∂U are consequences of recurrence and exactness of σ^* the boundary map of φ. By Theorem 2.10 we have.

$$1 - |\varphi^n(z)| \geq \frac{1}{n^\alpha}.$$

where $\alpha > \frac{1}{2}$. which implies recurrence and exactness of σ^*. ∎

Theorem 2.12 *Ergodicity of a finite Blaschke product is equivalent to $J(B) = S^1$.*

Proof. We are going to divide the proof into two cases:

1. If $B(z)$ has a fixed point in D. then by Proposition 2.3 the restriction of $B(z)$ to S^1 is ergodic. Moreover we discussed before that if $B(z)$ has a fixed point in D. then it is attracting and $J(B) = S^1$.

2. If $B(z)$ does not have a fixed point in D. then $B(z)$ has a fixed point on S^1. which can be either attracting or neutral.

    (a) If $B(z)$ has an attracting fixed point on S^1. then by Proposition 2.4 we showed that the restriction of $B(z)$ to S^1 is not ergodic. Moreover. we discussed before that if $B(z)$ has an attracting fixed point on S^1. then $J(B)$ is a Cantor like set of S^1.

    (b) If $B(z)$ has a neutral fixed point on S^1. then petals will appear and we will get one of the following:

        i. If the attracting petals are part of the unit circle. then by a previous discussion we found that $J(B)$ is a Cantor like set of S^1. Moreover. if

45

z_0 is the neutral point. then it is the end point of the complementary arcs α of $J(B)$. Furthermore. we can choose a small closed subarc $\eta \subset \alpha$ with $B^n(\eta)$ are all subarcs of α tending uniformly to z_0. $n = 1.2, \ldots\ldots$. Choose η small enough such that $g(\eta) \cap \eta = \emptyset$ and V be an open interval between η and $g(\eta)$. Let $A = \cup_{-\infty}^{+\infty} B^n(\eta)$. We want to show that $V \cap A = \emptyset$. Since $B^n(\eta)$ converges uniformly to z_0. no image of η will intersect V. Now. suppose that $V \cap A \neq \emptyset$. then $\exists r \in V \cap A$. i.e.. $r \in B^{-n}(\eta)$ for some n. Hence. $B^n(r) \in \eta$ which is impossible. It follows that $V \cap A = \emptyset$. Therefore. $m(S^1 \backslash A) > 0$. Also $B^{-1}(A) = A$. Hence. the restriction of B to S^1 is not ergodic in this case.

ii. The neutral point is attracting from one side and repelling from the other one. This case is similar to the previous one.

iii. The repelling petals are part of the unit circle. For this case we showed that $J(B) = S^1$. Moreover. by Corollary 2.3 we get that the restriction of $B(z)$ to S^1 is ergodic.

Therefore. ergodicity of a finite Blaschke product is equivalent to $J(B) = S^1$. $\blacksquare$

46

Chapter 3

Invariant Measures For Inner

Functions In The Upper Half Plane

In this chapter we will transfer the results from the circle to the line by a conformal map. That is, let $w = \Phi(z) = i\left(\frac{1+z}{1-z}\right)$ be the conformal map of D into the upper half plane $\mathbb{R}^{2+}$ of C, the complex plane. Φ carries S^1 onto $\mathbb{R}$ with $\Phi^{-1}(w) = \frac{w-i}{w+i}$. If F is an inner function in D, then $G = \Phi \circ F \circ \Phi^{-1}$ is an inner function in $\mathbb{R}^{2+}$ and vice versa. Φ carries the Poisson measure $P_z dt = \frac{1 - |z|^2}{|e^{it} - z|^2} dt$ to the Cauchy measure $Q_w(x) dx = \frac{1}{\pi} \frac{b}{(x-a)^2 + b^2} dx$ on the line, where $w = \Phi(z) = a + ib$, and dx is Lebesgue measure on the line.

47

3.1 Absolutely Continuous Invariant Measure

Theorem 3.1 *If $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ is an analytic function, then*

$$T(z) = \alpha_T z + \beta_T + \int_{\Re} \frac{1 + tz}{t - z} d\mu_T(t)$$

where $\alpha, \beta \in \mathbb{R}, \alpha \geq 0,$ and μ is singular with respect to Lebesgue measure if and only if T is an inner function.

Proof. Let $v = \mathrm{Im} T$, then $v : \mathbb{R}^{2+} \to \mathbb{R}_+$ is harmonic. Set $o(z) = o_{-1}(z) = \iota\left(\frac{1 - z}{1 + z}\right)$, then $o(e^{2\pi it}) = \tan \pi t$ and $v \circ o : D \to \mathbb{R}_-$ is harmonic. Since it is harmonic, then $\exists$ a positive measure μ on S^1 such that

$$v(z) = \int_0^1 p_{o^{-1}(z)} d\mu(t), \quad p_z = \frac{1 - |z|^2}{|e^{it} - z|^2}.$$

$$Now, \quad p_{o^{-1}(z)}(x) = \mathrm{Im} \frac{1 + \tan \pi x z}{\tan \pi x - z};$$

hence,

$$v(z) = \alpha_T z + \int_0^1 \mathrm{Im} \frac{1 + \tan \pi x z}{\tan \pi x - z} = \int_{\Re} \mathrm{Im} \frac{1 + tz}{t - z} d\mu_T(t)$$

where $\alpha_T = \mu\left(\left\{\frac{1}{2}\right\}\right)$ and $\int_{\Re} h(t) d\mu_T(t) = \int_{S^1 \setminus \{\frac{1}{2}\}} h(\tan \pi x) d\mu(x)$. Observe that,

$$z \mapsto \alpha_T z + \int_{\Re} \frac{1 + tz}{t - z} d\mu_T(t)$$

is an analytic function and has the same real part as T. Hence,

$$T(z) = \alpha_T z + \beta_T + \int_{\Re} \frac{1 + tz}{t - z} d\mu_T(t)$$

48

for some $\beta_T \in \mathbb{R}$.

The measure μ is singular if and only if $v \circ \phi$ has a boundary value zero, a.e. with respect to Lebesgue measure. This is true when $\phi^{-1} \circ T \circ \phi$ is an inner function of D. Also, μ_T is singular whenever μ is singular. Therefore, T is an inner function of $\mathbb{R}^{2+}$. ∎

Proposition 3.1 *Let* $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ *be an inner function. Let* $T : \mathbb{R} \to \mathbb{R}$ *be the restriction of* F *to* $\mathbb{R}$. *Then the Cauchy measure* $Q_w(x) = \frac{1}{\pi}\frac{b}{(x-a^2)+b^2}dx$ *is* T-*invariant if and only if* F *fixes some point in* $\mathbb{R}^{2+}$.

Proof. Consider the function $w = \Phi(z) = i\left(\frac{1+z}{1-z}\right)$, then $\Phi^{-1}(w) = \dfrac{w-i}{w+i}$ which maps $\mathbb{R}^{2+}$ conformally onto D. Therefore, the result follows by Corollary 2.1 and Proposition 2.2. ∎

Proposition 3.2 *Let* $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ *be an inner function with* $\alpha_T = 1$. *Then, the restriction of* T *to* $\mathbb{R}$ *preserves Lebesgue measure.*

Proof. Let $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an inner function, then

$$T(z) = \alpha_T z + \beta_T + \int_{\mathbb{R}} \frac{1+tz}{t-z}d\mu_T(t)$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$, and μ is singular with respect to Lebesgue measure. Recall, the Cauchy measure $Q_w(t)dt = \frac{1}{\pi}\frac{b}{(t-a)^2+b^2}dt$, where dt is Lebesgue measure on the real line. Then, for any measurable set $A \in \mathbb{R}$, $\pi b Q_{a+ib}(A) \to m(A)$, $\frac{|a|}{b} \to 0$ as

49

$b \to \infty$. Now, observe that

$$
\begin{aligned}
T(ib) &= ib\alpha_T + \beta_T + \int_{\mathbb{R}} \frac{1 + tib}{t - ib} d\mu_T(t) \\
&= ib\alpha_T + \beta_T + \int_{\mathbb{R}} \frac{(1 + ibt)(t + ib)}{t^2 + b^2} d\mu_T(t) \\
&= \beta_T + \int_{\mathbb{R}} \frac{t - tb^2}{t^2 + b^2} d\mu_T(t) + i\left(b\alpha_T + \int_{\mathbb{R}} \frac{b + bt^2}{t^2 + b^2} d\mu_T(t)\right).
\end{aligned}
$$

Therefore,

$$
\lim_{b \to \infty} \mathrm{Im} \frac{T(ib)}{b} \to \alpha_T.
$$

Hence, $\pi b Q_{ib}(T^{-1}A) \to m(T^{-1}A)$ and $\pi b Q_{T(ib)}(A) \to \dfrac{m(A)}{\alpha_T}$. But $\pi b Q_{ib}(A) = \pi b Q_{T(ib)}(A)$ and $\alpha_T = 1$. Therefore,

$$
m(T^{-1}) = m(A) \ \forall \ A \in \mathbb{R}.
$$

Therefore, the restriction of T to $\mathbb{R}$ preserves Lebesgue measure. ∎

Proposition 3.3 *Suppose that $T : \mathbb{R}^{+2} \to \mathbb{R}^{+2}$ is an inner function which is analytic around $x_0 \in \mathbb{R}$ and $T(x_0) = x_0$. $T'(x_0) = 1$. Then, the restriction of T to $\mathbb{R}$ preserves $d\mu_{x_0}(t) = \dfrac{dt}{(t - x_0)^2}$, where dt is Lebesgue measure on $\mathbb{R}$.*

Proof. Assume that $x_0 = 0$ and let $T''(x_0) = a$. Let $R(z) = \dfrac{-1}{T\left(\frac{-1}{z}\right)}$. Observe that

$$
\frac{1}{T(z)} - \frac{1}{z} \to -a \text{ as } z \to 0.
$$

It follows that $\exists K > 1$ such that

$$
R(z) = z + a + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \ \forall \ |z| \geq K.
$$

50

Hence. $\alpha_R = 1$ and

$$R(z) = z + a + \beta + \int_{\mathfrak{R}} \frac{1 + tz}{t - z} d\mu(t)$$

for some singular measure μ on $\mathbb{R}$ and $\beta \in \mathbb{R}$. It follows. by Proposition 3.2. that the restriction of R to $\mathbb{R}$ preserves Lebesgue measure. Since $R(z) = \dfrac{-1}{T\left(\frac{-1}{z}\right)}$. then

$$R'(z) = \frac{T'\left(\frac{-1}{z}\right)\frac{1}{z^2}}{T^2\left(\frac{-1}{z}\right)}.$$

Define $\Gamma_\theta = \{z : \theta < Arg(z) < \pi - \theta\}$. For $z \in \Gamma_\theta$ we get

$$R'(z) \to 1 \text{ as } z \to 0.$$

Since the restriction of R to $\mathbb{R}$ preserves Lebesgue measure. then the restriction of T to $\mathfrak{R}$ preserves the infinite measure $\frac{dt}{t^2}$ where dt is Lebesgue measure on $\mathfrak{R}$. Shifting the neutral fixed point again to x_0 we get that the restriction of t to $\mathbb{R}$ preserves the infinite measure $\frac{dt}{(t-x_0)^2}$. ∎

3.2 Ergodicity and Exactness

Theorem 3.2 *Let $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an inner function with boundary restriction T. Then T is exact if and only if $\delta(F^n(w). F^n(w')) \to 0$ as $n \to \infty$ for all $w. w' \in \mathfrak{R}^{2+}$. where $\delta(F^n(w). F^n(w')) = \left|\dfrac{F^n(w') - F^n(w)}{F^n(w') - \overline{F^n}(w)}\right|$ the Pioncaré distance in the upper half plane.*

51

Proof. Consider the function $w = \Phi(z) = i\left(\frac{1+z}{1-z}\right)$, then $\Phi^{-1}(w) = \frac{w-i}{w+i}$ which maps $\mathbb{R}^{2+}$ conformally onto D. Therefore. the result follows from Theorem 2.3. ∎

Theorem 3.3 (Pommerenke) *Let* $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ *be analytic with Denjoy-Wolff point* ∞. *Let* $T^n(i) = a_n + ib_n$. *then* $\exists \lim_{n\to\infty} \frac{T^n(z)-a_n}{b_n} = F(z)$. *Moreover.* $F \circ T \equiv aF + b$ *where* $b \in \mathbb{R}$. $a \geq \alpha_t$ *and* $F \circ T \equiv F \Leftrightarrow F \equiv i$.

Proof. Without loss of generality. T is not Möbius. Set $F_n(z) = \frac{T^n(z)-a_n}{b_n}$. Then. $F^n(i) = i$. Denote by δ the Pioncaré distance in $\mathbb{R}^{2+}$. Observe that

$$\delta(F_n(z). F_n(z')) = \delta(T^n(z). T^n(z')).$$

and

$$\delta(T^n(z). T^n(z')) \to 0 \text{ as } n \to \infty \; \forall \; z. z' \in \mathbb{R}^{2+}.$$

Then

$$\delta(T^n(z). T^n(z')) \to 0 \text{ as } n \to \infty \; \forall \; z. z' \in \mathbb{R}^{2+} \Leftrightarrow F_n(i) \to i \; \forall \; z \in \mathbb{R}^{2+}.$$

Suppose that $\delta(T^n(z). T^n(z')) \to L(z. z') = L(Tz. Tz')$. Then. if $F_{n_k} \to G$. we get

$$G(i) = i \text{ and } L(z. z') = \delta(G(z). G(z')).$$

Since $\delta(G(Tz). G(Tz')) = L(Tz. Tz') = L(z. z') = \delta(G(z). G(z'))$. then by Schwarz -Pick lemma we get

$$G \circ T = A \circ G$$

52

where A is Möbius. Suppose that A has a fixed point $p \in \mathbb{R}^{2+}$. Notice that.

$\mathrm{Im}T^n(z) \to \infty$ as $\mathrm{Im}z \to \infty$ because $\alpha_{T^n} = \alpha_T^n \geq 1$ for all $n \geq 1$. Hence.

$\mathrm{Im}G(z) \to \infty$ as $\mathrm{Im}z \to \infty$. Now. $\mathrm{Im}T^n(i) \to \infty$ as $n \to \infty$. This implies that

$\mathrm{Im}A^n(i) = \mathrm{Im}A^n \circ G(i) = \mathrm{Im}G \circ T^n(i) \to \infty$ as $n \to \infty$. Since $\delta(A^n(i).p) = \delta(i.p)$

for $n \geq 1$. we have that $A^2(i) = i$. Since $A^2(p) = p$. we must have $A(i) = i$. Now

let $\Phi : D \to \mathbb{R}^{2+}$ be an analytic bijection with $\Phi(0) = i$ and set $h = \Phi^{-1} \circ T \circ \Phi$.

$g = \Phi^{-1} \circ G \circ \Phi$ and $a = \Phi^{-1} \circ A \circ \Phi$. Then $g(0) = 0$. $g(h(z)) = \lambda g(z)$ for some

$\lambda \in S^1$. and $g(h^n(0)) = 0 \ \forall \ n \geq 0$. Then by Theorem 1.1 we get

$$|g(z)| \leq \Pi_{n=0}^{\infty} \frac{h^n(0) - z}{1 - \bar{z}h^n(0)}.$$

Hence. for every $N \geq 0$.

$$|g(z)| = |g(h^n(z))| \leq \Pi_{k=0}^{N} \frac{h^{k+n}(0) - h^n(z)}{1 - \overline{h^n(z)}h^{k+n}(0)} \to |g(z)|^N \text{ as } n \to \infty.$$

This implies that $g \equiv 0$. or $G \equiv i$. Hence. $L \equiv 0$ and $F_n(z) \to i$.

Now. suppose that A has no fixed point in $\mathbb{R}^{2+}$. Then $A^n(i) \to \infty$. Therefore.

$A(z) = az + b$ where $a \geq 1$ and $b \in \mathbb{R}$. In case A is not the identity map. claim that

$F_n \to G$. Suppose that $F_{n_l} \to H$. We have that $H \circ T = B \circ H$ with $B^n(z) \to \infty$.

Also. $\delta(H(z).i) = L(z.i) = \delta(G(z).i)$. Hence. $H = C \circ G$ where C is Möbius

and $C(i) = i$. But since $A^n(z).B^n(z) \to \infty$. we have in addition that $C(\infty) = \infty$.

Hence. C is the identity map and $G = H$.

53

Finally. we are going to show that $a \geq \alpha_T$. Clearly. $\frac{b_{n+1}}{b_n} \to a$ as $n \to \infty$. Using Theorem 3.1. we get

$$\frac{b_{n+1}}{b_n} = \alpha_T + \int \left(\frac{1 + t^2}{(t - a_n)^2 + b_n^2} \right) d\mu(t) \geq \alpha_T$$

which ends the proof. ∎

Theorem 3.4 [Aa] *Let $F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be analytic. and not Möbius. Then either*

1. $\delta(F^n(w) . F^n(w')) \to 0$ as $n \to \infty$ for all $w. w' \in \mathbb{R}^{2+}$.

    or

2. There is a non constant. bounded analytic function $h : \mathbb{R}^{2+} \to D$ such that

    $h \circ F \equiv h$.

Proof. Suppose that F 's Denjoy-Wolff point is in $\mathbb{R}^{2+}$ and assume that ∞ is the Denjoy-Wolff point or otherwise conjugate by a Möbius transformation. Then by the Pommerenke's theorem. either

$$\delta (F^n(w) . F^n(w')) \to 0 \text{ as } n \to \infty \ \forall \ w. w' \in \mathbb{R}^{2+}$$

or $\exists$ an analytic function $G : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ satisfying

$$G \circ F \equiv aG + b \text{ for some } a. b \in \mathbb{R}. \ a \geq \alpha_T. \ a \neq 1 \text{ and } b \in \mathbb{R}.$$

Thus $\exists \ H : \mathbb{R}^{2+} \to D$ analytic and non constant such that $H(aw + b) = H(w)$. Let $h = H \circ G$ then $h \circ F = h$. ∎

54

Remark 3.1 *In case 2, for a.e. x, $\lim_{y \to 0} h(x + iy) = k(x)$ exists. $k(Tx) = k(x)$*

a.e. and k is a non-constant because h is non-constant. Hence T is not ergodic.

Corollary 3.1 [Aa] *Ergodic inner functions are exact.*

Proof. Let $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an inner function. If T is ergodic, then by Remark 3.1 and Theorem 3.4 we get

$$\delta(T^n(w), T^n(w')) \to 0 \text{ as } n \to \infty \ \forall \ w, w' \in \mathbb{R}^{2+}.$$

Therefore, by Theorem 3.2 we get that T is exact. ∎

Theorem 3.5 [Aa] *Let $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ be an inner function with Denjoy-Wolff point ∞. If $\alpha_T > 1$, then the restriction of T to $\mathbb{R}$ is non-ergodic.*

Proof. By Theorem 3.4 $\exists F : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ analytic such that

$$F \circ T \equiv aF + b$$

where $b \in \mathbb{R}, a \geq \alpha_T > 1$. It follows that $\exists H : \mathbb{R}^{2+} \to D$ analytic and non-constant, such that

$$H(az + b) = H(z)$$

$H \circ F$ being bounded, analytic, non-constant and T-invariant. Therefore, by Remark 3.1, the restriction of T to $\mathbb{R}$ is non-ergodic. ∎

Theorem 3.6 [Aa] *If $T : \mathbb{R}^{2+} \to \mathbb{R}^{2+}$ is odd, inner function and $\alpha_T = 1$, then the restriction of T to $\mathbb{R}$ is exact.*

Proof. Write $T^n(i) = ib_n$, then

$$\frac{b_{n+1}}{b_n} = 1 + \int_{\mathbb{R}} \left(\frac{1+t^2}{t^2 + b_n^2} \right) d\mu(t) \leq 1 + \frac{\mu(\mathbb{R})}{b_n} ;$$

therefore.

$$\frac{b_{n+1}}{b_n} \longrightarrow 1 \text{ as } n \longrightarrow \infty.$$

It follows by Pommerenke's Theorem that

$$\frac{T^n}{b_n} \longrightarrow F(z) \text{ as } n \longrightarrow \infty \; \forall \; z \in \mathbb{R}^{2+}.$$

where $F : \mathbb{R}^{2+} \longrightarrow \mathbb{R}^{2+}$ is analytic and odd. and $F(Tz) = F(z)$. hence $F \equiv i$.

$\delta(T^n(w).T^n(w')) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $w. w' \in \mathbb{R}^{2+}$ where δ denotes the Pioncaré

distance in $\mathbb{R}^{2+}$. Therefore. by Theorem 3.2. the restriction of T to $\mathbb{R}$ is exact. ∎

3.3 Julia Set and Ergodicity for a Class of Meromorphic Functions

In this section. we are going to discuss a class of meromorphic functions which are represented in the form

$$g(z) = A + \varepsilon \left[Bz - \frac{C_0}{z} - \sum_s C_s \left(\frac{1}{z - p_s} + \frac{1}{p_s} \right) \right] \tag{1}$$

where $A. B. C_s. p_s. s = \pm 1. \pm 2. \ldots$ are real constants. $B. C_s \geq 0$ and $\varepsilon = \pm 1$. Observe

that $g(z)$ represents the inner functions in the upper half plane. In this representation.

the constant B is the same as a_T in the previous represntaion.

56

Proposition 3.4 *Let $g(z)$ be of the form (1). Then. $\sum_s \frac{C_s}{p_s^2} < +\infty$.*

Proof. There is no loss in generality if we assume that $C_0 = 0$. Then. $g(z)$ is the uniform limit of the sequence

$$g_n(z) = \sum_{s=-n}^{n} \frac{C_{s,n}}{p_s - z} + c_n$$

where c_n is a real constant. Taking the derivative at the point zero. we find that the sequence of sums

$$\sum_{s=-n}^{n} \frac{C_{s,n}}{p_s^2}$$

converges to $g'(0)$. On the other hand. it is evident that $C_{s,n}$ tends to the residue C_s at the point p_s. Passing to the limit as $n \to \infty$. we find from the inequality

$$\sum_{s=-N}^{N} \frac{C_{s,n}}{p_s^2} \le g'(0) \qquad (n \ge N)$$

that

$$\sum_{s=-N}^{N} \frac{C_s}{p_s^2} \le g'(0).$$

and accordingly the series

$$\sum_s \frac{C_s}{p_s^2}$$

57

converges. ∎

Now. we are going to discuss the behavior of g at ∞. Although g is not analytic at ∞. we still can explore its behavior in a neighborhood of ∞ with removed real line $\mathbb{R}$. There is no loss in generality if we assume that $C_0 = 0$. then. we conjugate g using the function $z \to \frac{-1}{z}$ and consider $G(z) = -1/g(\frac{-1}{z})$. infact. we have used the same kind of argument in the Proposition 3.3.

Lemma 3.1 *Let Γ_θ be the cone $\{z : \theta < Arg(z) < \pi - \theta\}$. Then. for $z \to 0.z \in \Gamma_\theta$. we have*

$$\sum_s \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z} \to 0 \qquad and \qquad \sum_s \frac{C_s}{p_s^2} \frac{z^2}{(\frac{1}{p_s} + z)^2} \to 0$$

Proof. Let us first notice that from the geometrical considerations it follows that for $z \in \Gamma_\theta$ and any p_s. we have

$$|\frac{1}{p_s} + z| > |z| \sin \theta \qquad and \qquad |\frac{1}{p_s} + z| > |\frac{1}{p_s}| \sin \theta.$$

Fix $\delta > 0$. Since $\sum_s \frac{C_s}{p_s^2}$ converges we can find s_0 such that $\sum_{s \geq s_0} \frac{C_s}{p_s^2} < \frac{1}{2}\delta \sin \theta$. Then. we have

$$|\sum_s \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z}| \leq |\sum_{s < s_0} \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z}| + |\sum_{s \geq s_0} \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z}|$$

The second summand is less than $\delta/2$ and the first can be made smaller by choosing $|z|$ sufficiently small. This proves the first part of the Lemma. Similarly. we have

$$|\sum_s \frac{C_s}{p_s^2} \frac{z^2}{(\frac{1}{p_s} + z)^2}| \leq |\sum_{s<s_0} \frac{C_s}{p_s^2} \frac{z^2}{(\frac{1}{p_s} + z)^2}| + |\sum_{s\geq s_0} \frac{C_s}{p_s^2} \frac{z^2}{(\frac{1}{p_s} + z)^2}|$$

The second summand is less than $\delta/2$ and the first can be made smaller by choosing $|z|$ sufficiently small. This proves the second part of the Lemma. ∎

Proposition 3.5 *Let $B \neq 0$. Then. For $z \in \Gamma_\theta$*

(1) $\lim_{z\to0} G(z) = 0$

(2) $\lim_{z\to0} G'(z) = \frac{1}{B}$

For $B = 0$. if $G(0)=0$. then $G'(0) = \infty$. where both values are calculated as appropriate limits for $z \to 0. z \in \Gamma_\theta$.

Proof. Let $B \neq 0$. We have

$$G(z) = -z(Az - B - \sum_s \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z})^{-1}.$$

By Lemma 3.1. $G(z) \to 0$ as $z \to 0$ for $z \in \Gamma_\theta$. This proves the first part of the Proposition. Moreover. we have

$$G'(z) = \frac{g'(-1/z)\frac{1}{z^2}}{g^2(-1/z)} = \frac{B + \sum_s \frac{C_s}{p_s^2} \frac{z^2}{(\frac{1}{p_s} + z)^2}}{(Az - B - \sum_s \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z})^2}$$

By Lemma 3.1. $G'(z) \to \frac{1}{B}$ as $z \to 0$. This proves the second part of the Proposition.

Now. for $B = 0$. assume that. in the sense we discussed above. $G'(0) = 0$. Then. for $z \in \Gamma_\theta$

$$G'(0) = \lim_{z \to 0} \frac{G(z)}{z} = \lim_{z \to 0} -(Az - \sum_s \frac{C_s}{p_s^2} \frac{z}{\frac{1}{p_s} + z})^{-1} = \infty.$$

This completes the proof of the Proposition. ∎

Corollary 3.2 *Let g be of the form (1). Then. the point ∞ is:*

(1) repelling. for $B < 1$:

(2) neutral. for $B = 1$. and

(3) attracting. for $B > 1$.

where the words repelling. neutral and attracting are understood in the sense defined in Proposition 3.5.

Example 1: Boole's Transformation

Boole's transformation is defined by $x \to x - \frac{1}{x}$. Since $B = 1$ in Boole's transformation. then by Corollary 3.2 we have ∞ as a neutral fixed point. Moreover. by Proposition 3.2. it preserves Lebesgue measure.

Example 2: Generalized Boole Transformation

The generalized Boole transformation is represented by

$$T.x = x + \beta + \sum_{k=1}^{n} \frac{p_k}{t_k - x}$$

where $n \geq 1$, $p_1, \ldots, p_n \geq 0$, $\beta, t_1, \ldots, t_n \in \mathbb{R}$. Since $B = 1$ in the generalized Boole transformation, then by Corollary 3.2 we have ∞ as a neutral fixed point. Moreover, by Proposition 3.2, it preserves Lebesgue measure.

Let $g(z)$ be of the form (1). Baker, Kotus and Yinian proved in [BKY] that the Julia set, $J(g)$, equals to $\mathbb{R}$ or it is an unbounded Cantor like set of $\mathbb{R}$. As a consequence of this work, and from the results we obtained on ergodic theory for inner functions, we got the following result:

Theorem 3.7 *Let $g(z)$ be of the form (1) with $B < 1$. Then, $J(g) = \mathbb{R}$ if and only if the restriction of g to $\mathbb{R}$ is ergodic.*

In Theorem 3.7, ergodicity is meant in a more general sense: whenever $g^{-1}(A) = A$, then either $m(A) = 0$ or $m(\mathbb{R} \backslash A) = 0$ where A is a measurable set.

Proof. By the Denjoy-Wolff theorem, $g(z)$ has a unique fixed point z_0 in $\overline{\mathbb{R}^{2+}}$ such that $g^n(z) \to z_0 \ \forall \ z \in \mathbb{R}^{2+}$. We have two cases:

1. $z_0 \in \mathbb{R}^{2+}$. By the Denjoy-Wolff theorem z_0 is an attracting fixed point. Therefore, $g^n(z) \to z_0 \ \forall \ z \in \mathbb{R}^{2+}$ and $\mathbb{R}^{2+} \subset F(g)$. Now, since g is symmetric with respect to $\mathbb{R}$, then $C \backslash \overline{\mathbb{R}^{2+}}$ is attracted to a fixed point in $C \backslash \overline{\mathbb{R}^{2+}}$ and $C \backslash \overline{\mathbb{R}^{2+}} \subset F(g)$.

Moreover. $g(z)$ preserves the real line: therefore. $\mathbb{R}^{2+}$ is completely invariant in $F(g)$. Suppose that $\mathbb{R}$ contains a subset of Fatou set. then this subset will be eventually attracted by the fixed point in $\mathbb{R}^{2+}$ or $C\backslash\overline{\mathbb{R}^{2+}}$. However. $g(z)$ preserves the real line. Hence. no subset of $\mathbb{R}$ can be attracted to the fixed point. Therefore. $\mathbb{R}$ cannot contain a part of Fatou set. It follows by [BKY] that $J(g) = \mathbb{R}$.

Using the map $w \longrightarrow \dfrac{w-i}{w+i}$ which maps $\mathbb{R}^{+2}$ conformally onto D. and Proposition 2.6. we get that the restriction of $g(z)$ to $\mathbb{R}$ is ergodic.

2. $z_0 \in \mathbb{R}$. Here we have two cases:

i. z_0 is an attracting fixed point. then $z_0 \in F(g)$. By [BKY]. $J(g) = \mathbb{R}$ or it is a Cantor like set of $\mathbb{R}$. and since $z_0 \in F(g)$ the latter follows.

Using the map $w \longrightarrow \dfrac{w-i}{w+i}$ which maps $\mathbb{R}^{2+}$ conformally onto D. and Proposition 2.7. we get that the restriction of $g(z)$ to $\mathbb{R}$ is not ergodic in this case.

ii. z_0 is a neutral fixed point. By Theorem 2.8 our neutral fixed point is rationally indifferent. Hence. petals will appear. Observe that

$$g'''(z_0) = 6\frac{C_0}{z_0^4} + 6\sum_s C_s \frac{1}{(z_0 - p_s)^4}$$

is strictly greater than zero. Therefore. z_0 is of multiplicity 3 or 2. and in both cases z_0 eventually attracts all points in $\mathbb{R}^{2+}$.

Consider the first case. which is shown in Figure 2. the repelling petals are on the real line. According to [BKY]. $J(g) = \mathbb{R}$ or it is a Cantor like set of $\mathbb{R}$. Since z_0 eventually attracts all points in $\mathbb{R}^{2+}$. then $\mathbb{R}^{2+} \subset F(g)$ and similarly for $C\backslash\overline{\mathbb{R}^{2+}}$. z_0 is the only possible attractor on $\mathbb{R}$. Since $\mathbb{R}$ contains the repelling petals. no piece of

$\mathbb{R}$ will be attracted to z_0: moreover. $g(z)$ preserves the real line: therefore. no piece of $\mathbb{R}$ will be attracted to $\mathbb{R}^{+2}$ or $C\backslash\overline{\mathbb{R}^{2+}}$. Hence. $J(g) = \mathbb{R}$. Now. by Corollary 2.3 we get that $g(z)$ is ergodic in this case.

Consider the second case. which is shown in Figure 3. the attracting petal is part of the real line. We know that the attracting petal is subset of a component of Fatou set. Therefore. the real line contains some parts of Fatou and it follows that . by [BKY]. $J(g)$ is a Cantor like set of $\mathbb{R}$.

z_0 is the end point of complementary intervals α of $J(g)$. We can choose a small interval $\eta \subset \alpha$ with $g^n(\eta)$ being subsets of α tending uniformly to z_0. $n = 1.2.\ldots$ Choose η small enough such that $g(\eta)\cap\eta = \emptyset$ and V be an open interval between η and $g(\eta)$. Let $A = \cup_{-\infty}^{+\infty}g^n(\eta)$. We want to show that $V \cap A = \emptyset$. Since $g^n(\eta)$ converges uniformly to z_0. no image of η will intersect V. Now. suppose that $V \cap A \neq \emptyset$. then $\exists x \in V \cap A$. i.e.. $x \in g^{-n}(\eta)$ for some n. Hence. $g^n(x) \in \eta$ which is impossible. It follows that $V \cap A = \emptyset$. Therefore. $m(\mathbb{R}\backslash A) > 0$. Also $g^{-1}(A) = A$. Hence. g is not ergodic in this case. ∎

Corollary 3.3 *Let $g(z)$ be of the form (1). Then $J(g) = \mathbb{R}$ if and only if the restriction of g to $\mathbb{R}$ is exact.*

Proposition 3.6 *In case 2 of Theorem 3.7. if $g'(z_0) = 1$ and the attracting petal is part of the real line. then the absolutely continuous invariant measure μ_{z_0} has the following property: for any g-invariant set A of positive μ-measure. the measure $\nu = \mu_{|A}$ is not ergodic.*
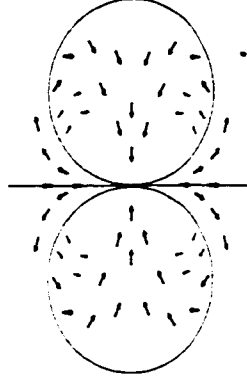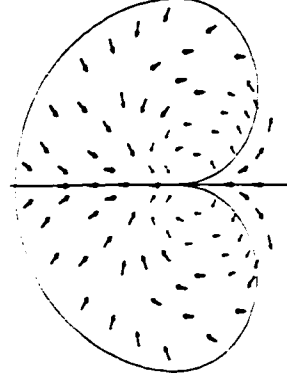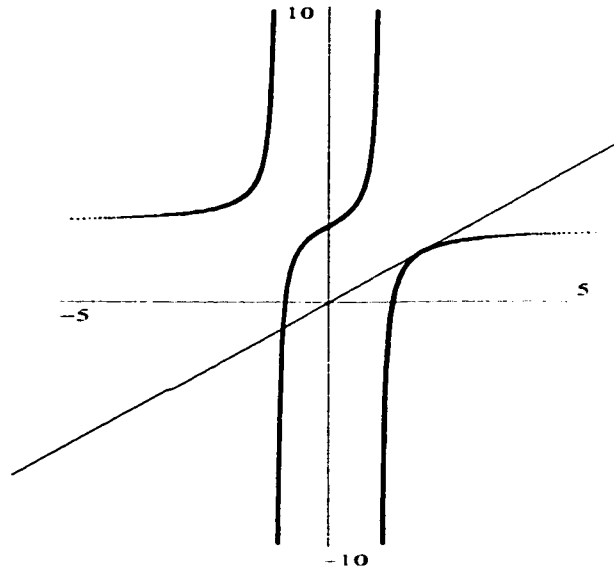
Figure 2   Figure 3

Proof. Let A be a g-invariant set. Choose $\gamma \subset A$ such that $\gamma \cap g(\gamma) = \emptyset$ with $g^n(\gamma)$ being subsets of the invariant set tending uniformly to z_0. Moreover, choose γ small enough such that $\exists \delta \subset A$ with $\nu(\delta) > 0$ and δ lies between γ and $g(\gamma)$. Now, let $B = \bigcup_{-\infty}^{\infty} g^n(\gamma)$. We will prove that $B \cap \delta = \emptyset$. Since $g^n(\gamma)$ converges uniformly to z_0, no image of γ will intersect δ. Suppose that $B \cap \delta \neq \emptyset$, then $\exists x \in B \cap \delta$, i.e., $x \in g^{-n}(\gamma)$ for some n. Hence, $g^n(x) \in \gamma$ which is impossible. It follows that $B \cap \delta = \emptyset$, i.e., $\delta \subset (A \backslash B)$. We have chosen δ such that $\nu(\delta) > 0$; therefore, $\nu(A \backslash B) > 0$. Also, $g^{-1}(B) = B$. Hence, the measure $\nu = \mu_{|A}$ is not ergodic. ∎

Example 3: Let $g(z) = \frac{1}{2}\left(\frac{-1}{z-1} + \frac{-1}{z+1}\right) + \frac{3}{2}\sqrt{3}$. Observe that $g(\sqrt{3}) = \sqrt{3}$ and that $g'(\sqrt{3}) = 1$, i.e., $\sqrt{3}$ is a neutral fixed point. Hence, the restriction of $g(z)$ to $\mathbb{R}$ preserves the infinite measure $\frac{dt}{(t-\sqrt{3})^2}$. Also, $g''(z) = \frac{1}{2}\left(\frac{-2}{(z-1)^3} + \frac{-2}{(z+1)^3}\right)$ and $g''(\sqrt{3}) < 0$. Plotting the graph of our function, see Figure 4, we notice that $z_0 = \sqrt{3}$ is attracting from one side and repelling from the other. Therefore, the petal will be

64

$$g(x) = \tfrac{1}{2}(\tfrac{-1}{x-1} + \tfrac{-1}{x+1}) + \tfrac{3}{2}\sqrt{3}$$

Figure 4

the same as in Figure 3. Therefore. by Proposition 3.6. for any g-invariant set A of positive μ-measure. the measure $\nu = \mu_{|A}$ is not ergodic.

Now. we are going to study $g(z)$ when $B \geq 1$. This is the case when ∞ is the Denjoy-Wolff point. It is different from the above because $g(z)$ is not analytic at ∞.

Proposition 3.7 *Let $g(z)$ be of the form (1) with $B \geq 1$. If $J(g)$ is a Cantor like set of $\mathbb{R}$. then the restriction of $g(z)$ to $\mathbb{R}$ is non-ergodic.*

Proof. Suppose that $J(g)$ is a Cantor like set of $\mathbb{R}$. Then. there exist open intervals in $F(g) \cap \mathbb{R}$. Since $B \geq 1$. ∞ is the Denjoy-Wolff point and all the points are repelling on $\mathbb{R}$ by Corollary 3.2. Therefore. there is a sequence of intervals α_n of $F(g)$ converging uniformly to ∞. We can choose a small interval $\eta \subset \alpha_n$ with $g^n(\eta)$ being subsets of α_n tending uniformly to ∞. $n = 1. 2. \ldots$ Choose η small enough such

65

that $g(\eta) \cap \eta = \emptyset$ and V be an open interval between η and $g(\eta)$. Let $A = \cup_{-\infty}^{+\infty} g^n(\eta)$. We want to show that $V \cap A = \emptyset$. Since $g^n(\eta)$ converges uniformly to ∞. no image of η will intersect V. Now. suppose that $V \cap A \neq \emptyset$. then $\exists x \in V \cap A$. i.e.. $x \in g^{-n}(\eta)$ for some n. Hence. $g^n(x) \in \eta$ which is impossible. It follows that $V \cap A = \emptyset$. Therefore. $m(\mathbb{R} \backslash A) > 0$. Also $g^{-1}(A) = A$. Hence. the restriction of $g(z)$ to $\mathbb{R}$ is not ergodic. ∎

Example 4: Let $g(z)$ be of the form (1) with $B > 1$. Suppose that $\sum_s C_s < \infty$ and $p_s < 0$ for all s. For $B > 1$. we have

$$\text{Re } g(z) > Bx + A - \sum_s C_s$$

for all Re $z > 1$. This shows that Re $g(z) > \text{Re } z + 1$ in some right half-plane H_1. Therefore.

$$g^n(x) \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

uniformly in H_1. Hence. $H_1 \subset F(g)$. It follows that $J(g)$ is a Cantor like set of $\mathbb{R}$. Hence. by Proposition 3.7. the restriction of $g(z)$ to $\mathbb{R}$ is not ergodic.

Example 5: Let $g(z)$ be of the form (1) with $B = 1$. Suppose that $\sum_s C_s < \infty$ and $p_s < 0$ for all s and $A - \sum_s C_s > 0$. We have.

$$\text{Re } g(z) > Bx + A - \sum_s C_s$$

for all Re $z > 1$. This shows that Re $g(z) >$ Re $z + \delta$. for some $\delta > 0$. in some right half-plane H_1. Therefore.

$$g^n(x) \to \infty \text{ as } n \to \infty$$

uniformly in H_1. Hence. $H_1 \subset F(g)$. It follows that $J(g)$ is a Cantor like set of $\mathbb{R}$. Hence. by Proposition 3.7. the restriction of $g(z)$ to $\mathbb{R}$ is not ergodic.

Remark 3.2 *In Example 5. we constructed a non-ergodic inner function for $B = 1$.*

Proposition 3.8 *If $g(z)$ satisfies the assumptions of Proposition 3.7 with $B = 1$. then Lebesgue measure m has the following property: for any g-invariant set A of positive Lebesgue measure. the measure $\nu = m_{|A}$ is not ergodic.*

Proof. Let A be a g-invariant set. Choose $\gamma \subset A \cap a_n$ such that $\gamma \cap g(\gamma) = \emptyset$. where a_n is a sequence of intervals of $F(g)$ converging uniformly to ∞. Being subsets of a_n. $g^n(\gamma)$ converges uniformly to ∞. Moreover. we choose γ small enough such that $\exists \delta \subset A$ with $\nu(\delta) > 0$ and δ lies between γ and $g(\gamma)$. Now. let $B = \bigcup_{-\infty}^{\infty} g^n(\gamma)$. We will prove that $B \cap \delta = \emptyset$. Since $g^n(\gamma)$ converges uniformly to ∞. no image of γ will intersect δ. Suppose that $B \cap \delta \neq \emptyset$. then $\exists x \in B \cap \delta$. i.e.. $x \in g^{-n}(\gamma)$ for some n. Hence. $g^n(x) \in \gamma$ which is impossible. It follows that $B \cap \delta = \emptyset$. i.e.. $\delta \subset (A \backslash B)$. We have chosen δ such that $\nu(\delta) > 0$: therefore. $\nu(A \backslash B) > 0$. Also. $g^{-1}(B) = B$. Hence. the measure $\nu = m_{|A}$ is not ergodic. ∎

Now we are going to discuss some examples where Theorem 3.7 is invalid for $B > 1$. Theorem 3.7 will hold for $B > 1$ only when it is rational. i.e.. when it

has finite number of poles where we can conjugate ∞ to a real Denjoy-Wolff point. However. it is invalid "in general" when $g(z)$ has infinite number of poles.

Example 6: Let $g(z) = \tan(z) + Bz$. This is an example where Theorem 1 fails for $B > 1$. Since $B > 1$. ∞ is an attracting fixed point. Observe that the intervals of $\mathbb{R} \backslash \bigcup_s p_s = L$ have bounded lengths. Then. for x in some interval $I \in L$. $g'(x) > B$ which implies that $(g^n(x))' > B^n$. Suppose that $J(g)$ is a Cantor like set of $\mathbb{R}$ and let $I_n = g^n(I)$. Then. there exists $I \subset F(g)$. It follows that $I_n \subset F(g)$. Then we get

$$|I_n| > B^n|I| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

. which is a contradiction. It follows that $J(g) = \mathbb{R}$. By Theorem. $g(z)$ is non-ergodic.

Example 7: This a more general example where Theorem 1 fails for $B > 1$. Let $\{p_s\}$ be an increasing sequence and

$$g(z) = Bz + \sum_{s \geq 0} \left(-\frac{1}{z - p_s} - \frac{1}{z + p_s} \right).$$

Suppose that the intervals of $\mathbb{R} \backslash \bigcup_s p_s = L$ have bounded lengths. The rest can be proved in the same way as Example 6.

Remark 3.3 *In Examples 6 and 7. if $B = 1$. we have that the restriction of $g(z)$ to $\mathbb{R}$ is ergodic by Theorem 3.6. Moreover. for any x in some interval $I \in L$. $g'(x) > \delta > 1$. Following the proof of Example 4. we get that $J(g) = \mathbb{R}$ in this case.*

One is led to ask : Does Theorem 3.7 hold for $B = 1$? We could not prove or disprove this statement.

$g(z)$	$g(x)$ is ergodic $\Rightarrow J(g) = \mathbb{R}$	$J(g) = \mathbb{R} \Rightarrow g(x)$ is ergodic
$B < 1$	True	True
$B = 1$	True	Open Problem
$B > 1$	True	False

Bibliography

[Aa] J. Aaronson. *An Introduction to Infinite Ergodic Theory*. American Math. Soc. 1997.

[Be] A. Beardon. *Iteration of Rational Functions*. Springer-Verlag. 1991.

[BG] A. Boyarsky. P. Góra. *Laws of Chaos: Invariant Measures and Dynamical Systems in One Dimension*. Brikhauser. New York. 1997.

[BKY] I.N. Baker . J. Kotus. L. Yinian. *Iterates of Meromorphic Functions: I*. Ergod. Thy. and Dynam. Sys.. 1991. 241-248.

[DM] C. Doerning. R. Mane. *The Dynamics of Inner Functions*. Ensaisos Mat.. 1992.

[GO] P. Góra. N. Obeid. *Absolutely Continuous Invariant Measures for a Class of Meromorphic Transformations*. preprint.

[Mi] J. Milnor.*Dynamics in One Complex Variable: Introductory Lectures*. SUNY Stony Brook. 1990.

[Ne] J. H. Neuwirth. *Ergodicity of Some Mappings of The Circle and The Line*. Israel

Journal of Math.. 1978. 359-367.

[Ru] W. Rudin.*Real and Complex Analysis*. McGraw-Hill. New York. 1974.