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Absolutely Continuous Invariant Measures For Piecewise Linear Interval Maps Both Expanding And Contracting

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ABSTRACT

Absolutely Continuous Invariant Measures For Piecewise Linear Interval Maps Both Expanding And Contracting

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Let $\mathcal{F}$ be a family of piecewise linear maps $f : [-1, 1] \rightarrow [-1, 1]$ with a discontinuity at $x = 0$ such that f is expanding in one of the intervals $(-1, 0), (0, 1)$ and contracting in the other. We will study the dynamics of maps of sub-classes of $\mathcal{F}$. It will be shown that a map in such a sub-class of $\mathcal{F}$ either has a periodic attractor or is eventually expanding. In the latter case, there exists an absolutely continuous invariant measure (acim) and in many examples we will find the measure.

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Contents

Chapter I

Introduction

1.1 General introduction

Finding conditions on a transfirmation that will guarantee the existence of absolutely continuous invariant measure (acim) is an important question in ergodic theory and dynamical systems. Although of mathematical interest by itself, this problem has many applications in other areas, namely in physical and biological sciences. Renyi [Re] was the first to introduce a class of transformations of the unit interval and proved that it has an acim. Later, in 1973, A. Lasota and J.A. Yorke [L-Y] generalized Renyi's result and using this generalized result, many mathematicians proved the existence of acim for some transformations. Recently, M. A. Boudourides and N. F. Fotiades [B-F] considered the piecewise linear interval maps both expanding and contracting and proved the existence of acim for such maps. Even when it is known that there exists a unique acim, finding it may be a tedious chore. This thesis provides a generalization of A. Boudourides's and N. F. Fotiades's results.

Consider the family $\mathcal{F}$ of functions $f : [-1,1] \longrightarrow [-1,1]$ defined by

$$f(x) = \begin{cases} \lambda_1 x + a & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ \lambda_2 x + b & , \quad x \in (0,1] \end{cases}$$

where $a, b, \lambda_1, \lambda_2$ are constants such that $f([-1,1]) \subset [-1,1]$.

Moreover, we assume that the absolute value of the slope of f is greater than 1 (expansion) in one of the intervals $(-1,0), (0,1)$ and less than 1 (contraction) in the other.

A. Lasota and J.A. Yorke showed in [L-Y] that for a piecewise expanding map there exists an acim with respect Lebesgue measure. In our case, one branch of a map $f \in \mathcal{F}$ is expanding and the other branch is contracting. We will show that under certain conditions on λ_1 and λ_2, $f \in \mathcal{F}$ is eventually expanding and under certain conditions on λ_1 and λ_2, f has a periodic attractor. In the former case, we will show that there exists an acim for $f \in \mathcal{F}$. Finally, for many examples, we will find the acim. We will follow [B-F]. They considered f with one increasing branch and one decreasing branch and discussed the dynamics of such maps. We will also consider f with both increasing branches and will discuss the dynamics of such maps. We will show the existence of acim and find the acim for Markov case using the matrix representation of Frobenius-Perron operator and also using numerical computations.

1.2 Scope of the Thesis

In Chapter II, we review some basic concept from measure theory, dynamical system, ergodic theory and functional analysis.

In Chapter III, we discuss the Frobenius-Perron operator, Markov transformations and the matrix representation of the Frobenius Perron operator and acim for piecewise monotonic transformations.

In Chapter IV, existence of acim for piecewise linear interval maps both contracting and expanding with one increasing branch and one decreasing branch are given.

In chapter V, we will consider f with two increasing branches and will show the existence of an acim for such f.

In Chapter VI, C++ code and corresponding numerical results for finding the density function are given.

Chapter II

Preliminaries

In this chapter we review some basic concepts from measure theory, dynamical systems, ergodic theory and functional analysis. We follow [B-G], [M-S], [R1] and [R2].

Definition 2.1.1

Let $(X, \mathcal{B})$ be a space X and a σ-algebra of its subsets. Two measures μ and ν on $(X, \mathcal{B})$ are said to be mutually singular if and only if there exist disjoint sets A_μ and B_ν such that $X = A_\mu \cup B_\nu$ and $\nu(A_\mu) = \mu(B_\nu) = 0$. In this case we write, $\mu \perp \nu$.

Definition 2.1.2

Let μ and ν be two measures on $(X, \mathcal{B})$. We say that ν is absolutely continuous (a.c.) with respect to μ if for any $A \in \mathcal{B}$, $\nu(A) = 0$ whenever $\mu(A) = 0$. In this case we write $\nu << \mu$.

Definition 2.1.3

Let $(X, \mathcal{B}, \mu)$ be a measure space. By $(L^1, \|\bullet\|_1)$ we mean the family of all integrable functions f on X, i.e.,

$$(L^1, \|\bullet\|_1) = \left\{f : X \to R \mid \|f\|_1 = \int |f(x)| d\mu(x) < \infty\right\}.$$

By $(L^\infty, \|\bullet\|_\infty)$, we mean $(L^\infty, \|\bullet\|_\infty)$ = space of almost everywhere bounded measurable functions on $(X, \mathcal{B}, \mu)$, i.e.,

$$(L^\infty, \|\bullet\|_\infty) = \{f : X \to R \text{ s.t. } \|f\|_\infty = \operatorname{essup}|f(x)| = \inf\{M : \mu\{x : f(x) > M\} = 0\} < +\infty\}$$

Theorem 2.1.1 (Radon-Nikodym Theorem):

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space (that is, there exists a sequence $\{X_n\} \subset X$ such $X = \bigcup_{n=1}^\infty X_n$ and $\mu(X_n) < \infty$) and let ν be a measure on $(X, \mathcal{B})$ which is absolutely continuous with respect to μ. Then there exists a unique non-negative measurable function f such that for any set $E \in \mathcal{B}$, we have $\nu(E) = \int_E f d\mu$. The function f is called the Radon-Nikodym derivative of ν with respect to μ and is denoted by $f = [\frac{d\nu}{d\mu}]$.

Definition 2.1.4

A transformation τ: $X \to X$ is called a measurable transformation if for all measurable subsets A of X, $\tau^{-1}(A) = \{x \in X : \tau(x) \in A\}$ is a measurable subset of X.

Definition 2.1.5

Let $(X, \mathfrak{B}, \mu)$ be a normalized Lebesgue measure space and τ: $X \to X$ be a transformation. Then τ is non-singular if and if $\mu(\tau^{-1}(A)) = 0$ whenever $\mu(A) = 0$ for all measurable subsets A of X.

Definition 2.1.6

Let $(X, \mathfrak{B}, \mu)$ be a measure space and τ: $X \longrightarrow X$ be a measurable transformation. Then μ is invariant (τ preserves μ) if for any measurable subset A, $\mu(A) = \mu(\tau^{-1}(A)$.

Theorem 2.1.2

Let τ: $X \longrightarrow X$ be a measurable transformation on $(X, \mathfrak{B}, \mu)$. Then τ is invariant (τ preserves μ) if and only if

$$\int f(x)d\mu = \int_X f(\tau(x))d\mu$$

for any $f \in L^\infty$. If X is compact and above holds for any continuous function then τ is μ-preserving.

Definition 2.1.7

Let $(X, \mathfrak{B}, \mu)$ be a normalized measure space and

$$\mathcal{D} = \mathcal{D}(X, \mathfrak{B}, \mu) = \{f \in L^1(X, \mathfrak{B}, \mu) : f \geq 0 \text{ and } \|f\|_1 = 1\}$$

denotes the space of all probability density functions. A function $f \in \mathcal{D}$ is called a density function or simply a density. If $f \in \mathcal{D}$, then $\mu_f(A) = \int_A f d\mu << \mu$ is a measure and f is called the density of μ_f and is written as $\frac{d\mu_f}{d\mu}$.

Definition 2.1.8

$C^0(X) = C(X)$ denotes the space of all continuous real functions $f : X \longrightarrow \mathbb{R}$ with the norm

$$\|f\|_{C^0} = \sup_{x\in X} |f(x)|.$$

Let $r \geq 1$. $C^r(X)$ denotes the space of all r-times continuously differentiable real functions $f: X \to \mathbb{R}$ with the norm

$$\|f\|_{C^r} = \max_{0\leq k\leq r} \sup_{x\in X} |f^{(k)}(x)|,$$

where $f^{(k)}(x)$ is the kth derivative of $f(x)$ and $f^{(0)}(x) = f(x)$.

Definition 2.1.9

A transformation τ: $X \to R$ is called piecewise C^2, if there exists a partition $a = a_0 < a_1 < ..... < a_p = b$ of the closed interval $I = [a,b]$ such that for each integer i ($i = 1,2,...,p$) the restriction τ_i of τ to the open interval (a_{i-1},a_i) is a C^2 function which can be extended to the closed interval $[a_{i-1},a_i]$ as a C^2 function. τ need not be continuous at the points a_i.

Definition 2.1.10

Let $f: I \mapsto I$ be a map and $J \subset I$. The first return map of f on J is a map $R: J \longrightarrow J$ defined by $R(x) = f^k(x)$ where $k = \min\{i > 0$ such that $f^i(x) \in J\}$.

Definition 2.1.11

Let $f: I \mapsto I$ be a map and $x \in [-1,1]$. The point x is called a fixed point of f if $f(x) = x$ and it is called a periodic point of period p if $f^p(x) = x$ and $f^q(x) \neq x$ for $1 \leq q < p$. We denote by $O(x)$ the periodic orbit $\{x, f(x), f^2(x), ..., f^{p-1}(x)\}$. The set $B(x) = \{y : f^k(y) \longrightarrow O(x)$, as $k \longrightarrow \infty\}$ is called the basin of $O(x)$. If $B(x)$ contains a nontrivial interval then the orbit $O(x)$ is called attracting.

Definition 2.1.12

A map $f: [a,b] \mapsto [a,b]$ is called expanding if there exists a partition of $[a,b]$, $a = a_0 < a_1 < a_2 < ...... < a_r = b$ such that $|f'(x)| \geq \lambda > 1$ for all $x \in (a_{i-1},a_i)$. The function f is called eventually expanding if f^k is expanding for some $k > 1$.

Definition 2.1.13

Let $I = [a,b]$. The transformation τ: $I \longrightarrow I$ is called piecewise monotonic if there exists a partition of I, $a = a_0 < a_1 < a_2 < \ldots\ldots < a_q = b$ and any integer $r \geq 1$ such that

(i) $\tau|_{(a_{i-1},a_i)}$ is a C^r function, $i = 1,2,3,\ldots\ldots,q$ which can be extended to a C^r function on $[a_{i-1},a_i]$, $i = 1,2,3,\ldots\ldots,q$ and

(ii) $|\tau'(x)| > 0$ on (a_{i-1},a_i), $i = 1,2,3,\ldots\ldots,q$.

If in addition to (ii), $|\tau'(x)| \geq \alpha > 1$ whenever the derivative exists, then τ is called piecewise monotonic and expanding. Note that (ii) implies that τ is monotonic on each (a_{i-1},a_i).

Definition 2.1.14

Let $I = [a,b]$ and $\tau : I \mapsto I$. Let P be a partition of I given by the point $a = a_0 < a_1 < \ldots\ldots < a_n = b$. For $i = 1,2,\ldots\ldots,n$ let $I_i = (a_{i-1},a_i)$ and denote the restriction of τ to I_i by τ_i. If τ_i is a homeomorphism from I_i onto some connected union of intervals $(a_{j(i)},a_{k(i)})$, then τ is said to be Markov. The partition $P = \{I_i\}_{i=1}^n$ is referred to as Markov partition with respect to τ. If, in addition , $|\tau'(x)| > 0$ on each I_i, we say that τ is in class τ_M. If each τ_i is also linear on I_i, we say that τ is piecewise linear Markov transformation.

Example 2.1.1

Let $0 = a_0 < a_1 < a_2 < \ldots < a_n = 1$ be any partition of $I = [0,1]$. Let $I_i = (a_{i-1},a_i)$, and define the piecewise linear, continuous transformation $\tau : I \mapsto I$ by the conditions

(i) $\tau(I_i) = I_{i+1}$, $1 \leq i \leq n-1$ and

(ii) $\tau(I_n) = \cup_i^n I_i$.

Then τ is a Markov transformation. In particular, the transformation f defined by

$$f(x) = \begin{cases} x + \frac{1}{4} & , \quad 0 \leq x < \frac{3}{4} \\ \frac{1}{4}(x - \frac{3}{4}) & , \quad \frac{3}{4} \leq x \leq 1 \end{cases}$$

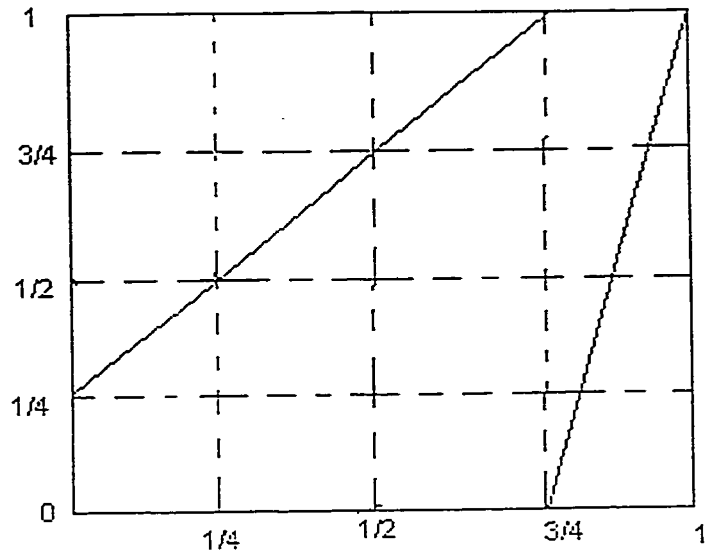


Figure 2.1.1

with the partition $\{(0,\frac{1}{4}),(\frac{1}{4},\frac{1}{2}),(\frac{1}{2},\frac{3}{4}),(\frac{3}{4},1)\}$ is a Markov transformation.

Chapter III

Frobenius-Perron operator and absolutely continuous invariant measures for piecewise monotonic transformations.

In this chapter we introduce the notion of Frobenius-Perron operator, a very useful tool in the study of acim. We shall also present the matrix representation of Frobenius-Perron operator for Markov transformations that we shall use in the next chapters to find the acim. Finally we shall show the existence of acim for piecewise monotone transformation using the method of bounded variation and Frobenius-Perron operator. In this chapter we follow [B-G], [D-S], [P] and [L-Y].

3.1 Frobenius-Perron Operator

Definition 3.1.1

Let $\tau : I \mapsto I$ be a measurable non-singular transformation and $f \in L^1(I)$. For $A \subset I$ measurable , define the measure μ by

$$\mu(A) = \int_{\tau^{-1}(A)} f d\lambda.$$

From the definition of non-singularity of τ we see that

$$\lambda(A) = 0 \Rightarrow \lambda(\tau^{-1}(A)) = 0 \Rightarrow \mu(A) = 0.$$

Thus $\mu << \lambda$. By the Radon Nikodym theorem, there exists $\phi \in L^1(I)$ such that for all measurable set $A \subset I$, $\mu(A) = \int_A \phi d\lambda$ and ϕ is unique almost everywhere (a.e.). Set $P_\tau f = \phi$. The probability density function f has been transformed to a new probability density function $P_\tau f$. P_τ obviously depends on the transformation τ. Thus P_τ maps L^1 into L^1 and for all measurable sets $A \subset I$,

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda.$$

P_τ is referred to as the Frobenius-Perron operator associated with τ. Clearly f is invariant under τ if and only if $P_\tau f = f$, i.e., f is a fixed point of the Frobenius-Perron operator. Letting $A = [a,x]$ and differentiating both sides of the above equation, we obtain:

$$P_\tau f = \frac{d}{dx}\int_{\tau^{-1}([a,x])} f(t)dt\ .$$

Example 3.1.1

$$\tau(x) = \begin{cases} -2x+1 & if \ \ 0 \le x < \frac{1}{2} \\ 2x-1 & if \ \ \frac{1}{2} < x \le 1 \end{cases}$$

If $x < 0$,

$$\tau^{-1}[0,x] = [0, \tfrac{1-x}{2}] \cup [\tfrac{1}{2}(1+x), 1].$$

For $f \in L^1(I)$,

$$\int_{\tau^{-1}([0,x])} f d\lambda = \int_0^{\frac{1-x}{2}} f d\lambda + \int_{\frac{1-x}{2}}^1 f d\lambda.$$

Therefore

$$P_\tau f = \frac{d}{dx}\int_{\tau^{-1}([0,x])} f(t)dt = \tfrac{1}{2}\{f(\tfrac{1-x}{2}) + f(\tfrac{1+x}{2})\}.$$

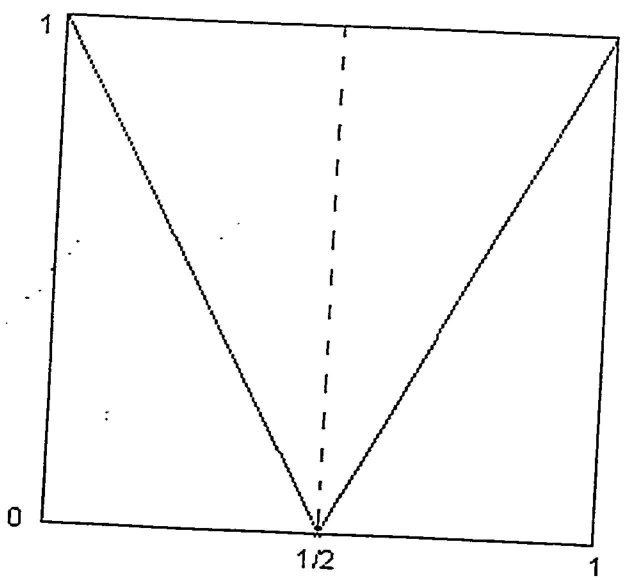


Figure 3.1.1

3.2 Properties of the Frobenius-Perron operator:

Property 3.2.1

$P_\tau : L^1(I) \mapsto L^1(I)$ is a linear operator;

Property 3.2.2

if $f \in L^1(I)$ and $f \geq 0$, then $P_\tau f \geq 0$;

Property 3.2.3

P_τ preserves integrals;

Property 3.2.4

$P_\tau : L^1(I) \mapsto L^1(I)$ is a contraction, i.e.,

$$\|P_\tau f\|_{L^1} \leq \|f\|_{L^1} \text{ for any } f \in L^1.$$

Moreover, $P_\tau : L^1(I) \mapsto L^1(I)$ is continuous with respect to the norm topology;

Property 3.2.5

Let $\tau : I \mapsto I$ and $\sigma : I \mapsto I$ be non-singular. Then

$$P_{\tau\circ\sigma} f = P_\tau \circ P_\sigma f;$$

Property 3.2.6

Let $\tau : I \mapsto I$ be a nonsingular transformation. Then $P_\tau f^* = f^*$ a.e if and only if the measure $\mu = f^*\lambda$, defined by $\mu(A) = \int_A f^* d\lambda$, is τ-invariant, i.e, if and only if $\mu(A) = \mu(\tau^{-1}(A))$ for all measurable set A, where $f^* \geq 0, f \in L^1$ and $\|f\| = 1$.

Proof:

(i) Let $A \subset I$ be measurable and α, β be constants. If $f, g \in L^1$, then

$$\begin{aligned}
\int_A P_\tau(\alpha f + \beta g) d\lambda &= \int_{\tau^{-1}(A)} (\alpha f + \beta g) d\lambda \\
&= \alpha \int_{\tau^{-1}(A)} f d\lambda + \beta \int_{\tau^{-1}(A)} g d\lambda \\
&= \alpha \int_A P_\tau f + \beta \int_A P_\tau g d\lambda \\
&= \int_A (\alpha P_\tau f + \beta P_\tau g) d\lambda.
\end{aligned}$$

Since A is arbitrary we have

$$P_\tau(\alpha f + \beta g) d\lambda = \alpha P_\tau f + \beta P_\tau g \text{ almost everywhere.}$$

(ii) For any $A \subset I$,

$$\int_A P_\tau f = \int_{\tau^{-1}(A)} f \geq 0. \text{ Hence } P_\tau f \geq 0.$$

(iii) Observe that

$$\tau^{-1}[a,b] = [a,b].$$

Now $$\int_I P_\tau f d\lambda = \int_{\tau^{-1}(I)} f d\lambda = \int_\tau f d\lambda.$$

(iv) Let $f \in L^1$ and

$$f^+ = \max(f,0),\ f^- = -\min(0,f).$$

Then

$$f = f^+ - f^- \text{ and } |f| = f^+ + f^-.$$

By linearity we get,

$$P_\tau f = P_\tau(f^+ - f^-) = P_\tau f^+ - P_\tau f^-.$$

Now

$$|P_\tau f| \leq |P_\tau f^+| + |P_\tau f^-|$$

$$= |P_\tau f^+| + |P_\tau f^-| = P_\tau(f^+ + f^-) = P_\tau |f|.$$

Thus

$$\|P_\tau f\|_{L^1} = \int_I |P_\tau f| d\lambda \leq \int_I P_\tau |f| d\lambda$$

$$= \int_{\tau^{-1}(I)} |f| d\lambda = \int_I |f| d\lambda = \|f\|_{L^1}$$

Hence

$$\|P_\tau f\|_{L^1} \leq \|f\|_{L^1}.$$

It follows from this result that $P_\tau : L^1 \to L^1$ is continuous with respect to the norm topology, since

$$\|P_\tau f - P_\tau g\|_{L^1} \leq \|f - g\|_{L^1}.$$

(v) Let $f \in L^1$ and define the measure μ by

$$\mu(A) = \int_{(\tau\circ\sigma)^{-1}(A)} f d\lambda.$$

Now $$\lambda(A) = 0 \Rightarrow \lambda(\tau^{-1}(A)) = 0 \text{ (non singularity)}.$$

$$\Rightarrow \lambda(\sigma^{-1}(\tau^{-1}(A)) \quad = 0 \text{ (non singularity again).}$$

Thus $\mu(A) = 0$ whenever

$$\lambda(A) = 0 \Rightarrow \mu << \lambda.$$

Therefore

$$v(A) = \int_{\tau\circ\sigma(A)} P_\tau f d\lambda \text{ is well defined.}$$

Now
$$\int_A P_{\tau\circ\sigma} f = \int_{(\tau\circ\sigma)^{-1}(A)} f = \int_{\sigma^{-1}(\tau^{-1}(A))} f$$
$$= \int_{\tau^{-1}(A)} P_\sigma f = \int_A P_\tau P_\sigma f$$

Hence

$$P_{\tau\circ\sigma} f = P_\tau P_\sigma f \text{ a.e.}$$

It is clear that $\quad P_{\tau^1} f = P_\tau^1 f.$

Assume that $P_{\tau^n} f = P_\tau^n f$. Now $P_{\tau^{n+1}} f = P_{\tau\circ\tau^n} f = P_\tau P_{\tau^n} f = P_\tau(P_{\tau^n} f)$.

Hence $P_{\tau^n} f = P_\tau^n f$.

(vi) Assume that $\mu(\tau^{-1}(A)) \quad = \mu(A)$. Then

$$\int_{\tau^{-1}(A)} f^* d\lambda = \int_A f^* d\lambda$$

and therefore

$$\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda.$$

Since A is arbitrary, $P_\tau f^* = f^*$ a.e.

Conversely, assume that $P_\tau f^* = f^*$ a.e. Then $\int_A P_\tau f^* d\lambda = \int_A f^* d\lambda = \mu(A)$.

By definition,

$$\int_A P_\tau f^* d\lambda = \int_{\tau^{-1}(A)} f^* d\lambda = \mu(\tau^{-1}(A)).$$

Hence $\qquad \mu(\tau^{-1}(A)) \quad = \mu(A).$

3.3 Representation of the Frobenius-Perron Operator

In this section we proceed to find P_τ for τ piecewise monotonic. By the definition of P_τ, we have

$$\int_A P_\tau f d\lambda = \int_{\tau^{-1}(A)} f d\lambda$$

for any Borel set A in I. Since τ is monotonic on each (a_{i-1}, a_i), $i = 1, 2, ..., q$, we can define an inverse function for each $\tau|_{(a_{i-1}, a_i)}$. Let $\phi_i = \tau^{-1}|_{B_i}$, where $B_i = \tau([a_{i-1}, a_i])$. Then $\phi_i : B_i \longrightarrow [a_{i-1}, a_i]$ and

$$\tau^{-1}(A) = \cup_{i=1}^{q} \phi_i(B_i \cap A)$$

where the sets $\{\phi_i(B_i \cap A)\}_{i=1}^{q}$ are mutually disjoint. Note also that, depending on A, $\phi_i(B_i \cap A)$ may be empty.

Thus
$$\begin{aligned}\int_A P_\tau f d\lambda &= \sum_{i=1}^{q} \int_{\phi_i(B_i \cap A)} f d\lambda \\ &= \sum_{i=1}^{q} \int_{(B_i \cap A)} f(\phi_i(x)) |\phi_i'(x)| d\lambda \\ &= \sum_{i=1}^{q} \int_A f(\phi_i(x)) |\phi_i'(x)| \chi_{B_i}(x) d\lambda \\ &= \int_A (\sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1}, a_i)}(x) d\lambda.\end{aligned}$$

Since A is arbitrary,

$$P_\tau f(x) = \sum_{i=1}^{q} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1}, a_i)}(x)$$

for any $f \in L^1$.

The operator P_τ is not one to one. To see this let us consider τ , the symmetric triangle transformation on $[0, 1]$. Let $f = 1$ on $[0, \frac{1}{2}]$ and -1 on $[\frac{1}{2}, 1]$. Then $P_\tau f = 0$ a.e. Thus P_τ is not a one to one operator.

3.4 Piecewise linear Markov transformation and the matrix representation of Frobenius-Perron operator:

Now we shall study a special class of piecewise monotone transformation known as Markov transformation. These transformation map each interval of partition onto a union of intervals of the partition. Of particular importance are the piecewise linear Markov transformation whose invariant densities can be computed easily since the Frobenius-Perron operator can be represented by a finite dimensional matrix. Furthermore, the piecewise linear Markov transformations can be used to approximate the long term behavior of more complicated transformations. Therefore, the fixed points of the Frobenius-Perron operator associated with general transformations can be approximated by the fixed points of appropriate matrices.

Definition 3.4.1

Let $\tau : I \longrightarrow I$ be a monotonic transformation and let $P = \{I_i\}_{i=1}^{n}$ be a partition of I. We define the incidence matrix A_τ induced by τ and P as follows :

Let $A_\tau = (a_{ij})_{1 \le i,j \le n}$ where $a_{ij} = \begin{cases} 1 \ , & I_j \subset \tau(I_i) \\ 0 \ , & \text{otherwise} \end{cases}$

Notation 4.3.1

Let us fix a partition P of I and let S denote the class of all functions that are piecewise constant on the partition P, that is, the step functions on P. Thus , $f \in S$ if and only if $f = \sum_{i=1}^{n} \pi_i \chi_{I_i}$ for some constants $\pi_1, \pi_2, \ldots\ldots \pi_n$. Such an f will also be represented by the column vector $\pi^f = (\pi_1, \pi_2, \ldots\ldots\ldots\pi_n)^T$, where T denote the transpose.

Theorem 3.4.1

Let τ: $I \to I$ be a piecewise Linear Markov transformation on the partition $P = \{I_i\}_{i=1}^{n}$. Then there exists an $n \times n$ matrix M_τ such that $P_\tau f = M_\tau \pi^f$ for $f \in S$ and π^f is the column vector obtained from f. The matrix M_τ is of the form

$$M_\tau = (m_{ij})_{1 \le i,j \le n}$$

where

$$m_{ij} = \frac{a_{ij}}{|\tau_i'|} = \frac{\lambda(I_i \cap \tau^{-1}(I_j))}{\lambda(I_i)}, 1 \le i,j \le n$$

and

$$A_\tau = (a_{ij})_{1 \le i,j \le n}$$

is the incidence matrix induced by τ and P.

Proof: We know , for $f \in L^1$ and $\tau : I \to I$,

$$P_\tau f(x) = \sum_{i=1}^{n} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|} \chi_{\tau(a_{i-1},a_i)}(x).$$

Let $f = \chi_{I_k}$ for some $1 \le k \le n$. Then

$$P_\tau f(x) = \sum_{i=1}^{n} \chi_{I_k(\tau_i^{-1}(x))} |\tau'(\tau_i^{-1}(x))|^{-1} \chi_{\tau(I)}.$$

Since the range of τ_i^{-1} is I_i, $\chi_{I_k}(\tau_i^{-1}(x))$ will be zero for all $i \ne k$. Thus

$$P_\tau f(x) = \frac{1}{|\tau'(\tau_k^{-1}(x))|} \chi_{\tau_k(I_k)}(x).$$

Since $\tau_k^{-1}(x) \in I_k$ and τ' is constant on I_k, we can write

$$P_\tau f(x) = |\tau_k'|^{-1} \chi_{\tau_k(I_k)}(x).$$

Now let $f \in S$, that is

$$f = \sum_{k-1}^{n} \pi_i \chi_{I_i} = (\pi_1, \pi_2, \dots\dots\dots\dots \pi_n)^T.$$

Since P_τ is a linear operator, we have

$$P_\tau f = \sum_{k=1}^{n} \pi_k P_\tau(\chi_{I_k}), \text{that is,}$$

$$P_\tau f(x) = \sum_{k=1}^{n} \pi_k |\tau_k'|^{-1} \chi_{\tau_k(I_k)}(x).$$

This proves that $P_\tau f \in S$. Let us write $P_\tau f$ as $(d_1, d_2, \dots\dots, d_n)^T$.

Let $x \in I_j$ and $P_\tau f(x) = d_j$. Now the kth term on the right hand side of the above equation equals $\pi_k |\tau_k'|^{-1}$if and only if $x \in \tau_k(I_k)$, that is, if and only if $I_j \subset \tau_k(I_k)$. Let $\Delta_{jk} = 1$ if $I_j \subset \tau_k(I_k)$ and 0 otherwise [that is $(\Delta_{jk})_{1 \le j,k \le n} = A_\tau^T)$]. Define the $n \times n$ matrix

$$M_\tau^T = (\overline{m}_{jk}) = \Delta_{jk} |\tau_k'|^{-1}.$$

Then

$$d_j = \sum \pi_k \overline{m}_{jk} \text{ and } P_\tau f = \begin{pmatrix} d_1 \\ d_2 \\ \cdot \\ \cdot \\ \cdot \\ d_n \end{pmatrix} = M_\tau^T \begin{pmatrix} \pi_1 \\ \pi_2 \\ \cdot \\ \cdot \\ \cdot \\ \pi_n \end{pmatrix}.$$

Note that τ determines M_τ uniquely, but converse is not true , that is, τ is not the only transformation that induces M_τ. On any segment I_i, the transformation τ_i can be replaced by a linear transformation with the same domain and range as τ_i but having slope $-\tau_i'$. Thus there exists 2^n piecewise linear Markov transformations which induce the same matrix M_τ.

Example: 3.4.1

Let $\tau : [0,1] \longrightarrow [0,1]$ be defined by

$$\tau(x) = \begin{cases} 2x+\frac{1}{2}, & x \in I_1 = [0,\frac{1}{4}) \\ -x+\frac{5}{4}, & x \in I_2 = (\frac{1}{4},\frac{1}{2}) \\ -2x+\frac{7}{4}, & x \in I_3 = (\frac{1}{2},\frac{3}{4}) \\ -x+1 & x \in I_4 = (\frac{3}{4},1] \end{cases}$$

It is Markov on $\{I_1, I_2, I_3, I_4\}$. The matrix representation of Frobenius-Perron Operator is

$$M_\tau = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Let $\pi = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \end{pmatrix}$. Then solving $\pi M_\tau = \pi$,we get

$$\begin{pmatrix} x_1, & x_2, & x_3, & x_4 \end{pmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 2 & 0 & 0 \end{bmatrix} = \begin{pmatrix} x_1, & x_2, & x_3, & x_4 \end{pmatrix}$$

$\Rightarrow \pi = (2,1,2,2)$

Thus the density function for f is given by

$$g(x) = \begin{cases} \frac{8}{7} \ , & x \in [0,\frac{1}{4}) \cup (\frac{1}{2},1] \\ \frac{4}{7} \ , & x \in (\frac{1}{4},\frac{1}{2}) \end{cases}$$

3.5 Absolutely Continuous Invariant Measures For Piecewise Monotonic Transformation.

Let $I = [a,b] \subset R$ be a bounded interval and λ denote Lebesgue measure on I. For any sequence of points $a = x_0 < x_1 < \ldots\ldots < x_{n-1} < x_n = b, \ \ n \geq 1,$ we define a partition

$P = \{I_i = [x_{i-1}, x_i) : i = 1, 2, ..., n\}$ of I. The points $\{x_0, x_1, ..., x_n\}$ are called endpoints of P. Sometimes we will write $P = P\{x_0, x_1, ..., x_n\}$.

Definition 3.5.1

Let $f : I \to R$ and $P = P\{x_0, x_1, ..., x_n\}$ be a partition of I. If there exists a positive number M such that $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M$ for all partitions P, then f is said to be of bounded variation on [a,b]. In this case $\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$ is called the variation of f with respect to P and we write $\bigvee_a^b(f, P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$. The number $\bigvee_{[a,b]} f = \sup \bigvee_a^b(f, P)$ is called the total variation or simply the variation of f on I.

Lemma 3.5.1

If $f \in C^1[a,b]$ with $|f'| > 0$, then f is monotone on $[a,b]$.

Proof: $f \in C^1[a,b] \Rightarrow f' \in C[a,b]$. Now $|f'| > 0 \Rightarrow$ either $-\infty < f' < 0$ or $0 < f' < \infty$. Since f' is continuous, it is only possible to have $f' < 0$ $\forall x \in [a,b]$ or $f' > 0$ $\forall x \in [a,b]$. In either case f is monotone on $[a,b]$.

Lemma 3.5.2

Let f and g be of bounded variation on $[a,b]$. Then

$$\bigvee_{[a,b]}(f+g) \leq \bigvee_{[a,b]} f + \bigvee_{[a,b]} g$$

and

$$\bigvee_{[a,b]} \sum_{k=1}^{n} f_k \leq \sum_{k=1}^{n} \bigvee_{[a,b]} f_k.$$

Proof: Let $P = P\{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$. Then

$$\bigvee_a^b((f+g), P) = \sum_{k=1}^{n} |(f+g)(x_k) - (f+g)(x_{k-1})|$$

$$\leq \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})|$$

$$= \bigvee_a^b(f, P) + \bigvee_a^b(g, P).$$

So $$\bigvee_{[a,b]}(f+g) = \sup(\bigvee_a^b((f+g), P) \leq$$

$$\leq \sup(\bigvee_a^b(f, P) + \bigvee_a^b(g, P))$$

$$\leq \sup \bigvee_a^b(f,P) + \sup \bigvee_a^b(g,P)$$

$$= \bigvee_{[a,b]} f + \bigvee_{[a,b]} g .$$

By induction, we get

$$\bigvee_{[a,b]} \sum_{k=1}^n f_k \leq \sum_{k=1}^n \bigvee_{[a,b]} f_k.$$

Lemma 3.5.3

$$x \in [a,b] \Rightarrow |f(a)| + |f(b)| \leq \bigvee_{[a,b]} f + 2|f(x)|.$$

Proof: For $x \in [a,b]$, $\{a,x,b\}$ is a partition of $[a,b]$ and call it P. Then

$$\bigvee_{[a,b]} f \geq \bigvee_a^b(f,P) = |f(x) - f(a)| + |f(b) - f(x)|$$

$$= |f(a) - f(x)| + |f(b) - f(x)|$$

$$\geq |f(a)| - |f(x)| + |f(b)| - |f(x)| = |f(a)| + |f(b)| - 2|f(x)|$$

Hence $\qquad |f(a)| + |f(b)| \leq \bigvee_{[a,b]} f + 2|f(x)|.$

Lemma 3.5.4

Let f_i be defined on $[\alpha_i, \beta_i] \subset [a,b]$ and

$$\chi_i(x) = \begin{cases} 1, & x \in [\alpha_i, \beta_i] \\ 0, & \text{otherwise} \end{cases} .$$

Then for $f = \sum_{i=1}^n f_i \chi_i$,

$$\bigvee_{[a,b]} f \leq \sum_{i=1}^n \bigvee_{[\alpha_i,\beta_i]} f_i + \sum_{i=1}^n (|f_i(\alpha_i)| + |f_i(\beta_i)|).$$

Proof:

$$f_i(x)\chi_i(x), \bigvee_{[a,b]} f_i \chi_i \leq \bigvee_{[\alpha_i,\beta_i]} f_i + |f_i(\alpha_i)| + |f_i(\beta_i)|,$$

equality holds if $\alpha_i \neq a$ and $\beta_i \neq b$.

From the previous lemma,

$$\bigvee_{[a,b]} f \leq \bigvee_{[a,b]} (\sum_{i=1}^n f_i)$$

$$\leq \sum_{i=1}^n (\bigvee_{[\alpha_i,\beta_i]} f_i + |f_i(\alpha_i)| + |f_i(\beta_i)|)$$

$$= \sum_{i=1}^n \bigvee_{[\alpha_i,\beta_i]} f_i + \sum_{i=1}^n (|f_i(\alpha_i)| + |f_i(\beta_i)|).$$

Lemma 3.5.5

Let f be differentiable and one to one and let $g = f^{-1}$. If $|f'| \geq \alpha$ then $|g'| \leq \frac{1}{\alpha}$.

Proof: $\qquad f(g(x)) = x \Rightarrow f'(g(x))g'(x) = 1$

$$\Rightarrow |f'(g(x))| = |\tfrac{1}{g'(x)}| \geq \alpha \Rightarrow |g'| \leq \tfrac{1}{\alpha}.$$

Lemma 3.5.6

If $f \in C^2[a,b]$ is monotone on $[a,b]$, then $\frac{d}{dx}|f'|$ exists everywhere on $[a,b]$.

Proof: f monotone$\Rightarrow f' > 0 \;\; \forall x \in [a,b]$ or

$$f' > 0 \; \forall x \in [a,b] \Rightarrow |f'| = f' \forall x \in [a,b]$$

or $\qquad |f'| = -f' \;\; \forall x \in [a,b].$

Now $\qquad f' \in C^1[a,b] \Rightarrow -f' \in C^1[a,b].$

Hence $|f'|$ is differentiable on $[a,b]$.

Lemma 3.5.7

If $\bigvee_{[0,1]} f \leq a$ and $\|f\|_1 \leq b$, where $\|f\|_1 = \int_0^1 |f|$, then

$$|f(x)| \leq a + b, \;\; \forall x \in [a,b].$$

Proof: $\|f\|_1 \leq b \Rightarrow \exists\alpha$ such that $|f(\alpha)| \leq b$. If not, then $|f(\alpha)| > b, \forall x \in [0,1]$ and therefore

$$\int_0^1 |f| > \int_0^1 b = b$$

and we have a contradiction.

Also

$$|f(x) - f(\alpha)| \leq \bigvee_{[0,1]} f \leq a$$

and thus $\qquad |f(x)| \leq a + |f(\alpha)| \leq a + b.$

Lemma 3.5.8

For $f \in L^1[0,1]$ and $\epsilon > 0$, there exists $r = r(\epsilon)$ such that

$$\int_0^1 (f^+ - r)^+ + \int_0^1 (f^- - r)^+ \leq \epsilon,$$

where $f^+ = \max(f,0)$, $f^- = -\min(0,f)$.

Proof: Let ψ be a simple bounded function such that $\psi \geq 0$ and let $M = \max\psi(x)$.

Then $\psi - r \le M - r$ and

$$(\psi - r)^+ \le (M-r)^+ = \begin{cases} M-r & \text{if } M-r>0 \\ 0 & \text{if } M-r \le 0 \end{cases}$$

Given $\epsilon > 0$, we set r such that $r \ge M - \epsilon$. Then $M - r \le \epsilon$. Therefore $(M-r)^+ \le \epsilon$ and

$$\int_0^1 (M-r)^+ = (M-r)^+ \le \epsilon.$$

Since $(M-r)^+ \ge (\psi - r)^+$, we have

$$\int_0^1 (\psi - r)^+ \le \epsilon.$$

Thus the theorem is proved for ψ simple and bounded.

The set of bounded simple function is dense in $L^1[0,1]$. Thus for $f \in L^1[0,1]$ and $\epsilon > 0$, we

can choose φ and ψ simple and bounded such that $\|f^+ - \varphi\| \le \frac{\epsilon}{4}$ and

$$\|f^- - \psi\|_1 \le \tfrac{\epsilon}{4},\ \varphi, \psi \ge 0.$$

For φ and ψ, we can choose r_1, r_2 such that

$$\int_0^1 (\varphi - r_1)^+ \le \tfrac{\epsilon}{4} \text{ and}$$

$$\int_0^1 (\psi - r)^+ \le \tfrac{\epsilon}{4}.$$

Then , for $r \ge \max(r_1, r_2)$, $\int_0^1 (f^+ - r)^+ + \int_0^1 (f^- - r)^+$

$$= \int_0^1 (f^+ - \varphi + \varphi - r)^+ + \int_0^1 (f^- - \psi + \psi - r)^+$$

$$\le \int_0^1 (f^+ - \varphi)^+ + \int_0^1 (\varphi - r)^+ + \int_0^1 (f^- - \psi)^+ + \int_0^1 (\psi - r)^+$$

$$\le \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} + \tfrac{\epsilon}{4} = \epsilon$$

Theorem 3.5.1 (Helly's Selection Principle)

Let $\mathcal{B}$ be a family of functions such that $f \in \mathcal{B} \Rightarrow \bigvee_{[a,b]} f \le \alpha$ and $|f(x)| \le \beta$, for any $x \in [a,b]$. Then there exists a sequence $\{f_n\} \subset \mathcal{B}$ such that $f_n \to f^* \forall x \in [a,b]$ and $f^* \in BV[a,b]$.

Theorem 3.5.2 (Mazur's theorem)

Let X be a Banach space with $A \subset X$ relatively compact. Then $\overline{co}(A)$ is compact where

$co(A)$ is the convex hull of A and $\overline{co}(A)$ denotes its closure with respect to the metric topology.

Theorem 3.5.3 (Kakutani-Yoshida)

Let $T : X \longrightarrow X$ be a bounded linear operator from a Banach space X into itself. Assume that there exists $M > 0$ such that $\|T^n\| \leq M$, $n = 1, 2, \cdots$. Furthermore, if for any $f \in A \subset X$, the sequence $\{f_n\}$, where $f_n = \frac{1}{n}\sum_{k=1}^{n} T^k f$, contains a sub-sequence $\{f_{n_k}\}$ which converges weakly in X, then for any $f \in \overline{A}$, $\frac{1}{n}\sum_{k=1}^{n} T^k f \longrightarrow f^* \in X$ (norm convergence) and $T(f^*) = f^*$.

Recall that A set $A \subset X$ of a Banach space X is called relatively compact if every infinite subset of A contains a sequence that converges to a point of X.

Theorem 3.5.4 (Lasota and Yorke)

Let $\tau : [0,1] \longrightarrow [0,1]$ be a piecewise C^2 transformation such that $\inf|\tau'| > 1$. Then for any $f \in L^1[0,1]$ the sequence $\frac{1}{n}\sum_{k=1}^{n} P_\tau^k f$ is convergent in norm to $f^* \in L^1[0,1]$. The limit function has the following properties:

$(i)\, f \geq 0 \Rightarrow f^* \geq 0.$

$(ii)\, \int_0^1 f^* d\lambda = \int_0^1 f d\lambda.$

$(iii)\, P_\tau f^* = f^*$ and consequently $d\mu^* = f^* d\lambda$ is invariant under τ.

$(iv)\, f^* \in BV[0,1]$. Moreover $\exists$ c independent of choice of initial f such that

$\bigvee_{[0,1]} f^* \leq c\|f\|_1.$

Proof: Let $s = \inf|\tau'|$ and choose N such that $s^N > 2$. Then the function $\varphi = \tau^N$ is piecewise C^2. Let $0 = b_0 < b_1 < \cdots < b_p = 1$ be the corresponding partition for φ and $\varphi_i = \varphi|_{[b_{i-1}, b_i]}$ be the corresponding C^2 functions. By the chain rule $|\varphi_i'| \geq s^N$ for all $x \in [b_{i-1}, b_i]$. To compute the Frobenius-Perron operator for φ, set $\psi_i = \varphi_i^{-1}$ and $\sigma_i(x) = |\psi_i'(x)|$. Observe that $\sigma_i(x) \leq \frac{1}{s^N}$. From the fact that $|\varphi_i'| \geq s^N$ and Lemma 3.5.1, we see that φ_i is monotone. Therefore using the representation of Frobenius Perron operator in section 3.3 we can write,

$$P_\varphi f(x) = \sum_{i=1}^{p} f(\psi_i(x))\sigma_i(x)\chi_{J_i}(x), \tag{1}$$

where $J_i = \varphi_i([b_{i-1}, b_i])$ and $\chi_{J_i}(x) = \begin{cases} 1, & x \in J_i \\ 0 & \text{otherwise} \end{cases}$

Let f be a given function of bounded variation over $[0, 1]$. Then

$$f(\psi_i(x))\sigma_i(x)\chi_{J_i}(x) = 0 \quad \text{for } x \notin [\varphi_i(b_{i-1}), \varphi_i(b_i)].$$

Thus by lemma 3.5.4 we have

$$\bigvee_{[0,1]} P_\varphi f \le \sum_{i=1}^{p} (\bigvee_{J_i} (f \circ \psi_i)\sigma_i + |f(b_{i-1})\sigma_i(\varphi_i(b_{i-1}))| + |f(b_i)\sigma_i(\varphi_i(b_i))|)$$

$$\le \sum_{i=1}^{p} \bigvee_{J_i} (f \circ \psi_i)\sigma_i + \tfrac{1}{s^N} \sum_{i=1}^{p} (|f(b_{i-1})| + |f(b_i)|). \qquad (2)$$

In order to the first sum in (2) we write

$$\bigvee_{J_i} (f \circ \psi_i)\sigma_i = \int_{J_i} |d((f \circ \psi_i)\sigma_i)|$$

$$\le \int_{J_i} |f \circ \psi_i| |\sigma_i'| d\lambda + \int_{J_i} \sigma_i |d(f \circ \psi_i)|$$

$$\le K \int_{J_i} ||f \circ \psi_i| \sigma_i d\lambda + \tfrac{1}{s^N} \int_{J_i} |d(f \circ \psi_i)|,$$

where $K = \frac{\max|\sigma_i'|}{\min(\sigma_i)}$.

Changing the variables we obtain

$$\bigvee_{J_i} (f \circ \psi_i)\sigma_i \le \int_{b_{i-1}}^{b_i} |f| d\lambda + \int_{J_i} \sigma_i |df|. \qquad (3)$$

In order to evaluate the second term in (2) we set $d_i = \inf\{ |f(x)|$ for $x \in [b_{i-1}, b_i]$. Using lemma 3.5.3 we have

$$|f(b_{i-1})| + |f(b_i)| \le \bigvee_{[b_{i-1}, b_i]} f + 2d_i.$$

Letting $h = \min_i(b_i - b_{i-1})$ we see that $d_i h \le \int_{b_{i-1}}^{b_i} |f| d\lambda$.

Thus $\sum_{i=1}^{p} (|f(b_{i-1})| + |f(b_i)|) \le \sum_{i=1}^{p} \bigvee_{[b_{i-1}, b_i]} f + 2h^{-1} \sum_{i=1}^{p} \int_{b_{i-1}}^{b_i} |f| d\lambda.$

$$= \bigvee_{[0,1]} f + 2h^{-1} \|f\|. \qquad (4)$$

Now using (3) and (4) we get from (2)

$$\bigvee_{[0,1]} P_\varphi f(x) \le K\|f\| + s^{-N} \bigvee_{[0,1]} f + s^{-N} \bigvee_{[0,1]} f + 2s^{-N} h^{-1} \||f\|. \qquad (5)$$

Set $\alpha = K + 2s^{-N} h^{-1}$ and $\beta = s^{-N}$.

Then from (5) we get

$$\bigvee_{[0,1]} P_\varphi f(x) \le \alpha \| f \| + \beta \bigvee_{[0,1]} f. \tag{6}$$

For the same f, let us write $f_k = P_\tau^k f$. Then

$$f_{Nk} = P_\tau^{Nk} f = P_\tau^N P_\tau^{N(k-1)} f = P_\varphi f_{N(k-1)}.$$

Therefore

$$\bigvee_{[0,1]} f_{Nk} = \bigvee_{[0,1]} P_\varphi f_{N(k-1)} \le \alpha \| f_{N(k-1)} \| + \beta \bigvee_{[0,1]} f_{N(k-1)}.$$

Since $\| f_n \| \le \| f \|$ for all $n = 1, 2, 3, \cdots$ we have

$\bigvee_{[0,1]} f_{Nk} \le \alpha \| f_{N(k-1)} \| + \beta(\alpha \| f \| + \beta \bigvee_{[0,1]} f_{N(k-2)}) \le \cdots \le \sum \alpha \beta^n \| f \| + \beta^k \bigvee_{[0,1]} f_0$, where $f_0 = f$

Consequently,

$$\limsup \bigvee_{[0,1]} f_{Nk} \le \tfrac{\alpha \| f \|}{1-\beta}. \tag{7}$$

Using lemma 3.5.7 we have for all k, $|f_{Nk}| \le \frac{\alpha \| f \|}{1-\beta} + \| f \|$. (8)

From (7) , (8) and theorem 3.5.1, we have that every infinite subset of $C = \{f_{Nk}\}_{k=0}^{\infty}$ contains a subsequence which converges in $L_1[0,1]$. So C is relatively compact in $L_1[0,1]$. Since P_τ is continuous, $P_\tau^N C$ is also relatively compact.

Now $\{f_k\}_{k=0}^{\infty} \subset \cup_{k=0}^{N-1} P_\tau^N C$. So $\{f_k\}_{k=0}^{\infty}$ is also relatively compact. By theorem 3.5.2, $\mathrm{co}(C) = \{\sum_{k=0}^{n-1} a_k f_k : a_k \ge 0, \sum_{k=0}^{n-1} a_k = 1\}$ is relatively compact. Finally, by theorem 3.5.3, $\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$ converges to $f^* \in L_1[0,1]$, where $P_\tau f^* = f^*$.

Property (i), (ii) and (iii) of the theorem follow from the properties 3.2.2, 3.2.3, 3.2.6 respectively. So it remains only to prove (iv), that is, we have to show that f^* is of bounded variation and there exists a constant c independent of the choice of intial f such that $\bigvee_{[0,1]} f^* \le c \| f \|_1$.

Set $\alpha_i = \tau_i^{-1}, \beta_i = | \alpha_i^{-1} |$. Then as before

$$P_\tau f(x) = \sum_{i=1}^{p} f(\alpha_i(x)) \beta_i(x) \chi_{I_i}(x),$$

where $0 = a_1 < a_2 < a_3 < \cdots < a_q = 1$ is the corresponding partition for τ and τ_i be the restriction of τ on (a_{i-1}, a_i), $I_i = \tau_i([a_{i-1}, a_i])$.

By lemma 3.5.4 and the same procedure as before, we get

$$\bigvee_{[0,1]} P_\tau f \leq \sum_{i=1}^{q} (f \circ \alpha_i)\beta_i + \frac{1}{s}\sum_{i=1}^{q}(|f(a_{i-1})| + |f(a_i)|)$$

$$\leq c_1 \bigvee_{[0,1]} f + c_2 \| f \|,$$

and $\bigvee_{[0,1]} f_{Nk+1} = \bigvee_{[0,1]} P_\tau f_{Nk} \leq c_1 \bigvee_{[0,1]} f_{Nk} + c_2 \| f \|$ for some $c_1, c_2 > 0$.

Moreover, $\bigvee_{[0,1]} f_{Nk+2} \leq c_1 \bigvee_{[0,1]} f_{Nk+1} + c_2 \| f \|$

$$\leq c_1 (c_1 \bigvee_{[0,1]} f_{Nk} + c_2 \| f \|) + c_2 \| f \|$$

$$= c_1^2 \bigvee_{[0,1]} f_{Nk} + (c_1 c_2 + c_2) \| f \|.$$

Consequently, for n=1, 2, 3, $\cdots, N-1$,

$$\bigvee_{[0,1]} f_{Nk+n} \leq c_1^n \bigvee_{[0,1]} f_{Nk} + c_2 \frac{1-c_1^N}{1-c_1} \| f \|,$$

and $\overline{\lim}_{k \to \infty} \bigvee_{[0,1]} f_{Nk+n} \leq c_1^n \, \overline{\lim}_{k \to \infty} \bigvee_{[0,1]} f_{Nk} + c_2 \frac{1-c_1^N}{1-c_1} \| f \|.$ (9)

With (9) and (7), we have

$$\overline{\lim}_{k \to \infty} \bigvee_{[0,1]} P_\tau f \leq c \| f \|, \ c > 0$$

for $f \in BV[0,1]$, where c is independent of f. For such an f and for any n,

$$\bigvee_{[0,1]} \left(\frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f\right)$$

$$\leq \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f \leq \frac{1}{n} \sum_{k=0}^{n-1} c \| f \| = c \| f \|$$

Let $T_n = \frac{1}{n} \sum_{k=0}^{n-1} P_\tau^k f$. Then

$$\bigvee_{[0,1]} T_n f \leq c \| f \|, \ f \in BV[0,1]. \tag{10}$$

Now

$$\| T_n f \| \leq \frac{1}{n} \sum_{k=0}^{n-1} \| P_\tau^k f \| \leq \frac{1}{n} \sum_{k=0}^{n-1} \| f \| = \| f \|$$

From (9), (10) and lemma 3.5.7, we get

$$|T_n f(x)| \leq (c+1) \| f \|. \tag{11}$$

From (10), (11) and theorem 3.5.2, there exists a subsequence $\{T_{n_k}\}$ which converges everywhere on [0,1] to a function of bounded variation. But $T_n f$ converges strongly. Thus

$$\bigvee_{[0,1]} Tf \leq c\|f\|, \tag{12}$$

where $T = \lim_{n\to\infty} T_n$

For $\varphi, \psi \in L_1[0,1], T(\varphi + \psi) = \lim_{n\to\infty}(\frac{1}{n}\sum_{k=0}^{n-1}(P_\tau(\varphi + \psi))$

$$= \lim_{n\to\infty}(\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1} P_\tau\varphi) + \lim_{n\to\infty}(\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1} P_\tau\psi)$$

$$= T\varphi + T\psi$$

So T is linear.

Observe that $L_1[0,1] = \overline{BV}[0,1]$ and therefore for any $f \in L_1[0,1]$ there exists a sequence $\{\varphi_n\} \subset BV[0,1]$ such that $\varphi_n \to f$. By (12) we get

$$\bigvee_{[0,1]} T\varphi_n \leq c\|\varphi_n\|.$$

Since $\| T\varphi_n\| \leq \|\varphi_n\|$ it follows by lemma 3.5.7 that

$$|T\varphi_n(x)| \leq (c+1)\|\varphi_n\|.$$

For $\varepsilon > 0$, there exists N such that for all $n > N$, $\|f - \varphi_n\| \leq \varepsilon$.

Therefore, for all $n \geq N$

$$\| \varphi_n\| \leq \varepsilon + \|f\|.$$

Thus, for all $n \geq N$, $\quad \bigvee_{[0,1]} T\varphi_n \leq c(\varepsilon + \|f\|)$

and $\quad |T\varphi_n(x)| \leq (c+1)(\varepsilon + \|f\|)$.

By using theorem 3.5.7 once more, we have the existence of a subsequence $\{\varphi_{n_k}\} \subset \{\varphi_n\}$ with $n_k > N$, such that $T\varphi_{n_k} \to f^*$ for some f^* of bounded variation. But also $T\varphi_n \to Tf$ and $\{T\varphi_{n_k}\} \subset \{T\varphi_n\}$, since T is continous, thus

$$Tf = f^* \in BV[0,1] \text{ for } f \in L_1[0,1].$$

This proves the theorem.

Chapter IV

Piecewise Linear Interval Maps both Expanding and Contracting with One Increasing and One Decreasing Branch.

Consider the family $\mathcal{F}$ of functions $f : [-1,1] \to [-1,1]$ defined by

$$f(x) = \begin{cases} \lambda_1 x + a & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ \lambda_2 x + b & , \quad x \in (0,1] \end{cases}$$

where $\lambda_1 > 0$ and $\lambda_2 < 0$ and $a, b, \lambda_1, \lambda_2$ are constants such that $f([-1,1]) \subset [-1,1]$.

Moreover, we assume that absolute value of the slope of f is greater than 1 (expansion) in one of the interval $(-1,0), (0,1)$ and less than 1 (contraction) in the other (Figure 4.1.1).

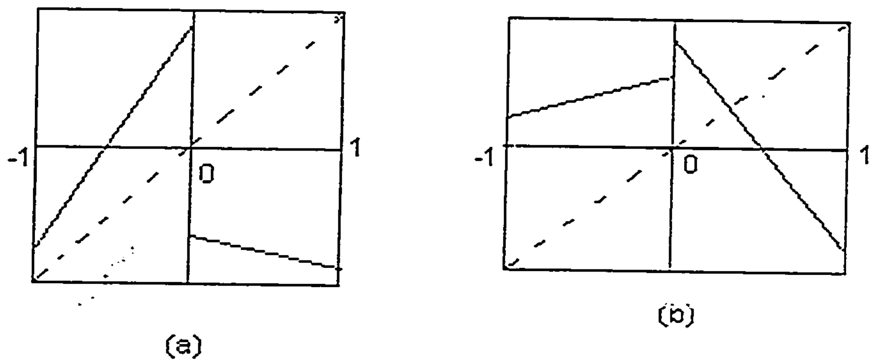


Figure 4.1.1

In this chapter we will consider a sub family $\mathcal{B} \subset \mathcal{F}$ in such a way that $f \in \mathcal{B}$ has one increasing branch and one decreasing branch. A Lasota and J.A.Yorke showed in [L-Y] that for a piecewise expanding map there exists an acim with respect Lebesgue measure. In our case, one branch of a map $f \in \mathcal{B}$ is expanding and the other branch is contracting. We will show that under certain conditions on λ_1 and λ_2, $f \in \mathcal{B}$ is eventually expanding and under certain conditions on λ_1 and λ_2, f has a periodic attractor. In the former case, we will show that there exists an acim for $f \in \mathcal{B}$. We will follow [B-F]. We will try to find the acim using the matrix representation of Frobenius-Perron operator when the map is Markov.

4.1 The case $1 < \lambda_1 \leq 2$ and $-1 < \lambda_2 < 0$.

In this section we will consider a subclass $\mathcal{B}$ of $\mathcal{F}$ by taking $1 < \lambda_1 \leq 2$ and $-1 < \lambda_2 < 0$. We will state and prove four lemmas and using these lemmas we will prove the main theorem.

Lemma 4.1.1

The first return map $R : [0,1) \to [0,1)$ defined by $R(x) = f^r(x)$ where $r = \min\{i > 0 : f^i(x) \in [0,1)\}$ is well defined. Moreover, there exists a partition $0 = b_0 < b_1 < b_2 < \cdots < 1$ of $[0,1)$, not necessarily finite, and an integer N such that the restriction $R|_{[b_k,b_{k+1})}$ is linear with slope $\lambda_1^{N+k}\lambda_2$.

Proof: If $x \in (-1,0) \Rightarrow f^n(x) \in (-1,0)$ for all $n \geq 1$, then we have

$$f(x) = \lambda_1 x + a = \lambda_1(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1}$$

$$\Rightarrow f^2(x) = \lambda_1(\lambda_1(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1}) + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1}$$

$$= \lambda_1^2(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1}.$$

By induction,

$$f^n(x) = \lambda_1^n(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1}.$$

Observe that $f([0,1)) \subset (-1,1)$.

Moreover, for each $x \in (-1,0)$ there exists $n_0 \geq 1$ such that $f^{n_0}(x) \in [0,1)$, since otherwise $f^n(x) = \lambda_1^n(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1} \to \infty$ as $n \to \infty$, which is a contradiction. This proves that R is well defined. Linearity is obvious.

To construct the required partition of $[0,1)$, we will first construct a partition $\{a_n\}$ which is defined inductively as follows: $a_0 = 0$, $a_{n+1} = f^{-1}(a_n) \cap [-1,0)$ for $n \geq 0$.

Observe that, $\quad a_0 = 0 \Rightarrow f^{-1}(a_0) = -\frac{a}{\lambda_1} = -\frac{a(\lambda_1-1)}{(\lambda_1-1)\lambda_1} = a_1$

$$\Rightarrow f^{-1}(a_1) = -\frac{a(\lambda_1^2-1)}{(\lambda_1-1)\lambda_1^2} = a_2.$$

Inductively, $\quad f^{-1}(a_{n-1}) = -\frac{a(\lambda_1^n-1)}{(\lambda_1-1)\lambda_1^n} = a_n,\ n \geq 0.$

Now, $b = \lim_{x\to 0^+} f(x) \Rightarrow b > -1$. From the definition of $\{a_n\}$ it follows that $b > a_n$ for some n. Let N be the smallest integer such that $b > a_N$. Then

$$\begin{aligned}
b > -\frac{a(\lambda_1^N-1)}{(\lambda_1-1)\lambda_1^N} &\Rightarrow (\lambda_1 - 1)\lambda_1^N > -\tfrac{a}{b}(\lambda_1^N - 1)\\
&\Rightarrow \tfrac{b}{a}(\lambda_1 - 1)\lambda_1^N > -(\lambda_1^N - 1)\\
&\Rightarrow [1 + \tfrac{b}{a}(\lambda_1 - 1)]\lambda_1^N > 1\\
&\Rightarrow N\ln\lambda_1 + (\ln(1 + \tfrac{b}{a}(\lambda_1 - 1))) > 0\\
&\Rightarrow N > -\frac{\ln(1+\frac{b}{a}(\lambda_1-1))}{\ln\lambda_1}.
\end{aligned}$$

Since N is the smallest integer such that $N > -\frac{\ln(1+\frac{b}{a}(\lambda_1-1))}{\ln\lambda_1}$, we have

$$N = [1 - \frac{\ln(1+\frac{b}{a}(\lambda_1-1))}{\ln\lambda_1}], \text{ where } [t] \text{ denotes the integer part of } t.$$

Now, we define the sequence $\{b_n\}$ inductively by

$$b_0 = 0,\ b_{n+1} = \begin{cases} ff^{-1}(a_{N+n}) \cap (0,1] & \text{if } f^{-1}(a_{N+n}) \cap (0,1] \neq \phi \\ 1 & \text{if } f^{-1}(a_{N+n}) \cap (0,1] = \phi \end{cases}$$

This is an increasing sequence. Let $x \in [b_k, b_{k+1})$. Then from the construction of $\{a_n\}, \{b_n\}$ we have

$f(x) \in [a_{N+k}, a_{N+k-1})$,

$f^2(x) \in [a_{N+k-1}, a_{N+k-2})$ and so on. Thus

$f^{N+k}(x) \in [a_1, a_0)$.

Finally $f^{N+k+1}(x) = R(x) \in (0,1)$

Now $R'(x) = (f^{N+k+1})'(x) = f'(f^{N+k}(x)) \bullet f'(f^{N+k-1}(x)) \cdots \bullet f'(f(x)) \bullet f'(x)$.

$$= \lambda_1 \bullet \lambda_1 \bullet \cdots \bullet \lambda_1 \bullet \lambda_2 = \lambda_1^{N+k}\lambda_2.$$

Hence the lemma is proved.

Lemma 4.1.2

Let $x \in (0,1)$. Then $f^n(x) \notin (0,1)$ for all $1 \le n \le N-1$. Moreover, if r of the points $x, f(x), f^2(x), \cdots, f^{n-1}(x)$ belong to $(-1,0)$ and s of them belongs to $(0,1)$, then $r \ge N(s-1)$.

Proof: $x \in (0,1) \Rightarrow f(x) \in (-1,0)$.

Le $x \in (b_i, b_{i+1})$. Then

$f(x) \in (a_{N+i}, a_{N+i-1}), f^2(x) \in (a_{N+i-1}, a_{N+i-2}), \cdots, f^{N+i}(x) \in (a_1, a_0)$ and

$f^{N+i+1}(x) \in (0,1)$.

Hence $x \in (0,1) \Rightarrow f^n(x) \notin (0,1)$ for all $1 \le n \le N$.

From above, observe that for one positive value we are getting N negative values. So negative values are at least $N(s-1)$, because it is possible that not the whole negative trajectory of the last positive point is included in the sequence. Hence $r \ge N(s-1)$. Hence $r \ge N(s-1)$.

Example 4.1.1 (Map and first return map)

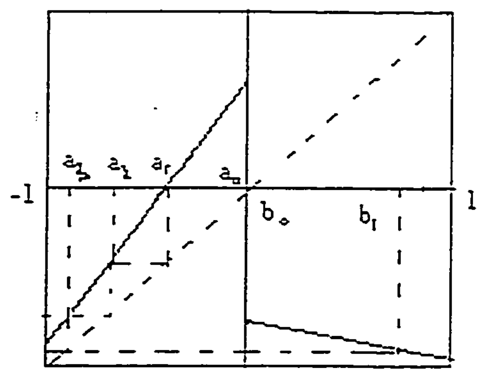


Figure 4.1.2(a)

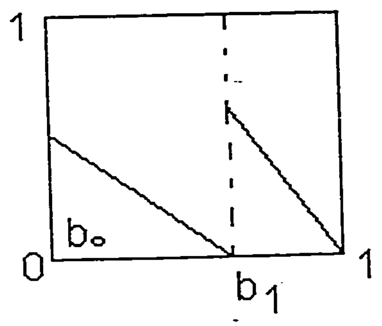


Figure 4.1.2 (b)

Consider the map f defined by

$$f(x) = \begin{cases} 1.5x + .6 & if \quad -1 \le x < 0 \\ 0 & if \quad x = 0 \\ -.25x - .7 & if \quad 0 < x \le 1 \end{cases}$$

Here $\lambda_1 = 1.5, \lambda_2 = -.25,\ a = .6, b = -.7;$

$N = [1 - \frac{\ln(1+\frac{-.7}{.6}(1.5-1))}{\ln 1.5}] = [3.1592] = 3$

$a_0 = 0, a_1 = f^{-1}(a_0) = -\frac{.6}{1.5} = -.4$

$a_2 = f^{-1}(a_1) = \frac{-.4-.6}{1.5} = -.66667$

$a_3 = f^{-1}(a_2) = \frac{-.66667-.6}{1.5} = -.84445 < -.7 = b$

$a_4 = f^{-1}(a_3) = \frac{-.84445-.6}{1.5} = -.96297.$

$a_5 = f^{-1}(a_4) = \frac{-.96297-.6}{1.5} = -1.042.$

Now $b_0 = 0$

$$b_1 = b_{0+1} = f^{-1}(a_{N+0}) = f^{-1}(a_{3+0}) = f^{-1}(a_3) = f^{-1}(-.84445)$$
$$= \frac{-.84445+.7}{-.25} = .5778.$$

$$b_2 = b_{1+1} = f^{-1}(a_{N+1}) = f^{-1}(a_4) = f^{-1}(-.96297) = \frac{-.96297+.7}{-.25} = 1.0519$$

So $\quad b_2 = 1$

$$R|_{[b_0,b_1)} = f^{3+0+1}(x) = f^4(x).$$
$$R|_{[b_1,b_2)} = f^{3+1+1}(x) = f^5(x).$$

If $x = .5$ then $f(x) = -.25(.5) - .7 = -.825.$

$$f^2(x) = 1.5(-.825) + .6 = -.6375$$
$$f^3(x) = 1.5(-.6375) + .6 = -.35625$$
$$f^4(x) = 1.5(-.35625) + .6 = 6.5625 \times 10^{-2} > 0$$

We can easily verify above lemmas with this example.

Lemma 4.1.3

If -1 is not a fixed point of f, then the first return map has a finite number of branches.

Proof: Observe that in Lemma 4.1.1, we constructed the sequence $\{a_n\}$ inductively by $a_{n+1} = f^{-1}(a_n) \cap [-1,0)$ for $n \geq 0$. In our case, there exists $m > 0$ such that $f^{-1}(a_m) \cap [-1,0) = \phi$ and we define $a_{m+1} = \frac{a_m - a}{\lambda_1} < -1$. So the sequence $\{a_n\}$ has $m+2$ terms.

In the same lemma we constructed the sequence $\{b_n\}$ inductively by

$$b_0 = 0, b_{n+1} = \begin{cases} f^{-1}(a_{N+n}) \cap (0,1] & \text{if } f^{-1}(a_{N+n}) \cap (0,1] \neq \phi \\ 1 & \text{if } f^{-1}(a_{N+n}) \cap (0,1] = \phi \end{cases}$$

where $N = [1 - \frac{\ln(1+\frac{b}{a}(\lambda_1 - 1))}{\ln \lambda_1}]$.

Since $\{a_n\}$ is finite so is $\{b_n\}$. Hence $R(x)$ has finite number of branches on $[0, 1)$.

Lemma 4.1.4

Suppose that there exists $b_k \neq b_0$ such that the slope of R on $[b_k, b_{k+1}], [b_{k+1}, b_{k+2}], \cdots$ is less than -1. Then the set of points of $[b_k, 1)$ whose orbit visits (b_0, b_k) is dense in $[b_k, 1)$.

Proof: Assume that there exists an open set $K \subset [b_k, 1)$ such that $R^n(K) \cap (b_0, b_k) = \phi$. Obviously, $b_j \notin R^n(K)$ for each $j \geq k$ and each $n \in N$; since otherwise $R^{n+1}(K) \cap (b_0, b_k) \neq \phi$. So for each $n \in N$, there exists $j_n \geq k$ such that $R^n(K) \subset (b_{j_n}, b_{j_{n+1}})$.

Let l_j be the slope of $R|(b_j, b_{j+1})$. Then $|R^{n+1}(K)| = |l_{j_n}||R^n(K)|$.

By induction, it follows that $|R^{n+1}(K)| = |l_{j_n}| \bullet |l_{j_{n-1}}| \bullet |l_{j_{n-2}}| \bullet \cdots \bullet |l_{j_0}|$.

Now $|l_k| \leq |l_j|$, so $|l_{j_n}| \bullet |l_{j_{n-1}}| \bullet |l_{j_{n-2}}| \bullet \cdots \bullet |l_{j_0}| \geq |l_k|^{n+1}$ and since $|l_k| > 1$ we have $|R^{n+1}(K)| \geq |l_k|^{n+1}|K| \to \infty$ as $n \to \infty$, which is a contradiction. This proves the lemma.

Theorem 4.1.1

Let $f \in \mathcal{B}$ with $1 \leq \lambda_1 \leq 2$ and $-1 < \lambda_2 < 0$, i.e., $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \lambda_1 x + a & , \quad x \in [-1, 0) \\ 0 & , \quad x = 0 \\ \lambda_2 x + b & , \quad x \in (0, 1] \end{cases}$$

Let $N = [1 - \frac{\ln(1+\frac{b}{a}(\lambda_1 - 1))}{\ln \lambda_1}]$. Then one of the following holds:

(i) If $\lambda_1^N \lambda_2 < -1$, then f is eventually expanding and there exists an absolutely continuous invariant measure with respect to Lebesgue measure.

(ii) If $\lambda_1^N \lambda_2 = -1$, then there exists $A > 0$ such that every $x \in (0, A)$ is periodic. Moreover, the set

$$\bigcup_{n \in N} f^{-n}(0, A) \cap [A, 1)$$

is dense in $[A, 1)$.

(iii)If $\lambda_1^N \lambda_2 > -1$, then there exists $A > 0$ such that every $x \in (0, A)$ except from a finite number of points, is in the basin of attracting periodic point. Moreover, the set

$$\bigcup_{n \in N} f^{-n}(0, A) \cap [A, 1)$$

is dense in $[A, 1)$.

Proof: We will prove that there exists an integer n_0 such that the iteration f^{n_0} is expanding

Let $\lambda > 1$ and take

$$n_0 > (\tfrac{1}{N} + 1)\frac{\ln(\lambda |\lambda_2|^{-1})}{\ln(\lambda_1 |\lambda_2|^{\frac{1}{N}})} + 1.$$

Suppose that r of the points $x, f(x), f^2(x), \cdots, f^{n_0 - 1}(x)$ lie in $(-1, 0)$ and s of them lie in $(0, 1)$.

Then $r+s = n_0$ and $r \geq N(s-1)$. This implies that

$$\begin{aligned} r &\geq N(n_0 - r - 1) \\ &\Rightarrow (\tfrac{1}{N}+1)r + 1 \geq n_0. \end{aligned}$$

Combining $n_0 > (\frac{1}{N}+1)\frac{\ln(\lambda|\lambda_2|^{-1})}{\ln(\lambda_1|\lambda_2|^{\frac{1}{N}})} + 1$ and the last inequality we get

$$\begin{aligned} &(\tfrac{1}{N}+1)r + 1 \geq (\tfrac{1}{N}+1)\frac{\ln(\lambda|\lambda_2|^{-1})}{\ln(\lambda_1|\lambda_2|^{\frac{1}{N}})} + 1 \\ &\Rightarrow r \geq \frac{\ln(\lambda|\lambda_2|^{-1})}{\ln(\lambda_1|\lambda_2|^{\frac{1}{N}})} \\ &\Rightarrow r\ln(\lambda_1|\lambda_2|^{\frac{1}{N}}) \geq \ln(\lambda|\lambda_2|^{-1}) \\ &\Rightarrow \ln(\lambda_1|\lambda_2|^{\frac{1}{N}})^r \geq \ln(\lambda|\lambda_2|^{-1}) \\ &\Rightarrow \lambda_1^r|\lambda_2|^{\frac{r}{N}} \geq \lambda|\lambda_2|^{-1} \\ &\Rightarrow \lambda_1^r|\lambda_2|^{\frac{r}{N}+1} \geq \lambda > 1 \end{aligned}$$

Let J be an interval of $(-1,1)$. The slope of f^{n_0} is $\lambda_1^r|\lambda_2|^s$. But we have from $r \geq N(s-1)$, $s \leq \frac{r}{N}+1$ and since $|\lambda_2| < 1$ we have $\lambda_1^r|\lambda_2|^s \geq \lambda_1^r|\lambda_2|^{\frac{r}{N}+1}$. So $\lambda_1^r|\lambda_2|^s \geq \lambda > 1$. Hence f^{n_0} is expanding which implies f is eventually expanding. The existence of acim follows from [L-Y].

(ii) In this case the first return map in $[b_0, b_1)$ is $R(x) = -x + b_1$. and $R^2(x) = x$ for all $x \in [b_0, b_1)$. and since $R|_{[b_k, b_{k+1})} = f^{N+k+1}$, we have $f^{2(N+1)}(x) = x$ for all $x \in [b_0, b_1)$. Hence f is periodic of period $2(N+1)$. From lemma 4.1.4 we have that the set

$$\bigcup_{n\in N} f^{-n}(0,A) \cap [A,1)$$

is dense in $[A,1)$, where $A = b_1$.

(iii) Let $A = b_k$ where b_k is as in Lemma 4.1.3. From this lemma we have that

$$\bigcup_{n\in N} f^{-n}(0,A) \cap [A,1)$$

is dense in $[A,1)$.We have to prove that every $x \in (0,A)$ except from a finite number of points is in the basin of a periodic attractor. Since $R(b_1) = b_0$ and the slope of l_0 of $R|(b_0,b_1)$ is such that $|l_0| < 1$ there exists a unique fixed point $p \in (b_0,b_1)$ of R. But $R|(b_0,b_1) = f^{N+1}$, so, p is an attracting periodic point of f of period $N+1$ and (b_0,b_1) is in the basin of p.

Since $R(b_2) = b_0$ we have that $R(b_1,b_2) \subset (b_0,b_1)$. If $R(b_1,b_2) \subset (b_0,b_1)$ then (b_1,b_2) is the basin of p. Otherwise there exists $y \in (b_1,b_2)$ such that $R(y) = b_1$. Then (y,b_2) is the basin of p and there exists $q \in (b_1,y)$ such that $R(q) = b_1$. But $R|_{(b_1,y)} = f^{N+2}$, so, p is an attracting periodic point of f of period $N+2$ and (b_1,y) is in the basin of q. We can use a similar argument for $(b_2,b_3), \cdots, (b_{k-1},b_k)$.

Example 4.1.2

Consider the one parameter family of maps $f : [-1,1] \longrightarrow [-1,1]$ defined by

$$f(x) = \begin{cases} 2x+1 & , \ x \in [-1,0) \\ 0 & , \ x = 0 \\ -(c+1)x+c & , \ x \in (0,1] \end{cases}$$

where the parameter c is a constant such that $0 < (c+1) < 1$ and $f([-1,1]) \subset [-1,1]$.
If $c = -\frac{1}{2^n}, n \geq 1$ we can find a partition of $[-1,1]$ such that f is Markov. In particular if

$$c = -\tfrac{1}{2}, f(x) = \begin{cases} 2x+1 & , \ x \in [-1,0) \\ 0 & , \ x = 0 \\ -\frac{1}{2}x - \frac{1}{2} & , \ x \in (0,1] \end{cases}$$

then $I_1 = [-1,-\frac{1}{2})$, $I_2 = (-\frac{1}{2},0)$, $I_3 = (0,1]$ is a Markov partition for f.

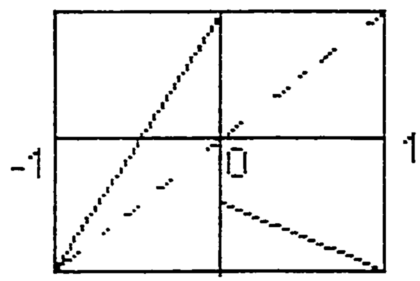


Figure 4.1.3

Observe that $f(I_1) = [-1,0), f(I_2) = (0,1), f(I_3) = (-1,-\frac{1}{2})$

Here $N = [1 - \frac{\ln(1+\frac{-.5}{1}(2-1))}{\ln 2}] = 2$ and $\lambda_1^N \lambda_2 = 2^2(-\frac{1}{2}) = -2 < -1.$

So by theorem 4.1.1 f has an absolutely continuous invariant measure with respect to Lebesgue measure. We are interested in finding the acim. To do that we have to find the matrix representation (M_f) of Frobenius-Perron operator P_f. The matrix M_f is of the form $M_f = (m_{ij}), \quad 1 \le i,j \le n,$ where

$$m_{ij} = m_{ij} = \frac{a_{ij}}{|f_i'|} = \frac{\lambda(I_i \cap f^{-1}(I_j))}{\lambda(I_i)}, 1 \le i,j \le n.$$

So in our case,

$$M_f = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 2 & 0 & 0 \end{bmatrix}. \text{ Let } \pi = \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix} \text{ Then solving } \pi M_f = \pi \text{ ,we get}$$

$$\begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 2 & 0 & 0 \end{bmatrix} . = \begin{bmatrix} x_1, & x_2, & x_3 \end{bmatrix}$$

$\Rightarrow \pi = (4,2,1)$

Thus the density function for f is given by

$$g(x) = \begin{cases} 1 & , \quad x \in [-1,-\frac{1}{2}) \\ \frac{1}{2} & , \quad x \in [-\frac{1}{2},0) \\ \frac{1}{4} & , \quad x \in [0,1) \end{cases}$$

where $\int_{-1}^{1} g(x) = 1$

Example 4.1.3

Consider the two parameters family of maps $f : [-1,1] \to [-1,1]$ defined by

$$f(x) = \begin{cases} 2x+1 & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ -(c-d)x + c & , \quad x \in (0,1] \end{cases}$$

where the parametters c and d are such that $0 < (c-d) < 1$ and $f([-1,1]) \subset [-1,1]$. If $c = \frac{1}{2^n}, n \geq 1$ we can find a partition of $[-1,1]$ such that f is Markov. In particular if

$c = -\frac{1}{8}$ and $d = -\frac{7}{8}$, that is

$$f(x) = \begin{cases} 2x+1 & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ -\frac{3}{4}x - \frac{1}{8} & , \quad x \in (0,1] \end{cases}$$

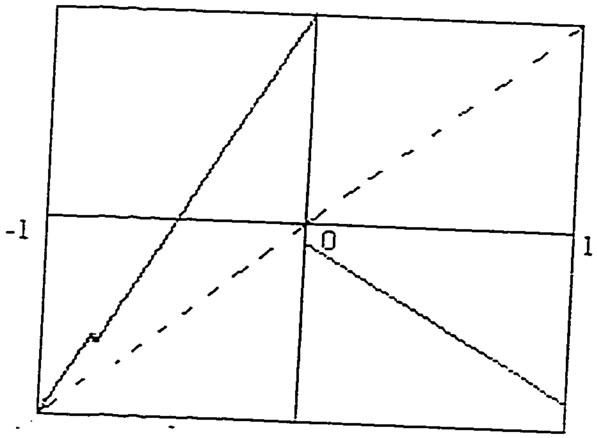


Figure 4.1.4

If we divide $(-1,0)$ into 16 intervals of equal length $\frac{1}{16}$, then

$$\{\{J_n = [-1 - \tfrac{1-n}{16}, -1 + \tfrac{n}{16})\}_{n=1}^{16}, (0, \tfrac{3}{4}), (\tfrac{3}{4}, 1)\}$$

is a Markov partition for f.

Observe that, here $N = [1 - \frac{\ln(1+\frac{-.25}{1}(2-1))}{\ln 2}] = [1.415] = 1$ and

$\lambda_1^N \lambda_2 = 2^1(-\frac{3}{4}) = -1.5 < -1.$

Here the map f is Markov and by theorem 4.1.1, there exist an acim .

Chapter V

Absolutely Continuous Invariant Measures For Piecewise Linear Interval Maps Both Expanding And Contracting With Two Increasing Branches.

In Chapter IV, we considered $f \in \mathcal{F}$ with one increasing branch and one decreasing branch and proved that f is either eventually expanding or f has a periodic attaractor. In this chapter, we will consider f with both branches increasing and will discuss the dynamics of such maps. We will prove the existence of acim. We also find the acim using the matrix representation of Frobenius-Perron operator. We are interested in considering the following two cases:

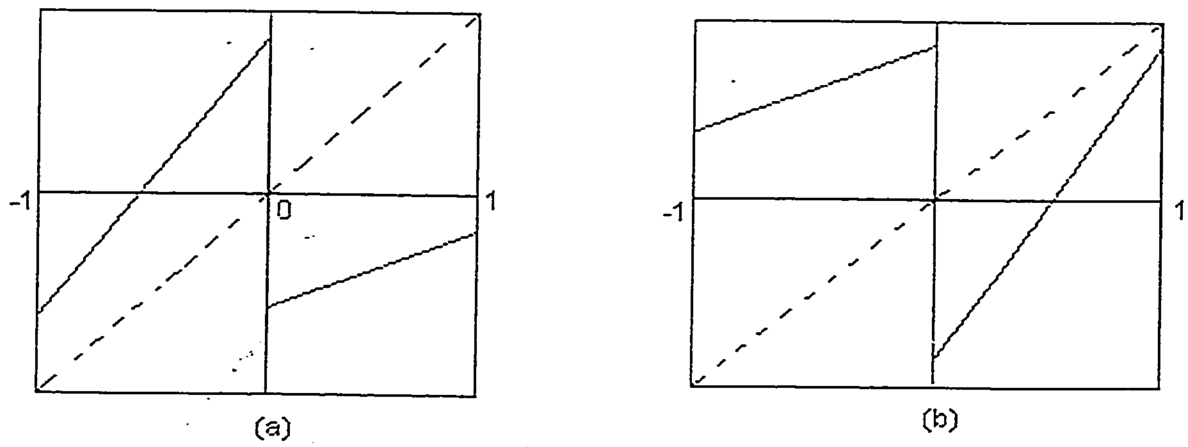


Figure 5.1.1

5.1 The case $1 < \lambda_1 \le 2$ and $0 < \lambda_2 < 1$.

In this case we will consider a subclass $\mathcal{B}_1$ of $\mathcal{F}$ by taking $1 < \lambda_1 \le 2$ and $0 < \lambda_2 < 1$. We will state and prove three lemmas and using these lemmas we will prove the main theorem.

Lemma 5.1.1

The first return map $R : [0,1) \longrightarrow [0,1)$ defined by $R(x) = f^r(x)$, where

$$r = \min\{i > 0 : f^i(x) \in [0,1)\}$$

is well defined. Moreover, there exists a partition $0 < \cdots < b_3 < b_2 < b_1 < b_0 = 1$ of $[0,1)$, not necessarily finite, and an integer N such that the restriction $R|_{[b_{k+1},b_k)}$ is linear with slope $\lambda_1^{N+k}\lambda_2$.

Proof: Observe that $f([0,1)) \subset (-1,1)$. If $x \in (-1,0)$ and $f^n(x) \in (-1,0)$ for all $n \geq 1$, then we have

$$f(x) = \lambda_1 x + a = \lambda_1(x + \tfrac{a}{\lambda_1 - 1}) - \tfrac{a}{\lambda_1 - 1}$$

$$\Rightarrow f^2(x) = \lambda_1(\lambda_1(x + \tfrac{a}{\lambda_1 - 1}) - \tfrac{a}{\lambda_1 - 1}) + \tfrac{a}{\lambda_1 - 1}) - \tfrac{a}{\lambda_1 - 1}$$

$$= \lambda_1^2(x + \tfrac{a}{\lambda_1 - 1}) - \tfrac{a}{\lambda_1 - 1}.$$

By induction, $$f^n(x) = \lambda_1^n(x + \tfrac{a}{\lambda_1 - 1}) - \tfrac{a}{\lambda_1 - 1}.$$

Moreover, for each $x \in (-1,0)$, there exists $n_0 \geq 1$ such that $f^{n_0}(x) \in [0,1)$, since otherwise $f^n(x) = \lambda_1^n(x + \frac{a}{\lambda_1 - 1}) - \frac{a}{\lambda_1 - 1} \longrightarrow \infty$ as $n \longrightarrow \infty$, which is a contradiction. This proves that R is well defined. Linearity is obvious.

To construct the required partition of $[0,1)$, we will first construct the sequence $\{a_n\}$ which is defined inductively as follows:

$$a_0 = 0,\ a_{n+1} = f^{-1}(a_n) \cap [-1,0) \text{ for } n \geq 0.$$

Observe that, $$a_0 = 0 \Rightarrow f^{-1}(a_0) = -\tfrac{a}{\lambda_1} = -\tfrac{a(\lambda_1 - 1)}{(\lambda_1 - 1)\lambda_1} = a_1$$

$$\Rightarrow a_2 = f^{-1}(a_1) = -\tfrac{a(\lambda_1^2 - 1)}{(\lambda_1 - 1)\lambda_1^2} \text{ and so on.}$$

Inductively, $$a_n = f^{-1}(a_{n-1}) = -\tfrac{a(\lambda_1^n - 1)}{(\lambda_1 - 1)\lambda_1^n},\ n \geq 0.$$

Now let $c = \lim_{x \to 1^-} f(x)$.Then $c > -1$. From the definition of $\{a_n\}$ it follows that $c > a_n$ for some n. Let N be the smallest integer such that $c > a_N$. Then

$$c > -\tfrac{a(\lambda_1^N - 1)}{(\lambda_1 - 1)\lambda_1^N}$$

$$\Rightarrow \tfrac{c}{a}(\lambda_1 - 1)\lambda_1^N > -(\lambda_1^N - 1), \text{ since } a > 0$$

$$\Rightarrow [1 + \tfrac{c}{a}(\lambda_1 - 1)]\lambda_1^N > 1$$

$$\Rightarrow N \ln \lambda_1 + (\ln(1 + \tfrac{c}{a}(\lambda_1 - 1))) > 0$$

$$\Rightarrow N > -\frac{\ln(1+\frac{c}{a}(\lambda_1-1))}{\ln\lambda_1}.$$

Since N is the smallest integer such that $N > -\frac{\ln(1+\frac{c}{a}(\lambda_1-1))}{\ln\lambda_1}$, we have

$$N = [1 - \frac{\ln(1+\frac{c}{a}(\lambda_1-1))}{\ln\lambda_1}].$$

Now, we define the sequence $\{b_n\}$ inductively by

$$b_0 = 1,\ b_{n+1} = \begin{cases} f^{-1}(a_{N+n}) \cap (0,1] & \text{if } f^{-1}(a_{N+n}) \cap (0,1] \neq \phi \\ 0 & \text{if } f^{-1}(a_{N+n}) \cap (0,1] = \phi \end{cases}$$

Let $x \in [b_{k+1}, b_k)$. Then from the construction of $\{a_n\}$ and $\{b_n\}$ we have

$$f(x) \in [a_{N+k}, a_{N+k-1}),$$

$f^2(x) \in [a_{N+k-1}, a_{N+k-2})$, and so on.

Thus $\quad f^{N+k}(x) \in [a_1, a_0)$ and finally $f^{N+k+1}(x) = R|_{[b_{k+1},b_k)}(x) \in [0,1)$

Now $(R|_{[b_{k+1},b_k)})'(x) = (f^{N+k+1})'(x) = f'(f^{N+k}(x)) \bullet f'(f^{N+k-1}(x)) \bullet \cdots \bullet f'(f(x)) \bullet f'(x).$

Thus the slope of R on $[b_{k+1}, b_k)$ is $\lambda_1 \bullet \lambda_1 \bullet \cdots \bullet \lambda_1 \bullet \lambda_2 = \lambda_1^{N+k}\lambda_2$.

Hence the lemma is proved.

Lemma 5.1.2

Let $x \in (0,1)$. Then $f^n(x) \notin (0,1)$ for all $1 \le n \le N$. Moreover, if r of the points $x, f(x), f^2(x), \cdots, f^{n-1}(x)$ belong to $(-1,0)$ and s of them belongs to $(0,1)$, then $r \ge N(s-1)$.

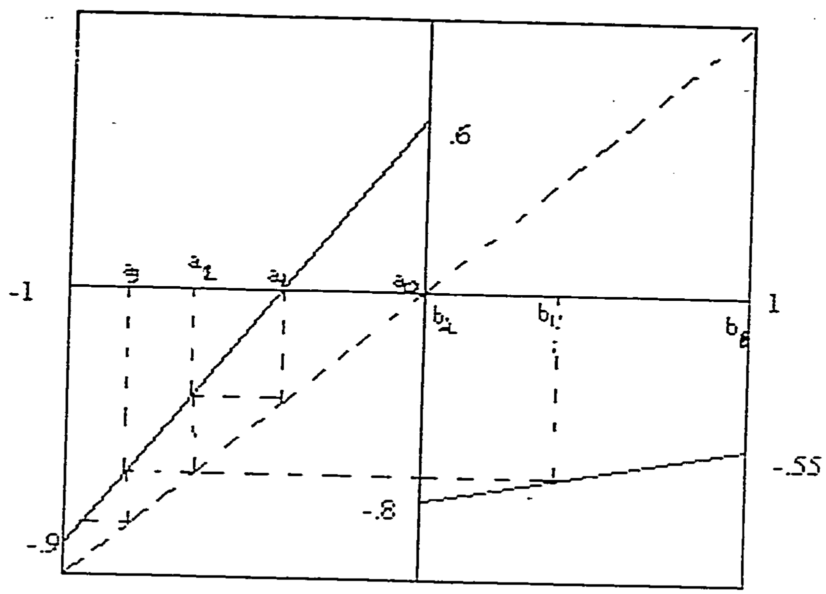
Proof: Le $x \in [b_{k+1}, b_k)$. Then $f(x) \in (a_{N+k}, a_{N+k-1}) \Rightarrow f^2(x) \in (a_{N+k-1}, a_{N+k-2})$ and so on. Thus $f^{N+k}(x) \in (a_1, a_0)$ and finally $f^{N+k+1}(x) \in [0,1)$. Hence $x \in (0,1) \Rightarrow f^n(x) \notin (0,1)$ for all $1 \le n \le N$.

From above, observe that for one positive value we are getting at least N negative values. So negative values are at least $N(s-1)$, because it is possible that not the whole negative trajectory of the last positive point is included in the sequence. Hence $r \ge N(s-1)$.

Example 5.1.1 (Map and first return map)

Consider the map

$$f(x) = \begin{cases} 1.5x + .6 & if \quad -1 \leq x < 0 \\ 0 & if \quad x = 0 \\ .25x - .8 & if \quad 0 < x \leq 1 \end{cases}$$



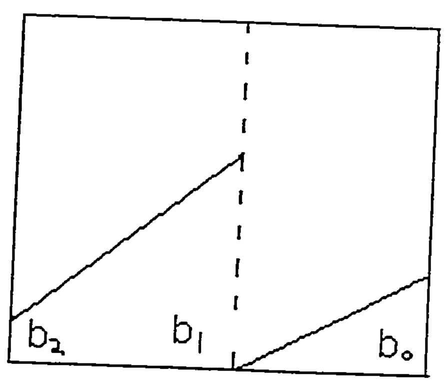


Figure 5.1.2

Here $\lambda_1 = 1.5, \lambda_2 = .25$, $a = .6, b = -.8$;

$c = \lim_{x \to 1^-} f(x) = .25 - .8 = -.55$

$N = [1 - \frac{\ln(1+\frac{-.55}{.6}(1.5-1))}{\ln 1.5}] = [2.5121] = 2$

$a_0 = 0, a_1 = f^{-1}(a_0) = -\frac{.6}{1.5} = -.4$

$a_2 = f^{-1}(a_1) = \frac{-.4-.6}{1.5} = -.66667 < -.55 = c$

So $N = 2$

$a_3 = f^{-1}(a_2) = \frac{-.66667-.6}{1.5} = -.84445$

$a_4 = f^{-1}(a_3) = \frac{-.84445-.6}{1.5} = -.96297.$

$a_5 = f^{-1}(a_4) = \frac{-.96297-.6}{1.5} = -1.042.$

Now $\quad b_0 = 0$

$b_1 = b_{0+1} = f^{-1}(a_{N+0}) = f^{-1}(a_{2+0}) = f^{-1}(a_2) = f^{-1}(-.66667) = \frac{-.66667+.8}{.25} = .53332$

$b_2 = b_{1+1} = f^{-1}(a_{2+1}) = f^{-1}(a_3) = f^{-1}(-.84445) = \frac{-.84445+.8}{.25} = -.1778 < 0$

So $\quad b_2 = 0$

Now $\quad R|_{[b_1, b_0)} = f^{2+0+1}(x) = f^3(x).$

$\quad R|_{[b2, b_1)} = f^{2+1+1}(x) = f^4(x).$

If $\quad x = .5 \quad$ then $\quad f(x) = .25(.5) - .8 = -.675, \quad f^2(x) = 1.5(-.675) + .6 = -.4125$

$, f^3(x) = 1.5(-.4125) + .6 = -.01875, f^4(x) = 1.5(-.01875) + .6 = .57188$

We can easily verify above lemmas with this example.

Lemma 5.1.3

If -1 is not a fixed point of f, then the first return map has a finite number of branches.

Proof: Observe that in Lemma 5.1.1, we constructed the sequence $\{a_n\}$ inductively by $a_{n+1} = f^{-1}(a_n) \cap [-1,0)$ for $n \geq 0$. In our case, there exists $m > 0$ such that $f^{-1}(a_m) \cap [-1,0) = \phi$ and we define $a_{m+1} = \frac{a_m - a}{\lambda_1} < -1$. So the sequence $\{a_n\}$ has $m+2$ terms. In the same lemma we constructed the sequence $\{b_n\}$ inductively by

$$b_0 = 1,\ b_{n+1} = \begin{cases} f^{-1}(a_{N+n}) \cap (0,1] & \text{if } f^{-1}(a_{N+n}) \cap (0,1] \neq \phi \\ 0 & \text{if } f^{-1}(a_{N+n}) \cap (0,1] = \phi \end{cases}$$

where $N = [1 - \frac{\ln(1+\frac{c}{a}(\lambda_1-1))}{\ln\lambda_1}]$.

Since $\{a_n\}$ is finite so is $\{b_n\}$. Hence $R(x)$ has finite number of branches on $[0,1)$.

Theorem 5.1.1

Let $f \in \mathcal{B}$ with $1 \le \lambda_1 \le 2$ and $0 < \lambda_2 < 1$. Let us have

$$N = [1 - \frac{\ln(1+\frac{c}{a}(\lambda_1-1))}{\ln\lambda_1}].$$

Then if $\lambda_1^N\lambda_2 > 1$, then f is eventually expanding and there exists an absolutely continuous invariant measure with respect to Lebesgue measure.

Proof: To show that f is eventually expanding we have to show that there exists an integer $n_0 > 1$ such that the iteration $f^{n_0}$is expanding. Suppose that r of the points $x, f(x), f^2(x), \ldots\ldots, f^{n_0-1}(x)$ lie in $(-1,0)$ and s of them lie in $(0,1)$. Then $r+s = n_0$ and $r \ge N(s-1)$. This implies

$$r \ge N(n_0 - r - 1)$$
$$\Rightarrow (1+\tfrac{1}{N})r + 1 \ge n_0$$

This also implies that $s \le \frac{r}{N} + 1$, which we will use latter.

Let $\lambda > 1$ and take $n_0 > (1+\frac{1}{N})\frac{\ln(\lambda\lambda_2^{-1})}{\ln(\lambda_1\lambda_2^{\frac{1}{N}})} + 1$ Note that $n_0 > 1$.

Combining the above two inequalities we get

$$(1+\tfrac{1}{N})r + 1 \ge (1+\tfrac{1}{N})\frac{\ln(\lambda\lambda_2^{-1})}{\ln(\lambda_1\lambda_2^{\frac{1}{N}})} + 1$$
$$\Rightarrow r \ge \frac{\ln(\lambda\lambda_2^{-1})}{\ln(\lambda_1\lambda_2^{\frac{1}{N}})}$$
$$\Rightarrow r\ln(\lambda_1\lambda_2^{\frac{1}{N}}) \ge \ln(\lambda\lambda_2^{-1})$$
$$\Rightarrow \ln(\lambda_1^r\lambda_2{}^{\frac{r}{N}}) \ge \ln(\lambda\lambda_2^{-1})$$
$$\Rightarrow \lambda_1^r\lambda_2{}^{\frac{r}{N}} \ge \lambda\lambda_2^{-1}$$
$$\Rightarrow \lambda_1^r\lambda_2{}^{\frac{r}{N}+1} \ge \lambda > 1$$

Let J be a sub-interval of $[-1,1]$. The slope of $f^{n_0}|_J$ is $\lambda_1^r\lambda_2^s$. Since $s \le \frac{r}{N} + 1$ and $0 < \lambda_2 < 1$, we have $\lambda_1^r\lambda_2^s \ge \lambda_1^r\lambda_2{}^{\frac{r}{N}+1} \ge \lambda > 1 \ge$. Hence f^{n_0} is expanding which implies that

f is eventually expanding. The existence of acim, we can conclude from [L-Y].

$\square$

From theorem 5.1.1 we know that if $\lambda_1^N \lambda_2 > 1$, then there exists an absolutely continuous invariant measure of f with respect to Lebesgue measure. Our next step is to find the measure for some examples. To do this we will consider f such that f is Markov. Recall that Frobenius-Perron operator for a piecewise linear Markov transformation can be represented by a matrix. Once we have the matrix representation, we can find the density function and then finally the acim.

Example 5.1.2

Consider the map $f : [-1,1] \longrightarrow [-1,1]$ defined by

$$f(x) = \begin{cases} 2x+1 & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ \frac{1}{2}x - 1 & , \quad x \in (0,1] \end{cases}$$

Here $c = \lim_{x \to 1^-} f(x) = -\frac{1}{2}$

$N = [1 - \frac{\ln(1 + \frac{-.5}{1}(2-1))}{\ln 2}] = 2.0$. Now the sequence (a_n) for the map f:

$a_0 = 0, a_1 = f^{-1}(a_0) = -\frac{1}{2} = -0.5,$

$a_2 = f^{-1}(a_1) = f^{-1}(-0.5) = \frac{-0.5-1}{2} = -.75 < -0.5 = c,$

$a_3 = f^{-1}(a_2) = f^{-1}(-.75) = \frac{-.75-1}{2} = -.875,$

$a_4 = f^{-1}(a_3) = f^{-1}(-.875) = \frac{-.875-1}{2} = -.9375,$

$a_5 = f^{-1}(a_4) = f^{-1}(-.9375) = \frac{-.9375-1}{2} = -.96875$ and so on.

Let $x_0 = \frac{1}{3}$. Then $f(x_0) = (\frac{1}{2}\frac{1}{3} - 1) = -\frac{5}{6}, f^2(x_0) = 2(-\frac{5}{6}) + 1 = -\frac{2}{3}$.

Here $\lambda_1^N \lambda_2 = 2^2 \frac{1}{2} = 2 > 1$ and hence f has an acim using theorem 5.1.1

Let $I_1 = [-1, -\frac{1}{2}), I_2 = (-\frac{1}{2}, 0), I_3 = (0,1]$ and $f(I_1) = [-1, 0]$, $f(I_2) = [0,1]$, $f(I_3) = [-1, -\frac{1}{2}]$

Observe that the map is Markov and the matrix representation M_f of Frobenius-Perron operator for f is $M_f = (m_{ij}), \quad 1 \le i,j \le n,$

where $m_{ij} = \frac{a_{ij}}{|f_i'|} = \frac{\lambda(I_i \cap f^{-1}(I_j))}{\lambda(I_i)}, 1 \leq i,j \leq n$ and $a_{ij} = \begin{cases} 1 & , \quad I_j \subset \tau(I_i) \\ 0 & , \quad \text{otherwise} \end{cases}$

Here $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 2 & 0 & 0 \end{bmatrix}$. Let $x = [x_1, x_2, x_3]$. *Then* solving $xM = x$, we get, $x = [4, 2, 1]$.

So the density function is $g(x) = \begin{cases} 1 & , \quad x \in [-1, -\frac{1}{2}) \\ \frac{1}{2} & , \quad x \in [-\frac{1}{2}, 0) \\ \frac{1}{4} & , \quad x \in [0, 1] \end{cases}$

where $\int_{-1}^{1} g(x) = 1$

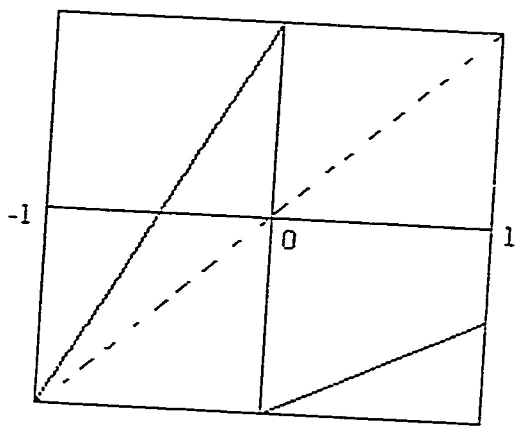


Figure 5.1.3

5.2 The case $0 < \lambda_1 < 1$ and $1 < \lambda_2 \leq 2$.

In this case we will consider a subclass $\mathcal{C}$ of $\mathcal{F}$ by taking $0 < \lambda_1 < 1$ and $1 < \lambda_2 \leq 2$. We will state and prove three lemmas and using these lemmas we will prove the main theorem.

Lemma 5.2.1

The first return map $R : (-1,0] \longrightarrow (-1,0]$ defined by $R(x) = f^r(x)$, where

$$r = \min\{i > 0 : f^i(x) \in [0,1)\}$$

is well defined. Moreover, there exists a partition $-1 = b_0 < b_1 < \cdots < 0$ of $(-1,0]$, not necessarily finite, and an integer N such that the restriction $R|_{(b_k,b_{k+1})}$ is linear with slope $\lambda_1\lambda_2^{N+k}$.

Proof: Observe that $f((-1,0]) \subset (-1,1)$. If $x \in (0,1)$ and $f^n(x) \in (0,1)$ for all $n \geq 1$, then we have

$$f(x) = \lambda_2 x + b = \lambda_2(x + \tfrac{b}{\lambda_2-1}) - \tfrac{b}{\lambda_2-1}$$

$$\Rightarrow f^2(x) = \lambda_2(\lambda_2(x + \tfrac{b}{\lambda_2-1}) - \tfrac{b}{\lambda_2-1}) + \tfrac{b}{\lambda_2-1}) - \tfrac{b}{\lambda_2-1}$$

$$= \lambda_2^2(x + \tfrac{b}{\lambda_2-1}) - \tfrac{b}{\lambda_2-1}.$$

By induction,

$$f^n(x) = \lambda_2^n(x + \tfrac{b}{\lambda_2-1}) - \tfrac{b}{\lambda_2-1}.$$

Moreover, for each $x \in (0,1)$ there exists $n_0 \geq 1$ such that $f^{n_0}(x) \in (-1,0]$, since otherwise $f^n(x) = \lambda_2^n(x + \frac{b}{\lambda_2-1}) - \frac{b}{\lambda_2-1} \longrightarrow \infty$ as $n \longrightarrow \infty$, which is a contradiction. This proves that R is well defined. Linearity is obvious.

To construct the required partition of $(-1,0]$ we will first construct the sequence $\{a_n\}$ which is defined inductively as follows:

$$a_0 = 0,\ a_{n+1} = f^{-1}(a_n) \cap (0,1] \text{ for } n \geq 0.$$

Observe that,

$$a_0 = 0 \Rightarrow f^{-1}(a_0) = -\tfrac{b}{\lambda_2} = -\tfrac{b(\lambda_2-1)}{(\lambda_2-1)\lambda_2} = a_1$$

$$\Rightarrow a_2 = f^{-1}(a_1) = -\tfrac{b(\lambda_2^2-1)}{(\lambda_2-1)\lambda_2^2} \text{ and so on.}$$

Inductively,

$$a_n = f^{-1}(a_{n-1}) = -\tfrac{b(\lambda_2^n-1)}{(\lambda_2-1)\lambda_2^n},\ n \geq 0.$$

Now let $c = \lim_{x \to -1^+} f(x)$. Then $c > -1$. From the definition of $\{a_n\}$ it follows that $c < a_n$ for some n. Let N be the smallest integer such that $c < a_N$. Then

$$c < -\frac{b(\lambda_2^N - 1)}{(\lambda_2 - 1)\lambda_2^N}$$

$$\Rightarrow \tfrac{c}{b}(\lambda_2 - 1)\lambda_2^N > -(\lambda_2^N - 1), \text{ since } b < 0$$

$$\Rightarrow [1 + \tfrac{c}{b}(\lambda_2 - 1)]\lambda_2^N > 1$$

$$\Rightarrow N \ln \lambda_2 + (\ln(1 + \tfrac{c}{b}(\lambda_2 - 1))) \ > 0$$

$$\Rightarrow N > -\frac{\ln(1+\frac{c}{b}(\lambda_2 - 1))}{\ln \lambda_2}.$$

Since N is the smallest integer such that $N > -\frac{\ln(1+\frac{c}{b}(\lambda_2-1))}{\ln \lambda_2}$, we have

$$N = [1 - \frac{\ln(1+\frac{c}{b}(\lambda_2-1))}{\ln \lambda_2}].$$

Now, we define the sequence $\{b_n\}$ inductively by

$$b_0 = -1,\ b_{n+1} = \begin{cases} f^{-1}(a_{N+n})\cap(0,1] & \textit{if } f^{-1}(a_{N+n})\cap[-1,0) \neq \phi \\ 0 & \textit{if } f^{-1}(a_{N+n})\cap[-1,0) = \phi \end{cases}$$

Let $x \in (b_k, b_{k+1}]$. Then from the construction of $\{a_n\}$ and $\{b_n\}$ we have

$$f(x) \in [a_{N+k-1}, a_{N+k}),$$

$f^2(x) \in [a_{N+k-2}, a_{N+k-1})$, and so on.

Thus $\quad f^{N+k}(x) \in [a_0, a_1)$ and finally $f^{N+k+1}(x) = R|_{(b_k, b_{k+1}]}(x) \in (-1, 0]$

Now $(R|_{(b_k,b_{k+1}]})'(x) = (f^{N+k+1})'(x) = f'(f^{N+k}(x)) \bullet f'(f^{N+k-1}(x)) \cdots \bullet f'(f(x)) \bullet f'(x)$.

Thus the slope of R on $[b_{k+1}, b_k)$ is $\lambda_2\lambda_2\cdots\lambda_2\lambda_1 = \lambda_1\lambda_2^{N+k}$.

Lemma 5.2.2

Let $x \in (-1,0)$. Then $f^n(x) \notin (-1,0)$ for all $1 \le n \le N$. Moreover, if r of the points $x, f(x), f^2(x), \cdots, f^{n-1}(x)$ belong to $(-1,0)$ and s of them belongs to $(0,1)$, then $s \ge N(r-1)$.

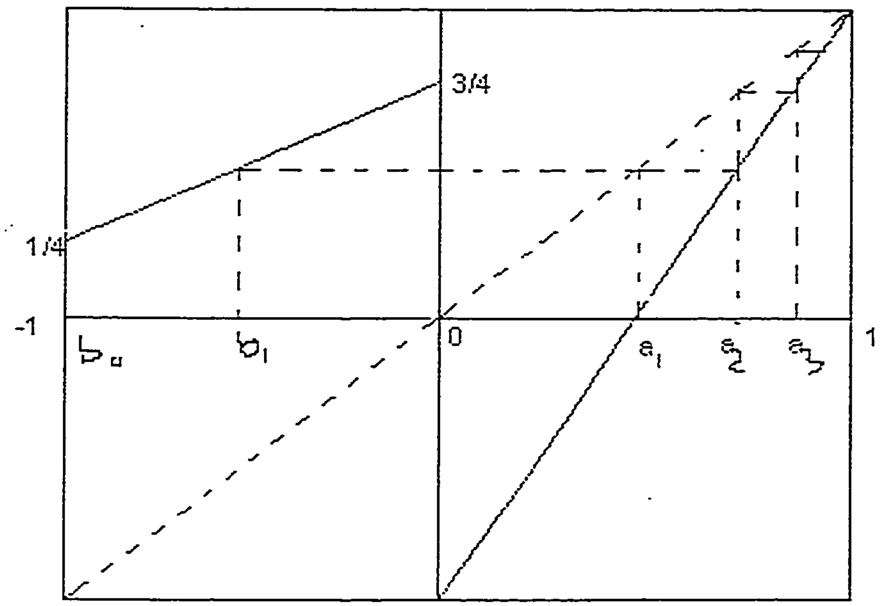
Proof: Le $x \in [b_k, b_{k+1})$. Then $f(x) \in [a_{N+k-1}, a_{N+k}) \Rightarrow f^2(x) \in [a_{N+k-2}, a_{N+k-1})$ and so on. Thus $f^{N+k}(x) \in [a_0, a_1)$ and finally $f^{N+k+1}(x) \in (-1,0]$. Hence $x \in (-1,0] \Rightarrow f^n(x) \notin [-1,0)$ for all $1 \le n \le N$.

From above, observe that for one negative value we are getting at least N positive values. So positive values are at least $N(r-1)$, because it is possible that not the whole positive trajectory of the last negative point is included in the sequence. Hence $s \geq N(r-1)$.

Example 5.2.1 (Map and first return map)

Consider the map

$$f(x) = \begin{cases} .5x + .75 & \text{if } -1 \leq x < 0 \\ 0 & \text{if } \quad x = 0 \\ 2x - 1 & \text{if } \ 0 < x \leq 1 \end{cases}$$



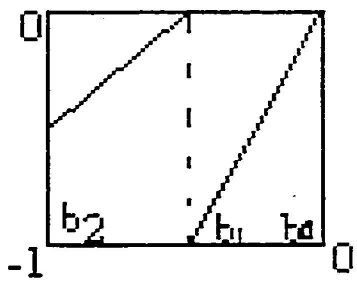


Figure 5.2.1

Here $\lambda_1 = .5, \lambda_2 = 2,\ a = .75, b = -1;$

$c = \lim_{x \to -1^+} f(x) = -.5 + .75 = .25$

$N = [1 - \frac{\ln(1 + \frac{.25}{-1}(2-1))}{\ln 2}] = [1.415] = 1$

$a_0 = 0, a_1 = f^{-1}(a_0) = \frac{1}{2} > .25 = c$

So $N = 1$

$a_2 = f^{-1}(a_1) = \frac{.5+1}{2} = .75$

$a_3 = f^{-1}(a_2) = \frac{.75+1}{2} = .875$

$a_4 = f^{-1}(a_3) = \frac{.875+1}{2} = .9375$

$a_5 = f^{-1}(a_4) = \frac{.9375+1}{2} = .96875$ and so on.

Now $\quad b_0 = -1$

$b_1 = b_{0+1} = f^{-1}(a_{N+0}) = f^{-1}(a_{1+0}) = f^{-1}(a_1) = f^{-1}(.5) = \frac{.5-.75}{.5} = -.5$

$b_2 = b_{1+1} = f^{-1}(a_{1+1}) = f^{-1}(a_2) = f^{-1}(.75) = \frac{.75-.75}{.5} = 0$

Now $\quad R|_{(b_0, b_1)} = f^{1+0+1}(x) = f^2(x).$

$R|_{(b1, b_1]} = f^{1+1+1}(x) = f^3(x).$

If $x = .7$ then $f(x) = 2(.7) - 1 = .4,\ f^2(x) = 2(.4) - 1 = -.2,$

$f^3(x) = .5(-.2) + .75 = .65$

We can easily verify above lemmas with this example.

Lemma 5.2.3

If 1 is not a fixed point of f, then the first return map has a finite number of branches.

Proof: Observe that in Lemma 4.2.1, we constructed the sequence $\{a_n\}$ inductively by $a_{n+1} = f^{-1}(a_n) \cap (0, 1]$ for $n \geq 0$. In our case, there exists $m > 0$ such that $f^{-1}(a_m) \cap (0, 1] = \phi$ and we define $a_{m+1} = \frac{a_m - a}{\lambda_1} > 1$. So the sequence $\{a_n\}$ has $m + 2$ terms.

In the same lemma we constructed the sequence $\{b_n\}$ inductively by

$$b_0 = -1,\ b_{n+1} = \begin{cases} f^{-1}(a_{N+n}) \cap (-1, 0] & \text{if } f^{-1}(a_{N+n}) \cap (-1, 0] \neq \phi \\ 0 & \text{if } f^{-1}(a_{N+n}) \cap (-1, 0] = \phi \end{cases}$$

where $N = [1 - \frac{\ln(1+\frac{c}{b}(\lambda_2-1))}{\ln\lambda_2}]$.

Since $\{a_n\}$ is finite so is $\{b_n\}$. Hence $R(x)$ has finite number of branches on $(-1,0]$.

Theorem 5.2.1

Let $f \in \mathcal{B}$ with $0 < \lambda_1 < 1$ and $1 < \lambda_2 \leq 2$. Let us have

$$N = [1 - \frac{\ln(1+\frac{c}{b}(\lambda_2-1))}{\ln\lambda_2}].$$

Then if $\lambda_1\lambda_2^N > 1$, then f is eventually expanding and there exists an absolutely continuous invariant measure with respect to Lebesgue measure.

Proof: To show that f is eventually expanding we have to show that there exists an integer $n_0 > 1$ such that the iteration $f^{n_0}$is expanding. Suppose that r of the points $x, f(x), f^2(x), \cdots, f^{n_0-1}(x)$ lie in $(-1,0)$ and s of them lie in $(0,1)$. Then $r+s = n_0$ and $s \geq N(r-1)$. This implies

$$s \geq N(n_0 - s - 1)$$
$$\Rightarrow (1+\tfrac{1}{N})s + 1 \geq n_0$$

This also implies that $r \leq \frac{s}{N} + 1$, which we will use latter.

Let $\lambda > 1$ and take $n_0 > (1+\frac{1}{N})\frac{\ln(\lambda\lambda_1{}^{-1})}{\ln(\lambda_2\lambda_1^{\frac{1}{N}})} + 1$ Note that $n_0 > 1$.Combining the above two inequalities we get

$$(1+\tfrac{1}{N})s + 1 \geq (1+\tfrac{1}{N})\frac{\ln(\lambda\lambda_1^{-1})}{\ln(\lambda_2\lambda_1^{\frac{1}{N}})} + 1$$
$$\Rightarrow s \geq \frac{\ln(\lambda\lambda_1^{-1})}{\ln(\lambda_2\lambda_1^{\frac{1}{N}})}$$
$$\Rightarrow s\ln(\lambda_2\lambda_1^{\frac{1}{N}}) \geq \ln(\lambda\lambda_1^{-1})$$
$$\Rightarrow \ln(\lambda_2^s\lambda_1{}^{\frac{s}{N}}) \geq \ln(\lambda\lambda_1^{-1})$$
$$\Rightarrow \lambda_2^s\lambda_1{}^{\frac{s}{N}} \geq \lambda\lambda_1^{-1}$$
$$\Rightarrow \lambda_2^s\lambda_1{}^{\frac{s}{N}+1} \geq \lambda > 1$$

Let J be a sub-interval of $[-1,1]$. The slope of $f^{n_0}|_J$ is $\lambda_1^r\lambda_2^s$. Since $r \leq \frac{s}{N} + 1$ and $0 < \lambda_1 < 1$, we have $\lambda_2^s\lambda_1^r \geq \lambda_2^s\lambda_1{}^{\frac{s}{N}+1} \geq \lambda > 1 \geq$. Hence f^{n_0} is expanding which implies that f is eventually expanding. The existence of acim, we can conclude from [L-Y].

Example 5.2.2

Consider the family of maps $f : [-1, 1] \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x + 1 & , \quad x \in [-1, 0) \\ 0 & , \quad x = 0 \\ 2x - 1 & , \quad x \in (0, 1] \end{cases}$$

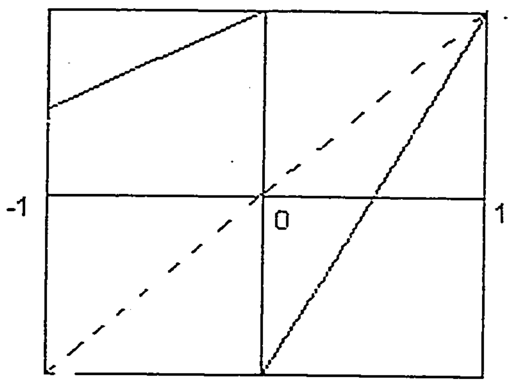


Figure 5.2.2

Here $c = \lim_{x \to -1^-} f(x) = \frac{1}{2}$

$N = [1 - \frac{\ln(1 + \frac{.5}{-1}(2-1))}{\ln 2}] = 2.0$ Now the sequence (a_n) for the map f:

$a_0 = 0, a_1 = f^{-1}(a_0) = \frac{1}{2} = 0.5,$

$a_2 = f^{-1}(a_1) = f^{-1}(0.5) = \frac{0.5+1}{2} = .75 > 0.5 = c,$

$a_3 = f^{-1}(a_2) = f^{-1}(.75) = \frac{.75+1}{2} = .875,$

$a_4 = f^{-1}(a_3) = f^{-1}(.875) = \frac{.875+1}{2} = .9375,$

$a_5 = f^{-1}(a_4) = f^{-1}(.9375) = \frac{.9375-1}{2} = .96875$ and so on.

Let $x_0 = -\frac{1}{3}$. Then $f(x_0) = (\frac{1}{2}(-\frac{1}{3}) \div 1) = \frac{5}{6}, f^2(x_0) = 2(\frac{5}{6}) - 1 = \frac{2}{3}$.

Here $\lambda_1 \lambda_2^N = \frac{1}{2} 2^2 = 2 > 1$ and hence f has an acim using theorem 5.2.1

Let $I_1 = [-1,0], I_2 = [0,\frac{1}{2}], I_3 = [\frac{1}{2},1]$ and $f(I_1) = [\frac{1}{2},1], f(I_2) = [-1,0], f(I_3) = [0,1]$

Observe that the map is Markov and the matrix representation M_f of Frobenius-Perron operator for f is $M_f = (m_{ij}), \quad 1 \le i,j \le n,$

where $m_{ij} = \frac{a_{ij}}{|f_i'|} = \frac{\lambda(I_i \cap f^{-1}(I_j))}{\lambda(I_i)}, 1 \le i,j \le n$ and $a_{ij} = \begin{cases} 1 & , \quad I_j \subset \tau(I_i) \\ 0 & , \quad \text{otherwise} \end{cases}$

Here $M = \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Let $x = [x_1, x_2, x_3]$. *Then* solving $xM = x$, we get, $x = [1,2,4]$.

So the density function is $g(x) = \begin{cases} \frac{2}{11} & , \quad x \in [-1,-\frac{1}{2}) \\ \frac{4}{11} & , \quad x \in [-\frac{1}{2},0) \\ \frac{8}{11} & , \quad x \in [0,1) \end{cases}$

Chapter VI

C++ Programs and Results

In Chapters IV and V we found the acim using the matrix representation of Frobenius-Perron operator for Markov *f*. In this chapter we present some C++ programs for finding the density function of absolutely continuous invariant measure in general. The programs are written in Turbo C++ version 4.5.

Algorithm 6.1.1

```
//—————————
// histo.cpp
//—————————
#include "w.h"
// Class for the function with parameter a1, a2, b1, b2
class gfun : public fun
{ private:
double a1, a2, b1, b2; // the parameters
public:
gfun( double a1i, double a2i, double b1i, double b2i): a1(a1i), a2(a2i), b1(b1i), b2(b2i){}
void seta1(double a1i) { a1 = a1i; }
void seta2(double a2i) { a2 = a2i; }
void setb1(double b1i) { b1 = b1i; }
void setb2(double b2i) { b2 = b2i; }
double geta1(){return a1;}
double geta2(){return a2;}
double getb1(){return b1;}
double getb2(){return b2;}
double f(double x) // Top level function
```

```
{ double y, a = -1, b=1; // [a , b] is the interval on which we will work.
if ((x >= a) & (x <= 0))
{if (x == 0) y=0;
else y = a1 * x + a2;}
else
{if ((x >=0)&(x<b))
{ if (x == 0) y=0;
else y = b1 * x + b2;}
}
return(y);
}
};
// io for arrays
void pl(int vi[], int k)
{ for (int j = 0; j < k; j++)
cout << vi[j] << " ";
nl(); }
void pl(double xi[], int k)
{ for (int j = 0; j < k; j++)
cout << xi[j] << " ";
nl(); }
void main()
{
double x; // x is the initial point
double a, b, a1i, a2i, b1i, b2i;
int n, N; // n= number of partitions, N= number of iteration
cout<<"enter the value of n, N , x , a , b , a1, a2, b1 , b2"<< "\n";
cin >> n >> N >> x >> a >> b >> a1i >> a2i >> b1i >> b2i;
```

```
double h = (b-a)/n;
double v1[100];
int v2[100];
for(int k = 0; k < = n; k++)
v2[k] = 0;
for (int j = 0; j < = n; j++)
{v1[j] = a + j * h ;}
int count = 0;
gfun *g = new gfun(a1i, a2i, b1i, b2i);
do{
for ( j = 0; j <= n; j++)
{ if (( v1[j] <= x ) & ( x < v1[j+1])) {v2[j]++;}
}
x = g->f(x); count++;
}while (count <= N);
nl(0); nl();
banner(" histo.cpp");
nl(); nl(0);
p("The vector v1 for the values of n , N, x, a1, a2, b1, b2 "); nl(); nl(); nl();
p("v1="); p(" [ "); pl ( v1 , n ) ; p( " ] ");
p("The vector v2 for the values of n , N, x, a1, a2, b1, b2"); nl(); nl(); nl();
p("v2="); p("["); pl( v2, n); p("]");
p("The vector v1 for the values of n , N, x, a1, a2, b1, b2"); nl(); nl(); nl();
p("v1="); p( " [" ); pl(v1, n); p(" ] ");
delete g;
}
```

Algorithm 6.1.2

```
//————————————————————
// graphics: histogram and density function
//————————————————————
/* In C++ 4.5 version invournment,for the graphics of histogram and density function,we need five files: gtiny.h, gtiny.cpp, gwtiny.h , gwtiny.cpp, gtiny.rc, gtiny.exe. The following is a part of gtiny.cpp
*/
void plot(gwin &w, int p)
{ w.open(p);
int wb = 20; // screen window bottom
if (p == 1) wb = 40; // higher for printer
w.locate(20,80,wb,80);
w.preframe();
int m=1;
int Max=1400;
w.scale(-1,m,0,Max,1);
double v1[]={-1, -0.96, -0.92, -0.88, -0.84 , -0.8, -0.76 , -0.72, -0.68 , -0.64 , -0.6 , -0.56, -0.52 , -0.48, -0.44, -0.4 , -0.36 , -0.32, -0.28, -0.24, -0.2, -0.16, -0.12, -0.08, -0.04, 2.08167e-17, 0.04 , 0.08, 0.12, 0.16, 0.2, 0.24, 0.28, 0.32, 0.36, 0.4, 0.44, 0.48, 0.52, 0.56, 0.6, 0.64, 0.68, 0.72, 0.76, 0.8, 0.84, 0.88, 0.92, 0.96,1};
int v2[]={229, 1152, 956, 1206, 1150, 1163, 1376, 1322, 1285, 879, 515, 664, 700, 560, 733, 717, 610, 850, 664, 791, 836, 677, 627, 330, 277, 325, 400, 429, 370, 306, 345, 419, 457, 397, 325, 371, 473, 501, 366, 371, 465, 523, 442, 383, 375, 388, 262, 176, 177, 150};
for (int i=0;i<50;i++)
{
w.move(v1[i],0);
w.line(v1[i],v2[i]);
```

```
w.line(v1[i+1],0);
}
w.p(-.5,-75,"Figure 6.1.1");
w.p(1,0,"1"); w.p(-1.075,10,"0");
w.p(-1.05,0,"-1");
double x =-1.025, y =0,d=100,e=.2;
for ( i=1;i<=14;i++)
{ w.move(x,y+i*d); w.line(-.98,y+i*d);
w.pf(x-e,y+i*d+5," %.6g",y+i*d);
}
w.unclip();
w.p(-.5,1450,"Histogram");
w.close();
}
```

Example 6.1.1

We consider the map f with two increasing branches defined by

$$f(x) = \begin{cases} 1.75x + 1 & , \quad x \in [-1, 0) \\ 0 & , \quad x = 0 \\ .35x - .97 & , \quad x \in (0, 1] \end{cases}$$

We have $N = [1 - \frac{\ln(1 + \frac{-.62}{1}(1.75-1))}{\ln(1.75)}] = [2.1177] = 2$ and $\lambda_1^N \lambda_2 = (1.75)^2(.35). = 1.0719$ So from theorem 5.1.1, there exists an absolutely continuous invariant measure. Let $x_0 = -.82$ and consider the first 30000 iterations of f with initial value x_0. We divide $[-1, 1]$ in 50 intervals of length 0.04. Using algorithms 6.1.1 and 6.1.2 we have the following histogram (the figure below) which displays the number of iterations that enter in each subinterval. From this histogram, we can get the density function for f.

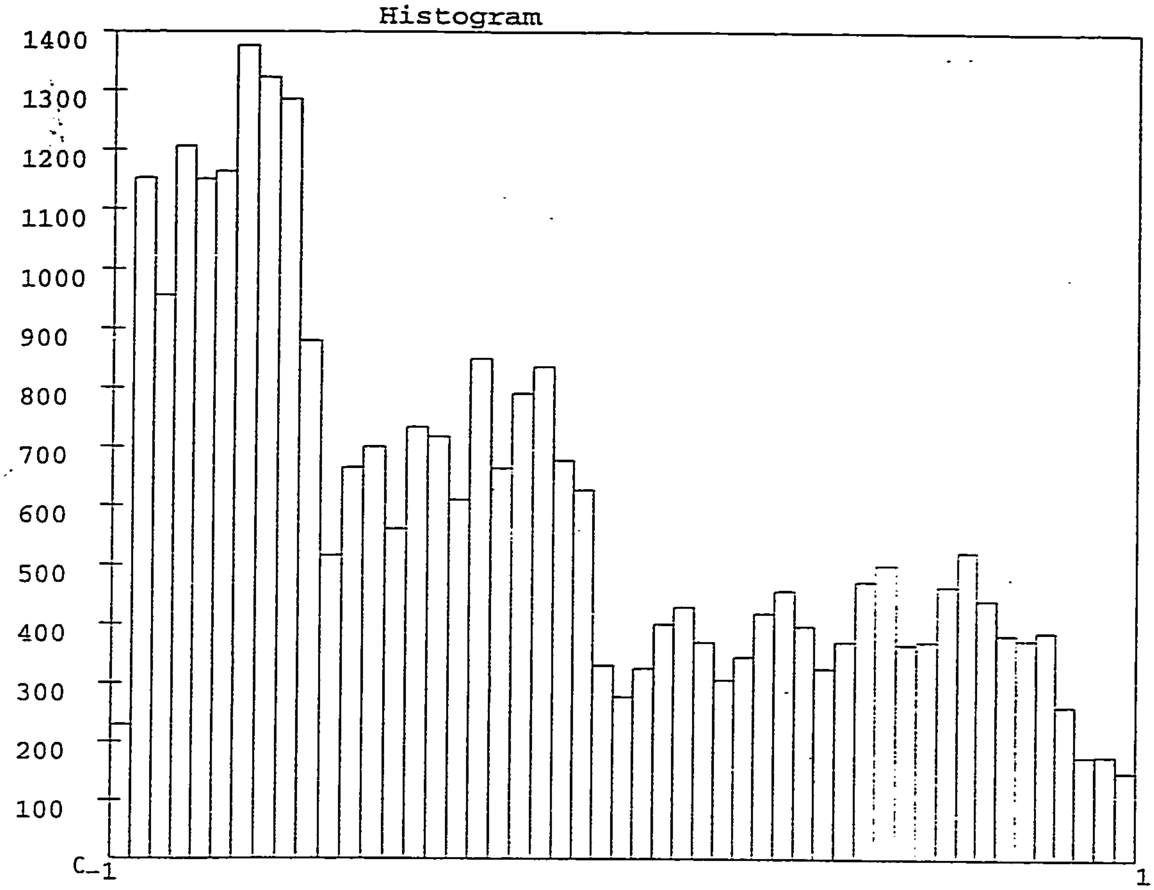


Figure 6.1.1

Example 6.1.2

We consider the map f with one increasing and one decreasing branch defined by

$$f(x) = \begin{cases} 2x+1 & , \quad x \in [-1,0) \\ 0 & , \quad x = 0 \\ -.43x - \frac{1}{2} & , \quad x \in (0,1] \end{cases}$$

We have $N = 2$ and $\lambda_1^N \lambda_2 = -1.72$. So from theorem 4.1.1, there exists an absolutely

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