The Role of the Graphing Calculator in Teaching and Learning of the Concept of Function: Observations in a Mathematics Classroom

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ABSTRACT

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This thesis, through research literature and a short study, explores the concept of function from didactic, pedagogical and psychological perspectives. The effects of recent curriculum reforms in Quebec, current mathematics textbooks, the implementation of the graphing calculator in the teaching and the learning of the concept of function will be studied. The research literature will set the theoretical basis for this paper and will be followed by a discussion of observations made in a graphing calculator complemented secondary 4 mathematics classroom of a local high school. Subsequent to these observations, an activity was designed and presented to the students as a test in the form of a questionnaire. For purposes of the study, this activity will provide an example of the role the graphing calculator plays when learning about function. Informal assessments were made concerning the role the graphing calculator when teaching and learning the function concept. One realization that was made concerned the necessary balance between traditional paper and pencil and graphing calculator methods needed in the classroom. Another dealt with the importance of the monitored integration of the graphing calculator in the classroom in order to be able to enhance the student's learning as well as to complement the teachers' methods.

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Richard.... We're finished!!!

TABLE OF CONTENTS

LIST OF TABLES

Introduction

Due to its reoccurring nature in high school mathematics, the notion of function has held its place in mathematics education research. With the advent of certain technological advances, those involved in research and in teaching have had to re-evaluate their methodologies. Mathematics education reform aims at making changes on many different levels. In having to adapt to new ideologies and philosophies, modifications in the curricula, changes of textbooks and offering teacher workshops, are some of the actions that have been taken in hoping to develop more appropriate ways of presenting mathematical topics.

The focus of this thesis will concern these issues on both theoretical and practical levels, specifically where the teaching and learning of the function concept is involved. Consideration of the curriculum, textbooks currently being used, as well as student and teacher difficulties will be discussed in this chapter. The subsequent chapter will mostly be in the form of a journal that was written during my observation period in a secondary 4 calculator-based mathematics class. It will help set the premise for the following chapter, which deals with the practical portion of this thesis. In that chapter III, links to research ideas regarding the notion of function, which were presented in the first chapter, will be made.

Possible issues concerning the difficulties involved with the implementation and the presence of the graphing calculator in the classroom will be illustrated. The graphing calculator has the potential of reducing or eliminating some of the procedures involved with constructions of the various representations of functions. Procedures such as the manual generation of tables of values and the plotting of points on the Cartesian plane may be lost. If used inappropriately, the graphing calculator may also do very little for concept development. This seems to be an unwanted result given the requirements found in the new mathematics curriculum reform in Quebec, specifically those emphasising concept development.

From personal experience when teaching mathematics, I have often thought about the moment at which a given amount of information that is shared with the student begins to get in the way of his or her learning. That moment often falls within a teaching sequence, regardless of its duration, leaving the students at a loss without them necessarily being aware that *something* is missing in needing to move to the next level of understanding. Will the use of the graphing calculator be responsible for creating more of these moments?

The new mathematics curriculum reform proposes that technology be used where appropriate. It may be observed in the textbooks currently being used that an example of the technology implied by the curriculum is the graphing calculator. As a consequence, most schools have seen the implementation of the graphing calculator in the classroom. Thus arise the aims of this thesis.

The purpose of this thesis is to assess the role of the graphing calculator during the conceptual development of the notion of function at the secondary 4 level of mathematics. Is the graphing calculator an asset or a liability to the students when learning the concept of function? This statement and question will be addressed and answered with the help of the classroom observations and the subsequent classroom activity. The classroom observations will emphasize the importance of achieving an appropriate balance between working with traditional paper-and-pencil and the graphing calculator. The brief study, in turn, will bring into question the role of the calculator as it pertains to the development of the concept of function within a specific mathematics classroom.

The thesis will be presented in the following fashion. Chapter I will discuss, from mathematical and psychological perspectives, pedagogical and didactic issues concerning the notion of function in mathematics education. Chapter II provides descriptions of observations I had the opportunity to make while attending a graphing calculator based classroom that was just beginning the topic of functions at the Math 436 level. In addition to these observations, a brief analysis of a test that they wrote will be presented. Chapter III deals with an activity regarding functions and the graphing calculator that was designed and conducted in the aforementioned class. The research tool was in the form of a questionnaire. Chapter IV will conclude this dissertation concerning the graphing calculator and the teaching and learning of the function concept with a conclusion and recommendations based on what has been seen through research, personal observations and analysis of the results of the activity that was conducted.

Chapter I

The Didactic, Mathematical and Psychological Perspectives on the Teaching and Learning Of the Concept of Function

Introduction

This chapter presents a review of several articles related to the teaching and learning of functions. Its aim is to provide the reader with a better sense of the different kinds of difficulties encountered by both teachers and students regarding this notion. Specifically, the context in which the teaching and learning of the function concept that will be discussed will be that of the most recent curriculum reforms in Quebec, referring particularly to the recommendation found in the guide which states, " using technology where appropriate in the teaching of mathematics". The reform directives and their realization in some textbooks are presented in section 1, which discusses the decisions made concerning the introduction of the concept of function at the different levels of schooling. This section addresses questions such as: What aspects of the mathematical concept are worth teaching? How are they expected to be taught? What problems are the students expected to learn to solve? Which teaching aids are recommended? And at which time? These decisions are then discussed from two perspectives; mathematical and psychological. Section 2 will consider the curricular decisions from a mathematical point of view about the meaning of the function concept as an element of the system of mathematical knowledge. In particular, the different representations of functions such as descriptions of properties, algebraic expressions, graphs, explicit enumerations of correspondences and tables are considered as forming parts of the meaning of the notion of function.

The psychological perspective discussed in section 3 considers the various intuitive notions of functions that students hold at the different stages of their cognitive and mathematical development. To what extent are these intuitions compatible with the mathematical notion of function? This perspective will also consider the obstacles encountered by students when learning functions. Are the pedagogical and didactic decisions in the teaching of functions building on students' intuition by considering possible obstacles encountered during these processes? How are teachers to help facilitate students in overcoming these obstacles?

1. Recent curriculum recommendations in Quebec related to functions and their textbook realization

What are the goals of mathematics education where the notion of the function is concerned? Answers to this and similar questions help to re-evaluate the current standards and consequently lead to curriculum reforms. Recently in Quebec, curriculum reform has had its place in primary and secondary level mathematics.

In relation to the notion of function, the reform has introduced some fairly radical changes. For example, at the Math 416 level, the use of the term *function* has been eliminated. Linear functions of the past are now referred to as zero variation, direct variation or partial variation linear relations.

1.1 The curriculum

The curriculum has a spiral organisation with respect to the notion of variable, tables of values, different representations of functions, as well as the determination of rules of relations. In addition to the expected increase in the level of difficulty from one level to the next, the topics are introduced at first informally, with a greater focus on the visual presentation. For example, a variable is first represented by a blank space, which is later replaced by a letter; relations and functions are first introduced as tables while the notion of variable still remains somewhat vague, and the notion of relation between two variables is introduced in higher grades. As one approaches the end of the curriculum, topics begin to tie together many of the previously introduced intuitive ideas through applications. This is done together with the presentation of functions other than linear functions, while the ability to perform algebraic procedures is expected to have become an acquired ability in the student. The curriculum guidelines concerning functions at the different levels in high school are provided in Appendix A.

It is interesting for the purposes of this research to consider the changes, with respect to functions, from the old Math 436 course to the new Math 436 course. The changes in the course content are summarised in the table below. This table is taken from *MAPCO's June 2000 Secondary Mathematics Teachers Support Session.* The Math Action Plan Committee is responsible for providing support on all fronts to the English sector of mathematics education in Quebec. A discussion regarding the current and predominant textbooks being used and the degree to which they adhere to the new curriculum standards will follow.

Table 1. Comparison of old and new Math 436 Course content on functions

Old	*New*
Sets	Sets removed
Relations: mathematizing, using ordered pairs, Cartesian products, inverses of relations	Removed
	Analyze situations involving functions using different modes of representation. Exploring the following types of functions: inverse, rational, square root, step and exponential.
Constant and affine functions. Inverse of a function.	Polynomial function of degree < 3. Removed. Functions of degree = 2. Operations on functions.

A goal behind the curricular changes appears to be the desire to make mathematical topics more conceptual and at the same time, less formulistic. Such an example is the removal of the set language. This could be interpreted from the, 'analysis of situations involving functions using different modes of representation' portion of the new program, suggesting that students should be expected to (i) conceptualise functions as models of real life relations between variable quantities, and also to (ii) abstract the notion of function from any particular form of it representation. This thesis focuses mainly on (ii), specifically where the use of the graphing calculator is concerned. The implementation of the graphing calculator comes as a result of the curriculum's recommendation concerning the use of "technology where appropriate". However, a general issue concerning the graphing calculator is, how its use can enhance conceptual understanding.

Traditionally, in high school mathematics, the focus has predominantly been on procedure, with little emphasis on conceptual understanding. Following the reform, a greater emphasis has been placed on understanding conceptually through analysis, almost eliminating procedure. The difficulty with these apparently one sided strategies on both theoretical and practical planes, relies on the fact that certain areas in mathematics require strong procedural understanding, whereas other areas require development towards more conceptual understanding. Why does an attempt at striking an appropriate balance between these two fundamental aspects of teaching and learning often result in the exclusion of one or the other?

1.2 The textbook realization of the curriculum

A curriculum is often implemented according to the textbooks rather than the official ministerial document. Let us consider the textbooks being used and the ways in which they handle definitions, explanations and examples related to the subject of functions.

Following a review of older textbooks, in addition to the experience I have had learning and teaching with them, a conclusion that may safely be drawn is that the intention of the authors of older textbooks was that *knowledge* be acquired through procedure. Emphasis on the acquisition of the concept was left implicit with a minimal inclusion of definitions and theorems. Examples in those older textbooks, as well as the examples done in classes were shown, not necessarily taught. This is a distinction I believe to be very important. At that time, perhaps concept development was not emphasized due to the belief that students at the high school level were not ready for certain mathematical notions that would be more formally introduced in post secondary mathematics courses. How true was this belief?

In the more recent textbooks, specifically the *Guy Breton Carrousel* and *Reflection* series (which are predominantly being used in Quebec), presentations of several topics are intended to help students learn more conceptually and reduce emphasis on procedure. It is difficult, however, to guarantee a conceptual understanding of a mathematical topic within a textbook. For example, attempts to explain topics such as factoring and operations on polynomials are discussed and illustrated using diagrams in the mentioned textbooks. Perhaps these attempts are based on the opinion that a visual representation by itself leads to, and enhances conceptual understanding. However, research does not support this opinion unconditionally, (Janvier, 1987). Another feature of the mentioned textbook series are lengthy verbal explanations. I do not think that an abundance of drawings, pictures and lengthy explanations lead to more conceptual understanding in students. I think that conceptual teaching and learning must be based on the dynamic nature of conversations and discussions in the classroom. Such activity cannot be transcribed into static text. The attempts of the authors to focus their emphasis on predominantly *conceptual* explanations often make the textbook presentation too lengthy, hence teachers and students may avoid reading them.

As a private math instructor at a learning centre working with students from various levels and from different schools, I observed that teachers use older texts when teaching certain topics they consider to be poorly covered in the newer texts. For example, the Guy Breton text introduces factoring pictorially, whereas older texts tend to introduce factoring by illustrating many examples. In either case, textbooks should act as tools in the classroom that complement the teaching of the subject and not act as a teacher replacement. Textbooks should be used as references and provide examples and exercises that would allow students of the subject to use them independently following some initial classes that would have, in theory, facilitated their learning process. The teacher's duty should be to explain and interpret the ideas being presented in the book to the students in trying to help them develop their knowledge in a more conceptual manner, and more procedurally whichever the case may be.

Although the curriculum and textbook objectives of all levels are included as appendices A and B, the following tables 2 and 3 will compare the objectives regarding the notion of function at the Math 436 level.

Table 2. Comparison of curriculum and textbook objectives concerning functions

	Curriculum Objectives concerning Functions	**Mathematical Reflections Textbook Objectives concerning Functions**
Terminal Objective 1.1 Chapter 1 Functions	To analyse situations involving functions, using different modes of representations.	To analyse situations involving functions, using different modes of representations.
Intermediate Objectives 1.1 Chapter 1 Functions	• To use symbols to represent a situation involving a function, indicating a source set, a target set and a rule of correspondence • To draw the Cartesian coordinate graph representing a situation involving a function, given an equivalent verbal description, tables of values or rules of correspondence • To prepare the table of values for a situation involving a function, given an equivalent verbal description, rule of correspondence or Cartesian coordinate graph • To describe the properties of a Cartesian coordinate graph representing a function: its rate of change, its x-intercept (zero), its y-intercept, its domain and range, its sign, whether it is constant, increasing or decreasing • To determine the relationships between changes in the parameters of the rule of correspondence of a function and changes in the equivalent Cartesian coordinate graph.	• To use symbols to represent a situation involving a function, indicating a source set, a target set and a rule of correspondence • To translate from one mode of representation to another • To describe the properties of a Cartesian coordinate graph representing a function • To determine the relationships between changes in the parameters of the rule of correspondence of a function and changes in the equivalent Cartesian coordinate graph.

Table 3. Comparison of curriculum and textbook objectives concerning polynomial functions

	Curriculum Objectives concerning Polynomial Functions	Mathematical Reflections Textbook Objectives concerning Polynomial Functions
Terminal Objective 1.3 Chapter 4 Polynomial Functions	To analyse polynomial functions of degree less than 3.	To analyse polynomial functions of degree less than 3.
Intermediate Objectives 1.3 Chapter 4 Polynomial Functions	• To draw the Cartesian coordinate graph (a straight line) of a real polynomial function of degree 0 or 1, given the equivalent rule of correspondence • To determine, from the rule of correspondence, the following information about a real function of degree 0 or 1: its rate of change, its x-intercept (zero), its y-intercept, its domain and range, its sign, whether it is constant, increasing or decreasing, and the member of its domain associated with a given image • To draw the Cartesian coordinate graph (a parabola) of a real polynomial function of degree 2, given the equivalent rule of correspondence • To determine, from its rule of correspondence, the following information about a real polynomial function of degree 2: its rate of change, its x-intercept (zero), its y-intercept, its domain and range, its sign, whether it is constant, increasing or decreasing, and the member of its domain associated with a given image • To use algebra to convert the rule of	• To draw graphs and determine the properties of functions of degree 0, 1 or 2 • To determine the relationships between changes in the parameters of the rule of correspondence for a function and changes in the corresponding Cartesian graph • To use algebra to convert the rule of correspondence for a quadratic function from general form into the standard form and vice versa • To determine the rule of a function of degree 0, 1 or 2 given the value of certain parameters, the coordinates of certain points, a table of values or graph • To graph the sum, difference and product of two polynomial functions.

	correspondence for a real polynomial function of degree 2 from the general form, $f(x)=ax^2+bx+c, \quad a\neq0$ into standard form, $f(x)=a(x-h)^2+k, a\neq0$ and vice versa • To determine the relationship between changes in parameters of the rule of correspondence for a real polynomial function of degree less than 3 and changes in the equivalent Cartesian coordinate graph • To determine the rule of correspondence of a real polynomial function of degree 0 or 1 represented by a straight line, given the slope of that line and a point on that line or two points on that line • To determine the rule of correspondence of a real polynomial function of degree 2 represented by a parabola, given the vertex of that parabola and another point on that parabola or given its zeros and another point • To graph the sum, difference and product of two real polynomial functions, given the graph or the rule of correspondence of each of these functions.	

One of the observations that can be made from the above comparison is that the objectives are identical. Distinctions between the two are more in the lines of wording. For example, the objectives appear to be summarised in the textbook. For the most part, it can be concluded that the objectives of the curriculum and the Math 436 textbook are consistent with each other.

2. The meaning of the concept of function in mathematics: A historical approach

As our understanding of any given topic develops, our need for more accurate definitions of objects becomes a priority, regardless of whether we are seeking to justify our thoughts, or simply wanting to proceed on to subsequent topics. Solow (1990), Borasi (1993), Norman and Prichard (1994) and Cuoco (1994), offer interesting discussions about the historical developments of the function concept based on the evolution of its definition. However, I was taken by the complexity of what at first seemed straightforward in mathematics; that is to say, definitions.

Solow (1990, p.29) describes a definition in mathematics as an agreement by all individuals concerned, as to the meaning of a particular term. He states that definitions are not made randomly and that they are usually motivated by re-occurring mathematical concepts. Those definitions, he continues, may consequently be viewed as abbreviations that are agreed upon for particular concepts. Possible difficulties involved in defining the notion of function depend on the level of rigor that is aimed at, as well as the multitude of ways in which definitions can be represented.

In Kelley's (1997) supplemental exercise book on the study of functions using the TI-83 graphing calculator, the author explores the function concept through a historical perspective on the highlights of its evolution in mathematics. His discussion is based on the development of the definition of function. Using a historical timeline, Kelley illustrates the progression of the increasingly generalized function definitions from three major time periods.

The first part introduces Euler's definition of 1734 as, "A function f(x) is any algebraic expression involving variables and constants defined by an equation or graph" (p. 7). This definition prevailed until 1807, when modified by Jean Joseph Fourier for the purposes of his own research. Fourier allowed himself to define functions by different algebraic expressions in different intervals. This would motivate mathematicians to formulate a more general definition of function.

As presented in Kelly's timeline, in 1837 Dirichlet provided the definition of a function as a correspondence. Dirichlet defined the permissible values of *x* as the domain of the function and those assumed by *y* as the range. "If two variables, x and y are so related that whenever a value is assigned to x there is automatically assigned by some rule of correspondence, a value to y, then we say y is a 'function' of x" (p.7). Kelley adds that this definition removed the need to define *y* as an algebraic expression in *x*.

As mentioned by Solow (1990), it is often the case that there appears to be two possible definitions for the same concept. In fact, definitions are often simply restated versions of past ideas. We can observe this, for example, from Cuoco (1994) who cited Brown's (1984) definition, which is in fact, a re-statement of Dirichlet's definition, in the context of modern day mathematics textbooks. (Function being considered is a real valued function of a real variable)

A function consists of the following:

- *A set of real numbers called the domain of the function.*
- *A rule that assigns to each element in the domain exactly one real number.* (p.126)

As confusing as the possibility of having more than one definition may seem, Borasi (1993) suggested that it might be beneficial for students to have alternative definitions for the same concept. This would allow them, she said, to make the important realization of taking into consideration the context of a situation when making a mathematical decision.

For the nature of this discussion and subsequent classroom observations, Dirichlet's definition of function will take precedence over any other, for, as Sierpinska (1992) proposed, Dirichlet's definition is sufficient for secondary level mathematics. For the students, Sierpinska continues, it does not make sense to introduce the general definition of a function before a certain level of education in mathematics is acquired because it will either be ignored or misunderstood. Specifically, the formal set-theoretical definition cannot be understood prior to the development, in students, of an awareness of the role and place of definitions in mathematics. This point brings into play, such as many other things in life, the question of timing, and particularly its importance in education.

This brings us to the third and final portion of Kelley's timeline. The generalizing of the definition of function came from George Cantor near the end of the 19th century. Cantor's definition generalized the function concept so that, "it is expressed in terms of elements of sets and is therefore independent of the concepts of number and variable", Kelley (p. 7, 1997). Cantor's definition was stated as follows, " a function, f, is any set of ordered pairs of elements such that if $(x_1, y_1) \in f$, $(x_2, y_2) \in f$ and $x_1 = x_2$, then $y_1 = y_2$. The set of all first elements of the ordered pairs is called the "domain" of the function and the set of all the second elements of the ordered pairs is called the "range" of the function" (p. 7).

This last definition, concerning the function concept, was the basis of a worldwide mathematics education reform of the mid 20th century, inspired by the work of a group of mathematicians called 'Bourbaki'. This set the stage for the infamous *New Math* movement of the 60's and 70's. This consortium defined function as a particular relation in the language of set theory form, similar to the one initially defined in the late 19th century.

More recently, in the 1980's, a new school of thought arose. The focus this time around would be on mathematics in context. Functions would be considered as models of *real-life* relationships. With a strong influence from psychology, the goal was perhaps constructivist in nature with hopes of making mathematics relevant to the students. Herein lies an inconsistency with Quebec's mathematics curriculum and Guy Breton's publications at the present time.

The current curriculum proposes to define a function as, "a relation in which each independent value from the source set is associated with only one dependent value from the target set". The Guy Breton Math 436 textbook states that, "a function is a set of ordered pairs that represents a relation. More specifically, a relation between two variables is a function when each value of the independent variable has no more than one corresponding value of the dependent variable". Explicit in both definitions is the idea that functions are relations. In the textbook, this notion is moved from an intuitive plane to the level of an independent mathematical concept. What is then done with this notion? I will further elaborate on this point in the subsequent section, referring to Sierpinska's (1992) essay, specifically about the importance of needing to distinguish between functions and relations as a condition on understanding functions. The decision, as she states, in the *New Math* movement, to reduce functions to relations could be neither didactically, or epistemologically justified. This opinion was supported with reference to Grize's (1968) ideas on the distinction between functions and relations. In fact, learning to discriminate between functions and relations appears in the third of the four stages of the didactic model of understanding as proposed by Bergeron and Herscovics (1982) that will be discussed later. Grize (1968) stated that, perhaps to a certain extent, where psychological operations are concerned, the notion of relation is more primitive than that of the function. Supported by these arguments, it would not seem to make sense that the authors of the Guy Breton textbooks series would, for example, eliminate the concept of function from the Math 416 level, by making strict reference to a variety of relations but omitting the term, hence notion of function. Furthermore, with the *math in context* philosophy, if the function concept, as supported by Grize, tends to rely on a more constructive approach, would it not make sense that a greater emphasis in the curriculum concerning functions would be made with hopes of creating contextual, thus more subject-relative materials for the students?

To continue with the discussion on definitions, it is also interesting how in both curricular and textbook definitions, the use of the term *set* is used; yet that same topic has been removed from the Math 436 program. There is an inherent inconsistency when the predominant school of thought regarding the focus of current mathematics education, on the meaning and context of mathematics, is contradicted by the use of mathematics definitions of past schools of thought. Specifically, with the explicit reference to the set-theoretic definition of the function that is used. An important lesson that was learned from the mistakes of the *New Math* reform of the 60's and 70's, involving the inappropriate balance between procedural work and conceptual work does not seem to have been taken into consideration during the development of the current mathematics curriculum, especially where the function concept is concerned. With this in mind, one cannot help but feel discouraged by the fact that such involved research in mathematics education often goes unseen by teachers and those studying to become teachers. If curriculum and textbook developers are inconsistent in their presentation of materials; how then are teachers and especially students expected to acquire a respectable level of understanding of the function concept? The danger with this is the unfortunate and faulty conceptual development achieved during the initial stages of function concept acquisition.

3. Teachers' and students' understanding of function

This section will discuss some of the difficulties encountered in the teaching and learning of the function concept. It will be divided into four sections. Continuing with the previous discussion on the definition of the function, section 3.1 will discuss the difficulties related to the formal definition of the function as well as its multiple representations in mathematics. Section 3.2 will be the first to explain students' difficulties using the Bergeron and Herscovics (1982) model of understanding followed by the APOS (action, process, object, schema) theory of learning in section 3.3. Finally, section 3.4 will consider Sierpinska's (1992) discussion, specifically concerning the prevalent epistemological obstacles encountered and the conditions of understanding when learning functions.

3.1 Difficulties related to the formal definition of function

As facilitators of learning, it is interesting to consider the ways in which teachers define function. Moreover, it is interesting to see how their knowledge is relayed to, and consequently interpreted by the students. Cuoco (1994) made reference to a six-week course for teachers, where discussions about the function concept arose. During informal conversations in that six-week period, most of the teachers felt compelled to give formal definitions to their students for the simple reason that most textbooks seem to adhere to those forms. Interestingly enough, when asked to define a function, Cuoco reported that an overwhelming majority of the participants described the vertical line test for $\Re$ to $\Re$ functions. A traditionally procedural method of indicating whether or not a relation is a function falls short of being anything more than a physical mnemonic device. Showing this to the students does not present a threat, as long as a conceptual link is made between this representation and the formal definition. Because of students' difficulties with the formal definition, the test is potentially substituted for the definition rather than acting as a tool for determining whether or not a given graph represents a function. If such superficial explanations are provided to the students with no association made to formal definitions, one should not be surprised of the re-occurring obstacles encountered during the teaching and learning of the function concept. Especially if some of these difficulties are experienced from possible teacher misconceptions, which could then lead to providing unclear and inaccurate explanations to the students. This statement is in no way directing blame to teachers, but to those responsible at higher levels, such as curriculum developers and authors of textbooks, who propose definitions that are often beyond the realm of most students' comprehension. How then, are we to expect students to perform at a level adequate enough to successfully complete high school mathematics, let alone expect them to have the potential to pursue a post-secondary education and career, where a fluency in mathematics may be necessary?

Definitions are, as Vinner (1992) states, often problematic in the learning of mathematics. Vinner conceives this possible instructional mishap, as the distancing of a mathematical structure from a student's cognitive process involved during the acquisition of a concept. From a constructivist viewpoint, Bergeron and Herscovics (1982) question how one can expect a student to understand a definition unless it is part of his or her existing knowledge. Furthermore, according to the didactic model of understanding, as per Bergeron and Herscovics, the function definition could be classified as some pre-requisite of the second stage of concept development, *initial mathematization*, thus forming some sort of basis for the development of the function concept. These didactic levels will be discussed in section 3.2 of this thesis.

Vinner pointed that the goals of mathematics education do not seem to be based on the understanding of the students' cognitive abilities. Losing sight of this important issue concerning students' abilities may help explain the deficiencies experienced in the teaching of mathematics, specifically with respect to the notion of function. Vinner also illustrated this point by describing the ever-growing gap between students' performance and expectations in the educational system.

Another difficulty, expressed by Eisenberg and Dreyfus (1990), concerns academic knowledge and its intricate networks of relationships. They suggest, that this difficulty is responsible for not allowing the presentation of subject matter to be presented holistically, but rather sequentially. The reason for this dilemma lies in the idea that teaching sequentially may make it easier for text development and classroom organization, whereas a more holistic approach may require very small group settings and the ability by the teacher to tailor programs according to the class that they are teaching on a daily basis. For example, while teaching functions, one teacher may observe that the class is, for the most part, grasping the material at a reasonable level and rate. On the other hand, another teacher may observe that the students are experiencing greater difficulty with the algebraic portion of the work. The latter would have to most likely add a review of algebra in order to bring the students back to par, which would take time away from the sequence of events while the former teacher may choose to introduce more difficult functions to the class in hoping to challenge them more.

3.2 Explanations of students' difficulties provided by the Bergeron and Herscovics didactic model of understanding

During a conference on functions in 1982, Bergeron and Herscovics presented a discussion and paper on the *Levels in the Understanding of the Function Concept.* Taking a constructivist approach, these two authors proposed four levels to describe both the processes leading to the construction of conceptual schemas and the results of such constructions, that is, the different levels of understanding (p. 39, 1982). These levels are important as a theory of learning because they "...start from the pupil's intuition and experience, each one of the following levels of understanding is built on the preceding one" (p. 45, 1982). As mentioned in their essay, their model of understanding described the construction of a specific function. Bergeron and Herscovics (1982) stated that their model could be generalized and therefore applied to the construction of the function concept. In fact, the conclusion to the study presented in chapter III will associate these four stages with the questions of the test administered to the students and how they can be viewed through this model of understanding. This model brings forth didactical issues and has pedagogical implications.

They termed the first stage *intuitive understanding*, as it only involves an informal level of mathematics characterized by pre-concepts. It is described as a first step towards a process of mathematization, which is responsible for the organization and coordination of the students' informal knowledge. At this level, the students are expected to think based on their visual perception where precision is not emphasized since students' actions simply provide them with estimations. They deem this level to be important as a starting point towards understanding, based on the conceptual schemas that are taught in schools.

The second stage is that of *initial mathematization*, which is "essentially a procedure which leads to the construction of mathematical concepts by which are meant mathematical objects such as numbers or mathematical transformations such as operations (p. 40)". This level is considered the first step towards the construction of a concept. It involves, with respect to the function, all the possible representations that allow for the construction of the concept. Furthermore, as Bergeron and Herscovics (1982) cited from Herscovics (1979), linear equations provide the students with the opportunity to practice moving from one representation to another. This, he continued, would allow students to generalize their knowledge to non-linear equations in future courses.

The third stage is that of *abstraction*. A distinction is made between empirical abstraction and reflective abstraction. The former deals with the physical properties of objects, whereas the latter is associated with the coordination of actions. At this level, notions of independent and dependent variables, and domain and range take precedence. Finally, the two main ideas involved here are, as discussed earlier, the necessary distinction between a relation and a function, and that of dependence.

The final level in Bergeron and Herscovics' didactic model of understanding is that of *formalization*, which considers the symbolic nature of mathematics. The use of symbols, logical justification or the discovery of axioms takes place at this level. However for these to occur in students, the authors emphasize the importance of having been able to abstract the information previously acquired. The authors believe that if abstraction does not take place, definitions and notation, for example, would be meaningless to the students since the different aspects of certain notions may not have been formed as part of their existing conceptual knowledge. The conclusion to the experiment presented in chapter III will associate these four stages with the questions of the activity and how they can be viewed through the perspective of this model of understanding.

3.3 Explanations of students' difficulties provided by the APOS theory of learning (action, process, object, schema)

With an underlying constructivist foundation, this theory is inspired by Piaget's work. It describes the initial actions taken towards processes that are aimed at constructing objects that would ultimately form a conceptual schema of a notion.

3.3.1 Action

An action occurs when a subject, through a single-step or multi-step sequence of responses, physically or mentally transforms one or more objects (Dubinsky, 1997). An example of a single-step action that may come as a direct response to a stimulus such as the question, "What are the zeros of $y = x^2 + 5x + 6$?", is to find the zeros of that quadratic function through a learned algorithm.

For a multi-step sequence of actions, Dubinsky believes that the *next step*, coming from the student, must be triggered from the previously performed steps, rather than from an overall conscious control. In the case of functions, for example, a student may be asked to determine the rule of correspondence of a parabolic function given its graph, a point on it, as well as the vertex. The student may have to recall or have to look up the general equation, $y = a\,(x - h)^2 + k$. Given the information in the problem, he or she may realize that the value of a needs to be determined. The result may be obtained with the student blindly substituting the given values into the equation. At this point, the equation involving a as the only unknown value, may trigger the need to perform some algebra, initially by plugging in the numerical value obtained, followed by the same action with the vertex back into, $y = a(x - h)^2 + k$. Dubinsky states that when an action becomes repeatable for a student, based on several similar experiences, it is referred to as an action scheme.

3.3.2 Process

Upon reflection of this action scheme by the student, the evolution of a process begins. Dubinsky continues by stating that when a student establishes control over this action scheme, it can be considered as a part of that individual. The construction of a process from an action is referred to as *interiorization*. Here are two of the characteristics of a process; when two or more processes are associated, a new process is obtained and a process is reversible in the mind of an individual. Concerning functions, an example of reversibility could be when the student is asked to determine whether or not the inverse of a function is itself a function. By either verifying this through a graphical representation through some algebra or using a table, the student is able to express the necessary conditions for accepting or rejecting the hypothesis. Dubinsky claims that, perhaps the most important point of a process is when an individual can think about a transformation by imagining it, without actually performing it. I view this as being able to verbally, through writing or mentally, paraphrase a specific item. In doing so, the individual is then able to generalize that specific item.

For example, in the context of functions, being able to interiorize an action to form a process may arise in the following situation. Given a contextual math problem, such as the parabolic trajectory of a baseball that has just been hit, the question may be to determine how long the baseball will be in the air. The student might initiate such a thought process, 'I need to express this situation using the equation of the parabola either in general or standard form. Given the information about the maximum height reached and the initial height of the ball, I will then find the specific equation, set it equal to zero and then factor it to find the zeros.' The student here is able to determine what the equation of this situation will be without necessarily having determined its parameter, a, or having substituted the given points into the equation, but having ultimately identified what the question was. These would be considered as constituting operations on processes. Further reflection on them will eventually lead to the construction of objects.

I believe that if the process described in this example is actually broken down into individual steps, and viewed as separate items, then these steps would be reduced to actions. The distinction between an action and process in this example can be made if, for example, a student initially performs the necessary steps without reflecting on the possible outcome. This would be considered to be an action as opposed to a process. That is, identifying the given values, substituting them into the general equation, performing some algebra and finding the zeros through some factoring are considered to be single or multi-step actions.

3.3.3 Object

The ability to reflect on operations that have been applied to a process tend to lead to a more holistic awareness of such a process. Followed by a realization that the actions or processes may act on and finally be able to construct that process, suggests that the student is thinking of this process as an object (Dubinsky, 1997). When such a sequence occurs, the subject is said to have *encapsulated* the process in the construction of an object. It is equally important to be able, as emphasized by Dubinsky, to *de-encapsulate* an object back to the process from whence it came (even doing so back and forth).

Encapsulation in the context of functions could be when a student is able to distinguish between a function and a relation. Specifically in being able to determine that functions are special types of relations where the conditions, for example, on the correspondence between two variables become more important when distinguishing between the two. The ability to provide examples of relations or functions using their different representations along with some knowledge of their properties indicates that students are closer to having constructed such objects in their mind.

Dubinsky introduces two difficulties that arise when distinguishing between the two initial stages of this theory, the action and the process. He considers a continuum involving actions and processes. The difficulty, he claims, is that during observation, mixtures of both the action and the process can be detected in individuals. These mixtures may be very difficult to observe in being able to make explicit distinctions between where the student is in terms of the action or the process. The second difficulty described by Dubinsky is,

> "... that the property of being an action or a process is not found completely in either transformation or the individual, but in the relation between the two. We shall analyze this relationship in terms of the extent to which the subject is part of, or is controlled by, the transformation (action) or the extent to which the transformation is part of, or is controlled by, the subject (process)." (p. 95)

Dubinsky concludes that the action/ process/ object/ schema is not to be considered a *one shot affair*. Upon construction, objects can undergo other transformations, in a spiral-like fashion, thus creating higher-level actions and then processes. He claims that lower level constructions are not lost but remain as part of an enriched conception developed from the reconstruction of specific actions, processes and objects based on new higher level experiences. Hence, allowing for the interiorization of actions and encapsulation of processes.

3.3.4 APOS theory and the function concept

Attempting to relate the APOS theory on learning to the function concept has been made reference to in many research papers. For example, Nichols (1992) asks perhaps the most obvious question of all, what is a function from a cognitive point of view? Nichols suggests four classifications for the function, the first being a prefunction, which constitutes very little of a concept. It is followed by the function as an action, basically consisting of procedures and having little to do with processes. The third views the function as a process, thus bringing into question the inverse of a function. Finally, Nichols considers functions as unknowns. According to Nichols, an action is any repeatable physical or mental manipulation that transforms objects (e.g. numbers, geometric figures, sets) to obtain objects. The process conception of a function involves a dynamic transformation of objects according to some repeatable means that, given the same original object, will always produce the same transformed object.

Based on teachers' conceptions, Cuoco (1994) reports that many of the methods used to present functions are usually action or process descriptions. This draws parallels to the ways in which teachers tended to define functions as described earlier from Cuoco's research. Cuoco describes students, who view functions as actions, as thinking of a function as a sequence of isolated calculations or manipulations. Cuoco continues by describing the students using the process conception as those who think of functions as dynamic transformations that can be composed with other transformations. These students seem to be leading towards the construction of objects.

Cuoco also seems to stress the importance of being able to move back and forth between viewing the function as a process, then as an object, and vice-versa, actions still remaining as part of this enhanced conception, as mentioned earlier in the concluding discussion referring to Dubinsky. Accepting these two ways of viewing a function creates an environment whereby a student is not constrained to a single perspective. Ultimately, Cuoco suggests that the view of a particular function by an individual depends on that specific function, how familiar it is and how it is being used. Based on a constructivist-type philosophy, this idea of relative subject matter is key for success in mathematics and is consistent with ideas presented using the APOS theory of learning. From my experience as both a teacher and a student, and based on the literature, it should not be surprising to read that generally, students blindly apply algorithms, (Eisenberg, 1992). As Williams (1998) illustrated, from her experiments concerning semantic mappings of the concept function, most students' maps consisted primarily of algorithms with very little indication of concepts or even the relationships connecting them. From such results, Tall (1996) reports that conceptual connections are less likely to be made when routines are ultimately reduced to being just that, routines. He continues by saying that students will find it more difficult to answer questions that are conceptually challenging. The examples described by these authors seem to indicate that students often perform only at the level of action. Concerning the notion of function, those algorithms that have been repeated several times by students, in developing an action-scheme, may perhaps indicate that processes are evolving. However, it is not encouraging when considering the potential construction, or lack thereof, of the function as an object.

3.4 Epistemological obstacles and cognitive difficulties of the function concept

The concept of function, as stated by Holler and Norwood (1999), is considered to be one of the most central concepts in all of mathematics. For this reason, striving to acquire a better understanding of this concept, from pedagogical, didactic and research perspectives is crucial. This section will bring forth some of the difficulties, made explicit through research, that are experienced by students and teachers during the learning and teaching of functions. The focus in this portion of the discussion is on conceptual and procedural aspects of the notion of function. What are our expectations, both procedural and conceptual, of our secondary level students' mathematical abilities? Ultimately, as Sierpinska (1992) questioned, what do we want understanding to be?

Concerning the understanding of the function concept, Sierpinska suggested that one of the very first conditions to be met is the recognition of the problems surrounding changes and the relationships between them, as subject matter worth studying. It is also important, she continued, to bring the students to perceive and verbalize the subjects of change by emphasizing how they change, and furthermore, what it is that changes.

In tying some ideas discussed earlier concerning definitions and theories of learning together, Sierpinska stated in her essay that one of the epistemological obstacles that must be overcome has to do with the conception of the definition. Although the function concept can be defined in a pure and formal symbolic manner, without the use of words from natural language, she stated that a definition is often taken as a description of an object: the definition does not determine the object; rather the object determines the definition. With respect to understanding, she suggested that the discrimination between mathematical definitions and descriptions of objects is necessary. This would indicate being at the level of formalization in the model of didactic understanding of Bergeron and Herscovics (1982), and would involve the ability to encapsulate and de-encapsulate objects and processes described earlier in the APOS theory.

Arguments presented by Nichols (1992) suggest that people perform better in producing something specific rather than understanding what they are actually doing. Similarly, the research of Meissner and Phillipp (1993) suggests that students tend to have a more global and intuitive approach to problem solving. Students are not conscious of a formula or function, Meissner and Phillipp continue, but they know procedures and how to compute to obtain results. These ideas support Eisenberg's (1992) suggestion concerning the ease of getting students to perform. As he concludes, however, getting them to understand is much more difficult.

Eisenberg made three distinctions between visual and analytic understanding. First, cognitively speaking, visual understanding is more difficult to attain. Second, from a sociological perspective, it is more difficult to teach visually. Third, where rigor is concerned, visual understanding is not necessarily considered to be mathematical. The epistemological obstacles and acts of understanding concerning the function concept discussed in Sierpinska's essay could, in fact, be viewed in either of these visual or analytic contexts.

Eisenberg explains the difficulty with visualization, using the example of function transformations, as being an action. The result of considering it as an action rather than as a process, he explains, is the reason for students' poor visual understanding. Although he suggested that students solving visually seem to demonstrate better understanding, various representations are only useful to those who know the appropriate computational processes for taking advantage of them (Dreyfus and Eisenberg, 1990, as cited from Larkin and Simon, 1988).

With respect to visual representations, however, Eisenberg shares the view that students do not necessarily have the appropriate visual understanding of the function concept. Lloyd and Wilson (1998) also suggest that, understanding in one representation does not necessarily imply understanding in another. Based on these arguments, visual understanding may predominantly be a result of visual representations. The ability to make sense of a visual representation may be partially responsible for the gradual visual understanding of a given concept.

As Piaget et al. (1977) assert, the construction of the function concept, particularly from a cognitive standpoint, is difficult. This is documented throughout the research, but as Tall (1996) optimistically suggests, the fundamental nature of the function concept is beginning to be seen in a more realistic light of cognitive development. The cognitive and conceptual difficulties are beginning to be better understood, Tall claims, even though research is proving to be more problematic in catering to them.

Vinner (1992) proposes that the acquisition of a concept derives from building a concept image from its name and that thought processes are guided by it rather than by the concept definition. To understand this concept, he continues, means to have a concept image for it. Vinner also claims that people remember visual aspects of a concept better than its analytical counterparts. This claim is quite general since remembering implies memory by rote. Although a definite skill necessary in mathematics, remembering does not necessarily provide one with a concept acquisition. In fact, as mentioned earlier, such procedure is responsible for stifling those students who consistently perform actions.

A major difficulty illustrated by Vinner (1992) describes the use of a concept as a tool. Whether visual or analytical, using a concept as a tool, as Sierpinska (1992) indicates, can and should only be used to serve functional purposes. Eisenberg (1992) suggests that a concept is internalized when doing procedural work. Procedural work cannot be ignored, yet it often appears to take the form of an algorithm that has little to do with conceptualization. Furthermore, Vinner (1992) states that the original meaning of a concept is forgotten when instrumental aspects of the concept take the place of the concept itself. This dilemma of internalization, risking conceptual loss, may be avoided by explicitly distinguishing between the tasks being performed mechanically and the implicit conceptual notion being considered.

As taken from Vinner (1989) and Tall (1991) in Aspinwall et al. (1997), although visualization is thought to be essential in understanding mathematics, studies have consistently shown that students' understanding is typically algebraic and not visual. How does one compensate for the seemingly natural and overwhelming student tendencies to favor algebraic work? With respect to this question, Dreyfus and Eisenberg (1990) suggest a reason as to why compensating is difficult, which may in turn help explain that question. They hypothesize that students experience discomfort with visualizations because they have not constructed the appropriate cognitive frameworks in which to think of them. Another possible reason suggested by these authors (1990) relies on the notion that visual processing is anything but linear, and as such, represents a higher level of mental activity than algebraic processing does.

Students have difficulties with representations, as stated by Sierpinska. Furthermore, based on their extensive research, two of the frontrunners concerning graphing and visual understanding of the function, Eisenberg and Dreyfus (1992, 1994), provide similar facts. Eisenberg and Dreyfus (1994) stress that the Cartesian graph is an important aspect for the understanding of functions. Since graphical representations help provide an overview, as suggested by Bergeron and Herscovics (1982), some understanding may be lost if the data being considered is left in tabular form. Representations of functions are obviously not independent of each other; therefore, at the initial stages of learning about functions, these representations must be explicitly associated.

Yerushalmy (1997) stated that the primary reason for using graphs is to provide a visual terminology with which to think and discuss. From her discussion, Yerushalmy believes that the uses of graphs are a reflection of students' personal goals and interests. This reveals thoughtful intention rather than rote operation on their part. As she describes, graphing is a *live* language that uses many dialects. Such a description is a further indication that different students approach graphing differently. Graphs should be used to complement students' thinking, in addition to associating them with other representations.

In many ways, aspects of the function concept need to be made more explicit. The difficulty, for teachers, in providing more information about these different aspects and their relationships with each other, is the consideration of the conceptual readiness of the students. This may be the reason why, traditionally, much of the work concerning the function remained procedural, and to a certain extent, superficial in nature. Because of the difficulty in teaching conceptually, teachers and students are often communicating on levels that are too far apart. A teacher's responsibility is to reduce this didactic gap. That is why teaching procedurally has always been easier, the *do not ask why, but how*, type of approach.

Algebraic constructions, as stated by Bergeron and Herscovics, have proven to play an essential role in the construction of functions. Sierpinska also wrote that the understanding of functions becomes very difficult, if not impossible, without an appropriate level of algebraic awareness. Algebraic representations of functions being expressed as equations, for example, allow a different level of understanding to be achieved. Such understanding, however, may be lost given procedural and conceptual difficulties in algebra. Awareness of the multiple representations of functions is important yet incomplete without some familiarity of the relationships existing between them.

Chazan (1993) discussed ideas relating functions and their equations. His focus was on functional relationships of equations, suggesting that the equation is a particular type of a comparison between two functions. On the other hand, he also implied that functions appear to be special kinds of equations. The functional relationship of equations such as $y = ax + b$, expressing a rule of correspondence, he suggested, indicates that functions are at a higher conceptual level than equations due to the existence of their multiple representations. He further made the distinction between comparisons of two functions such as, $ax + b = cx + d$, as being questions, and forms such as, $f(x) = ex + g$, as algebraic rules expressing the correspondence between the elements of two sets in a single function. Hence, $f(x) = ex + g$, is not an equation since it is not a comparison of two functions. As Chazan suggested, viewing equations as functions as opposed to expressions representing unknowns may make algebraic modeling more natural and coherent for the students, due to the dynamic dependence relationships of functions.

From an algebraic perspective, MacGregor and Stacey (1993) emphasized the ability to perceive a relationship followed by its algebraic formulation is fundamental in being able to use algebra. This formulation, as a result of algebraic knowledge, indicates a direct relationship between algebra and functions. From their discussion on pattern recognition, tables, algebraic rules, and the importance of verbal descriptions, these authors seem to suggest that students' recognition of functions can improve their algebraic sense. However, it is the students' lack of confidence with symbol manipulation, as Tall (1996) suggests, that is responsible for unnecessarily accentuating their difficulties while learning.

The consideration of the inverse of a function is necessary for progressive understanding of the function concept. As many concepts in mathematics, the inverse of a function cannot be understood in a simplistic manner, (Even, 1990). Moreover, considering the inverse, as an act of undoing, she continues, is powerful, yet insufficient when dealing with aspects of its conception. That is, both may remain as actions. Slightly more boldly, Tall (1996) suggests that the theory of functions (in calculus) can be summarized as the study of the doing and undoing of the process involved.

Typically, teachers have demonstrated, as have the textbooks by means of a basic algebraic procedure, how the inverse of a function is found. Aside from developing a student's ability to substitute and improve algebraic performance, overemphasizing these types of algorithms appearing in traditional textbooks does very little for concept development. What is the purpose and meaning of substituting the x for the y variable, followed by the isolation of the y variable? Apt pupils all know how to do this, but conceptually, what does it offer them? Do they *understand* what the inverse of a function really is?

3.5 The graphing calculator and the function concept

This section will summarize a paper by van Streun, Harskamp and Suhre (1999), which is the second analysis of a two-tiered experiment concerning the effect of the graphing calculator on students' solutions. Some of their observations and conclusions were confirmed in the observations conducted for this dissertation. The aim of their study was to determine whether teachers could integrate the graphing calculator properly in mathematics education in efforts to assess students' achievement. In their experiment the graphing calculator was used to explore graphs and to find graphic solutions for different kinds of functions. Their research would test and confirm their hypothesis that, "long-term use of the graphing calculator leads to enrichment while short term use leads students to substitute algorithmic and heuristic approaches by graphical approaches (p. 29)". In fact, "... after one year of graphic calculator use students not only use a graphic approach more often next to the algorithmic approach, but they also acquire a better understanding of functions (p. 32)".

The issue of the integration and implementation of the graphing calculator on a large scale in mathematics education brought up two points in their discussion. The first concerned the fears that teachers experienced about the negative impacts that the graphing calculator might have on students' mathematical knowledge. The other considered the high expectations of the potential of this tool by innovators. These concerns are well justified and I suspect they encompass many other restrictions and expectations of such an elaborate tool. The full impact of the effects of the graphing calculator remains to be seen in mathematics education.

Based on a study by Ruthven (1990) that was cited in van Streun et al., three types of approaches were used by the students while solving mathematics problems about graphs and functions. The first was algorithmic in nature, where students applied procedural knowledge from their textbooks without using the graphing calculator. The second was a graphical approach, in which the students would analyze graphs to try and find functions for them. The third approach, a purely heuristic one, saw the attempts of students to find functions for graphs, simply by trial and error.

Both Ruthven's study and van Streun et al.'s study observed that students using the first approach did not do as well as the others on visual representations. However, the students' mastery of algorithms in that group, allowed them to be, generally, more successful than the other two groups using graphic or heuristic approaches. The stronger students made use of the first two approaches most often. Another one of their observations was that the graphic approach was helpful in bridging the gap between the relation of functions and graphs. They claimed that the students using the graphing calculator were stronger on three fronts. They performed better on open questions about the properties of functions; secondly, they were better able to relate graphs and their respective functions; thirdly, they were more successful on word problems and equations.

The availability of the graphing calculator caused students to use the graphic approach more often. Given earlier discussions in this chapter, this may not necessarily be a good indication, since on the one hand, it was confirmed that graphical solutions lack mathematical rigor, and on the other hand, based on the extensive research of Eisenberg and Dreyfus, students consistently experience difficulties with visual representations where understanding is concerned. Van Streun et al. add that there are other problems that arise from the use of the graphing calculator. For example, difficulties in finding correct formulas, and determining ranges of graphs and their precision of estimates. Also, the use of the graphing calculator was generally less effective for students using heuristic approaches.

One of their recommendations concerning the role of the graphing calculator was that it would help bridge the gap between the procedural concept of function and the structural concept of function. They also found that the graphing calculator had positive impacts on generally weaker students. One of the pros of the graphing calculator as cited from Ruthven was that it enriches students' graphical repertoire.

The article by van Streun et al. (1999) reviewed in this last section of chapter I, was used to illustrate some of the ideas and results obtained, which relate strongly to this thesis. They were merely summarized and little analysis was made of their work because of similarities of this work that will become apparent in the subsequent chapters. Their article, however, will be discussed in the concluding chapter of this thesis.

Final Remarks

In conclusion, educators and researchers may find themselves in a bind, since on the one hand, it has been demonstrated that analytic knowledge has been reduced to procedures with little emphasis on concept. On the other hand, although visual aspects of understanding have been reported as providing better conceptual knowledge, they have also been shown to provide much difficulty in attempting to communicate knowledge in this fashion. The underlying question throughout the remainder of this thesis will be, "does the graphing calculator act as an asset or as a liability to the students learning about the function concept?" Its didactic role shall also be questioned.

Understanding the function concept, as Nichols (1992) suggests, must include a process conception if understanding is to overcome the mere manipulation of formulae or diagrams. It is evident that the mathematics in education is different from the mathematics done by mathematicians, (Cuoco, 1994). Attempts to reduce this gap, Cuoco continues, often depend on the differences between the way students work and the way researchers work, and ultimately, proposals for mathematics education reform concentrate on changing the ways students acquire mathematical knowledge.

There will have to be a shift, from procedural descriptions of functions, to more conceptual ones. The need to transfer a major portion of the emphasis on concept development, from traditional procedural work, can be done. However, this must be done at the level of curriculum development and, more specifically, in the preparation of materials, such as textbooks. In discussion with a colleague, we questioned the integrity of the current textbooks being used as a consequence of this reform. At the high school level, should not textbooks be intended for people who are not yet familiar with the given subject? The material in the series of textbooks currently being used in Quebec, suggest that it can be quite overwhelming if one places oneself in the place of students.

The following is a series of general questions that were inspired from the readings, the observations described in chapter II and especially from the test to be discussed in chapter III. They will question the integrity of the graphing calculator in the classroom. Will the graphing calculator improve the teaching and learning of the function concept? Will it allow students to learn faster; hence will teachers spend less time on this topic?

Will multiple representations on the calculator reduce the ever-present difficulties in visual understanding? Keeping in mind the ability of the calculator to do such work as the generating of tables and the drawing of graphs, thus potentially reducing procedural work on the part of the student; will this increase the potential for conceptual understanding? Will the use of this calculator, allow more time for discussion due to its ability to perform procedures? Will it tend to focus or favor one representation over another? Can certain aspects of the function concept be taken for granted? Will certain aspects be better emphasized? The research will attempt to address these questions through one teacher's methods of using the graphing calculator in the classroom.

Optimistically, Cuoco (1994) states that the learning environments evolving from the current interest in the function concept will eventually make the mathematicians' view of the function more accessible to a greater number of students. Will the graphing calculator be responsible for this increased accessibility? Such a significant aspiration of mathematics education research may reassure our confidence when answering one of Vinner's (1992) questions, "What will remain in our students' minds after the end of the course and final exam?"

Chapter II

Observations of a Graphing Calculator-Based Mathematics Classroom During the Introduction of Functions

Introduction

To describe the setting for this study, a total of seven fifty minute periods were spent observing a math 436 class of an English middle class suburban high school over a period of approximately twenty days. For the purposes of this thesis and subsequent activity, these observations began at a very opportune time as the class was beginning to study functions. The class was well equipped with a class set of TI-83 graphing calculators, in addition to the teacher's calculator whose display screen was capable of being projected on an overhead (with the appropriate adapter).

Set in a traditional classroom, this was a class of approximately twenty-three predominantly bilingual students. The teacher described the class as being *good*. It is important to note that the students had math everyday. The teacher has had at least fifteen years experience, was actually a pioneer in this community in the teaching of mathematics with the graphing calculator and co-authored a supplemental text concerning the use of the graphing calculator and functions.

Referring to his own experience, he told me that in the preliminary stages of working with the TI-80 several years ago, it was mostly being used as a tool to reduce procedural tasks as opposed to using it towards more conceptual goals in understanding functions. He stated that his mistake in the past was to use the calculator to introduce a topic. Doing this left much room for students' misunderstanding in mathematics. He found that, initially, it is more important to use traditional paper and pencil methods to develop a stronger foundation in correctly interpreting the calculator's output.

This teacher has definitely learned from experience. One of my first observations concerning his teaching dealt with the seemingly perfect balance between the use of the graphing calculator and traditional paper and pencil methods. From appropriately distributed time, to providing examples and demonstrations using each approach, students were consistently shown that problems could be solved with or without the graphing calculator, thus reducing the students' potential dependency on such a tool.

Most of the seven periods were spent using problems from stencils and not from the Guy Breton textbook. Some of which were from the respective textbook's teacher's guide. According to the teacher, "the topic of functions is not covered very well in this textbook". In fact, where the textbook is concerned, the teacher went from chapter 2 (algebra) to chapter 4 (polynomial functions) and omitting chapter 1 (functions). The intentions of the curriculum only meant for chapter one to be skimmed. In this specific class, as described by the teacher, the students have shown dissatisfaction with skimming topics; for this reason, chapter one was omitted.

Generally, students did not seem intimidated by the teacher's questions. His method of teaching was a perfect example of the *let's talk about it* approach. Very little time was spent on theory, rather the teacher hoped to explain many of the ideas through examples. I realized near the end of the seventh class that the teacher had a very interactive approach; he talked *with* the students and not *to* them. There consistently appeared to be appropriate mathematical discussion. Seldom did the teacher involve me in class discussions, and I was seated in different places on any given day in attempts to blend in.

Journal of observations

Day 1 (Tuesday, November 23, 1999)

The first question presented to the class, on the overhead projector as well as on a stencil, was a multiple choice item taken from a previous Cayley examination[1] (1998, #19).

Mr. Anderson has more than 25 students in his class. He has more than 2 but fewer than 10 boys and more than 14 but fewer than 23 girls in his class. How many different class sizes would satisfy these conditions? The choices of answers were 5, 6, 7, 3 and 4.

The teacher, however, did not discuss it to the class in that form. He provided the students with a graphical representation of this situation, which was in fact the solution. The teacher's intention with this activity was to graph these constraints as a system of inequations in the graphing calculator.

In preparing for this activity, the class appeared to be well organized and demonstrated a respectable level of teacher-students interaction. From retrieving the class set of calculators and distributing them, to setting up the overhead projector, everybody shared the tasks. The teacher, however, was not controlling the overhead calculator; this was the duty of the students who alternated daily, on a voluntary basis. This strategy allowed the teacher to focus on the class discussion and his explanations.

Approximately fifteen minutes were spent on the use of the graphing calculator for this activity. One of the first questions posed by the teacher involved the input of $x + y > 25$ in the calculator. The student response was that, "you have to input the

[1] The Cayley examination is part of the Canadian Mathematics Competition series of exams created by the University of Waterloo in Ontario.

equation $y = 25 - x$." The teacher suggested, in question form, the input of $y = -x + 25$. The teacher continued by verbally identifying the slope and the y-intercept.

Two of the constraints in this question involved the graphing of vertical lines. The teacher mentioned the difficulty in graphing these lines, $b = 2$ and $b = 10$. At this point, a student stated that an appropriate window must be set. The teacher set the window and continued with the graphing of these two relations using the **draw** function by selecting the **vertical line** option from the menu. The functions of the TI-83 graphing calculator will be in bold font throughout this thesis. In fact, this function makes the plotting of such relations quite simple on the graphing calculator. Since the solution to this problem could only involve integers, the next issue that was brought up by the teacher involved whether the constraints $g < 23$ and $g > 14$ should be represented by the lines $g = 22$ and $g = 15$. The teacher explained that this would be the most appropriate approach since there cannot be fourteen or twenty-three girls, due to the strict inequality. During all of this, an interesting point concerning the storing and recalling of graphs in the calculator was raised. A student, described as being very strong by the teacher, at the overhead, explained how one could store and recall such a graph using the **picture** function.

Day 2 (Thursday, November 25, 1999)

Moving towards discussions involving quadratic functions, the teacher used the textbook (p. 74), as well as the overhead projector as the main tools for this lesson. The first question to be discussed was number 18 on page 268. The goal of this item dealt with the interpretation of the obtained values and the necessary distinctions that must be made between them when considering their relativity to the given problem as opposed to their consideration as a mathematical model.

Veronica is hired by a cosmetics company as a representative. She must pay $340 for her product kit. The company pays her a 20% commission on her sales.

a) *How much must Veronica sell if she wants to recuperate her initial investment and earn a salary?*
b) *Can the rate of change of this relation be found by simply reading the statement of this situation?*
c) *Is this relation a direct variation linear function? Explain your answer.*
d) *Graph this function.*
e) *Determine the signs of this function over its entire domain.*

A few students' initial reaction was to take out the TI-83. At this point I also observed that at least half of the class owned a graphing calculator. As the function rule, the teacher suggested, $C(s) = 0.2s - 340$. He then continued by drawing the first quadrant of the Cartesian plane on the overhead projector with cost as the dependent variable and number of sales as the independent variable. The students then entered the function rule in the calculator using the **equation editor**. Pressing the **graph** button showed nothing on the screen. Some of the students' first reaction was, "it did not work". The reason for this as explained by the teacher was that an appropriate window was not yet selected. He suggested that the values of the domain lie between [0, 10000]. With respect to the range, the teacher proposed values of [-400, 1000]. In this example, the teacher was responsible for having selected the appropriate window. From this moment on, the teacher ceded his place to a student volunteer to take over at the overhead with the graphing calculator.

The discussion continued with the teacher predominantly leading. With a series of questions with the goal of analyzing this situation, the teacher began by asking, "where is the break-even point?" The *x-intercept* was to be interpreted as the break-even point. The **calculate** function with **x-intercept** from its menu was selected as per the teacher's suggestion. At this point, the student at the overhead projector was receiving verbal aid from a few other students from their desks. With an output of (1700, 0), the teacher showed dissatisfaction and suggested that the class change the significant figures of the result using the **mode** function and changing the rounding off feature from **float** to **two** decimal places. Following this, the teacher focused the discussion on the use of vocabulary. This was interesting in terms of necessary translations between mathematical language and that of the graphing calculator's functions. For example, he asked the class how the rate of change and initial value (at s = 0) might be interpreted in this real life situation. This line of discussion brought into question mathematical situations versus real life situations, where certain values considered in the former may not necessarily be valid in the latter.

A brief description by the teacher on transferring the graphical output from the calculator to paper followed. He did so using the intercept method of graphing. At his point the teacher directed a comment to me with regards to learning how to use the TI-83. He suggested that students are usually quite able in deciphering the calculator's functions very well on their own by trial and error and deduction. For this reason, the specific teaching of the functions of the graphing calculator may not be mandatory. It is most likely that the students will acquire the necessary skills in the graphing calculator environment as they move from topic to topic.

To clarify and bring into question some theory of functions to the class, the teacher asked, "What is the domain?" Such a question may be open to interpretation depending on the level of rigour expected. One student's response was, "it represents the possible values of x." The teacher accepted this response but then he asked for a numerical answer. This is what was meant by the intentions of the teacher's question. In studying more properties of this function, the teacher asked, "Where is the function positive? Negative?" Following a brief moment of silence, the teacher writes that this function is negative over the interval [0, 1700] and positive over [1700, ∞). A student then asked, "Why is the interval not open over the positive part?" That is, from 1700 onwards. The teacher also posed the question in a graphical manner as, " where is the graph above the *x-axis*?" A student then asked, "What is the range?" This was answered by another student as [-340, ∞[. It is important to keep in mind that these values for the domain and range as well as the positive and negative intervals were mathematical in nature.

This discussion concerning some of the properties of functions helped the teacher conclude question 18 with another series of questions and brief answers (due to the lack of time). These questions were, "Are there any extremes?" and "Is there a maximum or minimum?" He specified by asking, "In terms of *y*?" At this point, few distinctions were made between the properties considered from the mathematical model and the given problem. This concluding discussion to question 18, however, seemed to focus on the former.

The following question to be discussed was a modified question 19 on page 269. It originally asked to list all the properties of the function associated with the given situation. It is not necessary to state the problem since the teacher only used the given information for the purposes of determining the slope. The first series of questions given by the teacher were in paper and pencil form and he asked, "What is the equation? What is a reasonable window?" (The teacher used the **table set** function). And finally, "How do we calculate the rate of change from the calculator's table on paper?" As a review, the teacher stressed the use of $(y_2 - y_1) / (x_2 - x_1)$, as opposed to the simplified reply of a student of, $\Delta y / \Delta x$. With the introduction of this algebra, the class began to show some difficulty. The teacher then took some time to review this on the overhead. The purpose of this item was to calculate the slope using the above formula given values obtained from the table. It was surprising that there was no mention, by the students, that the slope was already obtained, given the equation in functional form describing this situation. I suspect that the teacher wanted to demonstrate the ability of the graphing calculator to generate tables.

Following the review, the teacher did some foreshadowing with the following question, "Can the table of values determine whether the function is linear?" This was followed with no response from the class. The teacher concluded with a paper and pencil calculation of the initial value, performing a substitution of a point from the table and the slope into the equation that was obtained. The intentions of the teacher here appeared to emphasize the role of algebra when determining the parameters of an equation of a function. The graphing calculator was used intermittently during the whole class period today, and the class ended with the teacher offering lunch help.

<u>**Day 3** (Tuesday, November 30th, 1999)</u>

The topic of the day was parallel, perpendicular and coincident lines. The old *Addison-Wesley* textbook, alongside some good old-fashioned paper and pencil work was used. The reason for this choice, according to the teacher, is that this topic is not covered very well in the *Guy Breton* text. From this textbook, the first question was a modified version of number 1f on page 193. *Find the slope of the line perpendicular to* $9x + 3y + 20 = 0$. Using algebra and properties of equations, the teacher put the original equation in the form, $y = -3x - 20/3$. Replacing the slope of –3 with its negative reciprocal, 1/3, completed this. Students did not demonstrate much difficulty with this problem. An observation that was made that day in addition to past experience at this level is that few students tend to take notes or copy the problems from the board. In illustrating another representation of this problem, the teacher continued by graphing the original function. At first he used the intercept method, but with too many questions from the students, he complemented his explanation with some algebraic work. For example, he substituted 0 for x in the equation to obtain the corresponding y value, $9(0) + 3y = -20$.

The next problem was number 7a on page 195. *Find the value of b if the line* $y = (3/2)x + b$ *passes through (4, 7).* During the resolution of this problem, which was done similarly as in exercise 1f using algebra, I observed that the teacher used an interactive approach with very little emphasis on theoretical questions. The third, a modified problem was, *is (3, 2) on the line* $2x - 5y + 4 = 0$*?* Done with a simple substitution of the respective x and y values, the class confirmed that this point does lie on the given line. Question 10a on page 195 was next on the agenda.

Find the equation of the line that passes through (3, 5) and (-5, -3).

This problem had similar goals as number 19 on page 269 of the previous day with respect to using coordinates to calculate the slope. The teacher mentioned that such a question would also be done at a later date using the graphing calculator. One student quickly gave a response by trial and error. The teacher, however, made it explicit to the rest of the class that this was not the most appropriate way to go about solving such a problem. An interesting observation at this point, given this students' response and the teachers' reaction, was that the teacher consistently allows the class to give reasons for their actions. This is important in allowing students the opportunity to prove or disprove their own reasoning. Doing so allows the teacher to determine whether the student's line of reasoning is procedural or whether it is more conceptual. After this moment, the teacher went on to solve the problem substituting the corresponding values of *x and y* from the given ordered pairs into the calculation of slope formula. This was then followed by the substitution of one of the points along with the value of the slope, leaving the initial value, *b,* the only value left to complete the problem. The final problem was number 11b on page 195.

Find the equation of the line which passes through (5, 5) that is perpendicular to the line 5x + 2y = 10.

During the resolution of this problem, I observed that, in the last ten minutes or so, many students began to use the graphing calculator. This problem was solved using algebra and being that there were no questions or comments, I suspect that it did not pose any problems to the class.

In conclusion, the teacher re-stated, based on his previous experience with graphing calculators, that it is very important to maintain a level of paper and pencil work. He found that many more problems arose when the majority of the work was initially done using the graphing calculator.

Day 4 (Thursday, December 2nd, 1999)

As a premise to the test that the students were to write, the teacher began a review session at the overhead calculator. He proceeded with an initial discussion warning the students to be cautious with the units. After writing the equation on the board, the class enters it in the calculator using the **equation editor**, as $y1 = 0.0476 * 60x$. The question was taken from page R-45 from the teacher's guide.

The length of a cassette tape can be calculated from the time it takes to play the tape. In fact, a cassette player unwinds the tape at a speed of 47.6 mm/s.

a) Determine the rule of correspondence used to calculate the length of the tape.
b) What is the length of the tape after 21 minutes? (Round your answer to the nearest meter. (The teacher changed the unit from mm to m.))
c) If the tape is 50 m long, how long, in minutes, does it take to unwind it?

In terms of selecting an appropriate window, the teacher proposed the **zoom fit** function. I suspect that the teacher did not allow much time for the students to arrive at an appropriate window on their own in order to allow them as much time as possible for their test. With respect to the problem, the teacher's goal at this moment was not to adjust the domain and range. In answering question *b*, the function **calculate** was used in graphical mode by entering a value of 21 at the **x** = prompt. At this point, students began to suggest different possible windows because the following message appeared on the screen using the **zoom fit** function; *INVALID DOMAIN*. This acts as an important preface to the experiment that will follow regarding the importance of obtaining an appropriate window when interpreting a graphical representation of a function. The teacher also suggested that, with the calculator, if y-values were given in order to determine x-values, then for example, an equation of $y = 50$ could be entered and a system of equations could be solved using the **intersect** function. The result to question *c* was 17.5 minutes using the method just described.

Due to a lack of time in wanting to leave sufficient time for the test, the teacher rushed through the next question taken from the same page.

The sum of the measures of a polygon's interior angles is equal to 2 less than the number of sides in the polygon times 180 degrees.

a) *Determine the rule used to calculate the sum of the measures of the interior angles of a polygon with n sides.*
b) *What is the rate of change of the relation defined by this rule of correspondence?*
c) *How many sides are there in a polygon whose interior angles add up to 2700 degrees?*

The equations $y1 = 180(x-2)$ and $y2 = 180x - 360$ were entered into the **equation editor** right away. The teacher chose to solve this problem using the table. He set the **change in** x in the table to one and the **table start** at three. Through some algebraic work, a student suggested that 2700 / 180 was the answer to *c*. During this, the teacher was scrolling down the table until he found the y-value of 2700 and the corresponding value of x = 17.

The third and final question of review, once again from the same sheet was the following.

A travel agency uses a table of values to calculate the costs of various trips to France. These trips last between 15 and 35 days. What is the rule of correspondence for this situation?

Duration (days)	**Cost ($)**
15	2160
18	2535
21	2910
24	3285
28	3785
...	...

Using the **stat** function, and entering this table in the calculator, the teacher identified constant change in x as well as in y. He then said to the class that this probably suggests a linear relation. From there, the **linear regression** function was used in determining the slope and initial value and hence the rule of correspondence,

$y = 125x + 285$. One student thought that the change in y between 3785 and 3285 was different from the others. Another student interjected and stated that the change in x was also different, still making the change constant. The final question asked by the teacher was with respect to the parameters, and their representation in this situation. Some of these questions posed by the students indicate that they demonstrate some understanding of functions based on their previous classes.

For the purposes of the test, the teacher stated that it was predominantly calculator based. As he was distributing the exams he also told the students that was not necessary to show their work, and that he was only interested in the results. The amount of time given for these four questions was fifteen minutes. Following the presentation of the test, tabular and descriptive summaries of the results will interrupt the journal.

The following questions taken from the summative evaluation section of *Guy Bretons'* teachers guide (page R-45) were assigned as a class test, following a brief lecture. The teacher also provided graph paper to the students.

1. *It is possible to determine normal weight **W** of a person (in kilograms) given his height **h** (in centimetres). The height must however be greater than 100 cm. The relation of weight as a function of height is given by the equation* $W = 0.75h - 62.5$.
   a) *What type of function is this?*
   b) *What would be the normal weight of an individual who is 1.72 m tall?*
   c) *What would be the normal height of an individual weighing 80 Kg?*

2. *The temperature of the Earth increases as the core is approached. The rule of correspondence for the function of this situation is* $T = 9.8d + 20.58$, *where **T** represents the temperature (in degrees Celsius) and **d** the depth (in kilometres).*
   a) *What is the temperature at a depth of 6.4 km?*
   b) *At what depth is the temperature equal to 279.3 degrees Celsius?*

3. *To carry out work on a construction site, a contractor must rent a giant crane. The table of values below provides several costs of renting as a function of time.*

Crane Rental

Time (h)	...	*5*	*7*	*9*	*13*	*16*	...
Cost ($)	...	*3800*	*4320*	*4840*	*5880*	*6660*	...

a) *What is the rule of correspondence for function* ***C*** *used to calculate the rental cost?*
b) *What type of function is this?*
c) *What is the cost of renting the crane for six forty-hour weeks?*
d) *Make a price statement for the customer? (The teacher added Question d).*

4. *Over a 24h period, the temperature outdoors (T) was found to vary according to the equation T = -0.85h +16, where* ***h*** *represents the time in hours.*

a) *List the properties and draw the graph. (The teacher added this question).*
b) *What is the range of function T in this situation?*
c) *For how long was the temperature above the freezing point?*

As the students were writing the test, I was unable to see more than just a few students at any given point not using the calculator. I even noticed one student working with a basic calculator alongside the graphing calculator. The only question posed by a student was concerning one of his windows. He questioned the different windows possible depending on whether you viewed the situation that was given or whether you simply looked at the mathematical model. Once again this type of question would confirm and justify the purpose of the activity to be conducted for this thesis.

The following table summarizes the students' results as compared to the expected results. The table will be followed with a brief conclusion.

Table 4. Summary of students' responses to class test

Question	Expected response	Students' correct response	Total correct responses	Percentage correct responses	Students' incorrect responses	Total incorrect responses	Percentage incorrect responses
1a	Linear partial variation relation	12 10	22	92	Constant-1 No answer-1	2	8
1b	66.5 Kg	19	19	79	63 68.75 61.21 61.23(x2)	5	21
1c	190 cm or 1.9 m	12 9	21	88	169.16cm 1.775m 24m	3	12
2a	83.3° C	19	19	79	85.26 89.18 635.236 -1.489362 no answer	5	21
2b	26.4 m	15	15	63	26 26.27 28.24 4.92 248.92 no answer(x4)	9	37
3a	C(h)=260h+2500 F(x)=260x+2500 Y=260x+2500 C=260x+2500	1 1 6 8	16	67	No answer (x8)	8	33
3b	Linear function or Partial variation relation	4 13	17	71	Constant-1 No answer(x6)	7	29
3c	The rental cost is $2500 and you are charged $260/hour	8	8	33	Various(x3) No answer (x13)	16	67
3d	Answers may have varied, appearing as receipts.	8	8	33	No answer	16	67
4a	Graph and properties of the function	0	0	0	No graph or properties were given	24	100
4b	[-4.4, 16]	1	1	4	Incomplete	23	96
4c	18.8 hours	1	1	4	Incomplete	23	96

(N = 24)

The students were given approximately 15 minutes to answer these questions. I believe this to have been insufficient. This may explain why the percentage of success from item 1 through 3 decreased as a whole. From item 1a to item 3b, the percentage of success was respectable. Students generally seemed to have difficulty with question 3c. Upon inspection, some of the incorrect answers were as a result of error prone order of operations. Such operations are considered to be basic at this level of math 436. For example, appearing in the incorrect column of item 1b, the student who obtained 61.21 Kg, actually obtained –61.21, but neglected the sign that was obtained.

For the most part, the results of this class test indicate that the teacher's methods combining traditional tools complemented by the graphing calculator are effective. Concluding or suggesting that his methods were effective towards conceptual development may be premature. In fact, some of the results of this test as well as those of the activity developed for this thesis (that will be presented in the subsequent chapter) indicate that few students are prone to understanding functions on a more conceptual level.

Day 5 (Tuesday, December 7th, 1999)

Moving into the more general polynomial functions, specifically the quadratic, the class began with a question from the summative evaluation section of the *Guy Breton's* teacher's guide. From page R-47, the following problem was selected;

From midnight to 11:00 a.m., the content of a tank varies as a function of the equation $C = -0.32\,(t-5)^2 + 16$, *where C is the quantity of water and t is the time. Assuming C is limited to the situation:*

a) graph this function;
b) find its range;
c) find the interval over which it is decreasing.

Initially, the students' main difficulty was in determining the vertex. They were briefly introduced to the general and standard forms of the quadratic equation on Friday; however, this was done with a substitute teacher. The teacher suggested approaching such a problem from different angles. By this I suppose that he meant, studying it through its properties and from its different representations. He began with a paper and pencil explanation. The first few guesses of the vertex by the students were the following, (-.32, 16), (-5, 16) and (5, 16). The teacher placed the general form, $ax^2 + bx + c = 0$, above the given equation in order to compare parameters with the given values. The teacher then began discussing its properties by illustrating them on the overhead projector. He stated the following; the vertex is (5, 16), the axis of symmetry is $x = 5$ and that the zeros could be found using $x = h \pm \sqrt{(-k/a)}$. With the parameters *a, h* and *k* from the standard form, $y = a(x - h)^2 + k$. The final property listed was identifying the *y-intercept* on the graph that he had entered on the graphing calculator. He asked, in attempting to illustrate the property of symmetry, "what are the coordinates symmetric to the *y-intercept*?"

Following a paper and pencil approach, the teacher used the **equation solver** function in the calculator to find the roots of the equation. At this moment, I heard one student ask another how this was done. This function allows the user to enter a function at the '*0=*' prompt, which should be followed by entering bounding values close to where the zeros might be. The teacher suggested that they use large negative and large positive values in obtaining the left and right roots, respectively.

The teacher also used the graph to find the roots from the **calculate** function. This function presents a sub-menu to the user that allows one to determine various values of interest. For example, with this function one can determine, a minimum, maximum, intersection point, zeros, and corresponding *y*-values to *x*-values that have been selected (one at a time).

Another question posed by the teacher concerning the calculator was, "How do we find the y-intercept with the calculator?" One student almost immediately replied correctly with, " use 2nd calculate, enter the value of x = 0, hit enter…" and the calculator yielded y = 8. In conclusion, to this lesson, the teacher assigned two more questions from the same sheet as class work. The total amount of time spent on the calculator today was slightly over fifteen minutes.

<u>**Day 6** (Friday, December 10th, 1999)</u>

Continuing with the quadratic function, the first problem of the class, number 8 on page 302, was taken from *Guy Breton's* chapter four.

The number of employees (N) working for an advertising agency increases steadily during its first six years in business, then decreases more and more rapidly. One of the agency's accountants explains that staffing has varied according to the relation defined by the rule of correspondence $N(t) = -5t^2 + 60t + 20$*, where t represents time in years.*

a) How many people did this agency hire when it first opened?
b) If the company is entering its tenth year in business, how many employees does it have?
c) In how many years will the agency close it doors if its growth continues according to the same rule?
d) At what point did it employ the most people?

Initially started with paper and pencil, for question *a*, some students answered, "y-intercept is 20". The vocabulary of the students indicated a stronger mathematical context as opposed to answering the question with respect to the number of employees. This may act as an example of the students neglecting the context of the problem due to its lengthy presentation. For question *b*, the teacher substituted x=*10* into the rule to obtain 120 with a simple calculation. It also begins to become noticeable at this stage that the teacher is emphasizing a greater use of the graphing calculator.

At this point, they began using the graphing calculator, right after the teacher asked the class how the window of the calculator should be set *according to the table of values*. Through trial and error, the teacher entered a few possibilities. In answering question *b*, the teacher used three different representations on the calculator in addition to the initial paper and pencil work in hoping to target as many students with his explanations. In doing question *c*, the teacher asked whether or not the equation was factorable. He did not wait for much of a response, and then solved it using the quadratic formula. Neglecting, of course, the negative result obtained, the answer was obtained with no trouble from the class. As a little exercise, in trying to obtain a more accurate value for the negative root, the teacher found its value using the **trace** function in graphical mode. A student suggested using the **zoom in** function to determine the negative root. The teacher took this opportunity to show the class how using the **zoom in** function alters the previous window. Finally, the teacher also used the **calculate zero** function in graphical mode to confirm the answer using a different method. However, the result obtained was $-3E^{11}$, which left many students asking, "What is that?" The teacher simply responded by helping them recall scientific notation and that the value obtained was, in fact, 0.

Returning to paper and pencil for question *d*, the teacher used the formula $y = (4ac - b^2) / 4a$. Two student questions that followed were, "Don't we need *h*?" and the other was, "Isn't it over $2a$?" The teacher here made an intentional mistake; he was about to calculate the wrong coordinate of the vertex. This was actually questioned by the first student, perhaps even by the second student who questioned the value of the denominator.

I can only assume that this last student was not referring to the quadratic formula. He then continued with the correct formula, $x = -b / 2a$.

In conclusion to this lesson, the teacher assigned a very similar question from the textbook, number 6 on page 315. Most, if not all the students began solving this problem accompanied by the TI-83. One student sitting next to me transcribed the resulting graph onto paper very well. Once again, approximately fifteen to twenty minutes were spent cumulatively using the graphic calculator.

Day 7 (Monday, December 13th, 1999)

On this, my last day of observation, I handed out the activity that was designed, for the purpose of this thesis, following a brief five minute lecture. More detail will be given concerning this activity and the results in a subsequent chapter. As a general introduction, however, the average time spent was between ten and twenty minutes. As suggested by the teacher, this activity was also given as a class test in hoping that most students would take it more seriously.

Following this activity, the teacher assigned two questions from the summative evaluation section of the teachers guide, where I observed most of the students working on their own with very little conversation, while the teacher was discussing some general ideas about functions.

Conclusion

The initial classroom observations focused primarily on the interaction between the teacher and the students during the teaching and learning of functions in a traditional paper and pencil environment complemented by the graphing calculator. Another aspect, which gained importance with the progression of my observations, involved the different goals of the teacher at the different stages of teaching. Where the topic of functions is concerned, the teacher's interpretations of the curriculum guidelines were well managed. More specifically, the identification of the teacher's motives in setting the particular classroom activities, exercises, and test questions seemed to provide the students with an appropriate structure.

The use of the graphing calculator appeared to be consistently well balanced with the use of the overhead projector along with the work done on the blackboard. This may, most likely, be explained from the comment directed to me by the teacher concerning his previous experience with the graphing calculator and how he felt that the time allotted to its use was unnecessarily overemphasized. Maintaining an appropriate balance between paper and pencil methods and the use of the graphic calculator provides the students with opportunities to target their subject-specific learning styles more effectively, without developing a dependence on any particular tool, such as a graphic calculator, the textbook or even the teacher's class notes.

In general, the use of the graphic calculator did not seem to be forced. The role of the calculator seemed to be founded on the basis that it should act as a supplemental tool, which may not always necessarily provide the user with the appropriate output. I even began to re-evaluate whether or not the graphic calculator provided a more dynamic working environment. The calculator's construction of graphic representations of functions on its view screen does not necessarily constitute a dynamic working environment. The calculator is dynamic to a certain extent, where a user may use the **trace** function, for example, to study the values of a function. Where the manipulation of parameters is concerned, the TI-83 does not allow one to do so dynamically. If the role of a parameter is being studied, the graphic calculator allows the graphing of multiple functions on the same screen, with the ability to distinguish between the various relations using different fonts. The function's graphical representations, however, remain static on the screen. Another function on the calculator that may allow for some dynamic presentation is the **zoom** function, which allows users to alter the windows at any time.

With respect to the curriculum, the teacher's goals were well aligned. In the seven classes that I attended, the students had an opportunity to write a test in addition to having one more week of classes prior to the mid-year exam. The teacher handled the time constraints set by the guidelines respectably. Perhaps a reason for this was his selection of classroom exercises, which may have allowed for an appropriate development of ideas. The use of the graphing calculator was definitely responsible for being able to adhere to time constraints, by allowing for the quick drawing of graphs and generating of tables of values.

The more specific goals of the teacher with respect to each class activity, as mentioned in the journal entries, allowed me to tie together ideas based on my observations and allowed me to better understand the reasons that the teacher may have had for doing or not doing certain things.

My initial observations seemed to indicate that the classes were proceeding as expected where the goals of the teacher and the curriculum were concerned. From a well-behaved and organized class to the teaching styles and strategies of the teacher, most of the factors seemed to indicate that an appropriate flow of mathematical ideas was present and effective in the teaching of functions. Upon inspection of my notes, however, I realized that most of the intellectual work was performed by the teacher. Often times, very little time was given to the students during class time to arrive at their own conclusions. Or as illustrated in the very first question of my first day of observations, the solution to the question to be worked on was provided. On the other hand, the teacher did allow for the reasoning of the students to be voiced. It often seemed that the reasoning expressed by the students came as a result of the teacher's comments and solutions, as opposed to giving them the opportunity to answer first.

I also re-evaluated some personal ideas that I was quite convinced about prior to the review of my notes. For example, I no longer consider the working environment of the graphing calculator as necessarily dynamic in nature. I will no longer assume that minimal classroom discussion is an indicator of understanding by the students. At times during the observation period, I did not realize that the teacher predominantly did the talking. The ability to complete the given tasks regularly from both a classroom and curriculum level was due to the little time that was given to the students to reflect and reply to the teacher's questions. In fact, the teacher often answered his own questions. At the time of this realization, I was under the impression that this meant that some students understood what the teacher said, or that others would understand in subsequent classes, and perhaps some unfortunate students did not understand and felt uncomfortable making their lack of understanding explicit to the class and to the teacher. Little communication, however, does not imply that there is no interaction present. Students often have a difficult time expressing their misunderstanding, or lack thereof. It is up to the teacher to make those judgements based on the knowledge one has of his or her students. One of the first comments by the teacher described the class as a good one. Perhaps this particular teacher was quite aware of the abilities of his students that I was unable to observe during my brief observation period.

Some of these ideas are confirmed through the results of the class test given by the teacher, which illustrated that, the communication of mathematical ideas was definitely present and as a consequence, the students were learning. In addition, the selection of classroom activities made by the teacher proved to be *good* questions, and the fact that students were given the opportunity to play the role of the teacher at the overhead projector allowed for such development of mathematical ideas that may have remained implicit for the greater part of my observations.

Chapter III

Analysis and Discussion of a Graphing Calculator Based Classroom Activity and the Students' Solutions Concerning Functions and Their Graphs

1. Methodology

1.1 Description of the research instrument: a questionnaire

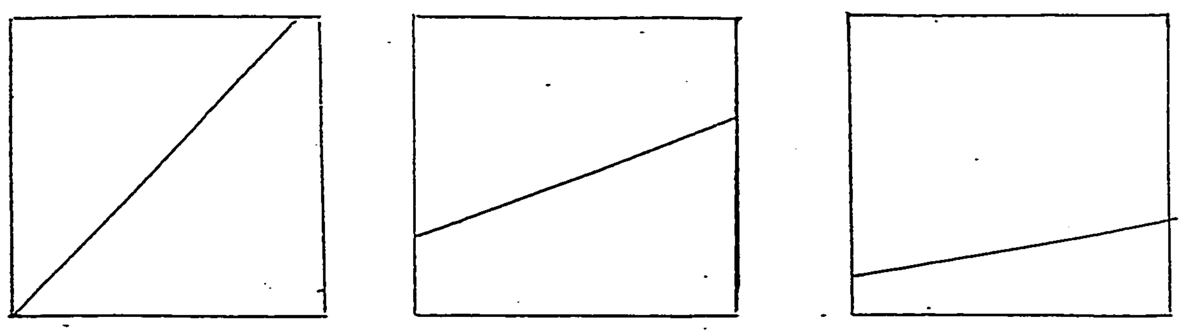
One of the goals of this research was to observe the teaching of the function concept in a graphing calculator based classroom. In order to complement the classroom observations that were described (in chapter II), an activity concerning the function concept and the possible use of the graphing calculator was developed. This activity was in turn given to the students, which was conducted in a test-like fashion and invigilated by me in the absence of the teacher.

Although the activity consisted of 3 questions illustrating the display window of the TI-83, the use of the graphing calculator was optional and not emphasized, either in writing or during the session. The following is the activity as it was presented to the class, less the space allotted for their responses.

Math 436

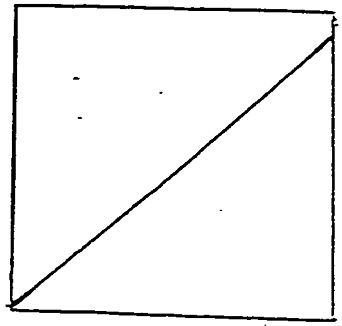
In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

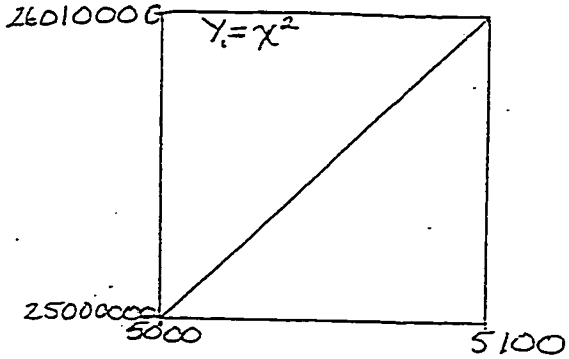


Please give;

i. a possible equation for this relation,
ii. include your domain and range, and also
iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

1.2 A priori analysis of the questionnaire

For the most part, the 3 questions focused on the graphical nature of the calculator. More specifically, emphasis was placed on the importance of the window selection in providing the most appropriate graphical representations of given functions. In the spirit of Dubinsky (1997), the goal was to determine whether or not the students were able to interiorize the visual representations of objects through their own actions and cognitive processes. The general tendency for students to experience difficulty with visual representations, as consistently demonstrated by Eisenberg's and Dreyfus' (1992, 1994) research, is also expected to be observed.

Question 1 presented three graphs that could potentially all belong to the same function. With either yes or no answers as a possibility, it was the students' reasoning that would prove to be of interest. The anticipated responses consist of the following reasoning (or any similar variation).

Yes, they are different because;

1. The slopes (rates) are different as observed from their different inclinations.
2. The initial values (y-intercepts) of each graph are different.
3. The leftmost graph is a direct variation, whereas the other two are partial variation relations.

No, they are not necessarily different from each other because;

1. The windows from each graph could be set differently so that the left and bottom borders may not necessarily represent the x and y-axes.
2. The scales could be different.

Given that this question did not necessarily have one correct answer, the reasoning behind each response would be in question. Students answering yes, to this question, are perhaps only considering how these graphs appear visually, without necessarily taking into account the connection between a function and its graph from different perspectives as well as simply viewing them as pictures.

Students answering no to this question may have interiorised the action of graphing a function into a process, and thus have a better idea of the dependence of the graph on the choice of the window (domain and range) and scale. Perhaps they realize that, with the graphing calculator, what you see is not always what you get.

Item 2 consisted of a 3-part question that potentially had many answers. This item questioned whether the graph being represented had a non-zero y-intercept. Based on their reaction to the initial question, in the next question they were asked was to determine a possible equation for this relation. The expected responses are equations of the form $y = ax + b$ or $y = ax$. Where the direct variation relation is concerned, that is $b = 0$, it will be surprising to find students describing this graph using $y = ax^2$. In either case, the students would be correct.

Questions ii. and iii. involve their ability to select appropriate windows for their interpretation of the algebraic representation for the graph. However, the questions are posed using some of the properties of functions and their graphs, specifically concerning the domain and range and the scales of the axes. The expectation that correct responses will generally be obtained for this question is optimistic. Incorrect answers will possibly indicate that the students do not have a proper grasp of some of the vocabulary in the question. For example, not knowing the difference between partial and direct variation relations. Or not being clear on what the meaning of domain, range or scales is. In this question, the use of the graphing calculator is expected.

The third and final question of this activity claimed that the given graph was a linear function, in addition to having a constant slope. The graph, appearing as a diagonal line with a positive slope, also included the equation at the top right part of what would be the display screen of the TI-83. The scales have also been included (which would in practice have to be obtained by viewing the window screen).

Answering 'yes' to the graph being linear would mean that the students did not consider the equation, $y_1 = x^2$, that was given. A 'yes' response could also be a result of simply visually considering the graph as a picture. Answering 'no, the function is not linear' could also be a result of direct observation of the given equation. I suspect that the more advanced students will take the given scale into consideration, and perhaps even enter the values in their graphing calculator.

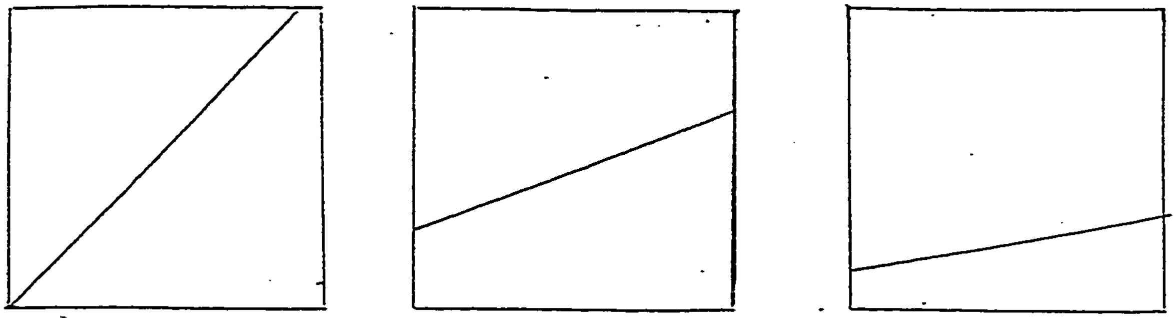
Students who answer 'yes' to the slope being constant are most likely to be the ones who determined the function to be linear, once again neglecting the fact that a quadratic equation is given. Those who will correctly answer that the slope is not constant will most likely determine that the given equation is quadratic. In contrast to the first two questions, each claim in this question has only one correct answer, given that the student also provides appropriate justification.

<u>2. Analysis of the student responses</u>

In the discussion that will follow, the quantitative and qualitative observations will be analyzed in attempting to draw some conclusions concerning the use of the graphing calculator during the activity, thus providing some insight concerning its use during the teaching and learning of the concept of function. The actual responses of the students will be included as Appendix C. In this discussion, students are referred to by the first page of their questionnaire. For example, C3, C6 and C9 refer to the first three students. However, the work of student C3 appears on pages C3, C4 and C5. The work of student C6 appears on pages C6, C7 and C8, and so forth. The only exception to this rule concerns student C78 who submitted the work on a single piece of paper.

The respective quantitative and the qualitative observations will be presented question by question. Twenty-six students answered the questionnaire. These results illustrating the distribution of students' answers will be tabulated in the concluding section.

1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

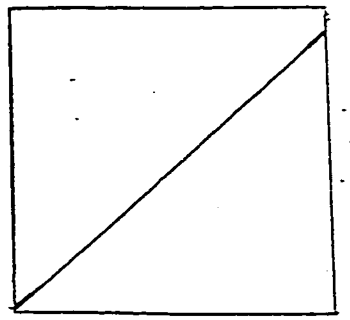
Of the nine students who answered that the graphs necessarily represented different functions, students C42, C51, C54, C57, C66 and C69 claimed that the graphs of the functions had different y-intercepts or different slopes or mentioned the origin. Thirteen students who answered that the graphs could represent the same function, used justifications based on the window settings of the calculator. Those were the most expected responses. Students C3, C9, C12, C21, C27 and C33, from the thirteen students who answered that the graphs could represent the same function, made reference to the **zoom** function. Those students were using the word "zoom" interchangeably with the term window (domain and range), as can be observed in Appendix C. Their language is strongly calculator based. In fact, none of the 26 subjects used much, if any, mathematical terminology, such as domain and range. Few, however, did use the term "scale". Student C36 actually contradicted himself by answering that they were not necessarily different by saying, "They just have different numbers but the basis of the equations is still the same". I believe that he was referring to the numbers a and b from the functional form of the equation, $y = ax + b$. His interpretation could be that the graphically represented functions are all the same because they are all linear functions. Perhaps it would have been more appropriate to classify this student as having provided a wrong reason, however, I gave him benefit of the doubt in classifying him in the *all-same* group.

Student C60 who answered, "yes" but gave no reason and student C57 who answered, "yes, because they have a different c value" were considered as having given incorrect or no reasoning.

Two students answered, based on slope and y-intercept arguments, that the leftmost function was different from the other two because it passed through the origin. They felt that the two rightmost graphs could be the same because the graphs did not pass through the origin, however; only one of them explained why they could be the same depending on the window being used. In fact, one of the students initially used similar arguments as described in this paragraph, with respect to the two rightmost graphs as being the same, but then concluded that all three graphs could all actually be the same function provided the **zoom 6** function of the calculator was used. The **zoom 6** function provides a *standard* window with both x and y-scales of 1 and a domain and range of [-10, 10].

Only two students drew axes on the graphs. One of them labelled the left side and lower side of the window as the y and x- axes, respectively, while the other centred the origin in the middle of the window.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



One of the students, Nina (C3), actually presented a worthwhile discussion on how the function could be graphically represented in several ways. "It could be either one of them or even neither. It could be a zoom into a part of a quadratic function or it could be a direct or partial variation. It all depends on how you see it". I would consider this student to have adequately understood the importance of the window settings when graphing with the TI-83. Nina had answered that the graphs in question 1 could all represent the same function. In fact, based on the a priori analysis, Nina was the only one who considered this graph as possibly representing a quadratic function. However, in defence of the other students, the wording of the question was strongly suggesting that the relations were linear.

I will continue to interpret the data based on the student's responses from the first question. This will allow me to draw conclusions regarding the consistency of the student's interpretations of the subsequent questions and also confirm whether or not students answering that the graphs could represent the same function as having been able to better understand this activity. This information will be illustrated more clearly in the conclusion of this section with a summary of results.

Of those who felt that the graphs in question 1 represented different functions, students C42, C45, C51, C54 and C48 predominantly categorized the graph in this question as representing a direct-variation relation. A large majority of the students, who justified their answers, specifically C6, C18, C21, C27, C30, C36, and C39 of the *all-same* group, and C42, C48, C51, C54, C69 and C72 from the *all-different* group, had arguments revolving around the idea of the origin. Student C78 spoke about the bottom left corner, which I interpreted as her reference to the origin.

The answers of those who stated that the graphs of question 1 could represent the same function were fairly well distributed in this item. One of these students felt that the graph given was necessarily a direct variation relation, three of which thought that the graph represented a partial variation relation. The majority, however, of the students in the *all-same* category stated that the graph could represent either. One other student, Catherine (C21) contradicted herself by stating that the graph was a partial-variation, but gave a direct-variation equation in question 2.ii. There were three students who did not answer question 2, but I classified them according to the equation they provided for item 2.ii. Once again, most of the justifications in this category revolved around the location of the origin in the window, even those who said that the graph could represent either direct or partial- variation relations. The most expected answer was that, either a partial or direct-variation relation could represent the graph.

Question 2 continued...

Please give;

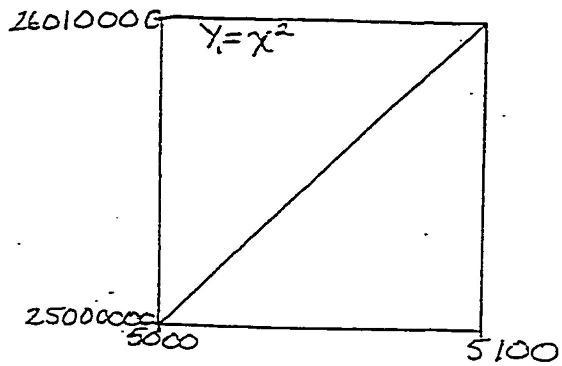
i. a possible equation for this relation,
ii. include your domain and range, and also
iii. describe the horizontal and vertical scales used to graph it.

Although question 2 was also open to interpretation, as mentioned earlier, answering that the graph could be partial or direct-variation, of which eleven students did, was the most anticipated response. Answers that were considered incomplete were a result of students not including any specified windows or scales that would allow me to graph their answers and view the output on the calculator. In answering this question appropriately, although it was not explicitly stated, it would have been expected that some students would have made use of the graphing calculator to make better sense of the problem. Some examples of incomplete answers given were, neglecting to provide an appropriate scale or domain and range. One student even wrote, "horizontal and vertical scales are 0." To a certain extent, this *incomplete* classification may also be considered as consisting of incorrect answers.

Many of the direct-variation equations given were either $y = x$ or $y = 2x$. As observed, fewer partial-variation equations were given. Many of the domain and ranges were given as, *negative infinity to positive infinity*, but in interval notation. With respect to the scales, I suspect that many students used the hint concerning their input in the TI-83, because many transcribed the values as they appear in the window screen. As mentioned earlier, these results will be tabulated and presented later.

Upon further inspection, this series of questions i, ii, and iii became quite interesting. After having entered each one of the equations along with the provided domain and range, and window settings (of those who did provide them) in the calculator (with the axes off), I realized that only students C3 and C36 of the *all-same* group and student C69 of the *all-different* group, provided sufficient information to possibly resemble the given graph. Based on my interpretations, it was difficult to categorize many of the others as having plausible results. The following are two examples of my interpretations of some responses. Domain and range values of negative infinity to positive infinity were interpreted as –1000 to +1000. A response such as $y = ax$ was interpreted as representing a basic direct variation relation which I entered as $y = 2x$.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

The results of this question indicate many inconsistencies on behalf of the student's reasoning. Only eleven of the twenty-six students answered that the function was not linear and that the rate was not constant, most importantly, they did this with appropriate justification. Where the given scale was concerned, it is important to note that none of the students made reference to it. As described in the a priori analysis, the expectation that the stronger students would make reference to the scales did not occur. Perhaps the presentation of the scale was not illustrated clearly on my behalf.

Nineteen students acknowledged that this graph was not linear based on the fact the equation $y = x^2$ was given. However, most of the arguments fell apart when discussing the rate of change. Those who answered correctly stated that the rate of a quadratic function could not be constant. I suspect, as in question 2, that there was not much use of the TI-83 in this item either.

3. Summary of Results

This section will contain a tabulated summary of the distribution of the results. A discussion relating the a priori analysis of the experiment with the a posteriori analysis of section 2 will attempt to determine whether the objectives of the experiment were met. The table will profile every student's response so that conclusions may be drawn based on the individual students' successive answers. The table predominantly categorizes the students within different classifications of possible answers within each question. Two of the columns are qualitative, in that, they actually list the student's responses to questions 2i, ii and iii.

The following table 5 summarizes the student responses to the questions in the activity. The number of students participating is 26.

Student	Q.1	Q.1	Q.1	Q.1	Q.2	Q.2	Q.2	Q.2i	Q.2ii	Q.2iii	Q.2	Q.2	Q.2	Q.3	Q.3	Q.3	Q.3	Q.3	Q.3
	all same	all diff.	wrong or no reason	some same	nec. dvr	nec. Pvr	poss. either	equation	domain / range	scales	plausible	implausible	Incomp	lin+cnst	lin+non-cnst	non-lin+cnst	non-lin+non-cnst	lin+unspc. rate	non-lin+unspec. rate
C3	1	0	0	0	0	0	1	y=ax	D+R=Real	0 to 50 by 5	1	0	0	0	0	0	1	0	0
C6	0	0	1	0	1	0	0	y=x	0	v.a. =1, h.a. =1	0	1	0	0	0	0	0	0	1
C9	1	0	0	0	0	1	0	x+1	[0,inf. / [1, inf.	zoom 6	0	1	0	0	0	1	0	0	0
C12	1	0	0	0	0	0	1	y=2x	(-inf., +inf)	nil.	0	1	0	0	0	0	1	0	0
C15	1	0	0	0	0	0	1	y=2x	(-inf., +inf)	x = 1, y = 1	0	1	0	0	0	1	0	0	0
C18	1	0	0	0	0	0	1	f(x)=2x	Real num.	x: 100, y: 100	0	1	0	1	0	0	0	0	0
C21	1	0	0	0	0	1	0	f(x)=11x	[0, 7] / [0, 150]	x = 1, y = 1	0	1	0	0	0	1	0	0	0
C24	1	0	0	0	1	0	0	y=2x	0 / 2	nil.	0	1	0	0	0	1	0	0	0
C27	1	0	0	0	0	0	1	y=x	(-inf., +inf)	1, 2, 5	0	1	0	0	0	0	1	0	0
C30	1	0	0	0	0	0	1	y=x+1	(0, inf.	x, y in 1000's	0	1	0	0	0	0	0	1	0
C33	1	0	0	0	0	0	1	2x=y	(-inf., +inf)	2	0	1	0	0	0	0	0	0	1
C36	1	0	0	0	0	1	0	.8x+1	(-inf., +inf)	x =0.8, y=1	1	0	0	0	0	0	1	0	0
C39	1	0	0	0	0	0	1	y=x	(-inf., +inf)	x=10, y=10	0	1	0	0	0	0	1	0	0
C42	0	1	0	0	1	0	0	y=x	0, 10	0	0	1	0	0	0	0	1	0	0
C45	0	1	0	0	1	0	0	y=x	0, inf.	nil.	0	1	0	0	0	0	1	0	0
C51	0	1	0	0	1	0	0	f(x)=x	(-inf., +inf)	10	0	1	0	0	0	0	1	0	0
C54	0	1	0	0	1	0	0	y=2x	[-10, 10]	1	0	1	0	0	0	0	0	1	0
C66	0	1	0	0	0	1	0	y=x+0.5	[-10, 10] / [-9.5, 10]	1	0	1	0	0	0	0	0	0	1
C69	0	1	0	0	0	0	1	y=0.9x+1	(-inf., +inf)	1	1	0	0	0	0	0	1	0	0
C72	0	1	0	0	0	0	1	y=2x+1	1, inf.	same scales	0	1	0	1	0	0	0	0	0
C63	0	1	0	0	0	1	0	y=x-2	nil	nil.	0	0	1	1	0	0	0	0	0
C48	0	1	0	0	1	0	1	2x	Real num.	zoom 6	0	1	0	1	0	0	0	0	0
C60	0	0	1	0	1	0	0	.8x	nil	nil.	0	0	1	0	0	1	0	0	0
C57	0	0	1	0	0	1	0	x	(0, inf)	nil.	0	1	0	0	0	0	1	0	0
C78	0	0	0	1	0	1	0	y=x	(-inf, inf) / [0, inf.)	zoom 6	0	1	0	0	0	0	1	0	0
C75	0	0	0	1	0	0	0	x-1	(-inf, inf)	1	0	1	0	0	0	0	0	1	0
Total:	12	9	3	2	8	7	11				3	21	2	4	0	5	11	3	3

Table 5. Summary of student responses to activity

The first thirteen students appearing in the table are those who answered that the graphs could possibly represent the same function in the first question. They were followed by the nine who answered that the graphs could possibly represent different functions, then by the two who replied incorrectly or did not provide reasoning. The remaining two students, who claimed that some graphs could represent the same function, appeared at the bottom.

With respect to the qualitative tabulations one observation was that some students gave expressions as opposed to equations, even though an equation was explicitly asked for. Most of the equations given by the students represented direct variation relations. Students C21, C57 and C78 contradicted themselves by saying that the graph was a partial variation relation, but gave a direct variation equation. Those three students as well as those who gave expressions instead of equations may not have properly understood such basic distinctions between expressions and equations, as well as distinctions between algebraic representations and their corresponding graphs.

To avoid repetition, values of the domain that were the same as those for the range were only entered once in the table. There were only two students who did not provide a domain and range, and six students did not provide scales. Student C3's responses to item 2i and 2ii were general, however, she did provide the values of the domain and range under her response to the scales. Perhaps she thought that the scales and the values of domain and range belong together, and she may have some difficulty where function notation is concerned. Three students described the scales with the **zoom 6** function, which actually provides a domain and range of [-10, 10] as well as scales of 1 unit. Responses of these students, as well as that of student C3, may indicate that you must consider the scales and the domain and range together. An issue that is implicitly raised deals with the distinction between mathematical language and calculator terminology. Teaching mathematical notions using graphing calculators pose a problem when the language of the students as well as their thought processes become calculator dependent. Visually speaking, this problem is compounded when students consider notions from the finite representations of the graphing calculator.

In order to be classed as plausible, it was important to provide enough information such that a graph could potentially be viewed on the calculator after the input of the equation, the domain and range and appropriate scale. In cases where the scale was left blank, I assumed it to be the scale obtained from the standard window. It is important to note, as discussed earlier, that few of their suggestions actually resembled the actual graph that was given. Unable to graph and interpret certain results because of not having included a domain and range or a scale, I deemed these students' work as incomplete. For example, student C24 was implausible because I was unable to interpret his response of "1 and 2" for the values of the domain and range, respectively.

Although question 3 was generally successful, with the correct response being that both the function and its rate were not constant, it was interesting to note that some did not take the given equation, $y = x^2$, into consideration and that some of those who did claimed that the rate was constant. I interpret these students' responses as not having fully understood the notion of rate of change. In fact, the arguments of those who answered that the graph and its function were not linear took a turn for the worst when discussing the rate of change. Only three of these students did not provide justification for the rate of change portion. Most arguments were centred on the given quadratic equation, but those four who claimed that it was linear did not take it into consideration.

4. Conclusions regarding the questionnaire results

Generally, the questionnaire was perhaps limited in terms of observing students' understanding. The summary to the experiment seemed to have confirmed the a priori analysis of the questionnaire with respect to those having answered in the *all-same* category. Those students demonstrated a greater potential in being able to associate different representations of functions, ultimately attaining a better conceptual understanding of this notion. The results did not appear to suggest any remarkably surprising outcomes where the mathematics was concerned. Where the understanding of the function concept is concerned, I feel that the level of questions posed to the students did not allow them to move further than the first two levels, *intuitive understanding* and that of *initial mathematization*, as taken from the didactic model of understanding from Bergeron and Herscovics (1982). The first level is said to involve more of a pre-concept rather than the concept itself that is attained from both physical and reflective abstraction, as cited from Piaget et al. (1968) by Bergeron and Herscovics (1982). Furthermore, the questions in this activity dealt mostly with ideas of types of variation and visual interpretations of graphical representations.

At the second level, *initial mathematization*, the representations of graphs as objects come into question. The importance of representations, as stated by these two authors, sets the premise for visual discrimination when learning to graph, and of equal importance, this is when the algebraic representations become an essential part during the development of the concept of function. This activity did not involve the generating of tables or drawing of graphs, nor did it involve much algebra. These were considered on my behalf as forming part of the student's prior knowledge. The activity did, however, ask for the subject's interpretations of what they perceived from the graphs.

The second question involved this level of *initial mathematization*. In fact, where the third level of *abstraction* is concerned, the authors propose an example of students attaining this level. It concerns the ability to abstract the domain and range in terms of the admissible values of the two variables. Many students using the graphing calculator selected and transcribed the appropriate windows. On a more mathematical level, the use of interval notation to describe the possible values of the two variables was given by some, regardless of the window that would have been selected.

The fourth level in this model of understanding is that of *formalization*. At this level, the authors state the importance of having to come from the third level of *abstraction*, otherwise definitions and notation become meaningless. As was observed from the results of the activity, this was absolutely the case especially where notation was concerned. Examples of this were, writing expressions instead of the required equations, and providing inadequate suggestions as to what the domain and range could have been.

It is important to keep in mind that this activity did not involve the definition of the function. Inclusion of the definition of the function may have been inappropriate, as the students were in the initial stages of learning about functions. Including the definition in such an activity may have been more appropriate in subsequent activities. As such activities, preparing a series of questions that would attempt to reach the higher levels of understanding as described by Bergeron and Herscovics would be a future goal.

The reason why such development towards questions involving the level of formalization is important comes from a question that these authors posed at the end of their essay concerning the importance of definitions. "How can one expect a student to understand a definition unless it is within the sphere of his existing knowledge? (p. 45, Bergeron and Herscovics (1982)). In addition, I do not believe that such development towards understanding may come from only one activity.

The most expected or appropriate responses for the succession of the questions were, for question 1, the graphs could represent the same function; for question 2, the graph could represent either a partial variation or a direct variation relation; and for question 3, the given graph was non-linear and did not have a constant rate. With respect to those twelve who answered under the *all-same* category, eight went on to answer under the *possibly either* category, and four of those eight answered the third question correctly. Student C36 who answered the first and third questions as expected, was the only one of the five classified as having answered item 3 correctly (from the *all-same* group) who claimed that the graph of question two necessarily represented a partial variation relation.

On the other hand, the students who answered that the three graphs in the first item represented different functions were not as consistent through the succession of the questions. Of the initial eight students in this group, only three stated that the graph of question 2 could represent either a partial or direct variation relation, and from those three, only student C69 remained in answering question 3 correctly. He was the only one, of the six who answered item three correctly, who developed through the second most expected response sequence. These results seem to confirm the a priori analysis discussed earlier in this chapter with respect to those students in the *all-same* who demonstrated a stronger ability where visual representations were concerned.

5. Improvements for Future Research Studies

Where the development of this activity is concerned, I was able to constructively criticize it as my research instrument in terms of its presentation in addition to gaining insight for future research. For example, upon inspection of the results of the first question, I found the answers to be quite confusing in terms of answering the question with respect to the phrase, *are they necessarily different from each other?* A clearer sentence such as, '*could these graphs represent the same function or could they represent different functions?*' would have been more appropriate and would have allowed for less confusion. Secondly, questions involving relations in addition to functions would have allowed for a higher level of abstraction, for as Bergeron and Herscovics stated, it is at this level concerning the notion of function that distinctions are made between relations and functions. Thirdly, bringing into question the definition of the function in addition to some of its properties may also have helped in the construction of the function as an object on which the students can act on and further develop through cognitive processes.

The strong focus on visual representations of this activity given the obtained results, allows me to conclude that they were not necessarily overemphasized. These visual representations may help make explicit to the teacher the conceptual difficulties experienced by their students. A stronger focus on algebraic and tabular representations would have been good, but may be included in subsequent activities of this type. The fact that I emphasized only one of the calculators' features on representing functions may have only complemented those students who have stronger visual tendencies while possibly neglecting those students with stronger algebraic capabilities.

A potential question for future experiments could be the following. The students could be shown a graph with a given window and equation. The question would show the students other views of this function and the students would be asked to determine what the scales, domain and range are. Such an open-ended question can allow students to approach this on an intuitive level or on a more formal level depending on their strength. Nonetheless, regardless of the response, a teacher may be able to assess the students understanding of properties of functions, and depending on if and how they justify their answers, further assessments about their conceptual understanding may be done. Such developments are a major responsibility of teachers and researchers.

My observations allowed me to confirm the degree of creativity that students express when it comes to answering non-traditional questions. Those people involved in the design of such mathematical activities, whether they are in the classroom, on computer or in a textbook, must parallel the student's creativity so that students of mathematics may be given the opportunity to formalize their knowledge within certain mathematical topics. Following this experiment, it is clearer to me how important the development of questions is, within an activity, when attempting to bring students to higher levels of thinking.

The purpose of this experiment was to question the role of the calculator during the development of the concept of function. In fact, as introduced to the students, it turns out that the activity was conducted at the initial stages of the development of the notion of function. For this reason, the discussion in this chapter concerning the inappropriate inclusion of the definition, as well as that of algebraic and tabular representations of functions, may have been overwhelming to the students.

With regards to the role of the graphing calculator, these results demonstrate that the students did not overly use it. Unfortunately, it appeared not to have been used at all to answer items on the questionnaire, even at times when it may have been. This indicates a parallel to the ways in which these students are taught, given the teacher's appropriate balance between the use of the calculator and traditional paper and pencil methods. This observation comes back to the comment, directed to me from the teacher during the observation period, concerning his prior experience with the overuse of the graphing calculator during the teaching of functions. It is important to experience this vicariously, through this teacher's experience. The role of the graphing calculator should be mainly to act as a tool. That is, it should be available to the students, but they must be aware that solutions can be obtained without it.

The use of the calculator must primordially be used in being able to provide the students with different representations of functions more quickly. For this to occur, it may be important for the manual generating of tables and graphs in the earlier grades, leaving more time for the development of the concept of function, when more appropriate, specifically starting from the Secondary 4 level. It needs to be clear that the ability of the calculator to provide multiple representations of functions does not imply that the students will develop conceptual understanding. The graphing calculator does not necessarily provide dynamic representations of functions, which may be misleading for the teacher when assuming that students have appropriately understood the notion of function conceptually. In conclusion, based on the observations and my experiment, that the monitored use of the calculator may aid in conceptual development of the notion of function, but only given that students fully understood the constructions of the different representations, which I believe may predominantly be achieved through their manual paper and pencil construction in the earlier grades. Or perhaps, given this technological age, the development of software may enable the students to construct various representations of functions and manipulate them dynamically. I would suggest that the use of Cabri Geometry or Geometer's Sketchpad, which are dynamic geometry software, might potentially provide such an environment.

Chapter IV

Conclusion and Recommendations

1. Conclusion

In attempting to understand the role of the graphing calculator in relation to the concept of function, this thesis provides the opportunity to do so on three levels. They are, a review of the research literature, observations in a graphing calculator complemented classroom, and finally through an analysis of a questionnaire designed for the purposes of acquiring better didactic, pedagogical and psychological understanding of the notion of function.

A series of questions were posed throughout this thesis based on observations of current didactical and pedagogical circumstances, from the literature as well as from the activity. Most importantly, however, is the consideration of the thesis statement concerning the assessment of the role of the graphing calculator during the conceptual development of the notion of function.

For all intents and purposes, this statement will be discussed through suggested answers to the series of questions appearing in this document. It was not the intention to arrive at any definite or formal conclusions, as such; they may have been premature and not justified given the scope of the study. For such and similar reasons, conclusions and recommendations that will be presented are, at most, humble.

Where the notion of the function is concerned, one of the first thoughts of this paper questioned what the goals of mathematics education were. It obviously deals with the knowledge we want students to construct. If we consider the role and perspective of the student, we may safely assume that their mathematical knowledge is a result of their experience in the classroom with the teacher as well as through their textbooks. The dependence on these two, which have been created during the elementary and secondary mathematics careers of students, suggests that their understanding may only be as good as that of the teacher, the effort they put in studying, or as formal as that of the textbook. Clarity in presentation in the classroom and textbooks as well as careful assumptions made by teachers about their students' prior knowledge are important if we expect students to be in parallel, yet not necessarily on the same level, as their teachers.

The Bergeron and Herscovics (1982) didactic model of understanding as well as the APOS theory of learning provided us with the grounds to better understand the levels which students are expected to attain when learning certain mathematical notions and whether they will be able to move onto subsequent levels. It is important to distinguish between subsequent levels of understanding and subsequent levels of mathematics. By remaining at initial levels of understanding, the promotion of students into subsequent courses may only constitute pedagogical growth and not the desired mathematical growth and development. The expectations on students' mathematical abilities must be considered through the perspectives of such models of understanding and theories of learning.

One aspect of mathematics education requires the curriculum and textbook developers to be consistent with each other. As a consequence, the teachers must also be consistent in interpreting and facilitating this information to the students. However, there is only so much that may be done by teachers with mathematical textbooks, for example, when such materials are presented at levels that are beyond the students' realm of understanding. This realm of understanding refers to the students' existing and prior knowledge. It is not so much the mathematical content of such textbooks that I seem to be concerned with, more so it is its presentation. At times it appears as though the authors of the textbooks try to hide the mathematics behind a wall of pictures, schematics, pseudoreal-life contextual problems while surrounding them all with overwhelming verbal descriptions that make the acquisition of the topics presented very difficult, even for the stronger students. As a consequence, the number of students that may realistically be able to walk away understanding the mathematics is reduced.

It is very important for developers and teachers to be able to answer the question posed by Nichols (1992), 'what is a function?' before they may even begin to expect students to know the meaning of this notion and all its aspects. Furthermore, when this question is answered, the question posed by Sierpinska (1992), 'what do we want understanding to be?' should then be the focus. The conditions of understanding and the epistemological obstacles concerning the notion of function such as described, for example, by Sierpinska, must be modelled by developers when reforming curricula and designing textbooks. They must also be able to generalize the conditions of understanding and prescribe ways in overcoming the epistemological obstacles that are prevalent to all topics in mathematics education.

I find it disheartening, in such a pragmatic field, when much research within certain topics in mathematics education is not taken into account in practice. In the end, developers of curricula and of educational materials generally do not seem to be making noticeable use of past and present research, that are made accessible and very pertinent to their works. Ultimately, their works should be intended to benefit and motivate, not discourage, students when learning mathematics.

As a premise to the activity conducted for this thesis, having been given the opportunity to be an observer in a graphing calculator-based mathematics class allowed me to see an example of its integration from a researcher's point of view. By complementing the theoretical aspects of the notion of function it was of equal importance to experience an appropriate setting involving the use of the graphing calculator.

Understanding the goals and the expectations of the teacher about his students helped clarify some of the readings described in chapter I. The most important didactic observation that was made during this observation period had to do with the seemingly perfect balance between traditional paper and pencil methods and the graphing calculator, utilized by the teacher. To be used as an example concerning his previous experience with the use of the graphing calculator in the classroom, this teacher was very clear about the problematic nature of overemphasizing the use of graphing calculator, especially during the initial stages of introducing the function concept to the students.

From the research, specifically that of van Streun et al. (1999), we must learn to discriminate between various students' solution approaches, for the use of the graphing calculator may be inappropriate with some of those students. Being able to use the graphing calculator to complement his teaching, this teacher would most likely be successful in targeting the various styles of learning as well as the different solution approaches of his students.

Through the analysis of the activity, this study was successful in assessing the role of the graphing calculator where one of the goals was to consider its relation to the teaching of the function concept at the high school level. We had the added opportunity to observe its presence at the initial stages in the development of this notion and some of it aspects. It was interesting and at times unfortunate, from the analysis and results of the activity, to see that the graphing calculator was sometimes not used. This was surprising based on the observed classroom conditions. This reason may help explain the higher success rate on the class test than on the activity conducted as the thesis questionnaire. Since they were not required to present any work on their class test, some inherent difficulties, such as admissible values of the domain and range, scales and graphical representations of functions, when using the graphing calculator and in the understanding of the function concept, were externalized during this activity.

At the end of the day, will the graphing calculator enhance the teaching and learning of the concept of function? We saw from the literature and from the experiment that it does demonstrate the potential to do so. Its ability to quickly provide various representations of functions can allow for in depth discussions about specific items during the time that would usually be required for their constructions. It is during these discussions where concept development should occur. A dynamic classroom environment is necessary for concept development in any subject. It is important to realize the potential of the graphing calculator and acknowledge that it does not provide the students with a dynamic tool, as much as it should play a role in an already dynamic classroom environment. Based on some concerns expressed in van Streun et al. (1999), the fears of teachers concerning their students' acquisition of knowledge of the notion of function, while using the graphing calculator, may be reduced and thus have a positive impact on their concept development.

The extensive body of research on difficulties with visual understanding of Eisenberg and Dreyfus (1990, 1992, 1994) brought to question whether the graphing calculator would improve visual understanding, specifically with respect to the graphs of functions. The possibility exists, and as reported in van Streun et al. (1999), students considered as being strong and tend to be algorithmic in nature improve their abilities with more graphical approaches upon long term use of the graphing calculator. Eventually leading into visual representations, perhaps this suggests the importance of initially having to learn algorithms to help complement their obvious relationship.

One important realization concerning procedural versus conceptual understanding was that, the acquisition of a notion might potentially be threatened by the integration of the graphing calculator. At first, I contemplated how the graphing calculators' ability to construct the various representations of functions would allow for a greater amount of time to discuss aspects of the function, as mentioned earlier. Upon reflection, however, I realized, from the observations and from personal experience that although the graphing calculator may allow for the latter situation to happen, it may also potentially be responsible for the elimination of procedure, regardless of the subsequent discussions. If constructions of representations are considered implicit abilities at the higher levels, than it is of paramount importance that the students, during the preceding levels of mathematics, develop such abilities as manually generating tables of values, plotting points and deriving rules of correspondence. So eventually when they are in the presence of the graphing calculator, its output is justified in their minds.

2. Recommendations

The opinion that the graphing calculator should not play a dominant role in the classroom suggests that its monitored use when learning about functions can definitely improve students' cognitive abilities during concept acquisition. Monitoring its use involves the knowledge of the different levels of students' abilities in the class. This may be achieved through pre-tests and more importantly through discussions that allow students to describe their thoughts and procedures that they performed, the latter having been observed during my observation period.

It is crucial and cannot be understated, that an appropriate balance between the use of the graphing calculator and traditional paper and pencil methods must be achieved in the classroom. Providing students with multiple approaches can help them complement their already existing solution methods and different learning styles. The graphing calculator can help provide them with such approaches when learning.

The graphing calculator does not necessarily have any bad aspects. What is problematic is its integration in mathematics education. As it stands, the graphing calculator seems to emphasize some aspects of the function, such as graphic representations and their tables over other aspects such as domain and range, which can potentially remain implicit. In such an example, the blame is not to rest on the graphing calculator. The teachers' role becomes important in making such information explicit to the students.

Although the graphing calculator has the ability to speed certain construction processes, it is very important to keep in mind that conceptual development cannot necessarily be sped up through these procedures. For this reason, moving through topics quickly may possibly give the teachers and students wrong impressions about what they have learned and what they will learn.

Careful inspection of textbooks prior to their implementation is mandatory. Although curriculum and textbook objectives on the function were consistent with each other, the inherent contradictions illustrated in chapter I suggest that revisions should be made in future editions. In fact, in discussion with the head of the mathematics department of a local high school, he informed me that, volumes 2 of the Math 416 and 436 textbooks are currently in reprint.

It is necessary for developers to keep in mind that the textbooks should be intended for students of all levels and abilities. As mentioned in chapter I, the current series of textbooks presents information at a level of difficulty that is generally superior, hence, possibly incomprehensible for a large portion of the students. In future selection and implementation of mathematics textbooks, presentations of mathematical topics need to be carefully considered and meticulously inspected on mathematical and didactic levels in order to help reduce possible inconsistencies and contradictions. These are responsible in forming misconceptions that students bring with them to the next levels of their mathematics education.

Consistent review of research literature as a resource tool by teachers and developers can aid in the strengthening of teachers abilities and improve the presentation of the mathematical content that is found in textbooks whose selection meets the criteria of the curriculum. People responsible for curriculum development should invite mathematics education researchers as consultants, not just experienced teachers.

In the end, it is not only important to keep in mind what we define a function to be, or what we want the students to learn, but also what they are to leave secondary mathematics education with. To reiterate one of Vinner's (1992) questions, "what will remain in our students minds after the end of the course and final exam?"

Curricula may undergo changes every year, textbooks may come and go, but are we preparing our students with the appropriate knowledge to pursue post-secondary mathematics studies? With or without the graphing calculator, one of the main goals is to help students develop their intuitive mathematical abilities towards procedural and conceptual understanding of notions.

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Appendix A

Curriculum Requirements Concerning The Notion of Function As Presented in MAPCO's June 2000 Teacher Support Session

The following explores the intentions of the curriculum as presented by MAPCO (Math Action Plan COmmittee). The following is a listing of the curriculum requirements where the function is concerned. The following is taken from MAPCO's implementation session of 2000.

Math 116:

- To explain a rule relating a number to its rank in a sequence.
- To express in symbolic language a rule relating a number to its rank in a sequence.
- To use the rule relating a number to its rank in a sequence.

Math 216:

- To give a comprehensive description of a situation represented by a table of values.
- To give a comprehensive description of a situation represented by a graph.
- To represent a situation, using a table of values.
- To represent a situation comprehensively, using a graph.

Math 314: i. To illustrate the type of dependence characterizing the relationship between the variables in the situation:

- To determine the dependent variable and the independent variable in a given situation.
- To represent the rule that applies in a given situation, using a table of values.
- To represent a situation and its corresponding rule by means of a graph, given a table of values.
- To express in their own words the relationship between the variables in a specific situation, given the description of that situation, a table of values or a graph.

ii. To solve problems related to a situations in which a linear relationship exists between the variables:

- Translate a situation involving direct or partial variation into an equation.
- To translate an equation involving direct or partial variation into a word problem.

- To determine the rate of change in a given situation involving direct or partial variation, given the corresponding equation or graph.

- To provide a qualitative description of how a parameter change will affect a graph, given the equation for a situation involving direct or partial variation.

Math 416: To analyze variations using different modes of representation:

- To determine the dependent variable and the independent variable in a given situation.

- To make a table of values for a given situation.

- To determine the most appropriate scale for the graph of a given situation.

- To draw a graph representing a particular situation, given a table of values.

- To compare different situations expressed by means of the same mode of representation.

Math 426: i. To analyze situations involving functions, using different modes of representation:

- To use symbols to represent a situation involving a function, indicating a source set, a target set and a rule of correspondence.

- To draw the Cartesian coordinate graph representing a situation involving a function, given an equivalent verbal description, table of values or a rule of correspondence.

- To prepare the table of values for a situation involving a function, given an equivalent verbal description, rule of correspondence or Cartesian coordinate graph.

- To describe the properties of a Cartesian coordinate graph representing a function:

    1. Increasing or decreasing function
    2. Sign
    3. Rate of change
    4. Axes of symmetry (if any)
    5. Maxima or minima (if any)
    6. X-intercept(s) (zeros)
    7. Y-intercept
    8. Domain and range

ii. To analyze polynomial functions of degree less than 3:

- To draw the Cartesian coordinate graph (a straight line) of a real polynomial function of degree 0 or 1, given the equivalent rule of correspondence.

- To determine from the rule of correspondence, the following information about a real function of degree 0 or 1: its rate of change, its x-intercept(s) (zeros), its y-intercept, its domain and range, its sign, whether its constant, increasing or decreasing, and the member of its domain associated with a given image.

- To draw the Cartesian coordinate graph (a parabola) pf a real polynomial function of degree 2, given the equivalent rule of correspondence.

- To determine from its rule of correspondence, the following information about a real polynomial function of degree 2: its extreme (vertex of the parabola), its zeros (if any), its y-intercept, its domain and range, the intervals of increase or decrease, its sign and the member(s) of it domain and range associated with a given image.

- To use algebra to convert the rule of correspondence for a real polynomial function of degree 2 from the general form,

$$F(x) = ax^2 + bx + c,\ a \neq 0$$

  Into the standard form,

$$F(x) = a(x - h)^2 + k,\ a \neq 0$$

  And vice versa.

- To determine the relationships between the changes in parameters of the rule of correspondence for a real polynomial function of degree less than 3 and changes in the equivalent Cartesian coordinate graph.

- To determine the rule of correspondence of a real polynomial function of degree 0 or 1 represented by a straight line, given the slope of that line and a point on that line or two points on that line.

- To determine the rule of correspondence of a real polynomial function of degree 2 represented by a parabola, given the vertex of that parabola and another point on that parabola or its given zero(s) and another point.

Math 436: i. To analyze situations involving functions, using different modes of representation:

- Same 4 points as part i. of Math 426.
- To determine the relationships between changes in the parameters of the rule of correspondence of a function and changes in the equivalent Cartesian coordinate graph.

ii. To analyze polynomial functions of degree less than 3:

- Same 8 points as part ii. of math 426.
- To graph the sum, difference and product of two real polynomial functions, given the graph or rule of correspondence of each of these functions.

Math 526: To solve problems by using functions involving one real variable to develop a model for a given situation:

The following objectives pertain to the absolute value, square root, exponential, logarithmic, rational, sine, cosine, tangent, and sinusoidal functions unless otherwise specified.

- To determine the relationships between the change in a parameter of the rule of correspondence of a function involving real variables and the change in the equivalent Cartesian coordinate graph (except for rational functions).
- To draw the Cartesian coordinate graph of a function given its rule.
- To describe the characteristics of the Cartesian coordinate graph of a function given its rule.
- To determine from its rule or its Cartesian coordinate graph the following information about a function.
    1. The domain and range, the element(s) of the domain associated within a given range.
    2. If they exist, extrema, and zeros, the y-intercept, the intervals within which it is increasing or decreasing, the sign, the equation of its asymptotes.
    3. The rule and the graph of its inverse (except for trigonometric and rational functions).
- To determine the rule of a function, given sufficient information (except for rational functions).
- To determine the rule of the function that represents a situation involving a problem to be solved, given the related model (except for rational functions).

Math 536: To solve problems by using functions involving one real variable to develop a model for a given situation:

- To determine the relationship between the change in parameter of the rule of correspondence of a function involving real variables and the change in the equivalent Cartesian coordinate graph.

- To draw the Cartesian coordinate graph of a function involving real variables, given its rule of correspondence[1].

- To describe the characteristics of the Cartesian coordinate graph of a function involving real variables, given its rule of correspondence[1].

- To determine from its rule of correspondence, the following information about a function involving real variables:

  a. The domain and range.

  b. The element(s) of the domain associated with a given range.

  c. If they exist, all extrema, zeros, the y-intercept, the intervals of increase or decrease, the sign, the equation of the asymptotes, its inverse, if it is a function[1].

- To determine the rule of correspondence of a function involving real variables, given sufficient information or its graph[2].

- To algebraically or graphically determine the sum, difference, product, quotient or composite of two functions involving real variables, given their graphs or rules of correspondence.

- To represent a situation by a function involving real variables.

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[1] This function can be an absolute value function, a step function ("the greatest integer less than or equal to," truncated, rounded), a square root function, a rational function, an exponential function, a logarithmic function, a sine, cosine, or tangent function, or a sinusoidal function.
[2] This function can be an absolute value function, a square root function, an exponential or logarithmic function, or sinusoidal function.

Appendix B

Intentions of the Authors of The Guy Breton Textbook Series Concerning the Notion of Function

The following are the intentions of the authors of the Guy Breton textbooks where the notion of the function is concerned at the various levels. The textbooks used for math 426 and math 526 are the same texts being used for math 436 and math 536, respectively.

Secondary II: *The main ideas*:

- The relationship between the components of a problem;
- Tables of values
- Graphs
- Rules or equations

> *Terminal Objective*: to translate one representation of a situation into another.

Secondary III: *The main ideas* (with respect to relations);

- The concept of a relation
- Dependent variables
- Various modes of representing variables
- Various relations
- Various characteristics of relations

> *Terminal Objective*: to illustrate the type of dependence between the variables in a given situation.

> *The main ideas* (With respect to linear relations):

- Rates of change
- Relations vs. equations
- Linear relations the role of parameters
- Combined relations linear equations

> *Terminal Objective*: to solve problems related to situations in which a linear relationship exists between the variables

Secondary IV (Math 416): *The main ideas*:

- Variation
- Step variation
- Exponential variation
- Comparing situations of variations

Terminal Objective: to analyze variation using different modes of representation.

Intermediate Objectives: for an exponential relation or step function between two variations:

- To determine the independent and the dependent variables
- To make a table of values for the relation
- To draw a Cartesian graph for the relation
- To compare different situations using tables and graphs for each

Secondary IV (Math 436): *the main ideas* (with respect to functions):

- The concept of function
- Function notation
- Modes of representation
- Properties of functions
- Roles of parameters in the rule of a function

Terminal Objective: to analyze situations involving functions, using different modes of representation.

Intermediate Objectives:

- To use symbols to represent a situation involving a function, indicating a source set, a target set and a rule of correspondence.
- To translate from one mode of representation to another.
- To describe the properties of a Cartesian coordinate graph representing a function.
- To determine the relationships between changes in the parameters of the rule of correspondence of a function and changes in the equivalent Cartesian coordinate graph.

The main ideas (with respect to polynomial functions):

- Polynomial functions
- Basic polynomial functions
- Transformed polynomial functions
- Constant function
- Linear functions quadratic functions
- Solving equations of degree one or two
- Operations on functions

Terminal Objective: to analyze polynomial functions of degree less than three.

Intermediate Objectives:

- To draw graphs and determine properties of functions of degree 0, 1 or 2.
- To determine the relationships between changes in the parameters of the rule of correspondence for a function and changes in the corresponding Cartesian graph.
- To use algebra to convert the rule of correspondence for a quadratic function from the general form into the standard form and vice versa.
- To determine the rule of a function of degree 0, 1 or 2 given the value of certain parameters, the coordinates of certain points, a table of values or graph.
- To graph the sum, difference and product of two polynomial functions.

Secondary V (Math 536): *the main ideas* (with respect to exponential and logarithmic functions):

- Laws of exponents and exponential functions
- Laws of logarithms and logarithmic function
- Solving exponential and logarithmic equations

Terminal Objective: to solve problems using exponential and logarithmic function as models for situations.

Intermediate Objectives:

- To construct a Cartesian Graph of an exponential or logarithmic function and describe its characteristics.
- To determine the properties of an exponential and logarithmic function.
- To make the connection between the change in a parameter of the rule for an exponential or logarithmic function and the change in the corresponding graph.
- To represent a situation using exponential or logarithmic function.
- To explore different operations on exponential or logarithmic functions algebraically and graphically.
- To apply the properties of logarithms in simplifying logarithmic expressions and finding the solution set of an equation in one variable involving logarithmic and exponential expressions.

The main ideas (with respect to functions in the domain of real numbers):

- Concept of a function
- Properties of a function
- Transforming functions
- Linear functions
- Quadratic functions

- Absolute value functions
- Square root functions
- Step functions
- Rational functions

Terminal Objective: to solve problems using functions involving real variables as models for given situations.

Intermediate Objectives:

- To represent a situation with a function involving real variables.
- To determine the properties of a function involving real variables.
- To determine the relationships between parameters in the rule of a function involving real variables and the corresponding Cartesian graph.
- To graph and to describe characteristics of certain real functions.
- To determine the rules of certain functions involving real variables, given sufficient algebraic or graphical data.
- To determine the sum, difference, product, quotient or composite of two functions in the domain of real numbers, given their graphs or rules.

Appendix C

Student Response Sheets to The Assigned Activity

The student responses are numbered in the form of C3, C4, etc. because reference has been made to some of the responses in the thesis.

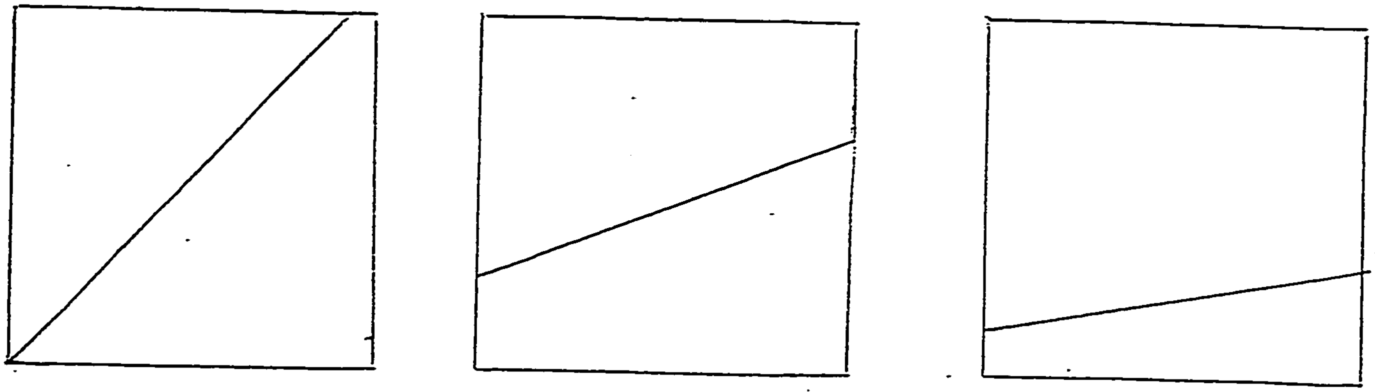
Name: Nina

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

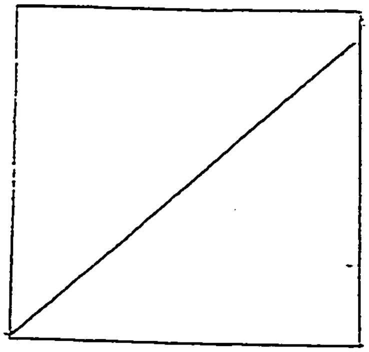
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No. They could very easily be the same graph but seen in a different way. The position of the graph could've been changed. The window could've been changed or even just the zoom.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



It could be either one of them or even neither. It could be a zoom into a part of a quadratic function or it could be a direct or partial variation. It all depends on how you see it.

Please give;

i. a possible equation for this relation,

$y = ax$

ii. include your domain and range, and also

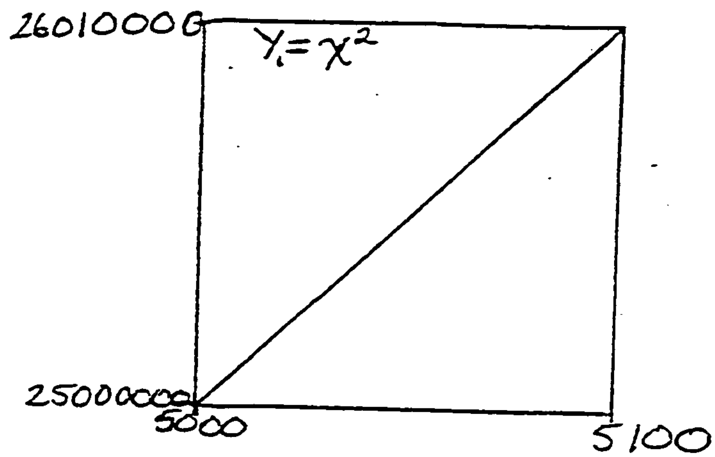
Domain + Range = $\mathbb{R}$

iii. describe the horizontal and vertical scales used to graph it.

ymin = 0 xmin = 0
ymax = 50 xmax = 50
yscale = 5 xscale = 5

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No they're both wrong. You're rule says that this is a quadratic function so it is impossible for it to be a linear function who's rate of change is constant.

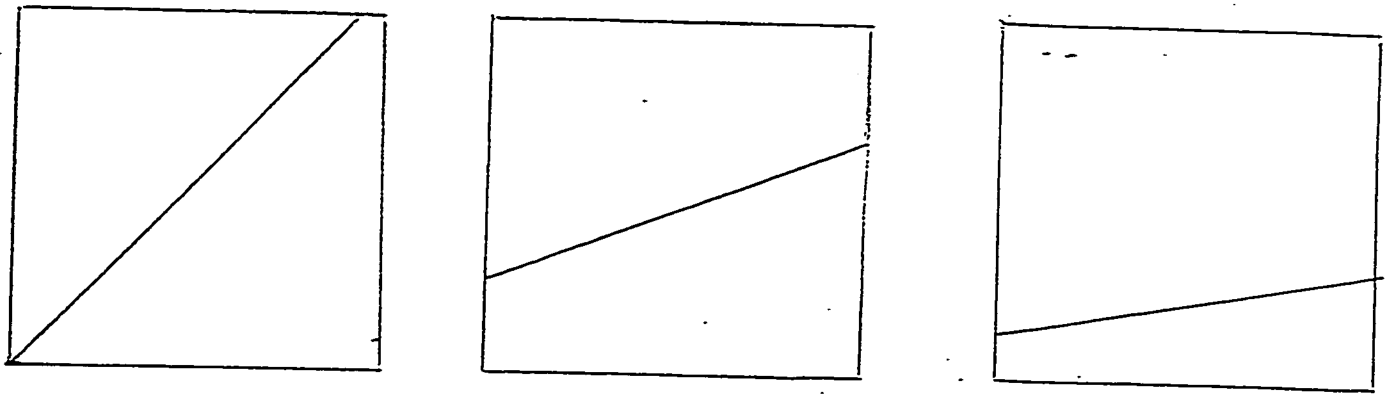
Name: Samuel

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

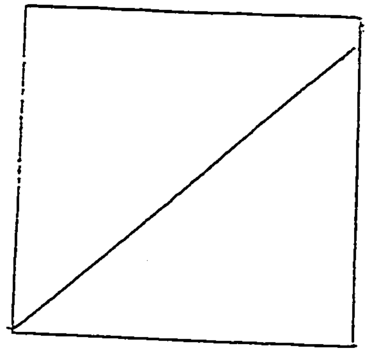
I. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

no they are all linear

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain. *it is a direct becaus the line cross through the zero [illegible]*



Please give;

i. a possible equation for this relation,

$y=x$

ii. include your domain and range, and also

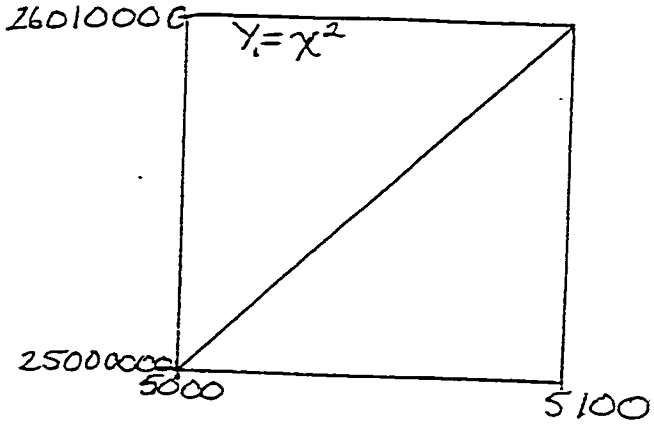
domain is 0
range is 0

iii. describe the horizontal and vertical scales used to graph it.

vertical scal 1
horizontal scal 1

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

no because your equatation will not be $y=X^2$ because if $y=X^2$ it will not be a linear function

Anthony

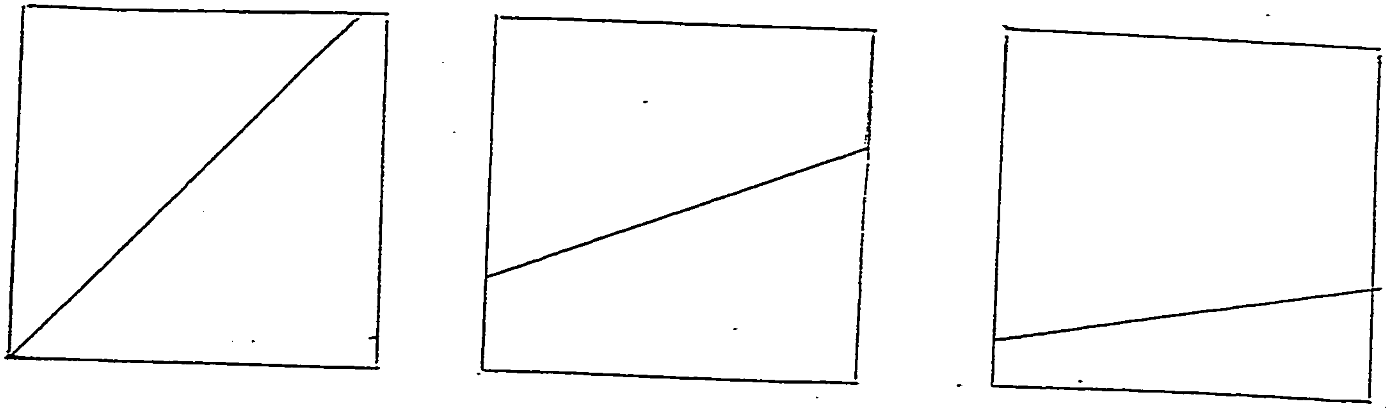
Name:

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

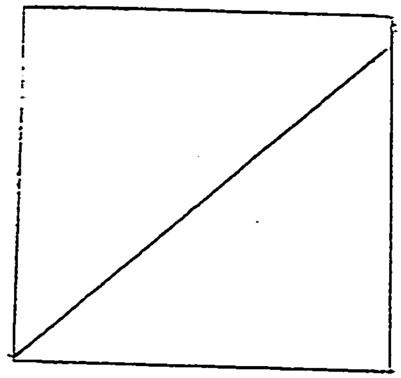


Are these functions necessarily different from each other? Please explain.

No because if you zoom in on a part of the line you could get different views of the line.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

It could represent a partial-variation relation because we could only be seeing a fragment of the line not knowing where the y-intercept is.



Please give;

i. a possible equation for this relation,

X+1

ii. include your domain and range, and also

domain = [0, ∞[

range = [1, ∞[

iii. describe the horizontal and vertical scales used to graph it.

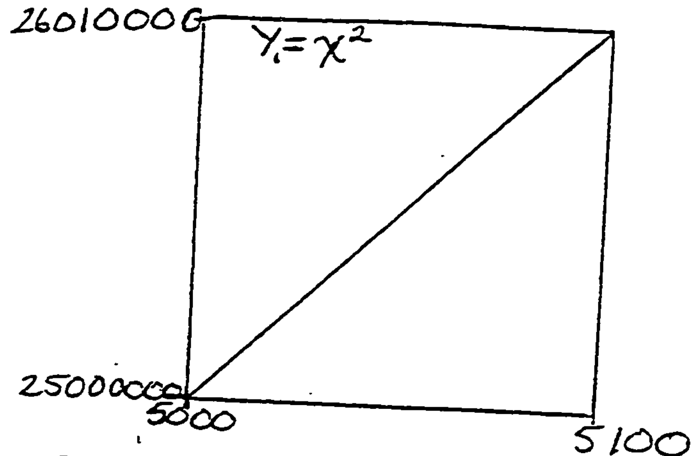
every time x goes up y goes up the same thing but you could have y starting at a certain point on the line giving it a y-intercept

For the last two questions, it may help to consider how you would enter them in the TI-83.

Use a zoom 6.

Xmin = -10
Xmax = 10
Xscl = 1
Ymin = -6
Ymax = 6
Yscl = 1
Xres = 1

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

It isn't a linear function because $y=x^2$ is a quadratic function but it could be a constant rate of chane because the rate of change is always $y=x^2$.

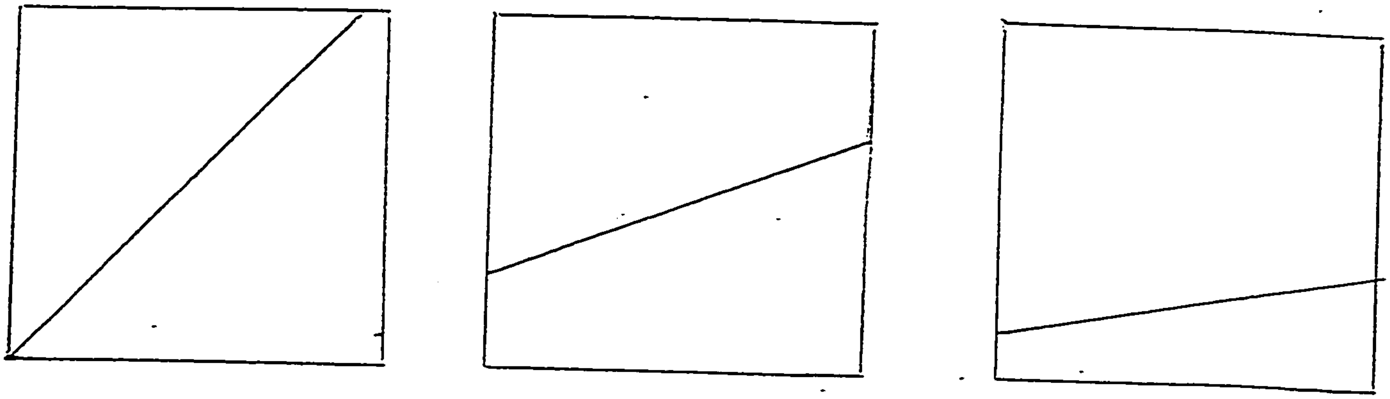
Name: MICHAEL

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

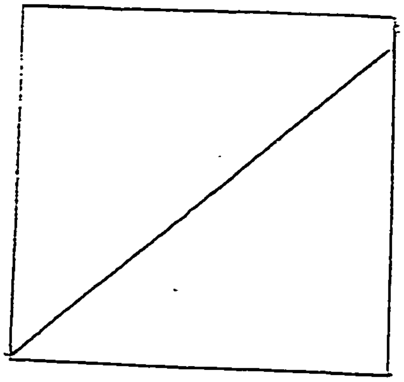
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No, since we dont see the axis and the scales, it could be the exact same function seen from differen zooms.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



IT DEPENDS WERE THE AXIS ARE.

Please give;

i. a possible equation for this relation,

$y = 2x$

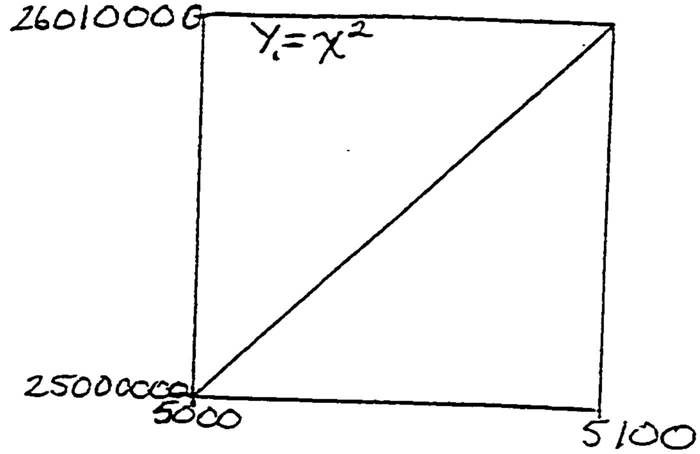
ii. include your domain and range, and also

$-\infty$ to $+\infty$ range $\quad$ $-\infty$ to $+\infty$ domain

iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No, I think the rate of change isnt a constant because the line is a parabola

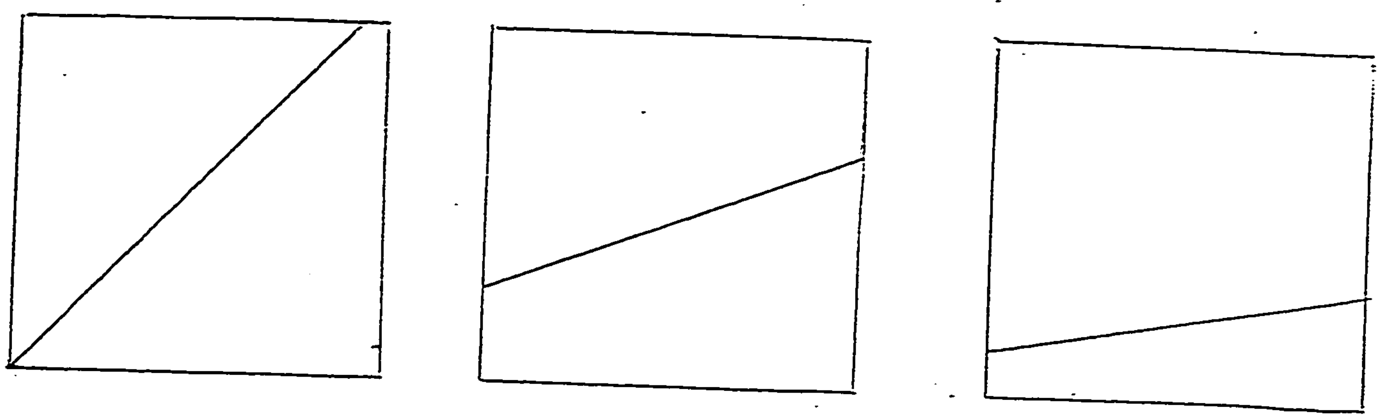
Name: Shawn

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

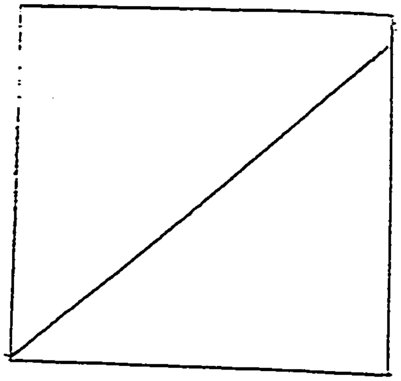


Are these functions necessarily different from each other? Please explain.

no. This is because the WINDOW settings could be set different causing us to see the line at a different angle and less steep

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

it could be direct-variation or partial variation depending on where the axes is located



Please give;

i. a possible equation for this relation,

y = 2x

ii. include your domain and range, and also

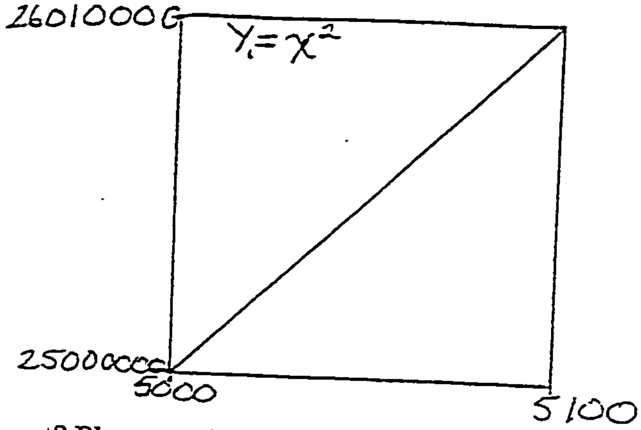
-∞, +∞ WINDOW: xmax = 100
xmin = 0
ymax = 1
ymin = 100

iii. describe the horizontal and vertical scales used to graph it.

xscl = 1
yscl = 1

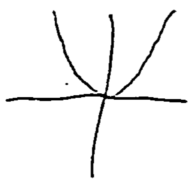
For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No it is not a linear function. This is because it has the exponent 2 (x^2) after it. This makes it a quadratic function and looks like this.



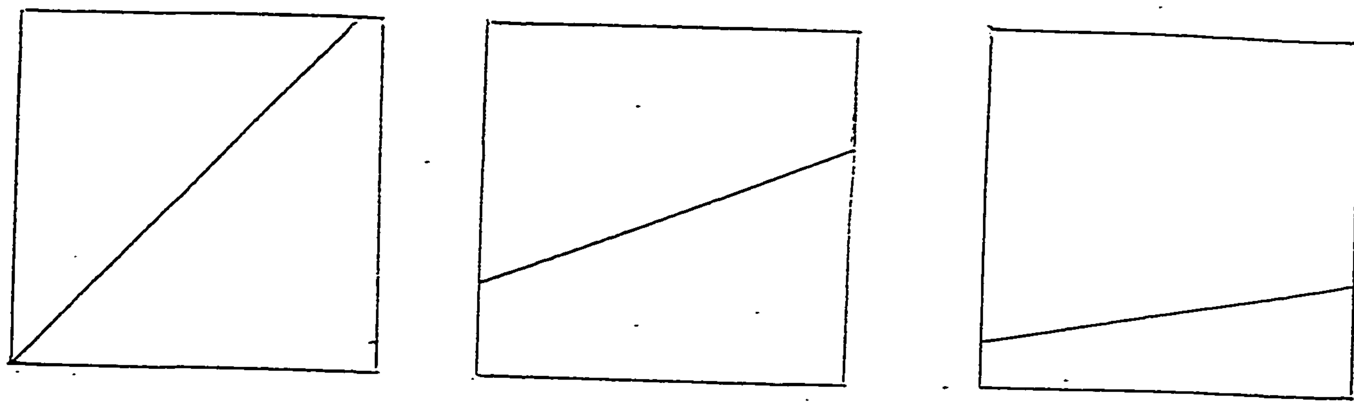
yes the rate of change is constant. It can go to $-\infty$ and $+\infty$ and have different x, y values the whole time.

Name: Marie-Eve December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

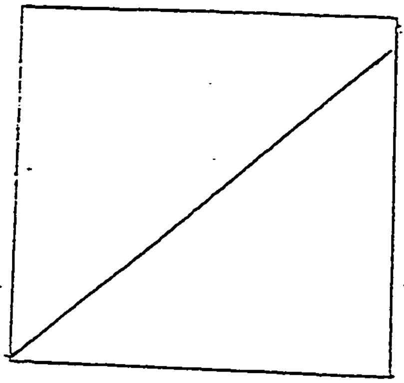


Are these functions necessarily different from each other? Please explain.

No they're not necessarily different. Some may be steeper than another if they have a different window (x and y scale). The smaller the Δ in x or y is, the steeper the line will be.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

It could represent both because it passes through the origin if you consider the corner of the box as the origin. But...



It could also be a zoom on a part of the graph and you can't even see where it crosses the y-axis.

Please give;

i. a possible equation for this relation,

$f(x) = 2x$

ii. include your domain and range, and also

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

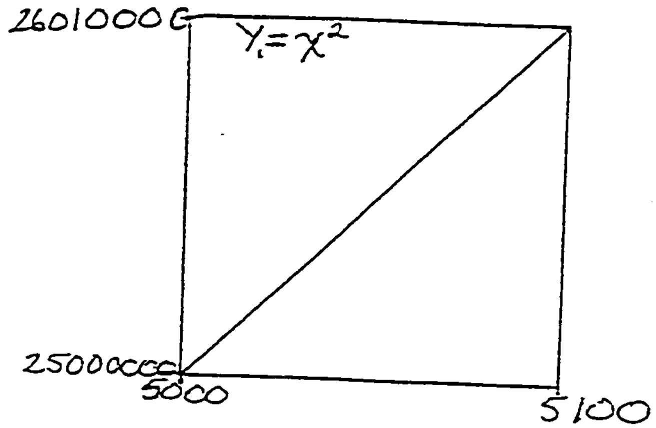
iii. describe the horizontal and vertical scales used to graph it.

y min.: -500 y max.: 500 Δ in y = 100

x min. -100 x max.: 100 Δ in x = 100

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

Yes, it's a linear function because the slope is straight and it rises (it could also decline) towards the right corner of the graph.

It has a constant rate of change because the line is straight.

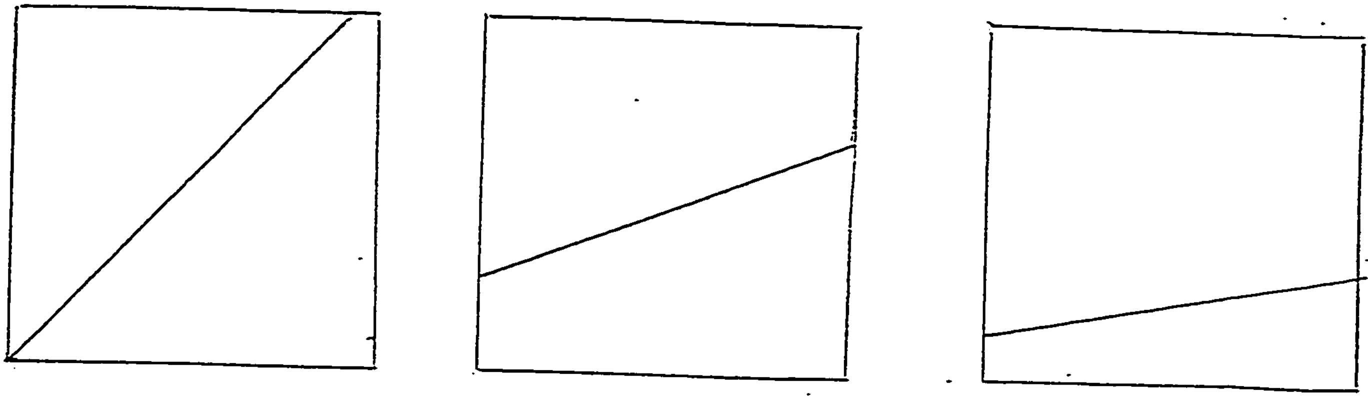
Name: Catherine

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

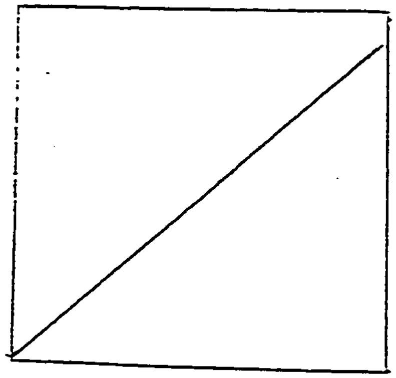
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No, It depends on what you put in the window (xmin, xmax, ymin, ymax) and where you zoom in.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



It could represe
a partial-variati
relation becaus
we cannot see
if the line
passes
through the
origin or no
It could be a
Zoom in on a
part of the
line.

Please give;

i. a possible equation for this relation,

$f(x) = 11x$

ii. include your domain and range, and also

Domain → [0, 7]

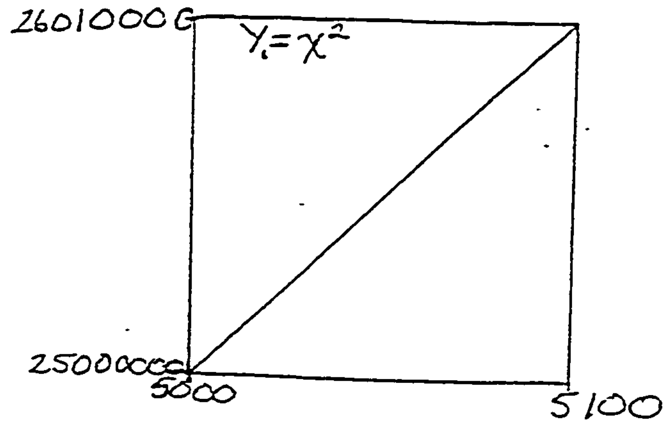
Range → [0, 150]

iii. describe the horizontal and vertical scales used to graph it.

Both scales go up by one

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

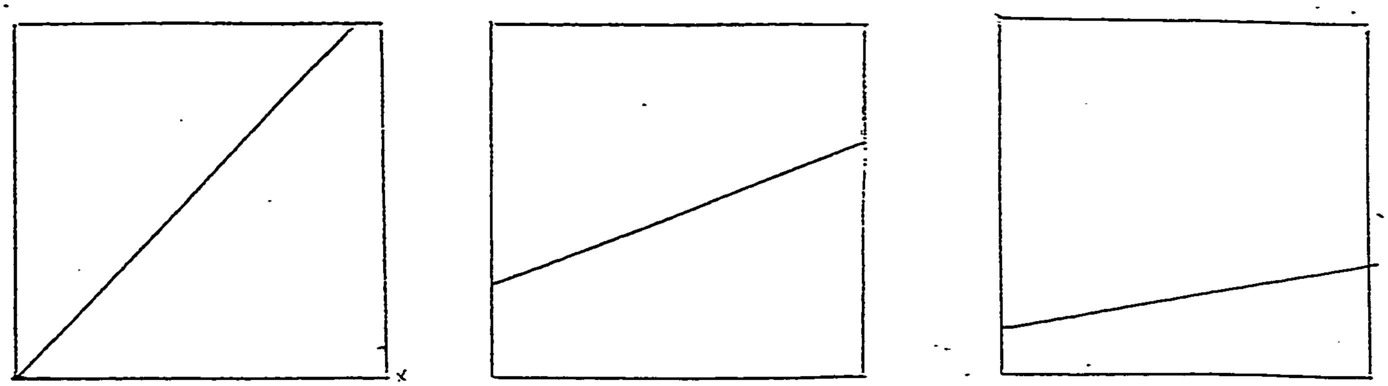
This is not a linear Function because a linear function is of degree 1 while this is of degree 2. The rate of change is constant because it is always "x" being squared.

Name: December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

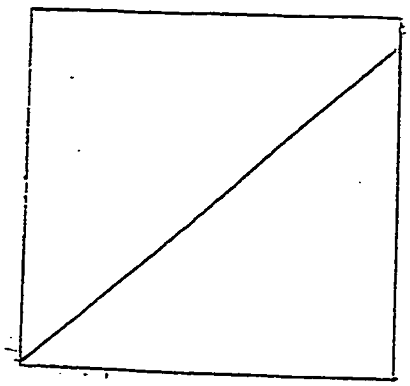
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No. because the lines vary according to how you set up your window on the graphic calculator.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



Please give;

i. a possible equation for this relation,

$y = 2x$

go to [Y=]

ii. include your domain and range, and also

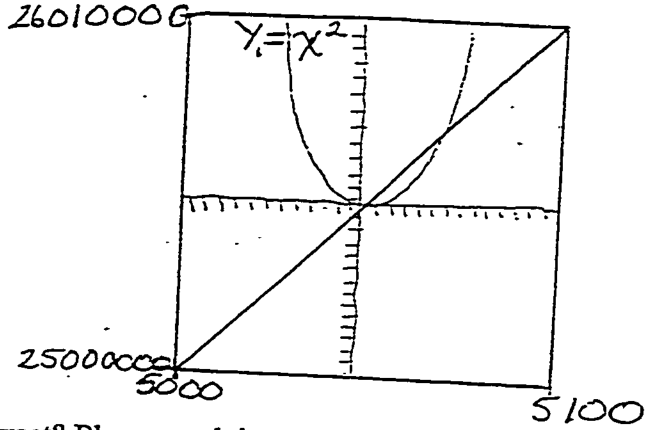
range = 2

domain = 0

iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

It is a quadratic function. x^2 would look like (above)

x	y	1st difference	2nd difference
0	0		
		1	
1	1		2
		3	
2	4		2
		5	
3	9		2
		7	
4	16		2
		9	
5	25		2
		11	
6	36		

rate of change is constant

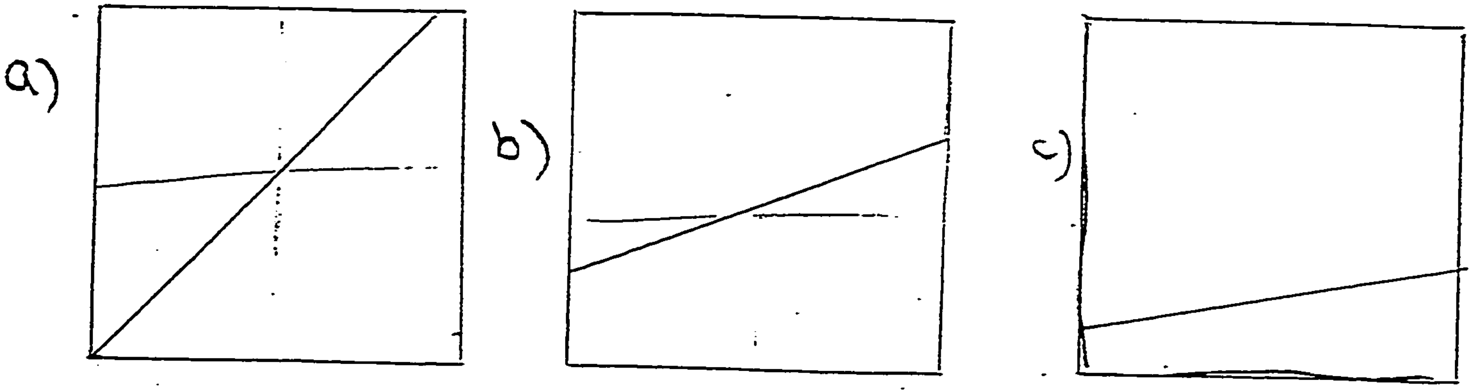
Name: Fany

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

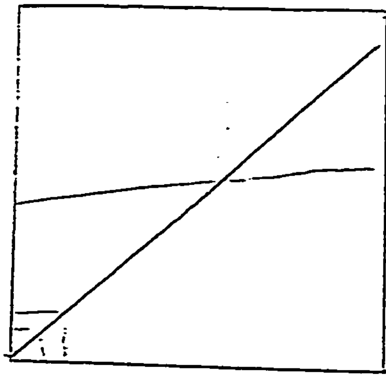


Are these functions necessarily different from each other? Please explain.

Yes! the angle of the lingnes are different and if I could belive that b and c could be the same graph, a doesn't look like it unless the ratio of the graph has been dramaticaly changed for the second two.

Actually, It could realy be the same graph just viewed at a zoom 6 then at a closer view of the origin and then of only the 1Q.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain. It could be eather! the screan dosen't nesescarly start a y=0 x=0 it could be higher up in the graph and then the linyn wouldn't go through the origin



Please give;

i. a possible equation for this relation,

$y=x$

ii. include your domain and range, and also

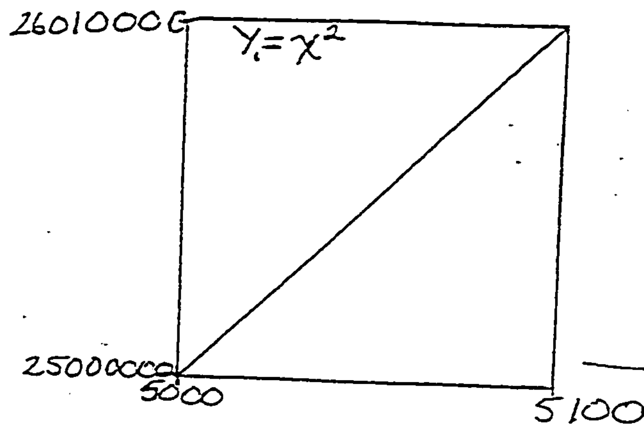
x [goes to the infinity both ways]
y [goes to the infinity both ways]
(assuming this situation alows it)

iii. describe the horizontal and vertical scales used to graph it.

scales of 1, 2, 5 or 5 for both if the equasion is $y=x$

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

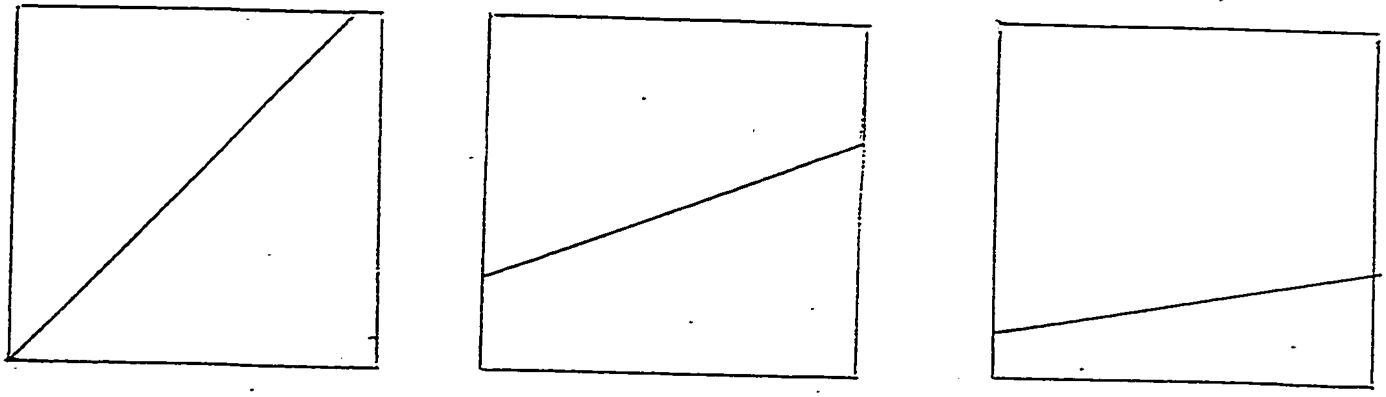
No! the may look liniar but $y=x^2$ will give you a parabola and not a straight line the rate of change isn't constante.

Name: December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

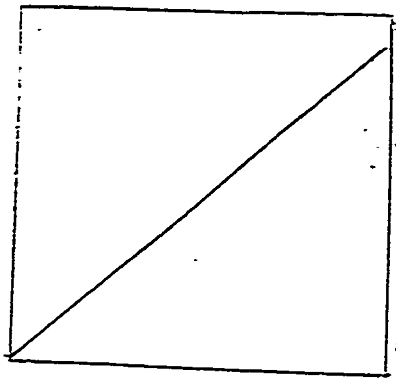
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

-yes because each function intercepts the y-axis at a different point and the rate of change are not the same but that is what it is at first glance. If you look at it more carefully, it is not necessarily different because the window format (or scale) may be different to show the situation at a larger scale

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



It could represent both because if the scale of the graph was bigger than normal, it may seem that the line intercepts the y-axis at (0,0) but it may actually be intercepting the y-axis at (0,1) but the scale is too huge to see it.

Please give;

i. a possible equation for this relation,

$y = x + 1$

$f(x) = x + 1$

ii. include your domain and range, and also

domain = $(0, +\infty($

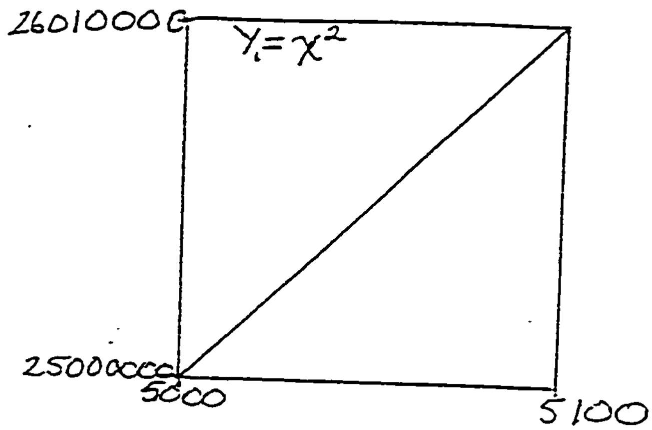
range = $(0, +\infty($

iii. describe the horizontal and vertical scales used to graph it.

both scale could be in the thousands/units or even greater

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

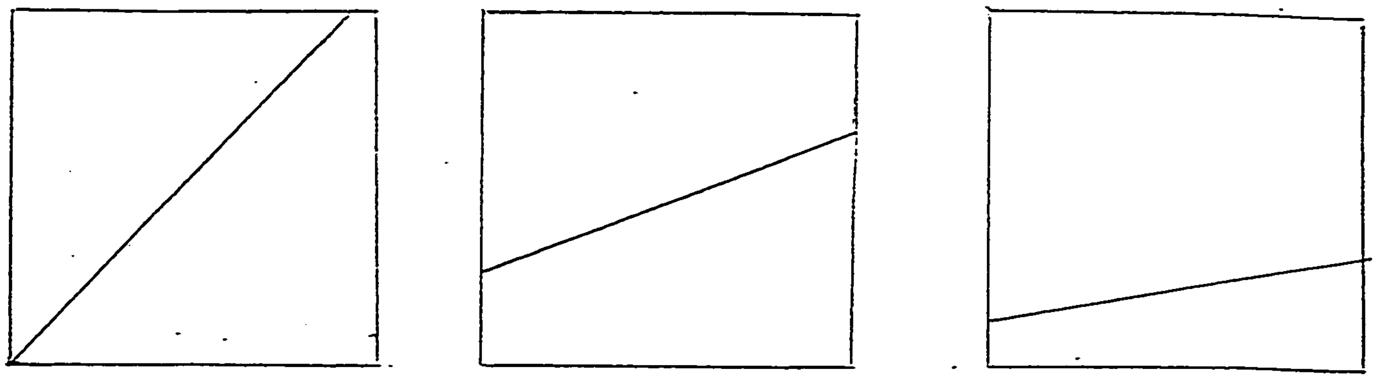
yes because the scale for the graph is proportional

Name: Nancy December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

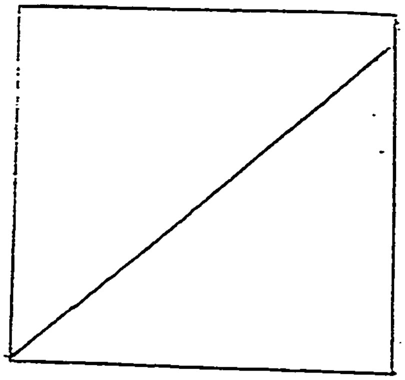
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No, because on the TI-83 calculator you can zoom onto a certain point, and see the line up close. These are probably all just the same line, but in different views.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



The graph could represent both of these because, no y-intercept is shown in the box.

Please give;

i. a possible equation for this relation,

$2x = y$

ii. include your domain and range, and also

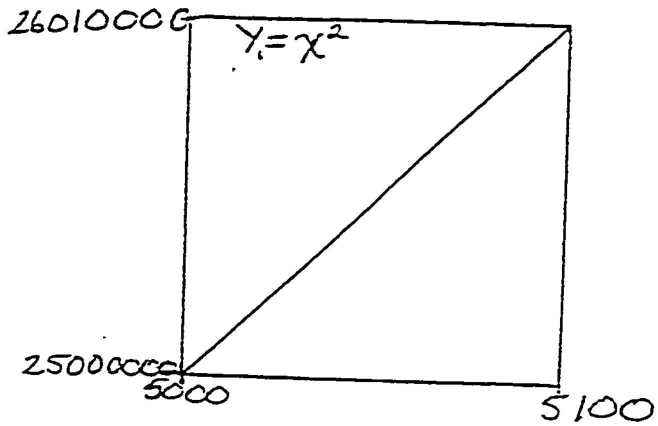
range $[-\infty, +\infty[$
domain $[-\infty, +\infty[$

iii. describe the horizontal and vertical scales used to graph it.

The scale would be 2, because the rate of change is 2.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No, you claim is not correct, because the X value cannot be squared (X^2) to obtain a linear function. I still don't see on the graph that the Y-intercep is "0", so you can never be sure.

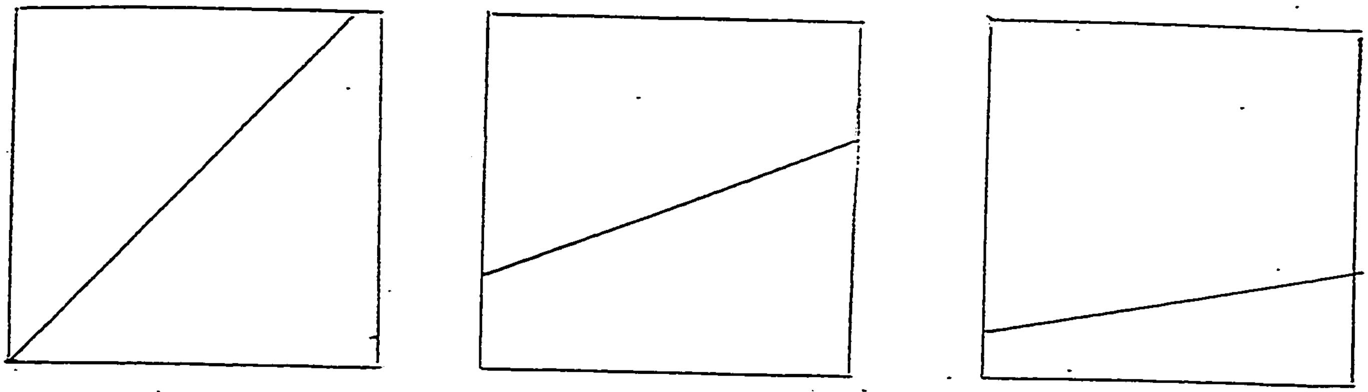
Name: Alex

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

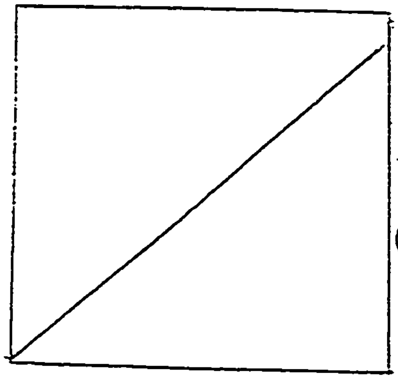
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

They are not necessarily different from each other because they are all partial variation linear relations. They just have different numbers but the basis of the equation is still the same.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



A Partial because It's linear but does Not Pass through (0,0).

Please give;

i. a possible equation for this relation,

.8x + 1

ii. include your domain and range, and also

-∞ to +∞ range

-∞ to +∞ domain

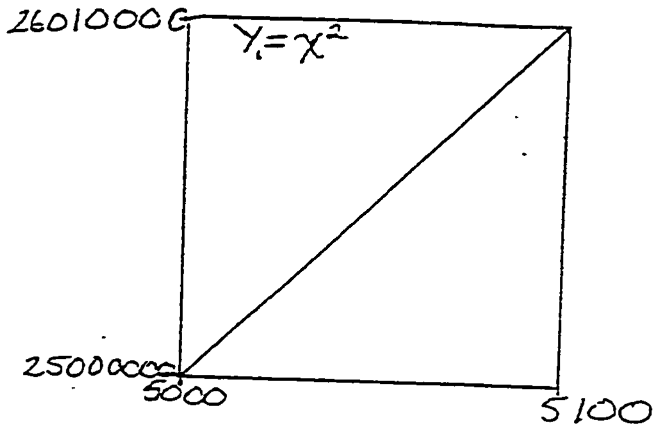
iii. describe the horizontal and vertical scales used to graph it.

vertical change of 1

horizontal change of .8

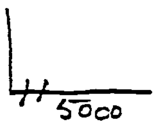
For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

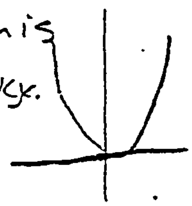
- Its not a Linear function because it doesn't start a (0,0). If you would of done it like this



it would of been okay.

2nd your equation is Quadratic. The Picture you Showed is for a Linear Function.

- By the look of this graph it's constant rate of Change but if we go with the equation $y=x^2$ the Graph would look like this



which would mean a fluctuating rate of change.

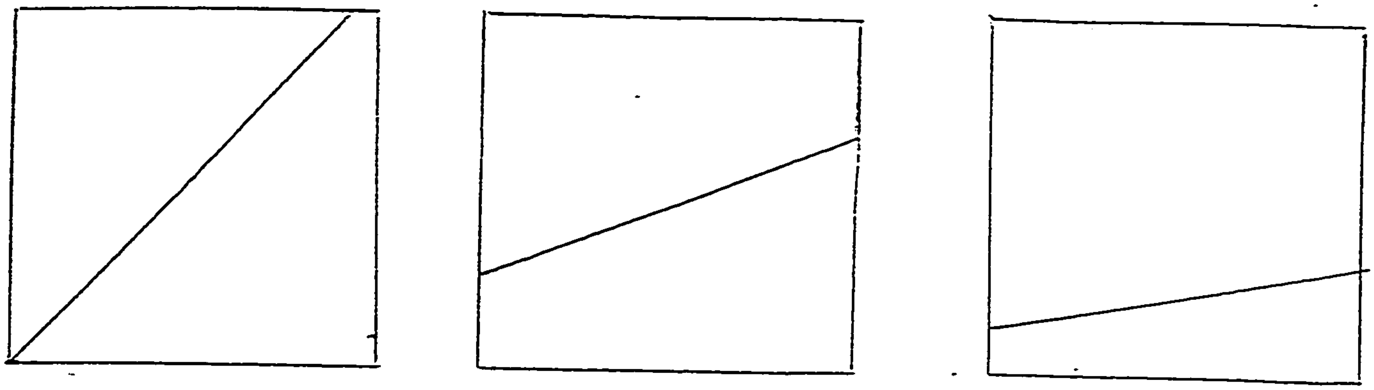
Name: M'gwl.

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

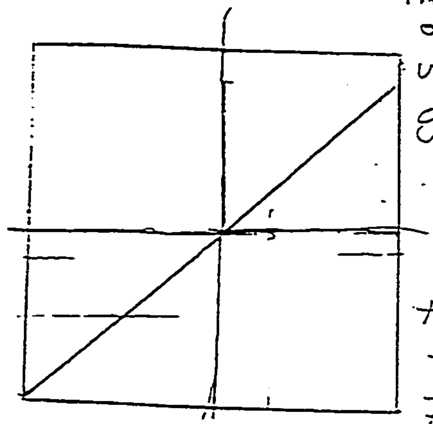
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

Not all of them are. The last two could be the same dependent on what type of window you are using to graph, i.e. The last one could be a closer view of the graph to the second one. Because you were using a smaller window the steepness of the graph wouldn't be as much so technically they could all be the same graph, but with different windows

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain. It could be both if your line of symmetry passed through the origin then it would be a direct variation, but if you had your graph window for the x axis is different from your y. the it could be a partial variation. If the axis are like the ones shown that of put in on the graph then it's a direct variation. It could also be the zoom.



Please give;

i. a possible equation for this relation,

$y = x$

ii. include your domain and range, and also

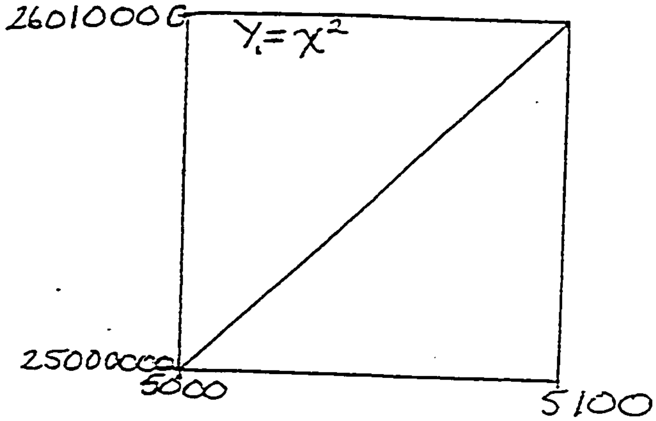
from $-\infty$ to $+\infty$

iii. describe the horizontal and vertical scales used to graph it.

I would use a scale up to 100 with increments of 10 for both the x and y axis, It would depend on the problem, though

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

Youre claim that it is a linear funtion isn't correct your equation is y = x2 which is a <u>quadratic funtion</u> The only reason it look linear is beause you are zoom in so far that you can only see it go up Like a liner funtion. The rate of chang is also wrong the increments aren't the same over the entire porabla.

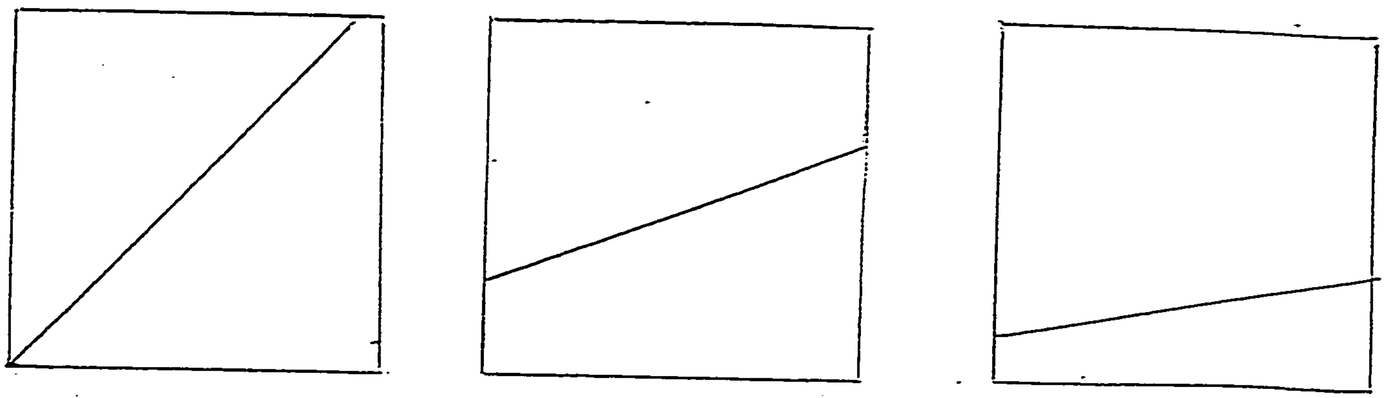
Name: Melissa

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

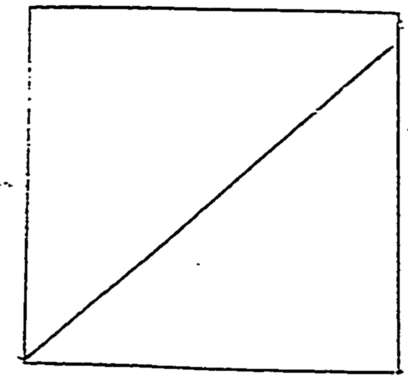
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

No, because obviously they go through different points. For example, the first one goes through the origin and the others don't.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



This graph represents a direct-variation relation, because if you were to label it properly, the line would pass through the origin. A partial-variation relation doesn't pass through the origin.

Please give;

i. a possible equation for this relation,

$y = x$

ii. include your domain and range, and also

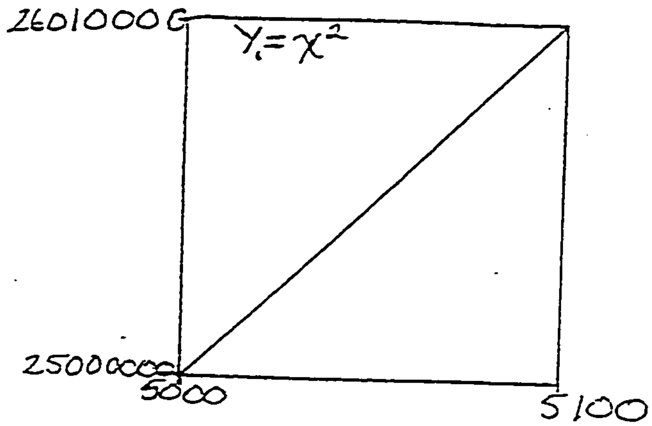
domain: [0, 10]
range: [0, 10]

iii. describe the horizontal and vertical scales used to graph it.

Both horizontal and vertical scales are 0.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

-No, because $y = x^2$ doesn't produce a linear function, and your rate of change is not constant in a quadratic function.

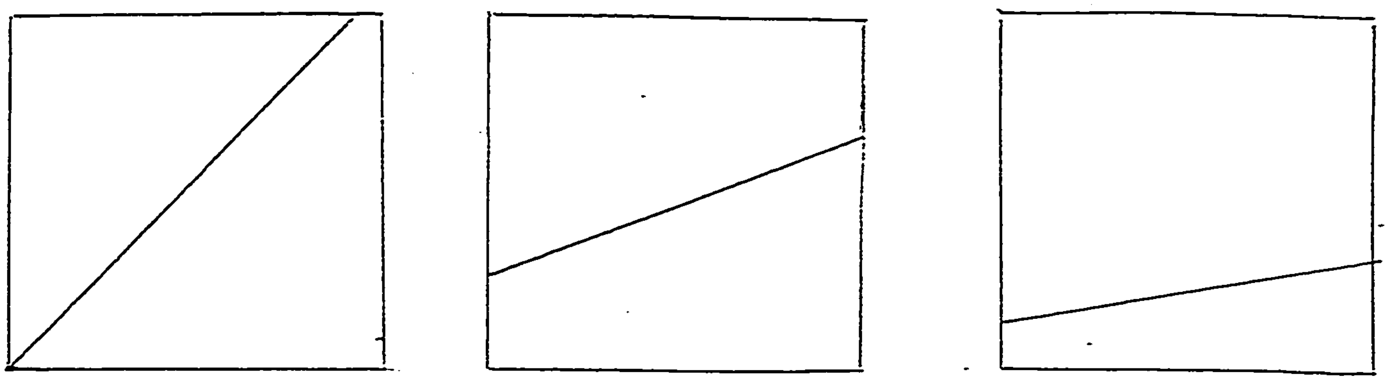
Name: Jason

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

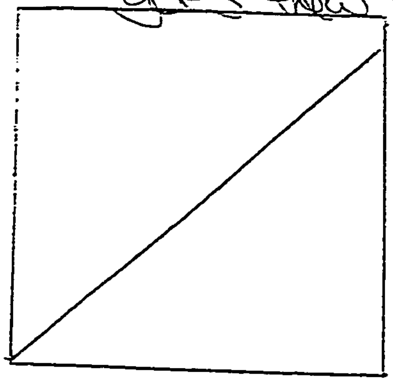


Are these functions necessarily different from each other? Please explain.

No they're no because they all go from negative to positive and the~~y all don't cross threw the 0~~ But box 1 goes threw the 0 of y and x so they are different

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

direct - variation because it goes threw the 0 of x and y



Please give;

i. a possible equation for this relation,

$y = x$

ii. include your domain and range, and also

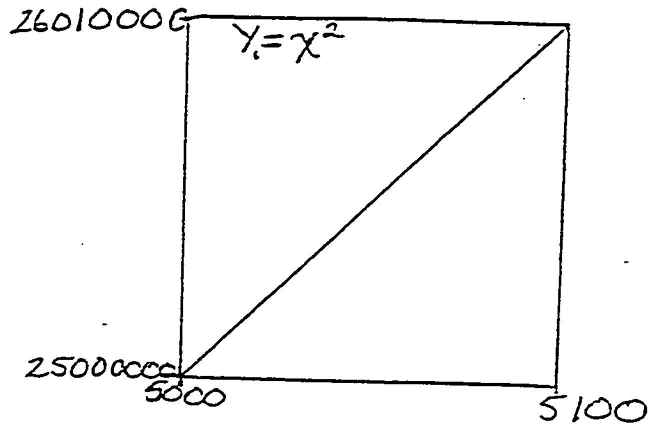
$[0, \infty[$, $[0, \infty[$

iii. describe the horizontal and vertical scales used to graph it.

what?

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

~~yes because~~

no, it's not linear because it does not go threw 0,0 but it's constant because the equation is $y=x^2$

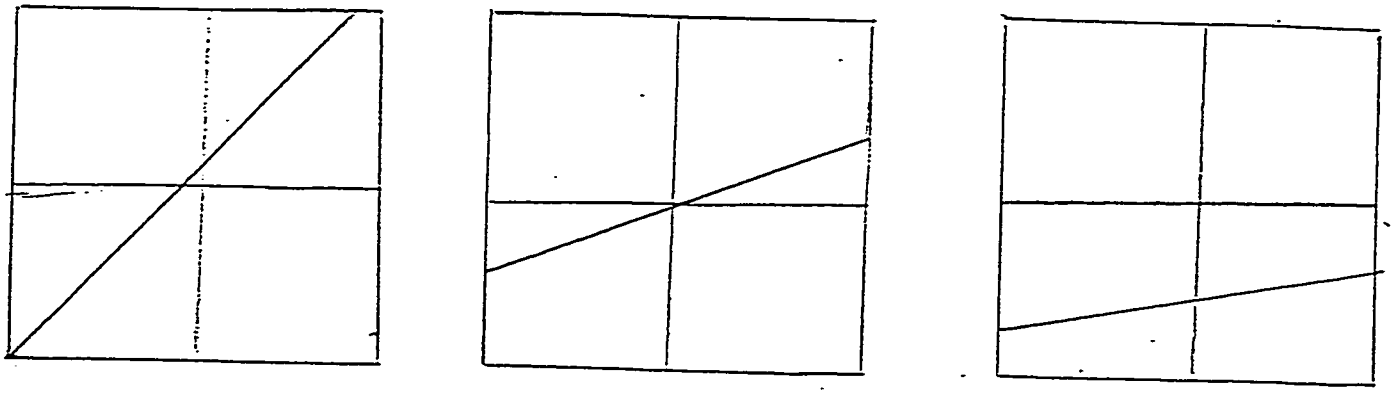
Name: Kim

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

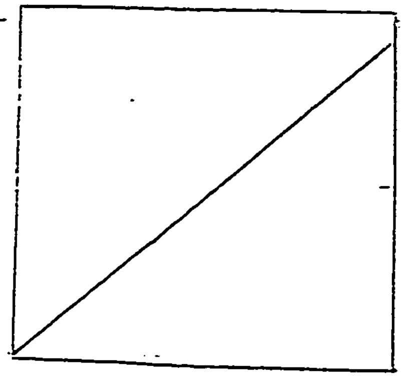
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

These functions are different from each other, but not by much. They are all partial linear functions because none of these go through the origin point. (0,0) Their only differences would be that they all have different points.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain. It looks to be a direct variation because it goes through the point of origin.



Please give;

i. a possible equation for this relation,

2x

ii. include your domain and range, and also

Domain + Range would be the IR (set of real numbers)

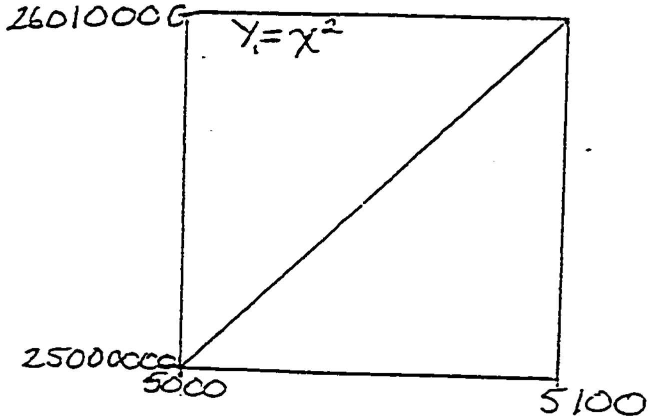
iii. describe the horizontal and vertical scales used to graph it.

The window used was

x min = -10   y min = -10
x max = 10   y max = 10
x scl = 1   y scl = 1

For the last two questions, it may help to consider how you would enter them in the TI-83:

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

This is correct point the line starts at the point of origin and it goes at a constant rate upwards, it doesn't alter at all.

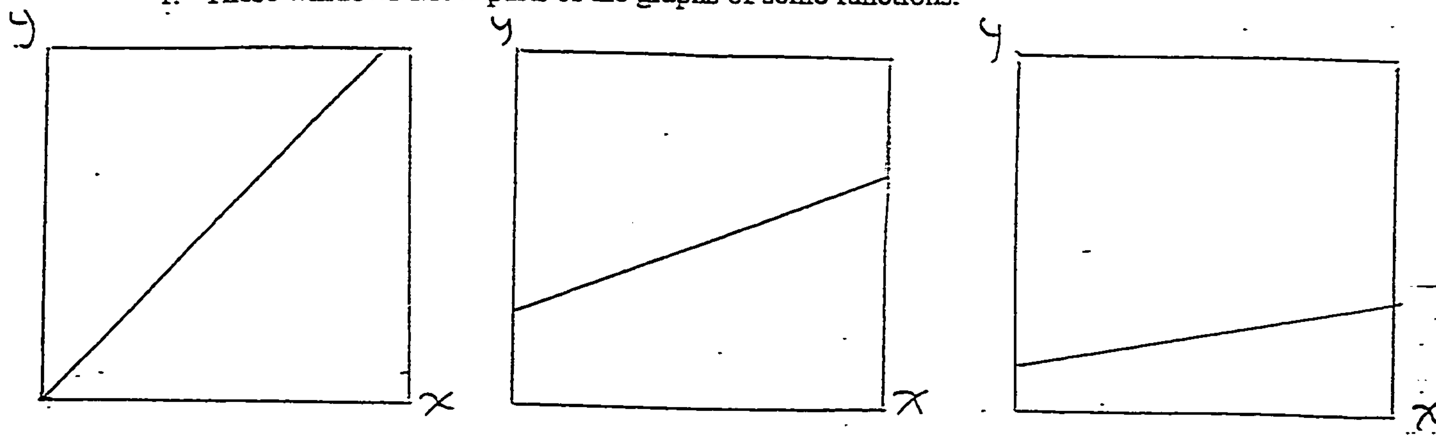
Name: Terry

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

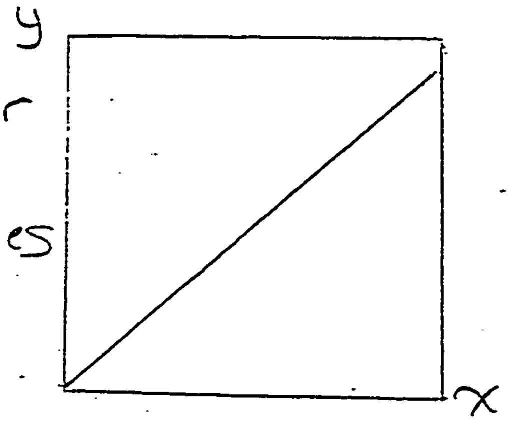


Are these functions necessarily different from each other? Please explain.

Yes. The functions are different from eachother because you can clearly see that there is a difference in the inclination of the slopes. This means that <u>all</u> three "a" values must be different, therefore so are the functions.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

A direct-variation because it is a linear graph which goes through the coordinates (0,0)



Please give;

i. a possible equation for this relation,

$f(x) = x$

ii. include your domain and range, and also

domain $=]-\infty, +\infty[$

range $=]-\infty, +\infty[$

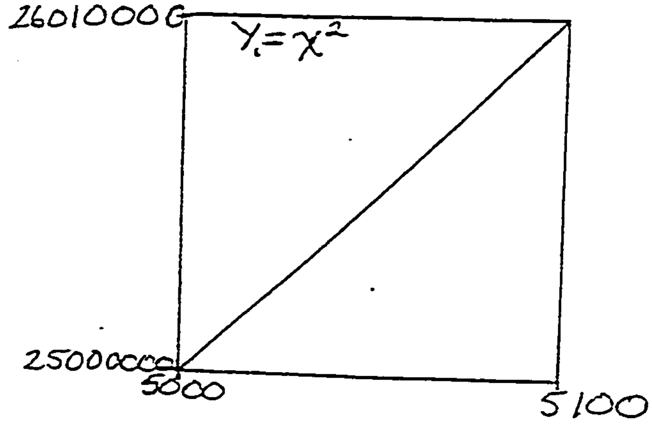
iii. describe the horizontal and vertical scales used to graph it.

horizontal → 0 to 100 by increments of 10

vertical = 0 to 100 by increments of 10

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No, The graph can not be linear because the equation for "Y" is $y=x^2$ which means you should have a parabola that opens upwards. The rate of of change isn't constant because as you go up the scale the difference between the numbers will get higher and higer. ie

	x	Y
5	4	2
	9	3
7	16	4
9	25	5

* If it were direct the table would look like this. →

x	Y
1	1
2	2
3	3

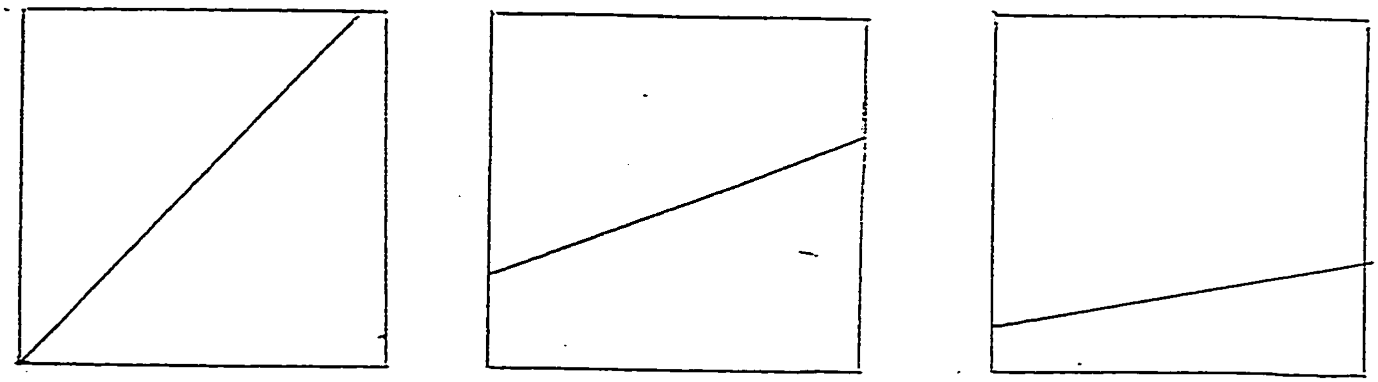
Y=

Name: Sarah December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

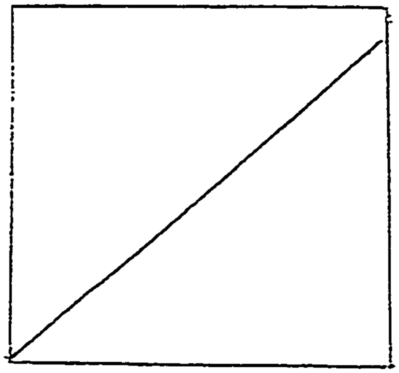


Are these functions necessarily different from each other? Please explain.

Yes because they have different y-intercepts.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

Yes because it is a straight line passing thrugh (0,0)



Please give;

i. a possible equation for this relation,

$y = 2x$

ii. include your domain and range, and also

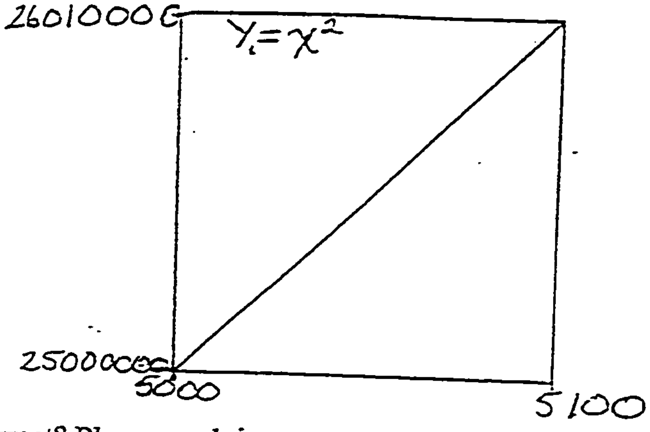
domain: (-10, 10)
range: (-10, 10)

iii. describe the horizontal and vertical scales used to graph it.

scale of 1

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

Yes because the graph is a straight line.

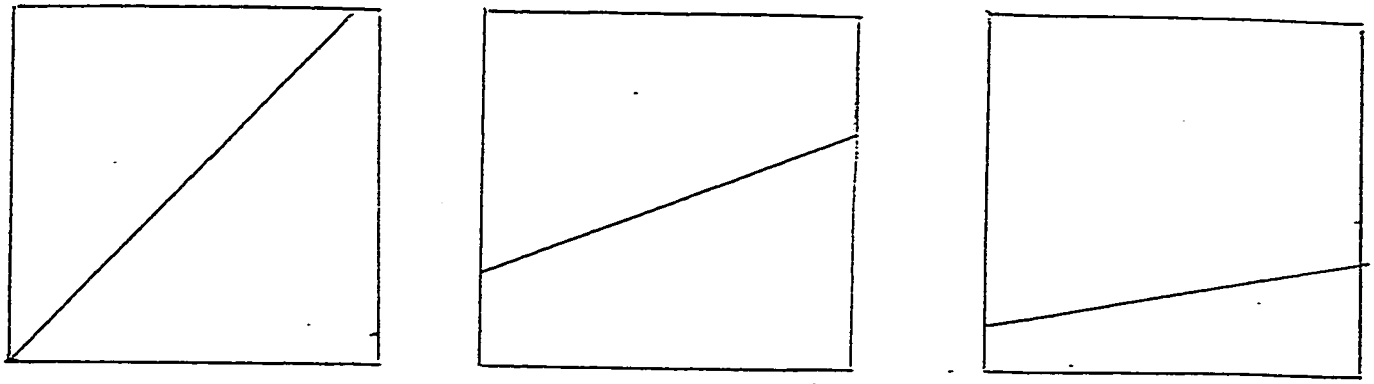
Name: David

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

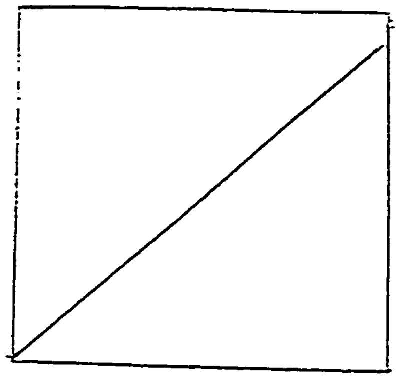


Are these functions necessarily different from each other? Please explain.

yes, because they have a different c value

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

it could represent a partial-variation relation



Please give;

i. a possible equation for this relation,

x

ii. include your domain and range, and also

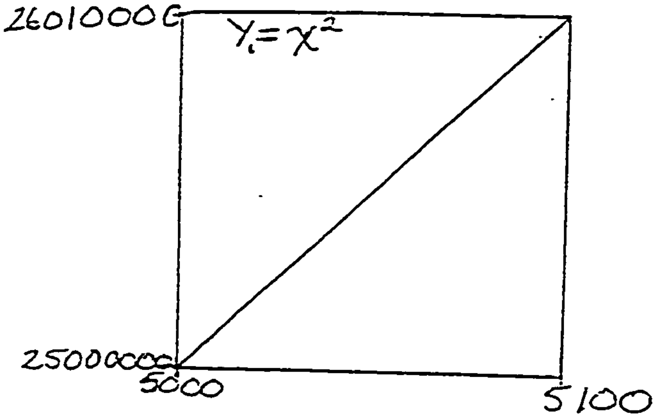
domain = [0, +∞[

range = [0, -∞[

iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

the graph is a quadratic function because $y=x^2$
the rate of change will not be constant because the will be a parabola

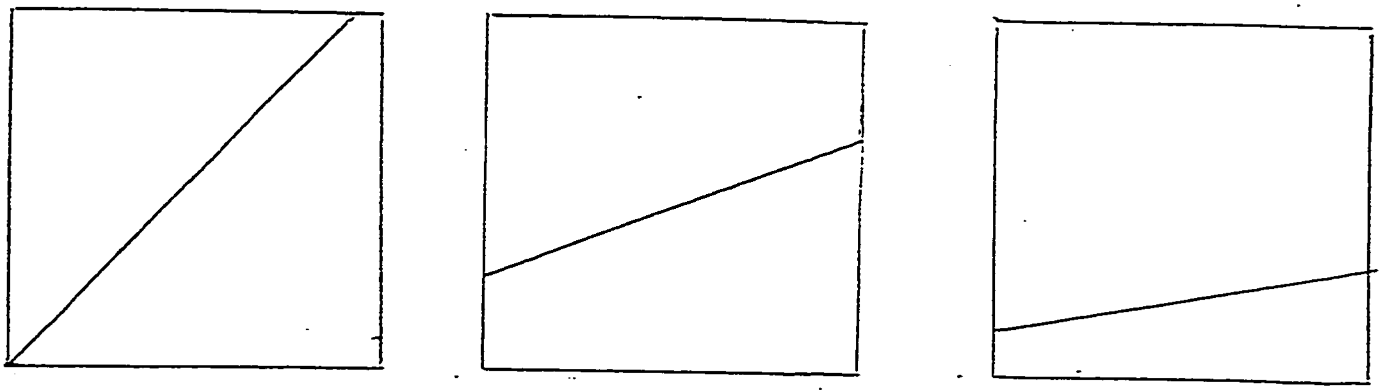
Lee

Name: December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

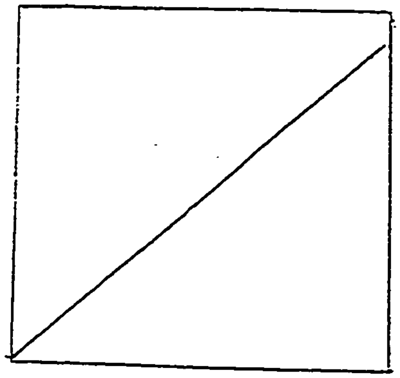
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

yes,

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



Please give;

i. a possible equation for this relation,

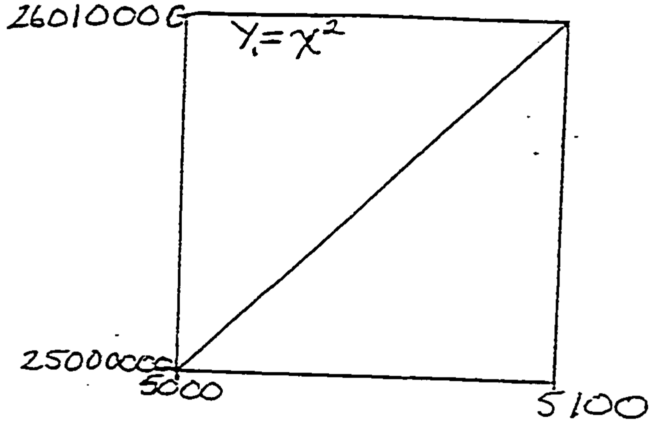
.8X

ii. include your domain and range, and also

iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

NO, The rate of change is constant, but it is not a linear function. when you put y1=X2 it to the TI83 it give a ∥ graph.

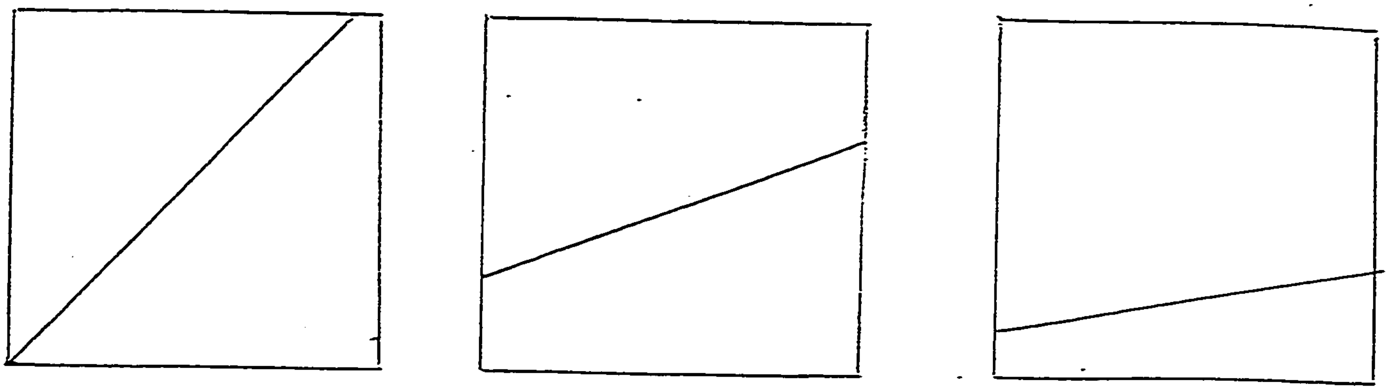
Name: Dick

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

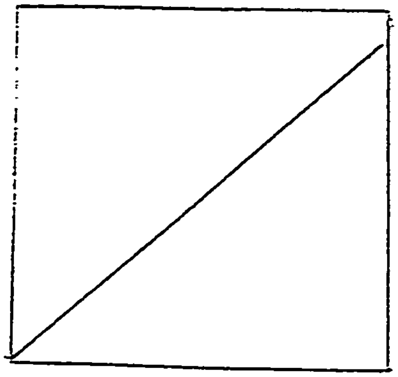
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

Yes, they are because if they weren't they would all represent the same thing, as in picture.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



Please give;

i. a possible equation for this relation,

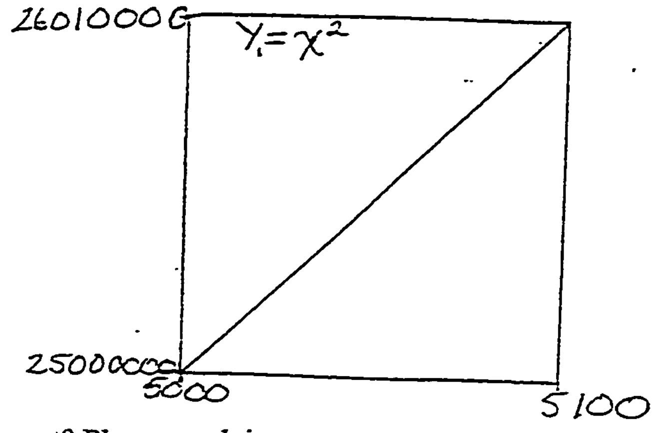
y = x - Z

ii. include your domain and range, and also

iii. describe the horizontal and vertical scales used to graph it.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

Yes, your claims are correct. It is a linear function because it passes through (0,0) and is a straight line. It is constant because the equation is $y=x^2$ and there is no c or b values.

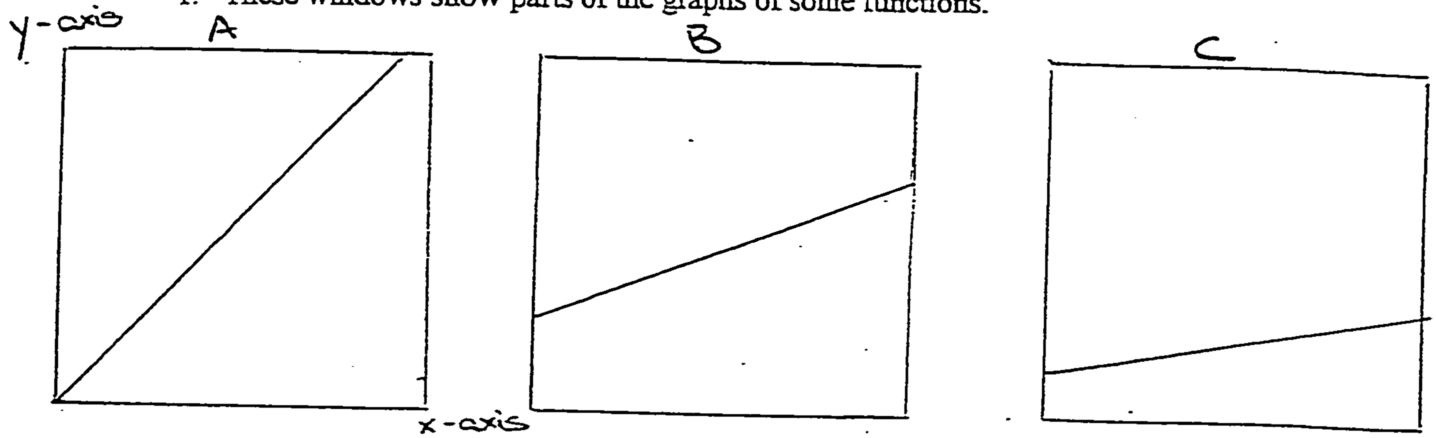
Name: Kristin

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

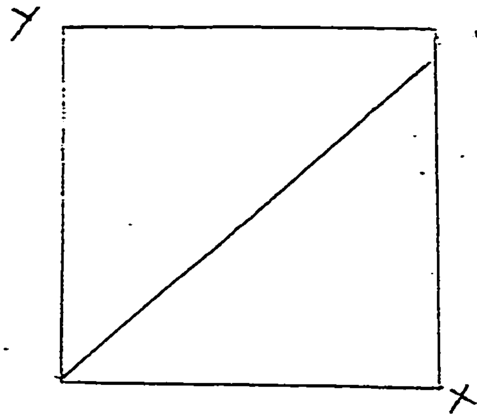
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

Yes, because A has an initial value that is smaller than that of B or C. Also, each function has a different slope.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.



It could represent a partial variation relation, with an initial value that is close to zero

Please give;

i. a possible equation for this relation,

$y = x + 0.5$

ii. include your domain and range, and also

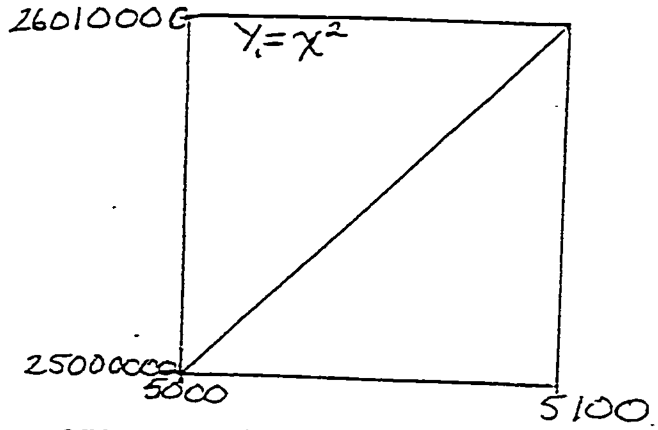
domain [-10,10] (]-∞,+∞[)
range [-9.5,10.5] (]-∞,+∞[)

iii. describe the horizontal and vertical scales used to graph it.

If I were entering this function into the calculator, I would use a scale of one for both the x and y axis.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

No, A linear function does not contain x^2, that is a quadratic function. If it were entered into a calculator it would give a parabola, and linear functions cannot be represented by lines that form parabolas.

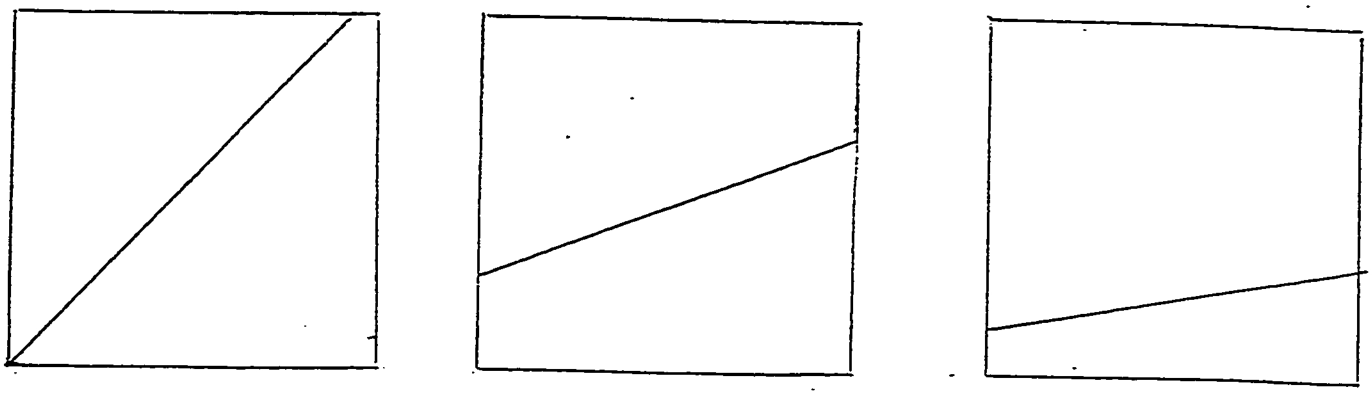
Name: Hugo

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

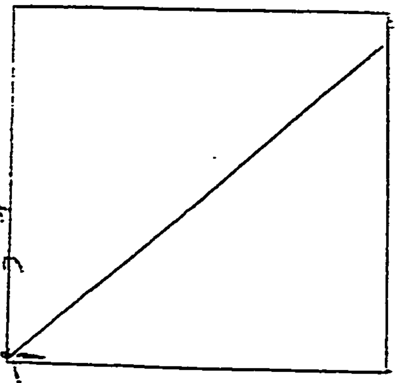


Are these functions necessarily different from each other? Please explain.

yes they are different because of there different slopes, means they have a different rate of change or they might be a partial variation meaning they start a different points

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

I think it should be a direct variation if it starts at the origin but it might be a partial variation relation if it has a different y-intercept than zero.



Please give;

i. a possible equation for this relation,

y= 0.9x + 1

ii. include your domain and range, and also

Domain [- ∞, + ∞] IR

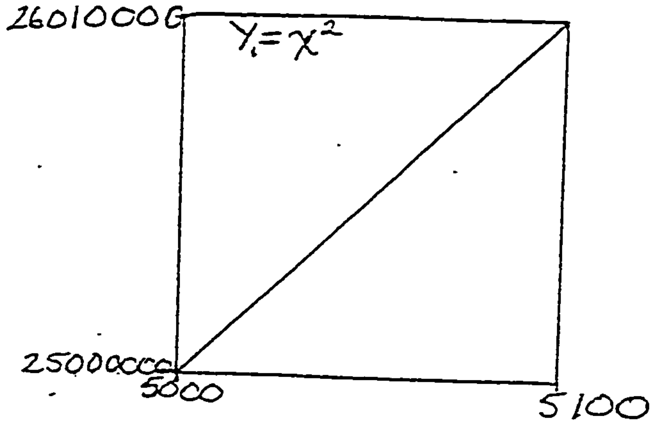
range [- ∞, + ∞] IR

iii. describe the horizontal and vertical scales used to graph it.

These scales could go up by alot of numbers, but I would like to see it go up by one.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

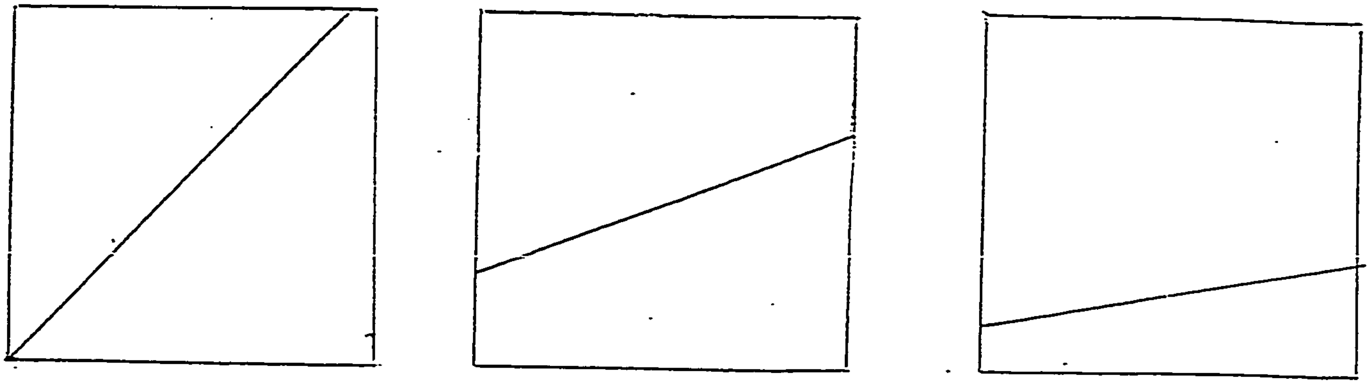
First of all x^2 these not match a linear fonction, so that false, The r fonction is not constant, I mean the rate of change is not constant.

Name: Michael December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

1. These windows show parts of the graphs of some functions.

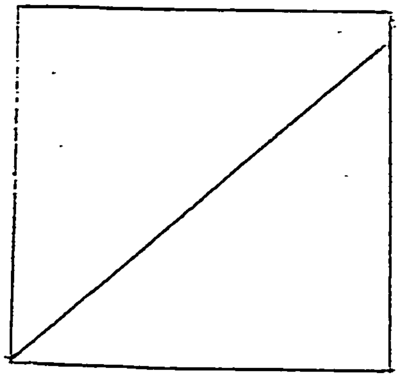


Are these functions necessarily different from each other? Please explain.

Yes, they have different equations & zeros probably

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain.

we don't ~~see~~ the origin but it could be both.



Please give;

i. ~~a possible equation for this relation,~~

$y = 2x + 1$

ii. include your domain and range, and also

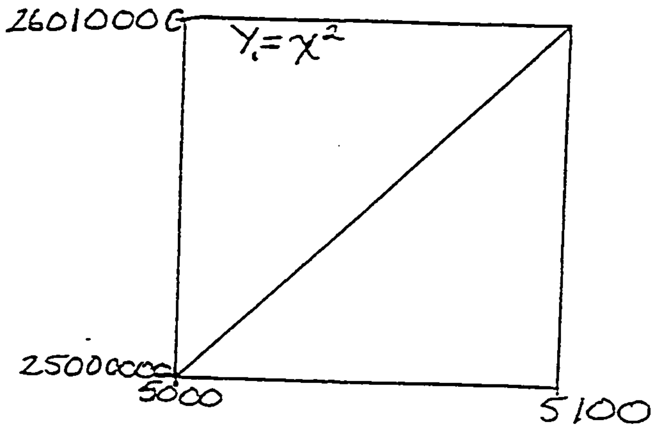
D = 1 ~~to~~ ∞
R = 1 ~~to~~ ∞

iii. describe the horizontal and vertical scales used to graph it.

any scale but equal for both x, y.

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

You have a linear function and yes your rate of change is constant.

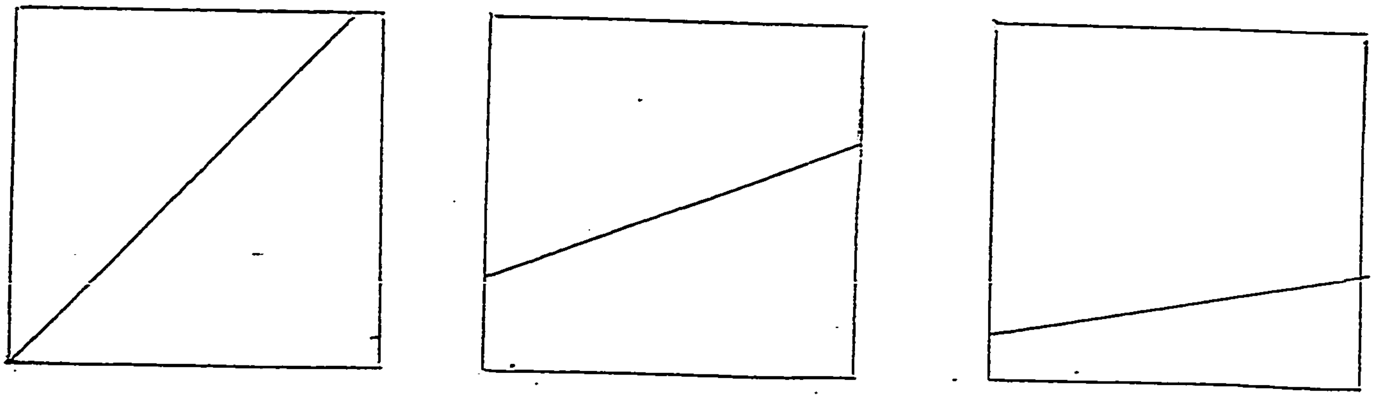
Name: Matt

December 6, 1999

Math 436

In each of the following questions, please explain your reasoning to the best of your ability.

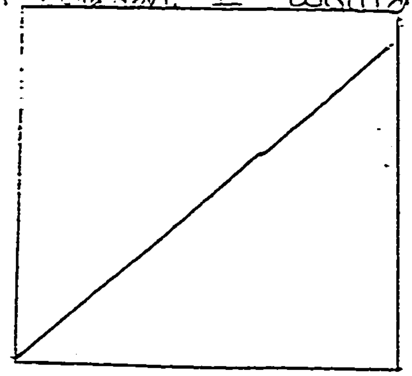
1. These windows show parts of the graphs of some functions.



Are these functions necessarily different from each other? Please explain.

They are all linear or first degree functions. The only thing that makes them different is that the first image is a direct variation relation and the last two are partial variation relations.

2. Does this graph necessarily represent a direct-variation relation, or could it represent a partial-variation relation? Please explain. no, depending on where the axis are, it could be a direct or partial variation relation. I would take this graph to be partial.



Please give;

i. a possible equation for this relation

$x - 1$

ii. include your domain and range, and also

$D =]-\infty, +\infty[$

$R =]-\infty, +\infty[$

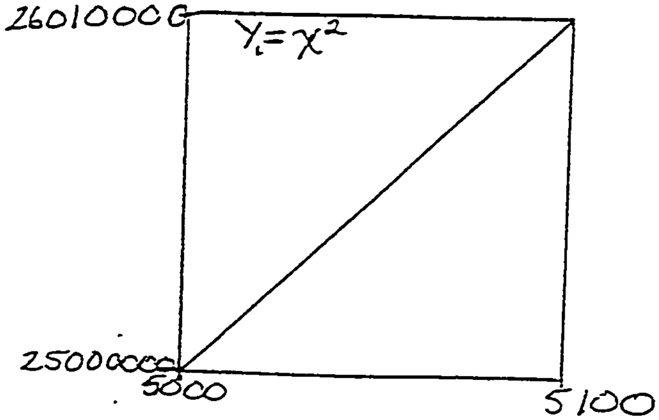
iii. describe the horizontal and vertical scales used to graph it

h = 1

V = 1

For the last two questions, it may help to consider how you would enter them in the TI-83.

3. I claim that the graph below is a linear function and that the rate of change is constant.



Are both my claims correct? Please explain.

The function is linear because it is a straight line.

I can't be sure of whether or not the function is constant or not because I can not see the origin. However, from the information given, I would guess that the function is partial.

Melanie December 13, [illegible]

Test.

(1) yes. The first is a direct linear relation and the other two, who might be the same, are parti[al] variation relations.

(2) it could be considered a partial variation relation because the line doesn't begin in the corner.

i $y = x$

ii ~~Range]-∞, +∞[~~
domain [0, +∞[

iii horizontal x min = 0
x Max = 10
x scl = 1
Vertical Ymin = 0
Ymax = 10
Yscale = 1

(3) No, because it is a quadratic function. X^2 is a 'degree of 2' and therefore falls under a quadratic function. Also your line is not constant. On a graph the ratio between the x and y is not a equality. The x is double of the y and making your statement for your graph incorrect.