A Performance Analysis of Wireless ATM Multiplexing System

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ABSTRACT

Performance Analysis of Wireless ATM Multiplexing System

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Wireless ATM is a direct result of the success of ATM on wired networks. Applying ATM over wireless channels is challenging since ATM was originally designed for band-width rich and nearly error-free medium, while in wireless environment, the radio channel is both bandwidth limited and the bit-error-rate is time-varying. It is therefore important to obtain good understanding of the statistical ATM multiplexing of the aggregate traffic on wireless channels.

In this thesis, we study the effects of channel error characteristics on a wireless ATM multiplexing system. The main characteristic of such a channel is that its error behavior is time-varying, with periods of low error rate (*Good*) transmission alternating with periods of high error rates (*Bad*) transmission. This variability is modeled as a two-state Markov chain with two states corresponding to *Bad* and *Good* states, respectively. The channel is modeled as being synchronous with the basic time unit being the slot. The transitions from *Bad* to *Good* states, and vice versa occur at the slot boundaries. We assume that during a *Good* state, the channel transmits a packet, while in a *Bad* state no packet is transmitted. The arrival process is modeled as the superposition of independent

binary Markov sources, each source alternates between *On* and *Off* periods. In the source *Off* state, no data is emitted while, in the *On* state, the source generates a random number of packets in a slot each of which fits into a channel slot. We present a discrete-time queuing analysis of such a system. Based on imbedded Markov chain analysis, a functional equation relating the joint probability generating function (PGF) of the system between two consecutive slots under the assumption of an infinite buffer is presented. The functional equation is transformed into a suitable form which makes it possible to derive the steady-state expressions for the marginal PGF's of the queue length, the state of the channel and the number of *On* sources. Then some performance measures such as mean queue length and mean packet delay are calculated. The relationship between the autocovariance of the channel state and the performance is also demonstrated. Finally, some numerical results are presented to show the effects of the wireless channel error characteristics on the behavior of the ATM multiplexing system.

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List of Abbreviations

ABR	Available Bit Rate
AMPS	Advanced Mobile Phone System
ATM	Asynchronous Transfer Mode
BER	Bit Error Rate
B-ISDN	Broadband Integrated Services Digital Network
bps	Bit per second
CBR	Constant Bit Rate
CCITT	Consultative Committee on International Telephone and Telegraphy
CDMA	Code Division Multiple Access
DAMPS	Digital Advanced Mobile Phone System
FCC	Federal Communications Commission
FDMA	Frequency Division Multiple Access
FM	Frequency Modulation
FSK	Frequency Shift Keying
GSM	Global System for Mobile Communications
HMM	Hidden Markov Model
IMT2000	International Mobile Telecommunications in the year 2000
IPP	Interrupted Poisson Process
ISM	Industry, Science, Medicine
ITU	International Telecommunications Union

Kbps	Kilobits per second
LAN	Local Area Network
NMT	Nordic Mobile Telephone
MAC	Medium Access Control
Mbps	Megabits per second
MMPP	Markov Modulated Poisson Process
PCN	Personal Communication Networks
PCS	Personal Communication System
PDC	Personal Digital Cellular
PGF	Probability Generating Function
QoS	Quality of Service
RHS	Right Hand Side
rt-VBR	Real-time Variable Bit Rate
TA	Terminal Adapter
TACS	Total Access Communication System
TDMA	Time Division Multiple Access
UBR	Unspecified Bit Rate
VBR	Variable Bit Rate
WAN	Wide Area Network
WAND	Wireless ATM Network Demonstrator
WLAN	Wireless Local Area Network

List of Symbols

a_k	Number of *On* sources in the system during slot k.
b_k	Number of packets that arrive during slot k.
c_s	It assumes the value of '1' if a source remains *On* in the next slot and '0' otherwise, given that the source is *On* in the present slot.
d_s	It assumes the value of '1' if a source becomes *On* in the next slot and '0' otherwise, given that the source is *Off* in the present slot.
$\bar{d}$	Mean packet delay.
$f_{j,k}$	Number of packets generated by the *j'th On* source during slot k.
$f(z)$	PGF of the number of packets generated by the *j'th On*-source during slot k.
g_s	It assumes the value of '1' if the channel remains *Good* in the next slot and '0' otherwise, given that the channel is *Good* in the present slot.
h_s	It assumes the value of '1' if the channel becomes *Good* in the next slot and '0' otherwise, given that the channel is *Bad* in the present slot
i_k	Number of packets in the system at the end of slot k.
m	Number of sources in the system.
n_k	The state ('1' for *Good* and '0' for *Bad*) of the server during slot k.

$p_k(i, l, j)$	Joint probability distribution of the queue length at the end of slot k, the state of the channel during slot k and the number of *On*-sources during slot k.
$p(i, l, j)$	Joint steady-state probability distribution of the queue length, the state of the channel and the number of *On*-sources. $(x)^{\dagger}$ max $(x, 0)$.
$(x)^{\dagger}$	max (0, x).
$A(y)$	Steady-state marginal PGF of the distribution of the number of *On*-sources.
$B(r)$	Steady-state marginal PGF of the distribution of the state of the channel.
$\overline{N}$	Steady-state average queue length.
$P(z)$	Steady-state marginal PGF of the queue length distribution.
$Q_k(z, r, y)$	Joint PGF of the queue length, the state of the channel and the number of *On*-sources distribution at the end of the *k'th* slot.
$Q(z, r, y)$	Joint steady-state PGF of the queue length, the state of the channel and the number of *On*-sources distribution.
α	Probability that a source is *On* in the current slot, given that it was *On* during previous slot.
β	Probability that a source is *Off* in the current slot, given that it was *Off* during previous slot.

γ	Probability that the channel is *Good* in the current slot, given that it was *Good* during previous slot.
σ	Probability that the channel is *Bad* in the current slot, given that it was *Bad* during previous slot.
ρ	Steady-state traffic load of the system.
$\pi_0(k)$	Probability of a source being *Off* during slot k.
$\pi_1(k)$	Probability of a source being *On* during slot k.
π_0	Steady-state probability of a source being *Off*.
π_1	Steady-state probability of a source being *On*.

Chapter 1

Introduction

Alexander Graham Bell's invention of the telephone over one hundred years ago launched a revolution in communications that enabled people to communicate efficiently over distance. Today, a new revolution - the wireless communication revolution - is taking place, which is taking us from a world where telephone subscribers were constrained to communicate over fixed telephone lines, to one where a tetherless and mobile communications environment has become a reality.

1.1 The Evolution of Wireless Networks

The field of wireless communication is growing at an explosive rate, covering many technical areas and its sphere of influence is beyond imagination. The worldwide activities in this growth industry are perhaps an indication of its importance. The demand of wireless communications is anticipated to expand steadily, the current annual activities exceed $100 billion.

Basically, wireless networks include voice-oriented wireless telephony networks and data-oriented wireless data networks. This split between voice-oriented and packet-data oriented wireless systems appears similar to the voice-network and data-network seen in wired networks.

Mobile cellular networks and cordless telephone, for instance, are wireless telephony systems. Cellular system allows the subscriber to place and receive telephone calls over the wireline telephone network wherever cellular coverage is provided. The basic cordless telephone provides a wireless counter-part to the standard telephone. Wireless data systems are designed for packet-switched rather than circuit-switched operation. Operators of wide-area messaging systems use licensed spectrum, and sell services to customers. Conversely, wireless local -area networks(WLANs) are usually privately owned, operated, and provide high-rate data communication over a small area. Figure 1.1 distinguishes the various categories of wireless networks [4].

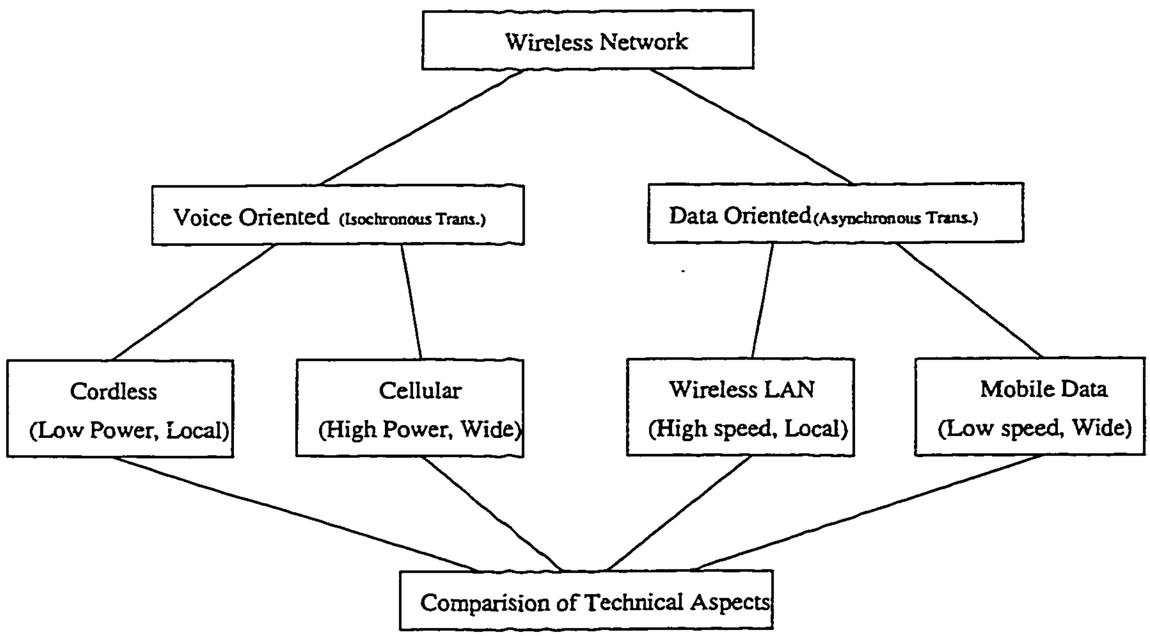


Figure 1.1 Categories of Wireless Networks

Whether wireless telephone systems or data systems, wireless networks are characterized by limited channel bandwidth. Because of this main problem, efforts in wireless communications have been focused on radio technology and channel resource management, where the goal is to increase bandwidth efficiency and radio channel capacity to meet the demands of users in wireless networks.

First and second generation wireless networks currently provide support for circuit-switched voice services as well as low-rate circuit-switched and packet-switched data services. In the 1980s, many cellular networks were implemented, such as Advanced Mobile Phone System (AMPS) in North America, Nordic Mobile Telephone (NMT) and Total Access Communications System (TACS) in Europe. All of these 'first-generation' cellular systems use analog frequency modulation (FM) for speech transmission and frequency shift keying (FSK) for signaling. Individual calls use different frequencies, this way of sharing spectrum is called FDMA. As the cellular subscribers increase dramatically, these networks are already reaching their capacity limits [1].

At the beginning of the 1990s, several digital technologies were introduced to increase spectrum efficiency and enhance wireless communications by adding attractive and innovative features and services such as data transmission. Thus, wireless networks have evolved from simple first-generation analog network systems to second-generation digital network systems. Digitalization allows the use of TDMA and CDMA as alternatives to FDMA. Digital systems can support more users per base station per Mhz of spectrum [2]. It is claimed that the capacity gains of digital systems is of ten times that of analog systems, which allow wireless network operators to provide service more economically. Second-generation digital cellular systems consist of Global System for Mobile Communications (GSM) in Europe, IS-136 or Digital AMPS (DAMPS) and IS-95 or North America CDMA (NA-CDMA) in North America, and Personal Digital Cellular (PDC) in Japan, etc. GSM, IS-136 and PDC are TDMA based systems, whereas, IS-95 relies on CDMA as its air interface. By the end of 1999, the number of subscribers has reached near 200 millions.

During the same period, in the other wireless areas also we saw great progress. The extraordinary success of the cordless-telephone spurred new standardization efforts for digital cordless and CT-2 TelePoint in the United Kingdom, wireless PBX, DECT in Sweden, the advanced cordless phone in Japan and the concept of Universal Digital Portable Communicator in the United States. The success of the paging industry led to development of low-speed wide-area packet data systems (Mobile Data) for commercial applications requiring longer messages. Motivated by the desire to provide portability and to avoid the high costs of installation and relocation of wired office information networks, Wireless LAN were suggested as an alternative. The FCC (Federal Communications Committee) announcements on unlicensed ISM (Industry, Science, Medicine) bands facilitate the development of a wide array of commercial devices from wireless PBX's and wireless LAN's to wireless devices using spread-spectrum technology.

The stated goal of wireless networks is ubiquitous communication. That is, to enable telecommunication services (voice, data, video and so on) without restrictions on time, the user's terminal (such as a simple handheld device or a laptop computer), location in the world, point of access to the network, access technology, or transport method.

The current second-generation wireless networks in use are not capable of efficiently supporting the growing requirements of video or Internet demands. Wireless networks have been evolving into the third-generation Personal Communication Networks (PCN) that can support multimedia services to mobile users. Although initial PCS (Personal Communication System) proposals have generally focused on voice-communication-related services, it is recognized that these systems will be required to evolve towards supporting a wider range of telecommunication applications involving packet data, video, and multimedia. At the same time, wireless local-area networks (WLANs), which were initially designed for conventional data, face a growing requirement to support computer applications incorporating image and/or video transfer. Thus the demand for multimedia-capable multi-service wireless networks is driven by parallel trends towards integration of voice, video, image and data in both telecommunication and computing environments [5].

The need to provide high-speed data services to support multimedia applications is driving the industry to look for solutions in broadband technologies. Many standards agencies around the world are working toward determining the next standard for this third-generation wireless system. International Mobile Telecommunications in the year 2000 (IMT2000) is the third-generation specification under development by the ITU that will handle heterogeneous traffic ranging from low-bit-rate voice signals to high-bit-rate data and image communications over wireless networks. It will provide up to 384kbps in wide areas and up to 2Mbps in limited local areas between service providers and end-users [7]. IMT-2000 effort envisions the provision of Asynchronous Transfer Mode(ATM) services in the wireless network, and as discussed in [5], wireless ATM maybe viewed as a general solution for the third-generation PCN capable of supporting multimedia.

1.2 Wireless ATM

Due to the success of ATM on wired network, wireless ATM is a direct result of the ATM 'everywhere' movement. It combines the advantages of both the wireless and the ATM networks. Wireless networks provide mobility or portability to the end users while the wired ATM networks allow broadband multimedia services with Quality of Service (QoS). The ATM Forum Wireless ATM Working Group is currently involved in defining the baseline of Wireless ATM system.

1.2.1 ATM basics

ATM is the technique that is becoming the common nominator for all types of services and networks. From the service point of view, ATM can combine the transmission of both speech and computer data into the wired networks, while scaling well into different network types such as LANs and WANs, and it is independent of the bit rate of physical medium. It is a connection-oriented, packet-switched network.

In ATM networks, the data is divided into small, fixed length units called cells. The cell is composed of 48 user bytes plus a 5-byte header containing the routing information and other cell-related information (priority, type). ATM does not provide any error detection operations on the user payload inside the cell, and also provides no retransmission services, and only few operations are performed on the small header. The small, fixed-sized cells allow fast and efficient multiplexing of sources with different QoS constraints. Due to its low queuing delay and delay variance, ATM technology networks are well suited for multimedia applications, and can handle any kind of traffic from circuit-switched voice to bursty video streams at any speed.

At ATM switches, capacity is shared by grouped sets of connections to save bandwidth through multiplexing efficiency. Thus, the switch needs to know the type of traffic generated by each new connection. Data streams are characterized by: destination, traffic type, bit-rate and QoS. If a new connection can be admitted without any adverse effect on the pre-established connections then it is admitted. If not, the request is turned down and the user has to place a new request with easier to meet characteristics.

There are five types of services that characterize connections in ATM networks. They are: CBR (constant bit rate), rt-VBR (real-time variable bit-rate), nrt-VBR (non-real-time variable bit-rate), ABR (available bit-rate) and UBR (unspecified bit-rate) [47].

1.2.2 Basic Concept of Wireless ATM

There are several factors that tend to favor the use of ATM cell transport for wireless networks, including: flexible bandwidth allocation and service type selection for a range of applications; efficient multiplexing of traffic from bursty data/multimedia sources; end-to-end provisioning of broadband services over wireless and wireline networks; suitability of available ATM switching equipment for inter-cell switching; improved service reliability with packet switching techniques. By preserving the essential characteristics of ATM transmission, wireless ATM systems offer improved performance and quality of service, which are not attainable by other wireless communication systems like cellular systems, cordless networks or wireless LANs. In addition, wireless ATM access provides location independence, which removes a major limiting factor in the use of computers and powerful telecommunication equipment over wired networks.

The general aim with most wireless ATM proposals is to design an integrated services wireless network that provide tetherless extensions of fibre-optic based ATM network capabilities in a relatively transparent, seamless, and efficient manner. This means that the proposed systems need to maintain uniformity of service classes (CBR, VBR, ABR and UBR), bit rates, and QoS levels associated with ATM [6]. It is recognized that there may be quantitative differences in achievable service characteristics due to the fundamental limitations imposed by the available wireless radio medium, but these are hoped to be kept to a minimum via innovative technical approaches [5]. Table 1 summarizes typical targets for wireless ATM service capabilities (based on access radio channel speeds

~10Mbps)[8].

Table 1: Typical service requirements for wireless ATM

Traffic class	Application	Bit-rate range	QoS requirement
CBR	voice digital TV	32kbps - 2Mbps	low cell loss low delay jitter
VBR	video conference Multimedia comm.	32kbps - 2Mbps(avg) 128kbps-6Mbps(peak)	low delay jitter statistical mux moderate cell loss
ABR and UBR	interactive data client-server	1 - 10Mbps	low cell loss high burst rate

Wireless ATM has the following components: WATM terminal, wireless terminal adapter, wireless radio port, mobility enhanced ATM switch and fixed ATM network. WATM terminal is the end-user device, and WATM terminal adapter (TA) is the wireless ATM network interface between Wireless ATM network and end user device. They both belong to the radio access part. They communicate with wireless radio port (also termed *base station*). The wireless radio port is connected with mobility enhanced ATM switch, which is the access switch with mobility support capabilities. Wireless radio port acts as an ATM-level bridge, and may also incorporate switching to deal with multiple radio and network ports. Mobility enhanced ATM switch will finally connect to fixed ATM switch[9]. Figure 1.2 shows the baseline reference model now under consideration within the ATM forum.

Clearly, in wireless ATM, statistical multiplexing has to be extended to the air interface. This can be done by using a TDMA scheme to divide the physical channel into slots that are able to carry one or several ATM cells. One basic approach is to use the standard ATM cell for network level functions, while a wireless header/trailer is added on the radio link protocol layers. ATM services with QoS are provided on an end-to-end basis, and standard ATM signaling functions are terminated at Wireless ATM terminals [9].

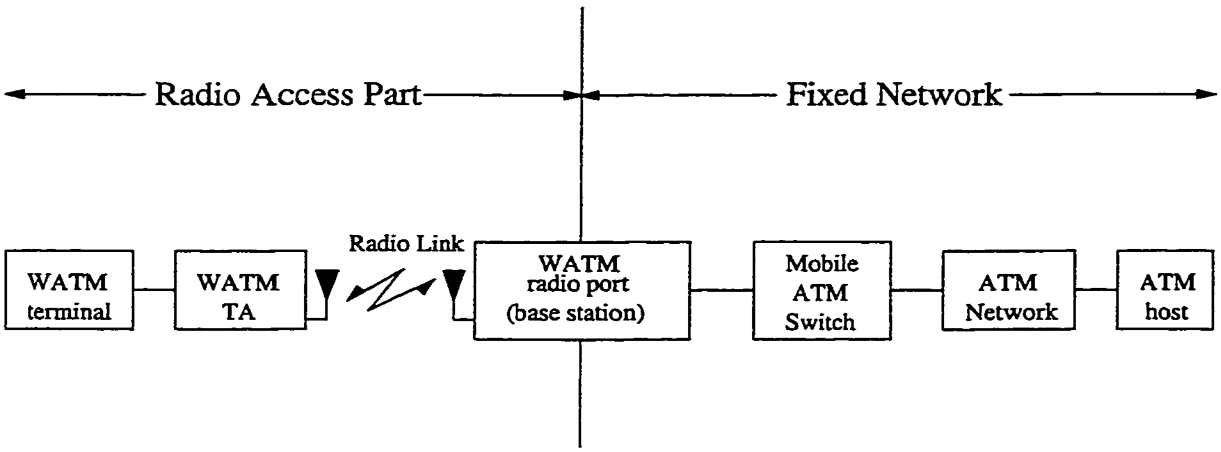


Figure 1.2 Wireless ATM Reference Model

In order to support ATM services, a high-speed radio modem (~10-25Mbps) is needed. Note that the use of a transmission bit-rate below the nominal B-ISDN specification of 155.5 Mbps for Wireless ATM access is similar in spirit to several current ATM forum proposals which aim to standardize a lower bit-rate (e.g. 25/50 Mbps) for twisted-pair copper access. Also new special wireless ATM medium access control (MAC), data link control and radio resource control are needed, responsible for radio link layer for wireless ATM access. In order to support mobility, appropriate extensions need to be added to the higher-layer control/signaling functions, which include handover, location management, routing, addressing and traffic management [10].

1.2.3 Wireless ATM Challenges

There are significant challenges for the design of wireless ATM [7, 11]. The two main issues under consideration are:

1. *Limited Spectrum*

The radio spectrum, and therefore the bandwidth available for wireless services, is generally limited by regulation. Thus, unlike wireline communications wherein an increasing user population can easily be served by deploying additional wire (or fiber) facilities, the available radio spectrum cannot be arbitrarily expanded, so in wireless networks, high bit rate transmission is hard to achieve and available frequencies are a rare resource. However, ATM was designed for bandwidth-rich media, and it effectively trades off bandwidth for simplicity in switching. Every ATM cell carries a header with an overhead of about 10%, this overhead is considered too high in the wireless environment, the inefficiency of the system may outweigh the advantages of wireless access.

2. *Time-Varying Quality of the Wireless Links*

Unlike the wireline networks, wherein the physical link between a remote user terminal and the end-office switch are of time-invariant high quality, the wireless link is subject to severe time-varying impairments arising from inherent user mobility and unavoidable changes caused by motion of the surrounding environment. These impairments are manifested in a time-varying bit error rate (BER) performance of the wireless link, with the BER often too high to meet the needs of the applications. Main causes of these time-varying impairments are multipath propagation, shadow fading, distance-dependent signal power path loss, and co-channel interference. Considering the fact that ATM was designed for media (optic fiber) whose bit error rate is very low, it is questionable whether it will ever work in the very noisy and time-varying wireless environment.

Currently, a number of research activities are focusing on the topic of wireless ATM to resolve the problems discussed above. These projects include, for instance, project Magic WAND (Wireless ATM Network Demonstrator) which belongs to ACTS program funded by the European Union [12], WATMnet by NEC [10], BAHAMA (a wireless ATM LAN) by Bell Laboratories [13] and so on. If these challenges are overcome, there will be significant advantages of wireless ATM, which will provide users ubiquitous ATM services across wireline and wireless interfaces.

1.3 Wireless ATM Performance Issues

Since Wireless ATM is shown to be a promising area, it is important to determine the performance that can be expected from a Wireless ATM system.

1.3.1 Statistical Multiplexing in Wireless ATM

One can view a Wireless ATM network as a collection of nodes that are connected by a set of wireline and wireless transmission links. Based on the cell header information, ATM cells are switched from a source node to their destination node, following the store and forward packet-switching principle. When a cell reaches a network node, it is temporarily stored there until the transmission channel to the next node becomes available. In wireless networks, since the quality of the wireless transmission channel is time-varying, when the BER is too high to meet the service requirements, the channel is not available to transmit data. For those cells which cannot be transmitted immediately, buffer space has been provisioned at each switching element.

In wireless ATM network, several traffic sources will be accessing a single link, which is the same as in wired ATM network. A key advantage of ATM is the efficient sharing of link capacities through statistical multiplexing of traffic streams. In other words, buffering is required to absorb traffic fluctuations when the instantaneous rate of the aggregate incoming streams exceeds the limited capacity of the outgoing wireless link, whose capacity is further reduced by the unreliable radio transmission.

In wireless ATM, broadband services must also be regarded as being similar to those provided in the wired ATM network, which means QoS guarantees should be provided to the users of the network. This counterbalances the multiplexing gain. QoS are often expressed in terms of packet loss, delay, delay jitter and so on. For example, voice traffic has the rate of several kilobits per second and is delay sensitive, while high speed data traffic, used for instance in file transfer, is of a few megabits per second and is loss sensitive. Therefore, in order to provide QoS, not only correct buffering has to be designed, but also efficient admission, bandwidth allocation and flow control policies are needed.

To deal with the above higher-level network management issues, it is essential to acquire good understanding of the statistical multiplexing of the aggregate traffic on the wireless ATM link.

From a modeling point of view, the choice of a connection-oriented fast-packet switching (with fixed-length packets) technique in a B-ISDN naturally leads to the choice of a slotted time axis with synchronized message transmission in the modeling of an ATM system [14]. In addition, the multiplexing of voice, data and video sources on wireless ATM links give rise to a discrete-time queuing problem, at the multiplexer's level, which involves a probabilistic discrete-time server and a special correlated discrete-time arrival process. It is obvious that the queuing performance depends on the combined effect of arrival and channel statistics. Thus, source characterization in ATM, as well as modeling of wireless channel are important for wireless ATM network performance analysis.

1.3.2 The Binary Markov On/Off Traffic Model

As mentioned above, wireless ATM networks must support various communication services, such as data, voice and video, each having different traffic characteristics. To evaluate the performance of such networks, appropriate source modeling is required.

There have been many models proposed in the literature for characterizing individual data traffic sources or a superposition of multiple sources. For instance, we have Poisson arrival process (continuous time case), geometric inter-arrival process (discrete time case) for data traffic, Interrupted Poisson Process (IPP) for voice traffic and Markov Modulated Poisson Process (MMPP) for data, voice and video traffic. A good review on traffic modeling can be found in [15].

Among those traffic models that have been used for the ATM sources, the most versatile one is the two-state Markov On/Off model. In this model, each source is characterized by ON (corresponding to active bursts) and OFF (corresponding to silence duration) periods, which appear in turn. During the silent periods, no cells are generated. Even though more complicated models, such as the three state model [16], have been proposed for more accurate modeling, analysis of a queuing model with these processes can be very complex and may not be tractable, therefore, these models have rarely been applied in mathematical analysis. On the other hand, the binary On/Off model is very popular and has been often used for the modeling of ATM traffic. For instance a binary Markov model has been successfully applied for the voice source [17, 18]. In addition, in [19], a video source is modeled as a birth-death process, which consists of the superposition of a number of independent and identical On/Off mini-sources. The CCITT has also provided parameter values for the On/Off sources that are to be used as traffic models for typical ATM sources. This is illustrated in Table 2 [20].

Because of its versatility and flexibility, the binary Markov source has been chosen as the basic model for the characterization of input traffic sources. Hence this thesis will be mainly concerned with the analysis of statistical multiplexing in Wireless ATM whose input processes consists of the superposition of many identical independent traffic streams generated by binary Markov sources.

Table 2: Parameter Values for Typical Traffic Sources, by CCITT

Representative Service	Mean burst length in number of cells	Average cell arrival rate
Connectionless data	200	700 Kbits/s
VBR video	2	25 Mbits/s
Connection-oriented data	20	25 Mbits/s
Background data/video	3	1 Mbits/s
VBR video/data	30	21 Mbits/s
Slow video	3	6 Mbits/s

1.3.3 Wireless Channel Model

In studying wireless communication systems, we need a model that captures the characteristics of the radio channel. As stated earlier on, the quality of wireless channel suffers from time-varying degradations such as fading, shadowing and interference, which lead to periods of correct transmission alternating with periods of high error rates.

Early work in modeling wireless channels focused on the stochastic behavior of channel at the physical layer [21, 22, 23], measured by received signal strength or bit error rate. Such physical-layer models can not be directly used to evaluate higher-layer network performance, such as queuing delay and loss. For example, a few bit errors within a packet, which is the basic data unit at the wireless link layer, will cause the loss of the entire packet. It is therefore imperative to develop packet-level wireless channel models, which can be used by network engineers to simulate and analyze the high-layer performance of wireless networks.

Two Markov models have been used as the packet-level wireless channel model, these are Gilbert-Elliot model and Hidden Markov Channel Model [24].

1) *Gilbert-Elliot model*

This model was proposed by Gilbert [25] and Elliot [26]. They study a two-state Markov chain to describe the channel. In their channel model, each state corresponds to a specific channel quality which is either noiseless or totally noisy, as “good” state or “bad” state respectively. This two-state channel model can be easily extended to a N-state Markov model with $N > 2$ for more accuracy [27, 28], where each state represents a different level of fading. Previous studies [29] and [30] show that these Markov chains provide a good approximation in modeling the error process at the packet level in fading channels. Based on the observation that mean residency time in states with high BER is longer than a single packet transmission time, which results in correlated packet losses, the authors in [31, 32] used the two-state Markov model to capture packet loss characteristics of the wireless links in their simulation study. In [33, 34], the Markov models were also used for the channel modeling of the system.

2) *Hidden Markov Channel Model*

The Gilbert-Elliot model can be seen as a special case of the more general Hidden Markov model (HMM). A HMM is a doubly stochastic process with an underlying stochastic process that is not observable but hidden. That is, by simply observing the output sequence, the state that generated the output cannot be determined at any given time. Recently, the HMM has been used to obtain extremely accurate models for the mobile channel [35].

In our analysis, we will assume that the channel behavior can be characterized by simple two-state Markov model, which has been commonly adopted [28] to match the average *Good* (success) and *Bad* (fail) periods of packet transmission on the wireless channel.

1.3.4 Steady-State Analysis of Wireless ATM Statistical Multiplexing

The wireless ATM research has already been active for some time. There are many papers written on wireless ATM. Some of them concentrated on the overview of issues and concepts in wireless ATM, the others described specific system architectures and reported on experiences based on designing such systems. Performance analysis of wireless ATM system is somehow limited, especially for statistical multiplexing behavior from the discrete-time queueing point of view. In this thesis, we present a performance analysis to study the wireless channel error characteristics on the wireless ATM multiplexing system.

Several techniques have been proposed to study the statistical multiplexing in ATM. In [37], the fluid approximation method was applied to analyze an infinite buffer ATM multiplexer, which is loaded with the superposition of statistically independent and identical On/Off sources. A Matrix-Analytic approach is used for a discrete-time queuing analysis of an ATM multiplexer in [38]. In [39], a Markov Modulated Poisson Process (MMPP) has been used to approximate the arrival process to a MMPP/G/1/K queue to derive the queue size. [40] and [41] have shown that a diffusion process provides a good approximation for the superposition of On/Off sources in ATM multiplexers. In [42], a model with binary On/Off Markov sources was presented and a discrete-time queueing analysis was developed, using a generating function approach. The functional equation describing the ATM multiplexer derived in [42] was solved in [43, 44] and the PGF of the queue length was obtained.

So far, in all the queueing analyses mentioned above the servers are assumed to be deterministic, which means the server is always available. This corresponds to the situation in wired ATM networks. In this thesis, we extend the performance analysis technique developed in [42, 43, 44] to the case of wireless ATM multiplexing system. Our objective is to study the combined effect of arrival and channel statistics on the performance of ATM multiplexer. Since the wireless channel is not always available due to the burst errors, the server in the queueing model will not be deterministic but probabilistic, which means it will alternate between *Good* and *Bad* state. Therefore, in this thesis, we will assume the two-state Markov model for the wireless channel.

1.4 Outline of the Thesis

The intent of this thesis is to obtain some understanding of how the characteristics of the wireless links affect an ATM multiplexing system with correlated arrival traffic. The rest of the thesis is organized as follows:

Chapter 2 describes the queueing analytical model for the wireless ATM multiplexing system. The two-state Markov model is chosen for both the traffic model and wireless channel model.

Chapter 3 derives a functional equation relating the joint probability generating function (PGF) of the system between two consecutive slots through an embedded Markov chain analysis. The functional equation is transformed into a suitable form which makes it possible to derive the steady-state expressions for the marginal PGFs of the queue length, the state of the channel and the number of *On* sources. The mean queue length and the mean packet delay are also given.

Chapter 4 studies the characteristics of the queueing system fed by correlated arrivals with a server not always available by presenting and discussing the numerical results of the queueing behavior.

Chapter 5 summarizes the research work and concludes the results.

Chapter 2

System Model

In this chapter, we describe the model of wireless ATM system under consideration. The channel model, source model and finally the queueing model will be given, as well as the definition of the main notations.

2.1 System Description and Notations

Shown in Figure 2.1, we consider a wireless ATM multiplexing system shared by m mutually independent and identical binary Markov traffic sources, each alternating between *On* and *Off* states, and the output wireless link itself alternates between *Good* and *Bad* states. Also we assume that there is a buffer shared by all the sources, and it is used for the temporary storage of the packets before they are transmitted on the shared output wireless link.

As a result, the system is modeled as a discrete-time queuing system with infinite queue length and a single stochastic server. The time-axis is divided into equal slots and packet transmission is synchronized to occur at slot boundaries. It is assumed that a packet can not leave the buffer at the end of the slot during which it has arrived and that a packet transmission time is equal to one slot.

Let us make the following definitions:

i_k: Number of packets in the system at the end of slot k.

b_k: Number of packets that arrive during slot k.

m: Number of sources in the system.

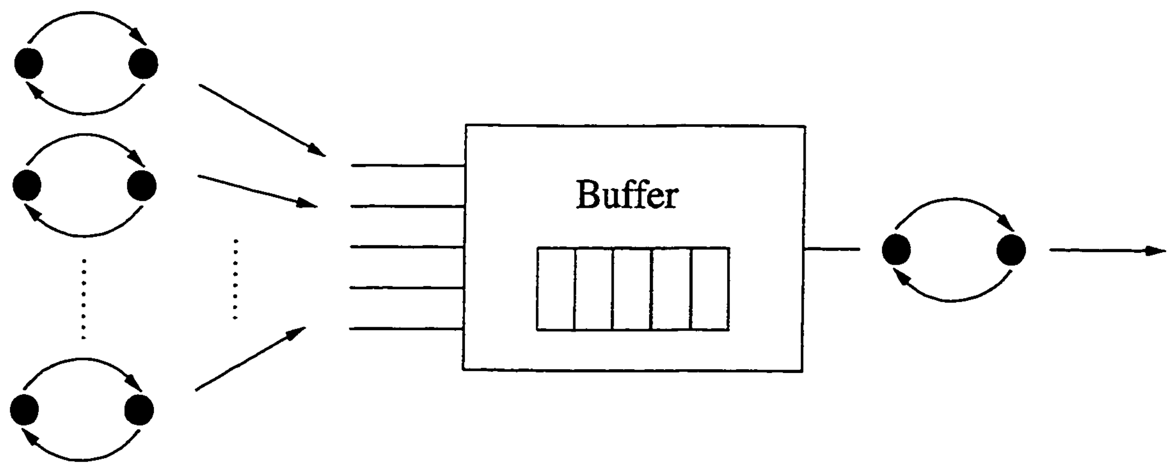


Figure 2.1: An Wireless ATM Multiplexing System Shared by m Homogeneous Traffic Sources

2.2 Channel Modeling Description and Notations

As stated earlier on, Markov chains provide a good approximation in modeling the error behavior of wireless channel, and the simple two-state Markov model has been commonly adopted to match the average good and bad periods of packet transmission on the wireless fading channel.

The shared outgoing wireless channel in our system corresponds to the server in the queuing analytical model and it is also modeled as a two-state Markov chain to characterize its burst error behavior. The channel switches between *Good* and *Bad* states. Since the error behavior of the wireless channel is time-varying, it will not always be available to transmit packets. We declare the channel to be in a *Bad* state when transmission is likely to fail due to errors, and in a *Good* state when transmission is likely to be successful. We assume that during a slot, if the channel is in *Good* state, it will transmit a packet if there are packets in the queue, while in the *Bad* state the channel will not transmit any packets whether there are packets in the queue or not. At the end of each slot, a transition from a *Good* to a *Bad* state occurs with probability $(1-\gamma)$, while the probability of a transition from a *Bad* to a *Good* state is $(1-\sigma)$, as shown in Figure 2.2. As a result, the number of slots that the channel spends in *Good* state is geometrically distributed with parameter γ, and the number of slots that the channel spends in Bad state is geometrically distributed with parameter σ.

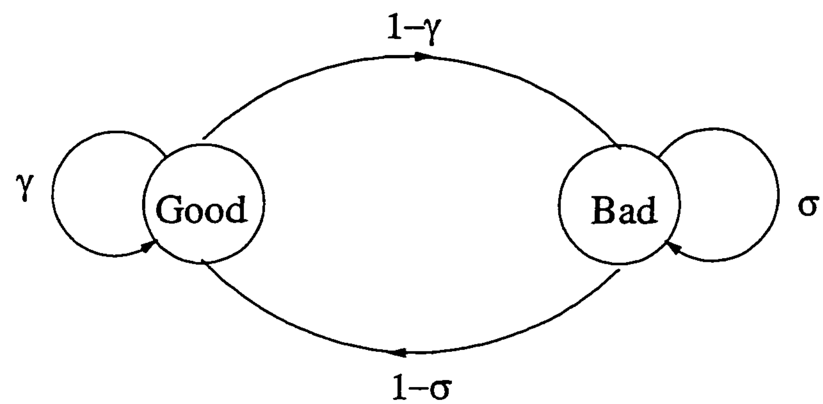


Figure 2.2 The two-state Markov Channel Model

Let us also define the following random variables,

n_k: the state of the channel during slot k ('1' for *Good* and '0' for *Bad* channel).

g_s: it assumes the value of '1' if the channel remains *Good* in the next slot and '0' otherwise, given that the channel is *Good* in the present slot.

h_s: it assumes the value of '1' if the channel becomes *Good* in the next slot and '0' otherwise, given that the channel is *Bad* in the present slot.

We note that g_s and h_s are similarly two sets of identically independent distributed (i.i.d.) Bernoulli random variables, and the corresponding probability generating functions (PGFs) of g_s and h_s are:

$$g(z) = 1 - \gamma + \gamma z \tag{2.1a}$$

$$h(z) = \sigma + (1 - \sigma)z \tag{2.1b}$$

thus the random variables n_{k+1} and n_k are related as,

$$n_{k+1} = \sum_{s=1}^{n_k} g_s + \sum_{s=1}^{1-n_k} h_s \tag{2.2}$$

In (2.2), we assume that a summation is empty if its upper limit is smaller than the lower limit. The first term in (2.2) represents the channel that was in *Good* state during slot k and remains in *Good* state during slot k+1, while the second term represents the channel that was in *Bad* state during slot k and becomes *Good* in the next slot.

2.3 Source Modeling Description and Notations

Because of the simplicity and capability to capture some of the correlation behavior, binary Markov sources have been widely used as basic building blocks to model broadband traffic, including voice and video.

We assume the arrival process of the system consists of m mutually independent and identical On/Off Markov traffic sources, each source alternating between *On* and *Off* states, as shown in Figure 2.3. During an *On* slot each source generates at least one packet, while during an *Off* slot no packet is generated. State transitions of the sources are synchronized to occur at the slots' boundaries. A transition from an *On* to an *Off* state occurs with probability $(1-\alpha)$ at the end of a slot, thus the number of slots that the source spends in *On* state is geometrically distributed with parameter α. Similarly, the probability of a transition from an *Off* to an *On* state is $(1-\beta)$ at the end of a slot, and the number of slots that the source spends in *Off* state is also geometrically distributed with parameter β. When α and β are high, then packets have tendency to arrive in clusters, alternatively when α and β are low, packet arrivals are more dispersed in time.

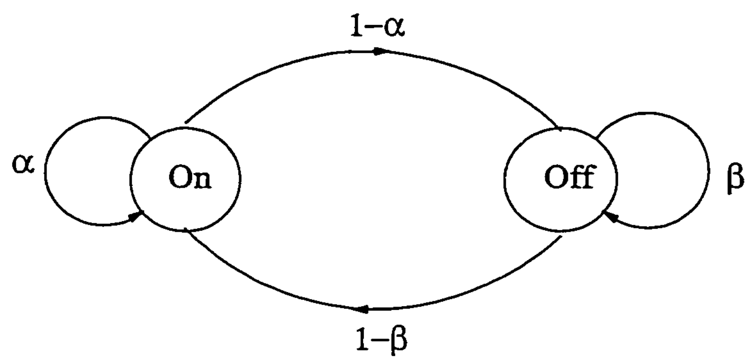


Figure 2.3 The On-Off Markov Source Model

Let us define the following Bernoulli random variables,

c_s: it assumes the value of '1' if a source remains *On* in the next slot and '0' otherwise, given that the source is *On* in the present slot.

d_s: it assumes the value of '1' if the source becomes *On* in the next slot and '0' otherwise, given that the source is *Off* in the present slot.

We can see that c_s's and d_s's are also two sets of i.i.d. Bernoulli random variables, and the corresponding PGFs of c_s and d_s are:

$$c(z) = 1 - \alpha + \alpha z \tag{2.3a}$$

$$d(z) = \beta + (1 - \beta) z \tag{2.3b}$$

Let us also define:

a_k: Number of *On* sources in the system during slot k.

$f_{j,k}$: Number of packets generated by the $j'th$ *On* source during slot k.

All the $f_{j,k}$'s are assumed to be i.i.d. random variables with PGF $f(z) = E[z^{f_{j,k}}]$. We also note that since, in this case, each source generates at least one packet per slot during an *On* state with PGF $f(z)$, then $f(0) = 0$. This also implies that if the random variable i_k is zero then a_k must also be zero, since packets can not leave the buffer at the end of the slot during which they have arrived. Thus if the buffer is empty at the end of slot k, all the sources should have been in *Off* state during slot k. The number of packet arrivals during slot k is given by,

$$b_k = \sum_{j=1}^{a_k} f_{j,k} \tag{2.4}$$

and the random variables a_{k+1} and a_k are related by,

$$a_{k+1} = \sum_{s=1}^{a_k} c_s + \sum_{s=1}^{m-a_k} d_s \tag{2.5}$$

In (2.5), we also assume that a summation is empty if its upper limit is smaller than the lower limit. The first term represents the number of sources which were in *On* state during slot k and remain in *On* state during slot $k+1$, while the second term represents the number of sources which were in *Off* state during slot k and change to *On* state during next slot.

2.4 Queuing System Model

The queuing system under consideration can be modeled as a discrete-time three-dimensional Markov chain. The state of the system is defined by the triplet (i_k, n_k, a_k), which denotes the length of queue at the end of slot k, the state (*Good* or *Bad*) of the channel during slot k and the number of the *On*-sources at the end of slot k in the system.

We focus now upon the number of packets in the queue at departure epochs, i.e., those time instants when a transmission is completed. Given the number of packets in the system at the end of slot k, the number of packets in the system at the end of slot $k+1$ depends only on new arrivals and not on past occupancies. Thus the succession of packet states forms an embedded Markov chain.

By definition from Section 2.1 we can assume that at the end of time slot k, the k'*th* departing packet leaves a non-empty system with i_k packets. At this epoch transmission of the next packet begins if the channel is *Good* during slot $k+1$ (i.e. $n_{k+1} = 1$), otherwise (i.e. $n_{k+1} = 0$) it will not transmit any packets. While this is in progress packets continue to arrive from those *On*-sources. The number of such arrivals is b_{k+1}, by definition from Section 2.1. Thus, the state of the system can be described by the following equation,

$$i_{k+1} = i_k - n_{k+1} + b_{k+1}$$

If the departure of the k'*th* packet leaves the system empty, the $(k+1)$'*th* departing packet has arrived to an empty queue, consequently it leaves behind only those packets that have arrived during slot $k+1$, thus we have,

$$i_{k+1} = b_{k+1}$$

then the state of the system can be summarized by,

$$i_{k+1} = (i_k - n_{k+1})^{\dagger} + b_{k+1} \tag{2.6}$$

where the notation $(x)^{\dagger}$ denotes max $(x, 0)$.

Chapter 3

Performance Analysis of the Wireless ATM Multiplexing System

In this chapter, we will present a performance analysis of the wireless ATM multiplexing system. The objective of the analysis is to determine the PGF of the queue length distribution, as well as distributions of the state of the channel and the number of *On*-sources. We will also present mean queue length and mean delay.

3.1 The Embedded Markov Chain Analysis

As stated in Section 2.4, the considered queuing system can be modeled as a discrete-time three-dimensional Markov chain. The state of the system is defined by the triplet (i_k, n_k, a_k). Let $Q_k(z, r, y)$ denotes the joint probability generating function (PGF) of the queue length, i_k, the state of the channel, n_k and the number of the *On*-sources, a_k, i. e.,

$$Q_k(z, r, y) = E[z^{i_k} r^{n_k} y^{a_k}] = \sum_{i=0}^{\infty} \sum_{l=0}^{1} \sum_{j=0}^{m} z^i r^l y^j p_k(i, l, j) \qquad (3.1a)$$

where, $$p_k(i, l, j) = Pr(i_k = i, n_k = l, a_k = j)$$

i.e., the joint probability that at the end of slot k, the queue length equals i, the state of the channel is l ('0' or '1') and the number of the *On*-source is j. So $Q_{k+1}(z, r, y)$ is given by,

$$Q_{k+1}(z, r, y) = E[z^{i_{k+1}} r^{n_{k+1}} y^{a_{k+1}}] \tag{3.1b}$$

Substituting for i_{k+1} from (2.6) in (3.1b) gives,

$$Q_{k+1}(z, r, y) = E[z^{i_{k+1}} r^{n_{k+1}} y^{a_{k+1}}] = E[z^{(i_k - n_{k+1})^+ + b_{k+1}} r^{n_{k+1}} y^{a_{k+1}}]$$

then substituting for b_{k+1} from (2.3), we have,

$$Q_{k+1}(z, r, y) = E\left[z^{(i_k - n_{k+1})^+ + \sum_{j=1}^{a_{k+1}} f_{j,k+1}} r^{n_{k+1}} y^{a_{k+1}}\right] \tag{3.1c}$$

Let us consider the right hand side (RHS) of (3.1c) by conditioning on i_k, n_{k+1} and a_{k+1} for the moment. This will enable us to take the expectations with respect to the random variables $f_{j,k+1}$, $j = 1, 2, \ldots, a_{k+1}$. Taking the constant factors outside the expectation symbol we have,

$$E\left[z^{(i_k - n_{k+1})^+ + \sum_{j=1}^{a_{k+1}} f_{j,k+1}} r^{n_{k+1}} y^{a_{k+1}} \middle| i_k, n_{k+1}, a_{k+1}\right]$$

$$= z^{(i_k - n_{k+1})^+} r^{n_{k+1}} y^{a_{k+1}} E\left[z^{\sum_{j=1}^{a_{k+1}} f_{j,k+1}} \middle| i_k, n_{k+1}, a_{k+1}\right]$$

Since $f_{j,k+1}$ are mutually independent,

$$z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} y^{a_{k+1}} E\left[z^{\sum_{j=1}^{a_{k+1}} f_{j,k+1}} \mid i_k, n_{k+1}, a_{k+1} \right]$$

$$= z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} y^{a_{k+1}} [f(z)]^{a_{k+1}}$$

thus (3.1c) becomes,

$$Q_{k+1}(z, r, y) = E[z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} [yf(z)]^{a_{k+1}}] \tag{3.1d}$$

Next substituting for a_{k+1} from (2.5) in (3.1d), we have,

$$Q_{k+1}(z, r, y) = E\left[z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} [yf(z)]^{\sum_{s=1}^{a_k} c_s + \sum_{s=1}^{m-a_k} d_s} \right] \tag{3.1e}$$

Next let us consider the RHS of (3.1e) again by conditioning on i_k , n_{k+1} and a_k. This will make it possible to take the expectations with respect to the random variables c_s and d_s. Taking the constant factors outside the expectation symbol we have,

$$E\left[z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} [yf(z)]^{\sum_{s=1}^{a_k} c_s + \sum_{s=1}^{m-a_k} d_s} \mid i_k, n_{k+1}, a_k \right]$$

$$= z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} E\left[[yf(z)]^{\sum_{s=1}^{a_k} c_s + \sum_{s=1}^{m-a_k} d_s} \mid i_k, n_{k+1}, a_k \right]$$

Since the random variables c_s and d_s are mutually independent, we have,

$$z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} E\left[[yf(z)]^{\sum_{s=1}^{a_k} c_s + \sum_{s=1}^{m-a_k} d_s} \mid i_k, n_{k+1}, a_k \right]$$

$$= z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} [c(yf(z))]^{a_k} [d(yf(z))]^{m-a_k}$$

where $c(z)$ and $d(z)$ have been defined in (2.3a) and (2.3b) respectively. Thus (3.1e) changes to,

$$Q_{k+1}(z, r, y) = E\{z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} [c(yf(z))]^{a_k} [d(yf(z))]^{m-a_k}\}$$

$$= [d(yf(z))]^m E\left\{ z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} \left[\frac{c(yf(z))}{d(yf(z))} \right]^{a_k} \right\}$$

$$= [d(yf(z))]^m E\{z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} Y^{a_k}\} \tag{3.2}$$

where,

$$Y = \frac{c(yf(z))}{d(yf(z))} \tag{3.3}$$

Next we will remove the operator $(x)^\dagger$ in (3.2). We see that the expectation in (3.2) depends on the random variables i_k, a_k and through n_{k+1} defined in (2.2) on n_k, g_s and h_s, Since the last two variables are independent of the other random variables, and also from the definition of expectation given in (3.1a) we may write,

$$E[z^{(i_k - n_{k+1})^\dagger} r^{n_{k+1}} Y^{a_k}] = \sum_{i=0}^{\infty} \sum_{l=0}^{1} \sum_{j=0}^{m} E[z^{(i - \bar{n}_{k+1})^\dagger} r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j)$$

$$= \sum_{i=1}^{\infty} \sum_{l=0}^{1} \sum_{j=0}^{m} E[z^{(i - \bar{n}_{k+1})^\dagger} r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j) + \sum_{i=0}^{0} \sum_{l=0}^{1} \sum_{j=0}^{m} E[z^{(i - \bar{n}_{k+1})^\dagger} r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j)$$

where, (3.4)

$$\bar{n}_{k+1} = n_{k+1}\big|_{n_k = l}$$

We may remove the $(x)^{\dagger}$ operator in (3.4), since $\bar{n}_{k+1}$ may only take the values of '0' and '1', thus (3.4) becomes,

$$E[z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} Y^{a_k}]$$

$$= \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} E[z^{i-\bar{n}_{k+1}} r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} E[z^0 r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j)$$

$$= \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} E\left[z^i\left(\frac{r}{z}\right)^{\bar{n}_{k+1}} Y^j\right] p_k(i, l, j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} E[r^{\bar{n}_{k+1}} Y^j] p_k(i, l, j) \qquad (3.5)$$

Let us define,

$$R = \frac{r}{z} \qquad (3.6)$$

From (2.2) we have,

$$\bar{n}_{k+1} = n_{k+1}\big|_{n_k = l} = \sum_{s=1}^{l} g_s + \sum_{s=1}^{1-l} h_s$$

Substituting above in (3.5) results in,

$$E[z^{(i_k - n_{k+1})^{\dagger}} r^{n_{k+1}} Y^{a_k}]$$

$$= \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} E\left[z^i R^{\sum_{s=1}^{l} g_s + \sum_{s=1}^{1-l} h_s} Y^j\right] p_k(i, l, j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} E\left[r^{\sum_{s=1}^{l} g_s + \sum_{s=1}^{1-l} h_s} Y^j\right] p_k(i, l, j)$$

$$= \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} z^i E\left[R^{\sum_{s=1}^{l} g_s + \sum_{s=1}^{1-l} h_s}\right] Y^j p_k(i, l, j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} E\left[r^{\sum_{s=1}^{l} g_s + \sum_{s=1}^{1-l} h_s}\right] Y^j p_k(i, l, j)$$

By taking the expectation with respect to random variables g_s and h_s, the above equation becomes,

$$E[z^{(i_k-n_{k+1})^{\dagger}}r^{n_{k+1}}Y^{a_k}] = \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} z^i[g(R)]^l[h(R)]^{1-l}Y^j p_k(i,l,j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m}[g(r)]^l[h(r)]^{1-l}Y^j p_k(i,l,j)$$

(3.7)

where $g(R), h(R), g(r)$ and $h(r)$ are determined from (2.1a) and (2.1b) through substitution of $z = r$ or R in the corresponding equations. Next, let us make the following definitions,

$$V = \frac{g(R)}{h(R)}, \qquad W = \frac{g(r)}{h(r)} \tag{3.8}$$

Substituting V and W into (3.7) gives,

$$E[z^{(i_k-n_{k+1})^{\dagger}}r^{n_{k+1}}Y^{a_k}]$$

$$= \sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} z^i h(R)V^l Y^j p_k(i,l,j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} h(r)W^l Y^j p_k(i,l,j)$$

Substituting above equation into (3.2) we have,

$$Q_{k+1}(z,r,y)$$

$$= [d(yf(z))]^m\left\{\sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} z^i h(R)V^l Y^j p_k(i,l,j) + \sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} h(r)W^l Y^j p_k(i,l,j)\right\}$$

$$= [d(yf(z))]^m\left\{h(R)\sum_{i=1}^{\infty}\sum_{l=0}^{1}\sum_{j=0}^{m} z^i V^l Y^j p_k(i,l,j) + h(r)\sum_{i=0}^{0}\sum_{l=0}^{1}\sum_{j=0}^{m} W^l Y^j p_k(i,l,j)\right\}$$

The first summation in above is $Q_k(z,r,y)|_{r=V, y=Y}$ defined in (3.1a) except that in the outermost summation, the lower limit is '1' instead of '0'. Next, we will complete this PGF by adding and subtracting the missing term. Thus the above equation can be rewritten as,

$$Q_{k+1}(z, r, y) = [d(yf(z))]^m \Big\{ h(R) \sum_{i=0}^{\infty} \sum_{l=0}^{1} \sum_{j=0}^{m} z^i V^l Y^j p_k(i, l, j)$$

$$- h(R) \sum_{i=0}^{0} \sum_{l=0}^{1} \sum_{j=0}^{m} z^i V^l Y^j p_k(i, l, j) + h(r) \sum_{i=0}^{0} \sum_{l=0}^{1} \sum_{j=0}^{m} W^l Y^j p_k(i, l, j) \Big\}$$

$$= [d(yf(z))]^m \Big\{ h(R) Q_k(z, V, Y) - h(R) \sum_{l=0}^{1} \sum_{j=0}^{m} V^l Y^j p_k(0, l, j)$$

$$+ h(r) \sum_{l=0}^{1} \sum_{j=0}^{m} W^l Y^j p_k(0, l, j) \Big\} \tag{3.9}$$

As we stated in Chapter 2, a packet can not be transmitted before the end of the slot that it has been generated. Thus if the queue length is zero at the end of a slot, there can not be any packet arrivals during that slot, since an *On* source generates at least a single packet during that slot. Thus we make the observation that there cannot be any sources in the *On* state during a slot at the end of which the queue length is zero. This means that if $i = 0$, then $j = 0$, so,

$$p_k(0, l, j) = p_k(0, l, 0)$$

and (3.9) becomes,

$$Q_{k+1}(z, r, y) = [d(yf(z))]^m \Big\{ h(R) Q_k(z, V, Y) - h(R) \sum_{l=0}^{1} V^l p_k(0, l, 0) + h(r) \sum_{l=0}^{1} W^l p_k(0, l, 0) \Big\}$$

$$= [d(yf(z))]^m \{ h(R) Q_k(z, V, Y) + [h(r) W^0 - h(R) V^0] p_k(0, 0, 0) + [h(r) W - h(R) V] p_k(0, 1, 0) \}$$

Substituting for V and W defined in (3.8) results in,

$$Q_{k+1}(z, r, y) = [d(yf(z))]^m\{h(R)Q_k(z, V, Y) + [h(r) - h(R)]p_k(0, 0, 0) + [g(r) - g(R)]p_k(0, 1, 0)\} \quad (3.10)$$

Equation (3.10) is the functional equation of the queue length, the state of the channel and the number of *On* sources in the system, as defined by i_k, n_k and a_k respectively. It describes the system at the embedded points and relates the joint PGF of the system at any two consecutive embedded points.

3.2 Transforming the Functional Equation into A More Tractable Form

In the above section we have obtained the functional equation describing the system, (3.10), but it is not mathematically tractable because on the right side of the equation we have a $Q_k(z, V, Y)$ instead of $Q_k(z, r, y)$. In this section, we will transform (3.10) into another form which lends itself to a solution. First let us develop a number of preliminary results that are needed for the transformation.

3.2.1 Preliminary Results

3.2.1.1 Definitions

Let us make the following definitions,

$$X(k+1) = X(1)X(k)|_{y=Y} \qquad k \geq 0 \quad (3.11a)$$

with,

$$X(0) = 1 \quad (3.11b)$$

$$X(1) = d(yf(z)) = \beta + (1-\beta)yf(z) \quad (3.11c)$$

Also we define,

$$B(k+1) = B(1)B(k)|_{y=Y} \qquad k \geq 0 \tag{3.12a}$$

with
$$B(0) = 1 \tag{3.12b}$$

$$B(1) = [d(yf(z))]^m \tag{3.12c}$$

From the above we have,

$$B(k) = [X(k)]^m \tag{3.12d}$$

Next let us make the following recursive equations,

$$H(k+1) = H(1)H(k)|_{r=V} \qquad k \geq 0 \tag{3.13a}$$

$$G(k+1) = H(1)G(k)|_{r=V} \qquad k \geq 0 \tag{3.13b}$$

$$\tilde{H}(k+1) = H(1)\tilde{H}(k)|_{r=V} \qquad k \geq 1 \tag{3.13c}$$

$$\tilde{G}(k+1) = H(1)\tilde{G}(k)|_{r=V} \qquad k \geq 1 \tag{3.13d}$$

where V has been defined in (3.8), and also assume the initial conditions,

$$H(0) = 1 \tag{3.14a}$$

$$G(0) = r \tag{3.14b}$$

$$H(1) = h(R) = \sigma + (1-\sigma)R \tag{3.15a}$$

$$G(1) = g(R) = 1 - \gamma + \gamma R \tag{3.15b}$$

$$\tilde{H}(1) = h(r) = \sigma + (1-\sigma)r \tag{3.15c}$$

$$\tilde{G}(1) = g(r) = 1 - \gamma + \gamma r \tag{3.15d}$$

where $R = \frac{r}{z}$, defined by (3.7). The RHS of (3.15a-d) are obtained from (2.1a) and (2.1b) through substitution of $z = r$ or R in the corresponding equations.

From definition (3.8) and (3.15a&b) we have,

$$V = \frac{G(1)}{H(1)} \tag{3.16a}$$

$$W = \frac{g(1)}{h(1)} \tag{3.16b}$$

Let $k = 0$, and substituting (3.14) into (3.13a) and (3.13b), and then substituting V obtained in (3.16a) we have,

$$H(1) = H(1)H(0)|_{r=V} = H(1) \tag{3.17a}$$

$$G(1) = H(1)G(0)|_{r=V} = H(1) \cdot \frac{G(1)}{H(1)} = G(1) \tag{3.17b}$$

which shows that (3.13a&b) are consistent with (3.14).

In the following sections, we will determine $X(k), B(k), H(k), G(k), \tilde{H}(k)$ and $\tilde{G}(k)$ explicitly.

3.2.1.2 Determining $X(k)$ and $B(k)$

From the definition of (3.10) it is possible to show that $X(k)$ satisfies the recurrence relationships given below,

$$X(k+1) = [\beta + \alpha f(z)]X(k) + (1-\alpha-\beta)f(z)X(k-1) \qquad k \geq 1$$

The above recurrence formula can be proved through induction, see [38,39]. It is a homogenous linear difference equation, with constant coefficients, and have the following characteristic equation,

$$\lambda^2 - (\beta + \alpha f(z))\lambda - (1-\alpha-\beta)f(z) = 0$$

The roots of the above equation are given by,

$$\lambda_{1,2} = \frac{\beta + \alpha f(z) \mp \sqrt{(\beta + \alpha f(z))^2 + 4(1 - \alpha - \beta) f(z)}}{2} \tag{3.18}$$

For $X(k)$ and $B(k)$ defined in (3.11) and (3.12), it is possible to show that (see Appendix I in [41]),

$$X(k) = C_1 \lambda_1^k + C_2 \lambda_2^k \tag{3.19a}$$

$$B(k) = (C_1 \lambda_1^k + C_2 \lambda_2^k)^m \tag{3.19b}$$

where C_1, C_2 are "constants" that may be determined from the initial conditions. From (3.11a&b) and (3.19a) we have,

$$X(0) = C_1 + C_2 = 1$$

$$X(1) = C_1 \lambda_1 + C_2 \lambda_2 = \beta + (1 - \beta) y f(z)$$

Solving the above two equations simultaneously, $C_{1,2}$ are given by,

$$C_{1,2} = \frac{1}{2} \mp \frac{2(y - y\beta - \alpha) f(z) + \beta + \alpha f(z)}{2\sqrt{(\beta + \alpha f(z))^2 + 4(1 - \alpha - \beta) f(z)}} \tag{3.20}$$

where C_1 and C_2 are taken with the negative and positive signs respectively. Thus $X(k)$ and $B(k)$ have been determined.

3.2.1.3 *Determining* $H(k), G(k), \tilde{H}(k)$ *and* $\tilde{G}(k)$

1. Difference equations of $H(k), G(k), \tilde{H}(k)$ and $\tilde{G}(k)$

We will show by using induction that $H(k), G(k), \tilde{H}(k)$ and $\tilde{G}(k)$ satisfy the following difference equations,

$$H(k+1) = \sigma H(k) + \frac{1-\sigma}{z} G(k) \qquad k \geq 0 \tag{3.21a}$$

$$G(k+1) = (1-\gamma)H(k) + \frac{\gamma}{z} G(k) \qquad k \geq 0 \tag{3.21b}$$

$$\tilde{H}(k+1) = \sigma H(k) + (1-\sigma)G(k) \qquad k \geq 0 \tag{3.21c}$$

$$\tilde{G}(k+1) = (1-\gamma)H(k) + \gamma G(k) \qquad k \geq 0 \tag{3.21d}$$

i) First, we consider $H(k)$. For $k = 0$, the difference equation in (3.21a) gives,

$$H(1) = \sigma H(0) + \frac{1-\sigma}{z} G(0) = \sigma + (1-\sigma)\frac{r}{z} = \sigma + (1-\sigma)R$$

Clearly it conforms to (3.15a) and from (3.17a) we note that (3.13a) also gives the same result. Next let us assume that (3.21a) is true for k and show that it will also be true for $k+1$. For $k+1$, (3.13a) yields,

$$H(k+2) = H(1)H(k+1)|_{r=V}$$

then substituting (3.21a) for $H(k+1)$ in the above equation, we have,

$$H(k+2) = H(1)H(k+1)|_{r=V} = H(1)\left[\sigma H(k) + \frac{1-\sigma}{z} G(k)\right]\Bigg|_{r=V}$$

$$= \sigma H(1)H(k)|_{r=V} + \frac{1-\sigma}{z} H(1)G(k)|_{r=V}$$

$$= \sigma H(k+1) + \frac{1-\sigma}{z} G(k+1)$$

which conforms to (3.21a) for $k+1$. This completes the proof of (3.21a).

ii) Next, let us consider $G(k)$. For $k = 0$, the difference equation in (3.21b) gives,

$$G(1) = (1-\gamma)H(0) + \frac{\gamma}{z}G(0) = 1-\gamma+\gamma R$$

Clearly it conforms to (3.15b) and from (3.17b) we note that (3.13b) also gives the same result. Next let us assume that (3.21b) is true for k and show that it will also be true for $k+1$. For $k+1$, (3.13b) yields,

$$G(k+2) = H(1)G(k+1)|_{r=V}$$

then substituting (3.21b) for $G(k+1)$ in the above equation, we have,

$$G(k+2) = H(1)G(k+1)|_{r=V} = H(1)\left[(1-\gamma)H(k) + \frac{\gamma}{z}G(k)\right]\Big|_{r=V}$$

$$= (1-\gamma)H(1)H(k)|_{r=V} + \frac{\gamma}{z}H(1)G(k)|_{r=V}$$

$$= (1-\gamma)H(k+1) + \frac{\gamma}{z}G(k+1)$$

which conforms to (3.21b) for $k+1$. This completes the proof of (3.21b).

iii) Next, let us consider $\tilde{H}(k)$. For $k = 0$, the difference equation in (3.21c) gives,

$$\tilde{H}(1) = \sigma H(0) + (1-\sigma)G(0) = \sigma + (1-\sigma)r$$

Clearly it conforms to (3.15c). Next we assume that (3.21c) is true for k and show that it will also be true for $k+1$. For $k+1$, (3.13c) yields,

$$\tilde{H}(k+2) = H(1)\tilde{H}(k+1)|_{r=V}$$

then substituting (3.21c) for $\tilde{H}(k+1)$ in the above equation, we have,

$$\tilde{H}(k+2) = H(1)\tilde{H}(k+1)|_{r=V} = H(1)[\sigma H(k) + (1-\sigma)G(k)]|_{r=V}$$

$$= \sigma H(1)H(k)|_{r=V} + (1-\sigma)H(1)G(k)|_{r=V}$$

$$= \sigma H(k+1) + (1-\sigma)G(k+1)$$

which conforms to (3.21c) for $k+1$. This completes the proof of (3.21c).

iv) Finally, we consider $\tilde{G}(k)$. For $k = 0$, the difference equation in (3.21d) gives,

$$\tilde{G}(1) = (1-\gamma)H(0) + \gamma G(0) = 1-\gamma+\gamma r$$

Clearly it conforms to (3.15d). Next let us assume that (3.21d) is true for k and show that it will also be true for $k+1$. For $k+1$, (3.13d) yields,

$$\tilde{G}(k+2) = H(1)G(k+1)|_{r=V}$$

then substituting (3.21d) for $\tilde{G}(k+1)$ in the above equation, we have,

$$\tilde{G}(k+2) = H(1)\tilde{G}(k+1)\big|_{r=V} = H(1)[(1-\gamma)H(k) + \gamma G(k)]|_{r=V}$$

$$= (1-\gamma)H(1)H(k)|_{r=V} + \gamma H(1)Gk)|_{r=V}$$

$$= (1-\gamma)H(k+1) + \gamma G(k+1)$$

which conforms to (3.21d) for $k+1$. This completes the proof of (3.21d).

Up to now we have obtained the expressions of $H(k+1)$, $G(k+1)$, $\tilde{H}(k+1)$ and $\tilde{H}(k+1)$ in terms of $H(k)$ and $G(k)$.

2. Solution of the difference equations

Next, we will determine the solutions of difference equations (3.21a-d). First let us define the following transforms of $H(k)$ and $G(k)$ with respect to discrete-time k,

$$H(\omega) = \sum_{k=0}^{\infty} H(k)\omega^k \tag{3.22a}$$

$$G(\omega) = \sum_{k=0}^{\infty} G(k)\omega^k \tag{3.22b}$$

Taking transforms of both sides of (3.21a) with respect to k and taking into account the above definition, we have,

$$\sum_{k=0}^{\infty} H(k+1)\omega^k = \sigma \sum_{k=0}^{\infty} H(k)\omega^k + \frac{1-\sigma}{z} \sum_{k=0}^{\infty} G(k)\omega^k$$

$$= \sigma H(\omega) + \frac{1-\sigma}{z} G(\omega) \tag{3.23}$$

Let $i = k+1$, substituting into the left-hand side (LHS) of (3.23), we have,

$$\sum_{k=0}^{\infty} H(k+1)\omega^k = \sum_{i=1}^{\infty} H(i)\omega^{i-1} \tag{3.24}$$

Since
$$\sum_{i=1}^{\infty} H(i)\omega^{i-1} = \frac{1}{\omega}\sum_{i=1}^{\infty} H(i)\omega^i$$

$$= \frac{1}{\omega}\left[\sum_{i=0}^{\infty} H(i)\omega^i - \sum_{m=0}^{0} H(i)\omega^i\right]$$

$$= \frac{1}{\omega}[H(\omega) - H(0)]$$

substituting above into (3.24), we have,

$$\sum_{k=0}^{\infty} H(k+1)\omega^k = \frac{1}{\omega}[H(\omega) - H(0)]$$

The above equation and (3.23) results in,

$$\frac{1}{\omega}[H(\omega)-H(0)] = \sigma H(\omega)+\frac{1-\sigma}{z}G(\omega) \tag{3.25}$$

Changing the form of (3.25) gives,

$$(1-\sigma\omega)H(\omega) = H(0)+\frac{(1-\sigma)\omega}{z}G(\omega) \tag{3.26}$$

Similarly, taking transforms of both sides of (3.21b) with respect to k and taking into account definitions from (3.22) we have,

$$\begin{aligned}\sum_{k=0}^{\infty} G(k+1)\omega^k &= (1-\gamma)\sum_{k=0}^{\infty} H(k)\omega^k+\frac{\gamma}{z}\sum_{k=0}^{\infty} G(k)\omega^k \\ &= (1-\gamma)H(\omega)+\frac{\gamma}{z}G(\omega)\end{aligned} \tag{3.27}$$

Let $i = k+1$, substituting into the LHS of (3.27), we have,

$$\sum_{k=0}^{\infty} G(k+1)\omega^k = \sum_{i=1}^{\infty} G(i)\omega^{i-1} \tag{3.28}$$

Since

$$\begin{aligned}\sum_{i=1}^{\infty} G(i)\omega^{i-1} &= \frac{1}{\omega}\sum_{i=1}^{\infty} G(i)\omega^i \\ &= \frac{1}{\omega}\left[\sum_{i=0}^{\infty} G(i)\omega^i-\sum_{i=0}^{0} G(i)\omega^i\right] \\ &= \frac{1}{\omega}[G(\omega)-G(0)]\end{aligned}$$

substituting above into (3.28), we have,

$$\frac{1}{\omega}[G(\omega)-G(0)] = (1-\gamma)H(\omega)+\frac{\gamma}{z}G(\omega) \tag{3.29}$$

Changing the form of (3.29) gives,

$$G(\omega) = \frac{1}{1-\frac{\gamma\omega}{z}}[G(0)+(1-\gamma)\omega H(\omega)] \tag{3.30}$$

Substituting $H(0) = 1, G(0) = r$ from definition (3.14), (3.26) and (3.30) change to,

$$(1-\sigma\omega)H(\omega) = 1+\frac{(1-\sigma)\omega}{z}G(\omega) \tag{3.31a}$$

$$G(\omega) = \frac{1}{1-\frac{\gamma\omega}{z}}[r+(1-\gamma)\omega H(\omega)] \tag{3.31b}$$

The simultaneous solutions of (3.31a) and (3.31b) give,

$$G(\omega) = \frac{(1-\gamma)\omega+(1-\sigma\omega)r}{1-\left(\sigma+\frac{\gamma}{z}\right)\omega-\left(\frac{1-\gamma-\sigma}{z}\right)\omega^2} \tag{3.32a}$$

$$H(\omega) = \frac{1-\frac{\gamma\omega}{z}+\frac{(1-\sigma)\omega r}{z}}{1-\left(\sigma+\frac{\gamma}{z}\right)\omega-\left(\frac{1-\gamma-\sigma}{z}\right)\omega^2} \tag{3.32b}$$

Let

$$D(\omega) = 1-\left(\sigma+\frac{\gamma}{z}\right)\omega-\left(\frac{1-\gamma-\sigma}{z}\right)\omega^2 \tag{3.33}$$

Then (3.32) can be rewritten as,

$$G(\omega) = \frac{(1-\gamma)\omega+(1-\sigma\omega)r}{D(\omega)} \tag{3.34a}$$

$$H(\omega) = \frac{1-\frac{\gamma\omega}{z}+\frac{(1-\sigma)\omega r}{z}}{D(\omega)} \tag{3.34b}$$

Next we will apply partial fraction expansion in the above expressions. First let us determine the roots of the denominator $D(\omega)$. Defining q_1, q_2 respectively as,

$$q_1 = \sigma + \frac{\gamma}{z}, \qquad q_2 = \frac{1-\gamma-\sigma}{z}$$

thus $D(\omega)$ can be written as,

$$D(\omega) = 1 - q_1\omega - q_2\omega^2 = 0$$

The roots of the above are given by,

$$\omega_{1,2} = \frac{-q_1 \mp \sqrt{q_1^2 + 4q_2}}{2q_2}.$$

where ω_1 and ω_2 are taken with the negative and positive sign respectively. Next letting $\lambda_{3,4} = 1/\omega_{1,2}$, then,

$$\lambda_{3,4} = \frac{q_1 \mp \sqrt{q_1^2 + 4q_2}}{2} = \frac{\gamma + \sigma z \mp \sqrt{\gamma^2 + \sigma^2 z^2 + 2z(2 - 2\gamma - 2\sigma + \sigma\gamma)}}{2z} \tag{3.35}$$

Now we can write (3.34) as,

$$G(\omega) = \frac{(1-\gamma)\omega + (1-\sigma\omega)r}{(1-\lambda_3\omega)(1-\lambda_4\omega)} \tag{3.36a}$$

$$H(\omega) = \frac{1 - \frac{\gamma\omega}{z} + \frac{(1-\sigma)\omega r}{z}}{(1-\lambda_3\omega)(1-\lambda_4\omega)} \tag{3.36b}$$

Applying the partial fraction expansion, $G(\omega)$ and $H(\omega)$ in (3.36) can be expressed as,

$$G(\omega) = \frac{C_3}{1-\lambda_3\omega} + \frac{C_4}{1-\lambda_4\omega} \tag{3.37a}$$

$$H(\omega) = \frac{C_5}{1-\lambda_3\omega} + \frac{C_6}{1-\lambda_4\omega} \tag{3.37b}$$

where,

$$C_3 = \left.\frac{(1-\gamma)\omega + (1-\sigma\omega)r}{1-\lambda_4\omega}\right|_{\omega = 1/\lambda_3} = \frac{r\lambda_3 + 1 - \gamma - \sigma r}{\lambda_3 - \lambda_4}$$

$$C_4 = \left.\frac{(1-\gamma)\omega + (1-\sigma\omega)r}{1-\lambda_3\omega}\right|_{\omega = 1/\lambda_4} = \frac{r\lambda_4 + 1 - \gamma - \sigma r}{\lambda_4 - \lambda_3}$$

Substituting $\lambda_{3,4}$ from (3.35) in above two equations we obtain C_3, C_4 as,

$$C_3 = \frac{r}{2} - \frac{2(1-\gamma-\sigma r)z + r(\gamma+\sigma z)}{2\sqrt{(\gamma+\sigma z)^2 + 4(1-\gamma-\sigma)z}} \tag{3.38a}$$

$$C_4 = \frac{r}{2} + \frac{2(1-\gamma-\sigma r)z + r(\gamma+\sigma z)}{2\sqrt{(\gamma+\sigma z)^2 + 4(1-\gamma-\sigma)z}} \tag{3.38b}$$

C_5 and C_6 may also be determined from the initial conditions, however as will be shown later on, these constants are not needed for the steady-state analysis and they will not be given here.

Inverting $G(\omega)$ and $H(\omega)$ gives,

$$G(k) = C_3\lambda_3^k + C_4\lambda_4^k \tag{3.39a}$$

$$H(k) = C_5\lambda_3^k + C_6\lambda_4^k \tag{3.39b}$$

Now that we have determined $G(k)$ (3.39a) and $H(k)$ (3.39b), together with (3.21c&d) which are repeated here,

$$\tilde{H}(k+1) = \sigma H(k) + (1-\sigma)G(k) \qquad k \geq 0$$

$$\tilde{G}(k+1) = (1-\gamma)H(k) + \gamma G(k) \qquad k \geq 0$$

we can also determine $\tilde{H}(k)$ and $\tilde{G}(k)$.

3.2.1.4 Determining special values of λ_i and C_i

Next, we will present all the values of λ_i, C_i and some of their derivatives that will be needed later on.

First let us define,

$$\tilde{\lambda}_{3,4} = \frac{\gamma + \sigma z \mp \sqrt{(\gamma + \sigma z)^2 + 4(1-\gamma-\sigma)z}}{2} \tag{3.40}$$

then because of (3.35) we have,

$$\lambda_3 = \frac{\tilde{\lambda}_3}{z}, \qquad \lambda_4 = \frac{\tilde{\lambda}_4}{z} \tag{3.41}$$

Also we define,

$$\tilde{C}_i = C_i\big|_{r=1,\, y=1} \tag{3.42}$$

with C_i as given in (3.20) and (3.38), we have,

$$\tilde{C}_1 = \frac{1}{2} - \frac{2(1-\beta-\alpha)f(z) + \beta + \alpha f(z)}{2\sqrt{(\beta + \alpha f(z))^2 + 4(1-\alpha-\beta)f(z)}} \tag{3.43b}$$

$$\tilde{C}_2 = \frac{1}{2} + \frac{2(1-\beta-\alpha)f(z) + \beta + \alpha f(z)}{2\sqrt{(\beta + \alpha f(z))^2 + 4(1-\alpha-\beta)f(z)}} \tag{3.43c}$$

$$\tilde{C}_4 = \frac{1}{2} - \frac{2(1-\gamma-\sigma)z + (\gamma + \sigma z)}{2\sqrt{(\gamma + \sigma z)^2 + 4(1-\gamma-\sigma)z}} \tag{3.43d}$$

$$\tilde{C}_4 = \frac{1}{2} + \frac{2(1-\gamma-\sigma)z + (\gamma+\sigma z)}{2\sqrt{(\gamma+\sigma z)^2 + 4(1-\gamma-\sigma)z}} \tag{3.43e}$$

Next we calculate $\tilde{C}_i\big|_{z=1}$, $\lambda_{1,2}\big|_{z=1}$ and $\tilde{\lambda}_{3,4}\big|_{z=1}$ based on the results from (3.43), (3.18) and (3.40),

$$\tilde{C}_1\big|_{z=1} = \frac{1}{2} - \frac{2(1-\beta-\alpha)+\beta+\alpha}{2\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}} = \frac{1}{2} - \frac{1}{2} = 0 \tag{3.44a}$$

$$\tilde{C}_2\big|_{z=1} = \frac{1}{2} + \frac{2(1-\beta-\alpha)+\beta+\alpha}{2\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}} = \frac{1}{2} + \frac{1}{2} = 1 \tag{3.44b}$$

$$\tilde{C}_3\big|_{z=1} = \frac{1}{2}\frac{2(1-\gamma-\sigma)+(\gamma+\sigma)}{2\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}} = \frac{1}{2} - \frac{1}{2} = 0 \tag{3.44c}$$

$$\tilde{C}_4\big|_{z=1} = \frac{1}{2} + \frac{2(1-\gamma-\sigma)+(\gamma+\sigma)}{2\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}} = \frac{1}{2} + \frac{1}{2} = 1 \tag{3.44d}$$

$$\lambda_1\big|_{z=1} = \frac{\beta+\alpha-\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}}{2} = \alpha+\beta-1 \tag{3.44e}$$

$$\lambda_2\big|_{z=1} = \frac{\beta+\alpha+\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}}{2} = 1 \tag{3.44f}$$

$$\tilde{\lambda}_3\big|_{z=1} = \frac{\gamma+\sigma-\sqrt{(2-\gamma-\sigma)^2}}{2} = \gamma+\sigma-1 \tag{3.44g}$$

$$\tilde{\lambda}_4\big|_{z=1} = \frac{\gamma+\sigma+\sqrt{(2-\gamma-\sigma)^2}}{2} = 1 \tag{3.44h}$$

By substituting $z = 1$ into (3.20), we can have,

$$C_1\big|_{z=1,r=1} = \frac{(1-\beta)(1-y)}{2-\alpha-\beta} \tag{3.45a}$$

$$C_2\big|_{z=1,r=1} = \frac{1-\alpha+(1-\beta)y}{2-\alpha-\beta} \tag{3.45b}$$

then, we substitute $z = 1$ and $r = 1$ into (3.38) and have,

$$C_3\big|_{z=1,\,r=1} = 0 \tag{3.45c}$$

$$C_4\big|_{z=1,\,r=1} = 1 \tag{3.45d}$$

Next, substituting $z = 1$ and $y = 1$ into (3.20) gives,

$$C_1\big|_{z=1,\,y=1} = 0 \tag{3.46a}$$

$$C_2\big|_{z=1,\,y=1} = 1 \tag{3.46b}$$

Again, substituting $z = 1$ into (3.38) results in,

$$C_3\big|_{z=1,\,y=1} = \frac{(1-\gamma)(r-1)}{2-\gamma-\sigma} \tag{3.46c}$$

$$C_4\big|_{z=1,\,y=1} = \frac{(1-\gamma)+r(1-\sigma)}{2-\gamma-\sigma} \tag{3.46d}$$

Also from (3.44e-f) we obtain,

$$\lambda_1\big|_{z=1,\,r=1} = \alpha+\beta-1 \tag{3.47a}$$

$$\lambda_2\big|_{z=1,\,r=1} = 1 \tag{3.47b}$$

$$\tilde{\lambda}_3\big|_{z=1,\,r=1} = \gamma+\sigma-1 \tag{3.47c}$$

$$\tilde{\lambda}_4\big|_{z=1,\,r=1} = 1 \tag{3.47d}$$

as well as,

$$\lambda_1\big|_{z=1,\,y=1} = \alpha+\beta-1 \tag{3.48a}$$

$$\lambda_2\big|_{z=1,\,y=1} = 1 \tag{3.48b}$$

$$\lambda_3\Big|_{z=1,\,y=1} = \gamma+\sigma-1 \tag{3.48c}$$

$$\tilde{\lambda}_4\Big|_{z=1,\,y=1} = 1 \tag{3.48d}$$

Next let us consider $\left.\frac{d\lambda_2}{dz}\right|_{z=1}$, $\left.\frac{d\tilde{\lambda}_4}{dz}\right|_{z=1}$, $\left.\frac{d\tilde{C}_2}{dz}\right|_{z=1}$ and $\left.\frac{d\tilde{C}_4}{dz}\right|_{z=1}$. First differentiating λ_2 from (3.18) with respect to z we have,

$$\frac{d\lambda_2}{dz} = \frac{1}{2}\left\{\alpha f'(z)+\frac{1}{2}\cdot\frac{2(\beta+\alpha f(z))\alpha f'(z)+4(1-\alpha-\beta)f'(z)}{\sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}}\right\} \tag{3.49a}$$

substituting $z=1$ in (3.49a) we have,

$$\left.\frac{d\lambda_2}{dz}\right|_{z=1} = \frac{1}{2}\left\{\alpha f'(1)+\frac{1}{2}\cdot\frac{2(\beta+\alpha)\alpha f'(1)+4(1-\alpha-\beta)f'(1)}{\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}}\right\}$$

but $\bar{f} = f'(1)$, which is the average number of the packets generated by an *On* source during a slot. Also substituting for the square root, we obtain $\left.\frac{d\lambda_2}{dz}\right|_{z=1}$ as,

$$\left.\frac{d\lambda_2}{dz}\right|_{z=1} = \frac{\bar{f}(1-\beta)}{2-\alpha-\beta} \tag{3.49b}$$

Now we differentiate $\tilde{\lambda}_4$ with respect to z from (3.40),

$$\frac{d\tilde{\lambda}_4}{dz} = \frac{1}{2}\left\{\sigma+\frac{1}{2}\cdot\frac{2\sigma(\gamma+\sigma z)+4(1-\gamma-\sigma)}{\sqrt{(\gamma+\sigma z)^2+4(1-\gamma-\sigma)z}}\right\} \tag{3.50a}$$

substituting $z=1$ in (3.50a) we obtain $\left.\frac{d\tilde{\lambda}_4}{dz}\right|_{z=1}$ as,

$$\left.\frac{d\tilde{\lambda}_4}{dz}\right|_{z=1} = \frac{1}{2}\left\{\sigma + \frac{1}{2}\cdot\frac{2\sigma(\gamma+\sigma)+4(1-\gamma-\sigma)}{\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}}\right\}$$

or, $$\left.\frac{d\tilde{\lambda}_4}{dz}\right|_{z=1} = \frac{1-\gamma}{2-\gamma-\sigma} \tag{3.50b}$$

Next let us differentiate $\tilde{C}_2$ from (3.44b) with respect to z,

$$\frac{d\tilde{C}_2}{dz} = \frac{2[2(1-\beta-\alpha)f'(z)+\alpha f'(z)]\sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}}{4[(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)]}$$

$$-\frac{2(1-\beta-\alpha)f(z)+\beta+\alpha f(z)}{4[(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)]}\cdot\frac{2(\beta+\alpha f(z))\alpha f'(z)+4(1-\alpha-\beta)f'(z)}{\sqrt{(\beta+\alpha f(z))^2+4(1-\alpha-\beta)f(z)}}$$

(3.51a)

substituting $z=1$ in (3.51a) we have,

$$\left.\frac{d\tilde{C}_2}{dz}\right|_{z=1} = \frac{2[2(1-\beta-\alpha)f'(1)+\alpha f'(1)]\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}}{4[(\beta+\alpha)^2+4(1-\alpha-\beta)]}$$

$$-\frac{2(1-\beta-\alpha)+\beta+\alpha}{4[(\beta+\alpha)^2+4(1-\alpha-\beta)]}\cdot\frac{2(\beta+\alpha)\alpha f'(1)+4(1-\alpha-\beta)f'(1)}{\sqrt{(\beta+\alpha)^2+4(1-\alpha-\beta)}}$$

but $\bar{f} = f'(1)$, also substituting for the square root, we have,

$$\left.\frac{d\tilde{C}_2}{dz}\right|_{z=1} = \frac{\bar{f}(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2} \tag{3.51b}$$

Finally differentiating $\tilde{C}_4$ from (3.44d) with respect to z we have,

$$\frac{d\tilde{C}_4}{dz} = \frac{2[2(1-\gamma-\sigma)+\sigma]\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}}{4[(\gamma+\sigma z)^2+4(1-\gamma-\sigma)z]} \tag{3.52a}$$

$$-\frac{2(1-\gamma-\sigma)z+(\gamma+\sigma z)}{4[(\gamma+\sigma z)^2+4(1-\gamma-\sigma)z]}\cdot\frac{2(\gamma+\sigma z)\sigma+4(1-\gamma-\sigma)}{\sqrt{(\gamma+\sigma z)^2+4(1-\gamma-\sigma)z}}$$

substituting $z=1$ in (3.52a) we have,

$$\left.\frac{d\tilde{C}_4}{dz}\right|_{z=1} = \frac{2[2(1-\gamma-\sigma)+\sigma]\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}}{4[(\gamma+\sigma)^2+4(1-\gamma-\sigma)]} - \frac{2(1-\gamma-\sigma)+(\gamma+\sigma)}{4[(\gamma+\sigma)^2+4(1-\gamma-\sigma)]}\cdot\frac{2(\gamma+\sigma)\sigma+4(1-\gamma-\sigma)}{\sqrt{(\gamma+\sigma)^2+4(1-\gamma-\sigma)}}$$

so we have,

$$\left.\frac{d\tilde{C}_6}{dz}\right|_{z=1} = \frac{(1-\gamma)(1-\gamma-\sigma)}{(2-\gamma-\sigma)^2} \tag{3.52b}$$

By differentiating (3.49a), (3.50a) and substituting $z=1$ in the resulting expressions we obtain,

$$\left.\frac{d^2\lambda_2}{dz^2}\right|_{z=1} = \frac{2(1-\alpha)(1-\beta)(\alpha+\beta-1)(\bar{f})^2}{(2-\alpha-\beta)^3} + \frac{(1-\beta)f''(1)}{2-\alpha-\beta} \tag{3.53}$$

$$\left.\frac{d^2\tilde{\lambda}_4}{dz^2}\right|_{z=1} = \frac{2(1-\gamma)(1-\sigma)(\gamma+\sigma-1)}{(2-\gamma-\sigma)^3} \tag{3.54}$$

3.2.2 Transformation of the Functional Equation

Now we are ready to transform the functional equation obtained in (3.10) into another form. Taking into consideration the definitions of $B(1)$, $H(1)$ $G(1)$, $\tilde{H}(1)$ and $\tilde{G}(1)$ from (3.11b) and (3.14), the functional equation (3.10) may be written as,

$$Q_{k+1}(z,r,y) = B(1)H(1)Q_k(z,V,Y) + B(1)[\tilde{H}(1)-H(1)]p_k(0,0,0) + B(1)[\tilde{G}(1)-G(1)]p_k(0,1,0) \tag{3.55}$$

By expanding $Q_{k+1}(z,r,y)$ in (3.55) for the first few values of k, we can prove by recurrence the following major result.

$$Q_k(z,r,y) = B(k)H(k) + \sum_{j=1}^{k} B(j)[\tilde{H}(j) - H(j)]p_{k-j}(0,0,0) + \sum_{j=1}^{k} B(j)[\tilde{G}(j) - G(j)]p_{k-j}(0,1,0) \quad (3.56)$$

Proof:

Since we have modeled the system as a Markov chain, the steady-state of the system will be independent of the initial conditions. We are only interested in the steady-state behavior of the system, therefore we will choose an initial condition that simplifies the analysis. We assume,

$$Q_0(z,r,y) = 1 \quad (3.57a)$$

which corresponds to zero queue length, *Bad* channel and zero number of *On* sources. Thus we also have,

$$Q_0(z,V,Y) = 1 \quad (3.57b)$$

Next, we will show that (3.56) holds for the first few values of k.

i) For $k = 0$, (3.55) yields,

$$\begin{aligned} &Q_1(z,r,y) \\ &= B(1)H(1)Q_0(z,V,Y) \\ &\quad + B(1)[\tilde{H}(1) - H(1)]p_0(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_0(0,1,0) \\ &= B(1)H(1) + B(1)[\tilde{H}(1) - H(1)]p_0(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_0(0,1,0) \end{aligned} \quad (3.58)$$

Equation (3.56) also gives the same result for $k = 0$.

ii) For $k = 1$, we will still calculate $Q_2(z,r,y)$ from (3.55).

First, by substituting $r = V$ and $y = Y$ in (3.58) we have,

$$Q_1(z,V,Y) = B(1)|_{y=Y}H(1)|_{r=v} + B(1)|_{y=Y}[\tilde{H}(1)|_{r=V} - H(1)|_{r=V}]p_0(0,0,0)$$

$$+ B(1)|_{y=Y}[\tilde{G}(1)|_{r=V} - G(1)|_{r=V}]p_0(0,1,0) \qquad (3.59)$$

Substituting $k = 1$ into (3.55) yields,

$$Q_2(z,r,y)$$

$$= B(1)H(1)Q_1(z,V,Y)$$

$$+ B(1)[\tilde{H}(1) - H(1)]p_1(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_1(0,1,0)$$

Substituting $Q_1(z,V,Y)$ from (3.59) into the above equation gives,

$$\begin{aligned}
Q_2(z,r,y) &= B(1)H(1)\{B(1)|_{y=Y}H(1)|_{r=V} \\
&\quad + B(1)|_{y=Y}[\tilde{H}(1)|_{r=V} - H(1)|_{r=V}]p_0(0,0,0) + B(1)|_{y=Y}[\tilde{G}(1)|_{r=V} - G(1)|_{r=V}]p_0(0,1,0)\} \\
&\quad + B(1)[\tilde{H}(1) - H(1)]p_1(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_1(0,1,0) \\
&= B(1)B(1)|_{y=Y}H(1)H(1)|_{r=V} \\
&\quad + B(1)B(1)|_{y=Y}H(1)\tilde{H}(1)|_{r=V}p_0(0,0,0) - B(1)B(1)|_{y=Y}H(1)H(1)|_{r=V}p_0(0,0,0) \\
&\quad + B(1)B(1)|_{y=Y}H(1)\tilde{G}(1)|_{r=V}p_0(0,1,0) - B(1)B(1)|_{y=Y}H(1)G(1)|_{r=V}p_0(0,1,0) \\
&\quad + B(1)[\tilde{H}(1) - H(1)]p_1(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_1(0,1,0) \\
&= B(2)H(2) + B(2)[\tilde{H}(2) - H(2)]p_0(0,0,0) + B(2)[\tilde{G}(2) - G(2)]p_0(0,1,0) \\
&\quad + B(1)[\tilde{H}(1) - H(1)]p_1(0,0,0) + B(1)[\tilde{G}(1) - G(1)]p_1(0,1,0)
\end{aligned} \qquad (3.60)$$

we can see that (3.56) will also give the same result as (3.60) for $k = 1$.

iii) For $k = 2$. Similarly, first we substitute $r = V$ and $y = Y$ in (3.60) and have,

$$\begin{aligned}
Q_2(z,V,Y) &= B(2)|_{y=Y}H(2)|_{r=V} \\
&\quad + B(2)|_{y=Y}[\tilde{H}(2)|_{r=V} - H(2)|_{r=V}]p_0(0,0,0) + B(2)|_{y=Y}[\tilde{G}(2)|_{r=V} - G(2)|_{r=V}]p_0(0,1,0) \\
&\quad + B(1)|_{y=Y}[\tilde{H}(1)|_{r=V} - H(1)|_{r=V}]p_1(0,0,0) + B(1)|_{y=Y}[\tilde{G}(1)|_{r=V} - G(1)|_{r=V}]p_1(0,1,0)
\end{aligned} \qquad (3.61)$$

Substituting $k = 2$ into (3.55) yields,

$$Q_3(z, r, y)$$

$$= B(1)H(1)Q_1(z, V, Y)$$

$$+ B(1)[\tilde{H}(1) - H(1)]p_2(0, 0, 0) + B(1)[\tilde{G}(1) - G(1)]p_2(0, 1, 0)$$

Substituting $Q_2(z, V, Y)$ from (3.61) into the above equation gives,

$$\begin{aligned}
Q_3(z, r, y) = {} & B(1)H(1)\{B(2)|_{y=Y}H(2)|_{r=V} \\
& + B(2)|_{y=Y}[\tilde{H}(2)|_{r=V} - H(2)|_{r=V}]p_0(0, 0, 0) + B(2)|_{y=Y}[\tilde{G}(2)|_{r=V} - G(2)|_{r=V}]p_0(0, 1, 0) \\
& + B(2)|_{y=Y}[\tilde{H}(2)|_{r=V} - H(2)|_{r=V}]p_1(0, 0, 0) + B(2)|_{y=Y}[\tilde{G}(2)|_{r=V} - G(2)|_{r=V}]p_1(0, 1, 0)\} \\
& + B(1)[\tilde{H}(1) - H(1)]p_2(0, 0, 0) + B(1)[\tilde{G}(1) - G(1)]p_2(0, 1, 0) \\
= {} & B(1)B(2)|_{y=Y}H(1)H(2)|_{r=V} \\
& + B(1)B(2)|_{y=Y}[H(1)\tilde{H}(2)|_{r=V} - H(1)H(2)|_{r=V}]p_0(0, 0, 0) \\
& + B(1)B(2)|_{y=Y}[H(1)\tilde{G}(2)|_{r=V} - H(1)G(2)|_{r=V}]p_0(0, 1, 0) \\
& + B(1)B(2)|_{y=Y}[H(1)\tilde{H}(1)|_{r=V} - H(1)H(1)|_{r=V}]p_1(0, 0, 0) \\
& + B(1)B(2)|_{y=Y}[H(1)\tilde{G}(1)|_{r=V} - H(1)G(1)|_{r=V}]p_1(0, 1, 0) \\
& + B(1)[\tilde{H}(1) - H(1)]p_2(0, 0, 0) + B(1)[\tilde{G}(1) - G(1)]p_2(0, 0, 0) \\
= {} & B(3)H(3) \\
& + B(3)[\tilde{H}(3) - H(3)]p_0(0, 0, 0) + B(3)[\tilde{H}(3) - G(3)]p_0(0, 1, 0) \\
& + B(2)[\tilde{H}(2) - H(2)]p_1(0, 0, 0) + B(2)[\tilde{G}(2) - G(2)]p_1(0, 1, 0) \\
& + B(1)[\tilde{H}(1) - H(1)]p_2(0, 0, 0) + B(1)[\tilde{G}(1) - G(1)]p_2(0, 1, 0)
\end{aligned}$$

we can see that (3.56) will also give the same result as (3.62) for $k = 2$.

Therefore (3.56) is verified for $k = 0, 1, 2$. Next let us assume that (3.56) is true for the order of k, i.e.,

$$Q_k(z, r, y) = B(k)H(k) + \sum_{j=1}^{k} B(j)[\tilde{H}(j) - H(j)]p_{k-j}(0, 0, 0)$$
$$+ \sum_{j=1}^{k} B(j)[\tilde{G}(j) - G(j)]p_{k-j}(0, 1, 0) \tag{3.62}$$

Next we will prove that it is also true for the order of $k+1$. First let us substitute $y = Y,\ r = V$ in equation (3.62),

$$Q_k(z, V, Y) = B(k)\big|_{y=Y}H(k)\big|_{r=V}$$
$$+ \sum_{j=1}^{k} B(j)\big|_{y=Y}[\tilde{H}(j)\big|_{r=V} - H(j)\big|_{r=V}]p_{k-j}(0, 0, 0)$$
$$+ \sum_{j=1}^{k} B(j)\big|_{y=Y}[\tilde{G}(j)\big|_{r=V} - G(j)\big|_{r=V}]p_{k-j}(0, 1, 0)$$

Next, substituting the above equation into (3.55) and taking into account (3.12a) and (3.13) gives,

$$Q_{k+1}(z, r, y) = B(k+1)H(k+1)$$
$$+ \sum_{j=1}^{k+1} B(j)[\tilde{H}(j) - H(j)]p_{k-j}(0, 0, 0)$$
$$+ \sum_{j=1}^{k+1} B(j)[\tilde{G}(j) - G(j)]p_{k-j}(0, 1, 0) \tag{3.63}$$

This shows that (3.63) is consistent with (3.56). This completes the proof of (3.56).

3.3 Solution of the Functional Equation

In this section, we present the solution of the functional equation. The objective is to determine the joint steady-state PGF of the queue length, state of the channel and the number of *On*-sources in the system by applying the final value theorem.

From this result we will determine the marginal PGF of the queue length and consequently the mean queue length and the mean delay.

First let us define the transforms of $Q_k(z, r, y)$, $p_k(0, 0, 0)$ and $p_k(0, 1, 0)$, with respect to discrete-time k as follows,

$$Q(z, r, y, \omega) = \sum_{k=0}^{\infty} Q_k(z, r, y)\omega^k \tag{3.64}$$

$$P_0(\omega) = \sum_{k=0}^{\infty} p_k(0, 0, 0)\omega^k \tag{3.65a}$$

$$P_1(\omega) = \sum_{k=0}^{\infty} p_k(0, 1, 0)\omega^k \tag{3.65b}$$

Substituting $Q_k(z, r, y)$ from (3.56) into $Q(z, r, y, \omega)$ defined in (3.64),

$$\begin{aligned} Q(z, r, y, \omega) = & \sum_{k=0}^{\infty} B(k)H(k)\omega^k + \sum_{k=0}^{\infty}\sum_{j=1}^{k} B(j)[\tilde{H}(j) - H(j)]p_{k-j}(0, 0, 0)\omega^k \\ & + \sum_{k=0}^{\infty}\sum_{j=1}^{k} B(j)[\tilde{G}(j) - G(j)]p_{k-j}(0, 1, 0)\omega^k \end{aligned}$$

Let us interchange the order of summations in the second and the third terms in above,

$$\begin{aligned} Q(z, r, y, \omega) = & \sum_{k=0}^{\infty} B(k)H(k)\omega^k + \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} B(j)[\tilde{H}(j) - H(j)]p_{k-j}(0, 0, 0)\omega^k \\ & + \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} B(j)[\tilde{G}(j) - G(j)]p_{k-j}(0, 1, 0)\omega^k \end{aligned}$$

Letting $n = k - j$, and substituting in the above equation gives,

$$Q(z, r, y, \omega) = \sum_{k=0}^{\infty} B(k)H(k)\omega^k + \sum_{j=1}^{\infty} B(j)[\tilde{H}(j) - H(j)]\omega^j \sum_{n=0}^{\infty} p_n(0, 0, 0)\omega^n$$
$$+ \sum_{j=1}^{\infty} B(j)[\tilde{G}(j) - G(j)]\omega^j \sum_{n=0}^{\infty} p_n(0, 1, 0)\omega^n$$

substituting $P_0(\omega)$ and $P_1(\omega)$ defined in (3.65a&b) for the innermost summations of the last two terms in the above equation,

$$Q(z, r, y, \omega) = \sum_{k=0}^{\infty} B(k)H(k)\omega^k + P_0(\omega) \sum_{j=1}^{\infty} B(j)[\tilde{H}(j) - H(j)]\omega^j$$
$$+ P_1(\omega) \sum_{j=1}^{\infty} B(j)[\tilde{G}(j) - G(j)]\omega^j \qquad (3.66)$$

Next, we will determine each of the terms in (3.66).

i) First, let us consider the first term $\sum_{k=0}^{\infty} B(k)H(k)\omega^k$ in the RHS of (3.66).

Substituting for $B(k)$ from (3.19b) and $H(k)$ from (3.39b), we can write the first term in the RHS of (3.66) as follows,

$$\sum_{k=0}^{\infty} B(k)H(k)\omega^k = \sum_{k=0}^{\infty} (C_1\lambda_1^k + C_2\lambda_2^k)^m (C_5\lambda_3^k + C_6\lambda_4^k)\omega^k$$

Substituting the Binomial expansion we have,

$$\sum_{k=0}^{\infty} B(k)H(k)\omega^k = \sum_{k=0}^{\infty} \sum_{i=0}^{m} \binom{m}{i} (C_1\lambda_1^k)^i (C_2\lambda_2^k)^{m-i} (C_5\lambda_3^k + C_6\lambda_4^k)\omega^k$$

Interchanging the order of summations in the above,

$$\sum_{k=0}^{\infty} B(k)H(k)\omega^k = \sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\sum_{k=0}^{\infty}(\lambda_1^i)^k(\lambda_2^{m-i})^k(C_5\lambda_3^k + C_6\lambda_4^k)\omega^k$$

$$= \sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\sum_{k=0}^{\infty}[C_5(\lambda_1^i\lambda_2^{m-i}\lambda_3\omega)^k + C_6(\lambda_1^i\lambda_2^{m-i}\lambda_4\omega)^k]$$

Evaluating the infinite summations gives,

$$\sum_{k=0}^{\infty} B(k)H(k)\omega^k = \sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_5}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega} + \frac{C_6}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right] \tag{3.67}$$

ii) Next, let us consider the second term $P_0(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{H}(j)-H(j)]\omega^j$ in the RHS of equation (3.66). Since we have obtained the following results in (3.21a&c) and (3.39a),

$$H(k) = \sigma H(k-1) + \frac{1-\sigma}{z}G(k-1)$$

$$\tilde{H}(k) = \sigma H(k-1) + (1-\sigma)G(k-1)$$

$$G(k) = C_3\lambda_3^k + C_4\lambda_4^k$$

we can determine the difference appearing in the second term of the RHS of (3.66) as,

$$\tilde{H}(j) - H(j) = (1-\sigma)\left(1-\frac{1}{z}\right)(C_3\lambda_3^{j-1} + C_4\lambda_4^{j-1}) \qquad j \ge 1$$

Thus, by substituting the above and $B(k)$ from (3.19b) the second term becomes,

$$P_0(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{H}(j)-H(j)]\omega^j$$

$$= (1-\sigma)\left(1-\frac{1}{z}\right)P_0(\omega)\sum_{j=1}^{\infty}(C_1\lambda_1^j + C_2\lambda_2^j)^m(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})\omega^j$$

Next, substituting the Binomial expansion we have,

$$P_0(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{H}(j)-H(j)]\omega^j$$

$$= (1-\sigma)\left(1-\frac{1}{z}\right)P_0(\omega)\sum_{j=1}^{\infty}\sum_{i=0}^{m}\binom{m}{i}(C_1\lambda_1^j)^i(C_2\lambda_2^j)^{m-i}(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})\omega^j$$

Changing the order of the summations,

$$P_0(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{H}(j)-H(j)]\omega^j$$

$$= (1-\sigma)\left(1-\frac{1}{z}\right)P_0(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\sum_{j=1}^{\infty}(\lambda_1^i)^j(\lambda_2^{m-i})^j\omega^j(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})$$

The inner summation in the above can be simplified as,

$$\sum_{j=1}^{\infty}(\lambda_1^i)^j(\lambda_2^{m-i})^j\omega^j(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})$$

$$= \sum_{j=1}^{\infty}\left[\frac{C_3(\lambda_1^i\lambda_2^{m-i}\lambda_3\omega)^j}{\lambda_3}+\frac{C_4(\lambda_1^i\lambda_2^{m-i}\lambda_3\omega)^j}{\lambda_4}\right]$$

$$= \frac{C_3(\lambda_1^i\lambda_2^{m-i}\lambda_3\omega)^j}{\lambda_3(1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega)}+\frac{C_4(\lambda_1^i\lambda_2^{m-i}\lambda_4\omega)^j}{\lambda_4(1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega)}$$

$$= \frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}$$

Substituting the above for the inner summation in the second term results in,

$$P_0(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{H}(j)-H(j)]\omega^j$$

$$= (1-\sigma)\left(1-\frac{1}{z}\right)P_0(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right]$$

(3.68)

iii) Finally, let us consider the third term $P_1(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{G}(j)-G(j)]\omega^j$ in the RHS of equation (3.66). Since we have obtained the following results in (3.21b&d),

$$G(k) = (1-\gamma)H(k-1)+\frac{\gamma}{z}G(k-1)$$

$$\tilde{G}(k) = (1-\gamma)H(k-1)+\gamma G(k-1)$$

similarly, by substituting $G(k)$ from (3.39a), we may determine the difference appearing in the third term as,

$$\tilde{G}(j)-G(j) = \gamma\left(1-\frac{1}{z}\right)(C_5\lambda_3^{j-1}+C_6\lambda_4^{j-1}) \qquad j\geq 1$$

substituting the above and $B(k)$ from (3.19b) into the third term and following the steps in the derivation of the second term, we have,

$$P_1(\omega)\sum_{j=1}^{\infty} B(j)[\tilde{G}(j)-G(j)]\omega^j$$

$$= \gamma\left(1-\frac{1}{z}\right)P_1(\omega)\sum_{j=1}^{\infty}(C_1\lambda_1^j+C_2\lambda_2^j)^m(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})\omega^j$$

$$= \gamma\left(1-\frac{1}{z}\right)P_1(\omega)\sum_{j=1}^{\infty}\sum_{i=0}^{m}\binom{m}{i}(C_1\lambda_1^j)^i(C_2\lambda_2^j)^{m-i}(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})\omega^j$$

$$= \gamma\left(1-\frac{1}{z}\right)P_1(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\sum_{j=1}^{\infty}(\lambda_1^i)^j(\lambda_2^{m-i})^j\omega^j(C_3\lambda_3^{j-1}+C_4\lambda_4^{j-1})$$

$$= \gamma\left(1-\frac{1}{z}\right)P_1(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right]$$

(3.69)

Finally, substituting (3.67), (3.68) and (3.69) into (3.66), we derive $Q(z, r, y, \omega)$ as,

$$\begin{aligned} Q(z,r,y,\omega) =& \sum_{k=0}^{\infty}B(k)H(k)\omega^k + P_0(\omega)\sum_{j=1}^{\infty}B(j)[\tilde{H}(j)-H(j)]\omega^j + P_1(\omega)\sum_{j=1}^{\infty}B(j)[\tilde{G}(j)-G(j)]\omega^j \\ =& \sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_5}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_6}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right] \\ &+ (1-\sigma)\left(1-\frac{1}{z}\right)P_0(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right] \\ &+ \gamma\left(1-\frac{1}{z}\right)P_1(\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right] \end{aligned}$$

(3.70)

Equation (3.70) gives the discrete-time transform of the joint PGF of the queue length, the state of the channel and the number of *On* sources. Our objective is to determine the steady-state joint PGF. Thus we need to apply the final value theorem [45] to (3.70),

$$Q(z, r, y) = \lim_{\omega \to 1}(1-\omega)Q(z, r, y, \omega) \quad (3.71)$$

Substituting (3.70) into (3.71) we have,

$$Q(z,r,y) = \lim_{\omega\to 1}(1-\omega)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_5}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_6}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right]$$

$$+ \lim_{\omega\to 1}(1-\omega)P_0(\omega)(1-\sigma)\left(1-\frac{1}{z}\right)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right]$$

$$+ \lim_{\omega\to 1}(1-\omega)P_1(\omega)\gamma\left(1-\frac{1}{z}\right)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3\omega}+\frac{C_4\lambda_1^i\lambda_2^{m-i}\omega}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4\omega}\right]$$

(3.72)

In $Q(z, r, y, \omega)$ the first term is transient while the second and the third term result in the steady-state PGF. Also since,

$$\lim_{\omega\to 1}(1-\omega)P_0(\omega) = p_\infty(0,0,0) = p(0,0,0) \tag{3.73a}$$

$$\lim_{\omega\to 1}(1-\omega)P_1(\omega) = p_\infty(0,1,0) = p(0,1,0) \tag{3.73b}$$

substituting (3.73a&b) into (3.72) gives,

$$Q(z,r,y) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)]\left(1-\frac{1}{z}\right)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}}{1-\lambda_1^i\lambda_2^{m-i}\lambda_3}+\frac{C_4\lambda_1^i\lambda_2^{m-i}}{1-\lambda_1^i\lambda_2^{m-i}\lambda_4}\right]$$

(3.74)

Substituting λ_3 and λ_4 from (3.41), (3.74) becomes,

$$Q(z,r,y) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)]\left(\frac{z-1}{z}\right)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{zC_3\lambda_1^i\lambda_2^{m-i}}{z-\lambda_1^i\lambda_2^{m-i}\bar{\lambda}_3}+\frac{zC_4\lambda_1^i\lambda_2^{m-i}}{z-\lambda_1^i\lambda_2^{m-i}\bar{\lambda}_4}\right]$$

$$= [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\sum_{i=0}^{m}\binom{m}{i}(C_1)^i(C_2)^{m-i}\left[\frac{C_3\lambda_1^i\lambda_2^{m-i}}{z-\lambda_1^i\lambda_2^{m-i}\bar{\lambda}_3}+\frac{C_4\lambda_1^i\lambda_2^{m-i}}{z-\lambda_1^i\lambda_2^{m-i}\bar{\lambda}_4}\right]$$

(3.75)

So equation (3.75) is the steady-state joint PGF of the queue-length, state of the channel and the number of *On* sources. We note that the above result still contains the unknowns $p(0, 0, 0)$ and $p(0, 1, 0)$, which will be determined in the following section.

3.4 Steady-State Analysis

In this section, we determine the marginal distributions of the queue length, the number of *On* sources and the state of the channel at the steady-state. From the steady-state PGF of queue length, mean queue length and mean delay are also calculated.

3.4.1 Steady-State Queue Length Distribution

In this section we will determine the steady-state distribution of the queue length. Let us define the marginal PGF of the queue length distribution, $P(z)$, as,

$$P(z) = Q(z, r, y)|_{r = 1, y = 1} \tag{3.76}$$

Then from (3.70), we have,

$$P(z) = [(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](z-1)\sum_{i=0}^{m}\binom{m}{i}(\tilde{C}_1\lambda_1)^i(\tilde{C}_2\lambda_2)^{m-i}\left[\frac{\tilde{C}_3}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_3} + \frac{\tilde{C}_4}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_4}\right] \tag{3.77}$$

where $\tilde{C}_i = C_i|_{r = 1, y = 1}$, as defined in (3.42).

From (3.77) we see that $p(0, 0, 0)$ and $p(0, 1, 0)$ are the only terms that we need to fully characterize the steady-state marginal PGF of the queue length. We have to derive two equations to determine these unknown probabilities. We obtain the first equation from the normalization condition, $P(z)|_{z = 1}$. Let us rewrite $P(z)$

from (3.77) as follows,

$$P(z) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)(\tilde{C}_2\lambda_2)^m\left[\frac{\tilde{C}_3}{z-\lambda_2^m\tilde{\lambda}_3}+\frac{\tilde{C}_4}{z-\lambda_2^m\tilde{\lambda}_4}\right]$$
$$+ [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\sum_{i=1}^{m}\binom{m}{i}(\tilde{C}_1\lambda_1)^i(\tilde{C}_2\lambda_2)^{m-i}\left[\frac{\tilde{C}_3}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_3}+\frac{\tilde{C}_4}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_4}\right] \tag{3.78}$$

It is convenient to rewrite the steady-state PGF of the queue length from (3.78) as follows:

$$P(z) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\left[E(z)+F(z)+\frac{M(z)}{z-S(z)}\right] \tag{3.79}$$

where,

$$E(z) = \sum_{i=1}^{m}\binom{m}{i}(\tilde{C}_1\lambda_1)^i(\tilde{C}_2\lambda_2)^{m-i}\left[\frac{\tilde{C}_5}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_3}+\frac{\tilde{C}_6}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_4}\right] \tag{3.80a}$$

$$F(z) = \frac{\tilde{C}_5(\tilde{C}_2\lambda_2)^m}{z-\lambda_2^m\tilde{\lambda}_3} \tag{3.80b}$$

$$M(z) = (\tilde{C}_2\lambda_2)^m\tilde{C}_6 \tag{3.80c}$$

$$S(z) = \lambda_2^m\tilde{\lambda}_4 \tag{3.80d}$$

We note that the limit of $(z-1)[E(z)+F(z)]$ is zero, while that of $\frac{(z-1)M(z)}{z-S(z)}$ becomes of the form $0/0$ as $z\to 1$. Thus we need to apply L'Hopital's rule to determine $P(z)|_{z=1}$.

Equation (3.79) is equivalent to,

$$P(z)(z-S(z)) = [(1-\sigma)p_{\infty}(0,0,0)+\gamma p_{\infty}(0,1,0)](z-1)[E(z)(z-S(z))+F(z)(z-S(z))+M(z)] \quad (3.81)$$

Differentiating both sides of (3.81) with respect to z yields:

$$P'(z)(z-S(z))+P(z)(1-S'(z)) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)]\{[E(z)(z-S(z))+F(z)(z-S(z))+M(z)]$$
$$+ (z-1)[E'(z)(z-S(z))+E(z)(1-S'(z))+F'(z)(z-S(z))+F(z)(1-S'(z))+M'(z)]\} \quad (3.82)$$

We note that $E(1) = F(1) = 0$ and $P(1) = M(1) = S(1) = 1$. Then substituting $z = 1$ in (3.82) yields,

$$1-S'(1) = (1-\sigma)p(0,0,0)+\gamma p(0,1,0) \quad (3.83)$$

In order to obtain $S'(1)$ and $M'(1)$, first we differentiate (3.80c&d) with respect to z,

$$S'(z) = m\lambda_2^{m-1}\frac{d\lambda_2}{dz}\tilde{\lambda}_4+\lambda_2^m\frac{d\tilde{\lambda}_4}{dz} \quad (3.84)$$

$$M'(z) = m(\tilde{C}_2\lambda_2)^{m-1}\left(\lambda_2\frac{d\tilde{C}_2}{dz}+\tilde{C}_2\frac{d\lambda_2}{dz}\right)\tilde{C}_4+(\tilde{C}_2\lambda_2)^m\frac{d\tilde{C}_4}{dz} \quad (3.85)$$

Substituting $z = 1$ as well as taking into account the following results from (3.44b,d,f&h),

$$\tilde{C}_2\big|_{z=1} = 1, \qquad \tilde{C}_4\big|_{z=1} = 1$$

$$\lambda_2\big|_{z=1} = 1, \qquad \tilde{\lambda}_4\big|_{z=1} = 1$$

we obtain,

$$S'(1) = m\frac{d\lambda_2}{dz}\bigg|_{z=1}+\frac{d\tilde{\lambda}_4}{dz}\bigg|_{z=1}$$

$$M'(1) = m\left(\left.\frac{d\tilde{C}_2}{dz}\right|_{z=1} + \left.\frac{d\lambda_2}{dz}\right|_{z=1}\right) + \left.\frac{d\tilde{C}_6}{dz}\right|_{z=1}$$

Substituting for the derivatives in the RHS from (3.49b), (3.50b), (3.51b) and (3.52b), we determine $S'(1)$ and $M'(1)$ as follows,

$$S'(1) = m\left.\frac{d\lambda_2}{dz}\right|_{z=1} + \left.\frac{d\tilde{\lambda}_4}{dz}\right|_{z=1} = \frac{m\bar{f}(1-\beta)}{2-\alpha-\beta} + \frac{1-\gamma}{2-\gamma-\sigma} \tag{3.86a}$$

$$M'(1) = m\left(\left.\frac{d\tilde{C}_2}{dz}\right|_{z=1} + \left.\frac{d\lambda_2}{dz}\right|_{z=1}\right) + \left.\frac{d\tilde{C}_4}{dz}\right|_{z=1}$$

$$= m\left[\frac{\bar{f}(1-\beta)}{2-\alpha-\beta} + \frac{\bar{f}(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\right] + \frac{(1-\gamma)(1-\gamma-\sigma)}{(2-\gamma-\sigma)^2} \tag{3.86b}$$

Substituting $S'(1)$ from (3.86a) into (3.83) gives us the first equation that relates $p(0, 0, 0)$ and $p(0, 1, 0)$,

$$(1-\sigma)p(0,0,0) + \gamma p(0,1,0) = 1 - \frac{m\bar{f}(1-\beta)}{2-\alpha-\beta} - \frac{(1-\gamma)}{2-\gamma-\sigma} \tag{3.87}$$

Next we determine the second equation relating $p(0, 0, 0)$ and $p(0, 1, 0)$ by using stability condition of the system. The state of the system given by (2.6) can be rewritten as,

$$i_{k+1} = \begin{cases} i_k - n_{k+1} + b_{k+1} & i_k > 0 \\ b_{k+1} & i_k = 0 \end{cases} \tag{3.88}$$

It is convenient to introduce shifted discrete unit step function,

$$U(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

Applying this definition to (3.88), we may summarize the state dynamics in the following equation,

$$i_{k+1} = i_k - U(i_k)n_{k+1} + b_{k+1} \tag{3.89}$$

Provided that the system is stable, the embedded Markov chain described by (3.89) is ergodic and a steady-state probability distribution exists. We take expectations on both sides of (3.89),

$$E[i_{k+1}] = E[i_k] - E[U(i_k)n_{k+1}] + E[b_{k+1}] \tag{3.90}$$

Since the random variable n_{k+1} is independent of $U(i_k)$, (3.90) is equal to

$$E[i_{k+1}] = E[i_k] - E[U(i_k)]E[n_{k+1}] + E[b_{k+1}] \tag{3.91}$$

Under the assumption that a steady-state solution exists,

$$\lim_{k\to\infty} E[i_{k+1}] = \lim_{k\to\infty} E[i_k] = E[\tilde{i}]$$

$$\lim_{k\to\infty} E[U(i_k)] = E[U(\tilde{i})]$$

$$\lim_{k\to\infty} E[n_{k+1}] = E[\tilde{n}]$$

$$\lim_{k\to\infty} E[b_{k+1}] = E[\tilde{b}]$$

Then, taking limit of (3.91) as $k \to \infty$ we have,

$$E[\tilde{i}] = E[\tilde{i}] - E[U(\tilde{i})]E[\tilde{n}] + E[\tilde{b}]$$

which yields,

$$E[U(\tilde{i})]E[\tilde{n}] = E[\tilde{b}] \tag{3.92}$$

Next we determine each of the quantities in (3.92). From the definition of expectation, $E[U(\tilde{i})]$ is given by,

$$E[U(\tilde{i})] = 0 \cdot Pr[\tilde{i} = 0] + 1 \cdot Pr[\tilde{i} > 0] = Pr[\tilde{i} > 0] \tag{3.93}$$

Now we determine $E[\tilde{b}]$. Let,

$\pi_0(k)$ = Pr (a source is *Off* during slot k at the steady state)

$\pi_1(k)$ = Pr (a source is *On* during slot k at the steady state)

and also let,

$$\pi_0 = \lim_{k \to \infty} \pi_0(k), \qquad \pi_1 = \lim_{k \to \infty} \pi_1(k)$$

From the definition given in Chapter 2 for the two-state Markov source, we have,

$$\begin{cases} \pi_0 + \pi_1 = 1 \\ \pi_0(1-\beta) = \pi_1(1-\alpha) \end{cases}$$

which gives us,

$$\pi_0 = \frac{1-\alpha}{2-\alpha-\beta}, \qquad \pi_1 = \frac{1-\beta}{2-\alpha-\beta}$$

Defining $\tilde{a} = \lim_{k \to \infty} a_k$, then the expected number of *On* sources at the steady-state is given by,

$$E[\tilde{a}] = m\pi_1 = \frac{m(1-\beta)}{2-\alpha-\beta}$$

The expected number of packets generated during a slot at the steady state is given by,

$$E[\tilde{b}] = E[\tilde{a}] \cdot \bar{f} = \frac{m(1-\beta)\bar{f}}{2-\alpha-\beta} \tag{3.94}$$

Next let us determine $E[\tilde{n}]$. An analysis similar to the derivation of $E[\tilde{b}]$ gives,

$$E[\tilde{n}] = \text{Pr (the channel is } Good \text{ at the steady-state)}$$

$$= \frac{1-\sigma}{2-\gamma-\sigma} \tag{3.95}$$

Substituting (3.93-3.95) into (3.92) results in,

$$Pr[\tilde{i}>0] = \frac{2-\gamma-\sigma}{1-\sigma} \cdot \frac{m(1-\beta)\bar{f}}{2-\beta-\alpha} \tag{3.96}$$

So we have,

$$\text{Pr [System is empty]} = p(0,0,0) + p(0,1,0)$$

$$= Pr[\tilde{i}=0] = 1 - Pr[\tilde{i}>0]$$

$$= 1 - \frac{2-\gamma-\sigma}{1-\sigma} \cdot \frac{m(1-\beta)\bar{f}}{2-\beta-\alpha}$$

which results in,

$$p(0,0,0) + p(0,1,0) = 1 - \frac{2-\gamma-\sigma}{1-\sigma} \cdot \frac{m(1-\beta)\bar{f}}{2-\beta-\alpha} \tag{3.97}$$

Thus (3.87) together with (3.97) gives us two equations that we may use to determine the unknown probabilities $p(0,0,0)$ and $p(0,1,0)$, which result in,

$$p(0,0,0) = \frac{1-\gamma}{2-\gamma-\sigma} - \frac{m(1-\beta)(1-\gamma)\bar{f}}{(1-\sigma)(2-\beta-\alpha)} \tag{3.98a}$$

$$p(0,1,0) = \frac{1-\sigma}{2-\gamma-\sigma} - \frac{m(1-\beta)\bar{f}}{(2-\beta-\alpha)} \tag{3.98b}$$

The above completes the derivation of the PGF of the queue length. Since we are dealing with a single-server system, also from the definition of the utilization factor ρ (traffic load), we have,

$$Pr[\tilde{i}>0] = \text{Pr [System is busy]} = \rho$$

thus,

$$\rho = \frac{2-\gamma-\sigma}{1-\sigma} \cdot \frac{m(1-\beta)\bar{f}}{2-\beta-\alpha} \tag{3.99}$$

3.4.2 Steady-State Mean of the Queue Length

Let $\overline{N}$ denote the steady-state mean of the queue length. By differentiating the equation (3.81) with respect to z and substituting $z=1$ in the resulting expression, we obtain the average queue length as,

$$\overline{N} = P'(z)\big|_{z=1} = \frac{S''(1)}{2(1-S'(1))} + M'(1) \tag{3.100}$$

The only unknown in (3.100) is $S''(1)$. Differentiate $S''(z)$ further from (3.84) and have,

$$S''(z) = m(m-1)\left[\frac{d\lambda_2}{dz}\right]^2 + m\cdot\frac{d\lambda_2}{dz}\cdot\frac{d\tilde{\lambda}_4}{dz} + m\cdot\frac{d^2\lambda_2}{dz^2} + \frac{d^2\tilde{\lambda}_4}{dz^2} \tag{3.101}$$

Substituting $z=1$ in (3.101),

$$S''(1) = m(m-1)\left[\frac{d\lambda_2}{dz}\bigg|_{z=1}\right]^2 + m\cdot\frac{d\lambda_2}{dz}\bigg|_{z=1}\cdot\frac{d\tilde{\lambda}_4}{dz}\bigg|_{z=1} + m\cdot\frac{d^2\lambda_2}{dz^2}\bigg|_{z=1} + \frac{d^2\tilde{\lambda}_4}{dz^2}\bigg|_{z=1}$$

Substituting (3.49b), (3.50b), (3.53) and (3.54) in the above we have,

$$S''(1) = \frac{m(m-1)(1-\beta)^2(\bar{f})^2}{(2-\alpha-\beta)^2} + \frac{2m(1-\alpha)(1-\beta)(\alpha+\beta-1)(\bar{f})^2}{(2-\alpha-\beta)^3} + \frac{m(1-\beta)f''(1)}{2-\alpha-\beta}$$
$$+ \frac{m(1-\beta)(1-\gamma)\bar{f}}{(2-\alpha-\beta)(2-\gamma-\sigma)} + \frac{2(1-\gamma)(1-\sigma)(\gamma+\sigma-1)}{(2-\gamma-\sigma)^3} \tag{3.102}$$

From (3.86a) and (3.86b) we have $S'(1)$ and $M'(1)$, together with (3.102) we will be able to determine the steady-state mean of the queue length $\overline{N}$.

3.4.3 Steady-State Mean of Packet Delay

From Little's result, the mean packet delay in the number of slots is given by,

$$\bar{d} = \frac{\bar{N}}{E[\bar{b}]} = \frac{\bar{N}(2-\alpha-\beta)}{m(1-\beta)\bar{f}} \qquad (3.103)$$

3.4.4 Steady-State Analysis of the Number of *On* Sources

In this section we will determine the steady-state distribution of the number of *On* sources.

Let $A(y)$ denotes the marginal PGF of the number of *On* sources,

$$A(y) = Q(z, r, y)\big|_{z=1,\, r=1} \qquad (3.104)$$

Substituting $Q(z, r, y)$ from (3.70) we have,

$$A(y) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\sum_{i=0}^{m}\binom{m}{i}(C_1\lambda_1)^i(C_2\lambda_2)^{m-i}\left[\frac{C_3}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_3}+\frac{C_4}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_4}\right]\Bigg|_{z=1,\, r=1}$$

(3.105)

In Section 3.2.1.3, we have obtained the following results in (3.45) and (3.47),

$$C_1\big|_{z=1,\,r=1} = \frac{(1-\beta)(1-y)}{2-\alpha-\beta}, \qquad C_2\big|_{z=1,\,r=1} = \frac{1-\alpha+(1-\beta)y}{2-\alpha-\beta}$$

$$C_3\big|_{z=1,\,r=1} = 0, \qquad C_4\big|_{z=1,\,r=1} = 1,$$

$$\lambda_1\big|_{z=1,\,r=1} = \alpha+\beta-1, \qquad \lambda_2\big|_{z=1,\,r=1} = 1$$

$$\tilde{\lambda}_3\big|_{z=1,\,r=1} = \gamma+\sigma-1, \qquad \tilde{\lambda}_4\big|_{z=1,\,r=1} = 1$$

In $A(y)$, since we have a factor of $(z-1)$, when we substitute $z = 1$, all the terms under the summation in (3.77) become zero except for $i = 0$, thus,

$$A(y) = [(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](z-1)(C_2\lambda_2)^m \left[\frac{C_3}{z-\lambda_2^m\tilde{\lambda}_3} + \frac{C_4}{z-\lambda_2^m\tilde{\lambda}_4}\right]\Bigg|_{z=1,\, r=1}$$

Also since $C_3\big|_{z=1,\, r=1} = 0$, we note that the term with C_3 disappears and it results in,

$$A(y) = [(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](z-1)(C_2\lambda_2)^m \frac{C_4}{z-\lambda_2^m\tilde{\lambda}_4}\Bigg|_{z=1,\, r=1} \tag{3.106}$$

Since $\lambda^m{}_2\tilde{\lambda}_4\big|_{z=1,\, r=1} = 1$, direct substitution results indeterminacy in the form of 0/0, thus we have to apply L'Hopitals' rule in (3.106),

$$A(y) = \frac{[(1-\sigma)p(0,1,0) + \gamma p(0,1,0)](C_2\lambda_2)^m C_4}{1 - m\lambda_2^{m-1}\frac{d\lambda_2}{dz}\tilde{\lambda}_4 - \lambda_2^m\frac{d\tilde{\lambda}_4}{dz}}\Bigg|_{z=1,\, r=1}$$

$$+ \frac{[(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](z-1)\left[m(C_2\lambda_2)^{m-1}C_4\left(\lambda_2\frac{dC_2}{dz} + C_2\frac{d\lambda_2}{dz}\right) + (C_2\lambda_2)^m\frac{dC_4}{dz}\right]}{1 - m\lambda_2^{m-1}\frac{d\lambda_2}{dz}\tilde{\lambda}_4 - \lambda_2^m\frac{d\tilde{\lambda}_4}{dz}}\Bigg|_{z=1,\, r=1}$$

The second term in the above goes to zero as $z \to 1$. Substituting $S'(z)$ from (3.84) in the denominator of the first term gives,

$$A(y) = \frac{[(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](C_2\lambda_2)^m C_4}{1 - S'(z)}\Bigg|_{z=1,\, r=1}$$

since $\lambda_2\big|_{z=1,\, r=1} = 1$, $C_4\big|_{z=1,\, r=1} = 1$ as given above, we have,

$$A(y) = \frac{[(1-\sigma)p(0,0,0) + \gamma p(0,1,0)](C_2\big|_{z=1,\, r=1})^m}{1 - S'(1)}$$

since $1 - S'(1) = (1-\sigma)p(0,0,0) + \gamma p(0,1,0)$ from (3.83), $A(y)$ is further simplified to,

$$A(y) = (C_2|_{z=1, r=1})^m$$

Finally, substituting the value of $C_2|_{z=1, r=1}$ presented above gives the steady-state distribution of the number of *On*-sources as,

$$A(y) = \left[\frac{1-\alpha}{2-\alpha-\beta} + \frac{y(1-\beta)}{2-\alpha-\beta}\right]^m = [\pi_0 + \pi_1 y]^m \tag{3.107}$$

where, $\pi_0 = \dfrac{1-\alpha}{2-\alpha-\beta}$, $\pi_1 = \dfrac{1-\beta}{2-\alpha-\beta}$

Therefore, in steady-state, the number of *On*-sources follows a Binomial distribution with mean $m\pi_1$. Since each source generates, on the average, $\bar{f}$ packets per *On* slot, then the average number of packet arrivals during a slot (packet arrival rate) is given by,

$$m\pi_1\bar{f} = \frac{m(1-\beta)\bar{f}}{2-\beta-\alpha}$$

This conforms with equation (3.94), thus provides a further proof on the correctness of the functional equation we have obtained earlier.

3.4.5 Steady-State Analysis of the State of the Channel

In this section we will determine the steady-state distribution of the state of the channel.

Let *B*(*r*) denotes the marginal PGF of the state of the channel, then,

$$B(r) = Q(z, r, y)|_{z=1, y=1} \tag{3.108}$$

Substituting $Q(z, r, y)$ from (3.70) we have,

$$B(r) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\sum_{i=0}^{m}\binom{m}{i}(C_1\lambda_1)^i(C_2\lambda_2)^{m-i}\left[\frac{C_3}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_3}+\frac{C_4}{z-\lambda_1^i\lambda_2^{m-i}\tilde{\lambda}_4}\right]\Bigg|_{z=1,\,y=1}$$

(3.109)

From equation (3.45) and (3.48) in Section 3.3.1, we have obtained,

$$C_1\big|_{z=1,\,y=1} = 0, \qquad C_2\big|_{z=1,\,y=1} = 1$$

$$C_3\big|_{z=1,\,y=1} = \frac{(1-\gamma)(r-1)}{2-\gamma-\sigma}, \qquad C_4\big|_{z=1,\,y=1} = \frac{(1-\gamma)+r(1-\sigma)}{2-\gamma-\sigma}$$

$$\lambda_1\big|_{z=1,\,y=1} = \alpha+\beta-1, \qquad \lambda_2\big|_{z=1,\,y=1} = 1$$

$$\lambda_3\big|_{z=1,\,y=1} = \gamma+\sigma-1, \qquad \tilde{\lambda}_4\big|_{z=1,\,y=1} = 1$$

Similar to the derivation of $A(y)$, in $B(r)$ all the terms under the summation in (3.109) become zero except for $i = 0$, thus we can simplify $B(r)$ as,

$$B(r) = [(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)(C_2\lambda_2)^m\left[\frac{C_3}{z-\lambda_2^m\tilde{\lambda}_3}+\frac{C_4}{z-\lambda_2^m\tilde{\lambda}_4}\right]\Bigg|_{z=1,\,y=1}.$$

Since $\lambda^m{}_2\tilde{\lambda}_3\big|_{z=1,\,y=1} \neq 1$, the term with C_3 disappears and results in,

$$B(r) = [(1-\sigma)p_\infty(0,0,0)+\gamma p_\infty(0,1,0)](z-1)(C_2\lambda_2)^m\frac{C_4}{z-\lambda_2^m\tilde{\lambda}_4}\Bigg|_{z=1,\,y=1} \qquad (3.110)$$

Since $\lambda^m{}_2\tilde{\lambda}_4\big|_{z=1} = 1$, direct substitution will result in the form of 0/0, thus we have to apply L'Hopitals' rule again to the equation (3.110) and have,

$$B(r) = \left.\frac{[(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](C_2\lambda_2)^m C_4}{1-m\lambda_2^{m-1}\frac{d\lambda_2}{dz}\bar{\lambda}_4-\lambda_2^m\frac{d\bar{\lambda}_4}{dz}}\right|_{z=1,y=1}$$

$$= \left.\frac{[(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](z-1)\left[m(C_2\lambda_2)^{m-1}C_4\left(\lambda_2\frac{dC_2}{dz}+C_2\frac{d\lambda_2}{dz}\right)+(C_2\lambda_2)^m\frac{dC_4}{dz}\right]}{1-m\lambda_2^{m-1}\frac{d\lambda_2}{dz}\bar{\lambda}_4-\lambda_2^m\frac{d\bar{\lambda}_4}{dz}}\right|_{z=1,y=1}$$

The second term in the above goes to zero as $z \to 1$. Substituting $S'(z)$ from (3.84) in the denominator of the first term gives,

$$B(r) = \left.\frac{[(1-\sigma)p(0,0,0)+\gamma p(0,1,0)](C_2\lambda_2)^m C_4}{1-S'(z)}\right|_{z=1,y=1}$$

since $\lambda_2\big|_{z=1,y=1} = 1$, $C_2\big|_{z=1,y=1} = 1$ as given above, we have

$$B(r) = \frac{[(1-\sigma)p(0,0,0)+\gamma p(0,1,0)]C_4\big|_{z=1,y=1}}{1-S'(1)} \tag{3.111}$$

but $1-S'(1) = (1-\sigma)p(0,0,0)+\gamma p(0,1,0)$, and substituting

$C_4\big|_{z=1,y=1} = \frac{1-\gamma+r(1-\sigma)}{2-\gamma-\sigma}$ given above into (3.111) we obtain the steady-state distribution of the state of the channel as,

$$B(r) = \frac{1-\gamma}{2-\gamma-\sigma}+\frac{r(1-\sigma)}{2-\gamma-\sigma} \tag{3.112}$$

Therefore, in the steady-state, the state of the channel follows a Bernoulli distribution with mean $\frac{1-\sigma}{2-\gamma-\sigma}$, which is the mean time that the channel is in *Good* state. This conforms with (3.95), and provides again another proof on the correctness of our analysis.

We note that the case of $\gamma = 1$ corresponds to the channel that is always

Good. This is the case that has been analyzed in [44, 45], and the results obtained in this thesis reduce to those of [44, 45]. This provides some assurance that the foregoing analysis is correct. Of course this is a necessary but not sufficient condition for the correctness of the foregoing analysis.

3.4.6 Autocovariance of the Channel State vs. Performance

From Figure 2.2, the transition probability matrix for the channel state n_k can be set up as,

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \sigma & 1-\sigma \\ 1-\gamma & \gamma \end{bmatrix} \tag{3.113}$$

It is possible to show that [48],

$$P^k = \frac{1}{2-\gamma-\sigma} \cdot \begin{bmatrix} 1-\gamma & 1-\sigma \\ 1-\gamma & 1-\sigma \end{bmatrix} + \frac{(\gamma+\sigma-1)^k}{2-\gamma-\sigma} \cdot \begin{bmatrix} 1-\sigma & \sigma-1 \\ \gamma-1 & 1-\gamma \end{bmatrix} \tag{3.114}$$

thus,

$$\lim_{k \to \infty} P^k = \frac{1}{2-\gamma-\sigma} \cdot \begin{bmatrix} 1-\gamma & 1-\sigma \\ 1-\gamma & 1-\sigma \end{bmatrix} \tag{3.115}$$

Equation (3.115) results in,

Pr [channel state n_k =1 at the steady state]

$$= \lim_{k \to \infty} P[n_k = 1] = \frac{1-\sigma}{2-\gamma-\sigma} \tag{3.116}$$

which is the same as (3.95).

The autocovariance of the state of the channel n_k can be expressed as,

$$
\begin{aligned}
R_k(i) &= E[(n_k - E[n_k])(n_{k+i} - E[n_{k+i}])] \\
&= E[n_k n_{k+i}] - E^2[n_k] \\
&= P[n_k = 1, n_{k+i} = 1] - \left(\frac{1-\sigma}{2-\gamma-\sigma}\right)^2 \\
&= P[n_k = 1]P[n_{k+i} = 1/n_k = 1] - \left(\frac{1-\sigma}{2-\gamma-\sigma}\right)^2
\end{aligned}
$$

Since (3.114) and (3.116), the autocovariance of the state of the channel at the steady-state is,

$$
\begin{aligned}
R(i) &= \frac{1-\sigma}{2-\gamma-\sigma}\cdot\left[\frac{1-\sigma}{2-\gamma-\sigma} + \frac{(\gamma+\sigma-1)^i}{2-\gamma-\sigma}\cdot(1-\gamma)\right] - \left(\frac{1-\sigma}{2-\gamma-\sigma}\right)^2 \\
&= \frac{(1-\gamma)(1-\sigma)(\gamma+\sigma-1)^i}{(2-\gamma-\sigma)^2} \qquad (3.117)
\end{aligned}
$$

We note that if the probability of the channel being in *Good* state $P[n_k = 1] = \frac{1-\sigma}{2-\gamma-\sigma}$ is given, $\frac{1-\gamma}{2-\gamma-\sigma}$ can be found. Also in reality, it is not practical that γ and σ are smaller than 0.5, thus we can see from (3.117) that when $\gamma + \sigma > 1$, the narrower the autocovariance $R(i)$, the smaller the value of σ (or γ), and if we consider the frequency domain, a narrower autocovariance implies a wider power density spectrum.

In Section 3.4.2 we obtained the steady-state mean queue length, which is repeated here as,

$$\overline{N} = \frac{S''(1)}{2(1-S'(1))} + M'(1) \tag{3.118}$$

where,

$$S'(1) = \frac{m\bar{f}(1-\beta)}{2-\alpha-\beta} + \frac{1-\gamma}{2-\gamma-\sigma} \tag{3.119a}$$

$$M'(1) = m\left[\frac{\bar{f}(1-\beta)}{2-\alpha-\beta} + \frac{\bar{f}(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\right] + \frac{(1-\gamma)(1-\gamma-\sigma)}{(2-\gamma-\sigma)^2} \tag{3.119b}$$

$$S''(1) = \frac{m(m-1)(1-\beta)^2(\bar{f})^2}{(2-\alpha-\beta)^2} + \frac{2m(1-\alpha)(1-\beta)(\alpha+\beta-1)(\bar{f})^2}{(2-\alpha-\beta)^3} + \frac{m(1-\beta)f''(1)}{2-\alpha-\beta}$$

$$+ \frac{m(1-\beta)(1-\gamma)\bar{f}}{(2-\alpha-\beta)(2-\gamma-\sigma)} + \frac{2(1-\gamma)(1-\sigma)(\gamma+\sigma-1)}{(2-\gamma-\sigma)^3} \tag{3.119c}$$

We note under given m, $\bar{f}$, $f''(1)$,α,β and $\frac{1-\sigma}{2-\gamma-\sigma}$, $\frac{1-\gamma}{2-\gamma-\sigma}$, $S'(1)$ as well as all the terms except the last term in $M'(1)$ and $S''(1)$ are also given. It is possible to show that $\overline{N}$ can be expressed as,

$$\overline{N} = A + \frac{(1-\gamma)(1-\sigma)(\gamma+\sigma-1)}{(1-S'(1))(2-\gamma-\sigma)^3} + B + \frac{(1-\gamma)(1-\gamma-\sigma)}{(2-\gamma-\sigma)^2} \tag{3.120}$$

where A and B are unchanged terms,

$$A = \Bigg[\frac{m(m-1)(1-\beta)^2(\bar{f})^2}{(2-\alpha-\beta)^2} + \frac{2m(1-\alpha)(1-\beta)(\alpha+\beta-1)(\bar{f})^2}{(2-\alpha-\beta)^3} + \frac{m(1-\beta)f''(1)}{2-\alpha-\beta} + \frac{m(1-\beta)(1-\gamma)\bar{f}}{(2-\alpha-\beta)(2-\gamma-\sigma)}\Bigg] / [2(1-S'(1))]$$

$$B = m\left[\frac{\bar{f}(1-\beta)}{2-\alpha-\beta} + \frac{\bar{f}(1-\beta)(1-\alpha-\beta)}{(2-\alpha-\beta)^2}\right]$$

according to (3.118) and (3.119). Further we may define x and y which are unchanged terms,

$$x = \frac{m\bar{f}(1-\beta)}{2-\alpha-\beta}, \qquad y = \frac{1-\sigma}{2-\gamma-\sigma} \tag{3.121}$$

Thus (3.120) can be written as,

$$\bar{N} = A + \frac{y(1-y)}{y-x} \cdot \frac{\gamma+\sigma-1}{2-\gamma-\sigma} + B - (1-y) \cdot \frac{\gamma+\sigma-1}{2-\gamma-\sigma}$$

$$= A + B + (1-y) \cdot \left(\frac{y}{y-x} - 1\right) \cdot \frac{\gamma+\sigma-1}{2-\gamma-\sigma}$$

$$= A + B + \frac{\gamma+\sigma-1}{2-\gamma-\sigma} \cdot \frac{x(1-y)}{y-x} \tag{3.122}$$

Under condition that the traffic load $\rho = \frac{x}{y} < 1$, we have,

$$\frac{x(1-y)}{y-x} > 0$$

From (3.122) we see that when $\gamma + \sigma > 1$, since $\frac{x(1-y)}{y-x} > 0$, A and B are given, the smaller the value of σ (or γ), the smaller the queue length $\bar{N}$.

We have concluded earlier on that when $\gamma + \sigma > 1$, the smaller the value of σ (or γ), the wider the power density spectrum.Thus from the above analysis we can conclude that the wider the power density spectrum, the smaller the value of σ (or γ), the smaller the queue length $\bar{N}$, which means a better performance.

Chapter 4

Numerical Results

In this section we present some numerical examples regarding the results of this thesis. The objective is to study the effect of different wireless link error characteristics on the behavior of ATM multiplexing system.

The different wireless channel characteristics are represented by the parameters γ and σ, where $1-\gamma$ and $1-\sigma$ are the probability of transition from *Good* to *Bad* state and the probability of transition from *Bad* to *Good* state for the channel at the end of each time slot, respectively. In the following, the mean time that the channel spends in *Good* and *Bad* states will also be referred as mean good duration and mean fade duration for the wireless link respectively. We assume that T is the length of a time slot corresponding to the transmission time of one ATM cell (53 bytes), thus the mean good and fade durations can be expressed as $T/(1-\gamma)$ and $T/(1-\sigma)$. Similarly, since the binary Markov source has been chosen as the model of input traffic, different traffic characteristics have been represented by different values of parameters α and β, and the mean durations of a source being in *On* state and *Off* state are $T/(1-\alpha)$ and $T/(1-\beta)$, respectively. It is also assumed that each *On* source generates a single packet during a slot, i.e., $f(z) = z$.

First we consider voice traffic (Figures 4.1 - 4.8). We choose the mean talk spurt as 1.32s and silence spurt as 1.82s [49]. Now we need to match these values to parameters α and β.

We assume that an active voice source generates information at the rate of 64 kbits/s. Thus during a talkspurt a voice source generates a total of $64 \times 1.32 = 84.48$ kbits, which is equivalent to 220 ATM cells. Since ATM is a very high speed network, clearly, a voice source does not generate enough information to fill a cell every slot. However, the presented analysis assumes that an *On* source generates at least a single cell every slot. Therefore, we will assume that a voice source generates on average 220 ATM cells consecutively and then it spends the remaining time in a cycle in the *Off* period. As a result, a voice source will spend on average $220 \cdot T$ seconds in *On* period, and $(1.82 + 1.32 - 220 \cdot T)$ seconds in *Off* period. Since the modeled ATM multiplexer will experience the same load as the one in practice, we expect that it will give close results.

$$\frac{T}{1-\alpha} = 220 \cdot T \qquad => \alpha = 0.995454$$

$$\frac{T}{1-\beta} = 3.14 - 220 \cdot T$$

Thus we determine $\alpha = 0.995454$ and different β's for different transmission rates.

In Figures 4.1 - 4.5, we plot the steady-state mean of the queue length in number of packets as a function of the number of sources m, with mean fade and mean good durations as parameters. As may be seen, for the same number of sources, different mean queue lengths are obtained for different wireless link error rates. From Figures 4.1, 4.2, as the mean fade period increases from 1ms to 20ms, the corresponding steady-state mean of the queue length also increases. This is expected, since the longer the fade duration, the longer that the channel is in *Bad* state, during which the queue builds up.

In Figures 4.3 - 4.5, we observe the effect of transmission rate on the queue length for two error characteristics. In Figure 4.3, we have three sets of curves for transmission rates of 800kbps, 2Mbps and 5Mbps, which are relatively low. We see that, for a given transmission rate, the mean queue length varies little with mean fade and mean good durations when we keep their ratio constant. On the other hand, as the transmission rate increases the mean queue length drops down for a given number of sources and constant mean good to fade duration ratio. Figure 4.4, 4.5 give the results for transmission rates of 10Mbps and 100Mbps, respectively, which are higher rates than those in Figure 4.3. We see that even though we keep the mean fade to mean good duration ratio constant, as the fade duration increases, the mean queue length increases. This reflects the fact that for the higher transmission rate, during the fade periods, the queue builds up faster than for the lower rates case under the same traffic load.

Also in Figure 4.4, since the ratio of mean fade and mean good durations is given, which means that at the steady state, the probability that the channel is in *Good* (or *Bad*) state is given. Figure 4.4A shows the channel autocovariance for the two error characteristics in Figure 4.4, and the corresponding power density spectrums for channel state are showed in Figure 4.4B and Figure 4.4C. We note that the performance is better (when $\sigma = 0.9576$, the queue length is smaller) for a wider power density spectrum (the curve in Figure 4.4C when $\sigma = 0.9576$), which is equivalent to a narrower autocovariance (the curve in Figure 4.4A when $\sigma = 0.9576$). This result conforms with the conclusion obtained in Section 3.4.6.

Table 3: Number of users (m) the system can accommodate under fixed mean delay of $\bar{d} = 200ms$ ($\alpha = 0.995454$)

Mean Fade	Mean Good	β	m
1ms	10ms	0.9998255	24
1ms	10ms	0.9999315	61
1ms	10ms	0.9999728	153
10ms	100ms	0.9998255	24
10ms	100ms	0.9999315	60
10ms	100ms	0.9999728	152

Figures 4.6, 4.7 present the mean packet delay in number of time slots as a function of the traffic load ρ , with the number of sources in the system, *m*, as parameter. In Figure 4.6 the results are given for the mean fade duration of 1ms, mean good duration of 10 ms, and the number of sources, $m = 2, 5, 10$ and 100. As may be seen, for any given load, an increase in *m* leads to a rise in the mean packet delay. We note that under given channel error characteristic, if we have

QoS requirement in terms of delay constraint, we can determine how many users the system can accommodate in order to meet the performance requirement, as shown in Table 3, for example.

In Figure 4.7, the results are given for the mean good duration of 100ms, mean fade durations of 1ms and 10ms, and the number of sources, $m = 2$ and 20. For each value of m, two mean delay curves have been plotted, which correspond to mean good to mean fade duration ratios of 100 and 10 respectively. It is observed that as the ratio decreases the mean delay curves shift upwards. The magnitude of the shift appears to be independent of the number of sources in the system.

From Figures 4.1 - 4.7, we also note that under heavy loading, there is a sharp increase in the mean queue length and the mean packet delay. This is the reason why we did not keep the part of the curves corresponding to loads higher than 0.8.

Figure 4.8 presents the approximate probabilities of buffer overflow under different buffer sizes. These probabilities correspond to the probabilities that the queue length will be greater than the chosen buffer size in the infinite queue length system that we have studied. This gives an upper bound to the corresponding finite buffer size system. As expected, for any given level of traffic, the probability of overflow decreases as the fade length decreases. Unfortunately, because of the finite precision problems we could not obtain probabilities of buffer overflow for systems with larger buffer sizes.

Figures 4.9 - 4.13 correspond to video input traffic. We have used the video source model from [19], which consists of the superposition of a number of 'mini-sources', where each mini-source consists of independent and identical On/Off sources. The number of mini-sources required for a good approximation of a video source was found experimentally to be 20 [19]. Based on these assumptions, we determine $\alpha = 0.99873445$ and we use different value of β's for different transmission rates. From Figures 4.9 - 4.13 we obtain conclusions for the video traffic similar to those for voice traffic.

Figures 4.14 - 4.19 correspond to data traffic. The source parameter values are chosen from Table 2 in Chapter 1. We chose the mean burst length in number of cells as 200, and the average cell arrival rate as 700 kbits/s. This results in $\alpha = 0.995$ and again we use different values of β's for different transmission rates. From the Figures 4.14 - 4.19 we obtain conclusions for the data traffic similar to those for the voice and video traffic.

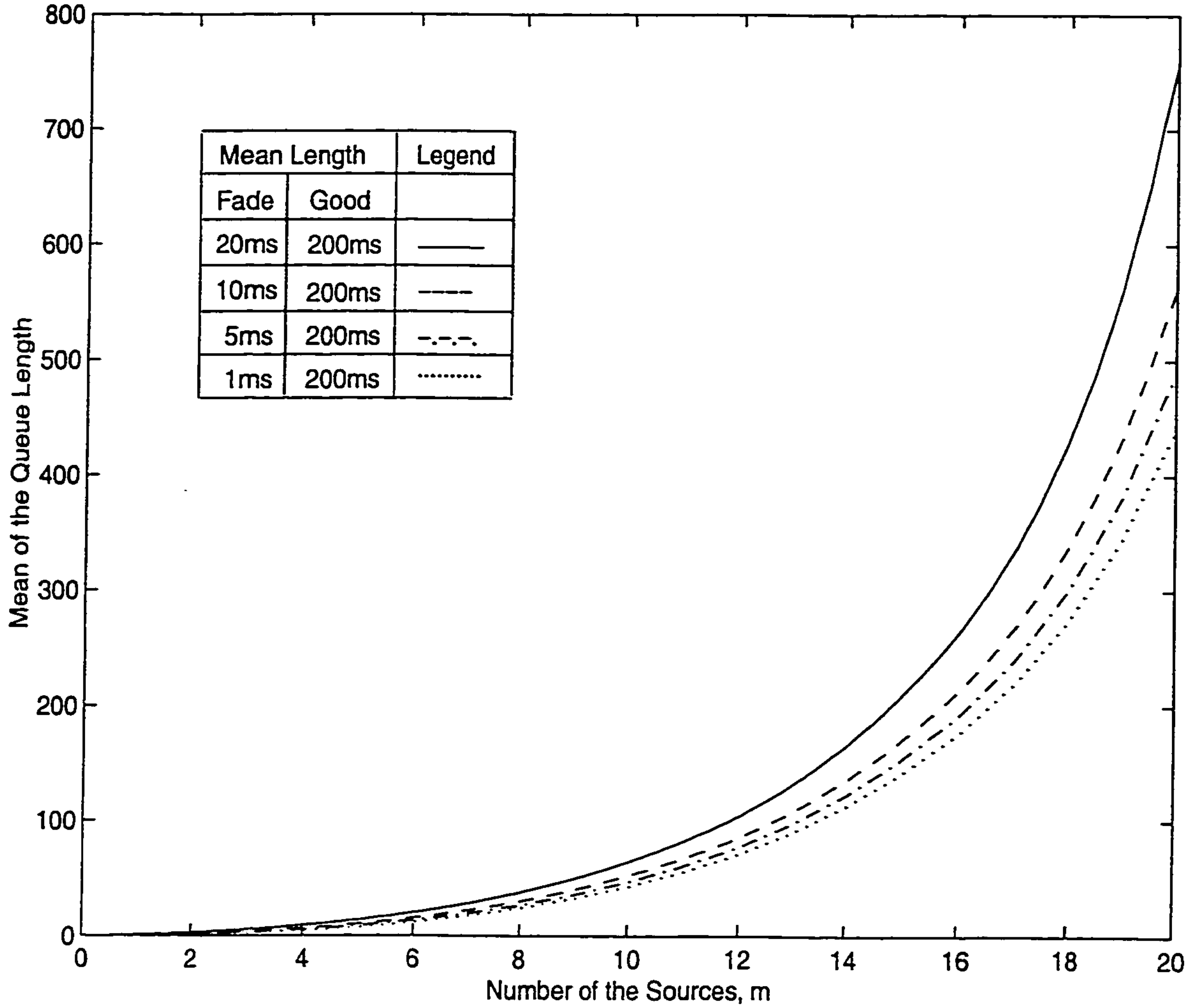


Figure 4.1 Steady-state mean of the queue length versus the number of sources, m, when the mean good duration = 200ms, voice traffic.

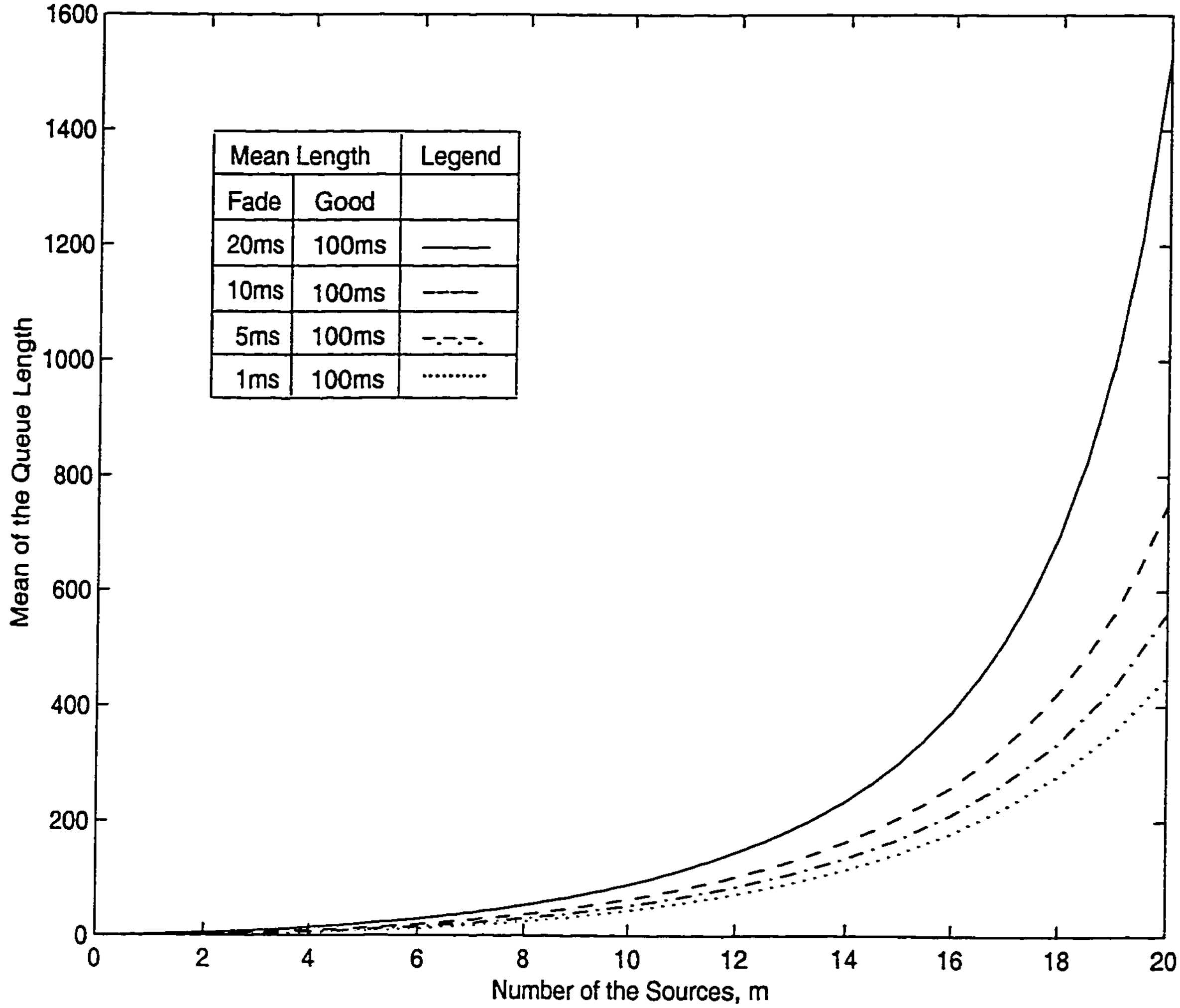


Figure 4.2 Steady-state mean of the queue length versus the number of sources, *m*, when the mean good duration = 100ms, voice traffic.

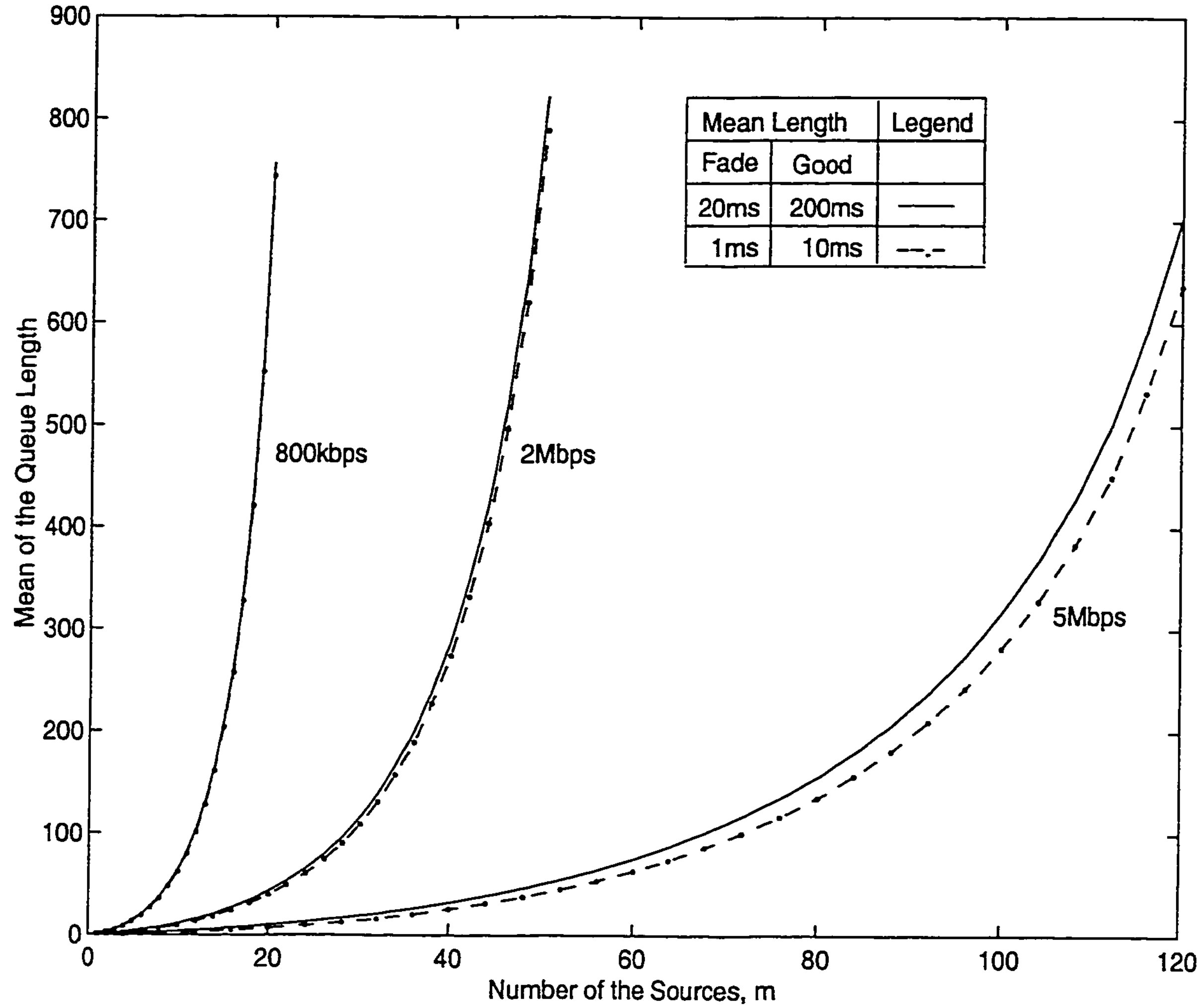


Figure 4.3 Steady-state mean of the queue length versus the number of sources, m, under different transmission rates for two error characteristics, voice traffic.

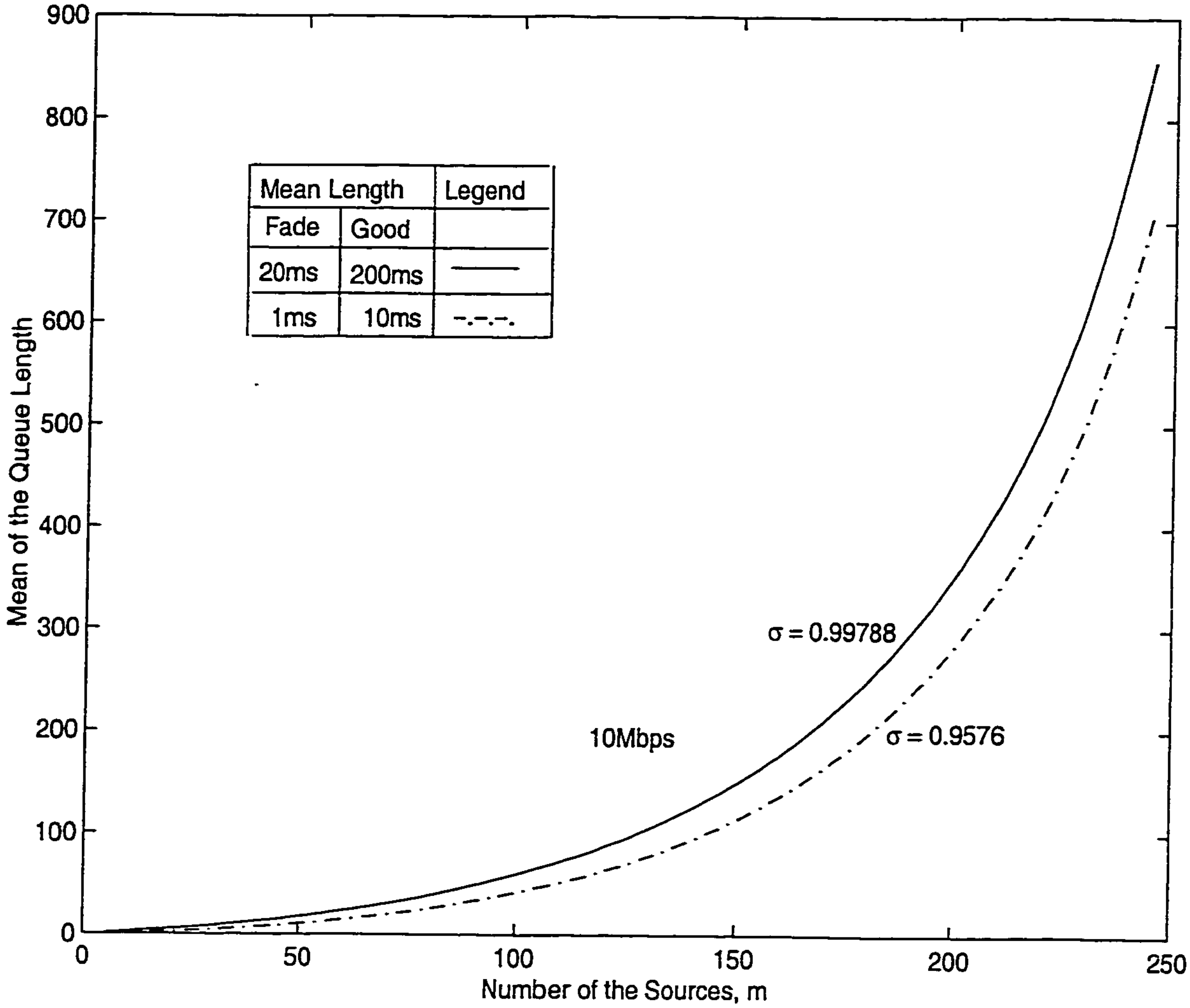


Figure 4.4 Steady-state mean of the queue length versus the number of sources, m, when transmission rate = 10Mbps for two error characteristics, voice traffic.

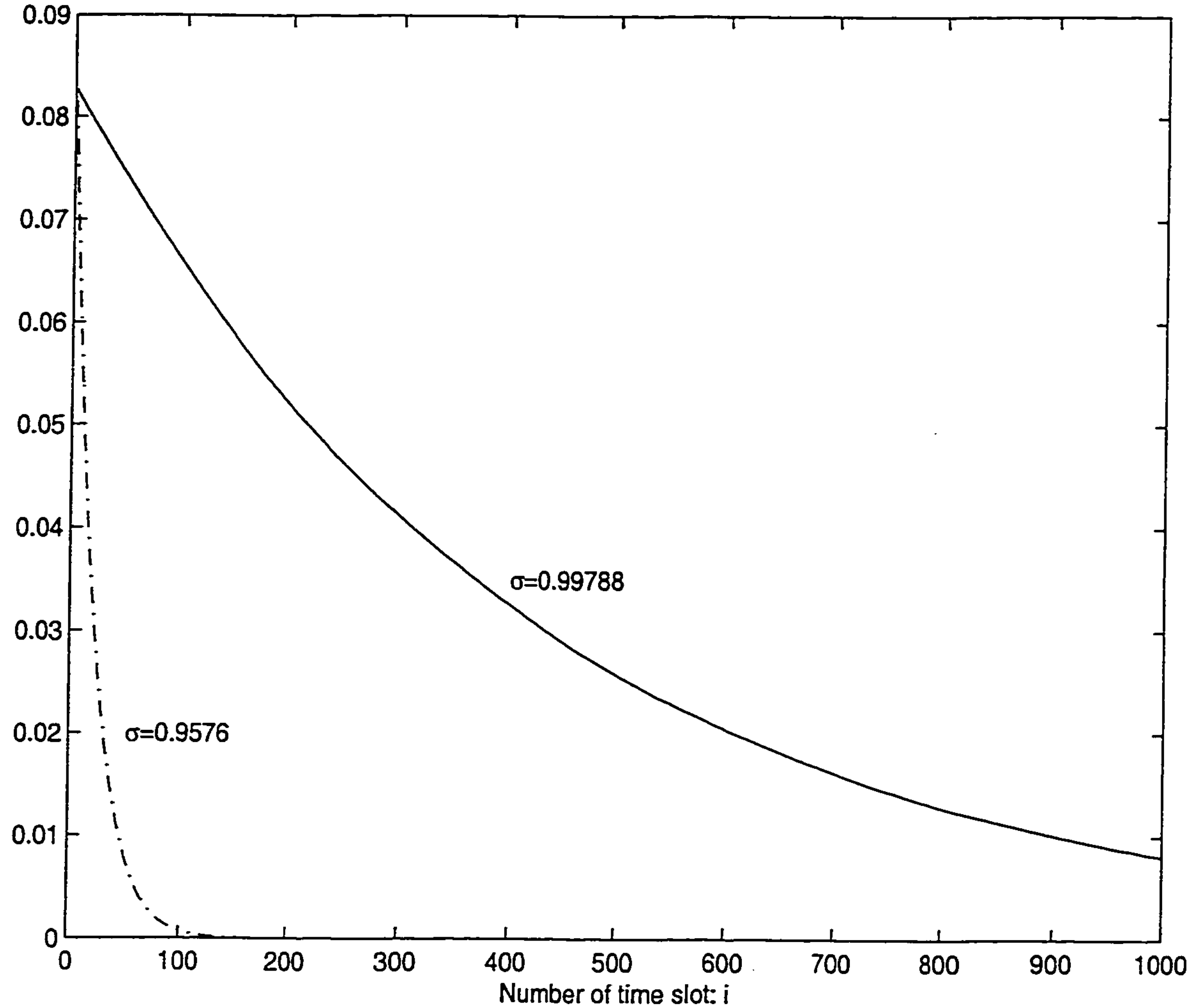


Figure 4.4A The autocovariance of the channel state corresponding to the two error characteristics in Figure 4.4.

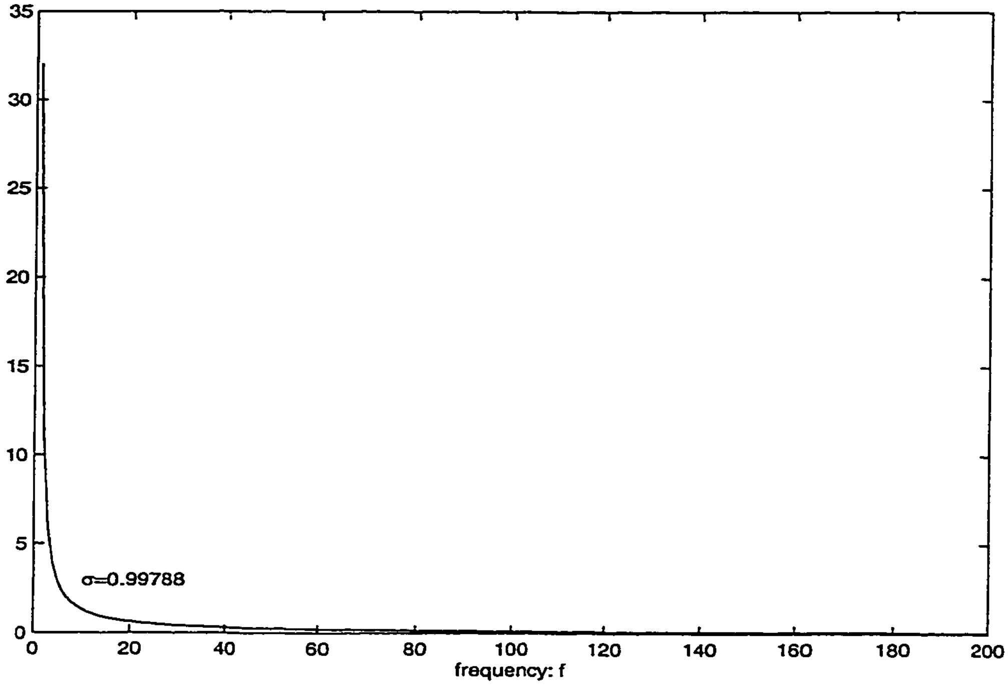


Figure 4.4B The corresponding power density spectrum of the channel autocovariance in Figure 4.4A when σ=0.99788

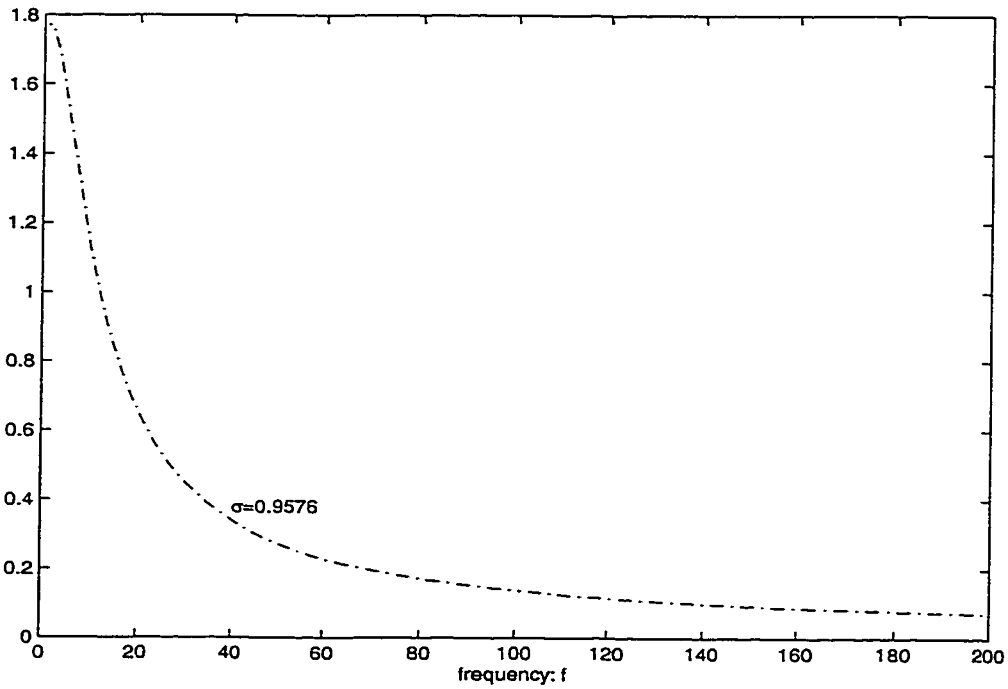


Figure 4.4C The corresponding power density spectrum of the channel autocovariance in Figure 4.4A when σ=0.9576

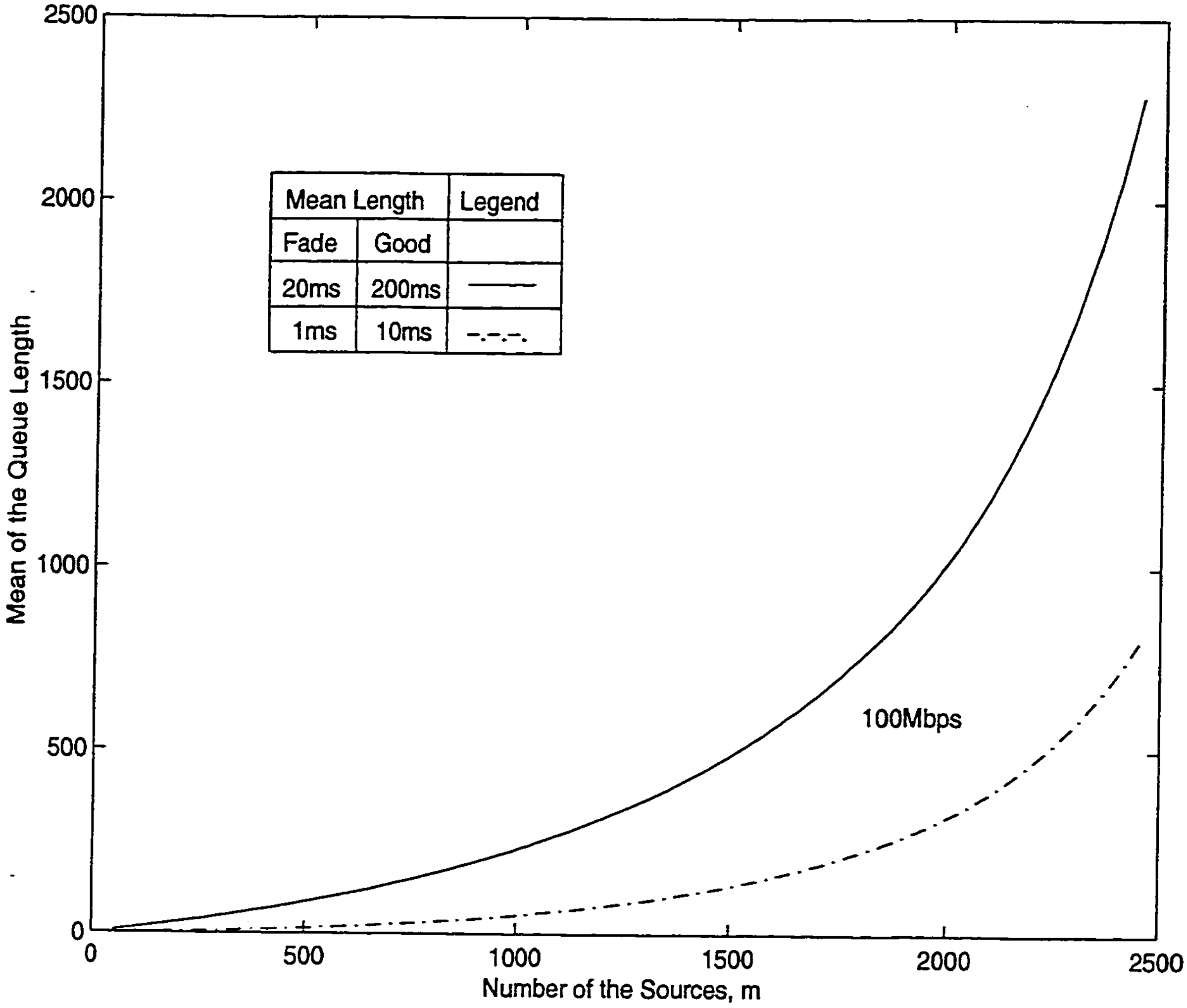


Figure 4.5 Steady-state mean of the queue length versus the number of sources, m, when transmission rate = 100Mbps for two error characteristics, voice traffic.

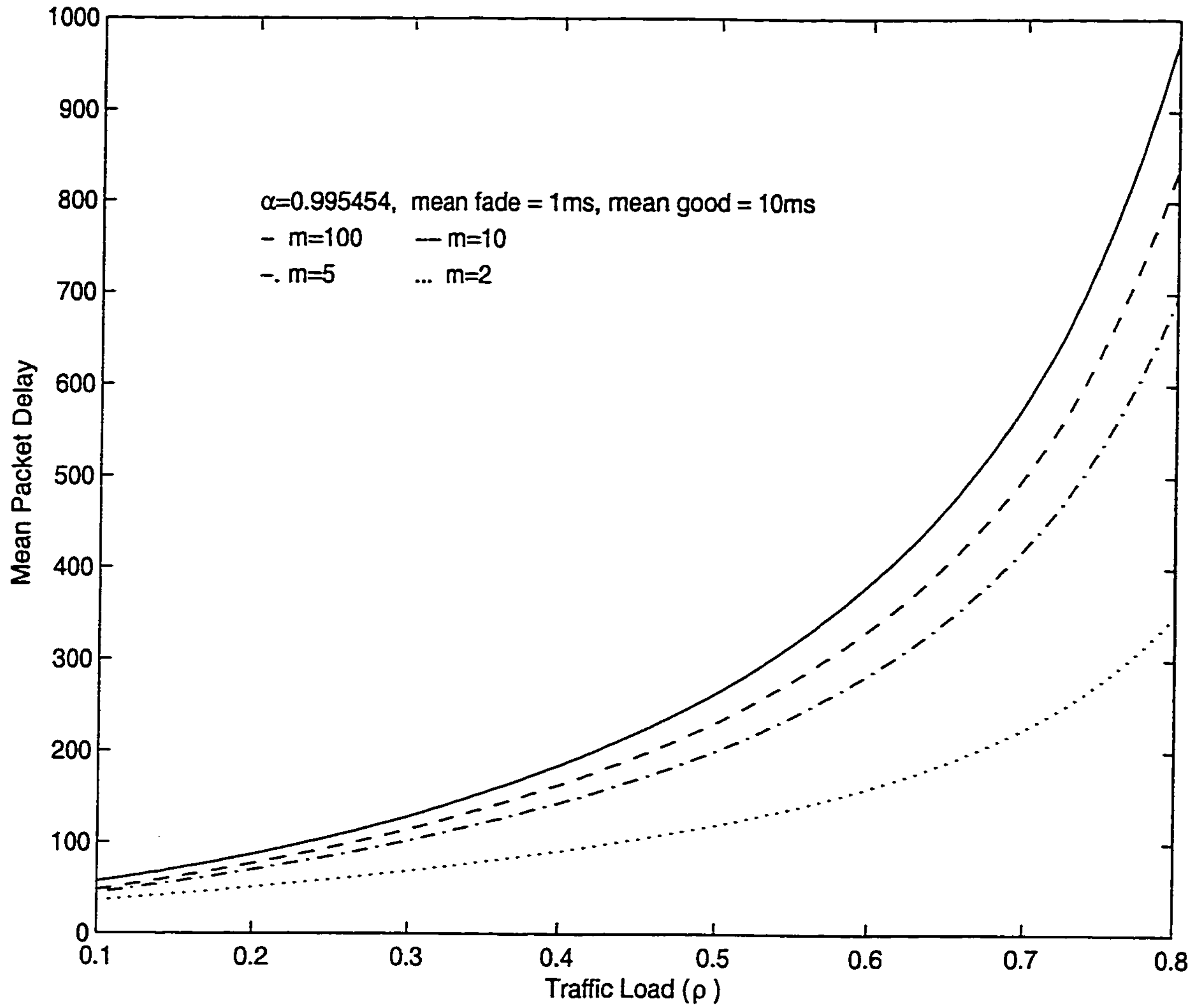


Figure 4.6 Mean packet delay in number of slots versus traffic load with different number of sources, under certain channel error condition, voice traffic.

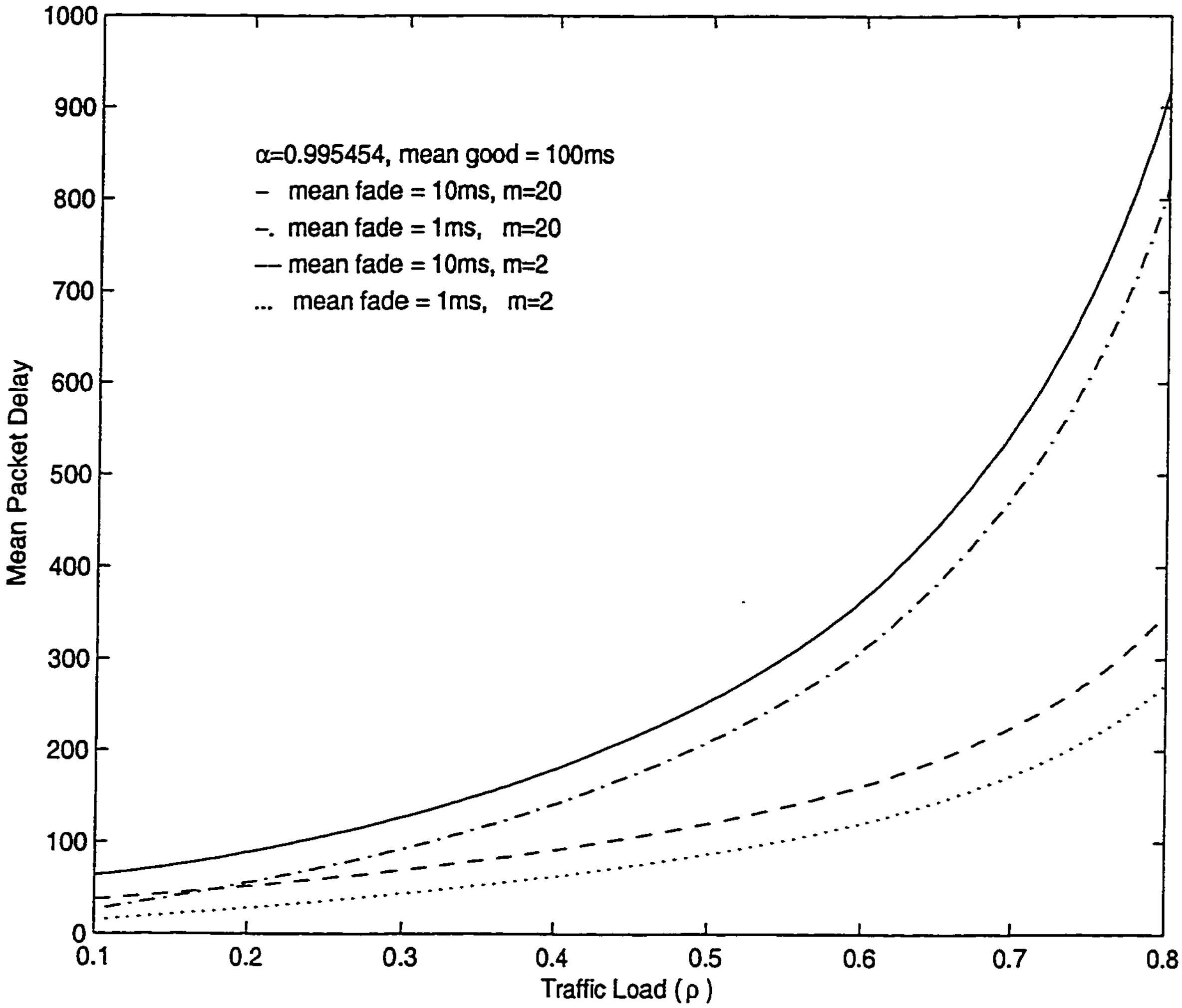


Figure 4.7 Mean packet delay in number of slots versus traffic load with different number of sources, under different channel error conditions, voice traffic.

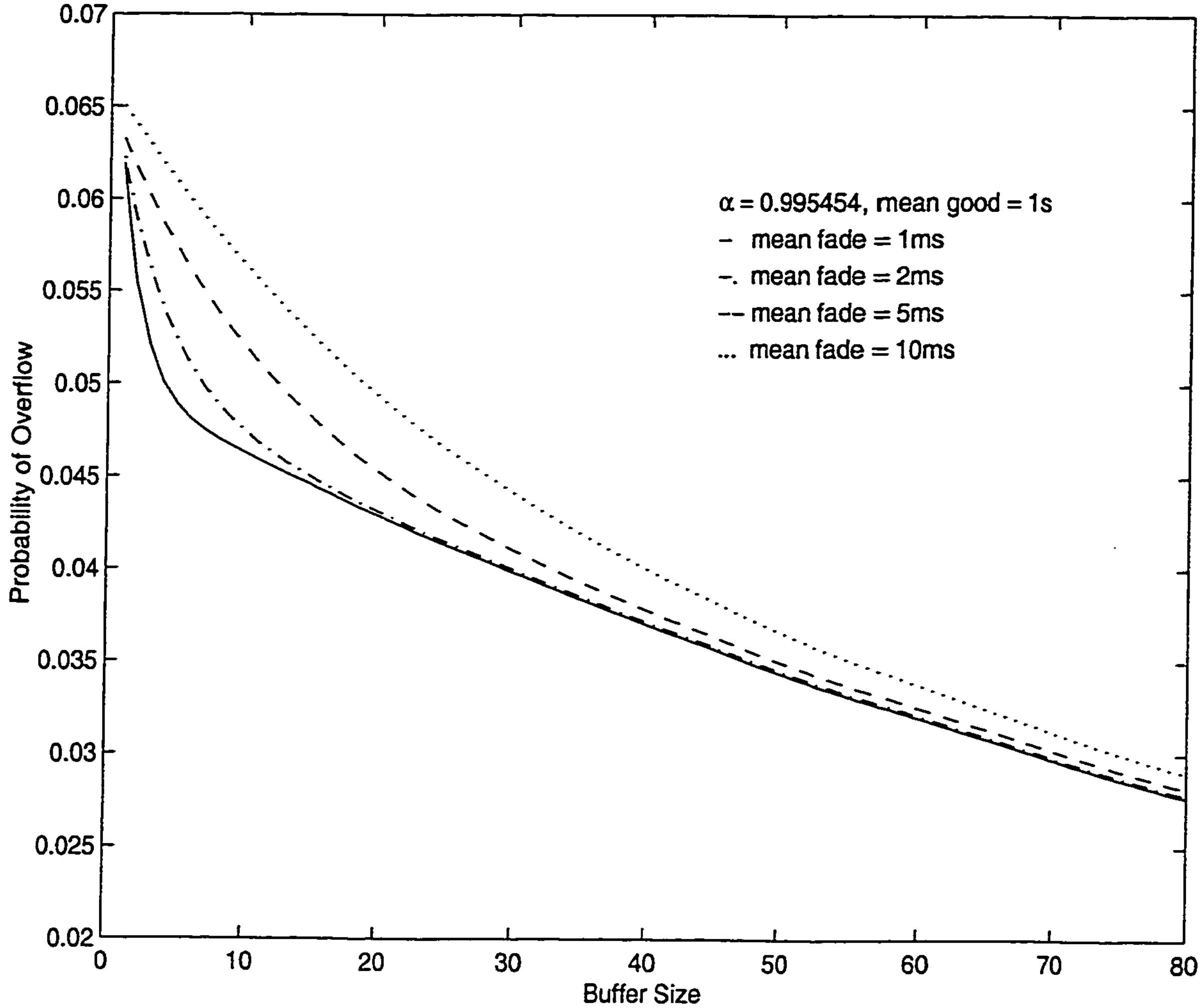


Figure 4.8 Probability of buffer overflow versus buffer size, voice traffic.

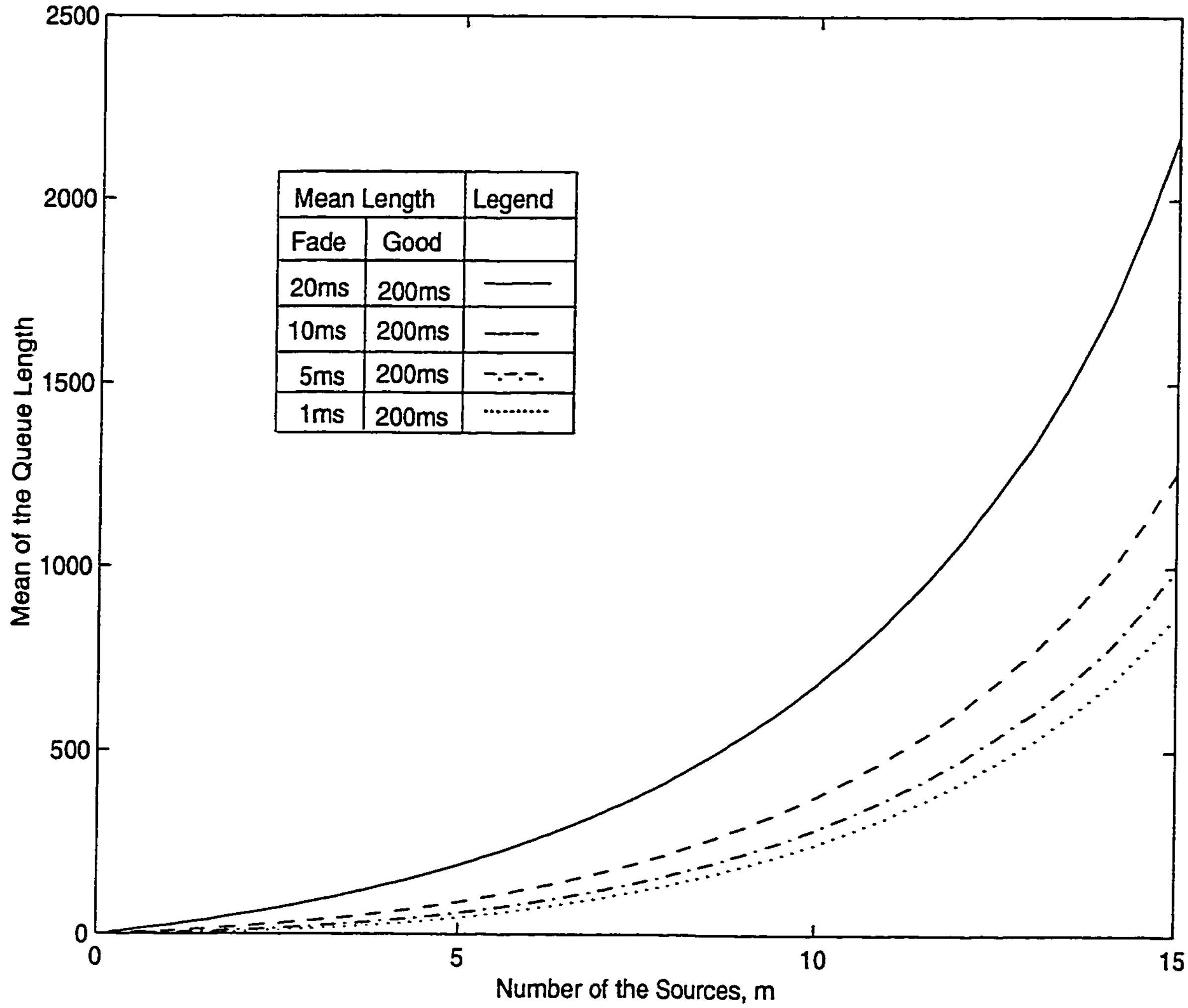


Figure 4.9 Steady-state mean of the queue length versus the number of sources, m, when the mean good duration = 200ms, video traffic.

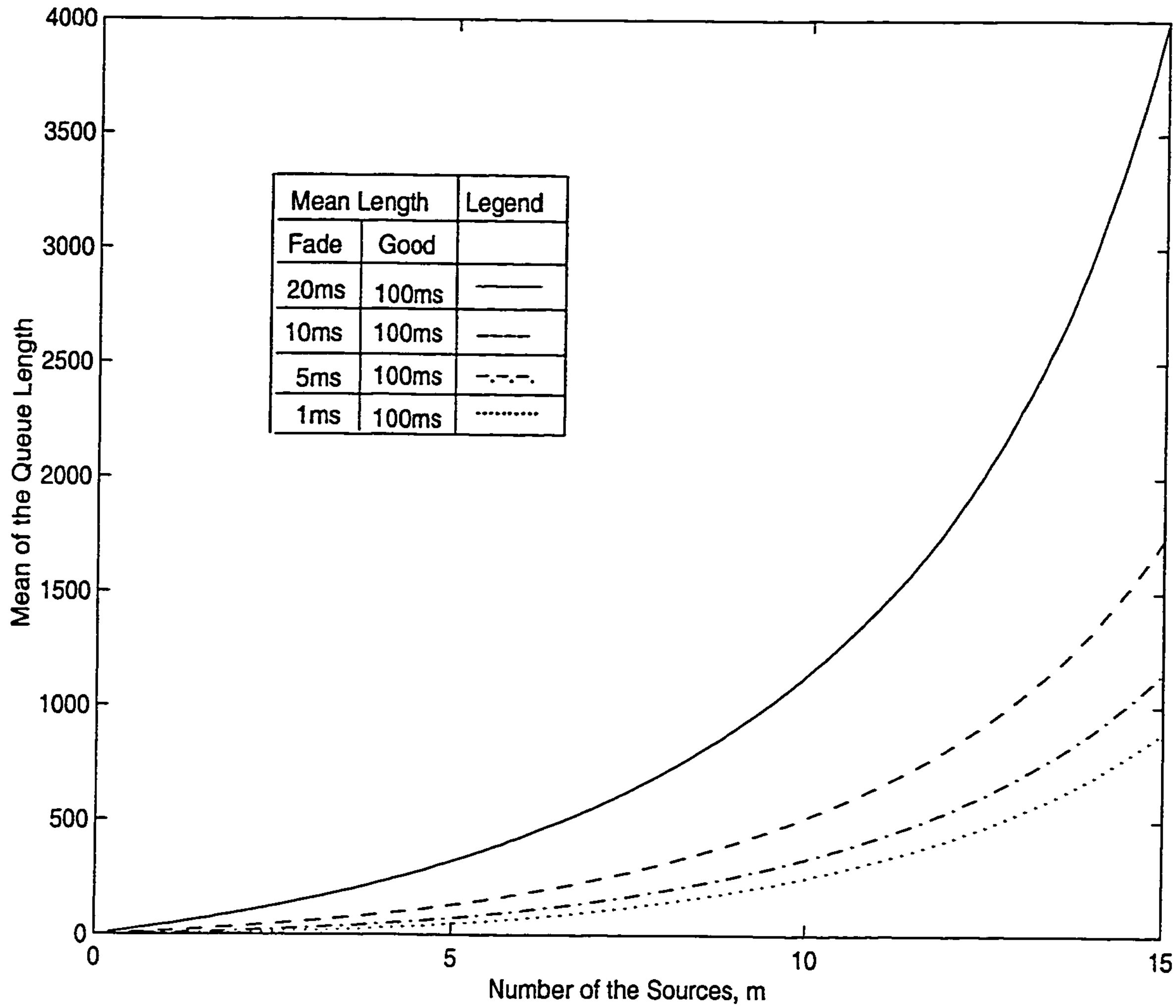


Figure 4.10 Steady-state mean of the queue length versus the number of sources, m, when the mean good duration = 100ms, video traffic.

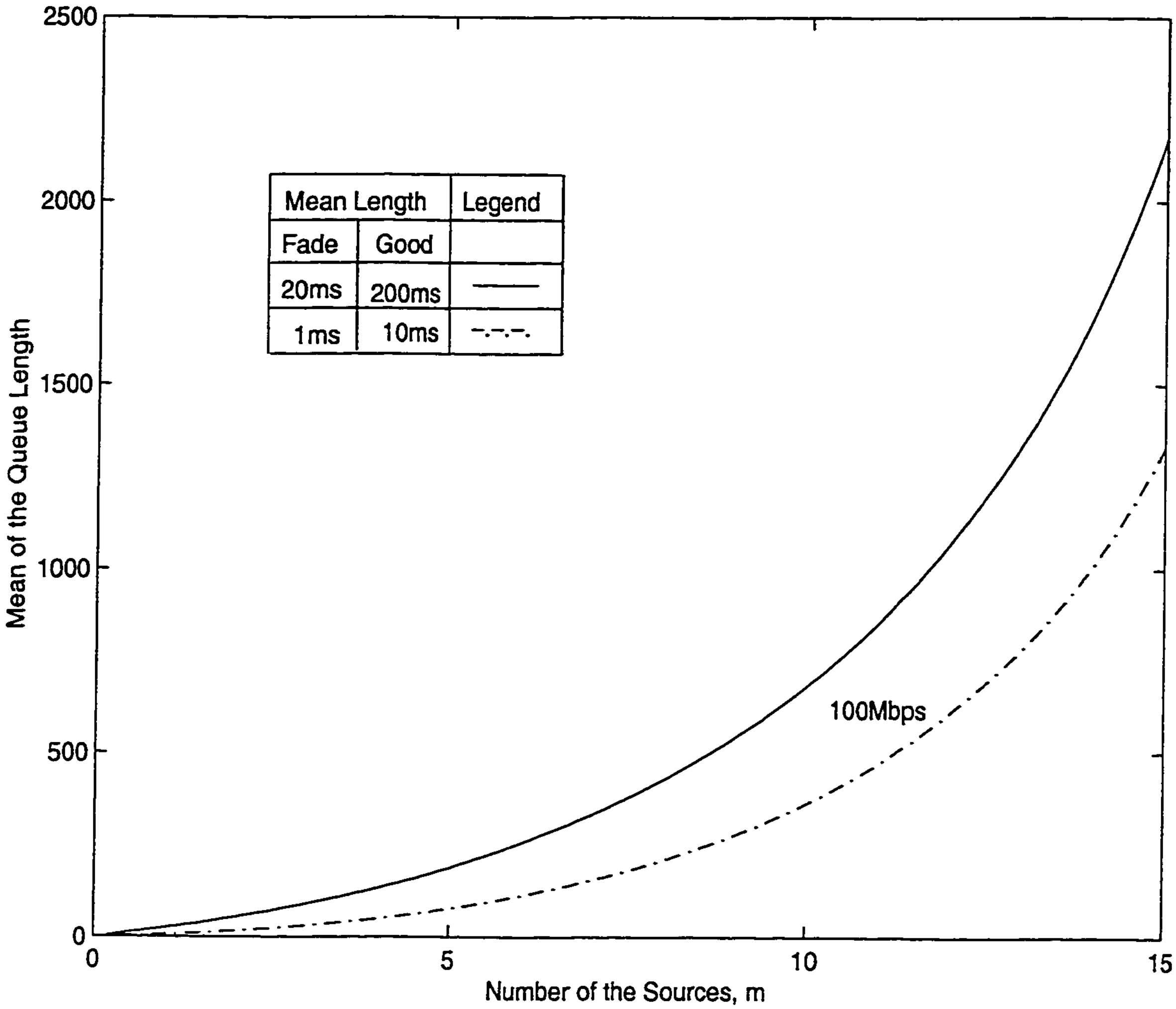


Figure 4.11 Steady-state mean of the queue length versus the number of sources, m, when transmission rate = 100Mbps for two error characteristics, video traffic.

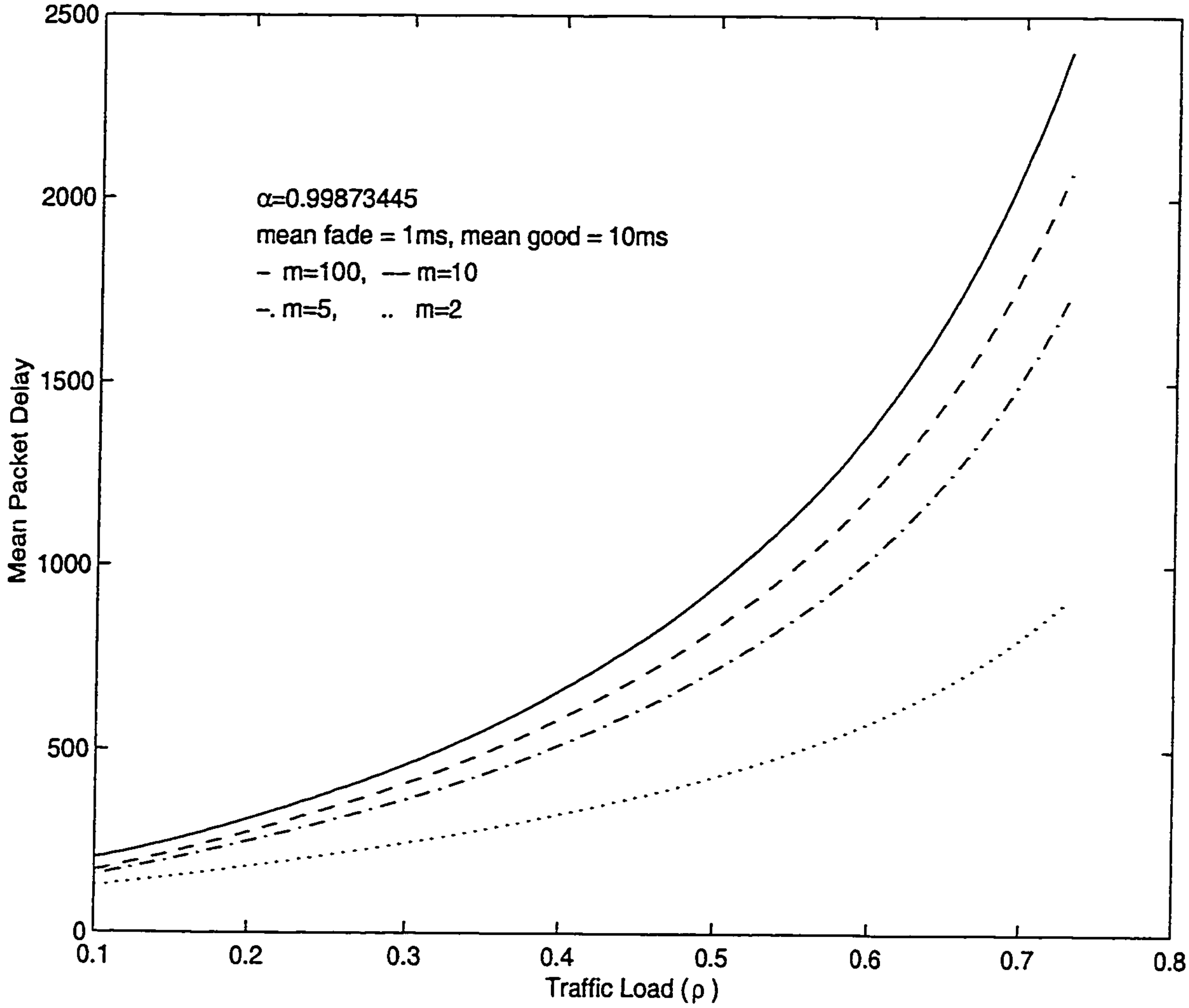


Figure 4.12 Mean packet delay in number of slots versus traffic load with different number of sources, under certain channel error condition,video traffic.

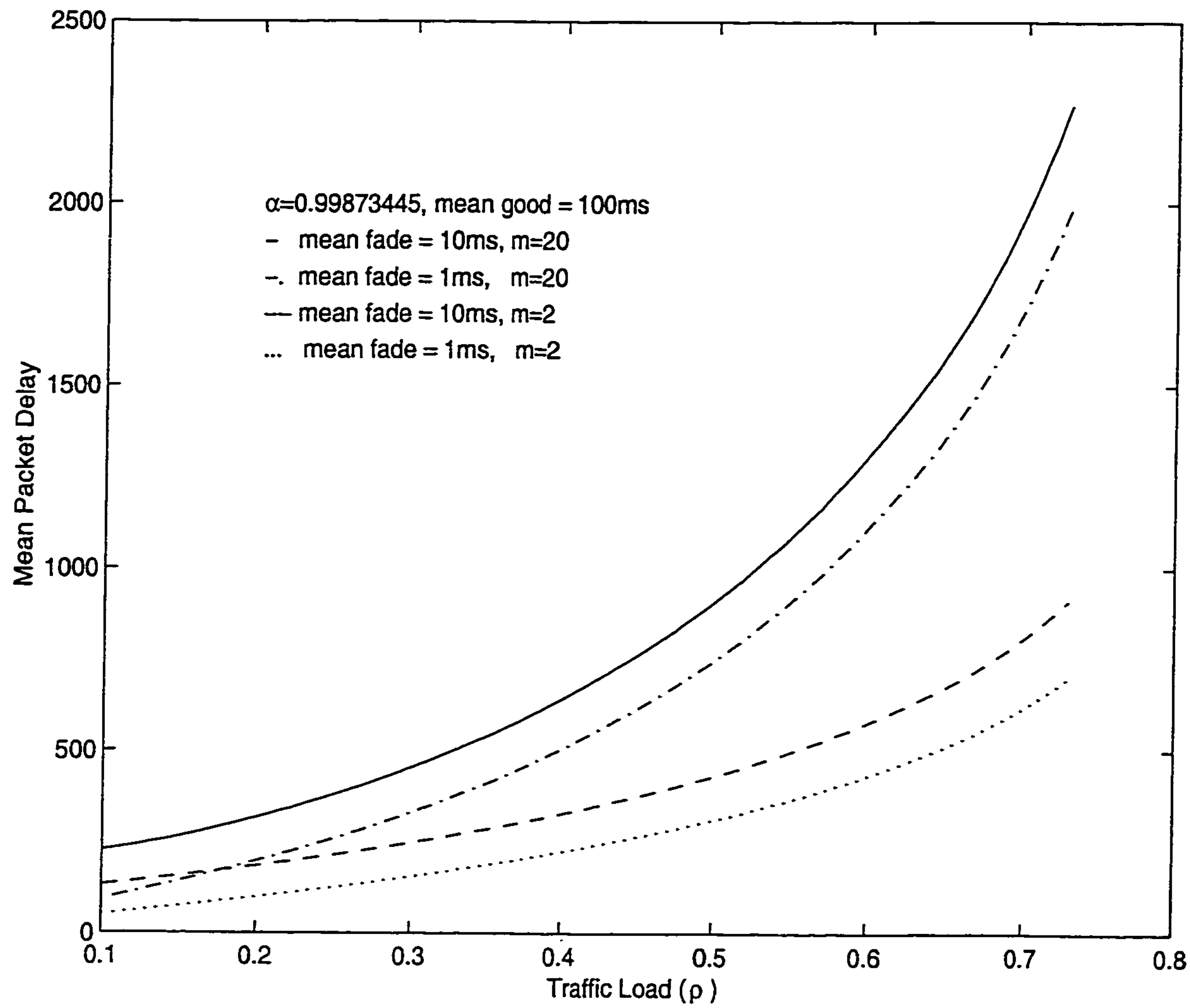


Figure 4.13 Mean packet delay in number of slots versus traffic load with different number of sources, under different channel error conditions, video traffic.

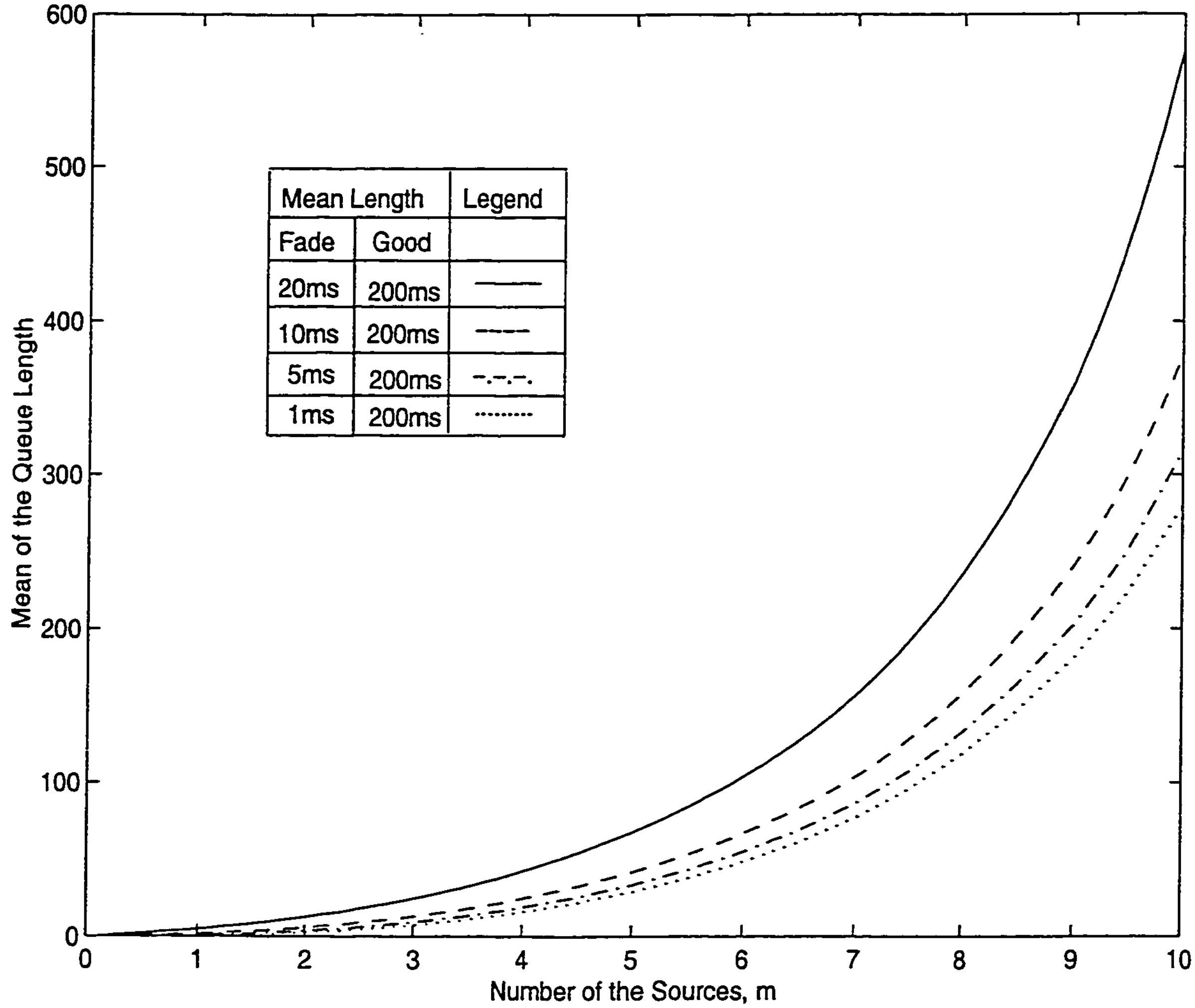
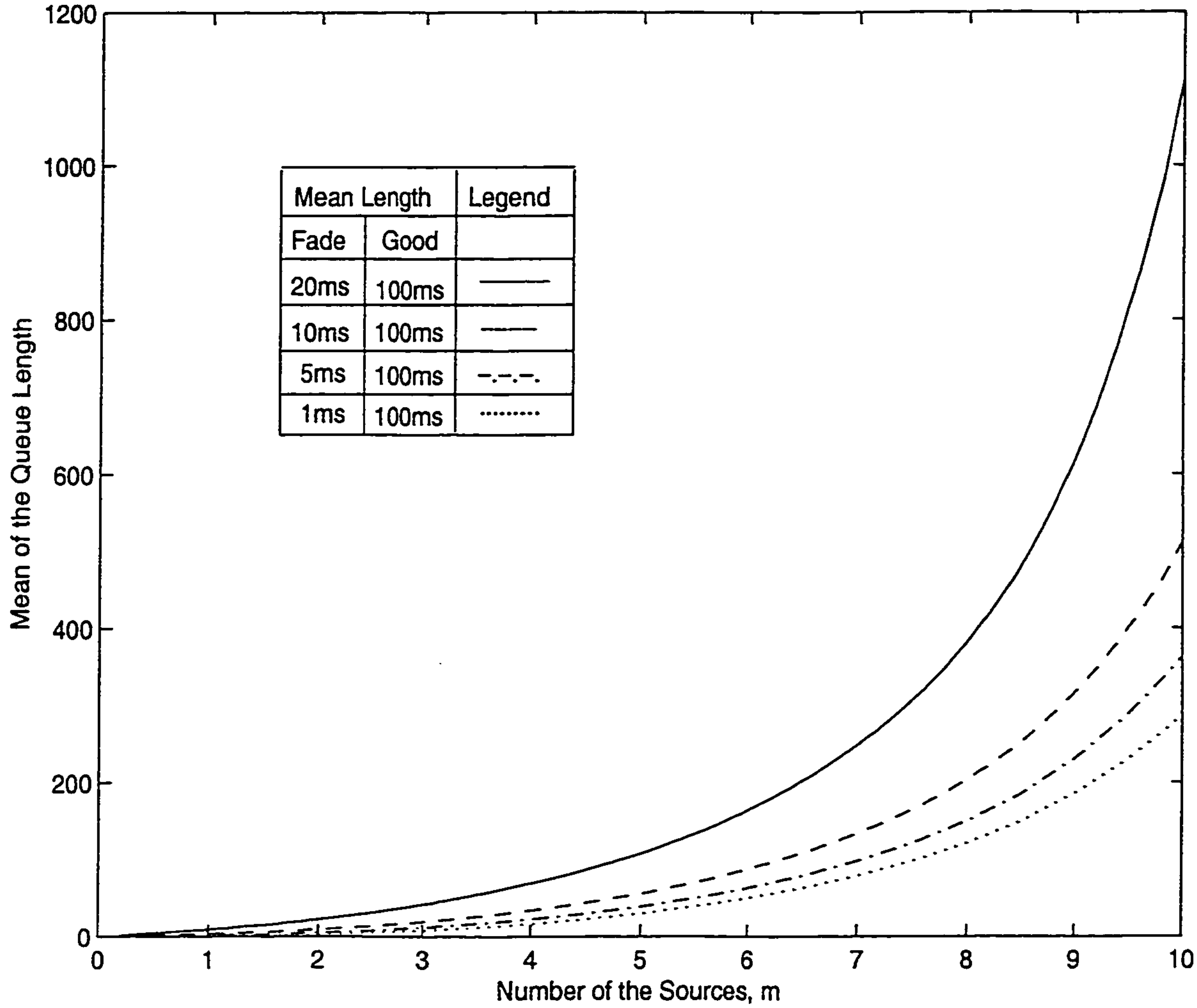


Figure 4.14 Steady-state mean of the queue length versus the number of sources, m, when the mean good duration = 200ms, data traffic



Mean Length		Legend
Fade	Good	
20ms	100ms	———
10ms	100ms	——
5ms	100ms	-.-.-.
1ms	100ms	....

Figure 4.15 Steady-state mean of the queue length versus the number of sources, m, when the mean good duration = 100ms, data traffic

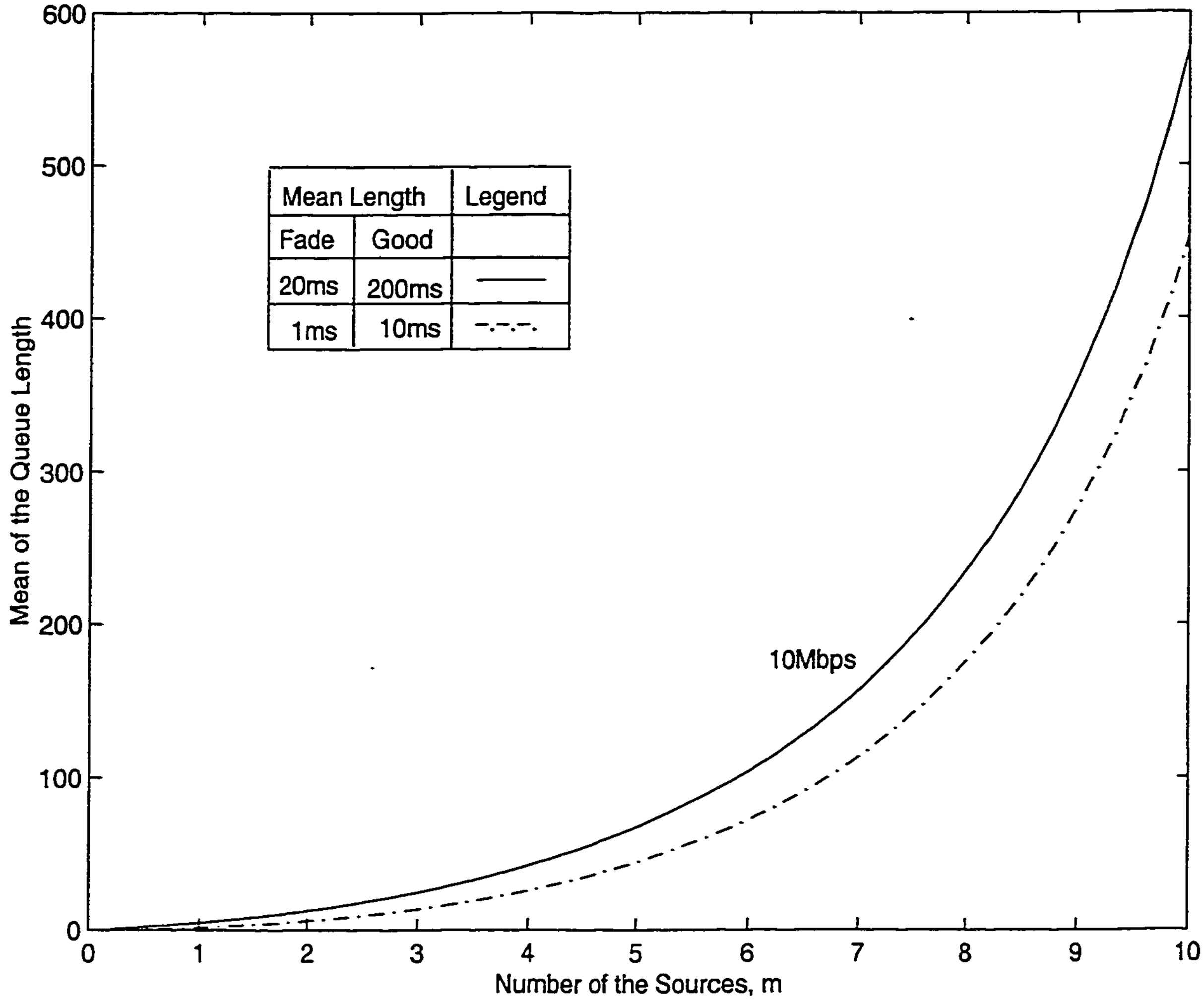


Figure 4.16 Steady-state mean of the queue length versus the number of sources, m, when transmission rate = 10Mbps for two error characteristics, data traffic

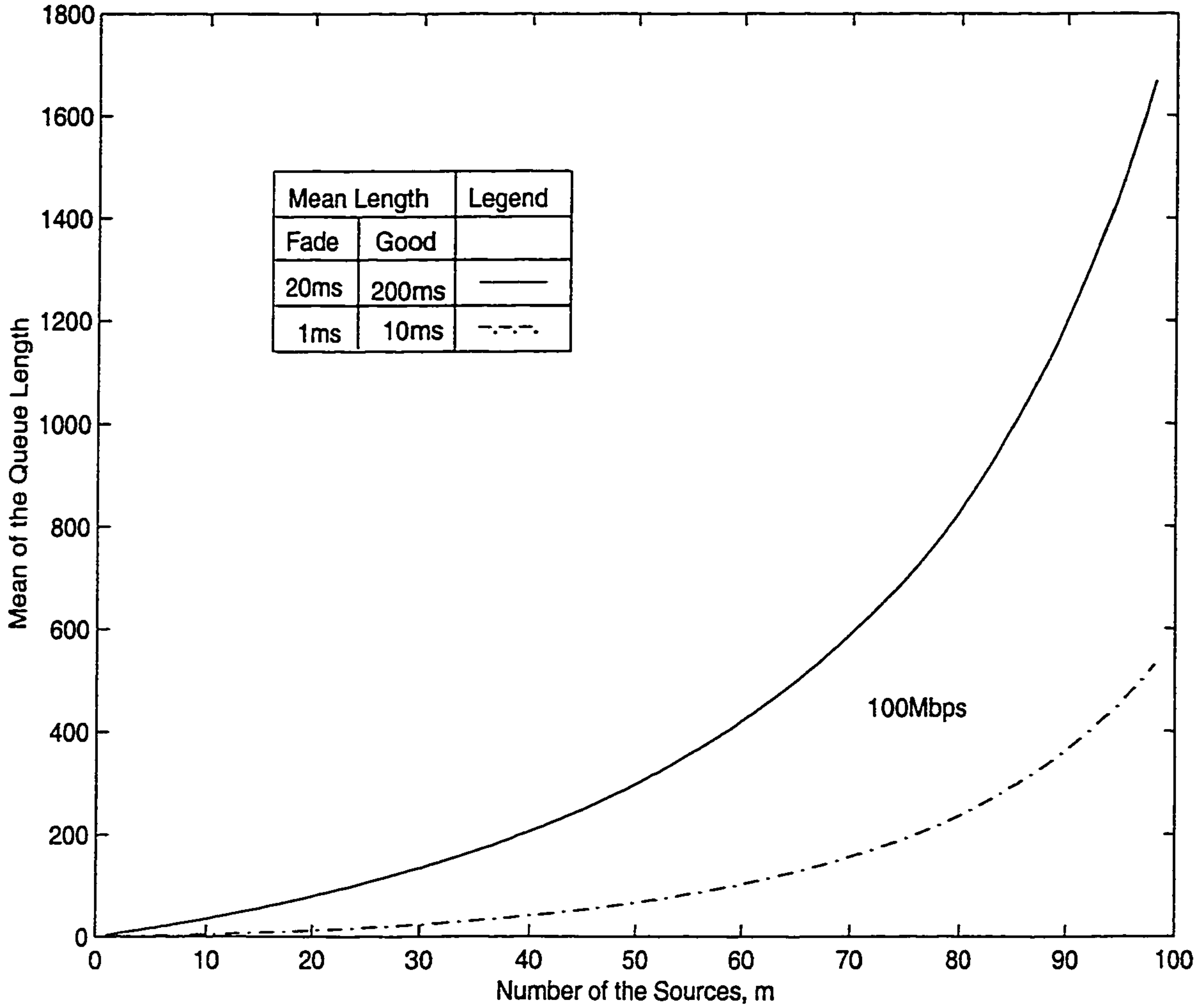


Figure 4.17 Steady-state mean of the queue length versus the number of sources, m, when transmission rate = 100Mbps for two error characteristics, data traffic

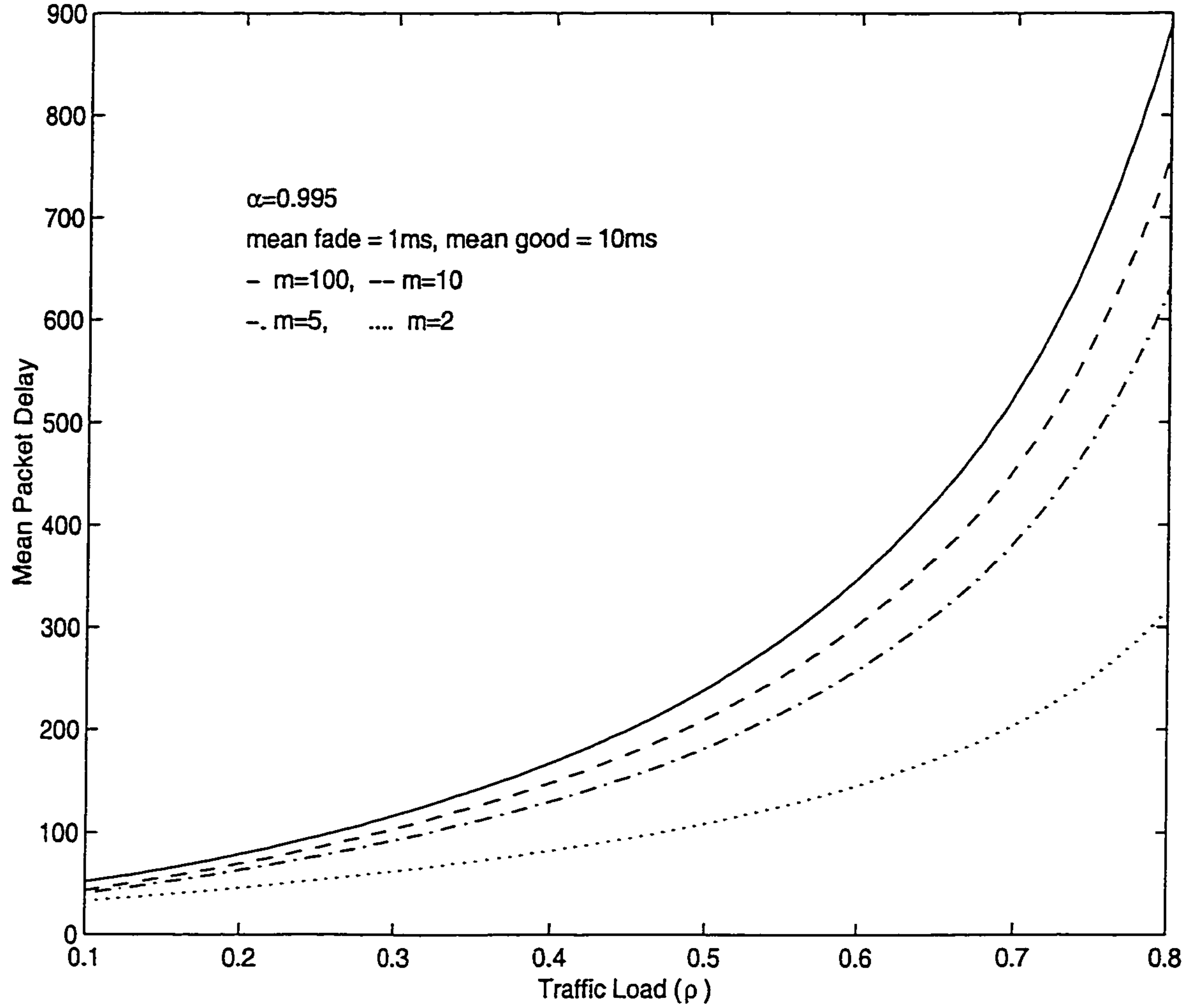


Figure 4.18 Mean packet delay in number of slots versus traffic load with different number of sources, under certain channel error condition, data traffic.

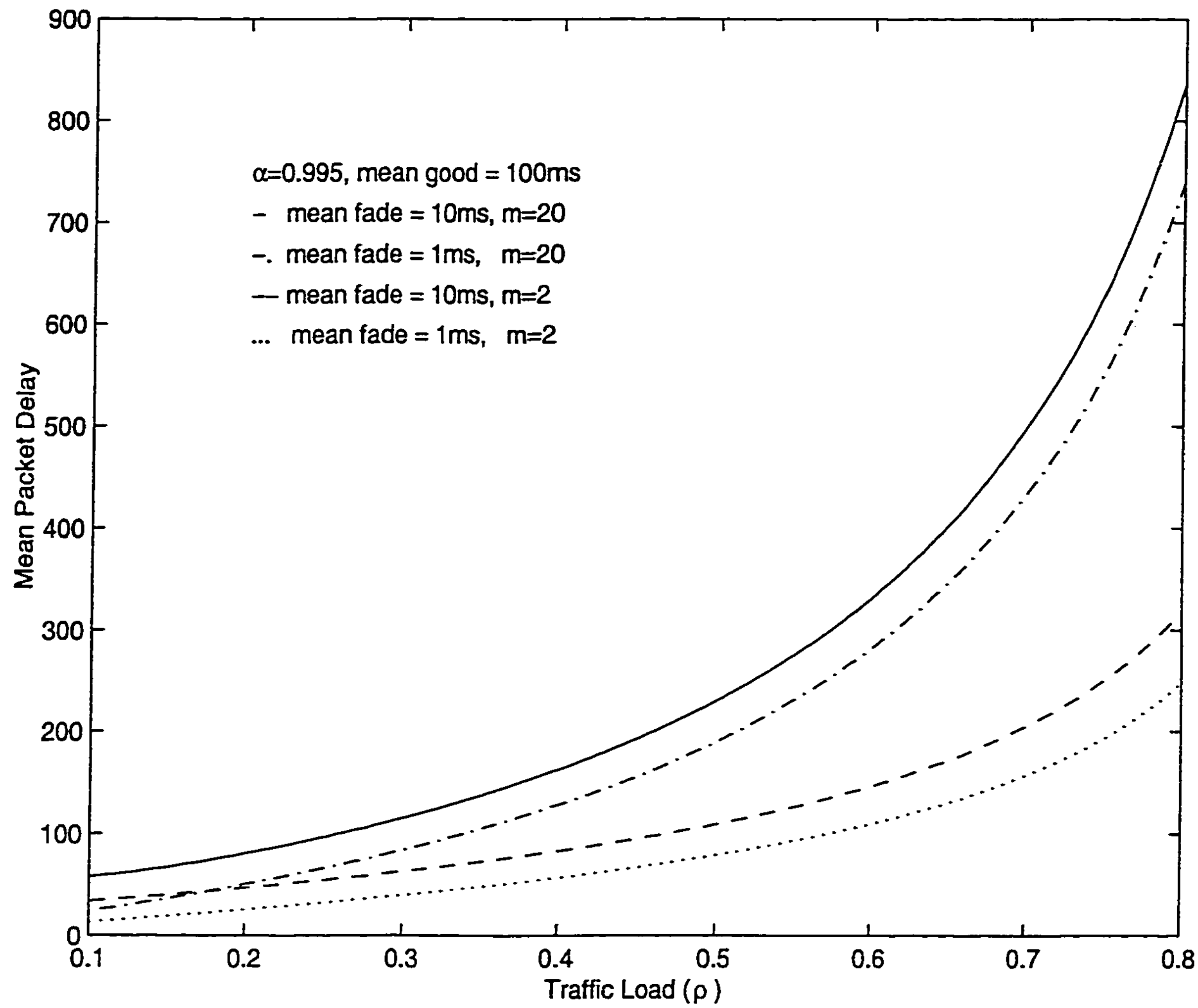


Figure 4.19 Mean packet delay in number of slots versus traffic load with different number of sources, under different channel error conditions, data traffic

Chapter 5

Conclusion

In this thesis, we have studied the effects of wireless channel errors on the ATM multiplexing system by presenting a discrete-time queuing analysis of the considered wireless ATM system with a two-state Markov channel and a correlated arrival process, which consists of the superposition of independent binary Markov On/Off sources.

In the analysis, we focused on the combined effects of the arrival and channel statistics on the discrete-time queuing performance. Through some mathematical manipulations we rewrote the joint probability generating function (PGF) of the system into a suitable form that lends itself to a solution. Then took transform with respect to discrete-time, and applied the final value theorem to extract the steady-state joint PGF of the buffer occupancy distribution, the number of the *On* sources and the state of the channel. After this, the expressions for some steady-state performance measures, such as mean queue length and mean packet delay were derived with ease.

The numerical results have shown that the channel error characteristics have substantial effect on the steady-state mean queue length and mean packet delay. Furthermore, when the transmission rate is higher, the channel errors have more effect on the queuing behavior. For a given transmission rate which is not very high, the mean queue length varies little with mean fade and mean good durations when we keep their ratio constant. On the other hand, at high transmission rate for a given mean good and fade durations ratio, the performance is better for a wider power density spectrum of the number of slots the channel spends in *Good* and *Bad* states, which is equivalent to a narrower autocovariance.

We also conclude that when the number of sources is large, the channel errors affect the performance more under a given traffic load. From this result we note that under a given channel error characteristic, if we have QoS requirement in terms of delay constraint, we can determine how many users the system can accommodate in order to meet the performance requirement.

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