On The Principle of MINQUE For The Estimation of Variance And Covariance Components

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Abstract

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In this thesis, we describe the method of MINQUE (C. R. Rao (1970)) and its various generalizations (C. R. Rao (1971, 1972), Chaubey (1977), P. S. R. S. Rao and Chaubey (1978)). This method can be used if some information about the variance components is available in the form of an a priori guess. Chaubey (1977) outlines the extension for estimating the elements of a covariance matrix using this principle. The method extends easily for the case when no a priori guess of the covariance matrix is assumed. However, for incorporating the a priori guess for estimating the distinct elements of a covariance matrix, we may need to consider a related but different minimization problem, whose solution is provided. A special case of the general model is considered for the numerical illustration.

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Contents

Chapter 1

Introduction

1.1. General Introduction.

In the usual analysis of variance model, interest lies mainly in estimating (and testing hypothesis about) linear functions of the effects in the models. These effects are called fixed effects, and the models containing these effects are called fixed effects models. There are, however, situations where we have no interest in linear functions of effects but where, by the nature of the data and their derivation, the prime interest concerning the effects are variances. Effects of this nature are called random effects, and certain of the models involving them are called random effects models. The variances associated with random effects are called variance components. A linear model may involve some fixed effects and some random effects. Such a model is called variance components model. The objective of using such models is to estimate the variance components. A linear regression model with a diagonal covariance matrix is a special case of the variance components model (described in Chapter 3).

The general variance components model is given by

$$Y = X\beta + U_1\xi_1 + \cdots + U_k\xi_k \tag{1.1.1}$$

where Y is an n-vector of observations, X is an $(n \times p)$ design matrix, β is a p-vector of unknown parameters, U_i is a given $(n \times c_i)$ matrix and ξ_i is a c_i-vector such that
$E(\xi_i) = 0$, $D(\xi_i) = \sigma_i^2 I_{c_i \times c_i}$, $i = 1, 2, ..., k.$ and $cov(\xi_i, \xi_j) = 0 \quad i \neq j$
Let $U = (U_1 \vdots \cdots \vdots U_k)$ and $\xi' = (\xi_1 \vdots \cdots \vdots \xi_k)$ then the model (1.1.1) can be written in a compact form as

$$Y = X\beta + U\xi \tag{1.1.2}$$

From (1.1.2)

$$E(Y) = X\beta \tag{1.1.3}$$

$$D(Y) = \sigma_1^2 V_1 + \cdots + \sigma_k^2 V_k \tag{1.1.4}$$

where $V_i = U_i U_i'$. The parameters $\sigma_1^2, \ldots, \sigma_k^2$, are called the variance components .

There is considerable literature on the subject of estimation of variance components in linear models. Anscombe and Tukey (1963) suggested an unbiased variance estimator based on residuals in the case of an additive two-way classification model with one-way variances, but the estimates they obtained are sometimes negative. Mandel (1964) presented a variance estimator for a general two-way model with one-way variances which was derived from a generalization of the estimator of Anscombe and Tukey, and it is always nonnegative. Hartley, Rao, and Kiefer (1969) presented an unbiased variance estimator which could be used with only one unit per stratum. It was very similar to the estimator of Anscombe and Tukey. The same special estimator was used by Duncan and Carroll (1962), discussed again by Duncan (1966), and extended in new ways by Chew (1970). None of these authors discussed optimality properties of their methods.

Rao (1970) introduced the method of minimum norm quadratic estimation (MINQUE) for regression models with heteroscedastic variances and later (1971a, 1971b, 1972) generalized it for variance and covariance components models. J.N.K. Rao (1971, 1973) has compared MINQUE and modified MINQUE (which is just the average of the squared residuals) estimators of heteroscedastic variances with the usual sample variances in the case of replicated data.

P. S. R. S. Rao and Chaubey (1978) considered some modifications of MINQUE and gave generalizations. These authors made it possible to estimate the distinct elements of the covariance matrix using similar methods as MINQUE in univariate as well as multivariate situations. The novelty behind this method of MINQUE is that it lays down a new optimality criterion of estimators and yields explicit estimators in complicated situations (cf. Rao, 1972

). Chaubey (1980b) used this method to estimate the variances and covariances arising from an unbalanced regression with residuals having a covariance matrix of intraclass form.

1.2. Outline of the thesis.

C. R. Rao (1970) introduced the method of MINQUE and the principles for this method. In Chapter 2 we discuss these principles. There is a major drawback of MINQUE that it may give negative values for estimates of non-negative variances, this was noted by J. N. K. Rao and Subrahmanium (1971), Hartley and Jayatillake (1971), J. N. K. Rao (1973), Horn, Horn and Duncan (1975). Horn, Horn and Duncan (1975) proposed another method for a simple replicated regression model and called it the Almost Unbiased Estimation (AUE). P.S.R.S. Rao and Chaubey (1978) derived the AUE and the Average of Squared Residuals (ASR) through modifications of the MINQUE principles for the unequal variances in a linear regression model. In Chapter 2 we discuss this ASR method and the related optimality principle. In this chapter we also discuss MINQUE and MINQE using apriori weights for the variance components model, called weighted MINQUE and MINQE.

P.S.R.S. Rao and Chaubey (1978) extended the principle of MINQUE for estimating the distinct elements of a covariance matrix of a linear model. They showed that the arguments involved in the principle of MINQUE for estimating the variance components can easily be adapted for the above problem by decomposing the covariance matrix into its distinct elements. They illustrated this technique by considering a specific model. The resulting minimization problem is similar to that discussed in C. R. Rao (1971, 1972), however this may not carry through in the weighted case, in general. In Chapter 3 we consider estimating the elements of a covariance matrix in a covariance component model.

Chaubey (1980b) used the method of MINQUE to estimate the variances and covariances arising from an unbalanced regression with residuals having a covariance matrix of intraclass form. In Chapter 4 we discuss intraclass correlation models and estimated weighted MINQUE, MINQE from the data given in Wiorkowski (1975).

Chapter 2

Minimum Norm Quadratic Estimation in Variance Components Model.

2.1. The Principles of MINQUE in Linear Models.

C.R. Rao (1971a) considered the estimation of a linear function of the variance components

$$p_1\sigma_1^2 + \cdots + p_k\sigma_k^2 \tag{2.1.1}$$

by a quadratic function $Y'AY$ of the random variable Y in (1.1.2). He developed the following criteria for determinig the matrix A .

1. Invariance under translation of the β parameter:

Instead of β, consider $\gamma = \beta - \beta_0$ as the unknown parameter, where β_0 is fixed. Then the model (1.1.2) becomes

$$Y - X\beta_0 = X\gamma + U\xi \tag{2.1.2}$$

and the estimator of $\sum_i p_i\sigma_i^2$ is

$$(Y - X\beta_0)'A(Y - X\beta_0). \tag{2.1.3}$$

But (2.1.3) should have the same numerical value as $Y'AY$ whatever β_0 may be. In fact (2.1.3) is

$$Y'AY - 2\beta_0'X'AY + \beta_0'X'AX\beta_0. \tag{2.1.4}$$

Thus (2.1.3) will have the same numerical value as $Y'AY$ under the restriction

$$AX = 0. \tag{2.1.5}$$

2. Unbiasedness:

We can write

$$
\begin{aligned}
Y'AY &= (X\beta + U\xi)'A(X\beta + U\xi) \\
&= \beta' X' AX\beta + 2\beta' X' AU\xi + \xi' U' AU\xi.
\end{aligned}
\tag{2.1.6}
$$

Now by (2.1.5) we get

$$
Y'AY = \xi' U' AU\xi \tag{2.1.7}
$$

in terms of the hypothetical vector variable ξ. $Y'AY$ is an unbiased estimator of $\sum p_i\sigma_i^2$ for all σ_i^2, $i = 1, 2, \cdots\cdots, k.$, if

$$
E(Y'AY) = E(\xi' U' AU\xi) = \sum_i p_i\sigma_i^2. \tag{2.1.8}
$$

But

$$
\begin{aligned}
E(Y'AY) &= E(\xi' U' AU\xi) \\
&= \sum_{i=1}^{k} E(\xi_i' U_i' AU_i\xi_i) \\
&= \sum_{i=1}^{k} tr\{U_i' AU_i E(\xi_i\xi_i')\} \\
&= \sum_{i=1}^{k} \sigma_i^2 tr(U_i' AU_i) \\
&= \sum_{i=1}^{k} \sigma_i^2 trAV_i.
\end{aligned}
\tag{2.1.9}
$$

So, equation (2.1.8) becomes

$$
\sum_{i=1}^{k} \sigma_i^2 trAV_i = \sum_{i=1}^{k} p_i\sigma_i^2 \tag{2.1.10}
$$

which implies

$$trAV_i = p_i\,, \quad i = 1,2,\cdots\cdots,k \tag{2.1.11}$$

3. Minimum norm:

If the hypothetical variable ξ were known, then a natural unbiased estimator of $\Sigma p_i \sigma_i^2$ would be

$$\frac{p_1}{c_1}\xi_1'\xi_1 + \cdots + \frac{p_k}{c_k}\xi_k'\xi_k = \xi'\Delta\xi, \tag{2.1.12}$$

where $\Delta = diag(\frac{p_1}{c_1}I_{c_1}, \ldots\ldots\ldots\ldots \frac{p_k}{ck}I_{c_k})$. But the proposed estimator is $\xi'U'AU\xi$, and the difference is

$$\xi'(U'AU - \Delta)\xi. \tag{2.1.13}$$

Therefore, we would like to choose A such that the difference $\xi'(U'AU - \Delta)\xi$ be small. Since ξ is unknown, equation (2.1.13) can be made small in some sense by minimizing

$$\| U'AU - \Delta \|, \tag{2.1.14}$$

where $\|.\|$ denotes the norm of a matrix. We may now state the principle of estimation as follows. The quadratic form $Y'AY$ is said to be the MINQUE (minimum norm quadratic unbiased estimator) of $\sum_i P_i \sigma_i^2$ if the matrix A is determined such that $\| U'AU - \Delta \|$ is a minimum subject to the conditions

$$\begin{aligned} AX &= 0 \\ trAV_i &= p_i\,, \quad i = 1,2,\cdots,k \end{aligned} \tag{2.1.15}$$

We may choose the Euclidean norm

$$\| U'AU - \Delta \|^2 = tr(U'AU - \Delta)(U'AU - \Delta). \tag{2.1.16}$$

Under the condition in (2.1.11), we get

$$trU'AU\Delta = trAU\Delta U' = trA\sum_{i=1}^{k} \frac{p_i}{c_i} U_i U_i'.$$

That is,

$$
\begin{aligned}
trU'AU\Delta &= \sum_{i=1}^{k} \frac{p_i}{c_i} trAV_i \\
&= \sum_{i=1}^{k} \frac{p_i^2}{c_i} \\
&= tr\Delta\Delta
\end{aligned}
\tag{2.1.17}
$$

and hence

$$
\begin{aligned}
tr(U'AU-\Delta)(U'AU-\Delta) &= trU'AUU'AU - 2trU'AU\Delta + tr\Delta\Delta \\
&= trU'AUU'AU - tr\Delta\Delta \\
&= trAVAV - tr\Delta\Delta,
\end{aligned}
\tag{2.1.18}
$$

where $V = UU' = \sum_{i=1}^{k} U_iU_i' = \sum_{i=1}^{k} V_i$. Since $tr\Delta\Delta$ does not involve A, the problem of MINQUE reduces to minimizing $tr(AVAV)$ subject to conditions (2.1.15). If V is non-singular, then the solution of A is given by the following theorem.

Theorem 2.1 (C.R.Rao (1972)):

Let A be a symmetric matrix and V be a symmetric and invertible matrix. Then the minimum of $tr(AVAV)$ subject to the condition (2.1.15) is attained at

$$
A = \sum_{j=1}^{k} \lambda_j R V_j R, \tag{2.1.19}
$$

where

$$
\begin{aligned}
R &= V^{-1}Q_V = Q_V'V^{-1} \\
Q_V &= I - P_V \\
P_V &= X\left(X'V^{-1}X\right)^{-1}X'V^{-1}
\end{aligned}
$$

with $\lambda = (\lambda_1, \cdots, \lambda_k)'$ obtained from

$$
S\lambda = p
$$

where $p = (p_1, \cdots\cdots, p_k)'$ and the (i,j)th element of S is $tr\left(Q_V'V^{-1}V_iV^{-1}Q_VV_j\right)$.

The MINQUE of $\sum p_i\sigma_i^2$ is

$$\begin{aligned} Y'AY &= \sum \lambda_j Y' Q_V' V^{-1} V_j V^{-1} Q_V Y \\ &= \sum \lambda_j e' V^{-1} V_j V^{-1} e \\ &= \sum \lambda_j u_j \\ &= \lambda' u \\ &= p' S^{-1} u, \end{aligned} \tag{2.1.20}$$

where $e = Q_V Y$, $u_j = e' V^{-1} V_j V^{-1} e$ and $u = (u_1, \cdots\cdots, u_k)'$. We can write

$$p'\hat{\sigma} = p' S^{-1} u, \tag{2.1.21}$$

where $\hat{\sigma}' = (\hat{\sigma}_1^2, \cdots, \hat{\sigma}_k^2)$ is a solution of

$$S\sigma = u.$$

2.2. Principle of MINQE

P.S.R.S. Rao and Chaubey (1976) showed that the Average of Squared Residuals (ASR) can be derived as a quadratic estimator for the case of the regression model through the arguments of MINQUE by dropping the unbiasedness condition from this principle. Such a principle can be called MINQE (minimum norm quadratic estimation). Thus $Y'AY$ is the MINQE of $\sum p_i\sigma_i^2$ where A is determined such that the trace in (2.1.16) is minimized subject to the condition $AX = 0$. Again since Δ is given, this minimization is equivalent to minimizing

$$tr(AVAV) - 2tr(U'AU\Delta), \tag{2.2.1}$$

subject to the condition $AX = 0$.

Theorem 2.2 **(P. S. R. S. Rao and Chaubey):**

Let A be a symmetric matrix and V be a symmetric and invertible matrix. Then the minimum of $tr(AVAV) - 2trAV\Delta V$ subject to the condition $AX = 0$ is attained at

$$A = Q_v' V^{-1} U \Delta U' V^{-1} Q_V.$$

Proof :

To prove the above theorem we have to minimize

$$\Phi = tr(AVAV) - 2tr(U'AU\Delta) - 2trMAX,$$

where M is a $(p \times n)$ matrix of Lagrange multipliers. Differentiating Φ with respect to A and equating the derivative to zero results

$$VAV = U\Delta U' + \frac{1}{2}(XM + M'X'). \tag{2.2.2}$$

Since $AX = 0 \Leftrightarrow Q_v A Q_V = A$ and $Q_V' V^{-1} X = V^{-1}X - V^{-1}X(X'V^{-1}X)^{-1}X'V^{-1}X = 0$. The solution of A from (2.2.2) is $A = Q_v' V^{-1} U \Delta U' V^{-1} Q_V$. This proves Theorem 2.2. Therefore, the MINQE of $\sum p_i \sigma_i^2$ is

$$\begin{aligned} Y'AY &= Y'Q_v' V^{-1} U \Delta U' V^{-1} Q_V Y \\ &= e'V^{-1} U \Delta U' V^{-1} e \\ &= \sum_{i=1}^{k} \frac{p_i}{c_i} e'V^{-1} U_i U_i' V^{-1} e \\ &= \sum_{i=1}^{k} \frac{p_i}{c_i} e'V^{-1} V_i V^{-1} e \end{aligned} \tag{2.2.3}$$

and the MINQE of σ_i^2 is

$$\overline{\sigma_i^2} = \frac{e'V^{-1}V_iV^{-1}e}{c_i}. \tag{2.2.4}$$

2.3. MINQUE and MINQE with a priori weights.

If some information about σ_i^2, in terms of their relative magnitudes, is available, it can be incorporated in the model through reparameterization. Let $\Lambda_1^2, \cdots\cdots, \Lambda_k^2$ be a priori weights reflecting the relative magnitudes of $\sigma_1^2, \cdots, \sigma_k^2$. Thus the model (1.1.1) can be written as

$$\begin{aligned} Y &= X\beta + \Lambda_1^{-1}U_1\eta_1 + \cdots + \Lambda_k^{-1}U_k\eta_k \\ &= X\beta + U_*\eta \end{aligned} \tag{2.3.1}$$

where $U_* = (U_{1^\bullet} \vdots \cdots\cdots \vdots U_{k^\bullet})$, $U_{i^\bullet} = \Lambda_i^{-1}U_i$, $\eta_i = \Lambda_i\xi_i$ and
$\eta' = (\eta_1' \vdots \cdots \vdots \eta_k')$.

We discussed in Section 2.1 that C.R.Rao considered a quadratic form $Y'AY$ as an estimator of $\sum_{i=1}^k p_i\sigma_i^2$ and that this quadratic form should be unbiased and invariant under translations of the β parameter. From these two criteria we again get the following conditions:

$$trAV_i = p_i \qquad i = 1, 2, \cdots, k \tag{2.3.2}$$

$$AX = 0. \tag{2.3.3}$$

The natural unbiased estimator in terms of η is

$$\sum \frac{p_i}{c_i}\xi_i'\xi_i = \xi'\Delta\xi = \eta'\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}\eta, \tag{2.3.4}$$

where

$$\Delta = diag(\tfrac{p_1}{c_1}I_{c_1}, \cdots\cdots, \tfrac{p_k}{c_k}I_{c_k}) \text{ and } \Lambda = diag(\Lambda_1^{-2}I_{c_1}, \cdots\cdots, \Lambda_k^{-2}I_{c_k}).$$

Also the quadratic form $Y'AY$ in terms of η using (2.1.7) is

$$Y'AY = \eta'\Lambda^{\frac{1}{2}}U'AU\Lambda^{\frac{1}{2}}\eta. \tag{2.3.5}$$

Hence the problem of norm-minimization leads to minimization of

$$tr(\Lambda^{\frac{1}{2}}(U'AU - \Delta)\Lambda^{\frac{1}{2}})^2, \tag{2.3.6}$$

which can be written as

$$trAV_*AV_* - 2trAU_*\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}U_*' + tr\Delta\Lambda\Delta\Lambda, \qquad (2.3.7)$$

where $U_* = \Lambda^{\frac{1}{2}}U = (\Lambda_1^{\frac{-1}{2}}U_1 \cdots\cdots \Lambda_k^{\frac{-1}{2}}U_k)$ and $V_* = U_*U_*' = \sum_{i=1}^{k}\Lambda_i^{-2}V_i$.
One may verify again that

$$trAU_*\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}U_*' = tr\Delta\Lambda\Delta\Lambda, \qquad (2.3.8)$$

because

$$\begin{aligned} trAU_*\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}U_*' &= trA\sum_{i=1}^{k}\frac{p_i}{c_i}\Lambda_i^{-4}V_i \\ &= \sum_{i=1}^{k}\frac{p_i}{c_i}\Lambda_i^{-4}trAV_i \\ &= \sum_{i=1}^{k}\frac{p_i^2}{c_i}\Lambda_i^{-4} \qquad (2.3.9) \end{aligned}$$

and

$$tr\Delta\Lambda\Delta\Lambda = \sum_{i=1}^{k}c_i\Lambda_i^{-4}\frac{p_i^2}{c_i^2} = \sum_{i=1}^{k}\frac{p_i^2\Lambda_i^{-4}}{c_i}. \qquad (2.3.10)$$

Therefore the expression in (2.3.7) becomes

$$trAV_*AV_* - tr\Delta\Lambda\Delta\Lambda. \qquad (2.3.11)$$

Since Δ and Λ are given, the minimization of $tr(\Lambda^{\frac{1}{2}}(U'AU-\Delta)\Lambda^{\frac{1}{2}})^2$ is same as the minimization of $trAV_*AV_*$. The resulting estimator, known as weighted MINQUE, can be given by using Theorem 2.1, as

$$p'\widehat{\sigma_*} = \lambda'u = p'S_*^{-1}u, \qquad (2.3.12)$$

where $(i,j)th$ element of S_* is $tr(Q_H'HV_iHQ_HV_j)$, $Q_H = I - X(X'HX)^{-1}X'H$ and $H = V_*^{-1}$, $u_j = e'HV_jHe$ and $e = Q_HY$. Then the weighted MINQUE of σ is

$$\widehat{\sigma_*} = S_*^{-1}u. \tag{2.3.13}$$

To get the MINQE, we have to minimize the equation

$$\Phi = trAV_*AV_* - 2trAU_*\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}U_*'. \tag{2.3.14}$$

In this case we get the solution of A from Theorem 2.2 as

$$\begin{aligned} A &= Q_H'HU_*\Lambda^{\frac{1}{2}}\Delta\Lambda^{\frac{1}{2}}U_*'HQ_H \\ &= Q_H'H\left(\sum_{i=1}^{k}\frac{p_i}{c_i}\Lambda_i^{-4}V_i\right)HQ_H. \end{aligned} \tag{2.3.15}$$

Thus the MINQE of $\sum_{i=1}^{k} p_i\sigma_i^2$ is

$$\begin{aligned} Y'AY &= Y'Q_H'H\left(\sum_{i=1}^{k}\frac{p_i}{c_i}\Lambda_i^{-4}V_i\right)HQ_HY \\ &= \sum_{i=1}^{k}\frac{p_i}{c_i}\Lambda_i^{-4}e'HV_iHe \end{aligned} \tag{2.3.16}$$

and that of σ_i^2 is

$$\overline{\sigma_{i*}^2} = \frac{e'HV_iHe}{c_i\Lambda_i^4}. \tag{2.3.17}$$

Chapter 3

Distinct Elements of a Covariance Matrix in Covariance Components Model

3.1. Introduction

C.R.Rao (1971b, 1972) extended the principle of MINQUE to more general models involving hypothetical variables with unknown covariances. He considered the model

$$Y = X\beta + U_1\xi_1 + \cdots + U_k\xi_k, \tag{3.1.1}$$

where Y, X and β are as described in model (1.1.1), U_j is a given $(n \times q)$ matrix and ξ_j is a q- vector such that $E(\xi_j) = 0$, $E(\xi_j, \xi_{j}') = 0$ for $j \neq j'$ and $D(\xi_j) = \Sigma$, $j = 1, 2, \cdots\cdots, k$. To formulate the MINQUE procedure for model (3.1.1), C.R.Rao (1971b, 1972) assumed that all the variances and covariances in Σ are distinct. Thus this formulation can not be adapted to model (3.1.2) when some of the variances and/or covariances are equal. Chaubey (1977) considered a special case of the model (3.1.1) with $k = 1$, $q = n$, $U_1 = I_n$ and $\xi_1 = \varepsilon$ giving a non-homoscedastic linear model

$$Y = X\beta + \varepsilon. \tag{3.1.2}$$

where $E(\varepsilon) = 0$ and $D(\varepsilon) = \Sigma$. He showed that the principle of MINQE and MINQE is adapted to this case by replacing σ_i^2 by α_i and V_i by T_i. P.S.R.S. Rao and Chaubey (1978) described the method of MINQUE and MINQE with *a priori* weights for the variance components model and indicated that its adaptation goes through for estimating the variances and covariances of a non-diagonal covariance matrix Σ by decomposing it as

$$\Sigma = \alpha_1 T_1 + \cdots\cdots + \alpha_d T_d, \tag{3.1.3}$$

where $\alpha_1, \ldots, \alpha_d$ are d distinct elements of Σ and $T_1, \ldots, T_d$ are $(n \times n)$ matrices having zeros and unities as elements at the appropriate places. However, we note here that this may not hold in general for the weighted case, and the solution is provided.

In the next section we show the adaptation of the principle of MINQUE (without any *a priori* weights) for the estimation of the distinct elements $\alpha_1,\dots,\alpha_d$ of Σ considering the general model (3.1.1).

3.2. Unweighted MINQUE

Consider the model (3.1.1) , where

$$E(Y) = X\beta,$$

$$\begin{aligned} D(Y) &= D(U\xi) = UD(\xi)U' \\ &= (U_1 \vdots \cdots\cdots \vdots U_k)\begin{pmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma \end{pmatrix}\begin{pmatrix} U_1' \\ U_2' \\ \vdots \\ U_k' \end{pmatrix} \\ &= \sum_{j=1}^{k} U_j \Sigma U_j'. \end{aligned} \tag{3.2.1}$$

Let us consider $\in = U\xi$, then we can write

$$D(\in) = \sum_j U_j \Sigma U_j' \tag{3.2.2}$$

and Σ as

$$\Sigma = \sum_{i=1}^{d} \alpha_i T_i \qquad i = 1,2,\cdots,d. \ , \tag{3.2.3}$$

where $\alpha_1,\dots\dots,\alpha_d$ are the d distinct elements of Σ and $T_1,\cdots,T_d$ are specific to the model. The problem is to estimate the distinct elements $\alpha_1,\cdots\cdots,\alpha_d$. From (3.2.2), we can write

$$
\begin{aligned}
D(\in) &= \sum_j U_j\left(\sum_{i=1}^{d} \alpha_i T_i\right)U_j' \\
&= \sum_i \alpha_i \sum_j U_j T_i U_j' \\
&= \sum_{i=1}^{d} \alpha_i T_i^*,
\end{aligned}
\tag{3.2.4}
$$

where $T_i^* = \sum_j U_j T_i U_j'$.

To estimate α_i, $i = 1, 2, \cdots, d.$, we may consider the estimation of a linear function $p_1\alpha_1 + \cdots + p_d\alpha_d$ of the elements $\alpha_1, \cdots, \alpha_d$ by a quadratic function $Y'AY$ of the observation vector Y.

To determine the matrix A , from the invariance principle under translations of the β parameter we get the condition

$$
AX = 0 \tag{3.2.5}
$$

and from the principle of unbiasedness we get the condition

$$
trAT_i^* = p_i \quad ; \qquad i = 1, \cdots, d., \tag{3.2.6}
$$

as

$$
\begin{aligned}
E(Y'AY) &= \sum_{j=1}^{k} E(\xi_j' U_j' A U_j \xi_j) \\
&= \sum_{j=1}^{k} tr(U_j' A U_j) E(\xi_j' \xi_j) \\
&= \sum_{j=1}^{k} tr(U_j' A U_j) \sum_{i=1}^{d} \alpha_i T_i \\
&= \sum_{i=1}^{d} \alpha_i trA \sum_{j=1}^{k} (U_j' T_i U_j) \\
&= \sum_{i=1}^{d} \alpha_i trAT_i^* .
\end{aligned}
\tag{3.2.7}
$$

$\xi_j' T_i \xi_j$ is a natural unbiased estimator (NUE) of $\alpha_i s_i$, where s_i is the number of 1's in T_i, based on ξ_j. Combining the information from all ξ_j 's for $j = 1, 2, \cdots\cdots k$, we get a NUE of $\sum_{i=1}^{d} p_i \alpha_i$ as

$$\sum_{i=1}^{d} \frac{p_i}{s_i} \frac{1}{k} \sum_{j=1}^{k} \xi_j' T_i \xi_j = \sum_{j=1}^{k} \xi_j' \left(\sum_{i=1}^{d} \frac{p_i}{s_i k} T_i \right) \xi_j$$
$$= \xi_j' B \xi_j, \qquad (3.2.8)$$

where

$$B = \begin{pmatrix} \sum \frac{p_i}{ks_i} T_i & 0 & \cdots & 0 \\ 0 & \sum \frac{p_i}{ks_i} T_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum \frac{p_i}{ks_i} T_i \end{pmatrix}.$$

This follows because,

$$E\xi_j' T_i \xi_j = tr(T_i \Sigma)$$
$$= \alpha_i s_i + 0.$$

Our proposed estimator is $Y'AY = \xi' U' AU \xi$. Thus we have to choose A such that the difference $\xi'(U'AU - B)\xi$ is minimized. This difference can be made small in some sense by minimizing

$$\| U'AU - B \| .$$

We may now state the principle of estimation as follows.

The quadratic form $Y'AY = \xi' U' AU \xi$ is said to be a MINQUE (minimum norm quadratic unbiased estimator) of $\sum_i p_i \alpha_i$ if the matrix A is determined such that $\| U'AU - B \|$ is a minimum subject to the conditions $AX = 0$ and $tr(AT_i^*) = p_i$, $i = 1, 2, \cdots, d$.

In particular, with the Euclidean norm, the matrix A of the quadratic form $Y'AY$ should be determined by minimizing

$$\| U'AU - B \|^2 = tr(U'AU - B)^2$$
$$= tr(AUU'AUU') - 2tr(U'AUB) + tr(B^2). \qquad (3.2.9)$$

We can show that $tr(U'AUB) = tr(B^2)$. To show this, note that

$$tr(B^2) = ktr\left(\sum \frac{p_i}{ks_i} T_i\right)^2$$
$$= k\sum \frac{p_i^2}{(ks_i)^2} tr(T_i^2) + k\sum_{i \neq j} \frac{p_i p_j}{k^2 s_i s_j} trT_iT_j. \qquad (3.2.10)$$

By definition

$$trT_i^2 = \Sigma_j \Sigma_k T_{ijk}^2 = \Sigma_j \Sigma_k T_{ijk} = s_i,$$

and

$$trT_iT_j = \Sigma_{r=1}^q \Sigma_{s=1}^q T_{irs} T_{jsr}$$
$$= \Sigma_r \Sigma_s T_{irs} T_{jrs}.$$

But if $T_{irs} = 1$, $T_{jrs} = 0$ for $i \neq j$, hence $\Sigma_r \Sigma_s T_{irs} T_{jrs} = 0$ and

$$tr(B^2) = \frac{1}{k} \sum_i \frac{p_i^2}{s_i}. \qquad (3.2.11)$$

Also

$$tr(U'AUB) = tr(AUBU')$$
$$= trA \sum_j U_j \left(\sum \frac{p_i}{ks_i} T_i\right) U_j'$$
$$= trA \sum_i \frac{p_i}{ks_i} T_i \left(\sum_j U_j T_i U_j'\right). \qquad (3.2.12)$$

That is, we get

$$tr(U'AUB) = \sum_i \frac{p_i}{ks_i} trAT_i^*$$
$$= \sum_i \frac{p_i^2}{ks_i},$$

since $trAT_i^* = p_i$. Using equations (3.2.11) and (3.2.12) we get

$$trU'AUB = trB^2, \quad (3.2.13)$$

hence

$$tr(U'AU - B)^2 = tr(AUU'AUU') - tr(B^2) = tr(AVAV) - tr(B^2) \quad (3.2.14)$$

where $V = UU' = \sum U_i U_i' = \sum V_i$.

Since $tr(B^2)$ does not involve A, the problem of MINQUE reduces to minimizing $tr(AVAV)$ subject to the conditions,

$$AX = 0 \quad (3.2.15)$$

$$tr(AT_i^*) = p_i \quad ; \quad i = 1, \cdots, d. \quad (3.2.16)$$

If V is non-singular, then the solution of A is obtained by Theorem 2.1 and thus the MINQUE $\hat{r}$ for $r = p'\alpha$, where $\alpha = (\alpha_1, \cdots, \alpha_d)$ is given by

$$\hat{r} = \lambda' u$$
$$= p' W^{-1} u, \quad (3.2.17)$$

where the $(i,j)th$ element of W is $tr(Q_V' V^{-1} T_i^* V^{-1} Q_V T_j^*)$ and $u_j = e' V^{-1} T_j^* V^{-1} e$. Therefore

$$\hat{\alpha} = W^{-1} u. \quad (3.2.18)$$

3.3. Unweighted MINQE

To get the MINQE we are to minimize

$$\Phi = tr(AVAV) - 2tr(U'AUB) \tag{3.3.1}$$

subject to $AX = 0$. The solution for A using Theorem 2.2 is

$$A = Q_V' V^{-1} UBU' V^{-1} Q_V \tag{3.3.2}$$

and

$$Y'AY = e'V^{-1}UBU'V^{-1}e. \tag{3.3.3}$$

Now

$$\begin{aligned}
UBU' &= (U_1 \vdots \cdots\cdots \vdots U_k)\begin{pmatrix} \sum \frac{p_i}{ks_i}T_i & 0 & \cdots & 0 \\ 0 & \sum \frac{p_i}{ks_i}T_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sum \frac{p_i}{ks_i}T_i \end{pmatrix}\begin{pmatrix} U_1' \\ U_2' \\ \vdots \\ U_k' \end{pmatrix} \\
&= U_1 \frac{1}{k}\sum_i \frac{p_i}{s_i}T_iU_1' + U_2\frac{1}{k}\sum_i \frac{p_i}{s_i}T_iU_2' + \cdots\cdots + U_k\frac{1}{k}\sum_i \frac{p_i}{s_i}T_iU_k' \\
&= \frac{1}{k}\sum_j U_j\left(\sum_i \frac{p_i}{s_i}T_i\right)U_j' \\
&= \frac{1}{k}\sum_i \frac{p_i}{s_i}\sum_j (U_jT_iU_j') \\
&= \frac{1}{k}\sum_i \frac{p_i}{s_i}T_i^*.
\end{aligned} \tag{3.3.4}$$

Therefore

$$Y'AY = \frac{1}{k}\sum_i \frac{p_i}{s_i} e'V^{-1}T_i^* \, V^{-1}e$$
$$= \frac{1}{k}\sum_i \frac{p_i}{s_i} u_i, \tag{3.3.5}$$

where $u_i = e'V^{-1}T_i^* V^{-1}e$. Therefore the MINQE of $r = \sum p_i \alpha_i$ is given by

$$\bar{r} = \frac{1}{k}\sum_i \frac{p_i}{s_i} u_i$$

and that of α_i is given by

$$\overline{\alpha_i} = \frac{u_i}{ks_i} \, . \tag{3.3.7}$$

3.4. Weighted MINQUE

Let δ_i be a priori estimate of α_i, then from Section 3.1 we have an a priori estimate of $D(\xi_j)$ as $\sum_{i=1}^{d} \delta_i T_i = \Lambda_0$ and that of $D(Y) = \sum_{i=1}^{d} \delta_i T_i^*$, $j = 1, 2, \cdots, k$.

Using the a priori estimate Λ_0, we consider the error terms $\eta_j = \Lambda_0^{-\frac{1}{2}} \xi_j$ which should be more homogeneous. This gives $U\xi = U_* \eta$ where $U_* = U\Lambda_*^{\frac{1}{2}}$, $\eta = \Lambda_*^{-\frac{1}{2}} \xi$ and

$$\Lambda_* = \begin{pmatrix} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Lambda_0 \end{pmatrix}, \tag{3.4.1}$$

which gives

$$Y'AY = \eta' U_*' A U_* \eta. \tag{3.4.2}$$

The natural unbiased estimator $\xi' B \xi$ can be written as

$$\xi' B \xi = \eta' \Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}} \eta \tag{3.4.3}$$

and we are led to minimize

$$tr\left(U_*' A U_* - \Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}}\right)^2 = tr(AU_*U_*'AU_*U_*') + tr\left(\Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}}\right)^2 - 2tr\left(\Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}} U_*' A U_*\right). \tag{3.4.4}$$

But $tr\left(\Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}}\right)^2 = tr(B\Lambda_*)^2$ and $tr\left(\Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}} U_*' A U_*\right) = trB\Lambda_* U' A U \Lambda_*$ hence

$$tr\left(U_*' A U_* - \Lambda_*^{\frac{1}{2}} B \Lambda_*^{\frac{1}{2}}\right)^2 = tr(AV_*AV_*) + tr(B\Lambda_*)^2 - 2tr(B\Lambda_* U' A U \Lambda_*), \tag{3.4.5}$$

where $V_* = U_* U_*'$. This can be simplified as

$$\begin{aligned} V_* &= \sum_{j=1}^{k} U_j \Lambda_0 U_j' \\ &= \sum_{j=1}^{k} U_j \sum_{i=1}^{d} \delta_i T_i U_j' \\ &= \sum_{i=1}^{d} \delta_i T_i^* \end{aligned} \tag{3.4.6}$$

where $T_i^* = \sum_{j=1}^{k} U_j T_i U_j'$ for $i = 1, 2, \cdots, d$ and $j = 1, 2, \cdots\cdots, k$.

We have seen in the unweighted case that the product term $trAU\Lambda_* B\Lambda_* U'$ does not appear in the minimization problem, however, for the weighted case, this may not hold, because, in general

$$trAU\Lambda_* B \Lambda_* U' \neq tr(B\Lambda_*)^2 . \tag{3.4.7}$$

We will see in Chapter 4, that the equality holds in the intraclass correlation model .
In (3.4.5), put $U\Lambda_* B \Lambda_* U' = D$, then minimize

$$\Phi = trAV_*AV_* - 2trAD, \tag{3.4.8}$$

subject to the conditions

$$AX = 0, \tag{3.4.9}$$

$$tr(AT_i^*) = p_i \quad i = 1, 2, \cdots\cdots, d. \tag{3.4.10}$$

This minimization problem is similar to that in Theorem 2.2 except that the extra condition (given by Equation 3.4.10) for unbiasedness is also required. Hence, a new result for this minimization problem is required. The following theorem provides the solution.

Theorem 3.1

Let A be a symmetric matrix and V_* be a symmetric and invertible matrix, then the minimum of $(trAV_*AV_* - 2trAD)$, subject to the conditions (3.4.9), (3.4.10) is attained at

$$A = Q'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*} + \Sigma\lambda_j RT_j^* R, \tag{3.4.11}$$

where $R = Q'_{V_*}V_*^{-1} = V_*^{-1}Q_{V_*}$, $Q_{V_*} = I - P_{V_*}$, $P_{V_*} = X(X'V_*^{-1}X)^{-1}X'V_*^{-1}$ and $\lambda = (\lambda_1, ..., \lambda_d)'$ is determined from

$$W_*\lambda = p - q, \tag{3.4.12}$$

with $W_{*(i,j)} = tr(Q'_{V_*}V_*^{-1}T_i^*V_*^{-1}Q_{V_*}T_j^*)$.

Proof:

A direct proof of this theorem using the method of Lagrange multipliers is as follows: Minimizing the expression in (3.4.8) subject to the conditions (3.4.9), (3.4.10) is equivalent to minimizing

$$\Phi = tr(AV_*AV_*) - 2tr(AD) - 2\sum_j \lambda_j(tr(AT_j - P_j)) - 2tr(MAX), \tag{3.4.13}$$

subject to the conditions in (3.4.9) and (3.4.10) where M is a $(p \times n)$ matrix of Lagrange multipliers. Differentiating Φ with respect to A and equating the derivative to zero results in the equation

$$V_*AV_* = D + \Sigma\lambda_j T_j^* + \frac{1}{2}(XM + M'X'). \tag{3.4.14}$$

Now since $AX = 0 \Leftrightarrow Q'_{V_*}AQ_{V_*} = A$ and $Q'_{V_*}V_*^{-1}X = V_*^{-1}X - V_*^{-1}X(X'V_*^{-1}X)^{-1}X'V_*^{-1}X = 0$. The solution of A from (3.4.14) is

$$A = Q'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*} + \Sigma\lambda_j Q'_{V_*}V_*^{-1}T_j^*V_*^{-1}Q_{V_*}. \tag{3.4.15}$$

Using (3.4.10) we have,

$$\begin{aligned} trQ'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*}T_i^* + \Sigma\lambda_j trQ'_{V_*}V_*^{-1}T_j^*V_*^{-1}Q_{V_*}T_i^* &= p_i \\ &\Rightarrow W_*\lambda = p - q, \end{aligned}$$

where $q_i = trQ'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*}T_i^*$, $\lambda = (\lambda_1, \cdots, \lambda_d)'$, $p = (p_1, \cdots, p_d)'$ and the $(i,j)th$ element of W_* is $tr(Q'_{V_*}V_*^{-1}T_i^*V_*^{-1}Q_{V_*}T_j^*)$. This completes the proof.

Now the weighted MINQUE becomes

$$\begin{aligned} Y'AY &= Y'Q'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*}Y + \Sigma\lambda_j Y'Q'_{V_*}V_*^{-1}T_j^*V_*^{-1}Q_{V_*}Y \\ &= e'V_*^{-1}DV_*^{-1}e + \Sigma\lambda_j e'V_*^{-1}T_j^*V_*^{-1}e \\ &= e'V_*^{-1}DV_*^{-1}e + \Sigma\lambda_j u_j \\ &= e'V_*^{-1}DV_*^{-1}e + \lambda'u, \end{aligned} \tag{3.4.16}$$

where $e = Q_{V_*}Y$, $u_j = e'V_*^{-1}T_j^*V_*^{-1}e$ and $u' = (u_1, \ldots, u_d)$. Hence the weighted MINQUE , $\widehat{r_*}$ for $r = p'\alpha$ is given by

$$\begin{aligned} \widehat{r_*} &= e'V_*^{-1}DV_*^{-1}e + \lambda'u \\ &= e'V_*^{-1}DV_*^{-1}e + (p-q)'W_*^{-1}u. \end{aligned} \tag{3.4.17}$$

3.5. Weighted MINQE

To get weighted MINQE we have to minimize equation (3.4.8) subject to the condition

$$AX = 0$$

This minimization is equevalent to minimizing

$$\Phi = tr(AV_*AV_*) - 2tr(AD). \tag{3.5.1}$$

The solution for A is obtained from Theorem 2.2 , i.e

$$A = Q'_{V_*}V_*^{-1}DV_*^{-1}Q_{V_*}. \tag{3.5.2}$$

We get

$$\begin{aligned}
D &= U\Lambda_* B\Lambda_* U' \\
&= (U_1 \vdots \cdots\cdots \vdots U_k)\begin{pmatrix} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Lambda_0 \end{pmatrix} B \begin{pmatrix} \Lambda_0 & 0 & \cdots & 0 \\ 0 & \Lambda_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \Lambda_0 \end{pmatrix}\begin{pmatrix} U'_1 \\ U'_2 \\ \vdots \\ U'_k \end{pmatrix} \\
&= \sum_{j=1}^{k} U_j\Lambda_0 B\Lambda_0 U'_j \\
&= \sum_{j=1}^{k} U_j\Lambda_0\left(\sum_{i=1}^{d} \frac{p_i}{ks_i}T_i\right)\Lambda_0 U'_j \\
&= \sum_{j=1}^{k} U_j\left(\sum_{i=1}^{d} \frac{p_i}{ks_i}\Lambda_0 T_i\Lambda_0\right)U'_j.
\end{aligned}$$

That means

$$D = \frac{1}{k}\sum_{i=1}^{d} \frac{p_i}{s_i}T_i^0, \tag{3.5.3}$$

where $T_i^0 = \sum_{j=1}^{k} U_j\Lambda_0 T_i\Lambda_0 U'_j$ for $i = 1, 2, \cdots, d$. Therefore, the weighted MINQE of $\Sigma p_i\alpha_i$ is

$$Y'AY = \frac{1}{k}\Sigma\frac{p_i}{s_i}Y'Q'_{V_*}V_*^{-1}T_i^0V_*^{-1}Q_{V_*}Y$$
$$= \frac{1}{k}\Sigma\frac{p_i}{s_i}e'V_*^{-1}T_i^0V_*^{-1}e$$
$$= \Sigma p_i[\frac{e'V_*^{-1}T_i^0V_*^{-1}e}{ks_i}] \quad (3.5.4)$$

and that of α_i is

$$\overline{\alpha_{i\cdot}} = \frac{1}{ks_i}e'V_*^{-1}T_i^0V_*^{-1}e\ . \quad (3.5.5)$$

In the unweighted case, i.e when

$$\Lambda_0 = I,$$

then

$$tr(B\Lambda_*)^2 = tr(B\Lambda_*U'AU_*\Lambda). \quad (3.5.7)$$

However this may not hold in general except in some special cases (see P. S. R. S. Rao & Chaubey (1978)).

In the next chapter, we consider Σ with an intraclass correlation structure and apply the results obtained in this section.

Chapter 4

Intraclass Correlation Model

4.1. Introduction

The intraclass correlation model is given by

$$Y = X\beta + \varepsilon, \tag{4.1.1}$$

where

$$E(\varepsilon) = 0 \tag{4.1.2}$$

and

$$D(Y) = D(\varepsilon) = \Sigma = \sum_{i=1}^{2} \alpha_i T_i. \tag{4.1.3}$$

This can be seen to be a special case (see Chaubey (1978)) of model (3.1.1) where

$$k = 1,\ U_1 = I_{n\times n},\ d = 2$$

and

$$T_1 = I_{n\times n},\ T_2 = E - I_{n\times n},$$

where

$$E = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times n}$$

Here $T_i^* = T_i$ for $i = 1,2$, hence

$$
\begin{aligned}
V_* = \Lambda_0 &= \sum_{i=1}^{2} \delta_i T_i \\
&= \delta_1 T_1 + \delta_2 T_2 \\
&= \delta_1 I + \delta_2 (E - I) \\
&= \left(\delta_1 - \delta_2\right) I + \delta_2 E
\end{aligned} \tag{4.1.4}
$$

and

$$
D = V_* B V_*, \tag{4.1.5}
$$

with given values of δ_1 and δ_2 such that V_* is non singular for

$$
\begin{aligned}
B &= \frac{p_1}{s_1} I + \frac{p_2}{s_2}(E - I) \\
&= \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) I + \frac{p_2}{s_2} E.
\end{aligned} \tag{4.1.6}
$$

Now we verify that in the case of intraclass correlation models the expression in (3.4.7) does not hold, i.e

$$
tr(BV_*)^2 = trAV_*BV_*. \tag{4.1.7}
$$

From (4.1.4) and (4.1.6) we get

$$
\begin{aligned}
BV_* &= \left(\delta_1 - \delta_2\right)\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) I + n\delta_2 \frac{p_2}{s_2} E + \left(\delta_1 - \delta_2\right) \frac{p_2}{s_2} E + \delta_2 \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) E \\
&= \alpha_1 I + \alpha_2 E,
\end{aligned} \tag{4.1.8}
$$

where

$$
\alpha_1 = \left(\delta_1 - \delta_2\right)\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) \tag{4.1.9}
$$

and

$$
\begin{aligned}
\alpha_2 &= n\delta_2 \frac{p_2}{s_2} + \left(\delta_1 - \delta_2\right) \frac{p_2}{s_2} + \delta_2 \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) \\
&= \left(\delta_1 - \delta_2\right) \frac{p_2}{s_2} + \delta_2 \left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}.
\end{aligned} \tag{4.1.10}
$$

We get

$$(BV_*)^2 = \alpha_1^2 I + \alpha_2^2 E^2 + 2\alpha_1\alpha_2 E$$
$$= \alpha_1^2 I + n\alpha_2^2 E + 2\alpha_1\alpha_2 E \tag{4.1.11}$$

and

$$tr(BV_*)^2 = n\alpha_1^2 + n^2\alpha_2^2 + 2n\alpha_1\alpha_2$$
$$= c_1\delta_1^2 + c_2\delta_2^2 + c_{12}\delta_1\delta_2. \tag{4.1.12}$$

For the co-efficient of δ_1^2 in $tr(BV_*)^2$ we get

$$\begin{aligned} c_1 &= n\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)^2 + n^2 * \frac{p_2^2}{s_2^2} + 2n * \frac{p_2}{s_2}\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) \\ &= \frac{np_1^2}{s_1^2} - \frac{2np_1p_2}{s_1s_2} + \frac{n^2p_2^2}{s_2^2} + \frac{np_2^2}{s_2^2} + \frac{2np_1p_2}{s_1s_2} - \frac{2np_2^2}{s_2^2} \\ &= \frac{np_1^2}{s_1^2} + \frac{p_2^2}{s_2^2}(n^2 + n - 2n) \\ &= \frac{np_1^2}{n^2} + \frac{p_2^2}{(n^2-n)^2}(n^2 - n) \\ &= \frac{p_1^2}{n} + \frac{p_2^2}{(n^2-n)}. \end{aligned} \tag{4.1.13}$$

The co-efficient of δ_2^2 in $tr(BV_*)^2$ is

$$\begin{aligned} c_2 &= n\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)^2 + n^2\left[\frac{p_2^2}{s_2^2} + \left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}^2 - \frac{2p_2}{s_2}\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}\right] \\ &\quad + 2n\left[\frac{p_2}{s_2}\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) - \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}\right] \\ &= \frac{p_1^2}{s_1^2}(n + n^2 - 2n) + \frac{p_2^2}{s_2^2}\{n + n^2 + (n-1)^2n^2 - 2n^2(n-1) - 2n + 2n(n-1)\} \\ &\quad + \frac{p_1p_2}{s_1s_2}\{-2n + 2n^2(n-1) - 2n^2 + 2n - 2n(n-1) + 2n\} \\ &= \frac{p_1^2}{s_1^2}n(n-1) + \frac{p_2^2}{s_2^2}\left[-3n + n^2\{(n-1)^2 - 2(n-1) + 1\} + 2n^2\right] \\ &\quad + \frac{p_1p_2}{s_1s_2}(2n^3 - 2n^2 - 2n^2 - 2n^2 + 2n + 2n), \end{aligned}$$

i.e

$$c_2 = \frac{p_1^2}{s_1^2}n(n-1) + \frac{p_2^2}{s_2^2}\{-3n + n^2(n-2)^2 + 2n^2\} + \frac{p_1p_2}{s_1s_2}(2n^3 - 6n^2 + 4n)$$
$$= \frac{p_1^2(n-1)}{n} + \frac{p_2^2}{n^2(n-1)^2}(-3n + n^4 - 4n^3 + 4n^2 + 2n^2) + \frac{2np_1p_2}{s_1s_2}(n^2 - 3n + 2)$$
$$= \frac{p_1^2(n-1)}{n} + \frac{np_2^2}{n^2(n-1)^2}(-3 + n^3 - 2n^2 + 6n) + \frac{2np_1p_2}{n^2(n-1)}(n-1)(n-2)$$
$$= \frac{p_1^2(n-1)}{n} + \frac{2p_1p_2}{n}(n-2) + \frac{p_2^2}{n(n-1)^2}(n^3 - 4n^2 + 6n - 3)$$
$$= \frac{p_1^2(n-1)}{n} + \frac{2p_1p_2}{n}(n-2) + \frac{p_2^2}{n(n-1)}(n^2 - 3n + 3). \tag{4.1.14}$$

The co-efficient of $\delta_1\delta_2$ in $tr(BV_*)^2$ is

$$c_{12} = n\left\{-2\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)^2\right\} - \frac{2n^2p_2^2}{s_2^2} + \frac{2n^2p_2}{s_2}\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}$$
$$+ 2n\left[-\frac{2p_2}{s_2}\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) + \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}\right]$$
$$= -2n[\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)^2 + \frac{np_2^2}{s_2^2} - \frac{np_2}{s_2}\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}$$
$$+ \frac{2p_2}{s_2}\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) - \left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right)\left\{\frac{p_1}{s_1} + (n-1)\frac{p_2}{s_2}\right\}]$$
$$= -2n[\frac{p_1^2}{s_1^2} + \frac{p_2^2}{s_2^2} - \frac{2p_1p_2}{s_1s_2} - \frac{p_1^2}{s_1^2} + \frac{np_2^2}{s_2^2} - \frac{np_1p_2}{s_1s_2} - \frac{n^2p_2^2}{s_2^2} + \frac{2p_1p_2}{s_1s_2} +$$
$$\frac{np_2^2}{s_2^2} - (n-1)\frac{p_1p_2}{s_1s_2} - \frac{2p_2^2}{s_2^2} + \frac{p_1p_2}{s_1s_2} + (n-1)\frac{p_2^2}{s_2^2}]$$
$$= -2n\left[\frac{p_2^2}{s_2^2}(1 + n - n^2 + n - 2 + n - 1) + \frac{p_1p_2}{s_1s_2}(1 - n + 1 + 2 - n - 2)\right]$$
$$= -2n\left[\frac{p_2^2}{s_2^2}(-n^2 + 3n - 2) - \frac{2p_1p_2}{s_1s_2}(n-1)\right]$$
$$= 2n\left[\frac{p_2^2}{s_2^2}(n-1)(n-2) + \frac{2p_1p_2}{s_1s_2}(n-1)\right]$$
$$= 2n\left[\frac{p_2^2(n-1)(n-2)}{n^2(n-1)^2} + \frac{2p_1p_2(n-1)}{n * n(n-1)}\right]$$
$$= 2\left[\frac{p_2^2(n-2)}{n(n-1)} + \frac{2p_1p_2}{n}\right]. \tag{4.1.15}$$

Now find $trAV_*BV_*$, we get

$$
\begin{aligned}
V_*BV_* &= \left[\left(\delta_1 - \delta_2\right)I + \delta_2 E\right]BV_* \\
&= \alpha_1\left(\delta_1 - \delta_2\right)I + \alpha_2\left(\delta_1 - \delta_2\right)E + \alpha_1\delta_2 E + \alpha_2\,\delta_2 E^2 \\
&= \alpha_1\left(\delta_1 - \delta_2\right)I + \left\{\alpha_2\left(\delta_1 - \delta_2\right) + \alpha_1\delta_2 + n\alpha_2\,\delta_2\right\}E \\
&= \beta_1 I + \beta_2 E
\end{aligned} \tag{4.1.16}
$$

where

$$
\begin{aligned}
\beta_1 &= \alpha_1\left(\delta_1 - \delta_2\right) \\
\beta_2 &= \alpha_2\left(\delta_1 - \delta_2\right) + \alpha_1\delta_2 + n\alpha_2\,\delta_2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
trAT_1 &= trAI = p_1 \\
trAT_2 &= trA(E-I) = p_2 \\
&\Rightarrow trAE = p_1 + p_2.
\end{aligned}
$$

Here

$$
\begin{aligned}
trAV_*BV_* &= \beta_1 trA + \beta_2 trAE \\
&= \beta_1 p_1 + (p_1 + p_2)\beta_2 \\
&= d_1\delta_1^2 + d_2\delta_2^2 + d_{12}\delta_1\delta_2,
\end{aligned} \tag{4.1.17}
$$

and

$$
\begin{aligned}
d_1 &= \text{Co-efficient of } \delta_1^2 \text{ in } trAV_*BV_* \\
&= p_1\left(\frac{p_1}{s_1} - \frac{p_2}{s_2}\right) + (p_1 + p_2)\left(\frac{p_2}{s_2}\right) \\
&= \frac{p_1^2}{s_1} - \frac{p_1p_2}{s_2} + \frac{p_1p_2}{s_2} + \frac{p_2^2}{s_2} \\
&= \frac{p_1^2}{n} + \frac{p_2^2}{n(n-1)} \\
&= c_1,
\end{aligned} \tag{4.1.18}
$$

while

$$
\begin{aligned}
d_2 = &\ \text{Co-efficient of } \delta_2^2 \text{ in } trAV_*BV_* \\
= &\ p_1\left(\frac{p_1}{s_1}-\frac{p_2}{s_2}\right)+(p_1+p_2)[\frac{p_2}{s_2}-\left\{\frac{p_1}{s_1}+(n-1)\frac{p_2}{s_2}\right\}- \\
&\left(\frac{p_1}{s_1}-\frac{p_2}{s_2}\right)-\frac{np_2}{s_2}+\frac{np_1}{s_1}+\frac{n^2p_2}{s_2}-\frac{np_2}{s_2}] \\
= &\ \frac{p_1^2}{s_1}-\frac{p_1p_2}{s_2}+\frac{p_1p_2}{s_2}-\frac{p_1^2}{s_1}-\frac{np_1p_2}{s_2}+\frac{p_1p_2}{s_2}-\frac{p_1^2}{s_1}+\frac{p_1p_2}{s_2}- \\
&\frac{np_1p_2}{s_2}+\frac{np_1^2}{s_1}+\frac{n^2p_1p_2}{s_2}-\frac{np_1p_2}{s_2}+\frac{p_2^2}{s_2}-\frac{p_1p_2}{s_1}-\frac{np_2^2}{s_2}+ \\
&\frac{p_2^2}{s_2}-\frac{p_1p_2}{s_1}+\frac{p_2^2}{s_2}-\frac{np_2^2}{s_2}+\frac{np_1p_2}{s_1}+\frac{n^2p_2^2}{s_2}-\frac{np_2^2}{s_2} \\
= &\ \frac{p_1p_2}{s_2}(-n+1+1-n+n^2-n)+\frac{p_1p_2}{s_1}(-1-1+n)+ \\
&\frac{p_2^2}{s_2}(1-n+1+1-n+n^2-n)+\frac{p_1^2}{s_1}(-1+n) \\
= &\ \frac{p_1p_2}{s_2}(n^2-3n+2)+\frac{p_1p_2}{s_1}(n-2)+\frac{p_2^2}{s_2}(n^2-3n+3)+\frac{p_1^2}{s_1}(n-1) \\
= &\ \frac{p_1p_2}{n(n-1)}(n-1)(n-2)+\frac{p_1p_2}{n}(n-2)+\frac{p_2^2}{n(n-1)}(n^2-3n+3)+\frac{p_1^2}{n}(n-1) \\
= &\ \frac{p_1^2}{n}(n-1)+\frac{2p_1p_2}{n}(n-2)+\frac{p_2^2}{n(n-1)}(n^2-3n+3) \\
= &\ c_2.
\end{aligned}
\tag{4.1.19}
$$

Finally

$$
\begin{aligned}
d_{12} = &\ p_1 \text{ co-efficient of } \delta_1\delta_2 \text{ in } \beta_1+(p_1+p_2) \text{ co-efficient of } \delta_1\delta_2 \text{ in } \beta_2 \\
= &\ p_1\left[-2\left(\frac{p_1}{s_1}-\frac{p_2}{s_2}\right)\right]+(p_1+p_2)[-\frac{2p_2}{s_2}+\left\{\frac{p_1}{s_1}+(n-1)\frac{p_2}{s_2}\right\}+ \\
&\left(\frac{p_1}{s_1}-\frac{p_2}{s_2}\right)+\frac{np_2}{s_2}] \\
= &\ p_1\left[-\frac{2p_1}{s_1}+\frac{2p_2}{s_2}\right]-\frac{2p_1p_2}{s_2}+\frac{p_1^2}{s_1}+\frac{np_1p_2}{s_2}-\frac{p_1p_2}{s_2}+\frac{p_1^2}{s_1}-\frac{p_1p_2}{s_2}+ \\
&\frac{np_1p_2}{s_2}-\frac{2p_2^2}{s_2}+\frac{p_1p_2}{s_1}+\frac{np_2^2}{s_2}-\frac{p_2^2}{s_2}+\frac{p_1p_2}{s_1}-\frac{p_2^2}{s_2}+\frac{np_2^2}{s_2} \\
= &\ \frac{p_1^2}{s_1}(-2+1+1)+\frac{p_2^2}{s_2}(-2+n-1-1+n)+\frac{p_1p_2}{s_1}(1+1)+ \\
&\frac{p_1p_2}{s_2}(2-2+n-1-1+n) \\
= &\ \frac{p_2^2}{n(n-1)}(2n-4)+\frac{2p_1p_2}{n}+\frac{p_1p_2}{n(n-1)}(2n-2) \\
= &\ \frac{2p_2^2}{n(n-1)}(n-2)+\frac{4p_1p_2}{n}
\end{aligned}
$$

That means that

$$d_{12} = c_{12}. \quad (4.1.20)$$

Thus from equations (4.1.18), (4.1.19), (4.1.20) , we can say that

$$tr(BV_*)^2 = trAV_*BV_*$$

and therefore using Theorem 2.1 we get the weighted MINQUE of $p'\alpha$ as

$$\begin{aligned}\widehat{\widehat{r_*}} &= \lambda' u \\ &= p'Z^{-1}u, \quad (4.1.21)\end{aligned}$$

where the $(i,j)th$ element of Z, $Z_{ij} = tr(Q'_{V_*}V_*^{-1}T_iV_*^{-1}Q_{V_*}T_j)$ and $u_j = e'V_*^{-1}T_jV_*^{-1}e$. Using Theorem 2.2 we get the weighted MINQE of $\sum p_i\alpha_i$ as

$$\begin{aligned}Y'AY &= e'Be \\ &= \Sigma p_i[\frac{e'T_ie}{s_i}] \quad (4.1.22)\end{aligned}$$

and the weighted MINQE of α_i is

$$\overline{\overline{\alpha_{i^*}}} = \frac{e'T_ie}{s_i}.$$

4.2. Numerical Illustration

4.2.1. The Data and The Model

The following data is taken from Wiorkowski (1975) and is about the biological activity of adenosine triphosphate (ATP) in red blood cells, measured in the parents and male progeny of fourteen randomly selected families. The objective of this study was to find if the ATP levels conformed to a simple genetic model which would be equivalent to the observed trial being controlled by a large number of statistically independent loci, each making a small additive contribution to the final observed level of ATP. The model used is given by,

$$y_{ij} = \beta_0 + \beta_1x_{1i} + \beta_2x_{2i} + \varepsilon_{ij}, \quad (4.2.1)$$

$j = 1, 2, ..., n_i$, $i = 1, ..., k$, where y_{ij} denotes the ATP level of the jth progeny in the ith family, x_{1i} denotes the ATP level of the father and x_{2i} denotes that of the mother in the ith family. Here, number of families, $k = 14$. The observations in a family are supposed to have an intraclass structure, *i.e.*

$$\Sigma = \sigma^2[(1-\rho)I + \rho E] \tag{4.2.1}$$

where ρ denotes the intraclass correlation coefficient. It is of interest to find if the intraclass correlation coeeficient is zero and contribution of the mother and father is equal, i.e., we may be interested in testing $\rho = 0$ and $\beta_1 = 0.5, \beta_2 = 0.5$. The data is provided in Table 4.1 below.

Table 4.1. Adenosine Tri Phosphate Activity Level

Family	Father	Mother	Male Progeny				
1	3.72	4.43	4.18	4.81			
2	4.54	3.79	4.72				
3	5.05	4.66	4.98	5.03	5.16		
4	4.10	5.42	5.30	4.48	4.85		
5	4.26	4.39	4.87	3.99	4.19	4.28	5.15
6	4.09	5.29	4.74	4.10			
7	4.83	4.99	4.53	4.77	4.77		
8	4.24	4.38	3.72	4.12			
9	5.43	4.73	4.65	4.62			
10	5.23	5.34	5.83	6.03			
11	4.56	5.29	4.86	5.58	5.99		
12	5.16	4.71	5.44	4.34	5.43		
13	3.77	5.13	4.70	5.00	4.63		
14	4.15	4.18	4.82	4.14			

4.2.2. Calculations and Results

In the model above, the MINQUE and MINQE can be obtained with the parametrization

$$\alpha_1 = \sigma^2, \alpha_2 = \sigma^2 \rho$$

as described in previous sections. We used SAS (Statistical Analysis System) IML procedures for matrix computations. We confirm the numerical results obtained in Chaubey (1980b). The covariance matrix of the estimators is obtained using the formulae given in Chaubey (1980b). These are given in Table 4.2 and those for unweighted MINQE are given in Table 4.3.

Table 4.2 Unweighted MINQUE and Their Estimated Covariance Matrix

$$\begin{bmatrix} \widehat{\alpha_1} \\ \widehat{\alpha_2} \\ \widehat{\beta_0} \\ \widehat{\beta_1} \\ \widehat{\beta_2} \end{bmatrix} = \begin{bmatrix} 0.217417 \\ 0.0292862 \\ 0.3929127 \\ 0.4084862 \\ 0.5343059 \end{bmatrix}$$

$$\hat{Cov} \begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \\ \widehat{\beta_2} \\ \widehat{\alpha_1} \\ \widehat{\alpha_2} \end{bmatrix} = \begin{bmatrix} 1.3861233 & -0.115021 & -0.179056 & 0 & 0 \\ -0.115021 & 0.0272305 & -0.001588 & 0 & 0 \\ -0.179056 & -0.001588 & 0.0387455 & 0 & 0 \\ 0 & 0 & 0 & 0.0031423 & 0.0012038 \\ 0 & 0 & 0 & 0.0012038 & 0.0024708 \end{bmatrix}$$

Table 4.3 Unweighted MINQE Their Estimated Covariance Matrix

$$\begin{bmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \\ \overline{\beta_0} \\ \overline{\beta_1} \\ \overline{\beta_2} \end{bmatrix} = \begin{bmatrix} 0.1951973 \\ 0.000387 \\ 0.3477224 \\ 0.4061869 \\ 0.5460642 \end{bmatrix}$$

$$\hat{Cov}\begin{bmatrix} \overline{\beta_0} \\ \overline{\beta_1} \\ \overline{\beta_2} \\ \overline{\alpha_1} \\ \overline{\alpha_2} \end{bmatrix} = \begin{bmatrix} 1.0471448 & -0.08654 & -0.135713 & 0 & 0 \\ -0.08654 & 0.0203289 & -0.001025 & 0 & 0 \\ -0.135713 & -0.001025 & 0.0291933 & 0 & 0 \\ 0 & 0 & 0 & 0.0023442 & 0.0002286 \\ 0 & 0 & 0 & 0.0002286 & 0.0014362 \end{bmatrix}$$

To see the effect of weights, we use the above estimates as a priori weights in computing the corresponding weighted estimators. These are given in Tables 4.4 and 4.5.

Table 4.4 Weighted MINQUE and Their Estimated Covatiance Matrix (Weights $\widehat{\alpha_1}$, $\widehat{\alpha_2}$)

$$\begin{bmatrix} \widetilde{\alpha_1} \\ \widetilde{\alpha_2} \\ \widetilde{\beta_0} \\ \widetilde{\beta_1} \\ \widetilde{\beta_2} \end{bmatrix} = \begin{bmatrix} 0.2171987 \\ 0.0278557 \\ 0.3905864 \\ 0.4083858 \\ 0.5348881 \end{bmatrix}$$

$$\hat{Cov}\begin{bmatrix} \widetilde{\beta_0} \\ \widetilde{\beta_1} \\ \widetilde{\beta_2} \\ \widetilde{\alpha_1} \\ \widetilde{\alpha_2} \end{bmatrix} = \begin{bmatrix} 1.3747989 & -0.114033 & -0.177639 & 0 & 0 \\ -0.114033 & 0.0269847 & -0.001562 & 0 & 0 \\ -0.177639 & -0.001562 & 0.03842490 & 0 & 0 \\ 0 & 0 & 0 & 0.0031094 & 0.0011334 \\ 0 & 0 & 0 & 0.0011334 & 0.0024038 \end{bmatrix}$$

Table 4.5 MINQE and their estimated variances and covariances (Weights $\overline{\alpha_1}$, $\overline{\alpha_2}$)

$$\begin{bmatrix} \widehat{\alpha_1} \\ \widehat{\alpha_2} \\ \widehat{\beta_0} \\ \widehat{\beta_1} \\ \widehat{\beta_2} \end{bmatrix} = \begin{bmatrix} 0.1951973 \\ 0.0003872 \\ 0.3477227 \\ 0.4061869 \\ 0.5460641 \end{bmatrix}$$

$$\text{Cov}\begin{bmatrix} \widehat{\beta_0} \\ \widehat{\beta_1} \\ \widehat{\beta_2} \\ \widehat{\alpha_1} \\ \widehat{\alpha_2} \end{bmatrix} = \begin{bmatrix} 1.0471463 & -0.08654 & -0.135713 & 0 & 0 \\ -0.08654 & 0.020329 & -0.001025 & 0 & 0 \\ -0.135713 & -0.001025 & 0.0291933 & 0 & 0 \\ 0 & 0 & 0 & 0.0023441 & 0.0002283 \\ 0 & 0 & 0 & 0.0002283 & 0.0014355 \end{bmatrix}$$

4.3. Summary and Discussion

This thesis reviews the principle of MINQUE and MINQE for estimation of variance and covariance components and fills in the gap by providing the solution in the weighted case in general for estimating the distinct elements of a covaiance matrix with some *a priori* information. The estimates are computed for a real example, which illustrates that the substitution of proper quantities in the original MINQUE/MINQE principles without any weights may provide the solution in some special cases.

This example also illustrates that the principle of MINQE and MINQUE present credible alternatives to numerically cumbersome maximum likelihood method. In the computations of Wiorkowski (1975), to have stable search for the maxima of the likelihood function, the observations from family 5 had to be deleted. However, for MINQUE and MINQE, such problems did not arise in the weighted and unweighted case. In the present example, weighted and unweighted estimators are not very different from each other. However, to see the effects of weights, in general, it would be a good idea to consider iterative estimators by using the new estimators as *a priori* values in the next iteration. The property of unbisedness may be lost in this scheme and a thorough investigation may be necessary.

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