The Readability of the Mathematics Textbook: with Special Reference to the Mature Student

Astrid Defence

A Project

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfilment of the Requirements
for the Degree of Master at
Concordia University
Montreal, Quebec, Canada

September 1994

0-612-44873-8

ABSTRACT

The Readability of the Mathematics Textbook: with Special Reference to the Mature Student

Astrid Defence

The readability of their mathematics textbook is of prime importance to the College or University level students. These students are no longer taught mathematics concepts as they were at school in a teacher-controlled environment. They must be able to confidently turn to their textbook for meaningful explanations of the mathematics they are learning. However many students find reading their mathematics textbook a fruitless task. They were never taught how to read mathematical text or indeed that reading such text needed special reading skills. Consequently they find that they do not understand the language it is written in and are constantly faced with vocabulary that is at the same time familiar yet has taken on a different meaning. Much research has been conducted into the problems with the ambiguities between Mathematical English and Ordinary English at the school level, but this study is an investigation into the ambiguities in language encountered by students who have returned to studies after several years in the work force. Do their added years of experience make their reading of mathematical text easier or harder? How much teacher input is still necessary? Can they learn the mathematics from the textbook alone? It is in an attempt to seek answers to these questions that this study was undertaken.

TABLE OF CONTENTS

Chapter One: Survey on the Research Literature on Problems with Reading Mathematical Texts...... p. 1

Chapter Two: Methodology of an Investigation into the Understanding of Mathematical Text by the Older Student.......................... p. 12

Chapter Three: Analysis of the Interviews

Part 1:...................................... p. 21
Part 2:...................................... p. 28
Part 3:...................................... p. 36
Part 4:...................................... p. 43
Part 5:...................................... p. 47

Chapter Four: Summary and Conclusions .... p. 57

References: .... p. 66

APPENDICES

A: Part 1: pp. 1 - 17
B: Part 2: pp. 18 - 34
C: Part 3: pp. 35 - 53
D: Part 4: pp. 54 - 64
E: Part 5: pp. 65 - 97
F: pp. 98 - 102
G: pp. 103 - 113

CHAPTER 1

SURVEY ON THE RESEARCH LITERATURE ON PROBLEMS WITH READING MATHEMATICAL TEXTS

"Take care of the sense and the words will take care of themselves."

. . . thus said the Duchess in L. Carroll's, Alice in Wonderland, as the implied moral behind the words,"Ah, well! It means much the same thing, . . ".

Are we to take this seriously in the context of students' reading of mathematical text? Are the words merely the trees that block our view of the forest? To some extent I would have to agree with the Duchess. A concentration on the linguistic and syntactical differences between M.E (Mathematical English) and O.E. (Ordinary English) is not to my mind the main issue in this topic. Teaching the meaning behind the words, the Duchess's, "sense" of the words is the more important task at hand. On the same theme of "moral" implications, the Duchess presents us with another sentence of quite remarable astuteness,

"I quite agree with you," said the duchess; " and the moral of that is - 'Be what you would seem to be' - or, if you'd like it put more simply- 'Never imagine yourself not to be otherwise than what it might appear to others that what you were or might have been was not otherwise than what you had been would have appeared to them to be otherwise.' "

To many students, reading a paragraph of mathematics text is as meaningful at first reading as the above sentence is to the reader of L. Carroll! Yet the meaning of this sentence comes across despite the way the words are used to express it! Of course it is not "more simply put" than the previous comment, "Be what you would seem to be" - at first glance. But does it not attempt to **explain** the deeper meaning behind these fewer words, and as such does it not simplify the thoughts expressed? There is something familiar in the above statements to the mathematical explanations that were encountered in this investigation.

Which is the more simply put?

"The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain",

or, "The range of f is $\{f(x) \mid x \in A\}$?

Does the fuller explanation not at least presume to explain more fully the meaning of the range of a function? Theoretically one would suppose so, however as will be seen, unfamiliarity with any of the above symbolism was a major stumbling block in the understanding of either form of definition pertaining to functions.

My research into the literature that had already been published on this topic of the "Language of Mathematics", if you will, led me to read (amongst others) an article by J. L. Austin and A. G. Howson (1979). In this article they themselves offered a review of the then current publications on this issue focussed on the "interaction of language and mathematics education". Their premise being, "that mathematics education centres upon attempts to understand how mathematics is created, taught and learned most effectively", their work was subdivided into three sections:

(i) the language of the learner,
(ii) the language of the teacher,
and **(iii) the language of mathematics.**

The first section **(i) the language of the learner,** was not an area to which I had paid a great deal of concern, since I was primarily interested in the interaction between the student and the text itself. However the first language of the student vis à vis his reading of a mathematical text in a language already foreign to him is a field for research in its own right. The reading comprehension ability of a student in his own language is another aspect that has to be born in mind when considering his ability to understanding mathematical text, particularly when reading word problems.

I examined more extensively the second section **(ii), the language of the teacher,** since that section necessarily included the language of the text. It was soon seen to overlap however with that of the first part, the language of the learner. This section was further sub-divided into three areas of concentration, ***(a)* readability, *(b)* removing reading difficulties**, and ***(c)* the language in the classroom.**

Under the sub-title, ***(a)* readability**, we read that tests originally devised to examine the readability of English prose material were being used in the classroom to examine the readability of mathematical "prose". These tests however were subsequently dismissed by mathematicians, (e.g. Kane, 1968, 1970) on the premise that one should not and could not compare M.E. with O.E., that they were essentially quite different. The reasons commonly quoted were that,

i) letter, word and syntactical redundancies differ,

ii) names of mathematical objects usually have a single denotation, unlike nouns in O.E.

iii) adjectives are unimportant in M.E.

and iv) grammar and syntax in M.E. are less flexible than in O.E.

It had not been my intention at that time to address the validity, or otherwise, of these statements, merely to record them.

As for ***(b)* removing reading difficulties**, recommendations came in many guises; use simpler and shorter sentences, repeat often the key words and phrases, introduce only a few new words at a time - to name but a few. Reactions to these recommendations have ranged from their acceptance to their dismissal as producing "artificial formulations". (Nesher, and Teubal, 1975). Counter measures have included the offering of special reading instruction to help students in their understanding of word problems. A third alternative to overcoming difficulties in reading text is to offer more pictorial representations. Again the validity of this recommendation is open to evaluation. Of particular note in this section, and of particular relevance to my investigation, was the claim made by Austin and Howson in reference to older students,

"In later years, the student's difficulties usually arise not so much from an inadequate grasp of the (language of instruction), but from the peculiar nature of the (language of mathematics)."

Or, one could add, the peculiar nature of the objects about which this language is speaking?

Research into *(c)* **the language in the classroom**, has been the subject of many articles by many authors from many approaches:

i) methods of observing and recording verbal classroom interaction - Proctor and Wright (1961), Flanders (1960, 1976), Amidon and Hunter (1966) and Fey (1969).

ii) use of language in learning and in particular in the classroom - Barnes (1969), Britton (1970), Henry (1971), Creber (1972), and Goeppert (1977) . . to mention but two of the approaches enumerated by Austin and Howson as part of their bibliography of research on the "Interaction of Language and Mathematics Education".

The distinctions between these above-named three areas became blurry upon reading the work of H. Shuard and A. Rothery (1984), "Children Reading Mathematics". Their Table of Contents included such topics as, "Reading and Mathematics", "Vocabulary and Syntax", "The Visual Appearance of text", and "Improving the Reading Ability of the Reader", among its subtitles. The first 3 Chapters of this book, (1) Reading and Mathematics, (2) Characteristics of Mathematical Writing and (3) Vocabulary and Syntax were the ones I found most relevant to the main theme of my previous essay since they expressed views that were found to be unacceptable to the **active** reading that is the more desirable mode of reading in which we would like to see our students enrol. A **passive** view of reading was the interpretation attributed by T. Wing, (more of whom later), to their statement, "In mathematics, for us, reading is 'getting the meaning from the page' ." Meaning is inherent in the text and it is for us to decode it? Their text-centered definition of readability as the "sum total of all those elements within a given piece of printed material that affects the success which a group of readers have with it", is seen to be in direct conflict with the student-centered viewpoint of Austin and Howson who had described readability as "the matching of reader with material." This definition was the main theme of their first chapter.

In Chapter (2), they expanded upon the purposes for which authors use mathematical writing, e.g

a) teaching concepts, principles, skills and problem solving strategies,
b) giving practice in their use,

c) providing revision of a) and b),

d) testing for acquisition of a),

and e) developing mathematical language.

This was followed by a discussion of their classification of types of mathematical text:

a) exposition of concepts - to be read and digested, but **not** necessarily acted upon,

b) instructions - to carry out the tasks described,

c) examples and exercises - problems solved by the reader on his own,

d) peripheral writing - to be read in a **passive** way,

and e) signals - to guide the reader through the page.

Plus two others that they had trouble categorising:

f) questions - as in exercises and the rhetorical type

and g) worked examples - mostly exposition.

This investigation will show that I was primarily concerned with the effect of type a) on my students - **the exposition of concepts.**

Chapter 3 opened up another whole realm of thought, that of Vocabulary and Syntax. Shuard and Rothery recognised 3 categories of words which have been known to give specific problems to the reader:

(i) words which have the same meaning in M.E. as in O.E.

(ii) words which have meaning only in M.E.

(iii) words which occur in both O.E. and M.E. but which have a different meaning in M.E. from their meaning in O.E.

Any problems in Case (i) are usually overcome by an appropriate and sufficient context which, however, many word problems fail to provide.

Problems in Case (ii) are to be found in the derivation of mathematical terms such as parallelogram, isosceles, equilateral etc. . . , words, that is, of Greek and Latin origin with which the younger student, or even the older, is unfamiliar.

Case (iii) was the one I found most interesting, to the point of reading further material on this aspect as presented by K. Durkin and B. Shire in the chapter, "Lexical ambiguity in mathematical contexts", in their book, "Language in Mathematics Education" (pp. 71 - 84)

Durkin and Shire confronted this problem on 4 fronts:

(i) types of lexical ambiguity in mathematics education,

(ii) examples of ambiguous words commonly used in school mathematics,

(iii) spatial terms in mathematical descriptions,

and (iv) children's understanding of spatial terms in mathematics.

My preferred field of exploration is in (ii). Mathematicians take for granted the mathematical meanings of words when dealing with mathematical concepts that they often forget that to the layman a quite different meaning, one that is relevant to his experience, is more likely to be the meaning extrapolated. This investigation supports these findings.

On a more positive level, Durkin and Shire not only pointed out the difficulties but also offered to the teacher ways by which he could overcome them. The teacher should firstly be aware of the problem, and secondly confront it, even to the extent of exploiting it. A child's first encounter with the word "volume", for example, is not going to be one of a geometric connotation! Nor will the word "mean" or even "operation". And to speak of a "BIG" number or a "high" number to a little child is equally likely to mean precisely that as shown. Even among adults, or I might say **especially** among adults, who have more experience with the usage of language, these types of ambiguities abound, as my interviews with my older students will show.

Difficulties of syntax also play a significant role in the layman's reading of mathematical text, but there we enter the field of linguistics, a vast area of research that is beyond the scope of most mathematical research into these problems with reading a mathematical text. Suffice it to quote Shuard and Rothery's summary remarks that "the importance of syntax is as great as that of vocabulary, and an informed assessment of the complexity of a piece of writing is a vital step in evaluating the readability of the text."

As already stated, all of these discussions on the readability of mathematical text and the difficulties encountered therein can be viewed from one of two viewpoints, that of the "activist" or that of the "passivist" - in a strictly mathematical context, of course! Do we adhere to the school of thought that believes that the reader should become actively involved in his reading or do we believe in a "sit back and absorb attitude" on the part of the reader? If we believe in the former, does it become necessary then for students to be taught how to read mathematical text? Is this possible? How can this be done? These questions and others were addressed in the two articles discussed in my previous paper, that of T. Wing (1985), "Reading Children Reading Mathematics", a critique of the above mentioned work by Shuard and Rothery, and that of Zofia Krygowska (1969).

Wing proposed that reading should be "an activity in which a reader puts meaning on to a page", that meaning should become "the creation of the reader", and that the role of the text should "no longer be taken for granted". He claimed support for his beliefs in the work of Valerie Walkerdine (1982) in which she argued that the reader is forced "to concentrate on internal (metonymic) relations within statements", rather than being offered the "possibilities of external (metaphoric) meaning". Wing contended that his active perception of the reading of text invited the reading to provide the student with "an opportunity to exploit (his) creativity".

Why not let the reader supply his own interpretation on what he reads? My own findings revealed that, whether it was intentional or not, that wa precisely what the reader did in any case. Each reader interpreted what he read according to his own background in mathematics or his work experience. So where does the meaning of the text reside? On the page or in the head of the reader? To me the meaning of the text originally resides in the head of the **author** of the text, a third party in this whole affair who does not appear to get much recognition. Is the text not the medium by which the writer hopes to transmit his ideas? And is the problem not that the writer, being a sophisticated mathematician, sees mathematics in all of its symbolic sophistication yet is obliged to translate this sophistication into a language comprehensible to the layman? And if he doesn't then the layman has difficulty understanding him? To me this would appear to be the difference between what students consider a "good" mathematics text and that which they consider "bad". Is it easy to understand, is it written in a language they, as laymen, can understand or is it written for mathematicians? The author of the text cannot, of course, omit all sophistication and what we are finally looking for is a balance between explanations which are understandable to the layman yet which introduce him to the symbolism and conciseness of expression of the mathematical language.

This very question was addressed in the article by Krygowska. She was a very strong believer in an active approach to the reading of text but, to that end, she believed that the students have to be taught how to read mathematical text. She found that her student under observation, "savait se servir au cours de la lecture mathématique d'une technique d'analyse du texte adaptée à ses possibilités et à ses besoins personnels". That is, a person's reading and understanding of a mathematics text will necessarily depend on his mathematics background, his experiences and "sa rapidité intellectuelle". If this is the case then it is not to be found surprising that each reader will understand what he is reading differently from another. But surely this is not what is intended by the author of the text? He has one concept in mind and he would like to transmit this concept to the reader, not a myriad of interpretations of this concept. Students then have to be able to get "behind" the words, or as Krygowska stated it, "la vitrine formelle du texte". And this is the area for instruction. The reading of a mathematical text should never be reduced "à la lecture 'linguistique' de ce texte".

Her article centered around her investigation into students' response to the statement, "Homothétie de centre O et de rapport s (s étant un nombre non nul) est une transformation $X \rightarrow X'$ du plan tel que $\overrightarrow{OX'} = s\,\overrightarrow{OX}$". The students' initial understanding of this statement was found to be lacking in substance. "Elles sont absolument passives au cours de cette lecture, elles ne savent pas ce qu'on doit faire pour comprendre". It was only after the teacher broke this statement down into graspable and actively seen fragments that the students were able to grasp its unified meaning. Subsequently it was not until the students became actively involved in the interpretation of this statement, under the guidance of their teacher, that it made any sense.

There are two key issues at stake now, the activity of the students **and** the guidance of the teacher. The teacher, after all, is more likely to interpret the text in the spirit in which it was written, especially if it is highly symbolic as this text was, and the teacher's role becomes one of directing the students to the "correct" interpretation of the text. This places a heavy burden on the shoulders of the teacher. He has not only to act as intermediary between the words of the text and the many forms of "misinterpretation" for which the many possible ambiguities in the text are responsible, not to mention the personal background of his students, but has also to play the role of "activator" on these passive-minded students.

What happens then when the student is left alone to read the text for himself? And is this not the very "raison d'être" of the mathematics textbook, that it be the medium of instruction for the student without the aid of a teacher present? Krygowska had one major recommendation to make, i.e. that students need to be taught from an early age how to read mathematics textbooks, "qu'ils doivent quitter l'école sécondaire sans l'horreur de l'abracadabra du livre mathématique". Students need to be made aware of the verbal, symbolic and graphic elements in mathematics, its own style and sentence construction, and to appreciate its precision and conciseness. Under such training Krygowska found that her students' reading comprehension ability greatly improved, they developed critical attitudes and saw the need for precision themselves. They even suggested improvements on, how the text should have been presented, to the publishers.

Krygowska made special mention that reading comprehension is not to be confused with "memorisation". Many students are led to believe that they have understood a concept (or concepts) if they can verbally repeat what they have heard or read. This memorisation often tends to play a greater role in mathematics than its comprehension, especially in High School and College level mathematics. There are formulae for area and volume to be memorised in arithmetic, methods of construction in geometry, not to mention the numerous names of shapes that have to be known, formulae again and precise steps to follow in solving equations in algebra, methods of integration and special derivatives in calculus etc, etc. . . all of which must be known to pass the appropriate exams. Fortunately, at the lower levels of instruction, less emphasis is now being placed on memorisation and more on the understanding of a concept by showing its application in a more realistic and meaningful context than was previously done. Although this would appear to be getting away from the main issue of the text itself it is all part and parcel of the whole complexity of this issue.

Krygowska concludes her article with 2 recommendations concerning mathematics textbooks:

(i) Textbooks should use diverse techniques of presentation, including, not only examples and introductions to intuitive thinking, but also that which forces the pupil to work things out for himself.

(ii) Textbooks should be concerned with their presentation but their conception should encompass the transition of the pupil from the brightly coloured books of the Elementary grades through to the more sombre text books of the upper grades.

The interviews that follow will show how my students' perception of their textbooks relate to these recommendations.

In my conclusion I gave my greatest support to the views expressed by Zofia Krygowska, but this necessitated a down playing of the views expressed by the other authors mentioned. Students have to be taught how to read mathematical text with meaning, that is with active mental involvement, not just to passively read the words. We do not witness students' acquiring any understanding of a story in their mother tongue without forming mental images of what they are reading. So reading itself, **with comprehension,** cannot be said to be passive. The problem, as I claimed to see it, was that the mental images that the students were required to form when reading mathematics were usually too far removed from reality to be easily visualised. This latter belief addresses the question raised earlier as to whether it is the "peculiar nature" of the language of mathematics that is the problem or the "peculiar nature" of the subject matter itself.

On re-reading my final comments I find myself now disagreeing with what I wrote about finding "a" meaning from the text as opposed to finding "the" meaning. I believe more strongly now that there is indeed "a" meaning to be derived from reading a mathematics text, i.e. that of the author of the text. This being said, I can no longer agree with T. Wing's interpretation of Shuard and Rothery's "getting the meaning from the page" as being a passive form of reading. Reading of a mathematics text is necessarily text-centered - this does not make it equally necessarily passive.

Even if we believe that meaning is inherent in the text surely our decoding of this meaning is something which is active. Austin and Howson's student-centered "matching of reader with material", must go hand-in-hand with the textual decoding for an entirely active response on the part of the reader. To derive the desired meaning however often entails the intervention of a physically present teacher of mathematics a) to help guide the student through symbolism of which he is unaware, b) to help him find the key words, c) to help him distinguish between what is important and what is peripheral, d) to help him make connections between concepts, and e) to help him relate, or *unrelate* as the case may be, his past experiences with what he is reading, etc. . . In an ideal world students would eventually, by the time they reach College or University level, no longer need this intervention by a teacher but be sufficiently trained in how to read the text that they could be their own teacher and know what questions to ask themselves as they read.

As my studies have found we are far from living in an ideal world.

* * * * * * * * * *

CHAPTER TWO

METHODOLOGY OF AN INVESTIGATION INTO THE UNDERSTANDING OF MATHEMATICAL TEXT BY THE OLDER STUDENT

My study took the form of the presentation of a page and a half of text introducing the student to the concept of function as outlined in his assigned mathematics textbook. This page and a half were not given to the student in their entirety but in 5 parts, each of which conveyed a certain notion of the concept of function. Each part was then submitted to a few questions to determine what precisely the student was able to understand from his reading of the text. (see Appendices A - E).

The level of this exercise was at a University "service course" level, Math 206, a pre-requisite course for the Math 208 and Math 209 of the B. Commerce program. It was an evening course consisting mainly of Commerce students many of whom held regular daytime jobs and many of whom had been out of school for some time. The average age of the class was 28 years.

My students were 3 volunteers from this class, Micheal, Brent and Peter. Micheal and Peter were family men in their 40's. Peter was in Real Estate. Brent was a banker, aged 29. If there is any indication of a more familiar or relaxed relationship between Brent and myself than between the other two, it could be due to the fact that I had been his homeroom teacher 15 years ago when he was in Grade 8!

It was my intention to conduct these interviews on as impartial a plane as possible. I did not wish to influence their understanding of the text by asking leading questions or asking questions in any way that helped get the meaning across. I was first and foremost concerned with their interpretation of the text totally unguided by any external help. Only then did I intervene by drawing their attention to certain aspects of the text of which they had been unaware. The resulting difference in interpretation on their part could then be noted.

The following extract is the page and a half from their mathematics textbook - in its entirety but showing where I divided it into the 5 parts.
The *italics* and **bold** print are as they were in the text. (see Appendix F. p. 98)

"SECTION 4.1
FUNCTIONS"

PART 1:

"The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r.

The number N of bacteria in a culture depends on the time t. If the culture starts with 5000 bacteria and the population doubles every hour, then after t hours the number of bacteria will be $N = (5000)2^t$. This is the rule that connects t and N. For each value of t there is a corresponding value of N, and we say that N is a function of t.

The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no single neat formula that connects w and C, the post office has a rule for determining C when w is known.

In each of these examples there is a rule whereby, given a number (r, t, or w), another number (A, N, or C), is assigned. In each case we say that the second number is a function of the first number."

PART 2:

"A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B."

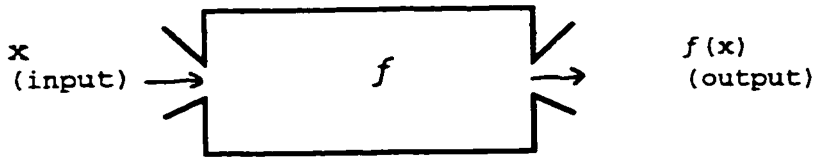
PART 3:

"We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The symbol $f(x)$ is read "f of x", or "f at x" and is called the **value of f at x**, or the **image of x under f**. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is, $\{f(x) \mid x \in A\}$.

The symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. The symbol that represents a number in the *range* of f is called a **dependent variable**. For instance, in the bacteria example, t is the independent variable and N is the dependent variable."

PART 4:

"It is helpful to think of a function as a **machine** (see Figure 1). If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs."



"Figure 1"

"The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the $\sqrt{x}$ key on your calculator is such a function. First you input x into the display. Then you press the key labeled $\sqrt{x}$. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input and the calculator wil indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ will appear in the display, correct to a certain number of decimal places. [Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$]."

PART 5:

"Another way to picture a function is by an arrow diagram as in Figure 2. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on."

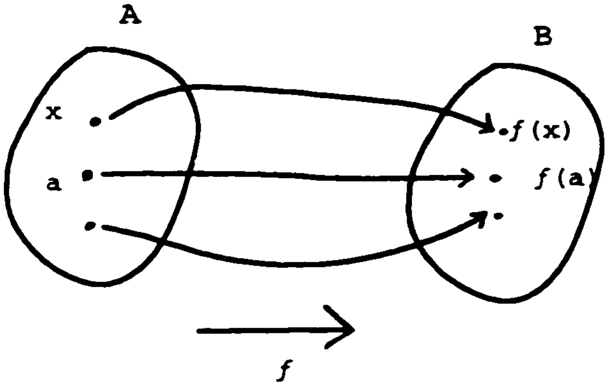


Figure 2.

THE CONCEPT OF FUNCTION

Although my main objective was to ascertain the readability of this portion of the text, a word or two must be said on the mathematical content involved. This very concept of function has historically been one of differing interpretations. The term "function" originated with the mathematician Leibniz in the 17th century in his definition of coordinates of points in the analytic plane as being "line segments fulfilling some function for the curve". This curve had come about as the loci of points moving at particular distances from certain lines of reference. (Sierpinska, 1992, p. 40).

This metaphoric expression underwent several shifts of meaning in the 18th century, finally stabilizing as referring to analytic expressions. Definitions of functions started to include pronouncements such as, "a quantity composed in any manner of a variable and any constants". (J. Bernoulli (1718), as quoted in C. Boyer, 1991, p. 422). Or in Euler's terms, "A function of a variable quantity is an analytic expression composed in whatever manner of this same quantity and numbers or constant quantities". (1797; as quoted in Sierpinska, 1992, p. 45). Euler's contemporary, Jean-Louis Lagrange (1736-1813), added an element of rigor to his definition of function but continued to define it as being an expression, "dans laquelle ces quantités entrent d'une manière quelconque", but tempered that of Euler by removing the condition on constants. ". . dans les fonctions on ne considère que les quantités qu'on suppose variables sans aucun égard aux constantes qui peuvent y être melées". (ibidem, p. 45)

Augustin Cauchy, however, in 1821, is quoted as having changed the focus from a function as being an expression to that of its being the result of operations that are involved in the expression. "On nomme fonctions d'une ou plusieurs quantités variables des quantités qui se présentent, dans le calcul, comme résultats d'opérations faites sur une ou plusieurs autres quantités constantes ou variables." For the earlier mathematicians it was the law of the function that was the important concept; for Cauchy it was the value. (ibidem, p. 45).

Then in 1837 came a less constrained definition of function again with the general definition of Lejeune Dirichlet. "If a variable y is so related to a variable x that whenever a numerical value is assigned to x, there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x" (after Boyer, 1991, p. 510). This last is basically the traditional definition of function as still taught in many classrooms today.

Modern mathematicians had cause, however, to change the focus of the concept once more with the introduction of set theory and the study of relations themselves. Peano, in 1911, suggested that a function was nothing more than a certain kind of relation - a "univocal relation". And from this widely accepted concept came yet another - the concept of an ordered pair - from which it was possible to operate on sets or spaces of functions. (Sierpinska, 1992, p. 48). The notion of function as a kind of relation is also to be found in today's textbooks.

A modern conception of function is that of an ordered triple $(\mathbf{X}, \mathbf{Y}, f)$, where $\mathbf{X}$ and $\mathbf{Y}$ are sets and f is a subset of $\mathbf{X} \times \mathbf{Y}$ such that if (x, y) belongs to f and (x, y') belongs to f then $y = y'$.

These historical differences aside, Sierpinska in her aforementioned article, "On Understanding the Notion of Function", outlined several "acts of understanding" that have to be experienced and "epistemological obstacles" that have to be overcome, before one can totally understand the concept of function. In the course of my investigation it could be seen that it was not just the readability of the text that was under scrutiny but the very understanding of the concept of function itself on the part of these 3 students. Several of Sierpinska'a "acts of understanding" came to the fore, in particular, the discrimination between the dependent and independent variable, the synthesis of the concepts of law and the concept of function, the discrimination between the concepts of function and relation, the discrimination between different means of representing functions and the functions themselves, the synthesis of the different ways of giving functions, representing functions and speaking about functions, the discrimination between mathematical definitions and descriptions of objects and the discrimination between the notions of functional and causal relationships,

The "epistemological obstacles" that accompanied the above "acts of understanding" also became evident throughout this study; regarding the order of variables as irrelevant, only relationships describable by analytic formulae are worthy of being given the name of function, and problems pertaining to their perception of definition.

These "acts of understanding" and "epistemological obstacles" will be discussed as they arose during the interviews, along with possible references to the historical development of the concept of function, in my commentary paragraphs on each part of the text.

As regards my own interpretation of the authors' intentions, in the college textbook used, in presenting this introduction to the concept of function at this level, it seemed that they were initially concerned with adhering to Dirichlet's general definition of function alluded to above. The last paragraph, aside from refering to the variables, r, t and w as numbers - Are we to assume the authors had already assigned numerical values to these variables? - matched that of Dirichlet by stating that one value is a function of another value, if "there is a rule whereby, given a number (r, t, or w), another number (A, N, or C), is assigned". The uniqueness of the assigned value was very much taken for granted due to the types of examples given.

However the definition which followed these opening paragraphs did not synthesize this concept. It had the flavour of the modern set theoretic concept with its reference to "each element x in a set A". But it missed its spirit by focussing on just one element of the triad (A, B, f), namely "the rule" through which the elements of f, or pairs (a, b), are chosen. In Part 1, this rule, in two cases out of three, is an algebraic formula. If a function is a rule, and the rule is in most cases a formula, then does it not follow logically that a function is just a formula, when it exists? It was, no doubt, not the authors' intention that such a logic be applied to this reading of the text. They would presumably have been trying to accomodate the maturity of the students at this level by offering them, not only a traditional meaning of function as usually found in high school textbooks, but also that of the more advanced mathematics student found in the theory of sets. However, the definition of the function as a rule connecting elements of two sets in Part 2, introduced a somewhat different concept than the one whose meaning was conveyed by Part 1. This did not look to me to be going to be a very successful introduction to this concept.

Part 3 contained more of the modern approach in its first paragraph followed by a return to the traditional approach in the second. And as for the pictorial representations that were supposedly introduced as aiding devices to the understanding of the concept of function, their consistencies (or inconsistencies), with the strictly textual explanations that preceded them, were certainly not going to prove conducive to this end.

THE FORMULATION OF THE QUESTIONS.

Since the formulation of the questions was as much experimental as this whole exercise, I inevitably found that when asked as first formulated, the format and type of questions revealed much room for improvement.

Michael was my first interviewee and as can be read from PART 1 of our conversation my questions had been far too "grammatical". (App. A, Michael, l. 77). Bearing in mind my interpretation of the authors' intent of describing a function as some form of three-fold concept: (i) the dependence of one value (or of one magnitude) on another, (ii) the rule that connected these two values (or magnitudes), and (iii) the association of one value with another, I had asked him. . .

"In each of these first 4 paragraphs there are 3 concepts involved.

1. The same verb is used in each of the first 3 paragraphs indicating the first of these concepts. What verb is it? Can you give a noun associated with this verb?
2. The same noun is used in all of the paragraphs to indicate the second concept. What noun is it? Another word very similar in meaning to this word is also used in paragraph 3. What word is it?
3. Different words, but with similar meanings are used in each of the 4 paragraphs to indicate the third concept. Can you say what they are?"

His problem became more one of understanding my questions than of understanding the text!

I subsequently changed these opening questions for Brent and Peter to the more simplified. . .

"1. Can you identify any key words that occur several times in each of these paragraphs?
2. Using these key words can you identify 3 main ideas about functions that the text is trying to convey?"

My objective in PART 1, (App. A), was to ascertain if the reader of these paragraphs was actually getting out of them what I deemed that the author of the text had intended.

For PART 2, (App. B), Michael's responses again prompted changes in the format of my questions. They took on a much more streamlined and specific layout when asked of Brent and Peter. My objective as regards this textbook definition of a function was to determine how a non-mathematician would interpret it. Did they understand this definition with its new focus on the set concepts involved? Did the uniqueness of the f(x) come across more clearly now that it was expressly stated? Did they know how to read the expression, "f(x)"? Could they relate this definition to the 3 paragraphs referred to in PART 1?

For PART 3, (App. C), there were less changes to be made. I realised after my session with Michael (App. C, Michael, ll. 385-400) that I had not specifically asked questions about the notational definition of range and that the understanding of this notation was not something that could be taken for granted, especially for students who had been out of school for quite some time, so I incorporated specific questions on it for Brent and Peter. Also the various terminologies used to describe *f(x)* posed such a problem for Michael (App. C, Michael, ll. 309 - 356) that this question was specifically included into the questions for Brent and Peter (Part 3, Qs. (ix) and (x)). My objective in this section was to determine if the concepts of domain and range were clearly enough explained in the text, and also the concepts of independent and dependent variables. I had not initially been too concerned with the various ways of expressing *f(x)* but as will be seen this turned out to be a major topic of difficulty in the reading of these 2 paragraphs.

No changes were made to the questions originally posed to Michael in PART 4. (App. D). My objective here was very simple. Did they really believe that picturing a function as a machine was helpful?

Likewise for PART 5. (App. E). How was the pictorial representation of a function by an arrow diagram found to be helpful? These 2 questions from

PARTS 4 and 5 concerning pictorial representations were in keeping with what had been briefly stated in Chapter One of this study, "A third alternative to overcoming difficulties in reading text is to offer more pictorial representations". (p. 3).

Changes were made to the questions for Brent and Peter because of Michael's insistent comparisons between the machine model and the arrow diagram (Michael, Part 5, lines 589 - 609).

A second objective, apart from the pictorial aspect question in PART 5, was to determine once more whether the uniqueness of the dependent variable in a function had come across sufficiently clearly for the reader to be able to recognise from a glance whether or not an arrow diagram represented a function. Since the text did not at this stage produce many examples of this type, I presented my students with a variety of arrow diagrams from which they were to determine whether or not they were functions.

I also went beyond the text in this section by introducting the concept of "reversibility" of the functions, and the one-to-one correspondence that this necessitated. This last part therefore was really teaching rather than investigating their understanding of the text, but I felt it added a nice finishing touch to the interview that they were coming away from it with an overall view of most that we would eventually be covering in class on the topic of functions.

Finally, these interviews lasted for 1 1/2 to 2 hours and were conducted prior to our discussion of this topic in the classroom.

* * * * * * * * *

CHAPTER THREE

ANALYSIS OF THE INTERVIEWS

I have chosen to analyse the interviews with Michael, Brent and Peter, not person by person in their entirety, but person by person within each PART, according to my divisions of the textual material to be read. This was to facilitate comparisons between the 3 students, since I was inevitably looking for specific similarities or differences in each of their interpretations of the text as well as for evidence of the difficulties in reading text that had been discussed in Chapter One.

PART 1 (see Appendix A, pp. 1 - 17)

Michael: (App. A, pp. 1 - 10)

As already mentioned in Chapter Two, Michael was the first to be interviewed and the questions asked of him were more of a trial run on my part. He was given time to read the 4 paragraphs by himself, without any interference from me, but his answering of the questions became more of a combined effort, a situation I attempted to remedy with Brent and Peter.

He did not respond well to my questions about verbs and nouns!(lines 8, 51 - 53, 77). This necessitated my rephrasing of the main question early on (l.13) to, "Can you pick out any key words?". M. (abbr. for Michael that will be used from now on) was still unable to come up right away with just one word that was being used consistently in these paragraphs until I prompted him some more. (ll. 16 - 17). He then gave me the word, "connects". (l.18). Prompted to look at the first line of the paragraphs (l. 23), he then found the word, "depends". (l. 24).

This relatively short portion of the interview was a surprise for me as to how *unobvious* the use of the word, "depends" was for M. To me it had been so obvious! Yet I had had to virtually force M. to see it.

We then entered a rather lengthy discussion about whether or not the dependency between the magnitudes worked only one way or both. (ll. 29 - 50) This originated with M.'s statement that the area and the radius depended on each other. I took him up literally on his use of the words, "each other" since this to me suggested a two way relationship, but on reading the transcript I wonder now if I hadn't really forced the issue somewhat. (l. 47) He had no problem seeing the area and the radius of the circle as being reversibly dependent, and possibly the weight and the cost of a letter, but the question on the bacteria and time was another matter. I eventually had to drop the issue and wait till the more appropriate time for this discussion would present itself in PART 5.

Despite my help in getting over my "grammatical hurdles" M. still had trouble hitting on the frequent occurrence of the word, "rule". (ll. 56 - 66). Note here what he answered to the question, "What is it that connects?": "The values", "Positive number", "The variables. . . the rule" (ll. 62 - 67).

Relating the word "formula" to "rule" was no problem from his reading of these paragraphs and as can be read in M.'s transcript the use of the word, "formula" in place of "rule" became a matter of great importance and significance for him especially after he had read the definition of function as given in the text.

My questions (l. 76, 78) were a mistake I did not repeat with Brent and Peter! I did not need to.

M. seemed to have caught on to what I was after with "my third concept" by producing the words, " Corresponding, associated" (l. 99) However when asked to then put the three concepts in his own words we were in trouble. (ll. 102 - 108). He began well enough by suggesting "relationships" for the concept of dependency, of giving his "governing law" for the rule, but then we diverged far from the main questions at hand with Michael's venting off his frustration with not only this textbook but the whole of the Math 206 course in general, up to this point anyway. (ll. 117 - 134). (We had actually only completed Chapters 1 - 3 and they had been for the most part a review of material already covered in Math 200, "Fundamental concepts of high school algebra"). I will return to what M. had to say in this section later.

When we got back on track we were able to settle on "corresponding value" as being a good expression to describe the third concept. (ll. 143 - 153)

His categorical answer of "No" to the last question would indicate that he read this question as, "Can you define a function in your own words?" rather than, "How would you. . . ?" In any case the answer was accepted as given and we moved on to PART 2.

Brent: (App. A, pp. 11 - 15)

I had interviewed Michael the same morning and had had time to change some of the questions, as already indicated in Chaper Two. So right away I asked for the key words. (No more verbs or nouns!).

I allowed time for B. (abbr. for Brent) to read the paragraphs and this time answer all the questions before any discussion took place.

The word, "function", was initially the only word that B. could find as a key word. He had however been able to come up with two of the main ideas. (I had changed the word "concept" to "idea"). These were "an associated value" and "a dependent and independent variable", and then, as an afterthought, a third, "a causal relationship". (ll. 14, 16). B. then gave me his definition of this "causal relationship" which he saw as something quite different from an associated value. The "causal relationship" referred to the situation in which a change in one variable causes a change in the other, whereas an associated value was just a direct relationship like $2x = 10$. (ll. 18 - 24).

I had to "chide" him on his using the words "dependent and independent variable" since these words had not been in the text that he was presently reading but, unfortunately for these interviews, B. had already read this part of the text and knew what was coming up next, and as he stated he already knew this from "his (past) history". (l. 26). Making the most of this prior knowledge I prompted him to find another key word in the four paragraphs - something related to dependent? He found the word "depends". He also now found the word "connects". Some more prompting, and eventually he found "rule". Note his first answer to, "What's doing the connecting?: "The variables". Sounds familiar? (ll. 37 - 42). The word "formula" was again interpreted as meaning something similar to a "rule".

So we re-addressed the answers that B. had given to Q. 2. Could he now relate these key words to his 3 main ideas? He related "connects" to idea #1, "depends" to idea #3, but did not see how the words "formula" or "rule" tied in with any of these ideas. I did not push this question any further at this stage.

I then checked his answers to Q. 3 and Q.4 to which he had virtually given the same answer. My experience that morning with M.'s answering of Q. 4 prompted me to ask this same question in 2 parts: Could you define a function in your own words? and if you could then, How could you do it?

B. answered these questions as one: "the causal relationship between two values" or "the causal relationship that one value has or is to another". (l. 64) and see p. 15.

Peter: (App. A, p. 16 -17)

P. (abbr for Peter) read the paragraphs but did not appear to be interested in writing down any answers to my questions, so we conducted the questioning orally. (I admit this was perhaps a mistake but his initial responses are all transcribed).

His initial response to Q. 1? Function. I pushed for more. He gave me "depends", "rule". He seemed to be stuck for another so I gave him "connects". He also concurred with me that the words "associated, corresponding", were key words that meant the same thing. (ll. 3 - 17)

P.'s response to Q. 2? See line 19. My response was noncommittal so he tried again. "Things are inter-related" (l. 21) I prompted him to use the key words he had just given me. He then gave me response 23: "So there's an inter-relationship between things. There's a connectivity between the two sides of the equation. One is dependent upon the other one. . . they will both grow or decrease according to each other. . . there's a set relationship between the 2. . " Could P. define a function in his own words? "Well a function would be . . it would be a function of another number if its value is connected in any way to that other number - in a sort of a logical sequence - so if there is a formulaic connectionship between the two . .".

Comments on PART 1:

Peter was certainly the man with fewer words when it came to answering my questions. He was the most capable of picking out the key words and came the closest to showing a Bernoulli - Euler understanding of the concept of function in his words "connected in any way". And to him the traditional "relationship" meaning of a function was very strong. I would hesitate to infer that he saw a set theoretic interpretation from his usage of the words, "a set relationship between the two". My reactions to P.'s words in this section of the interview are necessarily influenced by his comments in the other parts that have yet to be discussed, and in particular from his difficulties with the concept of set that formed part of the definition of function (see App. B, Peter, ll. 91). In his replies to my questions in this Part 1, he was groping for words that would express a relationship that was more than just "a relationship". In this case one might say that O.E. failed him in his search to express a concept that was new for him. He saw a relationship "of some kind", between a set (in a very general sense) of values, and that these values involved were connected by a formula. He did not see SETS of values in the mathematical sense of the word.

Brent kept his answers to the point and apart from his, "causal relationship" (on which I defer comment till the end of this commentary) explained himself very much in O.E. He was not a man for "technical jargon" - but again I am anticipating other parts of the interview.

Michael, on the other hand, was very verbose and of a much more nervous disposition. He was so anxious to unburden his frustrations with this present math course, especially the textbook, that it was remarkable that I was able to keep his attention on the questions I wanted answered for as long as I did. He was particularly unhappy with all my references to verbs and nouns, and in retrospect I don't blame him. But even when I asked him just to pick out some recurring key words he had difficulty finding them. Of particular note were the answers given by both B. and M. to, "What is it that connects?" Both gave the reply, "The variables." Did this come about because the way this question was worded followed the wording in the paragraphs that stated, "the rule that connects r and A" or "the rule that connects t and N."? They obviously saw the "that connects r and A" and "that connects t and N" in a stronger light than the fact that it was "the rule" that was doing the connecting. When the question was rephrased to ask what was connecting the variables then the correct answer came readily.

What surprised me the most in the above investigation was the difficulty they each had, M. more than B. and P., in seeing the key words. It is a little early to generalise but it does seem to support the views outlined in Chapter One that people read what for them is relevant and interpret according to their own experiences. Putting their own meaning on to the page, was how this was referred to earlier, was it not? But was their initial interpretation the one that was intended by the author? Were the ideas that he wished to convey the ideas gleaned by these 3 readers? Hardly. I felt all along that I was alone in my interpretation of this text, that my being the "expert" in this field enabled me to know what the author was getting at and that it was only after some, or much, prompting as the case had been, that they were able to interpret these paragraphs in the light that I deemed was intended by the author.

So my findings so far would appear to support the beliefs of Z. Krygowska, outlined in Chapter One, that students need to be taught how to read text, be it mathematical or otherwise it would seem. These paragraphs could hardly be deemed mathematical in their construction. There was no symbolism involved, and no terms that were specifically what has been referred to as M.E. There were some though that could be considered O.E. but used to convey another message in M.E., e.g. function, rule. But more on this in PART 2.

In other words there would seem to be a necessity for readers to be guided in their reading to see its intended meaning. This would be my main observation from PART 1 from the "linguistic" point of view.

A second observation would be regarding the question of active reading on the part of these students v. passive reading. I would have to contend that both M. and B.'s initial reading was more passive than active. They were slow in seeing the repeated occurrence of the words "depends" and "rule". It was not until they had been prompted to find them that they started to actively underline them or outline them in the text. Their increased understanding of the paragraphs was accompanied by much "active" annotations of their own on the text. On the other hand, however, the lack of such physical activity on the part of P. did not mean that he read these paragraphs passively. In fact his reading seemed to be extremely active mentally. He did not have the need to add underlining or otherwise to the text. He saw the repeated occurrence of the major words immediately and came the closest to understanding the underlying meaning intended by these paragraphs. I think P.'s case is more one of the exception rather than the rule. He was after all the best student in the class producing 100% in all his tests, including the final exam. Does this not say it all? Mental alertness does not need to be accompanied by physical alertness? But for those who are less mentally alert, the physical activities referred to are an aid to their mental thinking. And is not this something that is surely at the basis of most mathematics teaching? To have plenty of the concrete before moving on to the abstract?

A deep understanding of the concept of function from the reading of just these three or four paragraphs was not to be expected and all I was looking for was what meaning at all was derivable from these pieces of text. Incidentally several of the "acts of understanding" and "epistemological obstacles" outlined in Sierpinska's article, referred to in Chapter Two, surfaced during the course of this interview. A case in point here was the non-discrimination yet by P. between the concepts of function and relation and the necessity of there being a "formulaic connection" between the values. Also from M.'s comments we see that the order of the variables was to him irrelevant at this stage. The most significant "act of understanding" was yet to be faced by Brent as seen from his words "a causal relationship". B. referred often to his "history" of past experiences and although we did not actually discuss this in any detail here - he did mention later in the interview that he had recently taken Psychology courses at McGill - I would assume in retrospect that he had studied the philosophies of cause and effect in the course of these studies and was seeing these dependencies of one value on another as a change in one causing a change in the other. This concept, from the examples given of a change in radius "causing" a change in area, a change in time "causing" a change in the number of bacteria, is understandable and in fact plausible in these scientific circumstances where a direct formula is involved. However the modern mathematical concept of function does not require the necessity of a formula connecting the two sets of values involved and thus the changes in one set of values is not deemed to be the **cause** of the changes in the other.

* * * * * *

PART 2 (see Appendix B, pp. 18 - 36)

Michael: (app. B, pp. 18 - 22)

The only portion of the text that was presented to the students this time was the definition of a function as shown in Chapter 2, p. 13.

I deliberately refrained from offering any form of help during M.'s first reading of this definition and the result is right there on l. 164: "No, that doesn't make any sense to me at all".

My next approach was to see if by re-reading the first paragraph of PART 1 he could make any connections between this definition and what was being discussed there. What appears to be a multiple-type question (l.167) was intentionally so, in the hope that M. could put the pieces together on his own. Again this exercise did not produce the desired results. M. zeroed in on the "elements", considered that "elements" meant the same as variables, and therefore immediately associated "elements" with both the r and the A.. This, of course, was true but M. failed to distinguish between the two sets of elements, x and $f(x)$. In fact he initially referred to $A = \pi r^2$ as being the set. (l.168).

On being questioned on what he considered to be a set, however, M. answered that, to him, it was "a group of numbers". (l.170). How then, I returned, could he reconcile <u>this</u> notion of a set with the set as being $A = \pi r^2$? He struggled with this one for a few minutes (l.172) but clung to the notion that the expression $A = \pi r^2$ still represented "a set of numbers together" - in light of his seeing the r's and the A's as being sets of numbers? (l. 174)

I then took over the role of teacher and explained to him how this definition of function could be made to fit in with the question on the radius and area of a circle. (l.175 - 236).

This lesson was directed mostly by me but M., who had previously looked at these pages, and who had obviously been impressed by the machine diagram model of a function, redirected it along these lines for a while finding it easier to relate this definition of function to Fig. 3 on p. 158 of the textbook *(see Appendix F, p. 99)* - a section that I had not presented him with - than to the question on the radius and area. He had no problem relating the input to the set A, the output to the set B and what was happening in the machine as the function. (l.198). Being brought back to the radius/area question, he still had difficulty in differentiating between the r or the A as the input. (l. 202).

(This went back to the question already raised in PART 1 as to the dependency of r on A, or of A on r. M. still saw this as "both (depending) on each other to come up with an answer" (App. A, Michael, l. 34)

When pushed to decide on one of the elements, r or A, as an input in this situation he finally said what I had been hoping to hear, "I plug my radius in", even though he was still not sure that this was the correct thing to say. "I'm guessing!"(l.208). He was encouraged to, "Keep guessing!" (l. 209), because by doing so he was inevitably going to come up with the "correct" answers. He did. He eventually realised that the values for the radius would be the set A, those for the area the set B, and the rule that connected them both was the formula, $A = \pi r^2$, which according to the definition was in fact the function. "A function is a rule. . . " (l. 216 - 236). M.'s equating formula with function was well founded from this interpretation of the definition!

The final portion of this part of the interview was a discussion of the expression "$f(x)$" which had appeared for the first time in this definition with no explanation as to how to read it. (This was not dealt with till the next paragraph). M. had been reading it as "$f\,x$" and now revealed what was really quite an understandable misunderstanding of this expression, coming as he reminded me from Math 200. This was no different from an expression like $a(b)$ where we could just remove the brackets and simply write ab. The f and the x were just 2 variables multiplied together. I did not correct this interpretation at this stage in view of what was going to be read in PART 3.

Brent: (App. B, pp. 23 - 28)

From my experience with interviewing Michael that morning, the format of the questions for Part 2 were altered to deliberately include a comparison between what was described in paragraph 1, Part 1, with the textbook definition in Part 2.

Unlike Michael, Brent had been able to give some sort of definition of a function at the end of Part 1 and he was therefore first asked how his definition measured up to that of the text. B. claimed that he had said the same thing but "without the technical jargon". (l. 69) His definition, he claimed, was more of an explanation than a definition. (l. 73). "A causal relationship between 2 values" was what he had written as his definition and I wanted him to now try and rephrase this with some of the "technical jargon". He gave me "A function is a rule that shows the causal relationship between x in set A and the values in set B." (l. 77). This was good but I wanted him to realize that his "causal relationship" could be interpreted as the "rule", a point I had not pushed in Part 1. This I believed I had accomplished, even though it felt at one point as if I had been putting the words in his mouth! (l. 84 - 98). In retrospect I had. B. correctly contended all along that his "definition" was more an explanation. (l.99).

Question 2 asked him what the textbook definition really meant to him and for this B. gave me,"for every x in set A there is $f(x)$ in set B" (see App. B, p. 28). He was also able to relate this to a graphical representation on the coordinate plane with the $f(x)$ being replaced by y. He called this a "countervalue". (l. 111). And from previous work with this notation he knew to say $f(x)$ as "f of x".

Then we came to Q. 3 which required him to relate this definition of a function to the first paragraph of Part 1, and to my surprise he found this difficult to do. He admitted to finding this paragraph "very confusing", (l. 113) and it was only because he had recognised the formula for the area of a circle as being $A = \pi r^2$ that he had had some vague idea what they were getting at. (l. 119). I therefore had to resort to some prompting to get him to try and understand this paragraph in the light of the definition. He readily enough recognised that his x would be the r, his $f(x)$ would be the A, but insisted that his "causal relationship" between r and A produced the equation $A = \pi r^2$. I had to remind him that this "causal relationship" was what we wanted to identify as the rule. So now he was able to answer the three parts of Q. 3 successfully (see App. B, p. 28) However I remember having the feeling at that time that he had not completely seen the three aspects to a function that I had been driving at, and evidence of this can be read by much repetition from me on this concept in the transcript! Since Brent had already shown in Part 1 that he knew about dependent and independent variables I brought these concepts to the fore at this point to reinforce his understanding of what I was saying about the two sets and their being connected by a rule, the formula, in this case.

I pushed a little then to see if the last sentence of the radius/area question was imparting any special information that we had not really discussed but the significant words were only "A is a function of r" and the fact that r was "positive". I did not wish to "put the words in his mouth" this time by making him read any significance into the words "one value of A". My reaction now, after the fact, is that this concept of there being only one value of A was too obvious in this case to draw any special attention to itself and probably the word "one" was just read as "a". The uniqueness of this "one" was not made clear in the wording of this sentence. I did not pursue this idea any further at that time. There was more to come!

The last section of this Part 2 with Brent was only brought about because of what Michael had said that morning about believing that "f" was a variable. Brent's initial response was encouraging (l. 171) until I ruined everything by asking him what it stood for then. (l. 172). He replied, "a variable". But again in retrospect I believe he really meant that the whole expression "f(x)" was a variable and I was a little too anxious to make him see that his first reply had been the right one!

Peter: (App. B, pp. 29 - 34)

After our very encouraging beginning to this interview where it was clear that P. had understood a function to be some sort of relationship, it should not have come as a surprise to discover that Peter was not nearly as comfortable with the "mathematical parlance", as he called it, of the textbook definition of a function. (l. 33, 37, and 91). He could not now accept what, to him, appeared to be a whole new concept of a function as being a "rule". Since he had not written down his definition from Part 1, here again is what he had said: "a function would be . . it would be a function of another number if its value is connected in any way to that other number - in a sort of a logical sequence - so if there is a formulaic connectionship between the 2" *(see App. A, p. 17, l. 27)*. According to his meaning of "rule" (which will be apparent in the next paragraph) there was certainly no place for any "rule" in there! The strongest part of P.'s definition was the **connection** between the 2 numbers, albeit a "formulaic" one.

My intervention took the form of having him analyse piece by piece what the textbook definition was saying. "What is it really saying a function IS?" (l. 40). And having him reply, "A rule". Could he consider the "rule" as being part of a function? Not really. To him a function was more along the lines of a relationship than a rule. (l. 45). Was there a particular problem with the word "rule"? Well yes, a rule implied something that should be the same all the time. (l. 49) And here was my first open encounter in these interviews with O.E. interfering with M.E.

P. 's initial interpretation of the word "rule" was that of something that was "dogmatic, a law" (l. 77). This being the case he did not readily associate this word with a function which he had understood from the first 4 paragraphs to mean a relationship. He eventually conceded that a rule, as something that "inter-relates" the 2 sets of values, *could* be part of the concept of a function. (l. 51). He saw that if he had "A here and B here they would be related by a certain rule". (l. 55). We then entered into a discussion as to the relevance of this textbook definition of a function. I did not wish to get into a lengthy historical debate on the issue at this stage but felt that he needed some reassurance that his concept of a function as being more a relationship than a rule was well founded. (l. 60 - 64).

In the final analysis, the textbook definition of a function did not make him change his concept of function. (l. 86 -88). HIS definition meant more to him than the one in the book. He had not come to grips with the "terminology" being used, i.e. "the very nice mathematical parlance". (l. 91). He did not see where the notion of sets came into all of this. In fact he felt it would be much more to the point to "learn what it is that we're doing and then get at names afterwards". This textbook definition was not something he would use to explain a function to his children! (l. 99)

This last comment was taken by me as a cue to introduce him to what was considered to be more a "children's" version of function by having him compare what we had just been reading with an introduction to functions from a Grade 10 Mathematics book. *(See Appendix G, p. 103).* This he found to be much more in keeping with his meaning of function! "This is a lot clearer . . after having read this, this makes a lot of sense." (l. 107). This textbook was actually calling a function, **a relation**, not a rule. And it was more directly derived from some preceding examples of functions. I reminded him that the College Algebra definition also came after several examples but in this case the connection between the examples and the definition was too far removed to be meaningful. (l. 110 - 111).

From this Grade 10 definition I was able to get him to pick out the uniqueness of the second value, the "only one corresponding value", (l. 115) as something that we had not as yet discussed. When asked if this idea was not also in the College Algebra definition the reply was, "No, not necessarily". (l. 119) "What about the word 'exactly'", I asked, or words to that effect. (l. 120) Ah, now he saw it. He had been struck so forceably by the word "rule" that the other words had got blurred into the background. P. asked, "Is this a function of my inability to read properly?" (l. 129) My reply? "No"! (On re-reading this transcript I realised I missed an opportunity of discussing his use of the word "function" in this last comment!) He also preferred the way the Grade 10 Math text continued its more formal approach to functions in a later chapter *(see App. G, p. 104)* by relating it to a very concrete example involving parking meters and the dependence of the value of the coins on the number of quarters in the parking meter. He was surprised therefore to hear that another of my students would not have been so happy if I had presented them with the concept of function as explained at this Grade 10 level, at least in its introductory pages. *(This was Brent, but I hadn't shown him this Grade 10 text till the very end of the interview, so more of B.'s opinions on this topic later).*

We nearly got off-track on the now familiar topic of how much better the Math 200 textbook had been but thankfully it was kept brief! (l. 143 - 145). Getting back on track we had yet to face the problem of relating this "non-meaningful" textbook definition with the "meaningful" first paragraph of Part 1 *(meaningful and non-meaningful to P. that was).* First we had to establish what were the sets A and B in this example. As can be read (l. 154) he had it all in reverse. But after my explanations of set A being the radius and set B being the area and that the values of set B were coming from set A to form "$f(x)$ in a set B", which after all was "what it says", admitted P. (l. 172), then all was well. (l. 155 - 180)

We did not need any discussion on whether or not the "f" in the "$f(x)$" was a variable. Peter saw it as the "function f" , and only the x was the variable. (l. 182 - 192).

Comments on PART 2.

The similarities in the problems faced by each of these students come across more strongly than their differences. They each had the greatest difficulty relating the textbook definition of a function with the paragraphs that had preceded it - the first one at any rate. Reading and understanding the paragraphs in Part 1 was one problem. Reading and understanding the definition was another. But relating the two of them together was the biggest problem of them all. And of course this was all really quite understandable when you read the paragraphs and the definition from the point of view of Michael, Brent and Peter. Where in the opening paragraphs does it state that "a function is a rule"? Nowhere. It merely states that a second number is a function of a first number, and that the values of these numbers are **connected** by some rule, or formula. They did not state that the function **was** in fact the rule itself. Both Brent and Peter were perfectly justified in seeing this whole concept of function as being some kind of "relationship" from their reading of the paragraphs in Part 1. How could they now be expected to see it as a rule? The authors' placing side by side the traditional meaning of function as a special kind of relation with that of the modern set theoretic interpretation produced the feared consequences suggested in my Chapter Two. Their definition of a function as "a rule" coming after their examples of functions derived from relations between one set of values with another, without even this being necessarily a "formulaic" relation, was certainly doomed to confuse their readers.

It was no wonder therefore that Peter was delighted with the Grade 10 explanation of a function. It was certainly more in keeping with the concept of function he had gleaned from his reading of Part 1. "These relations are called **functions**." (see App. G) - *(my underlining)*.

Despite this very marked difference between what the paragraphs had been trying to convey and what was so neatly outlined as a definition of a function below them, we were finally able to at least relate the x and the $f(x)$ to the radius (r) and the area (A) of the first paragraph. The rule was finally accepted as the formula relating these 2 sets of values, i.e. $A = \pi r^2$.

This being said, the most interesting event, from the point of view of what was stated in Chapter One concerning ambiguities, was Peter's problem with the word "rule" itself. (see previous comments under "Peter" above).

There was also Michael's belief that the "f" in "$f(x)$" was a variable. How would one describe this misconception? It was not a case of a confusion between O.E. and M.E., or even a problem with M.E. itself, but a definite ambiguity in the mathematical symbolism. For someone who obviously took textual content very literally, as Michael did, it was again understandable that $f(x)$ should be no different from such an expression as $a(b)$, or ab. I did not wish to take Michael to task over this at that stage because the explanation of how in fact to read "$f(x)$" was just about to come up in Part 3. But seeing the f and the x as two variables was certainly not an asset to his understanding of the textbook definition. I wish now I had asked him what he thought these two variables f and x had been representing!

To conclude this section from the point of view of the reading of the text, therefore, the one fact that stood out the clearest was that the authors of this text obviously did not see any discrepancy between what the opening paragraphs were indicating about a function and their formal definition of a function that followed. Their readers did.

The other aspect of the development of the understanding of the concept of function again supported some of the views discussed in the article by Sierpinska. We see now M.'s growing understanding of the concept of function, <u>as defined in this text</u>, in his discrimination now between the dependent and independent variables. We see a discrimination on the part of P. between the mathematical definition - this mathematical definition at any rate - and the description of a function as he understood it. The conception of definition itself, on the other hand, is well placed under the heading of an epistemological obstacle.

* * * * * *

PART 3 (see Appendix C, pp. 35 - 53)

Michael: (App. C, pp. 35 - 43)

It should be remembered that we had concluded Part 2 with M. having some sort of understanding of the three parts of a function, i.e. the set A - our input values, the set B - our final answers, and the rule that connected them - which to M. meant the function. M. had identified the set B with $f(x)$, *(App. B, Michael, l. 236)*, but also had it in his mind from the definition that he had just read that "a function was a rule", another word for a rule was a formula, therefore to him a function was a formula. So $f(x)$ being the "function of x" also meant to him "the formula of x".

The above observations would explain M.'s confusion when he tried to relate these thoughts to what he was now reading as the definitions of domain and range. The concept of domain as being the set A, i.e. in this case the r, was no problem, but he was reading "The range of f is the set of all possible values of $f(x)$ etc. . " which was what he considered the middle part of his graphic setup of what was going on. *(see App. C, p. 43).* He felt however that the range was really the answer, the set B, but this he freely admitted was not what "they" were saying. This confusion led him from the belief that the range, according to what the text was stating and according to what his graphic setup was telling him, was the **formula** itself, $2\pi r$, to the belief, according to how he felt it should be, that the range was the set B, the **answers** themselves. *(l. 360, or rather paragraph 360, - l. 385).* We settled on how he felt it should be, "For me the range would be what comes out at the end." (l. 371).

Before we actually reached this discussion however we had had to overcome a few other hurdles. We had moved straight from the reading of the paragraphs to answering my "first" question which had not been formulated, i.e. Did the chart represent a function? Could he see a rule being depicted by these sets of numbers? (l. 261, 263). M.'s answers led to the graphic setup alluded to above. He had the set A, the formula, and the set B all neatly divided into their respective boxes, *(again see M.'s notes, p. 43),* and into each of these boxes he had placed the r, the $2\pi r$ and the C, (which was also the answer), without any problem. Everything was going well. "Well that makes sense", he claimed. (l. 290). I was about to direct him back up to the first paragraph, in order to establish what was meant by the domain and range, when he again brought up the question of the reversibility of this whole operation. Could not C be the set A, r the set B, and the rule whatever the formula was that enabled one to find r from C? (We ran into some algebraic manipulation problems here trying to get from $C = 2\pi r$ to $r = C/2\pi$, but that was not a major concern at this time.) I had to agree that indeed we could (l. 299) and left it at that for the time being.

I realise on reading the transcript that I hadn't given M. a chance to comment on what he had read before directing him to specific parts of these paragraphs - a possible error on my part. After directing him to establish the meaning of the domain as it pertained to the chart shown, I then directed him to the fact that the same paragraph told him how to read the expression $f(x)$. And the big question was, "Is f still a variable to you?" (l. 319) (Remember that at the end of Part 2, Michael had been left with the belief that the f and the x were both variables. *(Appendix B, p. 21, l. 250 - 256)).* Of course the answer was, "No." "It's more the function, more the formula"(l. 322). Especially the formula. "The formula x, is read, "f of x", or "formula at x". (l. 326). He didn't like the words, "f at x" however and embarked on some lengthy explanations on how it would be a lot more meaningful as "given x". From the point of view of this study, his comments on "getting used to the terminology" (l. 334) and "putting this in as layman's terms as possible" (l. 338) were particularly apt.

And then we came to "the image of x under f"!! Did the author of this text really find it necessary to throw that description of $f(x)$ in here? If so, he must have assumed that all his readers would be well versed in the language of transformations! This assumption was ill-founded! Michael being the "literal reader" that he was, imagined this as "x divided by f" and saw this as something else they had thrown at him! (l. 346). It was not within the scope of this interview to enlighten him as to it's true meaning at this time.

The difficulties faced by M. over what exactly the range should be, I have already discussed above, but again they can be seen as a direct consequence of M.'s literal interpretation of what he read.

The next part of this session was also problematic in that M. had no idea what the symbolic notation $\{f(x) \mid x \in A\}$ meant. "This I don't understand at all." (l. 385). I merely established for my own benefit what exactly he had not understood, and it was of course the { } themselves, the $\in$, and the $\mid$. I did not deem it necessary to explain all this to Michael at that time, although I did with Brent and Peter. I don't remember why I didn't with M. - possibly because of the amount of time we had taken explaining other matters!

The concepts of independent and dependent variables went down smoothly and now came the opportune moment to address the reversibility issue. If r was the independent variable and C the dependent variable then the whole operation had to be seen as a one-way process. If this was reversed then so would be the dependency. (l. 436 - 441). M. had no more problems with this issue.

The final part of this session was my showing M. the Grade 10 book already referred to previously with Peter in Part 2. M. reactions, "Thank you very much . . It's very easy reading. . this is very well put. . this is very simple . . I don't have to read it twice", say it all! There would have been no feelings of being insulted, as B. had indicated - still to come - if I had presented him with this material instead of that of his College Algebra.

Brent: (App. C, pp. 44 - 49)

Brent had already read these two paragraphs before this interview and claimed that they had in fact helped him understand the previous material. He had already encountered the concepts of independence and dependence and felt that he was on familiar ground. It may be remembered from Part 2 that B. had prefered to replace the $f(x)$ with y and he did this again here, in keeping with his graphic interpretation of function. (l. 194).

So all was well till we came to the "image of x under f". Since he had successfully understood everything up to this point, B. just, "mentally ignored it". (l. 212). He "skipped over it" (l. 212) claiming that "it was confusing". (l. 214). Made to face the words now and forced to say what they might mean, he tried for some form of visual representation - the "image". (l. 218). But since these words were obviously, to Brent, NOT to be taken literally, he eventually admitted that he had a problem with "the whole five words". (l. 240). We left this problem unresolved.

Our attention was then turned to the questions I had asked relating to the chart of numbers. B.'s answer to #1 was "yes" they did represent a function. What did I mean by "What is the function?" Brent immediately replied, "the relationship". What was this relationship? B. had forgotten the formula for circumference. (M., by the way, had initially come up with $r^2/4$ for this one!) So I refreshed B.'s memory with the formulas, $C = 2\pi r$, or

$C = \pi d$. He duly wrote them down on his papers. (see App. C, p. 49). The rest of the questions down to (ix) were answered without any serious problems. (l. 261 - 309).

But then we came to the symbolic notation. Here, unlike M. that morning, B. did have some idea what this symbolism alluded to. He recognised that the { } indicated a set, and depending on what his values for x were he would obtain values for y, i.e. the $f(x)$, his set B. *(see again p. 49 for the way in which B. labelled this notation on his paper).* When asked however what this was actually defining, he fell somewhat short of the correct answer by replying, "They were expressing the definition of a function." I asked him to re-read the text and this time he took notice of the words "**The range of f.** . . ". Because he had got halfway to knowing what all this symbolism referred to, I filled in the missing parts for him and he completed his labelling of this notation on his papers. (App. C, p. 49)

Peter: (App. C, pp. 50 - 53)

I had had an impressive Part 1 with Peter, a frustrating Part 2, and did not know what to expect from him in Part 3. It started out by surprising me again. If one of these 3 students, I thought, had all of this clearly understood up to this point it would have to have been Peter. In answer to my first question on the chart, "Do they represent a function?", he replied that only the C was the function because, did we not say that "C is a function of r"? We did not say that r was a function of C. I didn't argue with this, but had him tell me what this function C might be. "C would be the function of 2 times πr." (Finally someone who knew the formula for the circumference of a circle!) (l. 206).

We continued with the rest of the questions after that without any more startling revelations till the question about the textbook description of $f(x)$. This time I had anticipated the problem before it arose: "The image of x under f". "It's meaningless", claimed P. To him an "image" was "a picture", "a snapshot". Of course it would be, for anyone without the necessary background in transformations, as already stated in my paragraph on Michael. But unlike Brent, who had just ignored this part because it interfered with what he thought he had already understood, or Michael, who had read it quite literally and therefore couldn't figure out in what way this fitted in with all the rest that had been said, Peter realised that it must just be some other way of expressing *f(x)* and that, "the value of *f* at *x*" made a lot more sense than, "the image of *x* under *f*". (l. 240). He knew it was not meant to be taken literally, and he didn't just ignore it. To me he "absorbed it" along with the other descriptions of *f(x)*.

As for the symbolic notation, well I guess, like Michael, it had been a while since Peter had attended High School, and even then had probably not studied set theory, so the { } meant nothing special, the | even less and the $\in$ he had somehow confused with the notion of set. We straightened all this out, filling in the gaps and ended on a more inspiring note than we had begun.

Comments on PART 3:

If we are looking for ambiguities this section surely provided them. Even after all that had gone before this section about the definition of a function we were still encountering different understandings of what a function was. Only Michael, it would seem, was prepared to accept the textbook definition of a function, i.e. a rule. For better or worse, a **function** was a **rule**. A **rule** was another word for a **formula**. Consequently a **function** was a **formula**. This interpretation of a function served Michael well. Once he knew the correct formula to find the circumference of a circle, "given" the radius, then this formula was the function. $C = 2\pi r$. No more problem.

Brent, on the other hand still maintained that a **function** was a **relationship** (he had not entirely given up on his, "causal relationship") but this interpretation of a **function** still had him looking for the correct **formula**. So to him, the **relationship** was the **formula** that connected the sets of values.

Now Peter had *his* interpretation of function, that being that since one of the sets of numbers was said to be a function of the other, then only the **C** in $C = 2\pi r$ was a **function**. C was a function of r.

The interesting thing about all of the above is that all of these interpretations of function have some element of truth in them when one considers the historical development of the concept of function. One last word here on function would be M.'s comment about the formula "serving a function", which is remindful of the sense in which Leibniz had first used this word.

And on the topic of O.E., the O.E. version of the word "image" was inevitably going to clash with its mathematical meaning for anyone without the knowledge of its transformational interpretation. The very word "under" in this same 'offensive' expression, "an image of x **under** f" takes on a very different meaning than the one assigned by M. and B. It could also be said that they were not as ready as Peter to synthesize the different ways of representing functions. However, Peter showed that he was not yet at the stage of being able to discriminate clearly between the different means of representing functions and the functions themselves by his claim that only the C was the function, in the expression $C = 2\pi r$. "C is a function of r", were the words used, were they not?

Domain or "domaine", as M. spelt it, *(see App. C, M.'s notes, p. 43)* was about to cause similar problems, for M. anyway, in Part 4. His misspelling of the word here was a forewarning that I missed. P. also had difficulty with this word "domain" but was willing to accept it. (l. 222).

Which leads me to bring to the fore at this stage one of the major points at which this study was aimed: that of the ambiguities with language faced by the "older student" as opposed to school children. The High School student would merely have accepted "domain" as another mathematical term to be learned along with all the others. It is not a word used in their everyday vocabulary and to many it would in fact be the first occurrence of this word. But to the mature student the word "domain", though not necessarily an everyday word, would still have more likelihood to have been used by him in an everyday context - likewise for the word "function" itself.

The word "range" did not appear to produce any of these kinds of problems, no doubt because the notion of "a range of values" is a common one. But this very conception of "range" as "range of values" is an interference with the intended meaning of "range" in the context of a function. And on this topic of range, it is to be noted how both Brent and Peter had to be made to read the sentence preceding the symbolic notation twice in order to derive its meaning. Brent - to read that it was a definition of "**range**", and Peter - to read that it was saying that the range was a "**set**".

The above comments again reflect the findings in Sierpinska'a article concerning the discrimination between mathematical definitions and descriptions of objects. The student has to have the ability to read not only the words themselves but to read the definition in the light of its implications.

And finally there were all the problems faced by the M.E. or the symbolism of the notation $\{f(x) \mid x \in A\}$. These would not in this case be any worse for the older student than for school children, or would they? M.'s past experiences led him to believe initially that *f(x)* meant "*f* times *x*". P.'s past experiences led him to believe that the $\in$ indicated a set. B.'s past experiences, though not so past as those of his class mates, had him thinking along the right tracks as far as the "curly" brackets were concerned, but that didn't really help him very much. His problem was one of not reading the material closely enough. I would define him as a "skim reader". One who only reads that to which one can relate? If he recognised familiar words or expressions, fine. If the terminology did not fit in with his preconceived ideas of things he "skipped over it" - to use his own words. (l. 212).

* * * * * *

PART 4 (see Appendix D, pp. 54 - 64)

Michael: (App. D, pp. 54 - 59)

Michael's literal interpretation of the text was once more his near downfall. The text clearly stated, "If x is in the domain of the function f. . " or as M. would rather have had it, ". . the formula f. . ", then showed the inside portion of the function machine equally clearly labelled as "f", then where else would you have to consider x but inside the machine? But then this concept went contrary to the previous section where it had been explicitly explained that the x in question was the input.

Michael therefore now saw two domains involved in this whole setup. The domain as defined in Part 3 as the set A, the set of independent variables, the input, and this new "domain", the domain of the function, a "place" where the machine performed its operations on the input. Again attention should be drawn to the way in which M. spelt the word, "domain". He insisted on spelling it "domaine" - the French word for a place or region. This O.E. (or O.F. in this case!) meaning of the word was clearly affecting the interpretation M. was according it. *(App. D, p. 54 and half of p. 55)*

This apparent contradiction aside however, M. had been "happy" (l. 490) to read that the final sentence was in keeping with the way things had been explained in Part 3. He was still uncomfortable with the expression, "*f at x*" and now proposed another version which made more sense to him, "*f on x*". The formula was working **on** the value of x.

As for the paragraph about the preprogrammed functions in a calculator, this example of a function was found to be acceptable, but still not the word, "function". M. found it necessary to once again express his discomfort with this word. "It's just the function throws me off." (l. 506, and 508). His stronger concept of something "serving a function" was in no way letting up in favour of this mathematical usage of the word. All of which had nothing to do with the paragraph before us, so he had to be brought back to the questions being asked, in particular on the comment made in the square brackets at the end of the paragraph, i.e. Q. 2. He hadn't written anything down for this answer but his thoughts were recorded on tape to the effect that he did appreciate the fact that a calculator would only give the answer to a square root to an approximate number of decimal places depending on the capacity of the calculator, whereas the real answer could possibly have an infinite - or billions and trillions, according to M. - number of decimal places.

Michael reluctantly gave his answer to the last question - thinking it was too "unmathematical"? - as a mixer. Or, as I suggested, perhaps he meant a blender. He had. (l. 545 - 553)

Brent: (App. D, pp. 60 - 63)

Brent was definitely not open to any new suggestions that would interfere with his already preconceived ideas. His answer of "No" came as something of a surprise to me. Was not this graphic display of a function machine the very kind of display offered in Elementary School Mathematics textbooks to help explain the input/output aspects involved? But "No", B. had his mind set on a graphic display of another nature, that of an x/y coordinate system. These two displays did not converge onto the same thought. He did acknowledge however that the machine model might be useful to help people understand that, "depending upon your values of x, it's going to determine your outcomes of your $f(x)$". (l. 345).

The description of the calculator as an example of a function didn't impress B. too much either. He read the first part of this paragraph, accepted what it was stating, but then found the last part confusing. He was ready to ignore the parts he hadn't understood. Again I made him "face up to them" and discovered that he had had a misconception as to why the calculator would give an "error" message if $x < 0$. "Because you can't have zero square root." (l. 357) He was duly corrected on this point and agreed that the root was "noncomputable" (l. 359), hopefully for the correct reasons this time! The limitations on the value of a square root obtained on a calculator were readily seen. (l. 363, 365)

On re-reading the transcript I noticed that I failed to point out B.'s usage of the word "function" in his statement, "your calculator is an important function of math. . "(l. 353) It has now been noted!

Since B. had not been very enthusiastic about a machine model of a function, nor about the example of a calculator, I hesitated to ask for anything further along these lines. He became somewhat more enthused however in answering the last question by describing what he thought was a far better model of a function, i.e. a computer. His job at the bank apparently entailed a lot of work using computers and this was therefore a real-life example to which he could more readily relate. (l. 367 - 383)

Peter: (App. D, p. 64)

As can be seen, this section of my interview with Peter was very short. Yes, he found the machine model helpful. "It makes it more graphic". His graphic descriptions of putting guys in at one end, and "the other guy coming out the other side is going to be a little different", matched up to the graphic display of the machine! (l. 292)

He appreciated the example of a calculator and understood the implications expressed in the square brackets. (l. 294)

Another example of a function machine? An automobile. You put in so much gasoline and you go so many miles. "The miles would be the function of the gasoline". (l. 302)

Comments on PART 4:

In reverse order, there is little comment to make on Peter's interpretation of this part of the text. One might say that "all went according to plan", from the author's point of view. The graphic model that was designed to help, did precisely that for him. The example of a calculator that was offered as a realistic model of a function was well received. And finally P. showed his complete understanding of the concept of function up to this point by providing me with an excellent example of his own.

Brent's reactions to these paragraphs were quite different. As already indicated above, B. by now had a certain conception of function that served him well. He did not feel the need, nor did he have the desire, to look at it from another aspect. Anything that did not conform to this conception just added confusion, to his way of looking at things. So the graphic display, contrary to its helpful purpose did not in fact help him and the example of the calculator took a back seat to his own example of the computer. So much for the theories about the inclusion of pictorial representations in textual material! And real life models can only be considered "real" if one has first hand experience with them. What is "real" to one person may very well be foreign to another. Not that calculators were foreign to B. but he related his conception of a function much more to a computer with which he was more familiar.

Michael's interpretations of this text, however, continued to justify my labelling him a "literal reader". Every word, every phrase was analysed in detail. His hangup about the word "function" being better replaced by the word "formula" still followed him into this section. And now this was compounded by the apparent contradictions in what he was reading about "domaine". This whole problem was of course centered on the symbolic manner in which the function machine was labelled. Placing the "f" in the middle portion of the machine meant that this "f's", this "function's", this "formula's" abode was centered right there - to follow M.'s logical way of thinking about all of this. So the "domain of f" being the central part of the machine is totally understandable. He did not have an overall view of a function as being the whole relationship which Brent and Peter had clung to, despite how the definition in the textbook had read. To Michael the words, "A function is a rule. . etc" were the words that mattered.

If the author of this text had given a definition of function in its wider relational context, then it's my belief many of the problems encountered by Michael may have been avoided.

* * * * * *

PART 5 (see Appendix E, pp. 65 - 97)

Michael: (App. E, pp. 65 - 76)

In Part 4, it was seen that M. related well with the function machine model - he could "see" the input, "see" the output, and visualise that something was happening inside the machine to produce the change on the input - this something being his formula. His problem had been relating what he was **seeing** with what was **reading**. Now he was faced with another graphic representation of a functional relation, one that showed **how** the input values related to their respective outputs. This type of diagram was also offered at school level because it was felt that it reinforced the relation between the x and the $f(x)$ - the significant word here being the "relation". M. had not only proved how much of a literal reader of words he was but now showed again how much of a literal interpreter of diagrams he was, even to the extent of separating the "dots" from their "names". He had to circle the "x" himself and link it to the "$f(x)$", again circled, before he would accept the fact that they were linked. Likewise for the "a" and the "$f(a)$". *(see App. E, p. 75).* I had to actually tell him that the "dots" were representing these values and that they had in fact already been linked. As can be seen, he had to physically link the expressions themselves before he could accept what this diagram was telling him. (l. 566 - 569). He was very unhappy that this diagram did not represent the "rule" part of the function. (most of p. 66). He felt that, ". . we have to know something happened in between there." (l. 603). *(also read M.'s answer to Q. 1, on p. 75)* The ball fell in my court therefore to try and convince him, before this session was over, that there was a benefit to be gained from the arrow diagram.

His answers to Q. 2 and Q. 3 were recorded without any major conflicts of terminology or understanding, (ll. 614 - 649).

And then we came to Q. 4, the question concerning the reversibility of functions, the question that had been a recurring theme throughout this session. M. recalled what we had determined earlier in Part 3 that the formula $C = 2\pi r$ using r as the input to find C, could be re-written in terms of r if we had C as the input. However as was indicated in Part 3, M. had had difficulty there coming up with the correct "reversed" formula and it had never actually been recorded. *(see App. C, p. 36, where this was discussed, and p. 43, M.'s notes, where there is no sign of this formula in terms of r.)* So we ran into this same problem again here, only this time I allowed M. time to arrive at the correct formula himself. This had not been a particularly easy operation as can be seen from ll. 667 - 694.

The major question was then, could one determine from an arrow diagram whether or not a function was being depicted?

(This and the following question were not out of the text but had been felt by me to be a fair test of their understanding of what had been discussed from the textbook concerning the meaning of a function. So the following findings I believe are relevant to this whole topic).

This time M.'s "literalism" acted in his favour. Although he at first used the words "the same output" (l. 758), when questionned on this, he rephrased it as, "Well I have to come out with one. . there's only one answer". (l. 760) And this concept had actually been clear to him from the textbook definition. "That definitely comes across here. . Absolutely". (ll. 764, 766). He even readily came up with a mathematical example of how two inputs could have the same output by referring to the values obtained when finding points from which to draw a parabola, e.g. both a -2 and a +2 would produce a +4. (App. E, pp. 70 - 72). He was beginning to see the merits of the arrow diagram although he still thought, "it should show something happening in there". (l. 791). *(See M.'s notes, p. 75 and 76 where he physically showed "something happening in there" himself by drawing in the additional "zig-zags" and lines).*

And finally, could one determine from the arrow diagrams which ones would remain functions when reversed? M. had few problems with this question since he had readily seen the "functionality" of the diagrams from Q. 5. Figs, (i), (ii) and (iv) were not reversible, and only fig (iii) was.

(On re-examination of this question I realise that I had failed to eliminate fig. (ii) from this discussion of reversibility since it had been ruled out as being a function in the first place. I also inadvertently referred to it as a function (l. 836). I have indicated this with an, "Oops!!")

Brent: (App. E, pp. 77 - 89)

Brent, in this section, reinforced all my observations on the way he read and his attitude on confronting new material. "I already understand it. . Why are they asking me to go back and think it through again?" (l. 403).

He did see that the arrow diagram might be "beneficial" but "only for somebody that basically (had) no math. . or poorer than myself. . " (ll. 405, 407). It was incumbent on me again to try and convince him otherwise!

The only difference he really saw between the arrow diagram and the machine model was that the machine model stressed the **inputs** and the **outputs** whereas the arrow diagram stressed the two **sets** of values involved, the set A and the set B. He did **not** feel, like M. did, that there was something missing in the middle. He was satisfied with the arrow labelled "*f*" below the diagram indicating the function. As for Q. 3, he was able to identify the input with the x values and the output with the values of f(x). (l. 435)

(Again it is to be remembered that these questions were re-formulated after interviewing M. earlier that same day, so they were a reflection of some of the points raised spontaneously by M.)

Apart from some personal reflections, B. succeeded in answering questions 4 and 5. The main item of dialogue of consequence was B.'s reply that, "They would have to be", to the question of the reversibility of functions. (ll. 470, 471). This thought was held for the moment!

As for what now had become Q. 7, Which of the arrow diagram represented functions?, B. had recorded the answers, N. for figs. (i), (ii) and (iv) and Y. for fig. (iii). (see App. E, p. 89). When asked why this was so, he reinforced his misconception of function that had been about to surface on line 471 referred to above. Brent's understanding of a function was of a "one-to-one" nature, "variable per variable". (l. 501). Therefore fig. (i) did not compute, to paraphrase B.'s words. (l. 495). He was asked to re-read the definition of a function as presented in the text. His reaction, "Now I could say it is possible that 2 different elements in *x* could have the same element in *f(x),* but . ." he could not see this from a practical point of view. If he worked on an input value in a formula there was only one answer for the output. Or was there? I made him think again. It worked. He thought of a -5 and a +5 being squared to produce the same result, a +25. So now we were able to justify fig (i) as representing a function. In like fashion we were able to eliminate fig. (ii) from being a function because it showed 1 input having 2 outputs.

This section of the interview was followed by personal glimpses of how Brent studied and how today's session had opened his mind to reading textual material more closely for its intended meaning. (ll. 537 - 539). This was music to my ears!

We then tackled the question of which diagrams represented "reversible" functions and as can be read from line 553, this was now clearly understood. B.'s one-to-one concept of function could now be related to something specific. It was the necessary condition for a function to have an inverse.

I had not conducted the next part of our discussion - referring this concept of one-to-one correspondance back to the original paragraphs of Part 1 - with Michael but I did with Peter and the results were similar. So briefly stated, it was noted that the questions on the area of a circle and the bacteria were reversible but that perhaps that of the cost of mailing a letter according to its weight was not, since there was no direct formula involved.

The next two pages of dialogue with Brent took the form of B.'s reflections on how he had been used to studying and what he had learned from this present session. (p.84, 85). Again more music! "I think it will help me in the future pick out the key words in reading math texts. . relying more on diagrams. . " (l. 577). It continues, "Now with math courses, what I'm finding is that you can't rely on your text alone. It has to be accompanied with teaching in the classroom and practice. . " And as for textbooks, "They have to be alive and in colour". (l. 583). As for THIS textbook, "It is alive and in colour. They have a lot of graphics. Major points they have in boxes, which I like. ." (l. 589). If only there was some way of getting around the "technical jargon"! But then again, B. admitted, it wasn't really "technical jargon". It was the **language of mathematics**, just like any other foreign language. It had to be there, but in B.'s *ideal world* everybody would be taught in the language best suited to his own background. (l. 591)

A possible compromise between the mathematical language, which B. saw as necessary, and the exercises based on the theories should be, that these exercises themselves be based on "something more concrete. . something more real. . " (l. 595).

It was only then that I had Brent take a look at the Grade 10 textbook that had already been presented to Michael *(App. C, pp. 41, 42)* and Peter *(App. B, pp. 32, 33)*. The introductory part, p. 12 *(see App. G. p. 103)* was the part that B. had found "insulting" (l. 623). This was not written for a person who had been "out in the work force", (l. 607), but the excerpt from the later chapter, *(App. G, p. 104),* he liked. He found in it the type of concrete, realistic examples he had just been alluding to. He could relate to parking meters!

(We diverted somewhat on the matters on hand by wondering just how meaningful this particular example of a "real life" situation would be for school children who had never been to a city, e.g. his own background from the Gaspé. We concluded that what was "real" for city school children would not be "real" for their rural counterparts.) (l. 614 - 621)

Our conclusion of this session brought us back once more to the subject of graphics and visual representations. B. reaffirmed his commitment to "try to pay more attention to the graphics." To "try to correlate the graphics with the text." He agreed with me that the problem with visual representations was that what worked for some did not necessarily work for others, and went further to suggest that "some people aren't visual". (l. 633) "Some people are more theory minded". He again expressed his appreciation of this session by vowing that he would "get more value out of my graphs from hereon in. . " as long, that was, as he was given the opportunity to visualise them for himself.

Peter: (App. E, pp. 90 - 97)

It is difficult perhaps to understand how Peter could have sat through this whole discussion without putting pencil to paper. But although he did not write anything down, much pointing to relevant parts of the text or, in this case of the drawings, was done. There was no problem in seeing this graphic as depicting a function. *(He did not really answer the question as to whether it helped him understand it better).* But it is interesting to note that he had the same comments to make vis à vis the machine model as did M. Where had the middle part gone? "I don't see the work that's being done. ." (l. 317).

I thought for a moment that we were about to get into the same discussion about the O.E. meaning of the word "domain" that I had had with M. ". . from this domain to this domain. . "(l. 311) but P. corrected himself by rephrasing the latter part of this statement as "the range of numbers here." (l. 319) He appeared for a moment to be offended that I was questionning him on this! (ll. 327, 329). We let it pass as a "slip of the tongue".

Q. 3, concerning the inputs and the outputs, presented no further problems. (l. 335). And Q. 4 - 6 were relatively uneventful. (ll. 336 - 366).

Question 7 revealed, as it had done for Brent, some lingering misconceptions on the meaning of a function based on the textbook definition. The concept of the uniqueness of the value for *f(x)*, as opposed to *x*, had not yet really come across. (ll. 367 - 377). My noncommittal replies and insinuations that he was wrong, however, made him re-consider the definition. He then thought that perhaps it was feasible to have 2 different inputs produce the same output. He gave as an example a variation on the capital/province question we had just looked at by changing the function from "is the capital of" to "is a city in" to realise that both Montreal and Quebec City would be correct inputs for "the province of Quebec". (ll. 379 - 383). When pushed for a mathematical example, he gave me: $y = x^2 + 2$, again from recent work on parabolas. But still not too sure that he was on the right track! (l. 391).

This uncertainty turned out to be well-founded. (ll. 395 - 415). On re-reading this transcript I find I was particularly hard on Peter at this stage but I really wanted him to come to grips with this concept of the uniqueness of the *f(x)* from his own reading of the definition. My persistence was rewarded. He found the words I had been wanting to hear. ". . exactly one element. . " (l. 417). And so we were able to rule out fig. (ii) as being a function. But my persistence in having him read the definition of a function 'properly'? backfired when he insisted that fig. (iii) could not be a function because it did not include every value of x!! The definition states, " to EACH element x. . . " and "we're not doing that". (l. 437). He had a point! I quickly directed his attention to the last figure, fig. (iv). This time there were no more doubts. "It works. Exactly one element." (l. 439)

And finally the question on their reversibility. The slight confusions apparent here were, I believe, caused by his not writing things down. Basically he had things figured out correctly now. (ll. 441 - 457). My last comments on this section concerned one-to-one correspondence and the inverse function.

As stated above in my account of Brent's interview, we compared these recent findings with the functions of Part 1, only this time it was student-initiated. (l. 465 - 493).

Peter was then given the chance to comment freely on what he thought about mathematics textbooks. The highlights of these comments would have to be:

i) that he could relate to what the book was saying once it had been explained to him in different language in the class,
ii) that if he read it long enough it would sink in but that it didn't make very clear sense at first,
iii) that he had always found mathematics textbooks to be cumbersome, to be not very user friendly, and
iv) that for him the easiest way to learn a mathematical concept was to have it explained to him with examples.
(all of the above comments i) - iv) are paraphrased from Peter's own words, lines 497 and 499).

Comments on PART 5:

The major aspect of this section of the text was again whether or not the visual representation of a function, now as an arrow diagram, helped get the meaning of a function across. And how did this diagram relate to the machine model of PART 4?

For Michael, apart from the business of encircling the letters x and $f(x)$ and seing **them** linked as opposed to the dots, his main concern was that this diagram did not "show" that something was going on **in between** the x and the $f(x)$. This thought was echoed by Peter. The definition of a function as a **rule** should surely guarantee that one should always be made aware of the **rule** aspect of a function and not merely its components, the input values and the output values? The differences between the two diagrams were what came across the strongest for M. and P. For Brent, the reaction to this other view of a function was similar to his reaction on viewing the first. Why do I need this? I already understand what they mean without it.

So, as I indicated earlier, these three reactions to a supposedly helpful visualisation of a function have certainly put in question this whole concept of overcoming reading difficulties by offering more pictorial representations. Or as Brent himself said, "Some people aren't visual." (l. 633).

The next point of note was the apparent reversal of roles in initial understanding of the implications of the definition of function as given in the text. M.'s literal interpretation served him better than did the interpretations of B. and P. He had READ the words, "exactly one value of f(x)"! And so the rest of the questions on the reversibility of a function were readily understood by M. And he eventually admitted to seeing a value in the arrow diagram, even though it failed to show what was happening to the input values.

Both Brent and Peter, who had not been happy with the textbook definition of a function as just being a RULE and who had adhered more strongly to the relationship meaning, had in fact not really READ the definition in its entirety. For Brent, a function was a one-to-one relationship, until he was made to READ the definition for a second time. It took three or more re-readings on the part of Peter before he actually SAW the words, "exactly one value". To me, both these gentlemen had read the original definition of a function from their own biased preconceptions of what a function was. This bias had been well-founded however - their reading of the paragraphs of PART 1.

The final outcome of the interpretation of arrow diagrams ended successfully and included the significance of the special case of the one-to-one correspondence being the necessary condition for an inverse function.

Michael had already had (or rather had taken) the opportunity to voice his own frustrations as regards mathematics in general or mathematics textbooks in particular in PART 1 and I had deferred commenting on them. *(see App. A, Michael, ll. 117- 134, pp. 6 - 7).* But since B.'s and P.'s comments have been summarised above in this part of their interviews, I will go back now to what M. had to say. Briefly, his main objections were directed towards this particular mathematics textbook. He claimed to have had great success studying from the Math 200 text, but this one seemed to have left him in a state of deep confusion. But not only the textbook, the whole structure of the Math 206 course did not apear to be a direct consequence of the Math 200. "Is there a Math course that I missed in between here?", he asked. His method of working from the textbook explanations, to doing the problems, and to referring to the answers when in doubt etc. . . had served him well in Math 200, but this method was not working for him now. He could not find the explanations he was looking for from the text. "And I can't for the life of me figure it out from what they're saying in the text", to quote his own words. *(l. (or rather paragraph). 133, App. A. p. 7).* He was not happy with this textbook!

In comparison with these thoughts from M. we have B. who DID find the textbook "alive and in colour" and therefore good, and P. who echoed some of M.'s misgivings. "The (Math) 200 book was better than this". (App. E, Peter, l. 511, p. 97). All three of them either explicitly stated or implicitly implied that the best way to learn Mathematics was to have everything explained to them in class by the teacher, both the theory and how to do the examples, in "intelligible" language, and then they would *perhaps* be able to understand what they were reading in the textbook. At least they would stand a better chance of understanding it. Even B. who had not had any complaints about this particular text still found it advantageous to attend Math class. (see above). His main problem, with any mathematics text, was the "language". "It's like learning another language" *(App. E, Brent, l. 591, p. 85)*. Ideally if someone could explain the concepts to him in his own language, taking into account his own background then mathematics for him would be immensely more understandable. But he realised this was an idealistic view point.

(This is an interesting thought from the point of view of tutoring. In tutoring situations the student has the chance perhaps of this actually happening. In this kind of one-on-one setup the teacher can explain the concepts in the language best suited to his student and if he knows a little of his student's background, can relate the mathematics to this background in the most meaningful way to him.)

Tying in one more time the above discussion on the readability of the text with the actual concept of function itself, the students could be seen by the end of this session to have experienced some of the acts of understanding mentioned earlier. They were more readily able to discriminate between the dependent and independent variables, they became aware of the use of functions in modelling relationships, and they (Brent and Peter at least) became better able to discriminate between the different means of representing functions and the functions themselves. On the other hand they could be seen to have overcome some of the epistemological obstacles related to the understanding of the concept of function such as, regarding the order of the variables as irrelevant and thinking that only relationships describable by analytic formulae were to be considered as functions.

With these thoughts, I have come to the end of this Chapter 3 of presentations of what transpired during these 5-part interviews and my comments on them. My final chapter will be a summation of these findings and will compare their relevance to the introductory views outlined in Chapter One.

* * * * * * * * * *

CHAPTER FOUR

SUMMARY AND CONCLUSIONS DRAWN FROM THE INTERVIEWS

It now remains incumbent on my part to relate the findings of these interviews with the various aspects of the problems encountered with the reading of mathematical text outlined in Chapter One.

Firstly, one of the results of this investigation enables me to justifiably refute at least one of the reasons given for why readability tests were not compatible with M.E. i.e. (iii), or a variation of (iii), that adjectives are unimportant in M.E. Apart from the more obvious adjectives used in M.E. like, improper fraction, complex number, or closer to the topic under discussion, one-to-one functions, etc. . . instances where the adjectives in fact do play a very significant role, there occurred an incident involving a slightly different use of the adjectival form. In the opening paragraph the words "there is associated one value of A", the word "one" was not taken to mean "only one" or "exactly one", but it would seem it was simply read as "a value of A". This misinterpretation of the word "one" was, for me, the reason why B. and P. failed to pick up on the uniqueness of the $f(x)$ in PART 5. It was not until they were made to re-read the definition, where it more explicitly stated "exactly one", that this concept came through.

Under the section (b), removing reading difficulties, in Chapter One, it was suggested that repetition of key words and the offering of pictorial representations would be two ways of overcoming some of the reading difficulties. My findings here do not necessarily comply with these suggestions. First the key words had to be "coaxed" out of the text by me for M. and B. to see them. If this is how grown adults react to finding key words, what about school age children? But perhaps children in school are better equipped to do this kind of thing? To a certain extent they are "brainwashed" into picking out important words in their readings of other subjects, so this exercise may in fact be easier for them than for my adults. But this is pure conjecture.

And second, the pictorial representations, in this case the machine model and the arrow diagram, caused more problems, for M. anyway, than they solved. And B. didn't see the need for them in the first place. He preferred visualising things for himself. P. was the only one who appeared to benefit by them at first viewing. The last words here are significant - at first viewing. It took, in fact, several viewings to see the real implications behind the arrows in the arrow diagrams.

The third section (c), the language in the classroom, was addressed by both B. and P. who saw the classroom explanations, in everyday language, as a pre-requisite to being able to understand the language of the text afterwards.

On the issue of "active" v. "passive" reading, which arose out of the first chapter of the H. Shuard and A. Rothery book, "Children Reading Mathematics", I find myself disagreeing even more strongly with Wing's interpretations than before. Readers will automatically put their "own meaning onto the page", but is it the desired meaning? "Getting the meaning from the page", does not suggest a passive act, to my way of thinking. My students in this study were obliged to find the meaning of the concept of function from the opening paragraphs of PART 1. They had to find the meaning "behind the words", or Krygowska's "vitrine formelle du texte". Doing so passively would have resulted in their total non-comprehension of this text, and this certainly was not the case. Even their misunderstandings were in themselves active reading. Their underlinings and outlinings and other comments they wrote on their pages were other indications of active reading, for M. and B. in particular. But on the other hand, P.'s non-written attack on the text did not convey non-active thinking. He was very much the mental thinker, and highly active. One might even make the point that the more active mentally one is, the less apparent will be the activity associated with it.

Chapter 2 of this book was on the Characteristics of Mathematical Writing, which included a classification of the types of mathematical text. As indicated in Chapter One my main objective in this exercise was the study of the type of text that came under the heading (a) exposition of concepts. This extract from the College Algebra text was precisely that. "To be read and digested but not necessarily acted on", were the words accompanying this classification in my Chapter One. By "acted on", I understand these words to mean "no exercises performed" or "questions answered" directly on reading the text. But in order to ascertain what had been understood it had been necessary for me, of course, to ask questions, but not the kind alluded to in this context.

The third chapter of the Shuard/Rothery book was the one that had gained my greatest attention, in particular their section on the ambiguities of vocabulary. As stated in my first Chapter, these ambiguities are threefold, but the ones that interested me the most were those that concerned words that occurred in both O.E. and M.E. but with distinct meanings under each category. This study produced several of such words. First we read of P.'s identification of the word "rule" with something dogmatic; a law that is laid down and is the same for everyone or, in this case, for every function. Then we had M.'s problems with the word "domain" or, as he prefered to spell it "domaine", which carried with it all the connotations of a dwelling place. Next there was the word "function" itself. Were all these rules or formulas merely serving some useful "function" in mathematics? And as for the word "image", the message came through loud and clear that some additional background knowledge was necessary to know what was meant by "the image of x under f". I hesitate to include the word "range" in this list of ambiguous words since it didn't appear to have caused any misconceptions and also because its more commonly used meaning was close enough to its meaning in this functional context as a "range of values", but I'm quite sure another student would have inadvertently referred to the "range of values of the domain", as I have heard it said, in the same way as P. inadvertently referred to values being "transported from this domain to this domain".

One possible solution to this problem would be the recognition of the mathematical meanings of these types of words in a Glossary at the back of the textbook. This college Algebra book had none, of course, but the Grade 10 book referred to did. *(see a typical page of this Glossary in App. G, p. 109).*

I have already reaffirmed my disagreement with Wing's philosophies as regards passive and active forms of reading, but I do have to agree with him that the role of the text should not be taken for granted. Brent, in his final comments, echoes this sentiment when he says that "you can't rely on your text alone". The need is definitely still there for the teacher.

This brings me once again to the investigations and subsequent recommendations of Z. Krygowska. My own study supports three of her main ideas:

(i) that students can be taught to read mathematical text - in spite of their own mathematical background and past experiences,

(ii) that students can be made aware of the verbal, symbolic and graphic elements in mathematics, the style and sentence construction and the precision and conciseness of its language, and

(iii) that students should not be left under the impression that memorisation constitutes learning.

Examples of (i) came across strongest with B. who frequently referred to his "history" by which he meant his past experiences in mathematics, and who equally frequently related the mathematics in question with his work experiences. By "opening his eyes" to some new ideas he was able to benefit more from his reading of the text.

Examples of (ii) came across in all three interviews. The verbal implications of the definition of function as given in the text - "a rule", "exactly one" - the symbolic notation of $\{f(x) \mid x \in A\}$, and the graphic representations of a function as a machine or an arrow diagram were all areas in which the students needed coaching in order to appreciate their true significance.

The memorisation issue was only evident with my discussions with B. He talked of the way he had become accustomed to studying psychology - read the text, memorise the theory, apply the theory to the text. There was no need to visualise anything. This method no longer worked for him in his mathematics courses. Now he realised that he had to try and visualise what was happening since to him mathematics was more concrete than psychology. He could not just rely on memorisation of a bunch of facts.

I would concur partly with Krygowska's recommendations regarding textbooks in that they should include diverse techniques of presentation, especially those that evoke some active involvement (not just active reading!) on the part of the student. I have included some worksheets on functions that I happened to come across, unfortunately long after these interviews were over, that would certainly fit this recommendation. *(see App. G, pp. 105 - 108)*. However I do not wholly agree that students should be sensitised to the use of more sombre textbooks in their later years. I do not agree that advanced mathematics textbooks should necessarily be sombre! And neither did these adult students. B. was all in favour of a mathematics textbook that was "alive and in colour", M. wanted a mathematics textbook that explained things in layman's terms and P. liked his Math 200 textbook because it had, "bullet form recaps". *(see some sample pages of the Math 200 textbook in App. G, pp. 110 - 113).*

One of my objectives was to investigate the differences between young students' perceptions of text and that of the older student, especially those returning to studies after several years in the work force. We can be quite sure that the older student does not want a boring mathematics textbook any more than does the high school student. It's like eating a meal that has been artistically arranged on a plate. It stimulates the taste buds before one even starts to eat. If the textbook looks inviting, then the reader will want to read it.

Another aspect of the older student was the question of their more extended usage of O.E. as opposed to that of the teenager. Returning once more to the words "function" and "domain", it is possible that these words would have little prior meaning for the teenager before he met them in the mathematics classroom, especially "domain". (How something "functions" is not everyday talk but is more likely to have been known in this context.) But to the adult these words would possibly be much more commonplace and hence their greater chance of causing confusion.

And it is not only that a difference in age or maturity might cause different forms of ambiguity but also a difference of milieu. To a high school student, a "rule" may well make him think of the "rules" of the school, while from someone like M. or P. we hear comparisons to "governing laws" or things that are "dogmatic".

We have also to take into account the past history of the older student. M., who had been out of high school for twenty-some years, told me his math background was weak. He got through "on the Bell Curve". He was in fact only starting to gain some confidence in his mathematical abilities. B., on the other hand, a slightly more recent high school graduate and a graduate of CEGEP, could remember more math from his high school days but related his math very strongly with his work at the Bank. He was also well aware of his limitations in mathematics. P., like M., also in his forties, was coming back to math after a long absence and needed a certain amount of prompting in the right direction.

My own reflections, partly as a result of this investigation, and partly from my previous experiences in the classroom, lead me to believe that the whole process of reading a text, mathematical or otherwise, is in fact a three-way process. First there are the concepts in the mind of the author. Second there are the words used by the author to express these concepts. And third there is the reading and interpreting of these thoughts by the reader. This last could even itself be subdivided into three distinct actions, the reading, the interpreting, and the appreciation of the language used to express these thoughts. For B., the language of mathematics was just like any other foreign language. To a certain extent one would have to agree. The Frenchman states, or writes, "Il fait beau aujourd'hui". He doesn't have to think this out grammatically, syntactically or what vocabulary he needed to use. It is an everyday expression for him. The Englishman reads this statement, translates it (correctly or incorrectly as the case may be), then learns how to say it himself. It is only when he is obliged to reproduce this language for himself that he appreciates the vocabulary and the syntax that went into its formation. Is it not much the same scenario when the layman is faced with text written in the language of mathematics? He reads it, interprets it (correctly, or incorrectly), and it is only when faced with having to reproduce the mathematics for himself that the intricacies of the language come to the fore. For example, the reading of the symbolic set notation, $\{f(x) \mid x \in A\}$. The student may be able to read this, and even interpret its meaning in a global kind of way but does not really understand the language used if he cannot reproduce a similar statement on his own, thus taking into account all the symbolism involved.

To make one final comment on the text used in these interviews, it should perhaps be noted that the intentions of the author were no doubt good ones and the text itself was well structured. First there were the intuitive examples of one number being a function of another. Then there was a formal definition of a function. This was followed by further information accompanied by a graphical representation. We then had an application. And finally another visual aspect of a function. On the surface this looked very nice and structured along the lines one might expect a new concept to be introduced. The problems with it ran deeper, however, than its outward appearance.

(1) The formal definition did not send the same message as the intuitive examples.

(2) The further information wrought confusion with too much, "technical jargon". (Brent)

(3) The first visual representation was not found to be relevant for B., was confusing for M, and in fact was only acceptable by P.

(4) The application was not a particularly good one since in the end the author had to admit that the √ key was not the same as the function defined by $f(x) = \sqrt{x}$.

(5) The second visual representation lacked the "f" part for M. and P. and again was not seen as helpful for B.

Much of the above confusion and misunderstanding could, to my mind, have been avoided had the authors of this textbook just kept to the one form of meaning of a function at this stage. Their readers were after all basically College level students, but College level from the American perspective. In Canadian terms this means "entry-level" for CEGEP or University, and then mostly for students who needed a minimal mathematics background as a pre-requisite for Commerce or related programs. Students of Mathematics would not be learning their Mathematics from such a textbook. So why make this concept so hard to understand? My own three students' appreciation of how a function was introduced in the Grade 10 book confirms this view. There it stated that a function was a relation - a special kind of relation. This definition of function is the one most frequently taught at the high school level because it is the one that is the most understandable as an introduction to this whole concept. The equation or formula aspect of the function is the least significant of its components. As a relation, a function can be more readily seen to not even need a formula as its "rule of correspondence", as it is called at this level. And this interpretation of function would certainly have helped M. get over his fixation that a function was nothing other than a formula. (My example involving Provinces and their capitals was intended to disspell this notion too).

To attempt at some more advanced meaning of function as being a rule that assigns values to elements in sets was definitely not on track with the opening paragraphs. In fact one might say that this textbook definition of function seems to have reverted back to even earlier definitions of function as stated by Bernoulli and Euler where it was the <u>law</u> of the function that was the more important concept. (see Chapter Two, p. 15).

This apparent disagreement among mathematicians as to what definition of function best fits the occasion is echoed in several fields of reference such as Dictionaries, both of O.E. and M.E. . . and O.F. I quote. .

function: (a) A **variable** so related to another that for each value assumed by one there is a value determined for the other.

(b) A **rule of correspondence** between two sets such that there is a unique element in one set assigned to each element in the other.

- both from the one source, The New Book of Knowledge.

or again:

function (map, mapping): A **rule** that assigns to every element x of a set X a unique element y of a set Y, written $y = f(x)$ where f denotes the function.

A function can also be defined as the **set of all ordered pairs** (x, y), with x belonging to the domain X and y belonging to the range Y, where there is a many-to-one correspondence between the members of X and the members of Y. - both from The Penguin Dictionary of Mathematics.

And also:

fonction: **Relation** qui existe entre deux quantités, telle que toute variation de la première entraîne une variation correspondante de la seconde,

ou, en termes d'ensembles, étant donné deux ensembles X et Y, "**toute opération** qui associe à tout élément x de X un element y de Y que l'on note $f(x)$" (Choquet) - from Le Nouveau Petit Robert.

And comparing these definitions with the one quoted earlier in my Chapter Two, it would seem that there are two worlds of mathematics, or two languages perhaps, that of the Mathematician and that of the mathematics student, the layperson.

Our authors of this textbook cannot therefore bear sole responsibility for their definition of a function as a rule - see Penguin above - but their blame for the students' problems lie in their presenting this definition of function after "misleading" them in the opening paragraphs to seeing a function in the light of "one variable being a function of another", more along the lines of definition (a) of The New Book of Knowledge.

As can be seen it has been virtually impossible to divorce the readability of this piece of text from the actual understanding of the concept of function itself and how this concept was presented to the students.

But to finally return to the readability issue, several times throughout my analysis of the interviews I referred to M. as a "literal reader". I claimed at another time that B. was a "skim reader". I should not terminate this discussion without labelling P. as some form of reader as well. To me he was more along the lines of an "in-depth reader". Although he did tend towards the "literal" side at times, he knew when the literal meaning was not the one intended and could read beyond "la vitrine formelle du texte" much better than his class mates. It has become increasingly clear in this observation of adult students reading a mathematical text, that mathematical symbolism is essentially metaphorical in nature.

I therefore finish this essay off in the same vein as I began with Alice's reply to the Duchess.

> "I think that I should understand that better, " Alice said very politely, "if I had it written down: but I can't quite follow it as you say it." L. Carrol

To the reader to determine the meaning behind these words!

* * * * * * * * * *

REFERENCES

Austin, J. L. and Howson, A. G. (1979), Language and Mathematical Education, *Educational Studies in Mathematics*, 10 (3), pp. 161 - 197.

Boyer, C. ; Merzbach, U. , rev. ed. (1991), *A History of Mathematics*, New York, J. Wiley & Sons.

Daintith, J. and Nelson, R.D. (1989), *The Penguin Dictionary of Mathematics*, London, Penguin Books.

Durkin, K. and Shire, B. (1991), *Language in Mathematical Education*, Milton Keynes, Open Univ. Press

Flewelling et al. (1989), *Mathematics 10 - A Search for Meaning*, Toronto, Gage.

Gardner, M. (1970), *The Annotated Alice*, (Alice's Adventures in Wonderland, by L. Carroll), London, Penguin Books,

Krygowska, Zofia (1969), Le Texte Mathématique dans l'Enseignement, *Educational Studies in Mathematics*, Vol. 2, pp. 360 - 370.

Le Nouveau Petit Robert, (1993), Paris, Dictionnaires Le Robert

Sierpinska, A. (1992), The Concept of Function, *MAA Notes*, Vol. 25.

Shuard, H. and Rothery, A. (eds) (1984), *Children Reading Mathematics*,, London, J. Murray.

Stewart, Redlin and Watson, (1992). *College Algebra*, California, Brooks/Cole.

The New Book of Knowledge, (1979). Dictionary Vol., U.S.A., Houghton Mifflin.

Wing, T. (1985), Reading Children Reading Mathematics, *Mathematics Teaching 3,,* pp. 62 - 63.

Yoshiwara, K.F. and Drooyan, J. (1994). *Elementary Algebra for College Students (8th ed.),* Englewood Cliffs, N.J., Prentice Hall.

APPENDIX

A

APPENDIX A

INTERVIEW WITH MICHAEL: 94 - 2 - 23

PART 1:

1. A: I'd like you to have a look at this part first.
2. M: Do you want me to start reading this?
3. A: Yes. To read that, and then answer the questions and then I'll discuss it with you. *(M. reads the 4 paragraphs to himself)*
4. M: You want me to answer these questions right away?
5. A: Yes, I hope you understand my question.
6. M: I understand what was going on up here basically... the rule.. the function
7. A: No, function is not a verb
8. M: Well, what exactly do you mean by the verb?*(Reads the first paragraph in part one again.)*
9. A: If you look at the first sentence from each paragraph.
10. M: Right.
11. A: Okay, the same word appears in the first sentence in each paragraph.
12. M: I don't understand what you mean by verb I guess.
13. A: Okay. What is the idea that the text is trying to convey? Can you pick out any key words here that are appearing in each paragraph?
14. M: Here, or..
15. A: No, anywhere... anywhere at all now. Are there any key words that keep coming back?
16. M: The area.. *(reads the parags. again)* The connection between the two*(repeats)* the area, the radius and the area and the number of bacteria, with time. Those are the connections that I make, and the cost and weight. So, if it's all things that are related to each other, then that helped me come up with an answer.
17. A: Okay, they are related to each other. .... Is there a word they're using for... have they used the words "related to each other" in there?
18. M: Well, "connects".
19. A: They have got "connects".
20. M. *(reads through some of the paragraphs, finding the word "connects".)* Connects t and N.. connects again.. There you go. The relationship between the two.
21. A: But really, I was hoping that you could pick out ones on the first line of each paragraph.
22. M: Okay.
23. A: Look at the first line of each paragraph, not even just the first sentence, the first line. The same word appears.
24. M: "depends, depends, and depends".
25. A: That's the one. What does that word "depends" mean to you?

26. M: Depends, the area of the circle <u>depends</u> on the radius. So it's... it's... depends on each other, if you don't have one you don't have the other.
27. A: Okay
28. M: Depends. . they're connected. . they're associated. .
29. A: Is it really saying that they are dependant on each other? Does the A depend on . . It's saying the area A depends on the radius. Is it also saying that the radius r depends on the area?
30. M: Okay, hang on.
31. A: Is it working both ways or is it one way?
32. M: . . number of bacteria and time. They depend on each other. Now if I had just the time, I wouldn't be able to figure out bacteria. If I had just bacteria, I wouldn't be able to figure out time. Umm, or would I? I know for a fact. . I know that if I have the radius of a circle, I obviously can figure out the area, and if I have the area, I should be able to figure out the radius also.
33. A: Okay
34. M: There's definitely a relationship between both of them, but they both depend on each other to come up with an answer.
35. A: Is it just as strong in the question about the bacteria and the time?
36. M: *(reads the first line of the second paragraph)*. Yeah, absolutely. You would need **both** to come up with a.... the number of bacteria... okay, so, if I had just the number of bacteria, 5000, and it doubled every hour. Well I would have to know the time, and if I had the time I would be able to.... how would I be able to figure out the bacteria. I don't know, maybe I'm looking at this wrong, but I would not say so.
37. A: Do you think there is more of a dependence of the bacteria on the time here than there was between the area and the radius?
38. M: Umm... I haven't worked with bacteria, so. . I've worked with the radius and the area, so I know that for a fact. .
39. A: I mean the radius and the area are very closely connected with each other. .
40. M: Okay.
41. A: Right? But bacteria and time, well we're sure that the bacteria will keep on increasing with the length of time, but is the time dependent on the bacteria? I mean, is time dependent on bacteria?
42. M: Is time dependent on bacteria? Well, you have to put . . in order for the bacteria. . there has to be some time for it to increase. They are numbers, so time is relative to the bacteria...
43. A: Within this question.
44. M: Within this question, they are definitely relative to each other. Absolutely.
45. A: Okay, what about the third one?
46. M: *(reads the third paragraph)* Well they would have to be related, absolutely. You would need the weight to figure out the cost, and you could only figure out the cost if you have the weight.
47. A: So you see this very much as a two-way process.

48. M: Very much? Yeah, but based on the first two? The second one kind of threw me off. The number of bacteria in a culture depends on time. I could see that you. . if you gave me the number of bacteria and you gave me the time, and you told me that if the culture starts at 5000 bacteria and doubles every hour, then I could see how time relates to bacteria. You need both, you need one or the other to, I guess, to figure out that equation.
49. A: So, the way it's worded doesn't suggest to you that one is more dependent on the other? Do you see it very much as a-
50. M: Well the first and the third one I do, the second one is just kind of... I don't know why it's throwing me off. I mean they should both be. . depends on the time. .*(reads the first sentence of the second paragraph)*. . depends on the time, depends on time. If the culture starts at 5000, I understand what's happening here, and you need one or the other absolutely to figure it out. Doubles every hour. See if .... .
51. A: Alright, well anyway, the answer I was really after was, of course, was the idea of "depends", and the verb was the "depends" and I was after a noun associated with this verb. Can you give me a noun from the word "depends"? It's not in the paragraph, it's from English, the knowledge of English. What noun do you get from "depends"?
52. M: umm... I would look up "verb" and I would look up "noun". I know from school, but....
53. A: Okay, a noun is the name of something. If the verb is "depends", what would be a word to describe the name of that situation where something depends on something?
. . . *(no reply)*
I'll give it to you, "dependency".
55. M: Dependency. I would have said, "dependent".
56. A: Dependent, well that is more an adjective again. I've made this too grammatical! *(laughing)*. O.K. I was after, "dependency". Alright, next question. Here I am again with my grammar. "The same noun is used is all the paragraphs to indicate the second concept."
57. M: The second concept. .
58. A: It's used in all the paragraphs. There's a certain word that comes up in all of them.
59. M: Well there was "connects".
60. A: Connects. Well that's a verb. O.K. So that could have answered the first question. There's a name of something that comes up in every one of them. . . apart from the word, "function".
61. M: *(after a short pause for thought!)* Connects, associated. . I don't really know. .
62. A: Well what is it that connects? Beside the word, "connects", very close to it. It's telling you something that connects. What is it that connects?
63. M: The values.
64. A: What word are they using? They're saying *something* that connects.
65. M: Positive number. .
66. A: What word were they using. . with that, "connects"?

67. M: The variables. . the **rule** that connects. .
68. A: The rule. Do you see that "rule". .
69. M: Yes, I see the "rule" there, but I'm looking for it here. . . "bacteria. . .is the rule". . . there you go again.
70. A: And is it used in the third one?
71. M: Yes.
72. A: And in the fourth one which is trying to summarize the other three?
73. M: Yes.
74. A: I was after the word, "rule" there. Now I've also said in the question that there's, "Another word very similar in meaning to this word is also used in paragraph 3. Is there another word they use that's very similar to *rule*? In the third paragraph?
75. M: Formula.
76. A: Right. I'd like people to realise that this word *rule* is appearing in all three paragraphs. . in all four of them. . is there a better way I could have worded that? I mean I had you hunting for a noun. You know, it's like "hunt the word" game here!
77. M: Well, it's not so much the fact that I'm hunting for a noun, it's just the fact that . . I don't know if I was a good person to pick for this because the nouns and verbs and pick them out. . . I'm not. . eh. . Yes, I remember them from High School, but as for what exactly they're describing, like what exactly is a verb, I couldn't give you a good description which is pretty sad (*both laughing)*. . . considering that . .
78. A: O.K. But is there another way I could have asked those questions to get people to realize that this word *rule* is the concept here. It's not just the word. The word is being used for a particular concept. Is there some other way, you know, I could have made you pick out that word *rule*?
79. M: *Rule.* ". . *rule* that connects r and A (reading #1). .
80. A: Dependency is like the first concept here. Three things I'm trying to zero in on. The rule, every one of them. . all these dependencies has a rule that connects one with the other . .
81. M: O.K. Well can we go back to depends. Can we go back to the rule? I was just thinking about that. And the area of a circle depends on the radius of the circle. O.K. Is dependent. . .the area of the circle depends on the radius of the circle. . rule that connects. . the rule that connects r and A is given by the equation . . the rule that connects r and A is given by the equation $A = \pi r^2$ . . the rule. . how else could they refer to it. . how else . . what other . .
82. A: Is the rule important?
83. M: Absolutely.
84. A: Yes. Well what's the rule telling you?
85. M: . . the rule . . it gives me an equation, so whenever I come across radius or area I have that equation.
86. A: It's telling you how to find one in terms of the other.
87. M: Right.
88. A: What's it telling you to . . what does the equation enable you to find, in the first one there?

89. M: Well, the area of the circle. . or the radius.
90. A: Or the radius. O.K.
91. M: And the second one the time, the bacteria. . .the rule connects t and N. . the rule. . I guess that's a good word. .
92. A: The rule has to be there?
93. M: Yes. The rule has to be there. It has to be there.
94. A: O.K. Let's see if we can move on to the third part. . the third question I've asked. I'm looking for a third concept.
95. M: *(reads and re-reads Q.#3)* . . .the third concept being what?. . the third concept referring to? *(reads Q.#3 again with emphasis on the third concept).* I don't know what you're referring to by concept.
96. A: There was three. . the way I read this, there was 3 concepts that these paragraphs were trying to get across. Well we've picked out the dependency part. We've got down to the fact that this dependency is based on a rule.
97. M: O.K.
98. A: And there's something else there . . .there's another idea which has got to be. . there's another idea coming across.
99. M: Corresponding, associated . . .
100. A: You've got it!
101. M: . . . connects. . . *(reads paragraph 4)*. . . In each case we say that the second number is a function. . you see I would have picked function. . assigned. .
102. A: That's the right word. You're beginning to play my game now *(laughs!)* O.K. Now then, you've kind of picked out these three concepts. Can you in your own words, Q.4 there, "Can you state in your own words what the three concepts referred to are?"
103. M: . . .
104. A: Would you like to write down beside Q.4 what you think here?
105. M: O.K. ". . what the three concepts referred to are?" Um. . .just again. . just to understand what you're asking. . .*(re-reads Q.4)*. . in terms of what they are? . . how they relate to these sentences? . .
106. A: What they are trying to get across. To me there was 3 concepts that they. . 3 key ideas if you want that the 4 paragraphs were trying to get across. There's 3 important issues here.
107. M: Alright.
108. A: And we've kind of pin-pointed them all. Could you put this . . you know, kind of phrase it yourself? Or just give me back the same words there that we got.
109. M: Right. The relationships.
110. A: O.K.
111. M: The governing law. *(laughs!)*
112. A: O.K! Write that down. It's a good one. Write it down!
113. M: It's pretty powerful!
114. A: Yeah. What simpler word did they used?
115. M: For that I would say rule.
116. A: Right. O.K.

117. M: Um. . Have you seen the Math 200 book?
118. A: Yes. I taught Math 200 last term.
119. M: So I guess you saw that book?
120. A: mm . . mm
121. M: Did you think that was an excellent book.
122. A: Yes. I thought it was pretty good.
123. M: The only reason I say this which. . . . like looking at this book versus that book. . I only take one course per semester.
124. A: Yeah.
125. M: I sit down, and I do that. I always consider myself to be a math dummy. Like I never really listened in High School. I made it through on the Bell Curve. Now I'm taking the Math and I told myself, "Well it's time to sit down and get into this".
126. A: mm . . mm.
127. M: Well I sit down and I start reading the book. And the Math 200 was pretty clear.
128. A: Yeah.
129. M: In the sense that it. . . when you came. O.K. they. . when I read a book . . before I go ahead with any of this. . when I read a book I go through what they have to say before it starts. And then they give a couple of examples, and I do the examples and I try and come up . . and I follow the rules they give . . . cause right now I'm just getting into the. . .into the process. . . How do they go about doing this? How do they figure this out? And I go through. . I do all the examples and I go, "Great! Everything is going fine." This was with Math 200. I come to this book and they seem to jump. . . things that were well explained like. . I think for someone that really. . most people that take Math courses . . I would think. . everyone that I speak to. . so the Math 200 book was very explicit in the sense that they had examples. . then I would do some questions. . when I came across one that I had trouble with then I'd refer back to the reading . . I'd go through it, then I'd go through it again and then I'd try and do it, and then I'd look at the answer. . and then I'd try and work back from the answer to find out how they did it. . and that's basically how I found out. . like that's how . . I did very well on my last Math.
130. A: *(one of many, "O.K.'s")*
131. M: You know I did. . . I actually even got into the mathematics. . for once in my life I sat down and said,"I understand how. . ." I mean, I realise. . I don't necessarily understand the applications . . you know I'm just going through the mechanics of. . of a formula and getting excited about that . . and I don't necessarily know the application. . like you threw that question, the bamboo pole, at us. That's an application. That's when I know there's going to be an association between what I'm doing right now, with the math, the mechanics of it, and the actual applying it to something.
132. A: O.K. Yeah.
133. M: I understand that. It's to make the association. . the more word problems I do . . I find the word problems are very, " Oh s*** that's where I can apply this, that's where I can apply this." And I have this general gist of

what's going on. . but then I come to this Math book and it's all . . for me just coming from Math 200 to Math 206 . . Is there a Math course that I missed in between here? Was there something that I . . is there something that I missed? Cause I did. . I got an A^+. I think I got 100% on my last Math course. And then I'm coming to here and I'm going,"Whow!" I'm really having a hard time . . . I'm having a hard time just. . not just understanding this but keeping up with the course to find out where I'm going. . I seem to be jumping from one point to another and that's why when I read these things and then I try to do the questions. . like here I told you I got up to #20 *(in reference to EX. 4.1, p.161)*. . . everything was going great to this point. . here, I came here. . oh, practice this . . I know I'm having a hard time working with the square roots. . . those are things I have to. . I'm going to have to work on but all the rest. . I do all the odd numbers only because the odd number answers are in the book. . no problem . . and then I get to this point . . and usually I "green out" the ones I'm having problems with . . like for instance on page 161, question 21. I tried Q. 21. I "greened it out", moved on to #23, "greened that one out", moved on to 25, "greened it out". . I said. . that's why I put my arrow in my book saying, "I'm having trouble from this point on". So from here I stopped. I go back into the book and I try and figure out how they got this $g(x) = 2 - x^2$ and I try and figure out how it came to the answer by looking in the back of the book . . . how the hell do they come up with that? And I can't for the life of me figure it out from what they're saying in the text.

134. A: O.K. I'll keep that in mind. The questions I'm asking in here are leading up to your understanding of, or getting up to what it is maybe you don't understand about these questions.

135. M: O.K.

136. A: O.K. So there's something in the text that maybe you just haven't realised the text was telling you.

137. M: mm. .

138. A: Alright?

139. M: Absolutely.

140. A: Because the questions you are referring to here are asking you to find the domain and range and you said everything was O.K. for the first 4 questions and then I notice they all have an f in them, the other ones have g's and h's and maybe that's what's throwing you off.

141. M: Maybe. That's quite possible. O.K. We'll get back to that.

142. A: Yes. We're coming back to that. Thanks anyway.

143. M: *(goes back to reading Q.4, p.2 of Part 1)* "Can you state. . . .3 concepts"

144. A: Well you had. .

145. M: . . relationships. .

146. A: You've got your relationship, your governing law . . I really like that. . and what was the last thing that we talked about?

147. M: Corresponding values . . .relationships again. . corresponding values. .

148. A: I think your corresponding value. . that would do for a third. .

149. M: Yes.
150. A: It's not quite the same as relationship, is it?
151. M: No. It's just the corresponding value. .
152. A: The corresponding value suggests that you've got one value corresponding to another one. . . not quite the same thing as a relationship.
153. M: No. I like that one actually. . .corresponding value. .
154. A: O.K. So we'll put that down for your third one then. *(reading what M. wrote as his answer to Q. 5 - re-"definition of a function in your own words")* No. O.K. That's a fair answer. *(laughing!)*

Part 1:
Please read carefully these 4 paragraphs then answer the questions that follow.

ELEMENTS ELEMENTS

"The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r. SET

The number N of bacteria in a culture depends on the time t. If the culture starts with 5000 bacteria and the population doubles every hour, then after t hours the number of bacteria will be $N = (5000)\ 2^t$. This is the rule that connects t and N. For each value of t there is a corresponding value of N, and we say that N is a function of t.

The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no single formula that connects w and C, the post office has a rule for determining C when w is known.

In each of these examples there is a rule whereby, given a number (r, t, or w), another number (A, N, or C) is assigned. In each case we say that the second number is a function of the first number."

Questions:

In each of these first 4 paragraphs there are 3 concepts involved.

1. The same <u>verb</u> is used in each of the first 3 paragraphs indicating the first of these concepts. What <u>verb</u> is it? Can you give a <u>noun</u> associated with this verb?

2. The same noun is used in all of the paragraphs to indicate the second concept. What noun is it? Another word very similar in meaning to this word is also used in paragraph 3. What word is it?

3. Different words, but with similar meanings are used in each of the 4 paragraphs to indicate the third concept. Can you say what they are?

4. Can you state, in your own words, what the 3 concepts referred to are?

relationships
Governing law.
Corresponding value

5. From your reading of these 4 paragraphs how would you define a function to somebody else, in your own words.

NO

INTERVIEW WITH BRENT: 94 - 2 - 23

PART 1:

1. A: You're being recorded. I want you to do what it says there, to read carefully these 4 paragraphs and then attempt, without any interference from me, to answer the questions that follow.
2. B: O.K.
3. A: And then we'll discuss things. But you're on your own for the first little while there.
4. B: O.K. *(B. reads the 4 paragraphs)*
. . .
5. B: I'm allowed to go back and verify the text to. . ?
6. A: Sure..
. . .
7. A: O.K. So number 1. . key words. . what did you identify as key words?
8. B: The only key word that I would state that stands out to me is 'function', in the context that, A is a function of B.
9. A: O.K.
10. B: In the 3 examples. .
11. A: Any word that came out to you as being sort of 'there all the time'?
12. B: It's a common theme in the 3. . whether it's cost of mailing, bacteria, or radius of a circle.
13. A: Yes. O.K. . for the moment. Now question 2 was, 'Using these keywords. . ' In a way I was looking for more than "function", but from your answer what have you done with it?
14. B: O.K. I've identified 2 main ideas, the third one I'm not sure. The first main idea is, with every value there is an associated value. .
15. A: O.K.
16. B: There is a dependent variable and an independent variable, and I guess my third. . I can go on . . there is a causal relationship.
17. A: What do you mean by that?
18. B: Um. . I use "causal relationship", whereas one is dependent on the other. If I increase i.e. g(x), it will increase the function of . . increase in one will cause, not necessarily an increase in the other, but will cause a change in the other.
19. A: O.K.
20. B: One will affect the other. If I touch . . if we change one variable the other one will change also.
21. A: How do you see that answer as being different from your first one?
22. B: Causal relationship? If you put. . with every value there is an associated value.
23. A: That's different? Do you see that as being something quite different?
24. B: Yes. Because this here one is just straight out, $2x = 10$. This other one here is describing the fact that if we change x, we're going to change 10. This is just looking at the absolute value for the two. This equals this. And then

I'm saying the relationship between the two is. .we change this, it's going to change. .
25. A: I'm going to have to pull you up on your second answer here; dependent variable and independent variable. You didn't get those words from those 4 paragraphs.
26. B: This is from my history and the fact that I had read the text previously. I was aware that there was a dependent variable and an independent variable.
27. A: O.K. Using the words dependent and independent, is there a word in the paragraphs that is related to this idea of dependency? You couldn't pick a key word out in all these paragraphs that link to this idea?
28. B: Depends on. . it probably says that here too . . association. . connection. . depends. . the word "depends".
29. A: The word "depends" was one of the other key words. It's there in the first paragraph as well. The very first line. "Depends". That's one of the key words that. .
30. B: I had kind of taken the word "depends" as a given . .
31. A: Is there no other word that keeps cropping up in all these 4 paragraphs?
32. B: "Connects". .
33. A: Yes. "Connects" is there as well.
34. B: . . and. .
35. A: Is there another one?
36. B: Depends, connects. .
37. A: There's one word that's used in all 4 paragraphs. . quite an important word. .
38. B: . . .
39. A: What's doing the connecting?
40. B: Variables.
41. A: Variables are being connected. But what is connecting them?
42. B: The rule.
43. A: A rule. O.K. You see that word "rule" several times used?
44. B: Yes.
45. A: It's there in every one of those paragraphs.
46. B: Yes. Agreed.
47. A: And in the third paragraph they even use another word very similar to the word "rule". Can you give me a word in the third paragraph which is kind of similar?
48. B: Formula?
49. A: Formula. Alright. O.K. So you've picked out several words. You've picked out the most important ones. Now, I'm looking for 3 main ideas. Does your answer that you have there, now that we've identified a few more key words. . would you still maintain that those . . that your answer there is the same?
50. B: I would probably change my wording. . to use the key words that I have. .

51. A: I still want you to use them. You can use them, but does it change at all your outlook on what the 3 main ideas are?
52. B: I will maintain my three. .
53. A: You don't feel you've left something out? Maybe there's a fourth idea perhaps? Maybe I'm missing something out? Look at your 3 main ideas that you wrote down. Do you see your key. . Are those key words included within what you're saying there?
54. B: I can associate my number 1 with "connects". . a causal relationship "depends". . and this is from. . "formula". . I haven't touched on those.
55. A: O.K. That's fair enough. Did you get around to . . what's your answer to number 3. You can give a "yes" or "no" to start of with. Is this a "yes" or "no" for number 3? Could you define the meaning?
56. B: Yes.
57. A: You said "yes". O.K. So what was it?
58. B: I don't know - that's probably not spelt right - the causal relationship.
59. A: It doesn't matter. Don't worry about the spelling.
60. B: . . between 2 values.
61. A: That's your definition of a . .
62. B: & A: . . a function.
63. A: What did you have written up here?
64. B: I answered here actually (for Q. #3). . Is a causal relationship that one value has or is to another.
65. A: O.K. Well, I just wanted to know what you got out of that. Neither right nor wrong. . just your interpretation. . alright?

Part 1:

Please read carefully these 4 paragraphs then answer the questions that follow.

> "The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r.
>
> The number N of bacteria in a culture depends on the time t. If the culture starts with 5000 bacteria and the population doubles every hour, then after t hours the number of bacteria will be $N = (5000)\, 2^t$. This is the rule that connects t and N. For each value of t there is a corresponding value of N, and we say that N is a function of t.
>
> The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no single formula that connects w and C, the post office has a rule for determining C when w is known.
>
> In each of these examples there is a rule whereby, given a number (r, t, or w), another number (A, N, or C) is assigned. In each case we say that the second number is a function of the first number."

Questions:

1. Can you identify any key words that occur several times in each of these paragraphs?

Function a function of 3

depends.

Connects. rule. formula

2. Using these key words can you identify 3 main ideas about functions that the text is trying to convey?

(connect)

① with every value there is an associated value.

② dependant variable and independant variable

③ ? causal relationship (depend)

3. Could you define the meaning of a function in your own words from just these 4 paragraphs? → Is the causal relationship that one value has (is) to another?

4. If "yes" to #3, then HOW would you define a function?

→ the causal relationship between two values. →

INTERVIEW WITH PETER: Mar. 2, 1994

PART 1:

1. P: I've read this.
2. A: O.K. Can you identify any word that occurs several times in each of these paragraphs?
3. P: First of all I'd say 'function'.
4. A: *(agrees)*
5. P: equation.. formula..one number is a function of another.
6. A: Yes, .. well, they're basically trying to give you 4 examples of functions. Functions are certainly going to be in there. Any other words?
7. P: unknown?.. I don't know if I'm reading between the lines..but..
8. A: I'm basically looking for words that crop up in each of the 4 paragraphs. at least 3 out of the 4.
9. P: Depends..
10. A: Yes.
11. P: Depends, O.K. . . Rule.
12. A: Rule, definitely.
13. P: Function, definitely. um..
14. A: Another word I was looking for was, 'connects'. That's a significant one. It occurs in all 4 of them.
15. P: 3 of them.
16. A: O.K. Those were the main ones I was hoping you'd pick out. Depends, rule, connects, also the corresponding, associate.. words that ...
17. P: . . mean the same thing.
18. A: Alright. The second question is that from these key words, can you identify 3 main ideas about functions that this text is trying to convey?
19. P: That any given value is going to be dependent on another value. So if we - let me try and put that into intelligent language - a value on one side of the equation will be a function - I'm using the word 'function' - I shouldn't be using the 'function' - will be a function of a value on the other side multiplied by a constant - to my way of thinking.
20. A: That's certainly part of everything.
21. P: 3 main ideas? Things are inter-related.
22. A: O.K. Based on the key words.
23. P: O.K. So there's an inter-relationship between things. There's a connectivity between the 2 sides of the equation. One is dependent upon the other one. They normally involve - they will both grow or decrease according to each other. There's a certain . . .not synergy. .but simply a set relationship between the 2 . .
24. A: O.K. That about does it. . .. using all that now, could you give me
25. A: & P: . . a definition of a function?
26. A: . . in your own words - just from your reading - of those 4 paragraphs.

27. P: Well, a function would be. . it would be a function of another number if its value is connected in any way to that other number - in a sort of a logical sequence - so if there is a formulaic connectionship between the 2. .
28. A: O.K.
29. P: Thank goodness I'm not writing this. .

APPENDIX

B

APPENDIX B

MICHAEL - PART 2:

155. A: Well, this is right out of the text. This is the definition that immediately followed those 4 paragraphs.
156. M: O.K.
157. A: And . . um. . I've asked you to compare your definition with that of the text. . you couldn't give me one so. . .
158. M: Well I knew a function is a relationship. . .
159. A: My second question here, instead of asking you, "Are they saying the same thing?". . your definition and the one in the text. . Can you tell me what this means to you? Do you understand their definition?
160. M: *(reads the definition)* " A function. . exactly one element. . "
161. A: Continue reading this.
162. M: ". . called 'f x' in a set B." No I don't. . it's . . eh. .
163. A: Does that suggest the same thing as what we've been talking about in PART 1?
164. M: *(reads the definition again)* "A function f is a rule," . .O.K. . "that assigns to each element x". . " . . each. . " See, "element" O.K. element or variable. . Is an element a variable? ". . each element, x. ." O.K. ". . each element. . ", each piece, ". . element x,. . " O.K. I assume it's a variable, ". . in a set A. . " O.K. the overall set, ". . exactly one element, called 'f x' in a set B". I don't understand this. *(re-reads it again)* No that doesn't make any sense to me at all.
165. A: O.K. If you were reading that and you're comparing it to the first paragraph of PART 1. . the question about the circle, the radius and the area, . .
166. M: Right.
167. A: Could you tell me, you know, in terms of the definition, what is the rule here, what's f. . what's x, what's A, what's B? Can you compare the two.
168. M: O.K. *(starts to read over the definition of a function again)* " A function f is a rule that assigns to each element x . . ", O.K. For me the element would be the area **and** the radius . . I would call these elements. . " in a set. ." For me the set would be $A = \pi r^2$ . . I'll try again. . "exactly one element called "f x" in a set B". Now where they threw the set B, I don't know. I don't know where that's coming from. The set B, that throws me off completely. ". .element. . " You see I would call this the set. The elements meaning variables, I guess, *A* and *r*, I would call them elements. But "the set of elements". . . Pretty bad!
169. A: It's. . . not good or bad! *(laughing)*. It's interesting, your reading of that. Because I'm interested that what you called the set was the $A = \pi r^2$. You didn't see that. . originally you saw that as being an equation or a formula, so to you is an equation or a formula the same thing as a set? What does set mean to you?

170. M: A set, for me is a group, a group of numbers . . for me a set of numbers . . the set of numbers . . .so I would have a group of numbers. . a set.
171. A: Where is your group of numbers if you're saying your set is $A = \pi r^2$?
172. M: This time it wouldn't be a set. This would be a. . so in this case I would look at it more as a set of elements. Now I define element here has . . is it a variable? An element? An element is one? Element *x* in the set of A, so one element within a set? So if this was an element it's within this set of numbers. I understand it's a formula but it's also a set of. .
173. A: You're seeing the element as being part of. . as being within the formula?
174. M: Yes. Like for me, *r*. . like for the first one, *A* is an element. . for me *r*. . like my interpretation of it, *A* would be an element as well as *r*, *A*'s for area and *r* for radius. And the set of numbers together, well I see a set of numbers, $A = \pi r^2$.
175. A: O.K. The interpretation that you're really supposed to get out of this is that the set A and set B are two distinct sets. .
176. M: Right. That's why I said I can't find. . I don't know where they get the set B from. .
177. A: O.K. You don't know set B. Your set A and your set B are two distinct sets. Within set A you have certain elements . .
178. M: Right.
179. A: Right?. And you do something to them, you apply a rule of some sort to the element in set A and that makes it an element in set B. *(silent reception of this explanation)* Right? You have these elements in set A, right?
180. M: This is the machine thing that they show in the . .
181. A: This is the machine, O.K. And, you know, there's two distinct sets here and there's something relating the two distinct sets. And your function is your rule. They're saying that, "A function is a rule, that assigns to your elements in set A exactly one element in your set B", so you've got to have. . you've got to think in terms of two distinct sets here. .
182. M: Right.
183. A: . . and each set is a collection of numbers. . a set of numbers.
184. M: Right.
185. A: The set. . . what you're calling a set here, $A = \pi r^2$, is not really the set. That is the **rule** that is connecting. .
186. M: . . the set. .
187. A: . . the set of numbers in A with the set of numbers in B. .
188. M: . . the machine. .
189. A: It's like the machine, yes. Now what variables are going into the first set, if you're talking about the circle question?
190. M: Well the area and the radius.
191. A: Well you're grouping two things together here. I just want like one set.

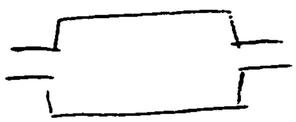
192. M: O.K. So I would . . area. . the machine being the, I guess, the radius, or no, A being the. .
193. A: The machine is the . . what's happening? Something is going into the machine.
194. M: Right. This is . .
195. A: What's going in? This is PART 3. . PART 4. . You've jumped away ahead here. .
196. M: This is what they have and judging by what . . can I refer to the book?
197. A: Yeah.
198. M: *(turns to page 158, Figure 3, see App. F, p. 98)* "Input . . square. . " O.K. being the function. . the input x and your x^2 you're saying . . . on page 158. . my understanding is the input, O.K. x, what it does is your function, the square and your output would be your answer. So you are telling me that this is set A and this would be set B.
199. A: Yeah
200. M: And this would be the function.
201. A: Right. Now in terms of the circle question, what is the input?
202. M: The input would be. . .*(reads part of the question)*. . would be either the A or the r.
203. A: It could be but the way the formula is written, is it more one than the other? From your reading. . just read that first paragraph again.
204. M: *(reads)* I'd just be guessing!
205. A: Well guess! *(both laughing!!)*
206. M: O.K.
207. A: You know you keep telling me you can get r from A and A from r, but is it not coming across stronger one situation than the other?
208. M: Well. . .I plug my radius in. . I'm guessing. . I don't. .
209. A: Keep guessing!
210. M: Keep guessing. I plug my radius in then my function would be the. um. . the π, I guess and then my second set I guess would be the area. . . would be my answer that I get out of that.
211. A: What are you saying now is your set A and your set B?
212. M: *(reading the definition again!)* "A function f. . .
213. A: What do you think is set A and set B in the circle question?
214. M: Well for me it would be radius and area, or area and radius. .
215. A: Which do you think is set A?
216. M: Set A for me would be the radius.
217. A: Radius. And set B is therefore going to be. .
218. M: . . the area.
219. A: The area.
220. M: Because the radius equals the area. .
221. A: What's the rule?
222. M: The rule is $A = \pi r^2$.
223. A: O.K. So you've got your function . .

224. M: Right.
225. A: . . is a rule, so that's like a formula. Whatever it is you have to do. .
226. M: Oh. O.K.
227. A: . . that assigns to each element *x* in a set A,
228. M. O.K.
229. A: So what's your x here is going to be your value for your. .
230. M: From . . from. . radius. .
231. A: Your radius, right. Your element here, your *r*, is in the set A so your set A is all the possible values for your radius, O.K?
232. M: Right.
233. A: But when you apply this rule to each of these *r*'s. . you apply the πr^2.. you get exactly one element . . you're going to get one answer. .
234. M: Which is the area.
235. A: . . which is going to be the area and this area produces another set of numbers - that's your set B - but they're calling it. .
236. M: 'f x'
237. A: 'f x'. Now they didn't tell you how to say that in the book at all did they? When you first read that did you just read that as . . When you first came across this expression in your book before you went any further, how did you read this? Did you just read it as. .
238. M: "*f* times *x*"
239. A: You read it as "*f* times *x*" ?
240. M: Yes. That's the way I looked at it.
241. A: "*f* times *x*". O.K. Because I wanted to know how . . it was important to me to know how you read this for the first time.
242. M: Well just coming out of Math 200, that's " *f* times *x*".
243. A: That's "*f* times *x*". You saw it. .
244. M: It could also be . . you could remove the brackets and put "fx".
245. A: So are you seeing that "f times x" the same way as, you know, a "2 times x", something like that? Are you seeing it like that?
246. M: Well whatever *f* stands for. .
247. A: O.K.
248. M: . . multiplied by the *x*. .
249. A: O.K. So *f* is just another variable?
250. M: Yeah. Yeah. It's just another variable.
251. A: O.K. So it's just like "*a* times *b*" or something which is. .
252. M: Which is *ab*.
253. A: Which is *ab*.
254. M: Right. For me I looked at it exactly the same way.
255. A: O.K. So you just see *f* as being another variable?
256. M: Right.

Part 2: This is the definition of a function from your text book.

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

Compare your definition of a function with that of the text. Are they saying the same thing?



Does this definition bring out the concepts referred to in Part 1? Did yours?

What is $f(x)$ and how do you say it?

BRENT - PART 2:

66. A: This is the definition from the book, alright? I'm asking you questions about it.
67. B: O.K.
68. A: I'm asking you to compare your definition. You did give one. How did your definition compare with the book's definition? You can write down your answer and we can discuss it out loud at the same time if you want.
69. B: I feel I was attempting to explain the same thing, but not using the technical jargon.
70. A: O.K.
71. B: (*records his answer*)
72. A: Could you explain then, you know, if you're looking at their technical jargon. . put that into your own words? The definition, I want you to keep comparing them. .
73. B: For, "A function is a rule that. .", mine's more of an explanation than a definition. . I'm looking back at it. .
74. A: Would you like to take your definition and attach some of the jargon to your words?
75. B: A function is a rule that assigns each element x. .
76. A: Your words were, "the causal relationship" there. Could you give that some technical jargon?
77. B: A function is a rule that shows the causal relationship between x in set A and the values in set B.
78. A: O. K. You talked about two values. Do you see two values in the textbook definition? Corresponding to your idea of the two values? Is that there?
79. B: Yeah.
80. A: Where?
81. B: In set A. .
82. A: mm. . mm
83. B: . . and there's one element also in set B.
84. A: O.K. There's just one word here in the definition that you haven't really pinned down. The word, "rule". What. . in your mind, what is it you're thinking of as the "rule"?
85. B: The "rule" is, " for every value of . .
86. A: No from your own definition. . in your own words, what are you using for "rule"? What words are you using for "rule"?
87. B: the "rule". . *(doesn't understand what I am getting at here!)*
88. A: To me you are. . . to me you're using the word "rule". . in your own words. . you're using other words for it.
89. B: *(still cannot reply!)*
90. A: You have a very simple definition there. You've got your two values.
91. B: Right.
92. A: What are you saying at the beginning?
93. B: The causal relationship.

94. A: The causal relationship.
95. B: O.K.
96. A: Is that your idea of the "rule"?
97. B: O.K. The rule. . yes. We could state that the rule was the causal relationship.
98. A: I'm not putting words in your mouth here!! *(convincing B. or A?)* This notion of rule and your causal relationship are the same thing?
99. B: Again I'd have to go back and state that I think that my definition is more of an explanation to what I envision is happening as opposed to a textbook definition.
100. A: O.K. You can see them saying the same thing though?
101. B: I'm still seeing them as saying the same thing. Yes.
102. A: O.K. So question 2, What does this definition of a function mean to you? And I mean by that the textbook definition. What does that really mean to you, all that up there?
103. B: *(records his answer)* It means for . . I know now that "*f of x*" will equal *y*. .
104. A: How did you know to say that as, "*f of x*"? *(in reference to $f(x)$)*
105. B: "f of x"?
106. A: The very first time that you read that, how did you say it? What . . .
107. B: It's just . . through education. Over the years, "*f of x*". .
108. A: "*f of x*". O.K. So you've heard that several times before?
109. B: Yes. As I answered for number 2, For every x in set A, there is an $f(x)$ in set B. $f(x)$ as I know it equals y. So for every value of x there is a value for$y$.
110. A: O.K. So you're immediately replacing the $f(x)$ by y.
111. B: To me that's how I'm picturing it. I can. . I'm picturing a graph and saying, "if I have a point, no matter where it is, there has to be a countervalue. . well it can be a zero, but. ."
112. A: O.K. Countervalue. That's interesting! Countervalue. *(pondering the use of this word!)*
Alright, the third one. Can you now **link** this definition to the first question from part 1? The question. . the first question from part 1. . the question about the . . the first paragraph there. . the area of the circle.
113. B: I'm starting to find this very confusing. This definition, I don't have a problem understanding what they're trying to tell me.
114. A: Really.
115. B: Because of my previous math, I understand that $A = \pi r^2$.
116. A: Right.
117. B: But the way it's given to me here, I'm reading it and saying. . eh!!. . When I read this paragraph originally. .
118. A: mm . .mm
119. B: . . all I basically picked out was the formula, which I recognised, because . . what . . how. . what they did to get to this formula. . confused me, and if I wouldn't have had a past history I would be saying, "I really don't know what they're trying to. . "
120. A: Why?

121. B: Why? I don't know if I can answer why, I just find it confusing.
122. A: Compare it now. . look at it now in the light of the definition of a function and. . you said, this definition of a function, you can understand it?
123. B: Right.
124. A: Well can you pick out the rule that's concerned here? What are the elements in set x? Sorry, the elements *x* in set A? What are the other elements *f(x)* in set B? Do you see these elements in these sets in there? Do you see rules in there?
125. B: (*after some hesitation*) As I understand it, the rule is claiming. . my *x* would be my *r*, and my *f(x)* would be my *A*, so it would be the . .back to the causal relationship between *r* and *A*, which therefore gives us the equation, $A = \pi r^2$.
126. A: O.K. Instead of using the words, "causal relationship", what am I trying to get you to see that as?
127. B: Rule.
128. A: The **rule.** Right. O.K. So, if we're answering these questions here now, (Q. 3, *Part* 2), are we really clear what's going on here? The elements of set A? What's the set A in the question about the circle?
129. B: Set A is going to be the values of *r*.
130. A: Right, the radius. O.K.
131. B: O.K. And...... set B.*(must have said "A", but inaudible)*
132. A: O.K.
133. B: . . and the only reason. . connection between the value of *r* and the value of *A*.
134. A: Which is?
135. B: Which is, $A = \pi r^2$.
136. A: Right. O.K.
137. B: *(records answers)*
138. A: So, you see, we're trying to arrive at looking at three parts to a function. You have an initial set of values which we're calling set A. .
139. B: O.K.
140. A: . . then we're doing things to these values. .
141. B: Right. That's. .
142. A: That's the rule, that we are applying to these values. And we're getting a new set of values which we're calling set B. So you've got like three parts to the function, O.K? And with your circle, do you see it now as coming out like that a bit better now? You've got your radius. The area, A, depends on the radius, r.
143. B: O.K.
144. A: So, you know, you're starting. . which one is your dependent variable there? I'm asking you this in the wrong place, but I'm asking you it now anyway!
145. B: The dependent variable will be. .
146. A: Which one is the independent and which one's the dependent, since you already know . .

147. B: The dependent is A, because it's dependent on r.
148. A: Exactly.
149. B: And if there is a change in r, it's going to affect the dependent variable.
150. A: Right. So there's the two setS of values, your r's and your A's, right?
151. B: Right.
152. A: And the rule that connects them is your formula, $A = \pi r^2$.
And the last sentence here *(from the first paragraph of Part 1)* is just emphasizing something else. It's not really saying anything new, but it's telling you something special about the way these values are associated.
153. B: A is a function of r, which means A is dependent on the value of. . the value of A is dependent on the value of r.
154. A: O.K. The part of the sentence right in front of that, the same last sentence though, is saying something rather special.
155. B: *(reading it again)*. . I. .
156. A: O.K. You're not seeing that as telling you something special there?
157. B: Well it's telling me if it's *positive* there's one value.
158. A: There's a value of A.
159. B: It doesn't tell me what happens if it's negative.
160. A: O.K. . So you're seeing it as *positive* as opposed to negative. Could it be negative?
161. B: . . uh. .
162. A: Could we have a negative number for r?
163. B: No.
164. A: No. We're talking of the radius of a circle. It would have to be positive.
165. B: O.K.
166. A: Alright. We'll hold these thoughts for the moment. We'll come back to them.
The fourth question there, you've already answered. (*how to say $f(x)$*). The symbolism here wasn't new to you. You already knew to say it as "f of x".
167. B: Right.
168. A: The "f" therefore was never in your mind as a variable? Like, you know, the "x" is a variable.
169. B: Right.
170. A: Was the "f" ever in your mind as a variable?
171. B: No.
172. A: No. How. . when you see that "f" there, what do you think it is standing for?
173. B: "f of x"? It is a variable. It's telling me that, depending on what this is, it's going to affect what my f is.
174. A: I'm glad you said that!! The idea is that the "f" is NOT a variable. So you were right in not thinking of it as being a variable. The "f" is in a way standing for the word "function". It's the "function of x".
175. B: O.K. Function of x. Agreed.

176. A: So it's not like an x, y, z, or, you know, g, f, h. It's not really a variable. I just wanted to know if you had at one time thought of it, and you said, "No, you hadn't", and then I guess I was confusing you by suggesting it might be. But. .
177. B: When I read ..., let's say $f(x)$ = whatever, so if we change $f(2)$, I was using it as this number changing, if it equals. . therefore . . now. . no this remains the same *(refering to $f(x)$ changing to $f(2)$ - only the x changes, not the f)*
178. A: So in a way the **whole thing** is a variable? If I said the **whole thing** was a variable, the f and the x all attached together is a variable, would that make sense?
179. B: Yes. Because it has to be together.
180. A: We've got our x's, right? And we do things to our x's and we get $f(x)$'s so we get variables here and we get variables there. So this whole thing becomes a bunch of variables. The f part is really, you know, telling you it's a function. The f on its own is not a variable.
181. B: O.K.
182. A: O.K. Right.

Part 2: This is the definition of a function from your text book.

A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

1. If you gave a definition of a function in Part 1, then compare your definition of a function with that of the text. Are they saying the same thing?

– attempt to explain same idea
less (technical jargon .

→

2. What does this definition of a function mean to you?

FOR every x in set A
there is f(x) = in set B -
(y)

3. Can you explain this definition of a function using the example from the first paragraph of Part 1, i.e. the relationship between the area and radius of a circle?

What are the elements of set A? r

What are the elements of set B? A

What is the rule? — Rule Between r & A = $A = \pi r^2$

4. What is $f(x)$ and how do you say it?

f(x)

f(x) > f(2)

PETER - PART 2:

30. A: Alright, I now hope you've got in mind what you just said there. . your definition. . we didn't write down anything. . because then Part 2 is for you to compare the text book definition with what you have just said and tell me if there is a difference. . Is it saying the same as what you just said? or .
31. P: This one here?
32. A: Yes. That's the definition right out of the text book.
33. P: *(reads it to himself)* . . Well I'll have to read this a few times in order for me to get the same meaning out of it.
34. A: It seems to match what you just said?
35. P: O.K. "A function is a rule. . . " There is. .
36. A: What part?
37. P: Well, the. .what I think I said was that any one given number is going to put down in consequence to whatever the other partner of it is on the other side of the equation. What they're saying here is that the function of x - I guess this is what this is - *(in reference to f(x))* - is going to be . . well will be exactly one thing every time that there's a different value on the other side of the equation. I'll be very honest with you, if I read this it's going to take me many readings to decipher from that what it is that I'm . .
38. A: . . what a function is?
39. P: . . what a function is.
40. A: O. K. What is it really saying a function IS?
41. P: Well, they're saying it's a rule. .
42. A: It's a rule?
43. P: Yes. .
44. A: O.K.
45. P: Whereas I was saying it was more an inter-relationship as opposed to a rule.
46. A: O.K. Is the rule part of the function? Part of your idea of a function then?
47. P: Um. . I wouldn't have called a function a rule, not the way that I read it from these 4 paragraphs here. I wouldn't consider it a rule. I would consider it a relationship more than a rule. .
48. A: Where does the idea of rule fit in then?
49. P: A rule would be something that would be the same all the time to me. A rule . . I suppose in any given function it would be a rule, yes in any given relationship it would be a rule for it to be a function. Yes, it would have to be. O.K.
50. A: So the rule is like part of the function then?
51. P: Yes. . well. . if there was a function there would have to be a rule that inter-relates the 2. Yes. Sorry, yes, there is a rule.
52. A: There is a rule.
53. P: O.K.
54. A: But the definition is saying here that the function IS a rule, you know, and it would seem to indicate a connection AND the 2 elements themselves.

55. P: Yeah, it is. . it is the connection between the 2 elements. If I have A here and B here they would both be related by a certain rule.
56. A: uh..uh
57. P: O.K. And that's what a function is. It's a rule.
58. A: It's a rule?
59. P: It's a rule. It wouldn't have fallen into my strict definition of a rule. I would have considered it more a relationship than a rule.
60. A: O.K. Well that's fair enough. There's a question whether this definition of a function is really meaningful to date. Because, definitions of functions have changed over the years.
61. P: Yes.
62. A: And this particular definition is slightly outdated.
63. P: O.K.
64. A: It's more a total. .concept of the elements themselves AND the rule. It's not just "the rule".
65. P: When I read 'rule', I'm sort of looking like. . looking at something where it's going to be the same for this one, this one, this one or this one.
66. A: uh..uh
67. P: And. . all . . I guess in the broad sense they all ARE the same, except that one could be related to . .I don't know. .
68. A: What do you mean by "the same"?
69. P: The first one. . the rule would be that for any value of r you've got to square it and multiply it by pi.
70. A: Right.
71. P: O.K. Then in this one here, I've got to multiply by a 2 to the value of t.
72. A: What do you mean by saying things were 'the same' then?
73. P: Well the rule. . there is a rule. . there is a different rule that applies to both of these. It's not the same rule, O.K?
74. A: O.K.
75. P: O.K. Whereas if I read rule I normally expect. . I would have expected both of these. . in other words, N to be a function of 5000 times pi. . 5000 squared times pi. . When I'm reading the word 'rule' I'm looking at something that is more. . a more continuous even type of situation. That's why the word 'rule' at first struck me as being incorrect. I was looking more at something that there is a. .a function is a relationship or is a . . .an association, a relationship, whatever, as opposed to rule.
76. A: What does the word 'rule' mean to you, just out of a mathematics context?
77. P: Dogmatic. A law.
78. A: O.K. So you expect it to be the same for all. .
79. P: Yeah. I'm reading this and I'm saying, Well O.K. That's fine. But now what is the rule? This is the first question I would ask myself. O.K. A function is a rule, that assigns . .O.K. so what is the rule? Is it. . and then I find out well the rule is not necessarily the same. I mean there is a rule. The FUNCTION is the rule. So whatever the relationship is has to be defined and then that becomes the function which is the rule.

80. A: Yeah.
81. P: But it is not the same to each one of these 4 paragraphs. It's a different thing.
82. A: So you're usual meaning of rule that. .
83. P: Yeah. The word 'rule' . .
84. A: . . meant something different. .
85. P: . . meant something totally different to me. I. . you know, once you explain it to me. . yes, it makes sense. . it is a rule and this makes sense once you read it enough times but on first reading this is not something. . if I was reading my text book tonight and I was trying to figure out how to do a problem and I went to read this, I'd say, "O.K., now what does that mean?" It would probably take me a few readings to in fact decipher what that meant.
86. A: O.K. . . .We won't talk anything more about that this now. This is the same thing - What does the definition of a function mean to you? Has this changed. . Has the reading of this definition changed your concept of function?
87. P: No.
88. A: No?
89. P: No, it hasn't. If anything I find this confusing. Why do I say confusing?
90. A: Because of the terminology they use?
91. P: Because of the terminology. There's a lot of terminology in here, and, you know, I'm looking at, you know, 'an element in a SET A, in a SET B'. . Why are we talking about sets? You know, I mean, I understand that that's probably very nice mathematical parlance, but for somebody that's trying to learn mathematics, you know, I think you. . it's easier for someone. . for me at any event, to learn WHAT it is that we're doing and then we can get at names afterwards. O.K?
92. A: O.K.
93. P: Because I don't really know what the names are all about right now. This is sort of like trying to define what this is, what that is, what's the rule? what's the set? O.K.
94. A: O.K. I think maybe you're confused with the notion of set here because you were distinctly saying all the time that you had ONE value in set A giving ONE value in set B, and what they're saying now is that, 'to EACH element in a set A'. Now when we start thinking of a SET. .
95. P: Yeah. .
96. A: . . you know, is there just one element in this set?
97. P: No, there are many elements in a set. .
98. A: There are many elements, so this is. . .
99. P: . . It's not. . If I were trying to teach one of my children what a function was I wouldn't use this kind of definition.
100. A: Well, its not at a children's level. .
101. P: Even for myself. . I would. . it would. . I would learn much better, I think I would be able to acquire a knowledge of functions much better by

reading these 4 paragraphs as examples as opposed to reading, 'a function is a rule'.
102. A: Yeah.
103. P: O.K. I mean this tells me much more. .
104. A: Well O.K. since you mentioned children's interpretation. This is a grade 10 book where they do functions very, very briefly. . functions. . introductions. . page 12 here. . introductions. Read this and tell me what you think. .
105. P: . . this here?
106. A: Yeah. You can read that and tell me what you think about this introduction compared to what we have just been looking at.
107. P: This is a lot clearer than this is. I mean after having read this, this makes a lot of sense.
108. A: Yeah. They're using the word, 'relations' there. These 'relations' are called 'functions'. See the difference? It's the relationship. Whereas in your other text book they're calling the function 'the rule', and here they're calling it the relationship. So the text books themselves don't seem to be too clear how. .
109. P: Again here they belong to an example and then they bring the example down to a definition, and that makes more sense to me rather than telling me the definition and then . . you know. .
110. A: They did attempt to give you some ex. .
111. P: This is fine, but it doesn't have a lot of relationship to this. .this is much more. . this makes a lot more sense. .*(reading)* . .'these relations are called functions". . yes, that makes sense.
112. A: Is there something special about this definition that is not coming across quite so loud and clear in the other text.
113. P: This. . this tells me. .
114. A: There's a point. . they're saying something special here which we haven't really touched on.
115. P: O.K. Well, what they're saying is that, "for every value, there's only one corresponding value".
116. A: Yeah. O.K.
117. P: O.K. And it's the relationship that . . the relationship, I guess, a rule.
118. A: When you think on it here- the 'only one corresponding value' , is that idea in the other definition?
119. P: No. Not necessarily. This one says, Yes, there's only one rule, from what I'm seeing here, but it doesn't tell me that there's only one corresponding value.
120. A: Oh, so the word 'exactly' there doesn't . .
121. P: . . doesn't hit me. O.K. Yes, I guess so. Hang on. . *(re-reads from my text)* " . . EXACTLY one element. . " Yes, O.K.
122. A: It's something. .
123. P: The word in this definition that hits the most is the word 'rule'.
124. A: Yes, because of the way it's . .
125. P: Yes.

126. A: A function is a RULE, and the other stuff. .
127. P: . . I didn't even look at the word 'exactly'.
128. A: O.K. that's what I thought.
129. P: Is this a function of my inability to read properly?
130. A: No, it's not. It's just the way things are. When they come to do functions per se, Chapter 8 devoted to functions (*Grade 10 book*), they tackle it from a slightly different viewpoint. Do you want to just have a glance over this (*page 208*). .
131. P: . . this and this? . . (*pp. 208, 209*)
132. A: No, just this part here (*p. 208*) . . and tell me if there is a difference.
133. P: (*reads p.208*)
134. A: Are they saying things any differently there?
135. P: Well yes. They're saying in this first paragraph here that . . any different variable. . I guess coins have value, they have weight, they have properties of those coins, um. . is a constant to a variable which is the number of coins which would equal. . whatever it is you are looking for. Whether you are looking at the total value, the total weight, the total. . O.K? Then they go on to talk about the parking meter where again they're saying, well we've got 2 different variables and one of them which is dependent on the other one. So there is a dependence of one variable on the other variable and that is through a constant which is the 25 cents, so whereas one of these variables is simply a variable the other one is a function of that variable.
136. A: O.K. So you find this definition. .
137. P: Well, it's. . it makes sense. .
138. A: It makes sense.
139. P: Yes. I guess I should be going to Grade 10 as opposed to. .
140. A: No, I showed this (text) to somebody else in my last interview and he felt that if I used this for University level he would be quite insulted.
141. P: Would he?
142. A: Yes, because he found it. . this was a school text. He wanted the hard words, and, you know, he wanted to feel that he was tackling something at a higher level.
143. P: Well. . I. . I . . Yes, it's nice to get to be able to use the, I guess, the parlance of the words, or whatever, but I think it's more important to learn the concepts first, and, speaking personally, I've always found it a lot easier to learn mathematics by looking at examples and having somebody show me something rather than sitting down with a book because quite frankly it takes too long. You eventually get to the same point, but it takes far longer. I took Math 200 last semester and I'm taking this one right now. . and the other (text) was better than this one here. .
144. A: I've heard that before from other students involved. .
145. P: Yes, far better. Because at least they had. . they pull out the important stuff and sort of summarize it in, you know, little bullet form. . the examples were clearer and easier to understand. . when they solved them for you as opposed to some of the ones that are solved for us in this text book. I hope you didn't write the text. .

146. A: Pardon!
147. P: I hope you didn't write that text because I'm not trying to insult. .
148. A: . . *(laughs!)*
149. A: O.K. Let's look at the questions here now on that page.
150. P: O.K.
151. A: We talked about what the definition of a function meant to you, O.K. Now question 3 then, FROM this definition, using the example from the first paragraph of Part 1, the relationship between the area and the radius of a circle, could you identify what they are talking about? These elements of set A? The elements of set B? What would they be if you were using this example of this circle question?
152. P: Well, the elements of set A - in this first example?
153. A: mm. .mm
154. P: . .would be the area. And the elements of set B would be the radius. A would be. . the value of set A would be dependent on the value of r in set B multiplied by. . the constant part of that would be the squaring of the variable r and multiplying it by pi.
155. A: O.K. Now that you've said that I have to correct you, . .
156. P: O.K.
157. A: O.K. The set A . .
158. P: mm . .mm
159. A: . . is the values of r . .
160. P: O.K.
161. A: Because what you're doing . . the values of set A is your first set.
162. P: Yes.
163. A: . . the set from which you are getting the other value. .
164. P: I see, O.K. So I have to put it in reverse.
165. A: Yeah.
166. P: O.K.
167. A: So your set A would be the values of your radius. .
168. P: Yes.
169. A: . . and your set B would be the values of the area.
170. P: O.K.
171. A: And you were getting those values of B from A. .
172. P: Yes, you're right. That's what it says, 'f(x) in a set B'
173. A: O.K?
174. P: O.K.
175. A: So what we're doing is we're starting of with our element x - it's going to be r in this case - in set A and we apply the rule to it, πr^2 . .
176. P: Right.
177. A: . . and we get our element in set B.
178. P: Yes.
179. A: O.K?
180. P: O.K. You're right. That's what it says. I didn't say it right but that's what . .

APPENDIX

C

APPENDIX C

MICHAEL - PART 3

257. A: Now then this is more out of the text. *(allows time for M. to read these two paragraphs)* O.K. So this is a. .
258. M: I find. .
259. A: . . the questions I've asked you here. . I've missed a very important first question.
260. M: O.K.
261. A: The first question should really be, "Is there a rule?. . .Do these two sets of numbers here suggest to you a function? *(pause)* Remember that the definition that we just read for a function was that it was a rule. .
262. M: Right.
263. A: So do these sets of numbers here? . . Are these sets of numbers producing a function? And what would it be?
264. M: Well assuming that this radius equals this circumference, by whatever the formula is . .
265. A: O.K. ....
266. M: There's definitely something went on here.
267. A: Yeah. Is there?. . You know the formula for?. .
268. M: . . circumference?
269. A: mm . .mm
270. M: . . $r^2/4$? . . I'm not too sure about the circumference.
271. A: $2\pi r$.
272. M: $2\pi r$, O.K.
273. A: $2\pi r$. . So . . . you see that as a function? Does all this follow our definition of function? Do we have here, you know, 2 sets?
274. M: Yeah. Absolutely.
275. A: O.K. What are they then?
276. M: The two sets? Well the radius would be your 4, 5. . the . . well, your set A. .
277. A: O.K.
278. M: Your function would be the formula that you use to get C. .
279. A: mm. . . mm
280. M: And C, or set B, as we called it before, would be 25.1, 31.4. . .
281. A: O.K.
282. M: I would call this A, this B *(see M.'s notes. p.4)* and the function I guess is the. . now is it C = 2 . .? No, the function would be $2\pi r$, in this case. .
283. A: Yeah.
284. M: So the function just represents a. . is the formula itself.
285. A: Pretty well.
286. M: O.K. So a function is the formula, *x* is the . . is one of the variables.
287. A: Right.
288. M: O.K.

289. A: And you apply the formula to it and you get another set of variables.
290. M: O.K. So. . .I understand. . so now we have set A, which is. . in this case would be *r*, then we have the formula. . O.K. . and then we have set B. Set B which is in effect the answer. Or in this case is the circumference which is also the answer. O.K. Well that makes sense.
291. A: It's beginning to make sense?
292. M: Yeah.
293. A: Now let's go back to this first paragraph here. .
294. M: Hang on. . Just a moment. If I was to make C. . if C was on this side. . the formula would be. . well I'd have to . . divide this side by πr, and that side by $C/\pi r$. No, hang on. . oh 2π. . O.K. Alright. Exactly.
295. A: O.K. So I mean, to you, does it. . you know, is the C necessarily coming from the *r*? Do you see it just as strongly as the *r* is coming from the C?
296. M: Absolutely.
297. A: You see it both ways?
298. M: Yes. The set A could also be the C.
299. A: Yeah. O.K.
300. M: Only with that formula. *(implying $r = C/2\pi$.)*
301. A: With that formula.
302. M: O.K. I understand. I see that.
303. A: But. . O.K. So that's the first paragraph. We've got our set A and B. I'd like you to read this again.
304. M: *(reading)* "We usually. . domain", O.K. whatever you want to call it, domain. . so here I am, my domain. .
305. A: What's the domain in reference to this question here? What would be the domain?
306. M: It would be the radius.
307. A: The radius. O.K. So it would be the numbers given there?
308. M: Right. Domain of the function. *(reading)* ". . domain is the function. . " The function is the formula, right?
309. A: Now it's telling you how to read this "f x" thing here. .
310. M: *(reading)* "The symbol f(x) is read 'f of x'. . " *(silent pause for a few seconds)*
311. A: Not quite the same. .
312. M: O.K. In this case *x* is *r*, right? In this formula where the . .
313. A: Yeah. It's the reading of this symbol that I'm interested in now, because you know you just told me you saw *f* as being like another variable.
314. M: Right.
315. A: Is this still . . is this still . .
316. M: . . still the case?
317. A: Is this still the case here?
318. M: *(reads)* "f(x) is read 'f of x' or 'f at x'. . function of x or function at x" O.K. That's alright.
319. A: Is *f* still a variable to you?

320. M: No. No. It's. .
321. A: What does *f* stand more for?
322. M: It's more the function, more the formula. .
323. A: O.K. So the *f* is not really a variable anymore?
324. M: No, not at all. Now it's a formula.
325. A: It's a formula.
326. M: The formula x, is read, 'f of x', or 'formula at x'.
327. A: O.K.
328. M: Or the function. O.K. the function. . what you do at x. . When you come to x that's your formula and it's called the value of "f at x". It's called the value of "f at x". *(a few seconds pause)* O.K. The value of the function given x. How about that?
329. A: Yes. O.K.
330. M: Given x, this function will be. . . ."f at x". . I don't know why they. .
331. A: Now do you think . . What do you think about all of that? I mean could you explain that simpler you think than they have done there?
332. M: Oh, yeah. I would do it the way I have it set up here. Well this is because. .
333. A: This particular sentence here, ". . the value of f at x. . "
334. M: The symbol f(x) is read "f of x" or "function of x" or "formula of x". . or the "formula at x". . or the "function at x" . . O.K. It's just a question of getting used to the terminology. . function. . O.K. *(reading)* "and is called the value of the function at x". . Why don't they just say "given x"? The value of the function given x?
335. A: Good point.
336. M: I just find that part throws me off . . "at x" . . Given x, what's the value of that function?
337. A: Did you find that their definition. . that their explanation of the symbol f(x) there is helpful? Is meaningful to you?
338. M: No. It's not. The symbol f(x), or function of x or . . Don't forget I'm putting this in as layman's terms as possible. .
339. A: That's what I want to hear. I mean people reading these books are not . . they don't have degrees in mathematics at this stage, right?
340. M: I guess not. . . Function. . I'm sure that's the math terminology they've been using for 3 billion years. . . or 350 B.C.!
341. A: Not really, actually!
342. M: But they could also call it a func. . . or a formula, just in keeping with what we've learned to this point. Why not call it a formula?
343. A: Why not!
344. M: But it is a function. I understand this serves a function. This formula serves a function, so yeah, but. . just looking at it the first time I found it a little confusing. O.K. "or the image of *x* under *f*. . " Now. . "the image of *x* under *f*. . "
345. A: Does that mean anything to you at all?
346. M: Well "the image of *x* under *f*. . " so I picture *f* over *x*. Now they've thrown something else at me.

347. A: O.K. What does "image" suggest to you?
348. M: An image. A mental image of f being on top of x. f/x.
349. A: f being on top of x?
350. M: Yes. f over x. f divided by x, how's that? Or the image of x under f...
351. A: You're seeing it as f over. .
352. M: I'm seeing it as f over x. .
353. A: f over x. .
354. M: Or f divided by x.
355. A: Or f divided by x. O.K. a mental image? That's all you're getting is your image here?
356. M: That's all I'm getting. Yes.
357. A: O.K. . . Let's try and get some more of these questions answered. Your domain therefore is your radius? Alright, we agreed on that?
358. M: O.K. Sure.
359. A: Your range? What's the range?
360. M: *(reading the last part of the first paragraph)*. . "The range. . the domain". *(repeats part of it)* "The range. . . **possible**", the **possible** answers, I guess, "the range of f is the set of all possible values". . all possible values, O.K., "of the formula x, or the function x, as x varies throughout the domain". O.K. So whatever the value of x is before. . in the domain, and I'm just looking at my picture, I tried to draw a mental image of it here, but everything that's in the domain. . whatever possible, whatever possible numbers I use, for r in this case, the function, the range, "the range of f is the set of all possible values". . but I don't see the range of f. . see it's this, assuming this middle part of the machine is f, I think they should say . . "varies throughout the domain" . . "The range of f is the set of all possible values of set B". Why in here? Nothing's really happening in here. We're just plugging in all different numbers . . cause for me the formula. . the machine. . let's . . what was it? $2\pi r$. . For me. . it says here . .*(reads the definition again)* "The range of f is. . . . domain", so if this is 1, 2, 3, 4 whatever . . the only thing that changes in here is this value here. It's actually set B that . . so whatever this is, the function's going to serve the same . . I mean it's just numbers together. We haven't calculated anything yet. This is ultimately where we're going to get the last answer. So what happens in here is just how you plug it into the . . so it's going to be 2π. . if this is 1, $2\pi 1$, if this is 5, $2\pi 5$, so it's really not relevant until you mix it all together, but to me this is what. . this is what the range for me at least would be.
361. A: O.K. So what would it be in this case here? What would it be in reference to this question?
362. M: In reference to this question?
363. A: Yeah. .
364. M: Well here the range. .
365. A: Your domain. . *(was going to help him relate the range to the domain)*

366. M: If this is 4, my range would just be 2π4. The formula itself, not the answer. I mean the answer's set B, right?
367. A: Yeah. . and you're saying. .
368. M: The 25.1 is your . .
369. A. & M. . . set B.
370. A: And what's the range then, according to you? Your range is what's happening in the machine?
371. M: For me, the range. . well the way they explain it. . *(re-reads the definition of range twice more)*. . For me the range would be what comes out at the end.
372. A: Well, what do you get in the end here then?
373. M: Well, in this case, it would be 25.1, If it's 5, 31.4.
374. A: O.K. So the range is all the values of the circumference?
375. M: Exactly.
376. A: That's not quite what you were saying a minute ago, is it? You were trying. .
377. M: I was trying to relate it to the machine. .
378. A: So is the range for you set B now?
379. M: Yeah. For me the range should be set B.
380. A: And the domain is your . .?
381. M: The domain is whatever set A is.
382. A: O.K. So. .
383. M: And the function would be whatever's in the middle.
384. A: Alright.
385. M: *(reading the definition of range again)*. ". . that is, . ." This, I don't understand at all *(in reference to the set notation)*. I've never come across this yet. I haven't come across this yet.
386. A: What's "this"?
387. M: Or this. This symbol right here.
388. A: Which symbols have you not come across? Say them for me because. .
389. M: The "∈". *(calls it "the Greek e")*. And the. .
390. A: . . the line going down?
391. M: Yeah, the line going down. f(x) . . the line I've never seen before. .
392. A: O.K.
393. M: And this "∈" I have no idea what it means.
394. A: So, what you. . . you know the f(x) . . What about the curly brackets there, does that suggest? . . say anything to you?
395. M: Well, whatever's within that. .
396. A: They're just brackets, like any other brackets?
397. M: Yes.
398. A: They have no special. .
399. M: No.
400. A: They have no special connotation?
401. M: Specific to. . ? No. No. Not at all.

402. A: Alright. Let's get back to the questions there. We've got our domain, correctly established as being the values for the radius, . .
403. M: O.K.
404. A: The range is the values you get after you've applied the rule to it . .
405. M: Right.
406. A: . . and you've got your circumference. Now the next two questions are talking about independent and dependent variables. Now if you go back up to the second paragraph it explains to you there what they mean by dependent and independent variables. And can you use this question to explain.
407. M: *(reads the second parag.)* For me this *r*. .
408. A: The *r*?
409. M: The *r* . . like in the domain. .
410. A: mm . . mm
411. M: . . is the symbol. . "The symbol. . . .number". . So *r* would be the variable and it represents whatever the arbitrary number is . . "in the domain. . . independent variable". O.K. So it's by itself.
412. A: O.K.
413. M: So whatever this *r* is in set A, or the domain, is that independent variable. . is an independent variable. *(reads the next sentence)* "The symbol. . . . dependent variable". O.K. This being your range, or your set B, over here. *(re-reads sentence)*. . O.K. The way I understand this is, this independent number, whatever this number may be. .
414. A: mm . . . mm
415. M: . . will. . this number will depend on whatever this independent number is.
416. A: Right.
417. M: So that's the way I understand that.
418. A: Right. Correct .
419. M: *(reads the last sentence of parag. 2)* "For instance. . . variable". O.K........*(inaudible comments)*
420. A: So which variable is the independent variable?
421. M: The independent variable would be the 4. .
422. A: O.K. So all the values for. .
423. M: All the values for *r*.
424. A: So, *r*.
425. M: For *r*, the radius.
426. A: And the dependent variable?
427. M: The dependent variable will be the range or the circumference. The answer after the formula. .
428. A: O.K. So which variable have they used?
429. M: In this case, 4 would be the domain. .
430. A: Yes, but which variable? I used a certain variable here.
431. M: Oh, C.
432. A: I should have that in brackets there. (in reference to the C in the table). So C is the dependent variable.

433. M: Right.
434. A: O.K. So you understand this idea of dependency then?
435. M: Yes. Absolutely.
436. A: Alright. Do you see how it was coming across in the first part where one variable was dependent on the other one?
437. M: Right.
438. A: Alright. And it's much more of a one-way thing now isn't it?
439. M: Absolutely.
440. A: If you start with your *r* as your independent variable then the formula is definitely going in one direction only, $C = 2\pi r$. The C is dependent on the *r*. If you want to see it the other way, you know, the $r = C/2\pi$ then there'd be a complete switch around of dependency, wouldn't there?
441. M: Absolutely.
442. A: O.K.
443. M: Yes. The radius would be dependent on the independent variable of C.
444. A: Of C, O.K. So it definitely depends on which one is the independent variable. It's very important this . . to know which one is the starting variable. . the independent variable.
445. M: O.K.
446. A: Alright, the last two questions you've already answered. What's set A and what's set B?
447. M: O.K.
448. A: Your set A you already established was the. . ?
449. M: Was the domain or the input. .
450. A: Which in this case was the . . ?
451. M: Was the *r*. The radius.
452. A: Right. And the set B?
453. M: Set B being C, for circumference.
454. A: O.K. And what is the function in this case? We've already established that answer too.
455. M: The function is $2\pi r$.
456. A: $2\pi r$. O.K.

(A diversion follows with a discussion on how functions were introduced in a High School Grade10 Mathematics text book)

Highlights of this conversation:

(in reference to the opening sentence of p.12)
457. M: Look how clear that was! ". . $A = \pi r^2$. ." Thank you very much. I understand that. It's very easy reading. As opposed to the. . *(first parag. of PART 1)* It's saying exactly the same thing but this is very well put. This is very simple. I understand exactly what they mean by that. I don't have to read it twice.

(in reference to the last parag. on p.12, "In mathematics. . . functions".)
458. M: I'd have to sit down and go through it. . It seems at first absolutely very straight forward. .
459. A: Do you have an idea what functions are after you've read that?
460. M: If I sat down with this, judging by what I've read so far. . . I would definitely feel more comfortable with this. It's a little more self-explanatory. .

(in reference to p.208-209)
461. A: What are your first impressions on reading this *real* chapter on functions? *(p.12 was only a brief introduction to the concept of functions)*. . . an overall general impression.
462. M: *(reading from the last parag. p.208)* ". . .there is only one value of v for each value of q" So that's, for every value of q there's only one value of v, and vice versa, I'm sure. If you were to switch it around?
463. A: Well, I'll leave that question unanswered at the moment.
464. M: Input-output, they have the same machine.
(from the last parag. of column 1, p. 209) "Name the function" would be, name the formula, "name the input, the independent variable", so the variable you can replace, "name the output, the dependent variable", which is dependent on the independent number, "represented by v."

Part 3:
Please read the next 2 paragraphs.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The **range** of f is the set of all ~~possible~~ values of $f(x)$ as x varies throughout the domain, that is, $\{f(x) \mid x \in A\}$.

The $\boxed{\text{symbol}}$ that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. The symbol that represents a number in the *range* of f is called a **dependent variable**. For instance, in the bacteria example, t is the independent variable and N is the dependent variable.

RANGE

SET B

FORMULA

SET A r $\boxed{2\pi r}$ = C

DOMAINE FUNCTION circ.

Suppose you are given the following chart:

(r) radius of circle (cm)	4	5	9	11	15	A
C circumference of circle (cm)	25.1	31.4	56.5	69.1	94.2	B

$(2\pi r$

From your reading of the above 2 paragraphs, can you identify
i) the domain
ii) the range
iv) the independent variable
v) the dependent variable
vi) the set "A"
vii) the set "B"

From your understanding of the first 4 paragraphs what is the function f in this case.

BRENT - PART 3:

183. A: Part 3. The technical part. . (*allows time for B. to read*). . . Basically I wanted to know what you found. . what you got out of that by reading it?
184. B: These 2 paragraphs were . . I read in the text originally. .
185. A: Uhuh.
186. B: . . is what helped me to more comprehend what was going on here. *(in reference to the definition)*
187. A: Yeah?
188. B: Cause this seemed more familiar to me than what they were trying to tell me here. *(again in reference to the definition)*
189. A: O.K.
190. B: And I have. . I feel because it's relating to a dependent and an independent variable of which I've previously seen and it was relating it to f, to x and y. .
191. A: O.K.
192. B: So I was. . I hate to say this but I was getting. . it was a visual again. .
193. A: Where's your concept of y coming in here? It doesn't say anything in here about y.
194. B: I'm going back to a visual of a graph.
195. A: O.K.
196. B: If I have a value of x. .
197. A: Uhuh. .
198. B: I change x, y is going to change.
199. A: O.K. So you had no trouble with the symbol there being called all these different names? (*refering to "f at x", "f of x", etc*)
200. B: This here?
201. A: mm . .mm
202. B: No. Cause I've al. . I knew previously it was called "*f of x*", so I just kind of mentally didn't even . . Now this here part, "**or the image of x under f**". .
203. A: What does that mean to you?
204. B: What. .
205. A: . . the words, you know. .
206. B: What does x look like when its under f? um. .
207. A: What does that really mean?
208. B: What I'm reading personally, if I underst. . I understood this part, **the value of f at x**. .
209. A: Alright. .
210. B: And it says "or the . .
211. A: mm . .mm
212. B: I would have more mentally ignored it. I just would have skipped over it.
213. A: Does it mean anything to you at all really . . what it's saying there?
214. B: It confu. . I was understanding it and it just. . it was confusing so I just ignored it, because I was understanding what was being said. .

215. A: . . before that? O.K. But I've forced you now to look at it! Can you make any meaning out of it at all?
216. B: (*reading aloud*). . "**image of x under f**". .
217. A: What does "**image**" mean to you. . at this stage?
218. B: What it would look like if it was put on a graph . . what would be the visual representation of this.
219. A: O.K. What does it mean by saying, "**of x under f**"?
220. B: I have no idea.
221. A: Just by reading it. . just word. .
222. B: . . word for word. .
223. A: . . just word for word. . what's it saying there to you?
224. B: (*pause - no reply!*). .
225. A: You're not even taking a literal meaning out of it at all?
226. B: I could take the literal. . if I'm taking the literal meaning of it, "**the image of x under f**", the first thing I'd think of is "**x under f**". .
227. A: Yeah. .
228. B: . . which is x divided by. . f divided by x. .
229. A: You think that's what they mean there?
230. B: No.
231. A: No.
232. B: But that . . that's what it's telling me. . that's the first thing I'm thinking of.
233. A: O.K. And yet you're saying that the **image** would make you think of a graphical representation?
234. B: Yes.
235. A: Now how does **that** thought tie in with your other thought there that. .
236. B: It doesn't.
237. A: It doesn't. O.k. So. . you. . is there a problem there?
238. B: Right.
239. A: A big problem?
240. B: Agreed. I have a problem with this. . the whole five words.
241. A: The words. Alright. O.K. Now what about the last. . bunch of symbols in that paragraph? Never mind. . just hold that question. I've asked you that at the very end here.
242. B: O.K.
243. A: O.K. Let's go to the questions.
244. B: O.K.
245. A: Alright, that's giving you another chart of numbers there.
246. B: O.K.
247. A: Do these 2 sets of numbers represent a function?. . you've put a "yes".
248. B: O.K.
249. A: Do you know what the function is? What am I really asking you when I'm asking, "What is the function?"
250. B: What's the relationship?

251. A: What's it's relationship? O.K.
252. B: Um. . I can't relate that back to $A = \pi r^2$ because that's the area. .
253. A: (*agrees*) That's the area.
254. B: Right. The formula for circumference. (*puts emphasis on "cir. . .*)
255. A: Circumference (*pronouncing it correctly*). Forgotten?
256. B: I've forgot. I don't. .
257. A: O.K. It's $C = \pi d$ or $2\pi r$. $C = 2\pi r$ or the other one is $C = \pi d$ (*repeating the formulas as B. is recording them*) d is the diameter. .
258. B: Right.
259. A: . . and it's worth $2r$'s, 2 radiuses. (*I know it should be radii!!*).
260. B: O.K.
261. A: Alright. So, you should see therefore how I got these values. . (*refering to the values for r and C in the chart*)
262. B: Right.
263. A: . . here. How did I. . Can you explain now the 4, and the 25.1?
264. B: The 25.1 will be the 4 which is our independent variable, times by 2π.
265. A: O.K.
266. B: Which will give us C. . the whole way along.
267. A: O.K. Now then, from your reading of the above paragraph where they explain to you about domain and range. .
268. B: Right.
269. A: . . we haven't discussed that yet. .
270. B: O.K.
271. A: . . but just from your reading of them, without any discussion, can you identify what would be the domain in this question?
272. B: Domain will be the radius.
273. A: mm . .mm
274. B: Range the circumference. .
275. A: Why?
276. B: Because the range is dependent on the domain.
277. A: Right.
278. B: This is always. . x is this . . I'm visualizing as x is always the dependent variable. . x is the domain. .
279. A: x is the. . ?
280. B: x is the domain.
281. A: x is the. . ?
282. B: . . domain.
283. A: dependent or independent?
284. B: It's the independent.
285. A: Independent, right.
286. B: The range is dependent on the domain.
287. A: Alright. O.K. then, so . . the independent and the dependent?
288. B: The independent 's going to be my radius which is my domain and the dependent is going to be. .

289. A: In terms of variables? What would you put then for the variable, for. . What did you write down there? *(checking what answers B. gave for (v) and (vi))* . . *r* and C. O. K. *r* and C *(recording these answers on my own sheet)* For domain you just wrote down *r*, here?
290. B: Right.
291. A: *r* is the variable. .
292. B: Right.
293. A: . . we're using here. .
294. B: Right.
295. A: The domain is a bit more than just . . you know you can't just say *r* here. .
296. B: O.K.
297. A: What are the values that we're using. What are these independent variables? or independent values?
298. B: It would be a set. Is that what you are looking for?
299. A: Yeah. Which values would they be?
300. B: 4, 5, 9, 11, 15. *(recording them)*
301. A: Yeah. I was really after those. .
302. B: *(recording the values for the range now as well)* 25.1, 31.4,
303. A: O.K. You don't have to write them all. . dot, dot, dot. . Then we're down to the set "A" and the set "B" that they keep using in their definition. Are we quite clear what they are?
304. B: That would be the domain and the range.
305. A: O.K. So which is which?
306. B: "A" would be the "domain".
307. A: mm . .mm
308. B: and "B" would be the range.
309. A: O.K. You've already answered part (ix) here, the explanation of $f(x)$.
310. B: O.K.
311. A: You already answered that. Now the very last question was the one I said to hold a minute ago.
312. B: O.K.
313. A: This last symbolic notation. Do you understand any of that?
314. B: O.K. *(reading it over)* . . set of f(x) which is going to be. . *(thinking out loud)*. .
315. A: Why did you say "set" there?
316. B: It's in a set brackets and it is. .
317. A: O.K. so you understand that those brackets are identifying a set?
318. B: Yes.
319. A: Yes. O.K
320. B: . . um . .because I know that x is an independent variable, it's telling me, based on what this variable is, x will equal. . I'm not sure what that means. . but x. . my own terms, depending upon what the x equals, my set B. . . no. . depending upon my x, it's going to depend what my values. . this is my y, *(indicating the $f(x)$)* and this is going to be my values of x. .
321. A: So which part of that is really the answer?

322. B: This is my answer right here.(*see B.'s sheet*)
323. A: Yes, but, in this notation here, is there sort of a meaningful part?
324. B: x . . this is . .
325. A: O.K. Let's go back up to where they put this in here. They were saying this was. . this was another way of expressing something else. What are they saying that's a way of expressing?
326. B: They were expressing the definition of a function.
327. A: No. Read the beginning. .
328. B: "The range of f is the set of all possible values of $f(x)$. . this is the range?
329. A: Yes. This is a way of identifying the range. So which part of that notation there is the range?
330. B: This is the range?
331. A: Which part of the definition . . of the symbolic notation here?
332. B: This would be my range. . this here,(*indicating the* $f(x)$) depending upon what x was, would give me my range.
333. A: O.K. *(I explain to him what all the symbols mean and B. records this on his sheet).* Right. Part 4. . .

Part 3:

Please read the next 2 paragraphs.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x** or the **image of x under f.** The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is, $\{f(x) \mid x \in A\}$.

The symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable.** The symbol that represents a number in the *range* of f is called a **dependent variable.** For instance, in the bacteria example, t is the independent variable and N is the dependent variable.

Questions:

Suppose you are given the following chart:

(r) radius of circle (cm)	4	5	9	11	15
(C) circumference of circle (cm)	25.1	31.4	56.5	69.1	94.2

i) Do these 2 sets of numbers represent a function? Y

ii) What is this function? $C = 2\pi R$ or $C = \pi d$

From your reading of the above paragraphs can you identify:

iii) the domain → r. 4, 5, 9, 11, 15

iv) the range – C. 25.1 31.4 56.5 69.1 94.2

v) the independent variable r

vi) the dependent variable C

vii) the set "A" –

viii) the set "B" –

ix) Is the above explanation of $f(x)$ meaningful to you? ✓

x) What does the symbolic notation $\{f(x) \mid x \in A\}$ mean?

y B f(x) such that x | x is element of A

PETER - PART 3:

193. A: O.K. Right. Part 3.
194. P: Yes. Read the next 2 paragraphs?
195. A: Yes. These are the next 2 paragraphs from the text book. . *(P. given time to read them).* We should be reading things here. Here I have given you a chart involving radius and circumference. . and I'm asking you, 'Do these 2 sets of numbers represent a function?'
196. P: These here?
197. A: Yeah. When I said 2 sets of numbers, I'm talking about the SET of numbers representing the radius and the SET of numbers representing the circumference.
198. P: Yes.
199. A: Do they represent a function? These 2 sets?
200. P: I would consider. . I wouldn't say they were a function. I would say that the C is the function - not the upper numbers but the lower numbers are the function? Yeah, the. . the. . any of these numbers are a function - anything that falls under here are a function - in my way of looking at it - this would be the function - whatever these numbers are. In other words, the relationship. . the function. . C is a function of r. .
201. A: Right. .
202. P: O.K. . . in this relationship here.
203. A: . .the whole thing together. To you the function is the C part and not really the r part as well?
204. P: When I say it, I say, 'C is a function of r'. I don't say 'r is a function of C'. So to me it's not . . the r is not the function. The C is the function OF something.
205. A: O.K. Right. Well, what is the function precisely then? If C is a function of r?
206. P: C would be the function of 2 times πr.
207. A: $2\pi r$.
208. P: $2\pi r$.
209. A: So that shows you that C is the function of r?
210. P: Yes.
211. A: O.K. Now, from the paragraphs you just read, the first paragraph defines what they mean by a domain. .
212. P: Yes.
213. A: . . from your reading, which. . what is the domain of this. . ? *(I hesitate to say 'function'!)*
214. P: The domain is the set A, which is the 4, 5, 9, 11 and 15.
215. A: mm. . mm. And the range?
216. P: The range, well in this particular case, would be between 25.1 and 94.2.
217. A: O.K. So that comes across clearly enough in that paragraph?
218. P: Yeah. It describes it. 'all possible values', 'throughout the domain'. .

219. A: So you had no trouble with this word 'domain'?
220. P: No. It's not a word I would use to describe it but. . yes . .
221. A: . . you accept it?
222. P: I accept it.
223. A: O.K. Then in the second paragraph they talk about independent and dependent variable.
224. P: Yes.
225. A: Is it clear from their description of them which is which in this situation?
226. P: Yeah. *(hesitatingly!)* So in this case the independent variable would be the radius and the dependent one would be the circumference.
227. A: O.K. Well, you really answered already the next 2 questions. What's set. .
228. P: The set A is the radius, and the set B is the circumference.
229. A: Yeah. O.K.
230. P: You've explained that to me before.
231. A: Yes, I know. .*(laughs)* O.K. Then we come to the explanation of this symbol, the function of x, $f(x)$, reading all what they're saying about it there. . What do you get out of it?
232. P: To me this means . . the important things. . the value of this number at x, in other words, the function of x at this particular moment. Whatever x is, there's going to be a value for f. 'The image of x under f. . I'm not too . .
233. A: Yeah. What does that?. . what kind of mental picture do you get when you read that. . those words, 'the image of x under f ?
234. P: Exactly. I get a picture. To me an image is a picture.
235. A: What kind of a picture?
236. P: A picture. A snapshot of x under f? Doesn't mean anything to me. It's meaningless.
237. A: Totally meaningless?
238. P: Totally. .
239. A: Any idea at all comes to your mind when you read that?
240. P: Yes, there's got to be some sort of a definition of x at the value of f, at the function of f, so there's a relationship between these 2. 'The value of f at x' makes a lot more sense to me than, 'the image of x under f.
241. A: O.K. Now the last bit there . . inside the curly brackets. How do you say all that? What does it mean? Does it mean anything at all to you?
242. P: Well, function of x. . and this is x in the set of A. .
243. A: Right.
244. P: O.K. and the brackets, I don't know if these mean the same things that we made when we were doing the annotation . . *(?)*
245. A: No. The brackets here have a . .
246. P: . . special. .
247. A: . . have a special significance.
248. P: O.K. It doesn't . . What it . . I'm guessing here. O.K. If I were to look at this, I'm looking at it and I'm reading what it says before something,

alright. . the function of x. . all the values of the function . . all of the values of f, O.K. the function of x, where x is a value in a set of A.
249. A: O.K. And that little line going down there, you said 'where' for that? You think that line means anything?
250. P: I'm sure it's not there for nothing. I don't know why it's there though.
251. A: O.K.
252. P: O.K.
253. A: O.K. And the brackets don't mean anything?
254. P: No. I was just assuming that perhaps they meant the same thing that anything within that. .
255. A: O.K. Let's go back to the beginning of that sentence there. What is it telling you the range is?
256. P: The range of f is the SET of all possible values. . as x varies through the domain. So I guess x, set of A is the domain?
257. A: Yes, so x varies throughout the domain. That's your x. It belongs to A, right?
258. P: And $f(x)$. . the line must mean ALL possible values?
259. A: No. The line is a short hand way of saying 'such that'.
260. P: Such that.
261. A: So we have $f(x)$ 'such that' x is a member of A. But it also said the range of f is the . . 'SET'. .
262. P: Set. O.K. So this is a set.
263. A: And your curly brackets are always used to indicate a set.
264. P: *(Indicating $\in$)* Is that not a set?
265. A: That means 'an element of a set'.
266. P: I see.
267. A: But when . . as soon as you use curly brackets. .
268. P: That's a set. .
269. A: . . in mathematics and you put things inside them, it becomes a set.
270. P: I thought a set of numbers was this thing here. *(indicating the $\in$ again)*
271. A: No that would say like, say I had, you know, the numbers, 1, 2, 3, . .
272. P: Mm . . mm
273. A: . . inside curly brackets, I'm saying I'm talking about the SET of numbers 1, 2, 3. Then I could say, let's call this set something, let's call it. . well A. Letters usually denote a set. So A is my set 1, 2, 3. I can then say that 1 is an element of set A.*($1 \in A$)*
274. P: I see. O.K.
275. A: O.K?
276. P: Alright.
277. A: Alright?
278. P: Yup.
279. A: I didn't want to bring this up here, but that's what that means, that each one is an element of that set.

280. P: O.K. So this is a set. .
281. A: . . x there is an element of the set A. .
282. P: . . such that . . the functions of x. . the set of the functions of x, such that x is an element of A.
283. A: So there are really 2 sets involved here. You've got your set A, and the whole thing . . when you work on this . . you get your set B, which is your f(x).
284. P: O.K. So B would be this?(*indicating the {......}*
285. A: Right. Good.

APPENDIX

D

APPENDIX D

MICHAEL - PART 4:

465. A: PART 4 is talking about the function as a machine, and again these two paragraphs are out of your text book.
466. M: I would not just write, 'function'. I would also write, in brackets, 'formula'. *(see p. 58 of M.'s notes)* Just the way I understand what a function is.
467. A: O.K. Can you just read that and give me. . just talk out loud if you like, to give me your general impressions of, you know, how you reacted when you first read this anyway.
468. M: . . output. . $f(x)$. . the $f(x)$ is in the output. . this would be the answer. The $f(x)$ is actually happening in here *(indicating the middle part of the machine)* for me.
469. A: For you. But that's not what they're saying, is it?
470. M: No.
471. A: What are they saying?
472. M: *(starts reading the first parag.)* "It is helpful. . If x is in the domain of f. . .". So if x is in f here. . see that's where f and x are together. "If x is in the domain of the function f. . ". O.K. Now see, I'm mixing up. . domain is over here. *(indicating the input area of the machine)* Now they're saying, "domain of the function f", so here's another domain then *(indicating the middle part of the machine)* So domain in another sense, another sense of the word. . "If x is in the domain of function f, then when x enters the machine. . accepted as an input and the machine produces an output $f(x)$" *(as M. read this he illustrated the motions of x entering the machine and $f(x)$ coming out of the machine by means of arrows going into and out of the 'f' part of the machine - see M.s notes, p. 58)* I see f, I see the output here. . I see this whole thing as being the function. . in here. . I see the $f(x)$ in here. . I see the output. . I see the input, I see the output, or set A, set B, but I see more the. . *(sighs)*. O.K. Just reading this, "If x is in the domain of the function f. ." So I see x being inputted into here *(the f part)*, through here, and becomes part of it.
473. A: So you see the domain as being something that's happening inside. .
474. M: . . inside here, but I know the domain. . from what we spoke about before. . I realize the domain is on the. . in set A or the input. I understand that.
475. A: Is the word domain having a different meaning for you here?
476. M: For me, yeah. It's saying, now I'm going into a different domain. "If x is in the domain of the function f". . So that means if it's in here. That means it went from this domain to this domain *(indicating the 'input' area into the 'f' area)*. But domain before was something else. Domain was whatever the numbers were here. Whatever I chose x to be.
477. A: Right.

478. M: So coming in here now, I'm changing domains cause now I'm in the domain of f. Just as well. . so then I could also determine from that that I could also transfer from the domain of the function, or the inside the machine, to the outside. . and I could also be in. . *(end of side 1 of tape so some words were lost here)* . . It is accepted as an input and the machine produces an output, O.K. " . . according to the rule of the function" *(repeats)* So whatever the rule of the function was happening in here, then I'll get my answer out here.
479. A: mm . . mm
480. M: But isn't the $f(x)$ inside of here? The function of x. The function of what's going on?
481. A: The function, the function of x, is the final output. The f, you see they've separated the f from the $f(x)$.
482. M: O.K.
483. A: The f, the way they've done it in the machine, is the f is the function, the f is the rule, the f is your formula. .
484. M: Right.
485. A: . . and the final output that you get is the result of this formula on x.
486. M: Formula on x.
487. A: The way, you know, to use your words. But do you notice the last sentence here uses the word 'domain' again. What domain are they refering to now?
488. M: "Thus we can think of the domain as the set of all possible inputs and the range. ." *(repeats)* "the domain as the set of all possible inputs". O.K. That's fine. That's what we said before.
489. A: That's what we said before.
490. M: Which is domain. O.K. I'm happy. ". . and the range as the set of all possible outputs." O.K. That's exactly what we said before.
Range. . the output of x. . O.K. The relation that f. . this function had on x. *(looks back to Part 3)* See here we have '*f of x*', '*f* ***at*** *x*', now I have '*f* ***on*** *x*' . .
491. A: '*f* ***on*** *x*'?
492. M: I'm . . .*(reads part of the paragraph from Part 3 again)* . . yes we'd said f. . maybe this is a better word than. .remember when we came back here we said '*f of x*', or '*f* ***at*** *x*',
493. A: mm . . mm
494. M: Maybe it should be '*f* ***on*** . .
495. A: ***On*** *x*!
496. M: The relation between the function **on** x, in other words the formula **on** x. .
497. A: O.K.
498. M: Maybe that's a better word. . I don't know. I'm just saying that!
499. A: O.K!!
500. M: *(starts reading parag. 2,)* "The preprogrammed. . . machine". O.K. I read this in the book. "For example. . error" Yeah, I put in -1 and it said 'error'! "If $x \geq 0$, . . . [. . . .]." *(M. had obvious difficulty understanding the [ ] part of this parag.)*

501. A: I'll come back to that, alright? Hang on to that thought for a moment. Question1. Can you answer that one for me?*(reads Q.1, p. 58)*
502. M: We keep calling it a function, and I understand that it serves a function. .
503. A: O.K.
504. M: . . but for me it's a formula.
505. A: Yeah.
506. M: So why do we keep. . "Is it helpful. . . " Like if I would have read this . . just based on what . . the math I've taken to this point. . "Is it helpful to think of a **formula** as a machine?" Yeah. Formula as a machine, and then maybe in brackets (function). So that I start associating function and formula. It's just the *function* throws me off.
507. A: The word, 'function'?
508. M: Yes. The word, 'function' itself throws me off.
509. A: Because you keep seeing it as something serving. .
510. M: Something else. When in fact all . . Is it just a formula? For me, it's just a formula. Does function = formula?
511. A: mm...That's part of it. I mean it's the rule part.
512. M: O.K. *(returns to Q.1)* . . .
513. A: Does this machine. . diagram. Does all that help?
514. M: Yeah. That definitely helps.
515. A: Do you want to write something down then.
516. M: *(records his comments- see p.58 - and reads them out to me) (answers his own question.)* Yes, my interpretation is, it is a formula.
517. A: O.K.
518. M: That's the only part that confuses me. And this. . the formula, like here, '*f on x*'. . Well, what is the formula **on** x? Do you know what I mean? Or x plugged into the formula? For me, I don't understand at all. .
519. A: Look at p.159 of your book. You have highlighted the fact that they **are** calling the function a formula. *(see bottom of p.159)*
520. M: O.K.
521. A: Why did you highlight that?
522. M: Because I wanted to remember that. *(reads)* "The convention is . . . real number".. . .
523. A: Did you highlight that because they used the word *formula* there which is the word that you like to think of all the time?
524. M: Yes. I think so. That's exactly why I did it. And it was in red also. So it was. .
525. A: Yes. It's was in red.
526. M: *(reads from the text)*. . .
527. A: They're using your word, *formula*.
528. M: Yeah.
529. A: O.K. Now then, let's go back to the thought that I told you to hold a minute ago. The very last sentence there of that paragraph. The one in the square brackets. What do they mean by the last sentence?
530. M: That's this one here?

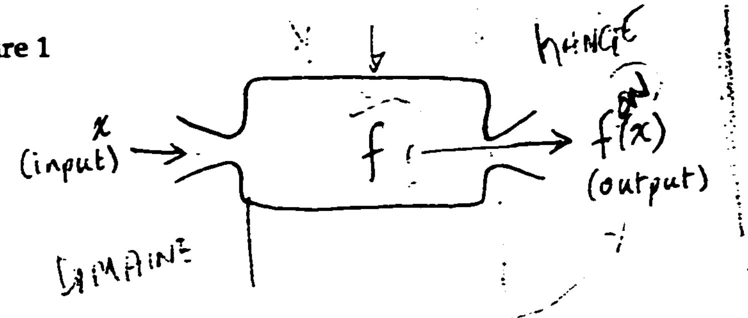
531. A: Yes.
532. M: *(reads it)* I don't understand what they mean by that. *(repeats with emphasis)* "is **not quite** the same". . So they're saying that it's not exactly the formula? It's not exactly? Why is it not exactly? It should be exactly, shouldn't it?
533. A: Well, it's saying it's **not** exactly. . .
534. M: But what . .?
535. A: . .What did they say before that?
536. M: *(reads with emphasis on the relevant words)* "If $x \geq 0$, then. . approximation to x. .
537. A: Yeah.
538. M: . . . . decimal places." O.K. "Thus the $\sqrt{x}$ key. . . exact. . ." So they're saying that they would go to more decimal places, or the number would be infinite? Their answer is more precise. .
540. A: Yeah.
541. M: . . to the billionth and billionth, and trillionth. .
542. A: O.K. . . And question 3, I don't know whether we should bother with really. *(reads out Q.3)*
543. M: I would keep this one.
544. A: This is not a real machine. It's just a diagram. They've given a calculator as a **real life**, if you want, machine. Can you think of anything else that's not just diagrammatic but something that's actually used like a calculator . . or anything else that makes you think would be a good example of what they're talking about here?
545. M: Well, the first thing that comes to mind, but I mean. .
546. A: What is it?
547. M: It's just, you know, like a mixer. .
548. A: Yeah.
549. M: . . throwing everything in at the top and then turning the mixer on then at the end you can pour it all into a glass.
550. A: O.K. *(both laughing!)*
551. M: But I mean there's got to be better. . there has to be something better. It's more or less the same thing.
552. A: O.K. That's fine. It's on tape, but would you just write down what you were thinking of? A mixer, or a blender. .
553. M: That's it. A blender.

Part 4:
The next 2 paragraphs and figure 1:

(formula)

It is helpful to think of a function as a **machine** (see figure 1). If x is in the domain of the function *f*, then when x enters the machine, it is accepted as an input and the machine produces an output *f(x)* according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

figure 1



The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the $\sqrt{x}$ key on your calculator is such a function. First you input x into the display. Then you press the key labeled $\sqrt{x}$. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input and the calculator will indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ will appear in the display, correct to a certain number of decimal places. *[Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function *f* defined by $f(x) = \sqrt{x}$].

Questions:
1. Is it helpful to think of a function as a machine? Explain why, or why not, in your own words.

Yes, it is a helpful visual aid, however I find the word "function" confusing. Is it a formula?

2. What do they mean by the last sentence in the [ ] brackets?

3. Can you give any other example of a machine that would be helpful in thinking about a function?

a blender.

PART 4: BRENT

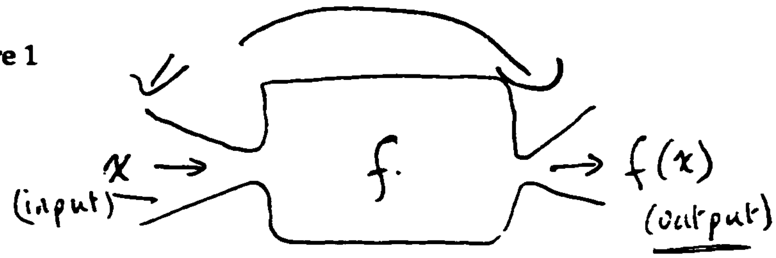
334. A: . . . the next part of the text which discusses the function as a machine.
335. B: *(reads the 2 paragraphs then records the answer, "No" to Q.1)* . . No.
336. A: O.K.
337. B: And why? I find it easier to understand when it relates back to your *x* and *y* variables.
338. A: mm. . mm
339. B: And your domain and range.
340. A: mm. . mm
341. B: I understand why they would want to do it, but for myself it didn't do anything.
342. A: O.K.....
343. B: It didn't do anything for me.
344. A: I've got all that on tape. You don't have to write it all down. O.K you said you could see why they would want to do it. What do you think, you know, they're really trying to make sure people understand by doing this?
345. B: That your. . depending upon your values of *x* it's going to determine your outcomes of your *f(x)*.
346. A: mm . .mm
347. B: So people . . people think, well if I put something in, it's going to come out the other side. .
348. A: mm . .mm
349. B: It's going to come out the other side. Depending on what I input .
350. A: mm . .mm
351. B: . . it's going to affect my output.
352. A: O.K. That's fair enough. Did you read the part about the calculator? Did you find their example there of a calculator helpful?
353. B: *(thinks about this for a few seconds)* I think they get too confusing as this makes you wonder. . . I think it's a nice touch to relate it to your calculator. Your calculator is an important function of math, but in reading it I find it confusing and my first reaction was to check it on my calculator and O.K., yes , I can see what's happening here and then they get into, "if x < 0 then x is not. . ." forget it!
354. A: What happens when x < 0? Do you get an answer to your square root?
355. B: No, it's going to give me, "error".
356. A: It's going to give you "error". Why?
357. B: Because you can't have zero square root. It's like "error". It's not. .
358. A: It's not the zero that's the problem. It's . .
359. B: . . . .minus. . the root is . . .noncomputable. .
360. A: It's the negative. O.K. Then. . the next question there. . What do they mean by the last sentence. .

361. B: . . in brackets?
362. A: Yeah. In that last paragraph. If you haven't found this helpful, then it's not. . Does that mean anything? Helpful or otherwise? Do you even know what they mean by it?
363. B: What I feel they're meaning by it is um . .my calculator has a limitation when I use this.
364. A: mm . .mm
365. B: This formula, as it stands as a formula, has no limitation if you have the computer capacity to compute it. Therefore it doesn't. . it's not quite exactly the same. In my view that's what it's saying, it does not equal cause depending on your process and capacity it will give you a different answer.
366. A: O.K. And question 3 there. Well since you're not too keen on machines I shouldn't force you to think of another machine! Can you think of any other practical type machine that **would** be helpful at all?
367. B: Computers.
368. A: Computers themselves?
369. B: Because I deal with computers that have calculating, re-evaluation facilities. .
370. A: mm . .mm
371. B: . . .where you just key in amounts. .
372. A: mm. . mm
373. B: . . and you key in a value date. .
374. A: mm. . mm
375. B: . . and it's going to base this on a formula. . depending on what I input and what value I input and on what my variables are, it will, as well as interest rates, it will re-calculate for me.
376. A: So you're using it all the time? in your work? so you're using functions all day?
377. B: Well it's set up for me. .
378. A: Yeah. . but you are inputting values . .
379. B: . . inputting values. .
380. A: There's a formula that. .
381. B: There's a formula that works it out and it comes back and gives you
382. A: & B: . . your answers.
383. A: So you're pretty familiar with that? O.K.

Part 4:
The next 2 paragraphs and figure 1:

It is helpful to think of a function as a **machine** (see figure 1). If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

figure 1



The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the $\sqrt{x}$ key on your calculator is such a function. First you input x into the display. Then you press the key labeled $\sqrt{x}$. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input and the calculator will indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ will appear in the display, correct to a certain number of decimal places. [Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$].

Questions:
1. Is it helpful to think of a function as a machine? Explain why, or why not, in your own words. NO —

2. What do they mean by the last sentence in the [ ] brackets?

3. Can you give any other example of a machine that would be helpful in thinking about a function?

PETER - PART 4:

286. A: Part 4, the next part which shows you this machine. .
287. P: You want me to read this again?
288. A: Yes please.
. . . *(P. reads)*
289. A: O.K. First question. Is it helpful to think of a function as a machine?
290. P: Well, its. .
291. A: Does it help?
292. P: Yeah. It makes it more graphic. It brings the domain. . it clears the relationship in that we see that something is happening in here, whatever that may be, and depending what the value is here, this one is going to be different. Depending on how many of these guys we put in there, what value this guy has, the other guy coming out the other side is going to be a little different.
293. A: Alright. O.K. . . What do they mean by the last sentence in the square bracket? This sentence here. Did you read this part about a calculator? At the very end it's saying something there. I just want to know if you know what they're getting at.
294. P: Well they're saying that the function key is not the same as the mathematical function because the function key is going to give me an approximation whereas the actual function is going to give me the exact value.
295. A: O.K. I just wanted to know what you got out of it, you know. And the other question I asked. The calculator here is an example of a machine, can you think of any other. .?
296. P: Machine?
297. A: Machine.
298. P: Well, an automobile. You put in 20 gallons of gasoline you're going to go x number of miles, you put in 10 dollars you going to go another. . um. . anything that requires. . I mean any machine requires input to give you output.
299. A: O.K.
300. P: Whether it be in terms of energy consumption. .whether it be in terms of. .
301. A: What would be your variables in this case?
302. P: The variables would be the gallons of gasoline, and the miles that were driven. The miles would be the function of the gasoline.
303. A: Yeah. Good.

APPENDIX

E

APPENDIX E

MICHAEL - PART 5:

554. A: Again a continuation of what was in the book. The last part. They're talking about arrow diagrams.
555. M: I didn't understand this at all from the book.
556. A: You didn't.
557. M: No. Looking at it now I'm going, "Oh, Geez. I remember that." *(reads)* "Another way to picture a function. . " Let me read it again. I just. . I didn't even have any comment. . . . .*(M. reads the explanation over several times and appears to have particular difficulty with the sentence, "Each arrow connects an element of A to an element of B.")*. These are the elements?
558. A: The dots. .
559. M: This dot with that dot?
560. A: Yeah.
561. M: Those would be considered the elements?
562. A: Yeah.
563. M: O.K. "The arrow indicates that f(x) . . *(M. still reads this as "f x")* . .is associated. . . . .and so on." I don't see it at all. The machine was a better. . . *(tries again)* "Each arrow. . . element of B." So this. . this being the element, it connects these two elements together. Connects the elements together so they actually become . . mixed. "The arrow indicates. . with x". . O.K. But that's. . the way I look at it, this element is called "f x" and this element is called "x".
564. A: O.K.
565. M: So instead of these dots I could have just done this *(draws circles around x and f(x) and connects them - see notes p.7)* They're telling me the same thing. You go from dot to dot. "f a" is associated" . . the same deal.
566. A: The *a* and the *f(a)* there were just labelling the dots.
567. M: O.K. So that's what the name of these dots is then?
568. A: Yeah. Right.
569. M: O.K. So this dot is called "f x".
570. A: Yeah.
571. M: Or "f at x".
572. A: "f at x".
573. M: "f on x". . "f at x". .
574. A: Not, "on", "f of x".
575. M: f of x.
576. A: "f of x" is what I'm going to be calling it.
577. M: *(reading the first question)* ". . help you understand. . . better?" Oh, I know they connect them together, they connect them together, but it . . thinking of the machine, the machine gave me a more visual . . a better visual image of it than this does. Because, essentially what I see here is that *x*.

actually I see here. . *x* connecting . .I could also interpret it as *x*. . this would be *x* and this would be *f*, and together they become *f(x)*.
578. A: But where have I put the *f* though? The *f* is the . .
579. M: function.
580. A: . . is the link. The *f* is what you are doing to the elements in here. . producing elements in there.
581. M: Yeah. But something had to happen in between.
582. A: This is it. This is your function.
583. M: O.K.
584. A: Can you identify inputs and outputs here?
585. M: Yes. You can do it. But I would show that something happened in between here. They didn't just jump over. You know what I mean?
586. A: mm . . mm
587. M: So something definitely is happening here. This is the function. This is the *f*, O.K. *(reads question 1 again)* . .
588. A: No. You've said, "No." That is being recorded that you're saying, "No." And tell me exactly why not there. *(see M.'s notes, p.75)*
589. M: *(records his answer on p. 75 and reads it out.)*
590. A: Did you answer my question about inputs and outputs? No, it's not on here. I'm just asking you.
591. M: O.K
592. A: Can you still pick out the input and the output from the diagram though?
593. M: Input would be the . . the domain would be my *x*. That would be my A, assuming this is the function, what's going on in between here . .
594. A: mm. . mm
595. M: . . as this would indicate.
596. A: What's the output?
597. M: The output would be the B, or the range.
598. A: So you think that the other diagram emphasizes more what's happening **between** the two sets?
599. M: Absolutely.
600. A: Whereas the arrow diagrams are just emphasizing the two sets and not really saying what's happening between them.
601. M: Right.
602. A: O.K. So to you then the rule should be the more important link between the two sets rather than the two sets themselves? Is that what you are saying?
603. M: Absolutely. Yeah. Well we have to know something happened in between there.
604. A: Yeah.
605. M: Here I get the feeling they're just shooting over. It's just coming over and nothing indicates to me. . it might as well be just a straight line. .
606. A: Well we haven't specified what the *f* is here. Maybe that's not helping either. But we didn't in the machine diagram. In the machine, we didn't say what the *f* was. .

607. M: No, but at least you had a mental image that there was a machine. . that there was something happening there. . Something stopped on the way over. We know that it stopped for a break on the way over! How's that!
608. A: So there's more emphasis put here on the two sets themselves rather than on what's happening?
609. M: Something's happening . . my input, so I came in, something happened and I went out. Here I just go right over. O.K. There's a function, O.K. When the function actually starts to happen I couldn't tell you just by looking at that. This one gave me a better image.
610. A: Text books very often give arrow diagrams before they give the machine. The text books tend to feel that the arrow diagrams are helpful.
611. M: Well maybe they. .
612. A: Anyway, we'll continue answering these questions on them and then. . you might see a reason why the arrow diagram ends up being more helpful in the end than the function machine, by the time we're finished the next page.
613. M: O.K. Alright.
614. A: O.K. Look at the second question I've put here.
615. M: Right.
616. A: Do you understand what I'm asking here?
617. M: *(reads)* "If "f" is "is the provincial capital of", what are (1), (2), (3), (4)?"
618. A: It's a geography question!!
619. M: Whow!! I know the answers but . .
620. A: Well what are they?
621. M: Fredericton, New Brunswick, . .
622. A: Do you want to answer them in order. What's (1)?
623. M: (1) is Quebec, (2) is B.C., (3) is N.B., and (4) is Ontario.
624. A: Did you think you'd have a question like that on functions? Is this still a function?
625. M: Well I don't see where the mix-up was though. I don't see the . . I don't see. . I understand . . hang on. . Fredericton. . the function in this case would be, "is the capital of".
626. A: Yes. Is that . . is this question mathematical?
627. M: In a sense, yeah. If you assume the "f" is "is the capital of". . right? So for me this would be the function in this case, your input would be the cities, and your output would be the provinces. . but. . O.K. I understand that.
628. A: So does it surprise you that a question which really didn't have anything to do with numbers would be involved in all of this?
629. M: No. It's good. It's actually very good. But "is the provincial capital of" is the function. .I think that should be. . well you put it in brackets up there which is good. Or whoever did this . . but "is the capital of " is the function. Right?
630. A: Yeah.
631. M: O.K.

632. A: It's the rule that's connecting the two sets, O.K? Then I asked, #3. I won't say a thing about that one! What does mean to you now?
633. M: Well it's the answer before. . it's the. . it's the first variable. What is f(Fredericton)? O.K. f(Fredericton)? What is the function of Fredericton? "is the capital of"?
634. A: mm . . mm
635. M: "is the provincial capital of". . So Fredericton is . . is the provincial capital of . . and then your set B or whatever, the (1), (2), (3), (4). . in this case (3) was the answer. .
636. A: Which was?
637. M: Which was hopefully N.B.!
638. A: N.B. Right! *(both laughing!)* Alright, I just wanted to see if you understood the symbolism here.
639. M: Oh, O.K.
640. A: This is like. . this is like your f(x).
641. M: Right.
642. A: O.K. So this is the " f of Fredericton".
643. M: Right.
644. A: Right? And the "f of Fredericton" is what happens when you apply the rule *f*, which was, "is the provincial capital of".
645. M: Right.
646. A: You apply that to Fredericton.
647. M: Right.
648. A: And you get New Brunswick.
649. M: O.K.
650. A: O.K? Now question 4. Do you think that functions will "work in reverse"?
651. M: This is getting back to something that you said before.
652. A: Right.
653. M: Yes, based on that circumference. . it should work in reverse.
654. A: Let's have that in print!!
655. M: Yes, . . . based on the. . . based on algebra.
656. A: Based on algebra?
657. M: I mean if your function or your formula. .
658. A: You based it on the . .
659. M: . . on the circumference.
660. A: We'll stick with that one for just now.
661. M: O.K.
662. A: And I've asked you, "Can you say what the "reverse" rule would be?" . . maybe I should ask you that for the first question. Based on your circumference question, what would be the reverse rule?
663. M: It would be that formula again which was $C = 2\pi r$, you told me.
664. A: Yes. $C = 2\pi r$ was the original one. .

665. M: So $C/2\pi r$ would be. . so it would be . . *(tries, unsuccesfully at this stage, to put formula in terms of r)*. . whatever the formula is to get the reverse of that. *(tries again)*. . multiply each side by C. . or divide . .
666. A: What are you doing?
667. M: C. . $2\pi r$. . just coming back to that formula you had before.
668. A: Yeah. Well.
669. A: and M: *(both talking at once)*
670. M: And the function would be that new formula. .
671. A: Which was. . r is $C/2\pi$. .
672. M: Alright. r. . C divided by 2π. . because I want to find out r.
673. A: It all works in reverse. O.K. We'd still have a function?
674. M: Hang on! .... My function would be the C, $2\pi r$. . $C/2\pi r$. .
675. A: $C/2\pi$. .
676. M: $C/2\pi$. . sorry. .
677. A: r = . .
678. M: The r. . r again would be. .*(pause)*
679. A: r would equal C over 2π.
680. M: *(hesitantly)* . . right. .
681. A: In the reverse.
682. M: Right.
683. A: Right?
684. M: So $r = C/2\pi$. . so if I want to find C in this case. . we were looking for C before? What were we looking for? We were looking for C, so . . I've got to reverse.
685. A: O.K. Before we started off with *r*. . *r* was our input before. . and we applied the formula $2\pi r$ and we got C, right?
686. M: Right. So this time. .
687. A: Now, if we switch everything around, we could use C as the input. .
688. M: Over 2π, that's right.
689. A: And then we divide it by 2π. . our rule would be this time divide by 2π and we'd get . .
690. M: . . the *r*.
691. A: . . the *r*.
692. M: Right, so the C would be on this side. O.K.
693. A: So that works in reverse.
694. M: Yeah.
695. A: The question we just looked at . . the capitals. . does that work in reverse? *(corrects mistake in reference to fig. 2, should have been, fig. 3 - see p.76, Q. 4b)*
696. M: It should work the same, yes.
697. A: The capital. .
698. M: The capital of. . sure. . absolutely it will work the same.

699. A: It will work in reverse? What would be the reverse rule? How would you word it?
700. M: What is the capital. . or *f*. . what is the capital . .what is the provincial capital of . . or what is the. . what is the provincial capital of?
701. A: Yes. O.K. . . . Right. The following questions involve arrow diagrams and we want to know if they all represent functions?
702. M: um. . good question!
703. A: Is there something. . ?
704. M: Well no. Dependent. . this goes back to independent and dependent. I would think that's what this goes back to, so . . Is it possible that if you have one answer you can have the same possible. . . or if you replace one variable before the function that you can have the same answer with two different numbers? Sure, if you were into negative. . but it doesn't make sense with the Fredericton. . this one doesn't make sense with the Fredericton example.
705. A: No, it doesn't, but from the definition of function. . just purely from the definition of function, is there something about functions that we really haven't examined in full detail?
706. M: As to whether, well, can it equal the same number? or can it? . . here it jumps all over, but it doesn't. . I don't see the. .
707. A: O.K. Look at the first diagram.
708. M: Right.
709. A: To you is that a function? Does that represent a function?
710. M: Yes.
711. A: Yes. It doesn't bother you that two. .
712. M: . . two numbers, yes. I'm curious about that. Is it possible that Fredericton could be the capital of 2 cities or whatever, or 2 Provinces if you look at it from the geographic example . .
713. A: It wouldn't work for something like that , but. .
714. M: Would it work for a number? Is that possible? Yes, if this was a -2 and this was a 2, going back to our parabolas. .
715. A: mm. . . mm.
716. M: . .where you have the same output. . your input's different . .
717. A: O.K.
718. M: . . like -2 and 2, but your output is the same answer.
719. A: Right. O.K. So. .
720. M: So I would say it is possible they have the same answer.
721. A: It's possible they have the same answer. O.K. So, O.K. for the first one. Second one?
722. M: Um. . It looks odd but I don't see why. . I'd have to have numbers to actually be able to prove whether it's possible or not.
723. A: Well the idea of the arrows was supposed to bring this across.
724. M: O.K. *(reading from the question)*. . Why, or why not? Do all these arrow diagrams represent function?
725. A: What's the difference in this particular one?
726. M: Well, here you have one number "equalling" two, and here you have two "equalling" one. And you have one "equalling" one, which is fine.

That looks odd. It strikes me as being odd. Whether it's possible or not I can't tell you just by looking at it.
727. A: Where you've got one input having two different outputs. .
728. M: Right. I see that. I'm comfortable with that.
729. A: You're comfortable with that?
730. M: When I have one input, two different outputs? No. It's not possible. O.K. That's right. It's not possible. But when I have two inputs coming out with one output, I think that's possible. This I don't. . .
731. A: It's possible within the definition of a function.
732. M: Right.
733. A: But why is the other situation not possible within the definition of a function?
734. M: Because your input and your output. . whatever input you have you should always have the same output.
735. A: Same output?
736. M: Yeah. If I put in. . assuming I'm using the same function, formula, whatever I put into it I should always have the same. . although I could have a plus or minus number if it was the case of a. . if it was the case of a square root. . where I come up. . if I look . . again I'm associating it to parabolas, I don't know why, maybe because I was just working on it, but where I have 2 numbers on an x-axis, one being negative, one being positive, or whatever it works out to be. .
737. A: So that's says it's O.K. for. .
738. M: That's O.K. This one I like. This one . .
739. A: That one, you like?
740. M: Because if I have . . Oh, hang on. . if I have. .
741. A: I thought it was the other one you liked! Your -2 and your +2 . . if they were inputs they would both give you . . in each case you get 4.
742. M: O.K. This one here's a function. . *(fig. (i))*
743. A: So that's O.K. for two inputs giving you the same output.
744. M: O.K. That. . I'm comfortable . .
745. A: You're comfortable with that.
746. M: I'm comfortable with that.
747. A: But we're looking at the other possibility of one input giving you two different outputs. Is this within the definition of a function?
748. M: No, I don't. . I don't know. But I say that's not possible. That's why I'm saying that's not possible. . this is possible *(fig. (i))* . .
749. A: Yeah.
750. M: But I would say this is not possible *(fig. (ii))*
751. A: Well it could be possible but is it possible within the definition of a function?
752. M: No.
753. A: Does a function allow this to happen?
754. M: No. It doesn't.
755. A: No. Where. . How do you know this?
756. M: Well because. .

757. A: From your definition of function where has this notion that you cannot have this situation arise?
758. M: By the fact that whatever number I put in I always have to come out with the same number.
759. A: It's not going to be the **same** number.
760. M: Well I have to come out with one. . there's only one answer.
761. A: There's only one answer.
762. M: Yes.
763. A: O.K. Now has that definition. . has that come across clearly from everything we've been doing on functions?
764. M: Yeah. Yeah. That definitely comes across there. I would say there's only one answer, one possible answer, with an input. You can't have 5 different answers.
765. A: O.K. So that has come across quite clearly? All the ways we've been looking at it?
766. M: Yes. Absolutely.
767. A: O.K. The third one? *(fig. (iii))*
768. M: The third one?
769. A: mm . . mm
770. M: The fact that it's not saying. . it's going over. .you have to have. . there has to be some. . well if the answer is zero, then it's zero. . but. . yes.
771. A: Yes. O.K. Is the fact that we've left out some of the elements relevant?
772. M: Not really. But I mean there should be. .I mean if they're put in, they should have something come out though. But I mean besides that . . it just looks as if they weren't applied.
773. A: They were used but they weren't applied. O.K. So that was, "Yes", for that one. And what about #4?
774. M: O.K. Yes. Absolutely. I say that's possible. With negative numbers or whatever. .
775. A: O.K.
776. M: It seems like a lot. .
777. A: Yeah.
778. M: . . but. .
779. A: & M: . . it's possible!
780. A: O.K. Now then, do you see perhaps a benefit to the arrow diagrams as opposed to the function machine diagrams? I mean, what do the arrows. .
781. M: Yeah. Yeah. I understand that.
782. A: O.K. The arrows are actually showing you what is happening to the input values and the output values, how they are connected. Whereas the machine shows you more clearly that something is happening to them. .
783. M: Right.
784. A: . . it doesn't really tell you **what** is happening to them. But the arrow diagram is clearly indicating what is happening to each of the input values, whether it's going to one output value, or whether it's going to two, whether they've all been used or not. .

785. M: Right.
786. A: O.K. So. .
787. M: I understand.
788. A: . . there's a slight advantage to using the arrow diagram. O.K.
789. M: I just don't like the fact that it just jumps over, that's all.
790. A: Yeah.
791. M: I think it should show something happening in there. .
792. A: . . in between? You want something to happen in between?
793. M: No. . just. . but I mean the arrow. .O.K. the arrow's fine. The arrow's fine. It shows that there's something. But, it doesn't really show. . it just shows the direction that the arrows are going. .
794. A: Yeah.
795. M: For me this is saying exactly the same thing.
796. A: Yeah.
797. M: But the function. . I mean if they showed a . .
798. A: Something going on in there. . . O.K. The final question.
799. M: #6. *(reading)* Which of the above. . . . . .reversed? Why? This one would not. *(fig.(i))*.
800. A: Why?
801. M: Because you'd have one input "equalling" two outputs.
802. A: mm . . mm
803. M: The second one, *(fig. (ii))* would be the same.
804. A: mm . . mm
805. M: Would not be a function.
806. A: Right.
807. M: The third one *(fig.(iii))* possible, O.K. And the fourth one *(fig. (iv))* not possible.
808. A: So why is fig.(iii) the only one?
809. M: Because it just goes right across. There's only. . it's not . . whatever this input was the output was always the same. Even if these two came down to one number it would still be. . it would still be good.
810. A: Yeah. O.K. So we can only . . we can have functions then which cannot reverse. Which is coming back to a question that you brought up before that it would always work both ways.
811. M: Right. Yeah.
812. A: But it won't necessarily always work both ways.
813. M: No.
814. A: The question about the. . all the questions we've been doing have been of the type of fig.(iii).
815. M: O.K.
816. A: Is that. . can you see. . ?
817. M: Yeah. Absolutely
818. A: Why? Why have all the questions we've been doing been reversible of the type of fig.(iii)?
819. M: I guess because you don't want it to be so complicated for now. *(both laugh!!)*

820. A: Because for every input . .
821. M: It seemed a lot simpler. .
822. A: . .we had only one output.
823. M: Right.
824. A: And the output would work in reverse for the input.
825. M: Right.
826. A: We call these things one-to-one functions.
827. M: O.K.
828. A: And they're the only ones that will reverse and still remain functions.
829. M: O.K.
830. A: O.K? Which becomes an important concept to understand when we come towards the end of the Chapter when we talk about inverse functions. .
831. M: And then obviously it gets more difficult.
832. A: When we do them in reverse, we've got a special name for it. We call it an inverse function.
833. M: O.K. Which in fact is one-to-one also.
834. A: It has to be. You cannot invert a function unless it's one-to-one. It won't remain a function. It has to remain a function when you do it in reverse.
835. M: So I assume that the fig.(i), (ii) and (iv) are just more complex. .
836. A: They're functions but they will not . . you can't do them. . *(oops!!)*
837. M: & A: . . in reverse.
838. A: You would not be asked to find an inverse function for certain functions.
839. M: O.K.

END OF SESSION.

Part 5:
Final paragraph:

Another way to picture a function is by an **arrow diagram** as in figure 2.

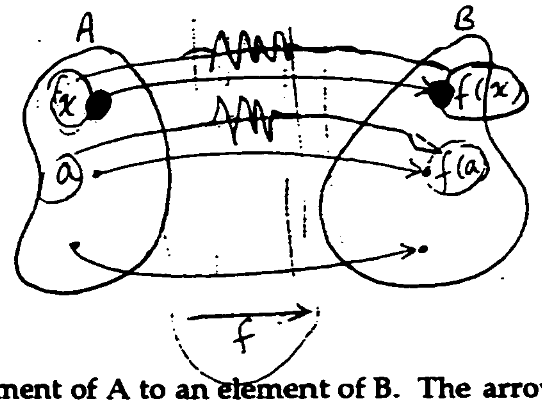


figure 2
Arrow diagram for f

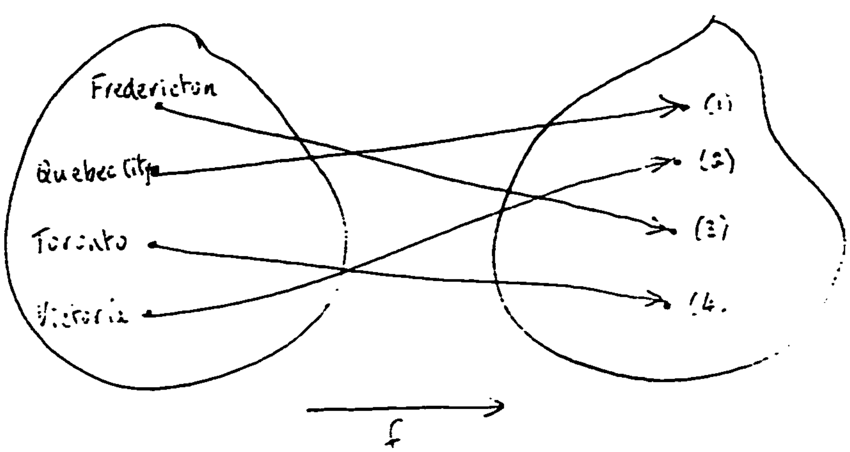
Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Questions:
1. Does this figure help you understand the meaning of function any better? Why, or why not?

No, I do not get a good visual image. The diagram does not show anything happening between A AND B

---

2. If "f" is "is the provincial capital of", what are (1), (2), (3), and (4)?



3. What is *f*(Fredericton)?

4. a) Do you think that functions will "work in reverse"?

yes, based on the circumference formula.

b) What would be the "reverse" rule for the function shown in figure ~~2~~ 3?

$\frac{C}{2\pi r} = 2\pi r \qquad r = \frac{C}{2\pi} \qquad C \to \boxed{c/2\pi} \to r$

5. Do all of these arrow diagrams represent functions? Why, or why not?

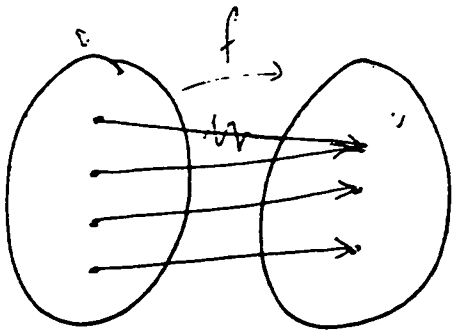


figure (i)

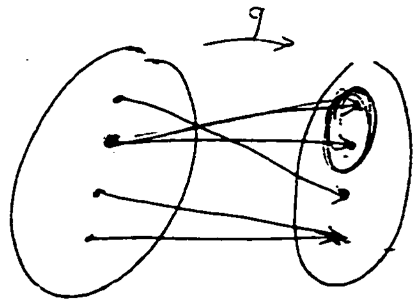


figure (ii)

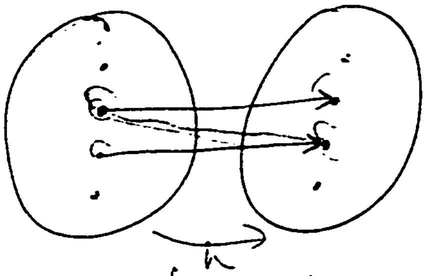


figure (iii)

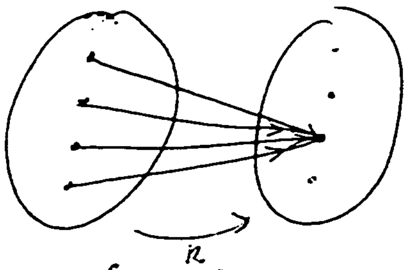


figure (iv)

6. Which of the above figures would remain functions when reversed? Why?

PART 5: BRENT

384. A: It deals with the representation of a function as an arrow diagram. And again, you know, does it help you understand. . ?
385. B: ". . . .why not?" *(reading question1.)* This comes after. .?
386. A: It comes at the end of all that introduction. It comes at the end here. It comes at the very end. *(showing B. where this section comes from in the text).*
387. B: I would find with this domain, range, independent variable, dependent variable definition, it helped me understand the concept. .
388. A: mm. . mm
389. B: . . to the best of my ability, so looking at another diagram after I already understand it. . um. .
390. A: You don't see the point to it?
391. B: It doesn't do anything for me. I know this is $f(x)$. The ... for x is $f(x)$, then a is $f(a)$, and then b is $f(b)$, c is. . .etc It's just a logical progression. For myself, I didn't think that it actually gave me anything.
392. A: O.K.
393. B: But . . if I can ask. . I. . um, as my marks show that my math is . . .my concentration and my effort that I put into math does not. . . no, how can I put this?. .my comprehension of the math behind math?. . sort of thing. .
394. A: O.K.
395. B: Because I work with computers that do most of this for me.
396. A: mm. . mm
397. B: . . my understanding of the relationships is there, it's just the bottom line, technicality points, the actual logical thinking of thinking the whole way through so my mind tends to try to go from here to the end. .
398. A: Yeah.
399. B: . . and not work its way through the middle, which is what I'm missing. .
400. A: O.K.
401. B: . . um. . so
402. A: You think. .
403. B: Yeah, this is my perception of it. So when I read this, I understood it. I said, "O.K. Great!" Now when I'm looking at this it's kind of making me go back and try and think it out again, and my mind's saying, "I already understand it. Why are they telling me. . Why are they asking me to go back and think it out through again? For somebody that doesn't have my background, it would probably. . like very helpful. . because it does give you a good visual. .
404. A: Yeah. .
405. B: Not that my background's that great, but for somebody that basically has no math. .
406. A: mm . . mm
407. B: . . or poorer math then myself, if that's possible, it would be beneficial.

408. A: O.K. By the time we've finished the next page you will hopefully find a more . . a use for it, O.K? Right, so you're saying, "No." for #1, basically?
409. B: Right.
410. A: How does it compare with the machine model? (#2)
411. B: It's more complex.
412. A: Is there any **major** difference between the two models that you could comment on? I mean, look at your machine model. .
413. B: It shows you that your set A will be your . . dependent. . no independent variables and your set B, which is your f(x) which is your dependent variables. .
414. A: mm. . mm
415. B: . . so it actually allows you to, not view what's going in and what's going out, but to actually visualise now two sets. .
416. A: Yeah.
417. B: . . as opposed to just inputs and outputs. So these are my inputs which is actually my set A. And these are my outputs which is actually my set B.
418. A: O.K. Was anything missing from the arrow diagram that was maybe emphasised more in the machine model?
419. B: Yeah. They didn't. . they should have said, "This is your input and this is your output", to relate it back. .
420. A: Yeah. That's true. But. . there's nothing else to you that's kind of missing. . ?
421. B: Well the *f* in the middle. .
422. A: The *f* in the middle?
423. B: . .I take that as A, x is a function of f(x) . .
424. A: The *f*'s down there.
425. B: You're looking for something else?
426. A: No, I'm just wondering if to you, you felt there was something missing there. .
427. B: um. . no.
428. A: O.K. You're quite happy, relatively happy with that? So, how does it compare with it? Can you maybe just give me one sentence to sum up what you just said?
429. B: It shows you the relationship between sets A and B and x and f(x) and a and f(a). .
430. A: Yeah. Does the machine model show you that?
431. B: No it does not. It shows me inputs and outputs. It does not relate it to sets.
432. A: O.K.
433. B: Or domain and range.
434. A: O.k. This is on tape. (*noting this because B. did not record this answer*). Alright. Can you identify the input and output?(#3)
435. B: My input will be my x value, my output will be my f(x). etc

436. A: O.K. And here I've given you another different type of example. Rather non-mathematical! (#4)
437. B: City (1)? Is that what you are looking for?
438. A: What's number (1), yes. What is number (1)?
439. B: Quebec City. . no. . yes. . Quebec City.
440. A: *(no reply from A. tells B. something's wrong!)*
441. B: This is my x. . the capital of . . $f(x)$. . f. . Quebec City is the provincial capital of (1).
442. A: But how do you finish this sentence? Is the provincial capital of? . . of what?
443. B: Of Quebec.
444. A: Of Quebec. So number (1) is Quebec.
445. B: O.K. Get you! And f. . Fredericton is . . number (3) is going to be, Fredericton is the provincial capital of N.B., f. . Toronto. . Toronto is the provincial capital of Canada. . or Ontario. .
446. A: Ontario.
447. B: Joke!!! And Victoria is the provincial capital of B. C.
448. A: O.k. Why did you put f up here? There's already an arrow down here with f on it. *(B. had written f above the second set.)*
449. B: This is the way I'm visualising it. This is not actually f.
450. A: It goes with the $f(x)$.
451. B: This is $f(x)$ *(writes $f(x)$ now above the second set.)*. This is going to be x (writes x above the first set).
452. A: Right. That's better. O.K. So does it bother you that we can do this without numbers? We can do this with names of cities? Is it O.K.?
453. B: No. *(in answer to the first question above!)* I'll tell you why something like this is familiar to me, not using numbers, cause I've set up LOTUS spreadsheets before. .
454. A: O.K.
455. B: We're stating where. . if cell 1 is, let's say, is a city in the province of Quebec, which is coded 2, . .
456. A: Yeah.
457. B: . .then go and get rate of 17% tax in cell 5, and times it by the amount in B and put my answer in C.
458. A: O.K.
459. B: That's my major problem. I can actually sit down and work on a spreadsheet and work it out and this is basically what I'm coming down to what I'm doing. .
460. A: Kind of like. .
461. B: . . but I'm missing. . that's what I was telling you previously, when it got down to too complicated, going back and forth, and different percentage values, I had to get my brother-in-law to write the formulas out for me, cause I didn't have the math background to back up what I was doing on the computer.
462. A: So you can see that this is a similar application. .
463. B: Right

464. A: . . to what you were doing. There's applications of functions all the time. O.K. Question 5 up there.
465. B: What is *f(Fredericton)*? *(reading the question)*
466. A: What is that supposed to be? What does that mean?
467. B: It's going to be N.B.
468. A: Right. So you understood what I meant by that?
469. B: Yes.
470. A: No problem! O.K. Now we come to, "Do you think that functions will 'work in reverse'?"
471. B: They would have to.
472. A: They would have to? O.K. Just hold that thought for a while. What would be the "reverse rule" then for the function shown in figure 3? The cities and the provinces. .
473. B: . . ."*f*" is "is the provincial capital of". . no, sorry. If **f** is the provincial capital of . . and *f(x)* is N.B. what is *x*?
474. A: The rule. . I mean the rule for it the way it stands at the moment is "is the provincial capital of". .
475. B: Right.
476. A: Now can you give me the correct words to use if you're putting this in reverse?
477. B: If *f* equals *x*. . no. .
478. A: No. Can you give me words like the words I've used. I mean I've used these words here. .
479. B: O.K.
480. A: . . "is the provincial capital of". .
481. B: Right.
482. A: Now if you would do the whole thing in reverse, how . .what would you say? You'd have to say these words first. . *(indicating the provinces)*
483. B: O.K.
484. A: . . that. . something something. . is that? What words would you use to put it backwards?
485. B: *f* of . .
486. A: Yes, the wording . .
487. B: *f(x)* is the province. . no. . *f(x)* equals the province and *x* equals the capital. What would *x* equal if *f(x)* equals Ontario?
488. A: *(Nervously laughing!)* I was looking for something a little bit more concise but. . you know the wording of the actual rule itself but. . it's more an English rather than a mathematic. . Right. We finish off here with a bunch of different representations of arrow diagrams and the question is, Are they functions?
489. B: *(reading)* Do all of these (arrow) diagrams represent functions? So,. . .I would say, "Yes", for this one. The rest of them, "No."
490. A: O.K Why are you saying, "Yes", for number 3 there?
491. B: Because it says 4. . as I understand it. . it doesn't seem logical. Give me one value of *x* I'll have a value of *f(x)*.
492. A: Yeah. Right.

493. B: In all the other cases we have one value of x, in this case *(in reference to fig (i))* it equals the value of $f(x)$, but there's also another value of x which would equal $f(x)$.
494. A: You think that's not allowed?
495. B: It does not compute! *(laughs)*
496. A: It does not compute! Does it compute with the definition of what a function is?
497. B: Yes it does. . no. . um. . if I take it literally that if I have variable per variable I'd have to have it matched off one for one.
498. A: One for one. .
499. B: Something doesn't seem . . it doesn't seem right to me . .
500. A: Yeah. .
501. B: But that's how. . if I'm taking the definition and thinking it out, I'm saying, O.K. I need. . I require one for one. . one variable for one variable. . but something tells me that's not right. something tells me that this is fine. .
502. A: Why?
503. B: I don't know.
504. A: Because my face. . *(laughs)*
505. B: I don't know. I really don't know. It. .
506. A: I want you to re-read the definition of a function from page 1, or wherever it was, and . . .
507. B: *(re-reads definition)*. . . Now I could say it is possible that 2 different elements in x could have the same element in $f(x)$, but. .
508. A: Alright then, so does that change your opinion about some of the other figures there?
509. B: Yes, they would all. . they would all be functions, because for every value of x we have a value of $f(x)$.
510. A: Yes, well. .
511. B: But it doesn't avoid my problem of the. .
512. A: It's O.K. then for two of them here to have the same value?
513. B: I'll tell you why I'm thinking this. This is my value of x and I have my function. . and my function is a formula, right?
514. A: mm. .mm . .
515. B: If I put. . if x is, let's go with 1 and 10. . if I. . no 1's a bad example, 2, if f . . if x is 2 and I put it into the same function, it will come out to this answer, $f(x)$. .
516. A: Right. .
517. B: . . but if I put 10 into the same function, my answer can't come out to the same answer when it was 2. .
518. A: You can't think of a case where this might be true? That different values will. . if you do something to different values you would get the same output?
519. B: If it was. . if the function was an x^2 and I was using -5 and +5 it would give me 25, . .
520. A: So is that not an example of where you could have different input values?

521. B: Agreed. .
522. A: giving you the same. .
523. B: Done. . Yes.
524. A: So it could be a function then. Is that right? It's a function? *(still re-fig.(i))*
525. B: Right.
526. A: Now, is there anything different about the second figure?
527. B: Two values. .
528. A: Is that O.K.? Can you have one input. .
529. B: . . giving you two . .?
530. A: . . giving you two outputs?
531. B: No, you can't. I'm thinking . . like if I factor something down to . . um. . let's go with x + 2 , x - 2, which equals x = -2, or x = 2. . no. . never mind. . I was thinking, Yes, I could have two variables . . I can only have ONE element, O.K. so this is not a function. *(see B.'s notes)*
532. A: Now then this whole concept of a function having to have ONE output answer never really came across, . . did it, . . till now?
533. B: *(no reply!) - (looking through his notes).*
534. A: I mean it's stated there in the definition, but does the wording really mean anything to you? Until we kind of forced the issue with these arrow diagrams?
535. B: I'll agree with that. Cause I was reading it more as. . one for one. . one for one. . one for one. .
536. A: O.K.
537. B: . . although I did have the knowledge which states that if I had a '+5' and a '-5' and my function had an x^2 in it that it would come out as the same answer . . I never actually visualised what was happening . . and it does work . . it's a good visualisation.
538. A: So this would be the justification for arrow diagrams? It's showing you what's happening, you know, what's being put . . what kind of an output situation you have from your input . . and certain situations are not going to be . . are not coming under the definition of function, right?
539. B: Yes, I can . . I can see the benefit. I have a tendency not to use it because my background was in Human Resources. That's what I studied at McGill, which is reading a text, memorising your theory and going back in and applying your theory in a written text, so there was never any need for me to actually visualise what was happening. I had the theory in my head and I'd say, "O.K. as per the theory this is what's happening", and I think because now I've gone from something abstract to something more concrete, which I consider math more concrete than psychology, then I'm having a problem in the way that my actual study methods. . So, yes, this is beneficial. To me, you just showed me. . the good trick, *(laughs!)* O.K. so I guess we learn something every day.
540. A: O.K. *(the above passage was also punctuated with many O.K's from A.)*

541. B: I. . my tendency was normally just to. . yes, it's there, very nice, colours and everything, great. .
542. A: But you didn't see any benefit to the arrow diagrams? But I wanted you to eventually see that. .
543. B: I see it now. .
544. A: You will understand also as we do the latter part of the chapter that the graphical representation of a function also is very meaningful, or just the graph itself will tell you whether things are functions or not. Right?
545. B: Yeah.
546. A: O.K. We'll come to that. Right, one last little question here, Which of the above figures would remain functions when reversed?
547. B: *(has trouble seeing this one!)* . . .
548. A: Remember what we said. .
549. B: Has one. . one for one. . no. .
550. A: For one input we can only have one output.
551. B: O.K. One input can only have one output. .
552. A: Will it be your fourth one? If you're doing this in reverse.
553. B: We have one input and 4 outputs, so that will be no, agreed. This would be O.K, (#3), one for one, so this here again is going to go under the principle, we have 1 input and 2 outputs (#1), and this one's going to be the same, so it's going to be, NO for #1, No for #2. .
554. A: So to be reversible, it can only be . . it can only be reversible when they are . . what you were calling one -to- one.
555. B: One-to-one.
556. A: So later on in the chapter we will be doing functions in reverse. We talk about doing. . finding **inverse** functions. .
557. B: Inverse functions, yes.*(he had obviously heard of them before!)*
558. A: . . and we can only work on inverse functions on one-to-one functions, because they must remain functions when you do them in reverse.
559. B: O.K.
560. A: So that will help you later on in the chapter too. The functions we'd been looking at, were they. . are they one-to-one functions?
561. B: To date, yes.
562. A: The ones we were looking at tonight, the area, the radius of the circle. . is that one-to-one?
563. B: Yes it is.
564. A: The bacteria, the time. . What about the cost of mailing a letter? By the weight of the letter? Is that one-to-one?
565. B: According to Canada Post? Yes. My cost is dependent on my weight. It's also dependent on my size actually but. .
566. A: Is there never a situaion you think where you might pay the same for different weights?
567. B: No.
568. A: You could have different weights but different sizes and maybe the size would balance off the weight so you. . although you've got different weights, you might end up paying the same. . Is that possible?

569. B: Yes.
570. A: Is it? I don't know. . I'm just asking. .
571. B: If you had a large size envelope and you wanted to send it across Canada, it would probably cost you the same as if you had a small envelope and folded up and heavy paper inside.
572. A: So different weights could end up with the same cost. So it's not necessarily one-to-one then.
573. B: O.K.
574. A: So, is there any last comment you want to make about all of this?
575. B: Yes, I do have a comment about this here text. .
576. A: About the paper. .what we've been working on tonight.
577. B: About this here paper tonight? No. It helped me. . I think it will help me in the future pick out the key words in reading math texts based on this formula, looking for keywords and saying, this is with this, relying more on diagrams which is something I never. . even though. . it's something you're always taught, look at your diagram, even with my brother-in-law, sorry I used him a lot, he has CAL1, CAL2, CAL3, so any time I asked him a question about my math the first thing he did was this, *(indicating drawing a diagram?)*
578. A: That's right. *(laughing!)*
579. B: I said, "No". I don't want to see that. I want to do it mathematically. . work it out, and he would do it, look at it and say, O.K. .this . . this. . .... and he would work with the tool(?), and I would say, "Look, you're confusing me. Get it away from me", and . . . The type of courses that I was used to and working full time. .
580. A: Right.
581. B: . . I could stay home and learn the theory myself and walk into an exam and feel comfortable walking out with about an 80.
582. A: mm . .mm
583. B: Now with math courses, what I'm finding is that you can't rely on your text alone. It has to be accompanied with teaching in the classroom and practice . . because one thing I noticed different about other texts and math texts is, . . I'm looking at Chapter 4, they give you as many examples as possible . . without, pardon the expression, but, "boring you to death" . . but then when you go to do your exercises, which are required as part of the plan, it's like you're required to go from a small step, and pull out the theory of this and apply it . . Now if you haven't seen anybody pulling the theory and applying it to more complex problems . . you lose your train of thought and you'll have a hard job trying to work through it . . You can do the simple problems when you have an example. O.K. They did this, they did that. But they do get into more complex problems and I can't see that the theory is there . . but I get lost in transfering over . . and the answers in the back of the book just give me the answer . . You can attempt to work back . . which is what I try to do when I give up, but working back is not as beneficial as seeing yourself work ahead . . But I personally don't think there's a solution

because if they attempted to explain in your text the most complex problem, students would just lose interest, they just wouldn't read it. It would be too dry, too technical. It has to be alive and in colour. .

(note: during the above comments from B., and in the ones which follow, the . . indicates an "mm . ..mm", or an "O.K", . . etc from A.)

584. A: O.K. And you find that this textbook is good?
585. B: This textbook?
586. A: mm . . mm. You find it explains things well enough for you to make some attempt on the exercises?
587. B: Oh, yes. I can use it. .
588. A: Do you find it alive and in colour enough?
589. B: It is alive and in colour. They have a lot of graphics. Major points they have in boxes which I like. ..... If there was some way that the text could give you the definition and then kind of draw back and put it in layman's terms, as opposed to like technical jargon, which is not really **technical** jargon, because this is a language. . but unfortunately mathematics is not my language. It's like reading a French text when your French is not that good . . You can understand what they're trying to say, then somebody says, "O.K. Write me 10 pages on it", and you're going to say, "Yeah, right!" So, I don't know if it can be done. .
590. A: So reading math to you is like reading a foreign language?
591. B: It's like learning another language . . and I'm saying, if it could be explained in my terms . . but then again my background is not somebody else's background so the terminology I'm wishing to see it in, I could only get from, if I had a teacher that had the exact same background as myself . . minus the math, that could explain everything back to me and say, "O.K. This is what you can relate to, therefore . . ", and then I could probably work from there. . But not having that, it's not . . I'm not living in a real world when I'm saying this, because you have a class of 30 people, everybody has different backgrounds, you have to use one common language, and they have to, of course, revert to the language of math which is what we're there to learn anyway. And that was basically it!
592. A: O.K. You find that the language here is not geared for the layman? Do you find he's too technical?
593. B: The technical is required . . I like to read it . .
594. A: You think it should be there?
595. B: When they do examples, it would be nice if they referred examples to something more concrete, something more . . real, I guess, if it's possible . . I don't know if you could . .
596. A: I want to show you something here briefly. This is from a school textbook, and this is how at the school level they introduce functions (Grade 10, A Search for Meaning, Gage (Pub)) I'd like you to comment on maybe just the wording that they're using here as opposed to the wording used in your College book.

597. B: *(reads page 12)*
598. A: The language . .
599. B: It's like a High School textbook.
600. A: Why are you saying that? It is a High School textbook, but why are you saying that?
601. B: Because. . I guess because I'm in Continuing Education, after the fact. This is much easier to understand than our textbook. .
602. A: Yeah. .
603. B: . . but um. .
604. A: You feel it's not going far enough?
605. B: It's. . no I feel at this level we require more the technical background and not just the basic. . here let me take your hand and walk you through it . . sort of description. .
606. A: You feel that's what it's doing?
607. B: Yeah. I think it is practical that somebody that has been more far. . that has been out in the work force as I have been. . you agree. . I'm reading High School text . . *(sounds of disgust!)* I'm not paying $200 plus $75 for books to read a High School text. I can go back to High School and get that.
608. A: You see a difference right away between the wording. .
609. B: Yes. Definitely, yes.
610. A: O.K. Would you still agree here where they're actually starting the whole chapter. . this whole chapter is devoted to functions. Again just skim read. .
611. B: This is what I'm looking for. . It's relating it to parking meters and coins. I know about parking meters and coins. .
612. A: Don't we all?!
613. B: The difference . . what would help me personally understand more what's happening, relating it to something that I say, "Ah! Parking meters and coins, and I can think about . . visualising myself putting money in a meter. .
614. A: And do you find it suitable for a **High School** text? You know, talking about parking meters, do you think this would be suitable for **High School** students?
615. B: If they were living in the city, yes. They're probably exposed to that in High School.
616. A: But not if they're living down on the Gaspé!!
617. B: Not relevant!!
618. A: Do you think that would be meaningful. . ?
619. B: No, not living on the Gaspé coast. Parking meters? What, you put money in to park? We can park for free!! It would be O.K. for somebody in the city.
620. A: Yeah, but textbooks are, you know, geared for the city. .
621. B: . . geared for the city.
622. A: But the wording? You didn't, sort of, you know, baulk at this as being .....?

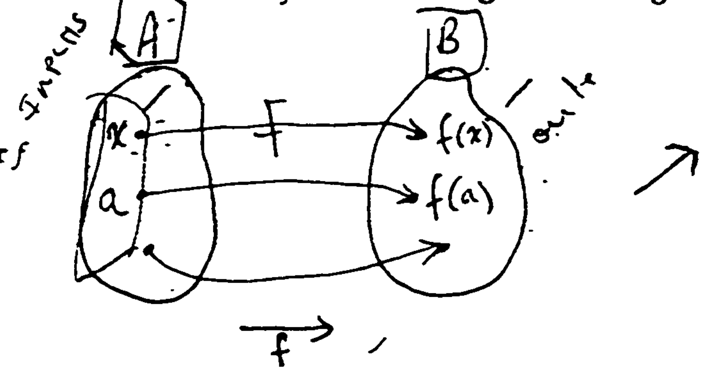
623. B: No. This I enjoyed. This was more pleasurable than reading. . while before, I found it kind of insulting. This here is like saying. . even though it is simple, it does relate back to the real world which is what I want to see. . which is what I need. . to be my visual of what's happening.
624. A: *(muffled comments about the input /output diagram)* One little word here. I'm surprised that you didn't say you liked this in some ways, because it's showing you the graph *(in reference to page 12)* and you said you liked visualising functions of graphs, and they're doing this here, and in a way I'm surprised that you didn't pick up on that.
625. B: I like visualising it **myself**. I don't like it when people give it to me.
626. A: O.K.
627. B: It's like. . I have to see it, but if somebody's showing me something. . to get back to my brother-in-law again. . he would go to the graph. . I don't want to see that..
628. A: You'd rather do it yourself?
629. B: I want to visualise it myself and go from there, which. .
630. A: . . is difficult. .
631. B: That's one of my problems was visualising myself. This session this evening was good because I'll try to pay more attention to the graphics. . I'll try to correlate the graphics to the text.
632. A: We tend to teach by giving you a visual representation of something cause we think it helps and, you know, we're often wrong in that, because what's helpful to one person visually is not necessarily helpful to somebody else.
633. B: Some people aren't visual. Some people are more theory minded. . to try and adapt one way of thought to . .
634. A: The whole idea of graphs is that graphs are supposed to make functions more meaningful because you can see a picture of it and this might not necessarily always be the case for some people.
635. B: Based on our session this evening, I'll get more value out of my graphs from hereon in, as long as I can make. .
636. A: . . make your own graphs. .
637. B: . . remember. . Oh yes, this is how it works. . And I'm saying ...teachers are always saying, "Use graphs, visualise, set yourself up. ." and for some reason in the classroom I said to myself, "I don't want to see it". Give me an A, B, C, D and then let me look at it. I don't know how you can get around that. It's just. . doesn't. . it's not. . it doesn't necessarily have to be a general problem. It's just a problem with the way I learn.
638. A: It's a problem you're going to have to overcome!! *(both laugh!)*

END OF SESSION

Part 5:
Final paragraph:

Another way to picture a function is by an **arrow diagram** as in figure 2.

figure 2
Arrow diagram for *f*



Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

Questions:

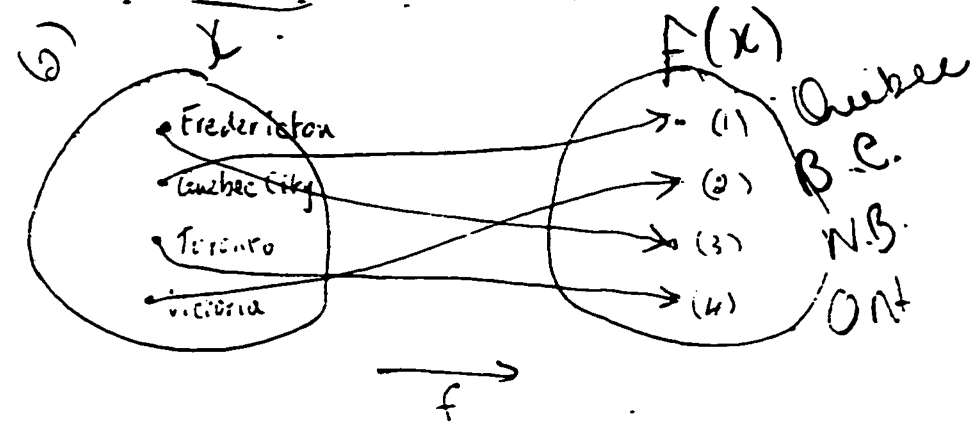
1. Does this figure help you understand the meaning of function any better? Why, or why not? NO →

2. How does this model of a function compare with the machine model?

3. Can you identify the input and the output?

4. If "*f* " is "is the provincial capital of ", what are (1), (2), (3), and (4)?

figure 3

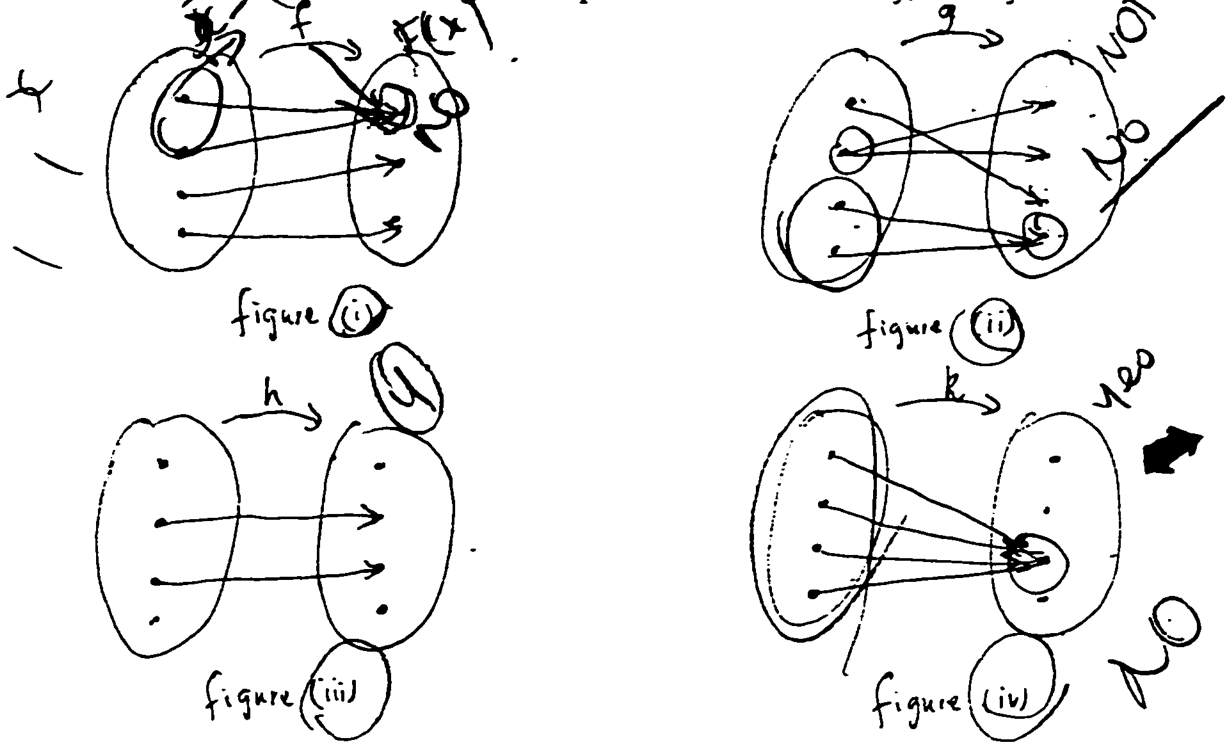


5. What is f(Fredericton)? NB

6. a) Do you think that functions will "work in reverse"?

b) What would be the "reverse" rule for the function shown in figure 3?

7. Do all of these arrow diagrams represent functions? Why, or why not?



8. Which of the above figures would remain functions when reversed? Why?

PETER - PART 5:

304. A: Last part.
305. P: O.K.
306. A: The final paragraph shows another way of representing a function using an arrow diagram.
307. P: Mm..mm
308. A: O.K.
309. P: *(reads)*. . . I see what it. . this is the domain. . function of *x*. . So that if this value were *a*, then it will be a function of *a*. This is *b*, then this is going to be a function of *b*.
310. A: O.K. Does the arrow diagram stress a different aspect of the idea of function than the machine diagram does?
311. P: Um. . being done here (*indicates inside the machine*). . whereas this one shows the work being done here (*outside the sets*) . . we're transporting something from this domain to this domain and doing something with it here. Here the result is what has happened by this in. . let me try and say this in a better way. .
312. A: O.K.
313. P: This one here tells me, or shows me a little more graphically that the function is something that is happening here. We go back to the word the 'rule'. So this is the rule. O.K?
314. A: O.K
315. P: . . this is going to determine what this is. .
316. A: Mm. . mm
317. P: I don't see that. I'm seeing that, yes, for any given value in here there is going to be a different value here, but I don't see the work that's being done. .
318. A: You said something about the domain of the numbers here and the domain of the numbers there. .
319. P: The RANGE of the numbers there
320. A: A slip of the tongue?
321. P: Yes. O.K.
322. A: O.K. Because, you know, we. . the word domain itself is used out of the mathematical context. .
323. P: Yes.
324. A: . . to mean, you know, . .
325. P: Well, no. .
326. A: . . a region of numbers here and a region of. . and I wondered if that was the way you meant it.
327. P: No. I remembered that it was a domain and a range.
328. A: O.K.
329. P: Excuse me, these are terms that I'm just learning this evening so. .
330. A: No. . It could have been a genuine slip. I just wanted to make sure whether it was or not.
331. P: No, it was a genuine slip.

332. A: O.K. So I asked you number 2 - How does this compare with the machine model? Can you identify the input and the output?
333. P: In this one?
334. A: Yeah.
335. P: Well. . again the input is this because it's flowing that way so the input's got to be here.
336. A: Right. O.K. Next one. If 'f' is 'is the provincial capital of'. . that's my function, what are (1), (2), (3), and (4)?
337. P: To you want me to tell you what they are?
338. A: Yeah. Why not?
339. P: New Brunswick is (1). Quebec City is (3) . .Sorry, no, Quebec City is (1), so Quebec city would be Quebec, Fredericton would be New Brunswick (3), Toronto is number 4, Ontario, and Victoria is B. C. number 2.
340. A: It's O.K. to have a question like this, non-mathematical, talking about functions?
341. P: I don't really see where any of these is a function. It doesn't relate to my understanding of what a function is.
342. A: O.K. Why?
343. P: Well I'm just saying this number. . well O.K. Yes, you're right, this number is a function of. . sorry. . this number is a function of the name over here, this Province that I'm going to write down here depends on what I'm being given on this side. .
344. A: O.K. So the rule doesn't necessarily have to be a formula?
345. P: No. You're right. You're right.
346. A: O.K.
347. P: Yes. You're right.
348. A: Right. Question 5, what is f(Fredericton)?
349. P: A function of Fredericton. .
350. A: Mm. .mm. Of what?
351. P: Of the number 3. Of the Province of N.B.
352. A: O.K. . .
353. P: . . . .So. . if we were to say then that the $f(x)$ in this particular instance would be that the capital of x, is the Province x.
354. A: Right. Now we come to something different. Do you think that functions will work in reverse?
355. P: Yes.
356. A: O.K. What would be the reverse rule for the function shown in figure 3, the one we were just talking about?
357. P: This one here? O.K. Well the function in this case, it would be f of this side here . . would be. .
358. A: Can you put it in terms of. . you know, using the words, 'Province' and 'capital'. .
359. P: . . the name of this Province will be a function of the capital city of the. . for instance, number 3, the function of the Province. . the Province is a function of the capital city. N.B. is going to be the capital. .

360. A: O.K. Could you try and word it like the way I've done here. I called it '*f*'. '*f* was the function the first time. '*f* is. .
361. P: O.K.
362. A: '*f*' is 'is the provincial capital of' . . Now let's say I use a different letter to represent my function. . I don't always have to use x either. .
363. P: O.K. '*f* is . .
364. A: If we use '*g*', what would '*g*' be?
365. P: '*g*' is 'is the Province of which the provincial capital is'. .
366. A: You got it!!. . . Now question 7, Do all of these arrow diagrams represent functions?
367. P: No. Because there's only . . I mean, sorry, maybe I'm wrong, to me. .
368. A: Can you say why?
369. P: In any function, if one thing is a function of the other and if you change the other then you've changed the function.
370. A: Is that what the definition was saying?
371. P: Yes. The definition at the beginning said that it's a rule and that there is exactly one element. . and here I see 2 elements. .
372. A: Where are you pointing to?
373. P: Sorry, right there, 2 elements. It says, each element x in a set A, right here, exactly one element called $f(x)$ in the set B. O.K. Now if I reverse that I've got to say that. .
374. A: We're not talking about the reverse at the moment. .
375. P: Well, O.K.. . . Well. . If this is. . If we're talking about the same rule here. . the same relationship. . the same. . if we're talking about the same relationship- I use the word 'relationship' - then different values here cannot give the same answer there.(*figure i*) . I don't think.
376. A: You don't think so? And you're saying that from reading that definition and from what we've been saying so far?
377. P: Yes. Let me give it some thought here. . I'm trying to think HOW it could be . . Well, O.K. Sorry, I'm wrong.
378. A: Why are you saying you're wrong?
379. P: I'll give an example. O.K. I'll go back to this example. .'*f* is 'is the provincial capital' . . If we had Quebec City and Montreal here and say, 'is a city in'. .
380. A: Mm. .mm
381. P: So therefore Quebec could be the right answer for both of those.
382. A: Right. . . . 'is a city in' is a function?
383. P: Yes. . would be a function. . yeah.
384. A: O.K.
385. P: But it wouldn't be a mathematical. .
386. A: Can you give me a mathematical example?
387. P: That would work for that?
388. A: That would work for that, yeah. I'll give you one hint. Think of the work you've been doing with parabolas. .
389. P: Yes. O.K. . . going on both sides, so for any value of y the expression $x^2 + 2$. .whatever . . you get 2 values of x for every value of y.

390. A: One *y*. . mm. .mm. . there's your 2 *x*-values and there's your one *y*-value.
391. P: Yes. O.K. . I guessed it and I'm wrong?
392. A: No, it's perfect. Alright what about this second one?
393. P: Do all these diagrams represent functions?
394. A: It doesn't have to be an *'f'* for function. In fact we can use any letter. So *'g'* is my function here.(*figure ii*)
395. P: O.K. Again I'm going by feeling here. It just doesn't look right. We've got 3 different relationships although this first one is not defined because there may be another value here whereby this could be a . . O.K.
396. A: mm. .
397. P: These. . we've got one value of *x* that gives you 2 values of the function of *x*.
398. A: mm . . mm
399. P: Here I've got 2 values of *x* that give me the same value of the function of *x*. *(in reference to fig. (ii)* I'm trying to think of a mathematical form. .
400. A: What's different about this diagram, from the first one?
401. P: Well, .
402. A: There's one major difference.
403. P: There's crossing over. .
404. A: That's not significant.
405. P: And then there's 2 ways that this is formulating. In this case either they were values here. Either they were singular values or they were 2 from the same singular value. Here you're looking at singular values on either side with 2 different. . with the variable on either side. .
406. A: mm . .mm. . O.K. Let's take this element here. That element is going to one element over here.
407. P: mm . . mm
408. A: That is crossing the other 2.
409. P: That's fine. mm . . mm
410. A: . . crossing is not important. This one here is going to 2 values over here.
411. P: Yes.. . . AH! Here we were going to one value, here we're going to two.
412. A: . . what are we talking about - the one value going to two. . !!
413. P: O.K. Alright. So we were saying. . because *y* is in fact a function of *x*.
414. A: mm. .mm
415. P: So here we're saying that for any given value of *y* there can be two values of *x*, but there can't be two different values of *y* to give the same value for *x*. Is that correct?
416. A: Is that part of the definition? Look at the definition again and see if the definition is saying that?
417. P: (*reads definition*) . . exactly one element. .
418. A: . . exactly one element. . they become THE most important words in that whole definition.

419. P: Yes. Yes.
420. A: . . exactly one element. .You haven't met . . giving you 2 values. .
421. P: Can this be done mathematically?
422. A: No. . yes it can, but it's not a function.
423. P: It's not a function.
424. A: It's not a function. A function has quite a special . . this is a very special relationship. . this is not allowed . . though mathematically it could be. An example would be if you take your x^2 and consider the square root. If you have $y =$ . . if you have $y^2 = x$. .
425. P: Yes.
426. A: . . and then you want y. y is going to be plus or minus. .
427. P: Yes.
428. A: . . you're going to have 2 values.
429. P: Yes. If x is a square root?
430. A: If x is a square root, you're going to get 2 different values. And then it no longer becomes a function.
431. P: O.K.
432. A: O.K. the third diagram? The thing about the third diagram was that some of the elements have been left out. Is that significant at all?
433. P: Is it significant? It depends on what the range of the set is? Is this the range of this set here? Is this the. . I mean if we're talking only about these 2 values then, yes, this makes sense. If we're talking about these two values then there should be something corresponding on the other side. It may be this one. It may be that one.
434. A: Yeah, but do we have to include every element in the set A?
435. P: If we're going to make a set, then yes, we should include every element.
436. A: O.K. . you think so. .
437. P: Well hang on a second, 'a function is a rule that assigns to EACH element x exactly one element', so we're not doing that.
438. A: And what about the last one?
439. P: It works. Exactly one element.
440. A: O.K. Alright then. So the other big question is, which ones of these functions can be reversed . . and still remain functions?
441. P: Which ones CAN be reversed, and still remain functions? O.K. 4, because if this is a function of that then we've got 4 values here.
442. A: Right.
443. P: This one could.
444. A: This one being number . .?
445. P: Number 3. Sorry. I'm sorry. I keep forgetting. .
446. A: You've got to talk to this guy. . *(indicating the tape recorder!)*
447. P: Figure 3 can. Figure 4 can. Figure 2 can't. . hang on . .
448. A: You said figure 4 can't. Figure 4 could not. .
449. P: No. 4 CAN. 3 can. 2 can't because 2 we're going. .

450. A: I thought you said 4 could not because one would be going to four values.
451. P: O.K. Start all over again. Figure 4 cannot. *(A: Cannot)* Figure 3 can. *(A: Can)* Figure 2 cannot.
452. A: Why not?
453. P: Because it's going to 2 values.
454. A: O.K. So therefore it's number 3, which in some way you may argue is not a function because they've left out some of the elements of A.
455. P: Yes.
456. A: But let's say we had them put them all in, O.K?
457. P: Yes.
458. A: We have a special name for a function that is reversible, and it's called one-to-one.
459. P: one-to-one.
460. A: Yeah. They are the only functions that we can do inverses for. .
461. P: . . that are reversible. .
462. A: . . and that's something that we're going to be, you know, covering, is what we call an inverse function.
463. P: O.K. . .
464. A: Inverse functions are functions which are reversible.
465. P: This would be a reversible function, wouldn't it? The area of a circle?
466. A: Yes.
467. P: They would all be. .
468. A: Would they? I don't know. The first one certainly would be.
469. P: The first certainly would be.
470. A: The bacteria? Would the bacteria one be?
471. P: Um. .
472. A: Right.
473. P: The third one would be, definitely.
474. A: I'm not so sure about the third one.
475. P: You have the cost. .
476. A: You may have a certain parcel of a certain weight. It's going to cost so much. .
477. P: Yes.
478. A: Somebody comes in with a parcel slightly heavier, would it necessarily cost more?
479. P: Yes.
480. A: O. K. Could somebody end up paying the same amount for different weights of something? Is it definitely going to cost more because somebody's coming in with something that may be 1oz more?
481. P: . . if I read this totally, and I see, "Although there's no single formula that connects w and C. . " You're right, we can't reverse it. . there's no single formula, but if there were one, if it was 25 cents per ounce, and I knew that it cost $2, then I'd know how many ounces I had.
482. A: Yeah. To me it's possible. I don't know, I haven't worked in a post office, but to me it's possible that you could end up paying the same for

something, you know, that had different weights, depending on how their scaling costs. .
483. P: O.K. If they were scaling by pounds it could be a pound and a half or a pound and a quarter . .
484. A: It could be between 1 pound and 2 pounds . . it could, you know. .
485. P: Yes. Yes.
486. A: . . you could have the same cost. .
487. P: I agree . .
488. A: . . for different weights. .
489. P: Yes, depending on what the scale. .
490. A: So you can have. .
491. P: Yes, yes. O.K.
492. A: It is possible.
493. P: In each of these examples. .
494. A: That's just summing it up. O.K. Do you have some general comments you would like to make, for the record, about. . Had you read this introduction already in the textbook?
495. P: No.
496. A: You hadn't read it?
497. P: No. I must be very honest with you. I tried going through the textbook at the beginning of the year, and it's been a lot easier just going to class, watching the examples, and learning from that and then doing the examples. I can relate to what the book is saying once it's been explained to me in different language in the class. I find it very difficult to, you know, sort of pioneer the book and go ahead and try and find. . after a while if I read it long enough it'll sink in but it doesn't make very clear sense at first.
498. A: Yeah. And yet the textbook is written for somebody to sit down and read and figure out the problems. .
499. P: Well, you know, again, I've never really been acquainted with terrific mathematics textbooks. I've always found them to be cumbersome, to be not very user friendly, you know. I don't know if I'm like everybody else. For me the easiest way to learn a mathematical concept is to have it explained to me with examples and then I'm going to be able to understand it but to read something that says. .
500. A: A function is a rule . .
501. P: It doesn't mean anything to me mathematically.
502. A: Yeah. Do you find that you go back to the text to figure out how examples are done, if you're getting stuck?
503. P: If I'm getting stuck, yes, I'll go back, but again it's after I've been. . you know, after we've done it in class. O.K. I've tried, I don't remember where, but there's one part where they have a lot of puzzles, almost sort of examples, where you have ABCD times F . . in one of the sections. . see this type of question. . I love that kind of stuff. . so I'm trying to go back here and say O.K. does this tell me how this is being done? *(see App. F, p. 101)*
504. A: No, it doesn't. It's not supposed to!

505. P: It's not supposed to. O.K. But even an example when we were doing inequalities and the chart, I read this, *(see App. F. p. 102)* and I did this before we had the class, and I thought I understood it, but I realised once you started doing it that I didn't really understand it, and what I didn't understand was that in fact this is what was the important part over here, and the relationship. .
506. A: It doesn't really come across. .
507. P: It doesn't really come across that well.
508. A: They don't tell you it's a positive . . greater than zero . .
509. P: I'm looking at it and saying, "Well that's very nice but what does all that mean?" I can make the diagram but I still don't know which one to pick.
510. A: O.K.
511. P: Um. . The 200 book was better than this.
512. A: Yes.
513. P: But the reason it was better is. . it explained examples a little more clearer and it had a nice little recap . . it was great for review, in other words.
514. A: Yes.

END OF SESSION.

APPENDIX

F

APPENDIX F

One of the most basic and important ideas in all of mathematics is that of a *function*. In this chapter we study functions, their graphs, and some of their applications.

SECTION 4.1 FUNCTIONS

The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r.

The number N of bacteria in a culture depends on the time t. If the culture starts with 5000 bacteria and the population doubles every hour, then after t hours the number of bacteria will be $N = (5000)2^t$. This is the rule that connects t and N. For each value of t there is a corresponding value of N, and we say that N is a function of t.

The cost C of mailing a first-class letter depends on the weight w of the letter. Although there is no single neat formula that connects w and C, the post office has a rule for determining C when w is known.

In each of these examples there is a rule whereby, given a number (r, t, or w), another number (A, N, or C) is assigned. In each case we say that the second number is a function of the first number.

> A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the **domain** of the function. The symbol $f(x)$ is read "f of x" or "f at x" and is called the **value of f at x**, or the **image of x under f**. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain, that is, $\{f(x) \mid x \in A\}$.

The symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**. The symbol that represents a number in the *range* of f is called a **dependent variable**. For instance, in the bacteria example, t is the independent variable and N is the dependent variable.

Figure 1

It is helpful to think of a function as a **machine** (see Figure 1). If x is in the domain of the function f, then when x enters the machine, it is accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.

The preprogrammed functions in a calculator are good examples of a function as a machine. For example, the $\sqrt{x}$ key on your calculator is such a function. First you input x into the display. Then you press the key labeled $\sqrt{x}$. If $x < 0$, then x is not in the domain of this function; that is, x is not an acceptable input and the

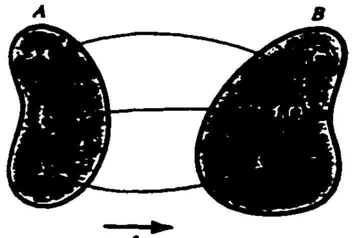


Figure 2
Arrow diagram for f

calculator will indicate an error. If $x \geq 0$, then an approximation to $\sqrt{x}$ will appear in the display, correct to a certain number of decimal places. [Thus the $\sqrt{x}$ key on your calculator is not quite the same as the exact mathematical function f defined by $f(x) = \sqrt{x}$.]

Another way to picture a function is by an **arrow diagram** as in Figure 2. Each arrow connects an element of A to an element of B. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.

EXAMPLE 1

The squaring function assigns to each real number x its square x^2. It is defined by the equation

$$f(x) = x^2$$

(a) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.
(b) Find the domain and range of f.
(c) Draw a machine diagram and an arrow diagram for f.

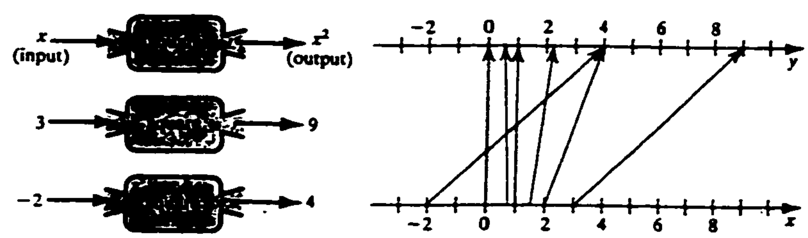
SOLUTION

(a) The values of f are found by substituting for x in the equation $f(x) = x^2$:

$$f(3) = 3^2 = 9 \qquad f(-2) = (-2)^2 = 4 \qquad f(\sqrt{5}) = (\sqrt{5})^2 = 5$$

(b) The domain of f is the set R of all real numbers. The range of f consists of all values of $f(x)$, that is, all numbers of the form x^2. But $x^2 \geq 0$ for all real numbers x, and any nonnegative number c is a square since $c = (\sqrt{c})^2 = f(\sqrt{c})$. Therefore the range of f is $\{y \mid y \geq 0\} = [0, \infty)$.
(c) Machine and arrow diagrams for this function are shown in Figure 3.

Figure 3



(a) Machine diagram (b) Arrow diagram ■

Note We previously have used letters to stand for numbers. Here we do something quite different. We use letters to represent *rules*. For instance, in Example 1 the name of the function is f and the rule is "square the number."

EXAMPLE 2

If we define a function g by

$$g(x) = x^2 \qquad 0 \leq x \leq 3$$

then the domain of g is given as the closed interval $[0, 3]$. This is different from the function f in Example 1 because in considering g we are restricting our attention to those values of x between 0 and 3. The range of g is

$$\{x^2 \mid 0 \leq x \leq 3\} = \{y \mid 0 \leq y \leq 9\} = [0, 9]$$ ■

EXAMPLE 3

If $f(x) = 2x^2 + 3x - 1$, evaluate the following:

(a) $f(a)$ (b) $f(-a)$ (c) $f(a + h)$ (d) $\dfrac{f(a + h) - f(a)}{h}$, $h \neq 0$

[Expressions like the one in part (d) occur frequently in calculus where they are called *difference quotients*.]

SOLUTION

(a) $$f(a) = 2a^2 + 3a - 1$$

(b) $$f(-a) = 2(-a)^2 + 3(-a) - 1 = 2a^2 - 3a - 1$$

(c) $$\begin{aligned} f(a + h) &= 2(a + h)^2 + 3(a + h) - 1 \\ &= 2(a^2 + 2ah + h^2) + 3(a + h) - 1 \\ &= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1 \end{aligned}$$

(d) $$\begin{aligned} \frac{f(a + h) - f(a)}{h} &= \frac{(2a^2 + 4ah + 2h^2 + 3a + 3h - 1) - (2a^2 + 3a - 1)}{h} \\ &= \frac{4ah + 2h^2 + 3h}{h} = 4a + 2h + 3 \end{aligned}$$ ■

In Examples 1 and 2 the domain of the function was given explicitly. But if a function is given by a formula and the domain is not stated explicitly, the convention is that **the domain is the set of all real numbers for which the formula makes sense and defines a real number.**

We should distinguish between a function f and the number $f(x)$, which is the value of f at x. Nonetheless, it is common to abbreviate an expression such as

the function f defined by $f(x) = x^2 + x$

to

the function $f(x) = x^2 + x$

In Problems 5 and 6 solve the equation.

5. $|2x - 1| - |x + 5| = 3$

6. $|x - 1| + |x - 2| + |x - 3| = 1$

7. Each letter in the following multiplication represents a different digit. Find the value of each letter.

$$\begin{array}{r} \text{ABCDE} \\ 4 \\ \hline \text{EDCBA} \end{array}$$

8. Each letter in the following subtraction represents a different digit. Find the value of each letter.

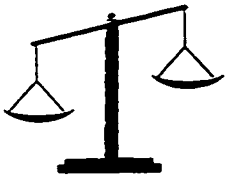
$$\begin{array}{r} \text{NINE} \\ -\ \text{FOUR} \\ \hline \text{FIVE} \end{array}$$

9. Find every positive integer that gives a perfect square if 132 is added to it and another perfect square if 200 is added to it.

10. A farmer has a fox, a goose, and a sack of grain. He wishes to cross a river in a rowboat that can hold only himself and one of these three. The farmer's problem is that foxes eat geese and geese eat grain. How can he safely ferry all three of these possessions to the other side?

11. Solve the following problem, which was first posed in the nineteenth century: "Every day at noon a ship leaves New York for Le Havre and another ship leaves Le Havre for New York. The trip takes seven full days. How many New York-to-Le Havre ships will the ship leaving Le Havre today meet during its trip to New York?"

12. In how many ways can 8 coins be divided into 4 groups so that there is at least one coin in each group?



13. One of nine eggs is lighter than the rest. The other eight all have exactly the same weight. How can you determine which is the lighter egg with exactly two weighings on a balance?

14. Among 12 similar coins there is one counterfeit. It is not known whether the counterfeit coin is lighter or heavier than a genuine coin. Using a balance three times, how can the counterfeit be identified and in the process determined to be lighter or heavier than a genuine coin?

EXAMPLE 2

Solve the inequality $x^2 + 3x > 4$.

SOLUTION 1

First we take all nonzero terms to one side of the inequality sign and factor the resulting expression.

$$x^2 + 3x - 4 > 0 \qquad \text{or} \qquad (x - 1)(x + 4) > 0$$

As in Example 1, we solve the corresponding equation $(x - 1)(x + 4) = 0$ and use the solutions $x = 1$ and $x = -4$ to divide the real line into the three intervals $(-\infty, -4)$, $(-4, 1)$, and $(1, \infty)$. On each interval the product keeps a constant sign as shown in the following chart. The signs in the chart could be obtained by using, for example, the test values -5, 0, and 2, respectively, for the three intervals.

Interval	$x - 1$	$x + 4$	$(x - 1)(x + 4)$
$x < -4$	$-$	$-$	$+$
$-4 < x < 1$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$

We read from the chart that the solution set is

$$\{x \mid x < -4 \text{ or } x > 1\} = (-\infty, -4) \cup (1, \infty)$$

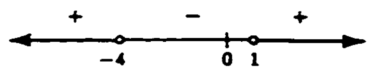


Figure 3

The solution is illustrated in Figure 3.

SOLUTION 2

As in the first solution we write the inequality in the form

$$(x - 1)(x + 4) > 0$$

Then we use the fact that the product of two numbers is positive when both factors are positive or when both factors are negative. We examine each of these two cases separately.

Case I When $x - 1 > 0$ and $x + 4 > 0$, we have $x > 1$ and $x > -4$. But if $x > 1$, then $x > -4$ holds automatically. So the solution in this case is given by $x > 1$.

Case II When $x - 1 < 0$ and $x + 4 < 0$, we have $x < 1$ and $x < -4$. But if $x < -4$, then $x < 1$ holds automatically. So the solution in this case is given by $x < -4$.

Combining both cases, we see, as before, that the solution set is

$$\{x \mid x < -4 \text{ or } x > 1\} = (-\infty, -4) \cup (1, \infty)$$ ■

In the remaining examples in this section we will use the method of the first solution in the preceding example. In the next example, we solve an inequality involving a quotient instead of a product.

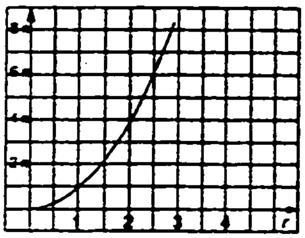
APPENDIX

G

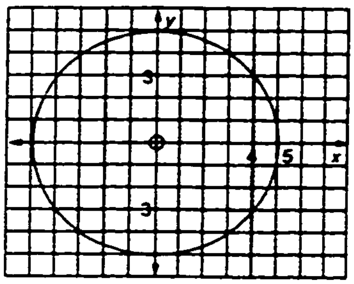
Introduction to Functions

The area A of a circle of radius r is given by the equation $A = \pi r^2$. If you choose different values for r and calculate the corresponding values of A, you can draw a graph to show this relation.

radius r (cm)	0	1	2	3	4
area, A(cm²)	0	π	4π	9π	16π



The following graph is the graph of a circle of radius 5. This graph can be represented by the equation $x^2 + y^2 = 5$.



If $x = 4$, what values can y be?

If you examine the graph, you will see that y can be either +3 or −3. For $x = 4$, y can have two different values.

Also, if $-5 < x < 5$, x will always have two corresponding values of y.

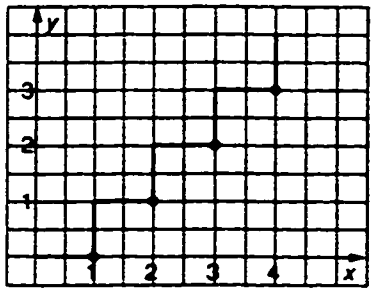
In mathematics a distinction is made for those relations whose equations have only one corresponding value of x. These relations are called **functions**.

If you are given a specific graph, there is an easy way to determine whether the graph represents a function.

If it is possible to draw a vertical line parallel to the y-axis, such that the line passes through the given graph in ~~more than~~ only *one* place, then the graph represents a function.

Example 1

Determine whether or not the following graph represents a function.



Solution

At $x = 2$ it is possible to draw a vertical line parallel to the y-axis such that it passes through the y values $1 < y \leq 2$. Since y has more than one value for $x = 2$, the graph does not represent a function.

Example 2

The table below represents a relation between the variables x and y. Determine if the relation is a function.

x	0	2	4	6	8	10
y	5	5	7	7	9	9

Solution

Since each value of x has one and only one corresponding value of y, the relation is a function.

The Function Concept

When you think of money, you usually think only of its value. Many people, however, are equally concerned with other aspects of money. A mint employee is interested in the amount of each metal needed to make coins. An architect designing a bank must know the mass of coins when designing storage for them. A vending-machine operator needs to consider both the mass and the volume of coins when purchasing bags or containers for emptying the machines. These properties of coins—amount of each metal, mass, volume, and value—depend on the number of coins being considered. You can describe the way these properties depend on the number of coins by using an important mathematical concept.

To illustrate this concept, consider a parking meter as a source of revenue for a municipality. If the meter takes only quarters, and can hold up to 200 coins, there are two quantities to consider: the number of quarters in a meter, represented by the letter q, and the monetary value of those quarters, represented by the letter v.

The parking-meter attendant is concerned with q, and wants to empty each meter before q exceeds 200. The municipality is interested in v, the amount, in dollars, of income collected each time the meter is emptied. The letter q is called a **variable** because it can represent the different whole number values from 0 to 200, that is, it can vary from 0 to 200. The letter v is also a variable because it can vary from 0 to 50 in multiples of 0.25. The variable q is called the **independent variable**. You need to know the value of q before you can determine v. Since the value of v depends on the quantity (q) of coins in the meter, the variable v is called the **dependent variable**.

The monetary value v is a **function** of the number of quarters q. When you say that v is a function of q, you are saying that v and q are somehow related; the value of v depends on the value of q; and there is only one value of v for each value of q.

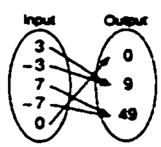
student math notes

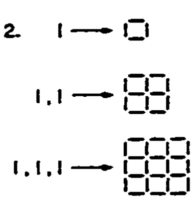
Familiar Functions

Mathematicians have always been interested in relationships between sets of numbers. These relationships have been studied for thousands of years. Some of these relationships are too complex for nonmathematicians to understand. However, many other relationships are interesting and simple enough for us to study.

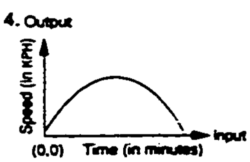
In mathematics class, Mrs. Perez introduced a new term, *function*. She said, "Some of the examples shown are special types of relations that can be called *functions*." Then she added, "To understand functions, you also need to see examples of relations that are *not* functions. Some of the examples shown are *not* functions."

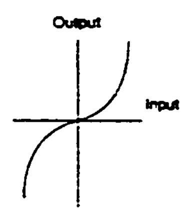
Functions

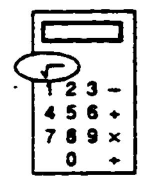
1.


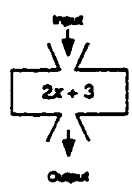
2.


3. $\{(1, 2), (7, 8), (16, 17), (-3, -2)\}$

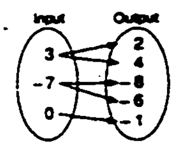
4.


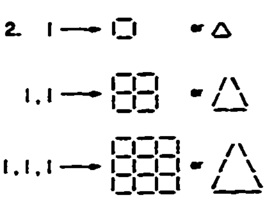
5.


6.


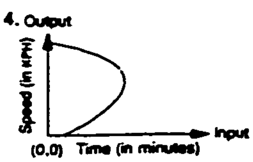
7.


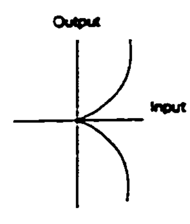
Not Functions

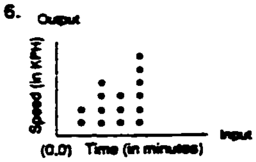
1.


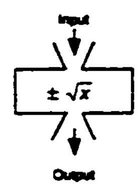
2.


3. $\{(1, 2), (1, 8), (1, 17), (1, -2)\}$

4.


5.


6.


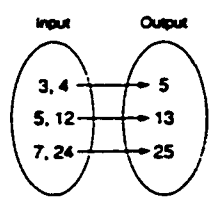
7.


What do you think is the difference between relations that are functions and those that are not? ______________________________

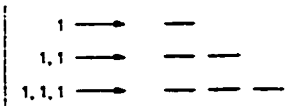
If you have not learned the definition of a function, the statement you made is a *conjecture*. Your conjecture states what you think is the definition of a function on the basis of the examples you were given. Check to see that your *conjecture* works to explain *all* the examples given.

1. Guy and Oma are trying to figure out which of the following examples are functions. Use your conjecture to help them decide by circling yes or no for each example.

a) Yes No



b) Yes No



c) Yes No

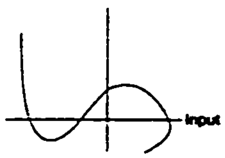
$\{(2, 4), (4, 2), (12, 13), (13, 12)\}$

d) Yes No

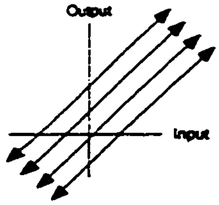
$\{(-1, 2), (1, 2), (-3, 8), (-3, 7)\}$

The editors wish to thank Ron Vavrinek and Susan K. Eddins, Illinois Mathematics and Science Academy, Aurora, IL 60506-1039, for writing this issue of the *NCTM Student Math Notes*.

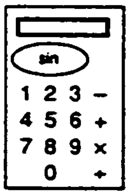
e) Yes No



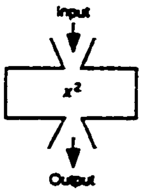
f) Yes No



g) Yes No



h) Yes No



2. Explain the reasoning you used for the examples in problem 1. ____________

Mrs. Perez told the students that to understand a function you must understand two things: (1) what it can accept as an input and (2) what *unique* output it gives for each input it can accept. She gave them this example:

Input	Output
$\frac{1}{3}$	$0.\overline{3}$
$\frac{3}{4}$	0.75
$\frac{1}{5}$	0.2
$\frac{2}{9}$	$0.\overline{2}$
$\frac{1}{11}$	$0.\overline{09}$

3. What inputs does it appear that this function can accept? ____________

4. How do you think the function gets an output from each input? ____________

5. What outputs would the function give for inputs of 1/6? ______ 4/7? ______ 1/8? ______ 9/10? ______ 1/30? ______

6. What input would the function need to give an output of 0.0625? ____________

Some functions use more than one input to compute each output.

Input	(5, 3)	(10, 4)	(6, 0)	(7, 7)	(2, 9)	$\left(\frac{5}{7}, \frac{3}{7}\right)$	(−5, 4)
Output	2	6	6	0	−7	$\frac{2}{7}$	−9

7. Describe the inputs it appears that this function can accept. ____________

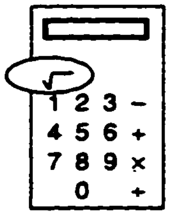
8. Describe the "rule" that you think the function uses to get an output from each input. ____________

9. Use this rule to complete the following table using these inputs.

Input	(98, 16)	$\left(\frac{5}{8}, \frac{1}{4}\right)$	(23, 23)	(3, 17)	(0.32, 0.478)	(−4, 7)	(4.
Output							

10. List three input pairs for which the function would give an output of 3. ____________

Calculators as Function Machines



11. Give the calculator's output when the √‾ function key is applied to each of these inputs. 25 ______, 45796 ______, 318 ______, 0.25 ______, 0.37 ______, 0 ______, −16 ______, −523 ______, −0.861 ______

12. For what type of input was the calculator not able to give a numerical output? ____________

13. Complete the table for the function x^y. (A scientific calculator may be needed for some entries.) (Using parentheses around negative numbers is a good idea with calculators.)

Input x	3	2	1	0	4	0	2	−4	36	32	−36	−32
Input y	2	3	17	4	0	−4	−3	0	$\frac{1}{2}$ or 0.5	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{5}$
Output x^y		8		0	1							

14. For what type of input pairs was the calculator *not* able to give a numerical output? ____________

15. If you wanted to test your conjecture from problem 14, what pairs of numbers would you input? ____________

16. For the same x^{y} function, complete the following table. (Multiple answers may be possible.)

Input x	2				1	1
Input y		3	2			
Output x^{y}	16	27	9	64	1	3

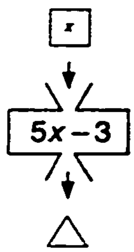
-7)	(−4, −7)	(−7, −4)

Function Machines

Functions can be defined through using function machines like those in problem 17. The input for machine 1 is whatever is in the square; for machine 2 it is whatever is in the hexagon. The output from machine 1 appears in the triangle; for machine 2, the output appears in the circle.

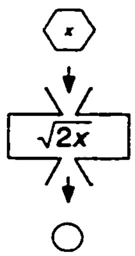
17. Complete the tables.

Machine 1



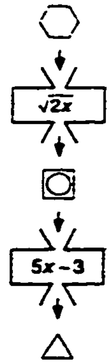
Machine 1	
Square	Triangle
−5	
	0
10	
0	
	−7
12	

Machine 2



Machine 2	
Hexagon	Circle
8	
	5
	0
−7	
	6
	−2

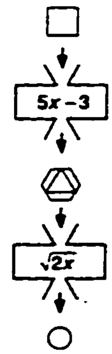
Linking Function Machines



Machine 1 and machine 2 have been linked so that the output of machine 2 becomes the input for machine 1.

18. Complete this table.

Hexagon	Machine 2 output = machine 1 input	Triangle
8		
5		
0		
−7		
18		



Next, machine 1 and machine 2 are hooked together in a different way. This time the output of machine 1 becomes the input for machine 2.

19. Complete this table.

Square	Machine 1 output = machine 2 input	Circle
−5		
0.6		
10		
0		
−0.8		
12		

20. Using the numbers 3, 0, 0.25, 2, and 0.75 as inputs, what would be the outputs for the following arrangements of machines?

	Inputs				
	3	0	0.25	2	0.75
Machine 1 → machine 1 → machine 1					
Machine 2 → machine 2 → machine 2					

21. What would happen if infinitely many machines of the same type were hooked together and the inputs from problem 20 were used?

	Inputs				
	3	0	0.25	2	0.75
Machine 1 → machine 1 → machine 1 →					
Machine 2 → machine 2 → machine 2 →					

Can you...

- identify the inputs that will lead to convergence in machine 2?
- list or describe all possible inputs and outputs for each of Mrs. Perez's functions?
- identify the function relating x, y, and z in the following set of ordered triples: $\{(1, 3, 2), (2, 4, 3), (3, 9, 6), (11, 17, 14), (40, 50, 45)\}$?
- list the first five outputs for the recursive function defined $f_1 = 1$, $f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$?
- name the function defined above?

Did you know that...

- $f(x)$ is often used to denote the output of a function where x is the input?
- functions that accept more than one input at a time are called *multivariable* functions?
- a *one-to-one* function has each output generated by a *unique* input?
- the *inverse* of a one-to-one function might be considered analogous to "running a function machine backward," in that it "returns" each output value for the input from which it originally came?
- the functions in problems 20 and 21 are called *iterative* functions, and this type of function often occurs in computer programming?
- the intermediate outputs of the functions in problem 20 are terms in a sequence?
- calculators and computers are function machines?

Mathematical Content

Functional relations and representations

Bibliography

Mathematical Investigations, a course written by the mathematics faculty of the Illinois Mathematics and Science Academy, Aurora, IL 60506-1039

Answers

1. (*a*) yes, (*b*) yes, (*c*) yes, (*d*) no, (*e*) no, (*f*) no, (*g*) yes, (*h*) yes
2. Answers will vary.
3. All inputs are positive fractions less than 1.
4. The outputs are decimal representations of the fractions, or divide the denominator into the numerator.
5. $0.1\overline{6}$, $0.\overline{571428}$, 0.125, 0.9, $0.0\overline{3}$
6. 1/16
7. Answers may vary. One possibility: The function accepts ordered pairs.
8. Answers may vary. One possibility: The output is the difference between the first and second number in the ordered pair.
9. 82, 3/8, 0, −14, −0.158, −11, 11, 3, −3
10. Answers will vary. Three possibilities are (3, 0), (4, 1), (0, −3).
11. 5, 214, 17.832554, 0.5, 0.6082762, 0, error, error, error
12. The calculator could not accept a negative number.
13. Output x^y: 9, 8, 1, 0 .1, error, 0.125, 1, 6, 2, error, −2
14. Answers may vary depending on calculator used. One possibility: Everything except $x < 0$ and $y = 1/2$. It should be noted that for some calculators any negative x input will cause an error message even though a solution exists. Some calculators can handle a negative x input provided that the y input is *not* a fraction whose denominator is even.
15. Answers will vary. One possibility: $x < 0$ and $y = 1/4$.
16. 4; 3; 3; 4^3, 2^6, 8^2; any real number; error
17. Machine 1: −28, 3/5, 47, −3, −4/5, 57; machine 2: 4, 12.5, 0, error, 18, nothing will work
18. 4, 17; ~ 3.16, ~ 12.8; 0, −3; error, error; 6, 27
19. −28, error; 0, 0; 47, ~ 9.695; −3, error; −7, error; 57, ~ 10.677
20. For machine 1 → 1 → 1: 282, −93, −61.75, 157, 0.75; for machine 2 → 2 → 2: 2.104, 0, ~ 1.542, 2, ~ 1.769
21. For machine 1 → 1 → 1 → : infinity, −infinity, −infinity, infinity, 0.75; for machine 2 → 2 → 2 → : 2, 0, 2, 2, 2

NCTM STUDENT MATH NOTES is published as a supplement to the NEWS BULLETIN by the National Council of Teachers of Mathematics, 1906 Association Drive, Reston, VA 22091−1593. The five issues a year appear in September, November, January, March, and May. Pages may be reproduced for classroom use without permission.

Printed in the U.S.A.

Difference of squares. A method used for factoring trinomials.
Dilatation. A transformation of a figure producing a larger, similar figure.
Discontinuous. Describing a line or curve that has discrete points.
Discrete. Separate and distinct.
Discriminant. For $ax^2 + bx + c = 0$, the discriminant is $b^2 - 4ac$.
Domain. The set of possible values of an independent variable.

Element. A term within a matrix.
Elimination. An algebraic method for solving systems of equations.
Entire radical. A number written entirely underneath a radical sign. The numbers $\sqrt{9a}$, $\sqrt{3}$, $\sqrt{a^2}$, $\sqrt{x}$, and $\sqrt{45}$ are entire radicals.
Equivalent rational numbers. Rational numbers having the same value.
Equivalent ratios. Ratios that can be simplified to the same ratio.

Face. A flat surface of a geometric solid.
Factor. **1)** A number, variable, term, or expression that divides exactly into another number, variable, term, or expression. **2)** To find the factors of.
FOIL. The rule stating the order in which to expand a product of two binomials.
Fraction form. A way of writing a ratio in the form of a fraction.
Function. A relation in which each element of the domain is associated with exactly one element of the range.

General form (of a linear equation). The equation of a line written in the form $ax + by + c = 0$, not both $a = 0$, $b = 0$.
General quadratic relation. A relation defined by the equation $y = ax^2 + bx + c$.
Glide reflection. A combination of a translation and a reflection in which the line of reflection is parallel to the direction of the translation.
Greatest common factor. The greatest number or expression that divides exactly into a group of numbers or terms.
Growth rate. The value g in an equation written in the form $y = gx + i$.

Heading. The direction in which an object moves.
Hypotenuse. The side opposite the right angle in a right-angled triangle.

Image. The point or figure resulting from a transformation.
Inequality. A mathematical statement relating quantities that are not equal.
Initial value. The value i in an equation written in the form $y = gx + i$.
Inverse Relation. A relation r with ordered pairs (x,y) has an inverse relation r^{-1} with ordered pairs (y, x).
Irrational number. A number that does not terminate or repeat when written in decimal form.
Isometry. Any transformation in which line segment lengths and angle measures do not change.
Isosceles triangle theorem. A theorem stating that if two sides of a triangle are equal, the angles opposite those sides are equal.

11

QUADRATIC EQUATIONS

In Chapter 7 we introduced quadratic equations and learned one method for solving them. Recall that a quadratic equation is one that can be written in the standard form

$$ax^2 + bx + c = 0,$$

where a, b, and c are constants, and $a \neq 0$. We solved quadratic equations by factoring the left side of the equation. If the left side of the equation cannot be factored, the method fails. In this chapter we will learn additional methods that enable us to solve any quadratic equation.

11.1 EXTRACTION OF ROOTS

Consider the quadratic equation

$$x^2 = 16. \tag{1}$$

We can solve the equation by writing it in standard form and factoring the left side.

$$\begin{aligned} x^2 - 16 &= 0 \\ (x + 4)(x - 4) &= 0 \\ x + 4 = 0 \quad &\text{or} \quad x - 4 = 0 \\ x = -4 \quad & \qquad\quad x = 4 \end{aligned}$$

The solutions are 4 and -4.

There is an easier way to solve equation (1). The original equation, $x^2 = 16$, tells us that x is a number whose square is 16. Therefore, x must be a square root of 16. Since 16 has two square roots, 4 and -4, these are the solutions of the equation.

We can think of the solution process in the following way. Since x is squared in the equation, we perform the opposite operation, or take square roots, in order to solve for x.

$$x^2 = 16 \qquad \text{Take square roots of both sides.}$$
$$x = \pm\sqrt{16} \qquad \text{Simplify.}$$
$$x = \pm 4$$

We must remember that every positive number has ***two*** square roots, and consequently the equation has two solutions.

This method for solving quadratic equations is called **extraction of roots.** It is convenient for quadratic equations in which there are no linear (first-degree) terms, that is, equations of the form

$$ax^2 + c = 0.$$

Notice that to apply this method, we must get the squared term on one side of the equation and the constant term on the other side.

EXAMPLE 1 Solve $3y^2 = 75$ by extraction of roots.

SOLUTION First divide both sides by 3 so that y^2 is isolated on one side of the equation.

$$\frac{3y^2}{3} = \frac{75}{3}$$
$$y^2 = 25 \qquad \text{Extract square roots of both sides.}$$
$$y = \pm\sqrt{25} \qquad \text{Simplify.}$$
$$y = \pm 5$$

The solutions are 5 and -5. ■

Common Error

In Example 1, do not forget that 25 has two square roots, 5 *and* -5. It would be incorrect to give the solution to Example 1 as "$y = 5$," because there are *two* solutions.

We can also solve the equation in Example 1 by factoring. However, extraction of roots can be used to solve some equations that cannot be solved by factoring. In the example below, the solutions are irrational numbers.

EXAMPLE 2 Solve $5x^2 - 3 = 2x^2 + 33$ by extraction of roots.

SOLUTION Collect all the terms with x^2 on one side of the equation and all the constant terms on the other side.

$$3x^2 = 36 \qquad \text{Divide both sides by 3.}$$
$$x^2 = 12 \qquad \text{Extract square roots of both sides.}$$
$$x = \pm\sqrt{12} \qquad \text{Simplify.}$$
$$x = \pm 2\sqrt{3}$$

The solutions are $2\sqrt{3}$ and $-2\sqrt{3}$. ■

Equations of the Form $(x + k)^2 = d$

We can also use extraction of roots to solve quadratic equations of the form

$$(x + k)^2 = d.$$

For example, to solve the equation

$$(x - 2)^2 = 9,$$

we first take the square root of both sides to obtain

$$x - 2 = \pm\sqrt{9}$$

or

$$x - 2 = \pm 3.$$

This is really *two* equations,

$$x - 2 = 3 \qquad \text{and} \qquad x - 2 = -3.$$

Solving each equation for x yields

$$x = 5 \qquad \text{and} \qquad x = -1.$$

The solutions are 5 and -1.

We summarize the method of extraction of roots as follows.

To Solve a Quadratic Equation by Extraction of Roots:

1. **Isolate the perfect square on one side of the equation.**
2. **Extract square roots of both sides of the equation; this gives two equations.**
3. **Solve each equation for the variable; there are two solutions.**

EXAMPLE 3 Solve by extraction of roots: $(2x + 3)^2 = 49$.

SOLUTION Since the left side of the equation is a perfect square, we begin by extracting roots.

$$2x + 3 = \pm\sqrt{49} \qquad \text{Simplify.}$$

$$2x + 3 = \pm 7 \qquad \text{Write as two equations.}$$

$$2x + 3 = 7 \qquad 2x + 3 = -7 \qquad \text{Solve each equation.}$$

$$2x = 4 \qquad 2x = -10$$

$$x = 2 \qquad x = -5$$

The solutions are 2 and -5. ∎

EXAMPLE 4 Solve by extraction of roots: $2(3x - 1)^2 = 36$.

SOLUTION First isolate the squared expression by dividing both sides by 2.

$$(3x - 1)^2 = 18 \qquad \text{Extract square roots.}$$

$$3x - 1 = \pm\sqrt{18} \qquad \text{Simplify.}$$

$$3x - 1 = \pm 3\sqrt{2} \qquad \text{Write as two equations.}$$

$$3x - 1 = 3\sqrt{2} \qquad 3x - 1 = -3\sqrt{2} \qquad \text{Solve each equation.}$$

$$3x = 1 + 3\sqrt{2} \qquad 3x = 1 - 3\sqrt{2}$$

$$x = \frac{1 + 3\sqrt{2}}{3} \qquad x = \frac{1 - 3\sqrt{2}}{3}$$

We can write the solutions as $\dfrac{1 \pm 3\sqrt{2}}{3}$. ■

Common Errors

In Example 4, note that we must divide both sides by 2 before taking square roots. In other words, it would be *incorrect* to write

$$2(3x - 1) = 6$$

because we have not taken the square root of 2. Also, recall from our work with radicals that

$$\frac{1 + 3\sqrt{2}}{3} \neq \frac{4\sqrt{2}}{3} \quad \text{and} \quad \frac{1 + 3\sqrt{2}}{3} \neq 1 + \sqrt{2}.$$

Formulas

We can use extraction of roots to solve some formulas with quadratic terms.

EXAMPLE 5 Solve for v: $K = \dfrac{v^2}{64}$.

SOLUTION First, isolate v^2. Multiply both sides by 64.

$$64(K) = 64\left(\frac{v^2}{64}\right)$$

$$64K = v^2 \qquad \text{Extract square roots.}$$

$$\pm\sqrt{64K} = v \qquad \text{Simplify the radical.}$$

$$\pm 8\sqrt{K} = v$$

■

Applications

In Section 7.6 we used the Pythagorean theorem to help solve problems about right triangles. Such problems often lead to quadratic equations that can be solved by extraction of roots.