Wigner Functions for a Class of Semidirect Product Groups

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ABSTRACT

Wigner Functions for a Class of Semidirect Product Groups

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We define and construct Wigner functions for the class of semidirect product groups $G = \mathbb{R}^n \rtimes H$ whose linear part $H \subset GL(n, \mathbb{R})$ is a closed Lie subgroup of $GL(n, \mathbb{R})$ admitting at least one open and free orbit in $\widehat{\mathbb{R}}^n$.
Such groups are classified up to conjugacy in dim $n = 3$ and in dim $n = 4$ under the further requirement that they possess a semisimple ideal.
The general construction is based on three main requirements:
i) the exponential map $exp : \mathfrak{g} \to G$ has a dense image in G, with complement of (left or right) Haar measure zero;
ii) the group admits a square-integrable representation;
iii) the Lebesgue measure $\mathrm{d}X^*$ in the dual of the Lie algebra can be decomposed as $\mathrm{d}X^* = \mathrm{d}\kappa(\lambda)\sigma_\lambda(X^*_\lambda)\mathrm{d}\Omega_\lambda(X^*_\lambda)$, $X^*_\lambda \in \mathcal{O}^*_\lambda$ where $\mathcal{O}^*_\lambda$ denotes a coadjoint orbit parametrized by an index λ , $\mathrm{d}\kappa(\lambda)$ is a measure on the parameter space, σ_λ is a positive function on that orbit and $\Omega_\lambda(X^*_\lambda)$ is the invariant measure under the coadjoint action of G.

We discuss in detail all these elements in the case of semidirect product groups $G = \mathbb{R}^n \rtimes H$ of the kind described above and give an explicit form of the generalized Wigner function related to them.

Cases of special interest are those for which the domain of the generalized Wigner function can be endowed with the structure of phase space: a sufficient condition for this to be is given in terms of purely geometrical properties of the coadjoint orbits.

Relevant examples are discussed with emphasis on the case of the quaternionic group as a 4-dimensional wavelet group; this is a natural non-abelian extension of the known notions of wavelet groups in 1 and 2 dimensions which have extensive applications in signal analysis.

Contents

0.1 Introduction

The study of quantum mechanical systems in the language of quasi-probability distribution functions had its beginnings in the early 30's when E.P.Wigner published his paper "On the Quantum Correction For Thermodynamic Equilibrium". In that paper Wigner defined a real function (now called the Wigner function) over a classical phase space of position and momentum variables, which was a Fourier transform of the shifted wave function $\psi(\vec{q})$ and its complex conjugate $\overline{\psi(\vec{q})}$.

$$W^{QM}(\psi \mid \vec{q}, \vec{p}; h) = \frac{1}{h^n} \int_{\mathbf{R}^n} \overline{\psi(\vec{q} - \frac{\vec{x}}{2})}\, e^{-\frac{2\pi i \vec{x}\cdot\vec{p}}{h}}\, \psi(\vec{q} + \frac{\vec{x}}{2})\, d\vec{x} \tag{1}$$

This function provides us with the same (complete) information about the state of a system as the wave function $\psi(\vec{q})$ itself in the Schrödinger picture of Quantum Mechanics. It is well known that according to the Heisenberg uncertainty principle, one cannot define a true phase space probability distribution for a quantum mechanical particle. Moreover, the Wigner function may assume negative values, which makes it clear that it cannot be interpreted as a joint probability distribution for position and momentum. However, if integrated over $\vec{p}$, the Wigner function gives the position distribution $|\psi(\vec{q})|^2$ and integrated over $\vec{q}$ it gives the momentum distribution $|\widehat{\psi}(\vec{p})|^2$ which explains the use of the quasi-probability distribution terminology. In spite of the problem of interpretation, the Wigner function has proven to be of great use in the study of quantum mechanical systems.

In signal analysis the search for a joint time-frequency description of signals is in principle very similar to the problem of finding a joint (quasi-)distribution of position and momentum in Quantum Mechanics. Thus, the Wigner function has been applied successfully also in signal analysis by L.Ville, where it is often called the Wigner–Ville distribution function.

There are many situations in signal analysis, when a classical Fourier analysis is not well adapted to represent a signal. In a piece of music or speech, for instance, the frequencies are constantly changing. This time evolution of frequencies calls for a joint time-frequency representation such as a wavelet transform or a Wigner function. These two concepts both arise from the theory of square-integrable group representations and are closely related as shown in [2] and as will be described later in this thesis. In many cases, however, the signal we would like to analyze has a built-in symmetry different from that of the Heisenberg -Weyl group. It seems natural to represent images as a function on a phase space related to the $SIM(2)$ group (a group of rotations, dilations and translations). In the case of an optical signal, a Wigner function should underline a symmetry of the optical system. This calls for a new definition of the Wigner function, which was the main idea in the work of S.T.Ali et al.([2]). In this paper the authors have defined a general Wigner function for a wide class of groups G, which posses a square-integrable representation and for which the range of the exponential map $exp : \mathfrak{g} \to G$ is dense in G. We discuss this construction in Chapter 3 and use it to build Wigner functions for groups of a special type, namely, semidirect product groups $G = \mathbb{R}^n \rtimes H,\ H \subset GL(n, \mathbb{R})$ such that H acts on $\widehat{\mathbb{R}}^n$ (dual space to $\mathbb{R}^n$) with open free orbits. Groups of this type, studied previously by D.Bernier and K.F.Taylor in the context of wavelets([9]), are of importance in signal analysis.

The rest of this thesis is organized as follows:

In the first chapter we classify all 3- dimensional connected Lie groups H, which act with open free orbits $\widehat{\mathcal{O}}_i$ on $\widehat{\mathbb{R}}^n$. The fact, which we prove, that each n-dimensional Lie algebra of such a group contain an $(n-1)$-dimensional ideal, greatly simplifies our work in dimension $n = 3$. In dimension $n = 4$ the classification is given for groups which contain a semisimple ideal.We also obtain the classification of irreducible ones (none of the cases in dim 3 is irreducible) .

In the second chapter we introduce all mathematical concepts and tools necessary to define generalized Wigner functions. The main objects are square integrable representations and the corresponding Duflo-Moore operators, which are defined there. In the case of a semidirect product group $G = \mathbb{R}^n \rtimes H$, whose linear part H acts on $\widehat{\mathbb{R}}^n$ with open free orbits, it is shown that each coadjoint G-orbit $\mathcal{O}_i^*$ in $\mathfrak{g}^*$, dual to the Lie algebra $\mathfrak{g}$, has the structure of cotangent bundle $\mathcal{O}_i^* = T^*\widehat{\mathcal{O}}_i$. The Lebesgue measure in the dual space $\mathfrak{g}^*$ can be decomposed as: $\mathrm{d}X^* = \sigma_i(X^*)\mathrm{d}\Omega_i(X^*)$, where $\mathrm{d}\Omega_i(X^*)$ is the invariant measure on $\mathcal{O}_i^*$ with respect to the coadjoint action of G. The simple structure of the groups under consideration allows us to relate Duflo-Moore operators to the Radon derivative $\sigma(X^*)$ and express them both using the structure constants of the Lie algebra $\mathfrak{g}$.

In the third chapter we recall the original Wigner function and its generalized version [2] together with their main properties. The explicit form of a general Wigner function for semidirect product groups $G = \mathbb{R}^n \rtimes H$ (as before) is given, together with a sufficient condition for its domain to be interpreted as a phase space.

In the last chapter we present some relevant examples of Wigner functions for semidirect product groups $G = \mathbb{R}^n \rtimes H$ such that H acts with open free orbits in $\widehat{\mathbb{R}}^n$. The interesting case of the quaternionic group is considered as a non-abelian extension of 1- and 2-dimensional wavelet groups.

Chapter 1

Classification of groups admitting open free orbits

1.1 Setting

Our convention will be to denote elements of $\mathbb{R}^n$ by column vectors and those of $\widehat{\mathbb{R}}^n$, its dual, by row vectors; generically, $\vec{u}$ will denote a column vector and $\vec{u}^T$ a row vector. In the following, H will be a closed connected Lie subgroup of $GL(n, \mathbb{K})$ (where $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) such that there are open and free orbits(i.e. a generic point has only trivial stabilizer) in $V = \widehat{\mathbb{K}}^n$. We notice at once that we must have $\dim(H) = \dim V = n$. Let $L^1, ..., L^n$ be generators of the Lie algebra $\mathfrak{h} = Lie(H)$. We adopt the following notation: For any $L \in \mathfrak{h}$ the vector field associated to its action is denoted by $\mathbb{L} := \sum_j^n (\vec{\omega}^T L)^j \partial_j = \sum_j^n \sum_i^n L_i^j \omega^i \partial_j$.

With this notation we can consider the map

$$\begin{array}{ccc} \bigwedge^n \mathfrak{h} & \rightarrow & \bigwedge^n TV \\ L^1 \wedge \cdots \wedge L^n & \rightarrow & \mathbb{L}^1 \wedge \cdots \wedge \mathbb{L}^n \end{array}$$

and hence define the function over V (for fixed basis $\{L^1, ..., L^n\}$

$$\Delta(\vec{\omega}^T) = d\omega^1 \wedge ... \wedge d\omega^n (\mathbb{L}^1 \wedge ... \wedge \mathbb{L}^n) = \det \begin{pmatrix} \vec{\omega}^T L^1 \\ ... \\ \vec{\omega}^T L^n \end{pmatrix} \tag{1.1}$$

For brevity we will suppress the dependence on $\mathfrak{h}$ and consider it as a function on V (after fixing a linear coordinate system). At this point , let us introduce the adjoint action Ad of the group H on its Lie algebra $\mathfrak{h}$:

$$Ad_h L := hLh^{-1}$$

The matrix of the adjoint action computed with respect to the basis $\{L^1, ..., L^n\}$will be denoted as M_h. We have the following

Lemma 1.1.1 *If $H \subset GL(n, \mathbb{R})$ is a group whose Lie algebra is $\mathfrak{h}$, then for any $h \in H \subset GL(n)$ we have*

$$\Delta(\vec{\omega}^T h) = \det(h) \det(Ad_h) \Delta(\vec{\omega}^T) \ .$$

Proof. We have

$$\begin{aligned} \Delta(\vec{\omega}^T h) = \det \begin{pmatrix} \vec{\omega}^T h L^1 \\ ... \\ \vec{\omega}^T h L^n \end{pmatrix} &= \det(h) \det \begin{pmatrix} \vec{\omega}^T L^1 M_h \\ ... \\ \vec{\omega}^T L^n M_h \end{pmatrix} \\ = \quad \det(h) \det(Ad_h) \Delta(\vec{\omega}^T) \ . \end{aligned} \tag{1.2}$$

This ends the proof. Q.E.D.

Clearly the expression $\chi := \det(h) \det Ad_h$ is a character of the group H. The corresponding infinitesimal character μ enters in a differential equation satisfied by the determinant:

Corollary 1.1.1 For any $L \in \mathfrak{h}$ we have

$$\mathbb{L}\Delta = \mu(L)\Delta, \qquad \mu(L) := \mathrm{Tr}_V(L) + \mathrm{Tr}_{\mathfrak{h}}(ad_L) \ . \tag{1.3}$$

Proof. It suffices to take $h = \exp(tL)$ in Lemma 1.1.1 and then $\mathbb{L}\Delta(\vec{\omega}^T) = \frac{d}{dt}\Delta(\vec{\omega}^T h)_{|_{t=0}} = \frac{d}{dt}\left(det(e^{tL})det(Ad_{e^{tL}})\Delta(\vec{\omega}^T)\right)_{|_{t=0}} = \frac{d}{dt}\left(det(e^{tL})det(e^{ad(tL)})\right)_{|_{t=0}}\Delta(\vec{\omega}^T) = (\mathrm{Tr}_V(L) + \mathrm{Tr}_{\mathfrak{h}}(ad_L))\Delta(\vec{\omega}^T)$. Q.E.D.

The first claim is that $\mu \neq 0$ and hence χ is not a trivial character: if this would be the case, the group would preserve the level surfaces of Δ (considered as an ordinary function) and hence the orbits would be neither open nor free.

Let F^α be a basis in the kernel of μ, with $\alpha = 1, .., n-1$. The corresponding vectors $\mathbb{F}^\alpha$ are then tangent to the level surface of Δ, since $\mathbb{F}^\alpha \Delta = \mu(F^\alpha)\Delta = 0$.

Lemma 1.1.2 *The subspace* $\mathcal{F} := \mathbb{K}\{F^1, ..., F^{n-1}\} \subset \mathfrak{h}$ *is an ideal.*

Proof. The subalgebra $\mathcal{F} = \ker(\mu)$ is the Lie subalgebra of the kernel of the character $\det(h)\det(Ad_h)$ (which is a normal subgroup), and hence it is an ideal. Q.E.D.

1.2 Classification in dim $n = 2$

The 2-dimensional case has been already investigated in [15], therefore we will only state the main result.

Theorem 1.2.1 *Suppose that* $H \subset GL(2, R)$ *acts on* $\mathbb{R}^2$ *with open orbits.*

(a) If H is connected abelian, it is conjugate to (exactly) one of the following three groups:

(i) $H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 > 0 \right\}$,

(ii) $H = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^+ \right\}$,

(iii) $H = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}$.

(b) If H is connected nonabelian, it is conjugate to (exactly) one element of the family H^c*, such that* $c \in \mathbb{R} - \{1\}$*,given by*

$$H^c = \left\{ \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} : a, b \in \mathbb{R}, a > 0 \right\}.$$

(c) If H is not connected, then after a suitable change of coordinates its connected component H_0 is equal to one of the groups from parts(a) or(b). H is a finite extension of H_0, more precisely $H \subset FH_0$, where $F \subset GL(2,\mathbb{R})$ is the finite group generated by

(i) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, *for the case (a)(i),*

(ii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *for the case (a)(ii),*

(iii) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, *for the cases (a)(iii) and (b).*

1.3 Classification in dim $n = 3$

In order to classify all groups H acting with open and free orbits in $\mathbb{R}^3$ we will classify their infinitesimal counterparts $\mathfrak{h}$. The task is equivalent under the assumption that H is connected. The classification is over $\mathbb{R}$, which clearly contains the complex case (i.e. some cases that are conjugated by a complex matrix can be distinct over $\mathbb{R}$) Let $L \in \mathfrak{h}$ with $\mu(L) = 1$. Since $\mathcal{F} = \ker(\mu)$ is a two dimensional ideal, in a suitable basis F^1, F^2 it has the commutation-relations

$$[F^1, F^2] = \epsilon F^2 \ ; \quad \epsilon = 0, 1.$$

First of all we notice that $\mathfrak{h}$ is solvable:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathcal{F}, \ [\mathcal{F}, \mathcal{F}] \subset span(F_2)$$

Since $\mathcal{F}$ is an ideal, $ad(L)$ is a derivation of $\mathcal{F}$, i.e.

$$(ad(L))[F^1, F^2]_{\mathcal{F}} = [(ad(L))F^1, F^2]_{\mathcal{F}} + [F^1, (ad(L))F^2]_{\mathcal{F}} \tag{1.4}$$

We can show that if $\epsilon = 1$ (i.e. $\mathcal{F}$ is not abelian) then all derivations of $\mathcal{F}$ are inner (i.e. can be represented by $ad(F)$ for some $F \in \mathcal{F}$). To see that we can write the general form of each derivation of $\mathcal{F}$ as :

$$DF^1 = a_1 F^1 + a_2 F^2$$

$$DF^2 = b_1F^1 + b_2F^2 \tag{1.5}$$

The condition that D is a derivation together with commutation relation $[F^1, F^2] = F^2$ gives $a_1 = 0 = b_1$.Thus (1.5) can be rewritten as:

$$\begin{aligned} DF^1 &= a_2F^2 \\ DF^2 &= b_2F^2 \end{aligned} \tag{1.6}$$

We can clearly represent each derivation D of $\mathcal{F}$ as $ad(F)$ where $F = b_1F^1 - a_1F^2$ since:

$$\begin{aligned} [b_2F^1 - a_2F^2, F^1] &= a_2F^2 \\ [b_2F^1 - a_2F^2, F^2] &= b_2F^2 \end{aligned} \tag{1.7}$$

Thus, we can always choose $L' \in \mathfrak{h}$ such that it commutes with $\mathcal{F}$, namely $L' = L - F_L$ (such that $adL = adF_L$). $\mathfrak{h}$ is then a trivial central extension of $\mathcal{F}$.

Case 1: $\epsilon = 1$

$$[L, F^1] = 0 \ ; \ [L, F^2] = 0 \ ; \ [F^1, F^2] = F^2 \ .$$

Case 2: $\epsilon = 0$ We have that $ad(L) \in End(\mathcal{F}) \simeq GL(2, \mathbb{R})$, and hence we can represent it by an arbitrary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In this case, in order to classify the groups with abelian ideal $\mathcal{F}$ we will proceed as follows. First we fix a conjugacy class of the representation for $\mathcal{F}$ in $GL(3, \mathbb{R})$. The normalizer $\mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F})$ of the abelian ideal (hence vector subspace) $\mathcal{F}$ in $GL(3, \mathbb{R})$ acts on $\mathcal{F}$ by automorphisms

$$Ad_{GL(3,\mathbb{R})} : \mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F}) \to GL(\mathcal{F}) \ .$$

Therefore we will act with it on $ad(L)$ in order to bring it into some canonical form.

We have described the abstract Lie algebras that may possibly occur: we still have to implement suitable three dimensional representations of them (clearly up to conjugacy). In order not to overburden the notation we will use the same symbols to denote both the Lie algebra elements and the matrices representing them.

Before going to special cases we notice that since $\mathfrak{h}$ is solvable then according to Lie's theorem [28, 16] its elements can be represented as lower triangular complex matrices.

Case 1

The Lie algebra reads

$$[L, F^1] = 0\ ;\ [L, F^2] = 0\ ;\ [F^1, F^2] = F^2\ .$$

It is clear that F^2 must be strictly lower–triangular (over $\mathbb{C}$), because if it had any diagonal part, the commutation relation $[F^1, F^2]$ would not hold. Therefore, since all eigenvalues are zeros, F^2 as a real matrix can be written in Jordan canonical form as 2×2 nilpotent Jordan block, in which case (**Case 1.a**) $(F^2)^2 = 0$ or a 3×3 nilpotent Jordan block and hence we have (**Case 1.b**) $(F^2)^3 = 0 \neq (F^2)^2$.

The matrix representing F^1 (denoted by the same symbol) belongs to the affine subspace

$$\mathcal{F}^1 = \left\{[F^1, F^2] = F^2\ ,\ \mathrm{Tr}(F^1) = -\mathrm{Tr}_{\mathfrak{h}}(ad(F^1)) = -1\right\}$$

Let $\mathcal{N}_{GL(3,\mathbb{R})}(\mathbb{R}\{F^2\})$ denote the normalizer of F^2 in $GL(3,\mathbb{R})$; we are to describe in each sub-case the quotient $\mathcal{F}^1/\mathcal{N}_{GL(3,\mathbb{R})}(\mathbb{R}\{F^2\})$.

Case 1.a: $(F^2)^2 = 0$.

We can assume (in a suitable basis) that F^2 has a 1 in the $(3,1)$ entry. After a direct inspection the affine space $\mathcal{F}^1$ has the following form:

$$F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\ ;\ \mathcal{F}^1 = \left\{ F^1 = \begin{bmatrix} A_{1,1} & 0 & 0 \\ A_{2,1} & -2A_{1,1} - 2 & 0 \\ A_{3,1} & A_{3,2} & A_{1,1} + 1 \end{bmatrix} \right\}$$

Notice that we can always assume that $A_{3,1}=0$ by shifting F^1 by F^2.

The normalizer subgroup of F^2 inside $GL(3)$ is

$$\mathcal{N}_{GL(3,\mathbb{R})}(\mathbb{R}\{F^2\}) = \left\{ \begin{pmatrix} \varphi_1 & 0 & 0 \\ \varphi_2 & \varphi_4 & 0 \\ \varphi_3 & \varphi_5 & \varphi_6 \end{pmatrix} \right\} .$$

There are three conjugacy classes in the quotient space $\mathcal{F}^1/\mathcal{N}_{GL(3,\mathbb{R})}(\mathbb{R}\{F^2\})$

1. **F^1 diagonalizable.** Suppose that $A_{1,1} \neq -1, -\frac{2}{3}$ or that $A_{1,1} = -1, A_{3,2} = 0$ or $A_{1,1} = -\frac{2}{3}, A_{2,1} = 0$, then F^1 is diagonalizable,

$$F^1 = \begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & -2(A_{1,1}+1) & 0 \\ 0 & 0 & A_{1,1}+1 \end{pmatrix},$$

by conjugating it by the matrix

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ -A_{2,1} & 3\,A_{1,1}+2 & 0 \\ A_{2,1}A_{3,2} & A_{3,2} & 3\,A_{1,1}+3 \end{pmatrix}$$

or (in the two other cases)

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ A_{2,1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & A_{3,2} & 1 \end{bmatrix}.$$

Notice that we can exclude the case the F^1 is diagonalizable and $A_{1,1} = -1$ for otherwise $F^1\mathbf{x}$ and $F^2\mathbf{x}$ would always be proportional to $[1,0,0]$ and the resulting group would not have open orbits.

We have now two sub-cases

(a) $A_{1,1} \neq -\frac{2}{3}$:

In this case the commutation relations imply that $L = \mathrm{diag}(L_{11}, L_{22}, L_{11})$.

By imposing the normalization condition $\mathrm{Tr}(L) + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = 1$ we have $L_{22} = 1 - 2L_{11}$

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & 1-2L_{11} & 0 \\ 0 & 0 & L_{11} \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$F^1 = \begin{pmatrix} A_{1,1} & 0 & 0 \\ 0 & -2(A_{1,1}+1) & 0 \\ 0 & 0 & A_{1,1}+1 \end{pmatrix}$$

$$\Delta = -{x_3}^2 x_2 (A_{1,1}+1)$$

(b) $A_{1,1} = -\frac{2}{3}$:

The commutation relations and the condition $\mathrm{Tr}(L) + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = 1$ give that

$$L = \begin{pmatrix} L_{11} & 0 & 0 \\ L_{21} & 1-2L_{11} & 0 \\ 0 & 0 & L_{11} \end{pmatrix}.$$

Now in the generic case, by acting with the normalizer of $\mathcal{F}$ we can diagonalize L and we have the same form as above (but with $A_{1,1} = -3/2$); conversely, if $L_{11} = -\frac{1}{3}$ and $L_{2,1} \neq 0$ we can only put it in Jordan form

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F^1 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\Delta = -\frac{1}{3}{x_3}^2 x_2$$

2. **F^1 not diagonalizable (eigenvalues $-1, 0, 0$).** If $A_{1,1} = -1$ and $A_{3,2} \neq 0$ then each F^1 can be put in the form

$$F^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

by a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ A_{2,1}A_{3,2} & A_{2,3} & 0 \\ -A_{2,1} & 0 & 1 \end{pmatrix}$$

The normalization condition $\mathrm{Tr}(L) + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = 1$ and the commutation relations give

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & L_{3,2} & \frac{1}{3} \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\Delta = \frac{1}{3} {x_3}^3 .$$

The normalizer of $\mathcal{F}$ is given by

$$\mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F}) = \left\{ \Phi = \begin{pmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_3 & 0 \\ \phi_2 & \phi_4 & \phi_5 \end{pmatrix} \right\},$$

and by conjugating L by such subgroup we can always bring it into canonical form, i.e. with $L_{32} = 0, 1$.

3. F^1 **not diagonalizable (eigenvalues** $-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$**).** If $A_{1,1} = -\frac{2}{3}$ and $A_{2,1} \neq 0$ then each F^1 can be put in the form

$$F^1 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 1 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix},$$

by a matrix

$$\begin{pmatrix} A_{2,1} & 0 & 0 \\ 0 & 1 & 0 \\ A_{2,1}A_{3,2} & A_{3,2} & 1 \end{pmatrix}$$

Imposing the commutation relations and normalization as before we obtain

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ L_{2,1} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 1 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$
$$\Delta = \frac{1}{3} x_2 {x_3}^2 .$$

The normalizer of $\mathcal{F}$ is now given by

$$\mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F}) = \left\{ \Phi = \begin{pmatrix} \phi_1 & 0 & 0 \\ \phi_2 & \phi_4 & 0 \\ \phi_3 & 0 & \phi_5 \end{pmatrix} \right\},$$

and by conjugating L by such subgroup we can -again- always bring it into canonical form, i.e. with $L_{21} = 0, 1$.

Case 1.b: $(F^2)^2 \neq 0 = (F^2)^3$

Up to conjugacy the matrix F^2 is given by

$$F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

In this case the equation $[F^1, F^2] = F^2$ and $\mathrm{Tr}(F^1) = -1$ can be solved to give (up -possibly- by a shift of F^2)

$$F^1 = \begin{pmatrix} -\frac{4}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ K & 0 & \frac{2}{3} \end{pmatrix} .$$

The normalizer of F^2 now is

$$\mathcal{N}_{SL(3,\mathbb{R})}(\mathbb{R}\{F^2\}) = \left\{ \Phi = \begin{pmatrix} \phi_1 & 0 & 0 \\ \phi_2 & \phi_1 + \alpha & 0 \\ \phi_3 & \phi_2 & \phi_1 + 2\alpha \end{pmatrix} \right\} .$$

By acting with such a matrix on F^1 we actually recognize that we can always bring K to zero.

Then the commutations $[L, F^1] = [L, F^2] = 0$ give necessarily that $L = \frac{1}{3}\mathrm{id}$ (we have used the normalization $\mathrm{Tr}(L) + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = 1$) and we find:

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} , \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \; F^1 = \begin{pmatrix} -\frac{4}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

$$\Delta = -\frac{1}{3} x_3 \left(-2\, x_3 x_1 + {x_2}^2\right)$$

Before passing to the analysis of the second case, a few words about the reality of the matrices.

The matrix F^2 spans $[\mathfrak{h}, \mathfrak{h}]$, therefore it belongs to the real Lie algebra of matrices and it can be put into Jordan form over $\mathbb{R}$. The other two matrices are fixed up to the action of the normalizer of F^2, which is in both cases upper–triangular: therefore, if any of the eigenvalues of other elements of $\mathfrak{h}$ are complex (and hence they come in conjugate pairs) we cannot put the matrices into real form. Therefore in all cases above if the algebra is a subalgebra of the real matrices, all free parameters that appear must be real.

Case 2

Now, the two matrices F^1, F^2 span an abelian subalgebra of $gl(V)$. Moreover, from the equation

$$\mathrm{Tr}(F^i) = -\mathrm{Tr}_{\mathfrak{h}}(ad(F^i)) = 0 \ ,$$

we obtain that actually they belong to $sl(V)$. There are only six conjugacy classes of such subalgebras:

(a.1)

$$F^1 := \mathrm{diag}(1,-1,0) \ ; \ F^2 := \mathrm{diag}(0,1,-1) \ ;$$

(a.2)

$$F^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \ ; \ F^2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ ;$$

(b)

$$F^1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ ; \ F^2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

(c)

$$F^1 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \ ; \ F^2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} .$$

(d)

$$F^1 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ ; \ F^2 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} ;$$

(e)

$$F^1 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \; F^2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} ;$$

We can deal right away with cases (a.1) and (a.2): first of all they belong to the same conjugacy class over $\mathbb{C}$. They can be simultaneously diagonalized (over $\mathbb{C}$) and then span a Cartan -subalgebra of $sl(3,\mathbb{C})$, namely the abelian subalgebra of traceless diagonal matrices. Now the adjoint action of the matrix L must leave this diagonal subalgebra invariant, ($ad(L)\mathcal{F} \subset \mathcal{F}$) which implies that L is diagonal too[1]. We have proved that L should belong to the set of diagonal matrices, which gives only an abelian action (corresponding to $ad(L) \equiv 0$). Since we can always shift L by $\mathcal{F}^{\mathbb{C}}$ in such a way that it is proportional to the identity matrix we can take it to be (imposing the normalization $\mathrm{Tr}(L) + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = 1$) equal to $\frac{1}{3}\mathbf{1}$, so that we have:

Case 2.a.1

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} , \; F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$
$$\Delta = x_1 x_2 x_3 \; .$$

Case 2.a.2

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} , \; F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} , \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} ,$$
$$\Delta = \frac{1}{2} x_1 ({x_2}^2 + {x_3}^2) \; .$$

We now look at the other three cases. For each of them we are to impose the commutation relations

$$\begin{cases} ad(L) \begin{pmatrix} F^1 \\ F^2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F^1 \\ F^2 \end{pmatrix} \\ 1 = \mathrm{Tr}_V L + \mathrm{Tr}_{\mathfrak{h}}(ad(L)) = \mathrm{Tr}_V(L) + (a+d) \end{cases} \tag{1.8}$$

[1] In other words, $ad(L)$ belongs to the normalizer of a CSA and hence belongs to the same CSA [28]

This is in principle an overdetermined system for the entries of L and hence we can expect some compatibility constraints on the values of the parameters a, b, c, d. Moreover, at each stage, it should be noted that we can always add any linear combination of F^1, F^2 to L without changing the commutation relations.

Finally we can act on L with the normalizer of the subalgebra $\mathcal{F}$ inside $GL(3,\mathbb{R})$ in order to reduce the free parameters.

Case 2.b Imposing the form (b) on the matrices F^1, F^2 and solving the system in Eq. (1.8) we get that L has the form

$$L = \begin{pmatrix} \frac{1}{3}-a & -b & 0 \\ -c & \frac{1}{3}-d & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \ , \ F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\Delta = -\frac{1}{3}{x_3}^3 \ .$$

It is clear that the normalizer of $\mathcal{F}$ is constituted by matrices Φ of the form

$$\Phi = \begin{pmatrix} \phi_{11} & 0 & 0 \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix} \ .$$

In particular one notices that the map

$$Ad_{GL(3,\mathbb{R})} : \mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F}) \to GL(\mathcal{F}) \simeq GL(2,\mathbb{R}) \ .$$

is surjective. Therefore we can conjugate L by a suitable element of $\mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F})$ and bring the action of $ad(L)$ into one of the three Jordan forms (over $\mathbb{R}$):

Case 2.b.1 $ad(L)$ diagonalizable over $\mathbb{R}$:

$$L = \begin{pmatrix} \frac{1}{3}+\mu_1 & 0 & 0 \\ 0 & \frac{1}{3}+\mu_2 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Case 2.b.2 $ad(L)$ diagonalizable over $\mathbb{C}$

$$L = \begin{pmatrix} \frac{1}{3}+\mu & -\nu & 0 \\ \nu & \frac{1}{3}+\mu & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Case 2.b.3 $ad(L)$ not diagonalizable.

$$L = \begin{pmatrix} \frac{1}{3}+\mu & 1 & 0 \\ 0 & \frac{1}{3}+\mu & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Case 2.c Imposing the form (c) on the matrices F^1, F^2 we find that compatibility imposes $d = 2a$ and $c = 0$ so that L has the form

$$L = \begin{pmatrix} \frac{1}{3}-2a & 0 & 0 \\ 0 & \frac{1}{3}-a & b \\ 0 & 0 & \frac{1}{3} \end{pmatrix} , \; F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\Delta = -\frac{1}{3}{x_3}^3 .$$

The normalizer of $\mathcal{F}$ consists of upper triangular matrices. Now the map

$$Ad_{GL(3,\mathbb{R})} : \mathcal{N}_{GL(3,\mathbb{R})}(\mathcal{F}) \to GL(\mathcal{F}) \simeq GL(2,\mathbb{R}) ,$$

is a surjection over (some) Borel subgroup of upper triangular matrices of $GL(2,\mathbb{R})$. One can check that then one can put L into the form (according to $a = 0$ or $a \neq 0$)

Case 2.c.1 $a \neq 0$, (then the eigenvalues of $ad(L)$ are a and $2a$, so that it is semisimple over $\mathbb{R}$)

$$L = \begin{pmatrix} \frac{1}{3}-2a & 0 & 0 \\ 0 & \frac{1}{3}-a & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Case 2.c.2 $a = 0 \neq b$, $ad(L)$ is not diagonalizable.

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Case 2.d Imposing the form (d) on the matrices F^1, F^2 we find that compatibility imposes $a = b = c = 0$ so that L has the form

$$L = \begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \; F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\Delta = -x_3{x_2}^2 .$$

It is clear that d can not be eliminated because it is one of the eigenvalues (i.e. its value is invariant under conjugation).

Case 2.e This does not give any group with open free orbits as one can check directly that the determinant Δ is identically zero (irrespective of the form of L). Indeed, $F^1\mathbf{x}$ and $F^2\mathbf{x}$ are always proportional to the vector $[1,0,0]$.

Summarizing the previous discussion, we have proved

Proposition 1.3.1 The closed connected subgroups of $GL(3,\mathbb{R})$ which act with open–free orbits on a three dimensional space fall, up to conjugacy, among the connected groups obtained by exponentiation of the algebra generated by the matrices below. We also write the form of the group element.

Case 1.a.1

$$L = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1-2\mu_1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$F^1 = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & -2(\mu_2+1) & 0 \\ 0 & 0 & \mu_2+1 \end{pmatrix}$$

$$\Delta = -{x_3}^2 x_2(\mu_2+1)$$

$$[L,F^1] = 0 = [L,F^2]\ ;\ [F^1,F^2] = F^2$$

$$H = \left\{ h = \begin{pmatrix} w^{\mu_1} z^{\mu_2} & 0 & 0 \\ 0 & w^{1-2\mu_1} z^{-2\mu_2-2} & 0 \\ t & 0 & w^{\mu_1} z^{\mu_2+1} \end{pmatrix} \quad w,z \in \mathbb{R}^\times_+,\ t \in \mathbb{R} \right\}$$

Case 1.a.2

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad F^1 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\Delta = -\frac{1}{3}{x_3}^2 x_2$$

$$[L,F^1] = 0 = [L,F^2]\ ;\ [F^1,F^2] = F^2$$

$$H = \left\{ h = \begin{pmatrix} wz^3 & 0 & 0 \\ 3\,wz^3 \ln(w\,z) & wz^3 & 0 \\ t & 0 & w \end{pmatrix},\ w,z \in \mathbb{R}^\times_+,\ t \in \mathbb{R} \right\}$$

Case 1.a.3 For $\mu = 0, 1$;

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \mu & \frac{1}{3} \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Delta = \frac{1}{3} {x_3}^3$$

$$[L, F^1] = 0 = [L, F^2] \ ; \ [F^1, F^2] = F^2$$

$$H = \left\{ h = \begin{pmatrix} w\, z^3 & 0 & 0 \\ 0 & w & 0 \\ t & w \ln(z^{-1} w^{\mu}) & w \end{pmatrix}, \ w, z \in \mathbb{R}_+^{\times}, \ t \in \mathbb{R} \right\}$$

Case 1.a.4 For $\mu = 0, 1$;

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \mu & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, F^1 = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 1 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\Delta = \frac{1}{3} x_2 {x_3}^2$$

$$[L, F^1] = 0 = [L, F^2] \ ; \ [F^1, F^2] = F^2$$

$$H = \left\{ h = \begin{pmatrix} w\, z^3 & 0 & 0 \\ 3\, w\, z^3 \ln(z^{\mu-1} w^{\mu}) & w\, z^3 & 0 \\ t & 0 & w \end{pmatrix}, \ z, w \in \mathbb{R}_+^{\times}, \ t \in \mathbb{R} \right\}$$

Case 1.b

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \ F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ F^1 = \begin{pmatrix} -\frac{4}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}$$

$$\Delta = -\frac{1}{3} x_3 \left(-2\, x_3 x_1 + {x_2}^2 \right)$$

$$[L, F^1] = 0 = [L, F^2] \ ; \ [F^1, F^2] = F^2$$

$$H = \left\{ h = w \cdot \begin{pmatrix} z & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2} \frac{t^2}{z} & \frac{t}{z} & z^{-1} \end{pmatrix}, \ w, z \in \mathbb{R}_+^{\times}, \ t \in \mathbb{R} \right\}$$

Case 2.a.1

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \ F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\Delta = x_1x_2x_3$$

$$[L,F^1]=0=[L,F^2]\ ;\ [F^1,F^2]=0$$

$$H=\left\{h=\begin{pmatrix} z & 0 & 0\\ 0 & w & 0\\ 0 & 0 & t\end{pmatrix},\ t,z,w\in\mathbb{R}_+^\times\right\}$$

Case 2.a.2

$$L=\begin{pmatrix} \frac{1}{3} & 0 & 0\\ 0 & \frac{1}{3} & 0\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\ ,\ F^1=\begin{pmatrix} 1 & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{1}{2}\end{pmatrix}\ ,\ F^2=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\ ,$$

$$\Delta=\frac{1}{2}x_1({x_2}^2+{x_3}^2)$$

$$[L,F^1]=0=[L,F^2]\ ;\ [F^1,F^2]=0$$

$$H=\left\{h=\begin{pmatrix} z & 0 & 0\\ 0 & w\cos(t) & -w\sin(t)\\ 0 & w\sin(t) & w\cos(t)\end{pmatrix},\ z,w\in\mathbb{R}_+^\times,\ t\in\mathbb{R}\right\}$$

Case 2.b.1

$$L=\begin{pmatrix} \frac{1}{3}-\mu_1 & 0 & 0\\ 0 & \frac{1}{3}-\mu_2 & 0\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\ ,\ F^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\ ,\ F^2=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$\Delta=-\frac{1}{3}{x_3}^3$$

$$[L,F^1]=\mu_1F^1,\ [L,F^2]=\mu_2F^2,\ [F^1,F^2]=0$$

$$H=\left\{h=\begin{pmatrix} w^{1+3\mu_1} & 0 & 0\\ 0 & w^{1+3\mu_2} & 0\\ t & z & w\end{pmatrix},\ w\in\mathbb{R}_+^\times,\ t,z\in\mathbb{R}\right\}$$

Case 2.b.2

$$L=\begin{pmatrix} \frac{1}{3}-\mu & -\nu & 0\\ \nu & \frac{1}{3}-\mu & 0\\ 0 & 0 & \frac{1}{3}\end{pmatrix}\ ,\ F^1=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\ ,\ F^2=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

$$\Delta=-\frac{1}{3}{x_3}^3$$

$$[L,F^1]=\mu F^1+\nu F^2,\ [L,F^2]=-\nu F^2+\mu F^2,\ [F^1,F^2]=0$$

$$H=\left\{h=\begin{pmatrix} e^{w(1-3\mu)}\cos(3w\nu) & -e^{w(1-3\mu)}\sin(3w\nu) & 0\\ e^{w(1-3\mu)}\sin(3w\nu) & e^{w(1-3\mu)}\cos(3w\nu) & 0\\ t & z & e^w\end{pmatrix} w,t,z\in\mathbb{R}\right\}$$

Case 2.b.3

$$L = \begin{pmatrix} \frac{1}{3}-\mu & 0 & 0 \\ 1 & \frac{1}{3}-\mu & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \; F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Delta = -\frac{1}{3}{x_3}^3$$

$$[L, F^1] = \mu F^1, \; [L, F^2] = F^1 + \mu F^2, \; [F^1, F^2] = 0$$

$$H = \left\{ h = \begin{pmatrix} w^{1-3\mu} & 0 & 0 \\ 3\ln(w) w^{1-3\mu} & w^{1+3\mu} & 0 \\ t & z & w \end{pmatrix} w \in \mathbb{R}_+^\times, \; t, z \in \mathbb{R} \right\}$$

Case 2.c.1

$$L = \begin{pmatrix} \frac{1}{3}-2\mu & 0 & 0 \\ 0 & \frac{1}{3}-\mu & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \; F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Delta = -\frac{1}{3}{x_3}^3$$

$$[L, F^1] = \mu F^1, \; [L, F^2] = 2\mu F^2, \; [F^1, F^2] = 0$$

$$H = \left\{ h = \begin{pmatrix} w^{1-6\mu} & 0 & 0 \\ t & w^{1-3\mu} & 0 \\ z & t w^{-3\mu} & w \end{pmatrix}, \; w \in \mathbb{R}_+^\times, \; t, z \in \mathbb{R} \right\}$$

Case 2.c.2

$$L = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \; F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\Delta = -\frac{1}{3}{x_3}^3$$

$$[L, F^1] = F^2, \; [L, F^2] = 0, \; [F^1, F^2] = 0$$

$$H = \left\{ h = \begin{pmatrix} w & 0 & z \\ t & w & 0 \\ z & -3w\ln(w) + t & w \end{pmatrix} w \in \mathbb{R}_+^\times, \; t, z \in \mathbb{R} \right\}$$

Case 2.d

$$L = \begin{pmatrix} \mu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; F^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \; F^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Delta = -x_3 {x_2}^2$$

$$[L, F^1] = 0,\ [L, F^2] = \mu F^2,\ [F^1, F^2] = 0$$

$$H = \left\{ h = \begin{pmatrix} w^{-\mu} t & 0 & 0 \\ z & t & 0 \\ 0 & 0 & \frac{w}{t^2} \end{pmatrix},\ w \in \mathbb{R}_+^\times,\ t, z \in \mathbb{R} \right\}$$

Corollary 1.3.1 The closed connected subgroups of $GL(3, \mathbb{C})$ which act with open-free orbits on a three dimensional complex space fall, up to conjugacy, among the connected groups obtained by exponentiation of the algebra generated by the matrices given in Prop. 1.3.1 (except case 2.a.2 and 2.b.2 which are the same as case 2.a.1 and 2.b.1 -respectively- over $\mathbb{C}$) with the following restrictions:
in cases 2.b.1, 2.c.1 and 2.d, the parameters μ_1, μ_2, μ appearing there must be rational (and hence we may assume integer by rescaling L).

Proof. The rationality condition follows from the requirement that the exponential group is a closed subgroup of $GL(3, \mathbb{C})$. Indeed each group is a subgroup of upper triangular matrices and the intersection with the subgroup of diagonal matrices must be closed. Then this implies that the ratios of the diagonal elements of L must be rational. The form of the group elements are the same given in Prop. 1.3.1 where one has to read instead of $\mathbb{R}_+^\times$ $\mathbb{C}^\times$ and $\mathbb{C}$ instead of $\mathbb{R}$. Q.E.D.

1.4 Classification in dim $n = 4$ with semisimple ideal

If $n = 4$ the ideal $\mathcal{F}$ is three–dimensional: in this case it can be semisimple (i.e. isomorphic to $sl(2, \mathbb{R})$ or $su(2)$) or not, in which case (since there are no semisimple algebras of dimension smaller than three) it must be completely solvable. We shall classify the cases in which $\mathcal{F}$ is (semi)-simple (hence isomorphic to either $sl(2, \mathbb{R})$ or $su(2, \mathbb{R})$).

Since $\mathcal{F}$ is simple and $ad(L)$ is a derivation of $\mathcal{F}$, then it must be inner (because all derivations of semisimple Lie algebras are inner), i.e. $\exists F \in \mathcal{F}$ s.t. $[L - F, \mathcal{F}] = 0$. In other words $\mathfrak{h}$ is a trivial central extension of $\mathcal{F}$.

Now, the matrix representation of $\mathcal{F}$ is a four dimensional real representation of a real form of $sl(2,\mathbb{C})$ (which are $sl(2,\mathbb{R})$, $so(3)$) i.e. (excluding the trivial and the two dimensional one for reasons that we would not get a open–free action), it must be either the direct sum of the two–dimensional one or the (embedding of the) three dimensional one or the four dimensional irreducible (over $\mathbb{C}$). We can exclude the three dimensional representation because each vector has a nontrivial stabilizer.

Over $\mathbb{R}$ we have the following two cases -irreducible or reducible- according to the representation of $\mathcal{F}^{\mathbb{C}}$.

Irreducible

It is well known that there are only two real forms of $sl(2,\mathbb{C})$, i.e. $so(3) \simeq su(2)$ or $sl(2,\mathbb{R}) \simeq so(2,1)$.

As there are no real, irreducible 4-dimensional representations of $so(3)$, the only possibility is that $\mathcal{F}$ provides a 4-dim. irrep. of $sl(2,\mathbb{R})$, which is given up to conjugacy by

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, F^{+} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}, F^{-} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since $[L, \mathcal{F}] = 0$ and the representation is irreducible, there is only one choice for L, i.e. the identity matrix. The corresponding determinant is (for the normalization $L = \frac{1}{4}\mathbf{1}$)

$$\Delta = -9\, x_1 x_2 x_4 x_3 + 4\, x_1 x_3{}^3 + 9/2\, x_1{}^2 x_4{}^2 - 3/2\, x_3{}^2 x_2{}^2 + 3\, x_4 x_2{}^3 \,.$$

As there are no real four dimensional irreducible representations of $so(3) \simeq su(2)$, this is the only real case.

Reducible

The representation is the sum of two two–dimensional ones, $V_{\mathbb{C}} = V_1 \otimes \mathbb{C} \oplus_{\mathcal{F}\mathbb{C}} V_2 \otimes \mathbb{C}$ As the two sub-representations must be equivalent, the commutant in $End(V)$ is easily described: the restriction to each of V_i must be proportional to the identity, and the maps $L_{ij} = \pi_i(L_{|V_j}) : V_i \to V_j$ must be intertwiners. It is convenient to represent $V = V_1 \oplus V_2$ as a tensor product $V = \mathbb{R}^2 \otimes V_1$. With this notation, if ρ is the two–dimensional representation of $sl(2, \mathbb{C})$, we have $\mathcal{F} = \mathbf{1} \otimes \rho$, and the commutant is easily described as $End(\mathbb{R}^2) \otimes \mathbf{1}$.

In more concrete terms, the matrices are

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, F^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$F^- = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix},$$

where we should add the normalization condition $\mathrm{Tr}(L) = 2(a + d) = 1$. In any case the determinant of the action is

$$\Delta = \frac{1}{2}(x_1 x_4 - x_2 x_3)^2 .$$

By conjugating with a matrix of the form $SL(2, \mathbb{R}) \otimes \mathbf{1}$ we can bring L into canonical Jordan form (over $\mathbb{R}$)

$$\begin{aligned} L &= \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \otimes \mathbf{1} \, , \; \mu_1 + \mu_2 = \frac{1}{2} \\ L &= \begin{pmatrix} \frac{1}{4} & -\alpha \\ \alpha & \frac{1}{4} \end{pmatrix} \otimes \mathbf{1} \, , \\ L &= \begin{pmatrix} \frac{1}{4} & 0 \\ 1 & \frac{1}{4} \end{pmatrix} \otimes \mathbf{1} \end{aligned}$$

The Lie algebra is always the same abstract algebra $\mathfrak{h} \simeq \mathbb{R} \oplus sl(2, \mathbb{R})$.

As for the other real form of $sl(2,\mathbb{C})$, namely $su(2) \simeq so(3)$, there is a 4-dimensional real representation which is irreducible over $\mathbb{R}$ but not over $\mathbb{C}$: such a representation is given by quaternions $\mathbb{H} \simeq \mathbb{R}^4$ we have the rep. of the other real form of $sl(2,\mathbb{C})$ as in

$$F^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, F^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, F^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The commutant is non-trivial (as Schur's lemma does not apply this being a reducible rep. over $\mathbb{C}$) and it is constituted by matrices of the form

$$L = a_1 \mathbf{1} + \begin{pmatrix} 0 & -a_2 & -a_3 & a_4 \\ a_2 & 0 & -a_4 & -a_3 \\ a_3 & a_4 & 0 & -a_2 \\ -a_4 & a_3 & a_2 & 0 \end{pmatrix}.$$

The normalization $\mathrm{Tr}(L) = 1$ fixes $a_1 = \frac{1}{4}$, but the other parameters are free and give inequivalent representations of the same Lie algebra, $\mathfrak{h} \simeq \mathbb{R} \oplus su(2)$. The particular choice $a_2 = a_3 = a_4 = 0$ gives exactly the action of quaternions on themselves on the right (at the group level the group is $H = \mathbb{R}_+ \cdot SU(2) = \mathbb{R}_+ \cdot Spin(3)$.
The determinant of the action is

$$\Delta = \frac{1}{4}\left({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2\right)^2$$

Proposition 1.4.1 The closed connected subgroups of $GL(4,\mathbb{R})$ which act with open–free orbits on a four dimensional space and whose subalgebra $\mathcal{F}$ is simple, fall –up to conjugacy–, among the connected groups listed below.

Irreducible The group is $\mathbb{R}_+ \cdot SL(2,\mathbb{R})$ where the representation of $SL(2,\mathbb{R})$ is the spin 3/2 irreducible one and $\mathbb{R}_+$ acts by dilations.

$$\Delta = -9\, x_1 x_2 x_4 x_3 + 4\, x_1 {x_3}^3 + \frac{9}{2} {x_1}^2 {x_4}^2 - \frac{3}{2} {x_3}^2 {x_2}^2 + 3\, x_4 {x_2}^3 .$$

Reducible 1 The abstract group is $R_+ \cdot SL(2,\mathbb{R})$ and the vector space has the structure of $\mathbb{R}^2 \otimes \mathbb{R}^2$ where $SL(2,\mathbb{R})$ acts reducibly and $\mathbb{R}_+$ is one of the following three subgroups depending on the real parameter $\mu \in \mathbb{R}$.

$$SL(2,\mathbb{R}) \longrightarrow \left\{ \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix} \;;\; ad - bc = 1 \right\} \tag{1.9}$$

(a) $\mathbb{R}_+^\times \rightarrow \left\{ \mathrm{diag}\left(w^\mu, w^\mu, w^{\frac{1}{2}-\mu}, w^{\frac{1}{2}-\mu} \right) \;;\; w \in \mathbb{R}_+^\times \right\}$

(b) $$\mathbb{R}_+^\times \rightarrow \left\{ \begin{pmatrix} w & 0 & \ln(w^4)w & 0 \\ 0 & w & 0 & \ln(w^4)w \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w \end{pmatrix} ; w \in \mathbb{R}_+^\times \right\} .$$

(c) $$\mathbb{R}_+^\times \rightarrow \left\{ \begin{pmatrix} w\cos\ln(w^{4\mu}) & 0 & w\sin\ln(w^{4\mu}) & 0 \\ 0 & w\cos\ln(w^{4\mu}) & 0 & w\sin\ln(w^{4\mu}) \\ -w\sin\ln(w^{4\mu}) & 0 & w\cos\ln(w^{4\mu}) & 0 \\ 0 & -w\sin\ln(w^{4\mu}) & 0 & w\cos\ln(w^{4\mu}) \end{pmatrix} ; w \in \mathbb{R}_+^\times \right\}$$

$$\Delta := \frac{1}{2} \left(x_1 x_4 - x_2 x_3 \right)^2 \tag{1.10}$$

Irreducible over $\mathbb{R}$ The group is $\mathbb{R}_+^\times \cdot SU(2)$ where $SU(2)$ acts in the real representation given by unit quaternions on themselves from the left and $\mathbb{R}_+^\times$ is given by exponentiation of the generator

$$L = \frac{1}{4}\mathbf{1} + \begin{pmatrix} 0 & a_2 & a_3 & -a_4 \\ -a_2 & 0 & a_4 & a_3 \\ -a_3 & -a_4 & 0 & a_2 \\ a_4 & -a_3 & -a_2 & 0 \end{pmatrix} ,$$

for any real a_1, a_2, a_3. The determinant is given by

$$\Delta := \frac{1}{4} \left({x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 \right)^2 .$$

Chapter 2

Some mathematical preliminaries

We begin with some basic properties of semi-direct product groups of the type mentioned above and in particular take a closer look at their non-trivial orbits,i.e. orbits of maximal dimension. Let $G = \mathbb{R}^n \rtimes H$ be the semidirect product group with elements $g = (\vec{b}, \mathbf{h})$, $\vec{b} \in \mathbb{R}^n$, and $\mathbf{h} \in H$ and the multiplication law :

$$(\vec{b}_1, \mathbf{h}_1)(\vec{b}_2, \mathbf{h}_2) = (\vec{b}_1 + \mathbf{h}_1\vec{b}_2, \mathbf{h}_1\mathbf{h}_2) \tag{2.1}$$

Here H is assumed to be a closed subgroup of $GL(n, \mathbb{R})$ and, as mentioned earlier, we will consider only the case where H is an n-dimensional subgroup of $GL(n, \mathbb{R})$ such that there exists at least one open free orbit, $\widehat{\mathcal{O}}_{\vec{k}^T} = \{\vec{k}^T\mathbf{h} \mid \mathbf{h} \in H\}$, for some $\vec{k}^T$ in $\widehat{\mathbb{R}}^n$(the dual of $\mathbb{R}^n$). An element $g \in G$ can be written in matrix form as

$$g = \begin{pmatrix} \mathbf{h} & \vec{b} \\ \vec{0}^T & 1 \end{pmatrix},$$

where $\mathbf{h} \in H$, $\vec{b} \in \mathbb{R}^n$ and $\vec{0}^T$ is the zero vector in $\widehat{\mathbb{R}}^n$. The inverse element is:

$$g^{-1} = \begin{pmatrix} \mathbf{h}^{-1} & -\mathbf{h}^{-1}\vec{b} \\ \vec{0}^T & 1 \end{pmatrix}.$$

Note that h is an $n \times n$ matrix with non-zero determinant, which acts on $\vec{x} \in \mathbb{R}^n$ from the left in the usual way, $\vec{x} \mapsto \mathbf{h}\vec{x}$, and similarly, it acts on $\vec{x}^T \in \widehat{\mathbb{R}}^n$ from the right, $\vec{x}^T \mapsto \vec{x}^T\mathbf{h}$.

The left invariant Haar measure, $d\mu_G$, of G is

$$d\mu_G(\vec{b}, \mathbf{h}) = \frac{1}{|\det \mathbf{h}|}\, d\vec{b}\, d\mu_H(\mathbf{h}), \tag{2.2}$$

$d\vec{b}$ being the Lebesgue measure on $\mathbb{R}^n$ and $d\mu_H$ the left invariant Haar measure of H. While it is the left invariant measure that we shall consistently use, it is nonetheless worthwhile to write down at this point the right invariant Haar measure $d\mu_r$ as well, in terms of the left Haar measure and the modular functions $\Delta_G,\ \Delta_H$, of the groups G and H, respectively:

$$d\mu_G(\vec{b}, \mathbf{h}) = \Delta_G(\vec{b}, \mathbf{h})\, d\mu_r(\vec{b}, \mathbf{h}) = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|}\, d\mu_r(\vec{b}, \mathbf{h}). \tag{2.3}$$

Let $\mathfrak{g} = Lie(G)$ be the Lie algebra of G and $\{L^1, L^2, ..., L^{2n}\}$ a basis of it. We choose this basis in a way such that the first n elements, $\{L^1, L^2, ..., L^n\}$, are $n \times n$ matrices, forming a basis in $\mathfrak{h} = Lie(H)$, and the last n elements, $\{L^{n+1}, L^{n+2}, ..., L^{2n}\}$, which are the generators of translations, form a basis in $\mathbb{R}^n$. An element $X \in \mathfrak{g}$ can be written in matrix form as:

$$X = x_1 L^1 + x_2 L^2 + ... + x_{2n} L^{2n} = \begin{pmatrix} X_q & \vec{x}_p \\ \vec{0}^T & 0 \end{pmatrix} \tag{2.4}$$

where X_q is an $n \times n$ matrix with entries depending on $x_i, \quad i = 1...n$, and $\vec{x}_p$ is a column vector with components $x_{n+1}, x_{n+2}, \ldots, x_{2n}$. Also it will be useful to introduce the vector $\vec{x}_q$, with components $x_i,\ \ i = 1...n$, and the vector of matrices $\mathfrak{X} = (L^1, L^2, \ldots, L^n)$. Next, for any $\vec{u} \in \mathbb{R}^n$, we define the matrix $[\mathfrak{X}\vec{u}]$ whose columns are the vectors $L^i\vec{u},\ \ i = 1, 2, \ldots n$,

$$[\mathfrak{X}\vec{u}] = [L^1\vec{u},\ L^2\vec{u},\ \ldots,\ L^n\vec{u}]. \tag{2.5}$$

The adjoint action of the group on its Lie algebra,as introduced in the previous chapter, is given by

$$X \mapsto \mathrm{Ad}_g X := gXg^{-1} = \begin{pmatrix} \mathbf{h}X_q\mathbf{h}^{-1} & -\mathbf{h}X_q\mathbf{h}^{-1}\vec{b} + \mathbf{h}\vec{x}_p \\ \vec{0}^T & 0 \end{pmatrix}. \tag{2.6}$$

Introducing the matrix $M(\mathbf{h})$ such that,

$$\mathbf{h}L^k\mathbf{h}^{-1} = \sum_{i=1}^{n} L^i M(\mathbf{h})^k_i, \tag{2.7}$$

the adjoint action of an element $g = (\vec{b}, \mathbf{h}) \in G$ may conveniently be written in terms of its action on the $2n$-dimensional vector $\begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix}$ as:

$$\begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix} \mapsto \begin{pmatrix} \vec{x}_q{}' \\ \vec{x}_p{}' \end{pmatrix} = M(\vec{b}, \mathbf{h}) \begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix}, \tag{2.8}$$

where $M(\vec{b}, \mathbf{h})$ is the $2n \times 2n$-matrix

$$M(\vec{b}, \mathbf{h}) = \begin{pmatrix} M(\mathbf{h}) & \mathbb{O}_n \\ -[\mathfrak{X}\vec{b}]M(\mathbf{h}) & \mathbf{h} \end{pmatrix}, \tag{2.9}$$

$\mathbb{O}_n$ being the $n \times n$ null matrix. Note that $M(\mathbf{h})$ is just the matrix of the adjoint action of $\mathbf{h}$ on $\mathfrak{h}$ (the Lie algebra of H) computed with respect to the basis $\{L^1, L^2, \ldots, L^n\}$. Similarly, $M(\vec{b}, \mathbf{h})$ is the matrix of the adjoint action of $g = (\vec{b}, \mathbf{h})$ on $\mathfrak{g}$, the Lie algebra of G. By abuse of notation, we shall also write,

$$\mathrm{Ad}_h\, X_q = M(\mathbf{h})\vec{x}_q, \qquad \mathrm{Ad}_g\, X = M(\vec{b}, \mathbf{h}) \begin{pmatrix} \vec{x}_q \\ \vec{x}_p \end{pmatrix}. \tag{2.10}$$

The coadjoint action of G on $\mathfrak{g}^*$, the dual space of its Lie algebra, can now be immediately read off from (2.9). Indeed, let $\{L^*_i\}_{i=1}^{2n}$ be the basis of $\mathfrak{g}^*$ which is dual to the basis $\{L^i\}_{i=1}^{2n}$ of $\mathfrak{g}$, i.e.,

$$\langle L^*_i \,;\, L^j \rangle = \delta^j_i, \qquad i, j = 1, 2, \ldots, 2n.$$

A general element $X^* \in \mathfrak{g}^*$ then has the form,

$$X^* = \sum_{i=1}^{n} \gamma^i L^*_i, \qquad \gamma^i \in \mathbb{R}$$

and again we introduce the row vectors,

$$\vec{\gamma}^T = (\gamma^1, \gamma^2, \ldots, \gamma^{2n}), \quad \vec{\gamma}_q^T = (\gamma^1, \gamma^2, \ldots, \gamma^n), \quad \vec{\gamma}_p^T = (\gamma^{n+1}, \gamma^{n+2}, \ldots, \gamma^{2n}).$$

Using the relation

$$\langle \mathrm{Ad}^\sharp_g X^* \,;\, X\rangle = \langle X^* \,;\, \mathrm{Ad}_{g^{-1}} X\rangle,$$

we easily obtain from (2.9),

$$\mathrm{Ad}^\sharp_{(\vec{b},\mathbf{h})} X^* = (\vec{\gamma}_q^{\,T},\ \vec{\gamma}_p^{\,T}) M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) = (\vec{\gamma}_q^{\,T},\ \vec{\gamma}_p^{\,T}) \begin{pmatrix} M(\mathbf{h}^{-1}) & \mathbb{O}_n \\ \mathbf{h}^{-1}[\mathfrak{X}\vec{b}] & \mathbf{h}^{-1} \end{pmatrix}. \tag{2.11}$$

Thus, under the coadjoint action, a vector $(\vec{\gamma}_q^{\,T},\ \vec{\gamma}_p^{\,T})$ changes to:

$$\begin{aligned} \vec{\gamma}_q^{\,\prime T} &= \vec{\gamma}_q^{\,T} M(\mathbf{h}^{-1}) + \vec{\gamma}_p^{\,T}\mathbf{h}^{-1}[\mathfrak{X}\vec{b}], \\ \vec{\gamma}_p^{\,\prime T} &= \vec{\gamma}_p^{\,T}\mathbf{h}^{-1}. \end{aligned} \tag{2.12}$$

Let us also note that the modular functions appearing in (2.3) can be written [19] in terms of the coadjoint operators as:

$$\Delta_G(\vec{b}, \mathbf{h}) = |\det\ \mathrm{Ad}^\sharp_{(\vec{b},\mathbf{h})}| = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|}, \qquad \Delta_H(\mathbf{h}) = |\det\ \mathrm{Ad}^\sharp_{\mathbf{h}}| = \frac{1}{|\det M(\mathbf{h})|}. \tag{2.13}$$

Before leaving this section we make a further important assumption on the nature of the group G. We require that the range in G of the exponential map be a dense set whose complement has Haar measure zero. (This includes, for example, groups of exponential type.) Thus, by exponentiating (2.4), we may write any element (up to a set of measure zero) of G as

$$g = e^X = \begin{pmatrix} e^{X_q} & e^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2}\vec{x}_p \\ \vec{0}^T & 1 \end{pmatrix}, \qquad X \in N, \tag{2.14}$$

where $N \in \mathbf{g}$ is the domain of the exponential map, which contains the origin and has the property that if $X \in N$ then $-X \in N$. The $n \times n$ matrix $\operatorname{sinch} A$ is defined as the sum of an infinite series:

$$\operatorname{sinch} A = \mathbb{I}_n + \frac{1}{3!}A^2 + \frac{1}{5!}A^4 + \frac{1}{7!}A^6 + \ldots \tag{2.15}$$

$\mathbb{I}_n$ being the $n \times n$ unit matrix. When the matrix A has an inverse, $\operatorname{sinch} A$ can also be formally written as:

$$\operatorname{sinch} A = \frac{e^A - e^{-A}}{2A} = A^{-1} \sinh A. \tag{2.16}$$

It will also be useful to introduce the matrix valued functions,

$$\begin{aligned} F(X_q) &= e^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2} \\ &= \mathbb{I}_n + \frac{X_q}{2!} + \frac{X_q^2}{3!} + \frac{X_q^3}{4!} + \dots \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} F(-X_q)^{-1} &= e^{X_q} F(X_q)^{-1} = \frac{e^{\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}} \\ &= \mathbb{I}_n + \frac{X_q}{2} + \sum_{k \geq 1} (-1)^{k-1} \frac{B_k X_q^k}{(2k)!}, \end{aligned} \tag{2.18}$$

where the B_k are the Bernoulli numbers, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, etc., and generally,

$$B_k = \frac{(2k)!}{\pi^{2k}\, 2^{2k-1}} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} .$$

Later we shall need to express the Haar measure $d\mu_G$ in terms of the coordinates of the Lie algebra, using the exponential map. Writing

$$d\mu_G(e^X) = m_G(\vec{x}_q, \vec{x}_p)\, d\vec{x}_q\, d\vec{x}_p, \tag{2.19}$$

the density function $m(\vec{x}_q, \vec{x}_p)$ is easily calculated, using (2.14). Indeed, from (2.2),

$$\begin{aligned} d\mu_G(e^X) &= d\mu_G(e^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2} \vec{x}_p, e^{\frac{X_q}{2}}) \\ &= \frac{1}{|\det e^{X_q}|} \,|\det(e^{\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2})|\, d\mu_H(e^{X_q})\, d\vec{x}_p \\ &= |\det(e^{\frac{-X_q}{2}} \operatorname{sinch} \frac{X_q}{2})|\, d\mu_H(e^{X_q})\, d\vec{x}_p \end{aligned} \tag{2.20}$$

It is also possible to write $\mathrm{d}\mu_H(e_q^X)$ in terms of the Lebesgue measure, $\mathrm{d}\vec{x}_q = \mathrm{d}x_1\mathrm{d}x_2...\mathrm{d}x_n$, times some density function m_H [20],

$$\begin{aligned}\mathrm{d}\mu_H(e^{X_q}) &= m_H(\vec{x}_q)\, d\vec{x}_q \\ &= \left|\det \frac{1-e^{-adX_q}}{adX_q}\right| \,\mathrm{d}\vec{x}_q = \left|\det(e^{-ad\frac{X_q}{2}}\,\mathrm{sinch}\left(ad\frac{X_q}{2}\right))\right| \,\mathrm{d}\vec{x}_q \quad (2.21)\end{aligned}$$

where adX is the linear map on $\mathbf{g}$ which is the infinitesimal generator of the adjoint action $\mathrm{Ad}_g,\ \ g \in G$:

$$adX(L) = [X, L], \quad \text{and} \quad \mathrm{Ad}_g = \mathrm{Ad}_{e^X} = e^{adX}. \tag{2.22}$$

Finally, the left Haar measure on G takes the form :

$$\mathrm{d}\mu_G(e^X) = \left|\det(e^{\frac{-X_q}{2}}\,\mathrm{sinch}\left(\frac{X_q}{2}\right))\ \det(e^{-ad\frac{X_q}{2}}\,\mathrm{sinch}\left(ad\frac{X_q}{2}\right))\right|\ \mathrm{d}\vec{x}_q\,\mathrm{d}\vec{x}_p, \tag{2.23}$$

and the density function $m_G(\vec{x})$ appearing in (2.19) is:

$$\begin{aligned}m_G(\vec{x}) &= |\det(e^{-\frac{X_q}{2}}\,\mathrm{sinch}\left(\frac{X_q}{2}\right))\ \det(e^{-ad\frac{X_q}{2}}\,\mathrm{sinch}\left(ad\frac{X_q}{2}\right))| \\ &= |\det F(-X_q)\ \det F(-adX_q)| \end{aligned}\tag{2.24}$$

2.1 Orbits and invariant measures

It is now possible to determine the non-trivial coadjoint orbits of G, which will be the main focus of our attention. These are orbits of fixed vectors in $\mathbf{g}^*$ under the coadjoint action (2.12). Consider first the vector $(\vec{0}^T, \vec{k}^T) \in \mathbb{R}^{2n},\ \ \vec{k} \neq \vec{0}$ and let $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}$ be its orbit under the coadjoint action, i.e.,

$$\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)} = \{(\vec{\gamma}_q^T, \vec{\gamma}_p^T) = (\vec{0}^T, \vec{k}^T)M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) \mid (\vec{b}, \mathbf{h}) \in \mathbb{R}^n \rtimes H\}. \tag{2.25}$$

Then, from (2.12),

$$\begin{aligned}\vec{\gamma}_q^T &= \vec{k}^T\mathbf{h}^{-1}[\mathfrak{X}\vec{b}] = \vec{\gamma}_p^T[\mathfrak{X}\vec{b}], \\ \vec{\gamma}_p^T &= \vec{k}^T\mathbf{h}^{-1}.\end{aligned}\tag{2.26}$$

The vectors $\vec{\gamma}_p^T$ generate the orbit $\widehat{\mathcal{O}}_{\vec{k}^T}$ of the subgroup H in $\widehat{\mathbb{R}}$. We now show that, for any $\vec{\gamma}_p^T$, the vector $\vec{\gamma}_q^T = \vec{\gamma}_p^T[\mathfrak{X}\vec{b}]$ is an element of the cotangent space of $\widehat{\mathcal{O}}_{\vec{k}^T}$ at this point. Indeed, for any $i = 1, 2, \ldots, n$, consider a curve, $\vec{u}^i(t)^T$ in $\widehat{\mathcal{O}}_{\vec{k}^T}$ of the type,

$$\vec{u}^i(t)^T = \vec{\gamma}_p^T e^{L^i t}, \qquad t \in [-\epsilon, \epsilon] \subset \mathbb{R}. \tag{2.27}$$

Then, $\vec{u}^i(0)^T = \vec{\gamma}_p^T$, and

$$\frac{d\vec{u}^i(t)^T}{dt}\bigg|_{t=0} = \vec{\gamma}_p^T L^i := \vec{t}_p^{\,i\,T}, \tag{2.28}$$

is a vector tangent to $\widehat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Recall that we are assuming that the action of H on $\widehat{\mathbb{R}}$ is open free. Hence the stability subgroup of the vector $\vec{k}^T$ under the action $\vec{k}^T \mapsto \vec{k}^T\mathbf{h}^{-1}$ is just the unit element of H and the orbit $\widehat{\mathcal{O}}_{\vec{k}^T}$ is an open set of $\widehat{\mathbb{R}}^n$, consequently of dimension n. This implies that the vectors $\vec{t}_p^{\,i\,T}$ are non-zero and linearly independent and hence form a basis for the tangent space $T_{\vec{\gamma}_p^T}\widehat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Let $\vec{t}_p^{\,i\,T} = (t^{i1}, t^{i2}, \ldots, t^{in})$, in components, and define the matrix

$$\Theta(\vec{\gamma}_p^T) = [t^{ij}] = [\vec{\theta}_1, \vec{\theta}_2, \ldots, \vec{\theta}_n], \tag{2.29}$$

where the vectors $\vec{\theta}_i$ are its columns:

$$\vec{\theta}_i^T = (t^{1i}, t^{2i}, \ldots t^{ni}), \qquad i = 1, 2, \ldots, n. \tag{2.30}$$

The vectors $\vec{\theta}_i$ form a basis for the cotangent space $T^*_{\vec{\gamma}_p^T}\widehat{\mathcal{O}}_{\vec{k}^T}$ of $\widehat{\mathcal{O}}_{\vec{k}^T}$ at $\vec{\gamma}_p^T$. Thus, if b^i are the components of the vector $\vec{b}$,

$$\vec{\gamma}_p^T[\mathfrak{X}\vec{b}] = \sum_{i=1}^{n} b^i\vec{\theta}_i^T, \tag{2.31}$$

implying that $[\vec{\gamma}_p^T[\mathfrak{X}\vec{b}]]^T = [\mathfrak{X}\vec{b}]^T\vec{\gamma}_p$ is just a cotangent vector at $\vec{\gamma}_p^T$. Letting $\vec{b}$ run through all of $\mathbb{R}^n$, these vectors generate the whole cotangent space at $\vec{\gamma}_p^T$. Thus,

$$\mathcal{O}^*_{(\vec{0}^T, \vec{k}^T)} = T^*\widehat{\mathcal{O}}_{\vec{k}^T}, \tag{2.32}$$

and if $\vec{k}^T$ is a vector such that the orbit $\widehat{\mathcal{O}}_{\vec{k}^T}$ is open and free, the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}^T)}$ has dimension $2n$. It is known [9] that if one such open free orbit exists, then there exists a finite discrete set of them, corresponding to vectors $\vec{k}_j^T$, $j = 1, 2, \ldots, N < \infty$, for which $\cup_{j=1}^N \widehat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in $\widehat{\mathbb{R}}^n$.

To see this it is sufficient to consider the function Δ defined in Eq. (1.1): the union of lower dimensional orbits coincides with the zero set of Δ, which is a subvariety of dimension $\leq n-1$ (remember that Δ is a homogeneous polynomial of degree n) and hence the union of open orbits is dense. The number of different open orbits is then equal to the number of connected components of $\widehat{\mathbb{R}}^n$ where Δ never vanishes and it is finite as a simple compactness argument shows on the projective space $\widehat{\mathbb{R}}^n/\mathbb{R}_+ \simeq S^{n-1}$.

Similarly, let us compute the coadjoint orbit of a vector $(\vec{x}^T, \vec{0}^T) \in \widehat{\mathbb{R}}^{2n}$. As before,

$$\mathcal{O}^*_{(\vec{x}^T,\vec{0}^T)} = \{(\vec{\gamma}_q^T, \vec{\gamma}_p^T) = (\vec{x}^T, \vec{0}^T) M(-\mathbf{h}^{-1}\vec{b}, \mathbf{h}^{-1}) \mid (\vec{b}, \mathbf{h}) \in \mathbb{R}^n \rtimes H\}, \qquad (2.33)$$

and again from (2.12),

$$\begin{aligned} \vec{\gamma}_q^T &= \vec{x}^T M(\mathbf{h}^{-1}), \\ \vec{\gamma}_p^T &= \vec{0}^T. \end{aligned} \qquad (2.34)$$

and these orbits all have dimension lower than $2n$. From the point of view of representation theory, these are the trivial orbits.

Using the coordinates γ^i to identify $\mathbf{g}^*$ with $\widehat{\mathbb{R}}^{2n}$, we arrive at the result:

Theorem 2.1.1 *If the action of H on $\widehat{\mathbb{R}}^n$ is open free, the set of non-trivial coadjoint orbits in $\mathbf{g}^*$ is finite and discrete and their union is dense in $\mathbf{g}^*$. Moreover, each nontrivial coadjoint orbit, $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, is the cotangent bundle, $T^*\widehat{\mathcal{O}}_{\vec{k}_j^T}$, of an open free* *orbit, $\widehat{\mathcal{O}}_{\vec{k}_j^T} \subset \widehat{\mathbb{R}}^n$, of a vector $\vec{k}_j^T \in \widehat{\mathbb{R}}^n$ under the action of H. Under the coadjoint action of $G = \mathbb{R}^n \rtimes H$ the dual space of its Lie algebra decomposes as*

$$\mathfrak{g}^* \;\simeq\; \widehat{\mathbb{R}}^{2n} \;=\; \left[\cup_{j=1}^N \mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}\right] \cup V \;=\; \left[\cup_{j=1}^N T^*\widehat{\mathcal{O}}_{\vec{k}_j^T}\right] \cup V, \tag{2.35}$$

where V is a set consisting of lower (than $2n$) dimensional orbits and therefore of Lebesgue measure zero in $\widehat{\mathbb{R}}^{2n}$.

The orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, being *homogeneous symplectic manifolds* [21], carry invariant measures under the coadjoint action (2.11 - 2.12). Indeed, if $d\vec{\gamma}^T$ denotes the Lebesgue measure $d\gamma^1\ d\gamma^2\ \ldots\ d\gamma^{2n}$, restricted to the orbit $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$, then using (2.12) and (2.13) it is easy to check that under the coadjoint action it transforms as,

$$d\vec{\gamma}'^T = \Delta_G(\vec{b}, \mathbf{h}) d\vec{\gamma}^T. \tag{2.36}$$

On the other hand, the mapping $\kappa_j : \widehat{\mathcal{O}}_{\vec{k}_j^T} \to H$,

$$\kappa_j(\vec{\gamma}_p^T) = \mathbf{h}, \quad \text{where} \quad \vec{\gamma}_p^T = \vec{k}_j^T \mathbf{h}^{-1}, \tag{2.37}$$

is a homeomorphism. Thus, it follows that the measure

$$\mathrm{d}\Omega_j(\vec{\gamma}_q^T, \vec{\gamma}_p^T) = \sigma_j(\vec{\gamma}^T)^{-1}\, \mathrm{d}\vec{\gamma}^T, \qquad \sigma_j(\vec{\gamma}^T) = \frac{\Delta_H[\kappa_j(\vec{\gamma}_p^T)]}{|\det\,[\kappa_j(\vec{\gamma}_p^T)]|}, \tag{2.38}$$

is invariant on $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ under the coadjoint action.

Note, finally, that each one of the orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ is homeomorphic to the group G itself. Indeed, using (2.26), (2.31) and (2.37) let us define a map,

$$\widetilde{\kappa}_j : \mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)} \longrightarrow \mathbb{R}^n \rtimes H, \qquad \widetilde{\kappa}_j(\vec{\gamma}_q^T,\ \vec{\gamma}_p^T) = (\vec{b}, \mathbf{h}) = (\Theta(\vec{\gamma}_p^T)^{-1}\vec{\gamma}_q,\ \kappa_j(\vec{\gamma}_p^T)), \tag{2.39}$$

where $\Theta(\vec{\gamma}_p^T)$ is the matrix of tangent vectors defined in (2.29). Then, $\widetilde{\kappa}_j$ is a homeomorphism and it is straightforward to verify that

$$\widetilde{\kappa}_j \circ \mathrm{Ad}^\sharp_{g_0} = L_{g_0} \circ \widetilde{\kappa}_j, \tag{2.40}$$

where $L_{g_0}(g) = g_0 g, \ \ g \in G$. More explicitly, if $(\vec{\gamma}_q^T, \vec{\gamma}_p^T) \mapsto g = (\vec{b}, \mathbf{h})$ under $\widetilde{\kappa}_j$, then

$$\mathrm{Ad}^\sharp_{g_0}(\vec{\gamma}_q^T, \vec{\gamma}_p^T) \longmapsto (\vec{b}_0, \mathbf{h}_0)(\vec{b}, \mathbf{h}) = (\vec{b}_0 + \mathbf{h}\vec{b}, \mathbf{h}_0\mathbf{h}). \tag{2.41}$$

In other words, the homeomorphism $\widetilde{\kappa}_j$, from the coadjoint orbit $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ to the group $\mathbb{R}^n \rtimes H$, intertwines the coadjoint action on the orbit with the left action on the group and furthermore, under this homeomorphism the invariant measure $\mathrm{d}\Omega_j$ on the orbit transforms to the left Haar measure $\mathrm{d}\mu_G$ on the group.

Before leaving this section, we describe a second, in a way more intrinsic, method for arriving at the invariant measure (2.38), using the fact that the orbits $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ are symplectic manifolds and thus carry G-invariant two-forms [21] which can be computed using the structure constants of the group. As before, $\{L^i\}_{i=1}^{2n}$ will be a basis for the Lie algebra $\mathbf{g}$ and $\{L^*_i\}_{i=1}^{2n}$ the dual basis of $\mathbf{g}^*$. The Lie algebra of the group G is determined by the commutation relations,

$$[L^i \ , \ L^j] = \sum_{k=1}^{2n} c_k^{ij} \ L^k, \tag{2.42}$$

where the c_k^{ij} are the structure constants. Thus, in this basis, the linear maps adL^i have the matrix elements $[adL^i]^j_k = c^{ij}_k$. Let $X^* = \sum_{i=1}^{2n} \gamma^i L^*_i \in \mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)} \subset \mathbf{g}^*$ and let us define a matrix $\Theta(\vec{\gamma}^T)$ at this point by

$$[\Theta(\vec{\gamma}^T)]^{ij} = \sum_{k=1}^{n} [adL^i]^j_k \ \gamma^k = \sum_{k=1}^{n} c^{ij}_k \ \gamma^k. \tag{2.43}$$

Using $\Theta(\vec{\gamma}^T)$ matrix, we now identify the Lie algebra $\mathbf{g}$ with the tangent space, $T_{X^*}\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$, to the orbit $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ at the point X^*. (Note that this tangent space is naturally isomorphic to $\mathbf{g}^*$ itself). Since the orbit $\widehat{\mathcal{O}}_{\vec{k}_j^T}$ is open free, it has dimension n and its cotangent bundle, i.e., the orbit $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$, has dimension $2n$. Thus, $T_{X^*}\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ has dimension $2n$ and in it we shall use the standard basis $\{\frac{\partial}{\partial \gamma^i}\}_{i=1}^{2n}$. Similarly, we shall use the dual basis $\{\mathrm{d}\gamma^i\}_{i=1}^{2n}$ for the cotangent space, $T^*_{X^*}\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$.

From (2.43) we see that for $i = 1, 2, \ldots, 2n$, the vectors $\sum_{j=1}^{2n}[\Theta(\vec{\gamma}^T)]^{ij}\frac{\partial}{\partial\gamma^j}$ form a linearly independent set of tangent vectors to the orbit $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ at the point X^* (under the coadjoint action). Thus, for $X = \sum_{i=1}^{2n} x_i L^i \in \mathbf{g}$, it follows that $\sum_{j=1}^{n}[\Theta(\vec{\gamma}^T)\vec{x}]^j \frac{\partial}{\partial\gamma^j}$ defines a vector in $T_{X^*}\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ and hence we have the identification map, $\phi_{X^*} : \mathbf{g} \longrightarrow T_{X^*}\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$,

$$\phi_{X^*}(X) = \sum_{i,j,k=1}^{2n} c_k^{ij}\, x_i\, \gamma^k\, \frac{\partial}{\partial\gamma^j} = \sum_{i,j=1}^{2n}[\Theta(\vec{\gamma}^T)]^{ij}\, x_i\, \frac{\partial}{\partial\gamma^j}, \tag{2.44}$$

as an isomorphism of vector spaces. The G-invariant 2-form (symplectic form) is then defined as:

$$\omega_{X^*}(\phi_{X^*}(X), \phi_{X^*}(L)) = < X^*; [X, L] >, \tag{2.45}$$

which using (2.42) and (2.44) can be expressed in the form,

$$\omega_{X^*} = \sum_{i,j=1}^{2n}[\omega_{X^*}]_{ij}\, \mathrm{d}\gamma^i \wedge \mathrm{d}\gamma^j = \sum_{i,j=1}^{2n}[\Theta(\vec{\gamma}^T)]_{ij}\, \mathrm{d}\gamma^j \wedge \mathrm{d}\gamma^i, \tag{2.46}$$

where $[\Theta(\vec{\gamma}^T)]_{ij}$ are the elements of the inverse matrix $[\Theta(\vec{\gamma}^T)]^{-1}$. From this the G-invariant measure on the orbit $\mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)}$ is computed to be

$$\mathrm{d}\Omega_j(\vec{\gamma}^T) = \lambda(\det[\omega_{X^*}])^{\frac{1}{2}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2\, \ldots\, \mathrm{d}\gamma^{2n} = \frac{\lambda}{(\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2\, \ldots\, \mathrm{d}\gamma^{2n},$$

where λ is a constant. By multiplying the basis vectors L^i by appropriate constants, λ can be made equal to one. We shall assume that this has been done and then write,

$$\mathrm{d}\Omega_j(\vec{\gamma}^T) = (\det[\omega_{X^*}])^{\frac{1}{2}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2\, \ldots\, \mathrm{d}\gamma^{2n} = \frac{1}{(\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}}\, \mathrm{d}\gamma^1\, \mathrm{d}\gamma^2\, \ldots\, \mathrm{d}\gamma^{2n}. \tag{2.47}$$

Comparing with (2.38) we find,

$$\sigma_j(\vec{\gamma}^T) = (\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}}. \tag{2.48}$$

2.2 Representations of G

In order to construct Wigner functions for the group $G = \mathbb{R}^n \rtimes H$ we shall use its quasi-regular representation. This representation acts via the unitary operators $U(\vec{b}, \mathbf{h})$ on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^n, d\vec{x})$:

$$(U(\vec{b}, \mathbf{h})f)(\vec{x}) = |\det \mathbf{h}|^{-\frac{1}{2}} f(\mathbf{h}^{-1}(\vec{x} - \vec{b})) \qquad f \in \mathfrak{H}. \tag{2.49}$$

This representation is in general not irreducible, but is always multiplicity free. Moreover, the existence of open free orbits implies that every non-trivial irreducible sub-representation of G is contained in U and each such representation is *square integrable* [9] in a sense to be made precise presently.

In order to obtain the irreducible sub-representations of U, it is useful to look at the unitarily equivalent representation $\widehat{U}(\vec{b}, \mathbf{h}) = \mathcal{F} U(\vec{b}, \mathbf{h}) \mathcal{F}^{-1}$, where $\mathcal{F} : L^2(\mathbb{R}^n, d\vec{y}) \to L^2(\widehat{\mathbb{R}}^n, d\vec{k}^T)$ is the Fourier transform operator:

$$(\mathcal{F}f)(\vec{k}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\vec{k}^T \vec{x}} f(\vec{x})\, d\vec{x}.$$

The action of $\widehat{U}(\vec{b}, \mathbf{h})$ on a vector $\widehat{f} \in \widehat{\mathfrak{H}} = L^2(\widehat{\mathbb{R}}^n, d\vec{k}^T)$ is easily seen to have the form,

$$(\widehat{U}(\vec{b}, \mathbf{h})\widehat{f})(\vec{k}^T) = |\det \mathbf{h}|^{\frac{1}{2}}\, e^{i\vec{k}^T \vec{b}}\, \widehat{f}(\vec{k}^T \mathbf{h}). \tag{2.50}$$

We shall also need the form of this representation, written in terms of Lie algebra variables, using the exponential map (2.14):

$$(\widehat{U}(e^X)\widehat{f})(\vec{k}^T) = |\det e^{-X_q}|^{\frac{1}{2}}\, e^{(i\vec{k}^T F(-X_q)\vec{x}_p)}\, \widehat{f}(\vec{k}^T e^{-X_q}). \tag{2.51}$$

Let $\vec{k}_j^T \in \widehat{\mathbb{R}}^n, \;\; j = 1, 2, \ldots, N$, be a maximal set of vectors whose orbits $\widehat{\mathcal{O}}_{\vec{k}_j^T}$ under H are open free and mutually disjoint. Then by Theorem 2.1.1, $\cup_{j=1}^N \widehat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in $\widehat{\mathbb{R}}^n$ and $\cup_{j=1}^N T^* \widehat{\mathcal{O}}_{\vec{k}_j^T}$ is dense in the dual, $\mathfrak{g}^*$, of the Lie algebra of G. Set

$\widehat{\mathfrak{H}}_j = L^2(\widehat{\mathcal{O}}_{\vec{k}_j^T}, \mathrm{d}\vec{k}^T)$ (the restriction of the Lebesgue measure to the orbit is implied). Then, it is not hard to see that each of these spaces is an invariant subspace for $\widehat{U}$. Moreover, the restriction $\widehat{U}_j$, of $\widehat{U}$ to $\widehat{\mathfrak{H}}_j$, is irreducible [9], and is in fact the representation of $\mathbb{R}^n \rtimes H$ which is induced from the character $\chi_j(\vec{x}) = \exp(i\vec{k}^T\vec{x})$ of the abelian subgroup $\mathbb{R}^n$. Thus,

$$\widehat{\mathfrak{H}} = \oplus_{j=1}^N \widehat{\mathfrak{H}}_j, \qquad \widehat{U}(\vec{b}, \mathbf{h}) = \oplus_{j=1}^N \widehat{U}_j(\vec{b}, \mathbf{h}), \tag{2.52}$$

and it follows from Mackey's theory of induced representations [22] for semidirect product groups that these irreducible representations exhaust all nontrivial irreducible representations of G.

Recall next that the unitary dual $\widehat{G}$ of the group G is defined to be the set of all equivalence classes of unitary irreducible representations of G. In the present case we see that $\widehat{G}$ is just the discrete set $\widehat{G} = \{1, 2, \ldots, N\}$. The Plancherel measure ν_G [13] of such a group is thus a simple counting measure and the left regular representation U_ℓ of G decomposes as the direct sum

$$U_\ell(g) = \oplus_{j=1}^N U_\ell^j(g), \qquad g \in G, \quad N < \infty, \tag{2.53}$$

of sub-representations U_ℓ^j carried by the Hilbert spaces $\mathfrak{H}_\ell^j$, such that,

$$L^2(G, d\mu_G) = \oplus_{j=1}^N \mathfrak{H}_\ell^j \ , \tag{2.54}$$

$d\mu_G$ denoting the left invariant Haar measure of G. Each sub-representation $U_\ell^j(g)$ is unitarily equivalent to a direct sum, of copies of the irreducible representation $\widehat{U}_j$, with infinite multiplicity (equal to the dimension of the carrier Hilbert space, $\widehat{\mathfrak{H}}_j$, of $\widehat{U}_j$). Recall, that the left regular representation U_ℓ is a unitary representation which acts on $L^2(G, \mathrm{d}\mu_G)$ in the manner,

$$(U_\ell(g)f)(g') = f(g^{-1}g'), \qquad f \in L^2(G, \mathrm{d}\mu_G), \quad g \in G, \tag{2.55}$$

and (2.53 - 2.54) then give its Plancherel decomposition [14]. The decomposition (2.54) also implies a similar decomposition of the right regular representation, U_r, of G:

$$(U_r(g)f)(g') = \Delta_G(g)^{\frac{1}{2}} f(g'g), \qquad f \in L^2(G, \mathrm{d}\mu_G), \quad g \in G, \tag{2.56}$$

in the manner,

$$U_r(g) = \oplus_{j=1}^{N} U_r^j(g), \qquad g \in G. \tag{2.57}$$

The operators $U_\ell(g)$ and $U_r(g)$ commute with each other for all $g \in G$ and combining them one has a *double representation*, $U_D(g_1, g_2) = U_\ell(g_1)U_r(g_2)$, of the direct product group, $G \times G$ on $L^2(G, \mathrm{d}\mu_G)$:

$$(U_D(g_1, g_2)f)(g) = (U_\ell(g_1)U_r(g_2)f)(g) = \Delta_G(g_2)^{\frac{1}{2}} f(g_1^{-1} g g_2), \qquad (g_1, g_2) \in G \times G. \tag{2.58}$$

It then follows that the decomposition (2.54) also accommodates the decomposition

$$U_D(g_1, g_2) = \oplus_{j=1}^{N} U_D^j(g_1, g_2), \qquad (g_1, g_2) \in G \times G, \tag{2.59}$$

of this double representation into the *irreducible* components $U_D^j(g_1, g_2)$, carried by the subspaces $\mathfrak{H}_\ell^j$ of $L^2(G, d\mu_G)$. The diagonal part of this representation, $U_D(g, g)$, is again a representation of G on $L^2(G, d\mu_G)$ for which the $\mathfrak{H}_\ell^j$ are also invariant subspaces, and the decomposition,

$$U_D(g, g) = \oplus_{j=1}^{N} U_D^j(g, g), \quad g \in G, \tag{2.60}$$

clearly holds. The sub-representations $U_D^j(g, g)$ are not, however, irreducible. Note also that $U_D(g, e) = U_\ell(g)$, $U_D(e, g) = U_r(g)$ and if $\mathbb{P}^j$ is the projection operator, $\mathbb{P}^j L^2(G, \mathrm{d}\mu_G) = \mathfrak{H}_\ell^j$, then

$$\mathbb{P}^j U_D(g_1, g_2) = U_D(g_1, g_2)\mathbb{P}^j, \qquad (g_1, g_2) \in G \times G. \tag{2.61}$$

Next consider the Hilbert spaces, $L^2(\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}, \mathrm{d}\Omega_j)$, of functions on the coadjoint orbits, where $\mathrm{d}\Omega_j$ is the invariant measure under the coadjoint action, obtained in (2.38). These spaces carry (reducible) unitary representations $U_j^\sharp$ of G of the type,

$$(U^\sharp{}_j(g)F)(\vec{\gamma}^T) = F(\vec{\gamma}^T M(\vec{b}, \mathbf{h})), \qquad g \in G, \quad F \in L^2(\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}, \mathrm{d}\Omega_j), \tag{2.62}$$

where $M(\vec{b}, \mathbf{h})$ is the matrix of the coadjoint action $\mathrm{Ad}^\sharp_{(\vec{b},\mathbf{h})^{-1}}$, as in (2.11). Let us write,

$$\mathfrak{H}^\sharp = \oplus_{j=1}^N L^2(\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}, \mathrm{d}\Omega_j), \qquad U^\sharp(g) = \oplus_{j=1}^N U^\sharp{}_j(g). \tag{2.63}$$

We shall see later, once the Wigner transform has been constructed, that using this transform it will be possible to embed the Hilbert space $L^2(G, d\mu_G)$ isometrically *into* $\mathfrak{H}^\sharp$ in a manner which will intertwine $U^\sharp(g)$ with $U_D(g, g)$.

Note, additionally, that in view of the homeomorphism between the orbits $\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}$ and the group G, it is possible to map each $U_j^\sharp$ unitarily *onto* the left regular representation U_ℓ. Indeed, it is easy to see, by virtue of (2.39) and (2.40) that the map:

$$V_j : L^2(\mathcal{O}^*_{(\vec{0}^T,\vec{k}_j^T)}, \mathrm{d}\Omega_j) \longrightarrow L^2(G, \mathrm{d}\mu_G), \qquad (V_jF)(g) = F \circ \widetilde{\kappa}^{-1}(g), \tag{2.64}$$

is unitary and

$$V_j U_j^\sharp(\vec{b}, \mathbf{h}) V_j^{-1} = U_\ell(\vec{b}, \mathbf{h}). \tag{2.65}$$

2.3 Square-integrability of representations

The irreducible representations $\widehat{U}_j$ in (2.52) all have one other property, of importance to us here. These representations are *square-integrable* [1]. Recall that a unitary irreducible representation U of a group G on a Hilbert space $\mathfrak{H}$ is square-integrable if there exists a non-zero vector $\eta \in \mathfrak{H}$, called an *admissible vector*, such that:

$$c(\eta) = \int_G | < U(g)\eta|\eta > |^2 \mathrm{d}\mu_G(g) < \infty \tag{2.66}$$

The existence of one such vector and irreducibility of the representation imply that the set of all admissible vectors $\mathcal{A}$ is dense in $\mathfrak{H}$. If the group is unimodular then $\mathcal{A}$ coincides with $\mathfrak{H}$, otherwise it is a proper subset of it. (For a more detailed description of square-integrable representations and their properties, see e.g., [1]). For any square-integrable representation U there exists a unique positive operator C on $\mathfrak{H}$ whose domain coincides with $\mathcal{A}$ and such that if $\eta_1, \eta_2 \in \mathcal{A}$ and $\phi_1, \phi_2 \in \mathfrak{H}$ the following *orthogonality relation* holds:

$$\int_G \overline{< U(g)\eta_2|\phi_2 >} < U(g)\eta_1|\phi_1 > \mathrm{d}\mu_G \;=\; < C\eta_1|C\eta_2 >< \phi_2|\phi_1 > \tag{2.67}$$

This result is due to Duflo and Moore [14] and the operator C is usually referred to in the literature as the Duflo-Moore operator. If G is unimodular, C is a multiple of the identity, otherwise, it is an unbounded operator.

For semidirect product groups $G = \mathbb{R}^n \rtimes H$ of the type discussed in the previous sections, with open free orbits, the irreducible representations $\widehat{U}_j(\vec{b}, \mathbf{h})$, appearing in the decomposition (2.52) are all square-integrable and one has the result [9]:

Theorem 2.3.1 *Let H be a closed subgroup of $GL_n(\mathbb{R})$ and let $G = \mathbb{R}^n \rtimes H$. Let $\widehat{\mathcal{O}}_{\vec{k}_j^T}$ be an open free H-orbit in $\widehat{\mathbb{R}}^n$. Then the restriction $\widehat{U}_j(\vec{b}, \mathbf{h})$, of the quasiregular representation to the Hilbert space $L^2(\widehat{\mathcal{O}}, \mathrm{d}\vec{k}^T)$, is irreducible and square-integrable. The corresponding Duflo-Moore operator C_j assumes the form:*

$$(C_j f)(\vec{k}^T) = (2\pi)^{\frac{n}{2}} [c_j(\vec{k}^T)]^{\frac{1}{2}} f(\vec{k}^T), \tag{2.68}$$

on $L^2(\widehat{\mathcal{O}}_{\vec{k}_j^T}, \mathrm{d}\vec{k}^T)$, where $c_j : \widehat{\mathcal{O}}_{\vec{k}_j^T} \longrightarrow \mathbb{R}^+$ is a positive, Lebesgue measurable function which transforms under the action of H as:

$$c_j(\vec{k}^T\mathbf{h}) = \frac{\Delta_H(\mathbf{h})}{|\det \mathbf{h}|} c_j(\vec{k}^T), \tag{2.69}$$

for almost all $\vec{k}^T$ (with respect to the Lebesgue measure). Furthermore, every irreducible representation of G is of this type and the quasi-regular representation is a multiplicity-free direct sum of these representations.

It has also been shown in [9] that $c_j(\vec{k}^T)$ is precisely the density function which converts the Lebesgue measure $d\vec{k}^T$, restricted to the orbit $\widehat{\mathcal{O}}_{\vec{k}_j^T}$, to the invariant measure $d\nu_j$ on it:

$$d\nu_j(\vec{k}^T) = c_j(\vec{k}^T)\, d\vec{k}^T \quad \text{and} \quad d\nu_j(\vec{k}^T\mathbf{h}) = d\nu_j(\vec{k}^T), \tag{2.70}$$

and can be defined simply to be the transform of the left Haar measure $d\mu_H$ of H under the homeomorphism (2.37),

$$d\nu_j(\vec{k}^T) = d\mu_H(\kappa_j(\vec{k}^T)). \tag{2.71}$$

If $\vec{\gamma}_p^T \in \widehat{\mathcal{O}}_{\vec{k}_j^T}$ is an arbitrary point and $\vec{\gamma}_p^T = \vec{k}_j^T\mathbf{h}^{-1}$, (see (2.37)), then in view of (2.69) we may set,

$$c_j(\vec{\gamma}_p^T) = \lambda\, \frac{|\det \mathbf{h}|}{\Delta_H(\mathbf{h})},$$

for almost all $\vec{\gamma}_p^T \in \widehat{\mathcal{O}}_{\vec{k}_j^T}$ (with respect to the Lebesgue measure), where λ is a constant. (Clearly, with this choice of of the density $c_j(\vec{\gamma}_p^T)$ the invariance condition in (2.70) is satisfied). In view of (2.71) we may, by multiplying $d\mu_H$ by a constant if necessary, make $\lambda = 1$. Assuming that this has been done, we may write (for almost all $\vec{\gamma}_p^T$),

$$c_j(\vec{\gamma}_p^T) = \frac{|\det[\kappa_j(\vec{\gamma}_p^T)]|}{\Delta_H[\kappa_j(\vec{\gamma}_p^T)]}, \tag{2.72}$$

Comparing with (2.3),(2.38) and (2.48), and using the homeomorphism $\widetilde{\kappa}_j : \mathcal{O}^*_{(\vec{0}^T, \vec{k}_j^T)} \to \mathbb{R}^n \rtimes H$ in (2.39), we have the result,

Theorem 2.3.2 *Let H be a closed subgroup of $GL_n(\mathbb{R})$ and let $G = \mathbb{R}^n \rtimes H$. Let $\widehat{\mathcal{O}}_{\vec{k}_j^T}$ be an open free H-orbit in $\widehat{\mathbb{R}}^n$ and let $T^*\widehat{\mathcal{O}}_{\vec{k}_j^T}$ be its cotangent bundle with invariant measure $d\Omega_j$. Then the following equalities hold:*

$$c_j(\vec{\gamma}_p^T)^{-1} = \sigma_j(\vec{\gamma}^T) = (\det[\Theta(\vec{\gamma}^T)])^{\frac{1}{2}} = \Delta_G[\widetilde{\kappa}_j(\vec{\gamma}^T)], \tag{2.73}$$

(except at most on a set of measure zero), where c_j is the function defining the Duflo-Moore operator of the unitary irreducible representation $\widehat{U}_j$ of G, associated to the orbit $\widehat{\mathcal{O}}_{\vec{k}_j^T}$, $\sigma_j(\vec{\gamma}^T)$ is the Radon-Nikodym derivative, $\mathrm{d}\vec{\gamma}^T/d\Omega_j$, at the point $\vec{\gamma}^T = (\vec{\gamma}_q^T, \vec{\gamma}_p^T) = (\gamma^1, \gamma^2, \ldots, \gamma^n, \gamma^{n+1}, \ldots, \gamma^{2n}) \in T^*\widehat{\mathcal{O}}_{\vec{k}_j^T}$, $[\Theta(\vec{\gamma}^T)]^{ij} = \sum_{k=1}^{2n} c_k^{ij}\gamma^k$, c_k^{ij} being the structure constants of G, and $\widetilde{\kappa}_j$ the homeomorphism between $T^*\widehat{\mathcal{O}}_{\vec{k}_j^T}$ and $\mathbb{R}^n \rtimes H$, normalized so that $\widetilde{\kappa}_j(\vec{0}^T, \vec{k}_j^T) = (\vec{0}, e)$.

2.4 The Plancherel transform

It is known (see, for example, [1]) that the orthogonality relations (2.67) for a square-integrable representation U of the group G have an extension to Hilbert-Schmidt operators on $\mathfrak{H}$. Let $\mathcal{B}_2(\mathfrak{H})$ denote the Hilbert space of all Hilbert-Schmidt operators ρ on $\mathfrak{H}$. This Hilbert space is equipped with the scalar product,

$$\langle \rho_1 \mid \rho_2 \rangle_{\mathcal{B}} = \mathrm{Tr}[\rho_1^* \rho_2].$$

Then there exists a dense set $\mathcal{D} \subset \mathcal{B}_2(\mathfrak{H})$ such that for any $\rho \in \mathcal{D}$, the (closure of) the operator $U(g)^* \rho C^{-1}$ is of trace class (C being the Duflo-Moore operator). Furthermore, the function

$$f_\rho(g) = \mathrm{Tr}[U(g)^* \rho C^{-1}], \tag{2.74}$$

is an element of $L^2(G, \mathrm{d}\mu_G)$ and moreover,

$$\|f_\rho\|^2_{L^2(G,\mathrm{d}\mu_G)} = \|\rho\|^2_{\mathcal{B}}. \tag{2.75}$$

Thus, we may define an isometric linear map, $\widetilde{\mathfrak{W}} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(G, \mathrm{d}\mu_G)$ which, for $\rho \in \mathcal{D}$, is given by (2.74) and is then extended by continuity to all of $\mathcal{B}_2(\mathfrak{H})$. Additionally, as a consequence of the relation,

$$CU(g) = [\Delta_G(g)]^{\frac{1}{2}}\, U(g) C, \tag{2.76}$$

$\widetilde{\mathfrak{W}}$ intertwines the representation $U(g_1) \wedge U(g_2)$, of $G \times G$ on the Hilbert space $\mathcal{B}_2(\mathfrak{H})$, where

$$U(g_1) \wedge U(g_2)(\rho) = U(g_1)\rho U(g_2)^*, \qquad \rho \in \mathcal{B}_2(\mathfrak{H}), \quad (g_1, g_2) \in G \times G, \tag{2.77}$$

with the double representation $U_D(g_1, g_2)$ (see (2.58)) on $L^2(G, \mathrm{d}\mu_G)$. For the group $G = \mathbb{R}^n \rtimes H$, with open free orbits, each irreducible representation $\widehat{U}_j$ gives rise in this way to a double representation $U_D^j(g_1, g_2)$ on $\mathcal{B}_2(\widehat{\mathfrak{H}}_j)$ and an isometric map $\widetilde{\mathfrak{W}}_j : \mathcal{B}_2(\widehat{\mathfrak{H}}_j) \longrightarrow L^2(G, \mathrm{d}\mu_G)$, $j = 1, 2, \ldots, N$. The range of this map is the subspace $\mathfrak{H}_\ell^j$ (see (2.53) - (2.53)). The inverse is a unitary map $\widetilde{\mathfrak{W}}_j^{-1} : \mathfrak{H}_\ell^j \longrightarrow \mathcal{B}_2(\widehat{\mathfrak{H}}_j)$, which on a dense set of elements $f \in \mathfrak{H}_\ell^j$ is defined as,

$$\widetilde{\mathfrak{W}}_j^{-1} f = \widehat{U}_j(f) C_j^{-1}, \qquad \widehat{U}_j(f) = \int_G f(g) \widehat{U}_j(g) \; \mathrm{d}\mu_G(g), \tag{2.78}$$

and is just the restriction of the *Plancherel transform* [14] to the subspace $\mathfrak{H}_\ell^j$ of $L^2(G, \mathrm{d}\mu_G)$. Let us write $U_D^{\oplus}(g_1, g_2) = \oplus_{j=1}^N U_D^j(g_1, g_2)$ and $\mathcal{B}_2^{\oplus} = \oplus_{j=1}^N \mathcal{B}_2(\widehat{\mathfrak{H}}_j)$. The Plancherel map is now defined as $\mathcal{P} = \oplus_{j=1}^N \widetilde{\mathfrak{W}}_j^{-1}$, and the Plancherel theorem [14] may be stated for this case as:

Theorem 2.4.1 *Let $G = \mathbb{R}^n \rtimes H$ admit open free orbits. Then the map*

$$\mathcal{P} : L^2(G, \mathrm{d}\mu_G) \longrightarrow \mathcal{B}_2^{\oplus}, \qquad \mathcal{P}(f)_j = \widetilde{\mathfrak{W}}_j^{-1} f = \widehat{U}_j(f) C_j^{-1}, \tag{2.79}$$

defined initially on a dense set of elements $f \in \mathcal{D} \subset L^1(G, \mathrm{d}\mu_G) \cap L^2(G, \mathrm{d}\mu_G)$, is an isometry and hence can be extended as a unitary map to all of $L^2(G, \mathrm{d}\mu_G)$. The inverse of this map is $\mathcal{P}^{-1} = \widetilde{\mathfrak{W}} = \oplus_{j=1}^N \widetilde{\mathfrak{W}}_j$, where

$$(\widetilde{\mathfrak{W}}_j \rho)(g) = \mathrm{Tr}[\widehat{U}_j(g)^* \rho C_j^{-1}], \tag{2.80}$$

again defined initially on a dense set of vectors $\rho \in \mathcal{B}_2(\widehat{\mathfrak{H}}_j)$ and then extended by continuity.

The map $\mathcal{P}$ intertwines the double representation $U_D(g_1, g_2)$ of $G \times G$ on $L^2(G, \mathrm{d}\mu_G)$ with the representation $U_D^{\oplus}(g_1, g_2) = \oplus_{j=1}^{N} \widehat{U}_j(g_1) \wedge \widehat{U}_j(g_2)$ on $\mathcal{B}_2^{\oplus}$, where for each $j = 1, 2, \dots N$, the representation $\widehat{U}_j(g_1) \wedge \widehat{U}_j(g_2)$ of $G \times G$ is irreducible on $\mathcal{B}_2(\widehat{\mathfrak{H}}_j)$. Furthermore, $\mathcal{P}$ maps the left regular representation $U_\ell(g)$ of G unitarily to the representation $\widehat{U}_j(g_1) \wedge I_j$, on $\mathcal{B}_2(\widehat{\mathfrak{H}}_j)$, ($I_j$ being the identity operator on $\widehat{\mathfrak{H}}_j$), and hence the restriction of $U_\ell(g)$ to $\mathfrak{H}_\ell^j$, in the decomposition (2.53) - (2.54), is an infinite direct sum of representations equivalent to $\widehat{U}_j(g)$.

Chapter 3

Wigner function

3.1 The standard Wigner function

Let us begin with a revision of some basic properties of the function defined in [26]. The quasiprobability distribution function W^{QM} is defined for any quantum mechanical state $\phi \in L^2(\mathbb{R}^n, \mathrm{d}\vec{x})$ on the flat phase space $\Gamma = \mathbb{R}^{2n}$ as

$$W^{QM}(\phi \mid \vec{q}, \vec{p}; h) = \frac{1}{h^n} \int_{\mathbb{R}^n} \overline{\phi(\vec{q} - \frac{\vec{x}}{2})}\, e^{-\frac{2\pi i \vec{x}\cdot\vec{p}}{h}}\, \phi(\vec{q} + \frac{\vec{x}}{2})\, \mathrm{d}\vec{x}\,. \qquad (3.1)$$

The vector $\vec{q}$ represents the position of the system, $\vec{p}$ its momentum at the point $\vec{q}$ and h is Planck's constant. The phase space Γ can also be viewed as an orbit $\mathcal{O}^*$ under the coadjoint action of Heisenberg-Weyl group G_{HW} in the dual space $\mathfrak{g}^*_{HW}$ to its Lie algebra $\mathfrak{g}_{HW}$, and this is the starting point of [2] for constructing a generalization of the Wigner function. Canonical transformations of the phase space Γ of the form:

$$(\vec{q}, \vec{p}) \to (\vec{q} - \vec{q}_0, \vec{p} - \vec{p}_0), \qquad (3.2)$$

can be viewed as the coadjoint action of G_{HW} on an orbit $\mathcal{O}^*$ in $\mathfrak{g}^*_{HW}$ and lead to unitary transformations on the space of wave functions ϕ:

$$\phi \longrightarrow U(\vec{q}_0, \vec{p}_0)\phi = e^{\frac{2\pi i}{h}(\vec{Q}\cdot\vec{p}_0 - \vec{P}\cdot\vec{q}_0)}\phi, \qquad (3.3)$$

where $\vec{Q}$ and $\vec{P}$ are the usual n-vector operators of position and momentum respectively. It can be shown that the Wigner function satisfies the following covariance condition related to (3.2) and (3.3):

$$W^{QM}(U(q_0,p_0)\phi \mid \vec{q},\vec{p};h) = W^{QM}(\phi \mid \vec{q}-\vec{q_0},\vec{p}-\vec{p_0};h) \tag{3.4}$$

The other important property of the Wigner function is the existence of the so called marginality conditions, which build a bridge to the theory of classical probability distributions. The two conditions are:

$$\int_{\mathbb{R}^{2n}} W^{QM}(\phi \mid \vec{q},\vec{p};h)\, \mathrm{d}\vec{p} = |\phi(\vec{q})|^2 \tag{3.5}$$

and

$$\int_{\mathbb{R}^{2n}} W^{QM}(\phi \mid \vec{q},\vec{p};h)\, \mathrm{d}\vec{q} = |\widehat{\phi}(\vec{p})|^2 \tag{3.6}$$

where $\widehat{\phi}$ is the Fourier transform of ϕ.On the contrary, in general for a given ϕ there exist in general regions of phase space over which the function $W^{QM}(\phi \mid \vec{q},\vec{p},h)$ can also assume negative values and hence W^{QM} cannot be probability density in the usual sense, whence the denomination of quasi-probability.

3.2 Group -Theoretical approach to the original Wigner function

It is possible to derive the original Wigner function from a unitary irreducible representation U^{λ}_{HW} of the Heisenberg-Weyl group G_{HW}, parametrized by the value of the central element λ.

$$(U^{\lambda}_{HW}(\theta,\xi,\eta)\phi)(k) = e^{i\lambda\theta}e^{\lambda\eta(k-\xi/2)}\phi(k-\xi) \tag{3.7}$$

Let us consider the case $\theta = 0$ and $\lambda = 1$ (equivalent to setting $\hbar = 1$). The representation $U_{HW}(\eta,\xi) = U^{\lambda=1}_{HW}(0,\eta,\xi)$ is irreducible on $\mathfrak{H} = L^2(\mathbb{R},\mathrm{d}k)$ and square integrable with respect to the homogeneous space $G_{HW}/\theta = \mathbb{R}^2$. The corresponding orthogonality relation is then:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \overline{< U_{HW}(\eta,\xi)\psi_1|\phi_1 >} < U_{HW}(\eta,\xi)\psi_2|\phi_2 > \, \mathrm{d}\eta\mathrm{d}\xi =< \phi_1|\phi_2 >< \psi_2|\psi_1 > \tag{3.8}$$

At this point let us stress the fact that this is not the same square-integrability condition we discussed in Chapter 2, since we integrate here over homogeneous space G_{HW}/Θ and not over the whole group.

Each element g in G_{HW} can be written as an image of some element X in $\mathfrak{g}_{HW} = Lie(G_{HW})$ under the exponential map $exp : Lie(G_{HW}) \rightarrow G_{HW}$: $g = e^X$

Adopting the following matrix notation for g and X:

$$X = \begin{pmatrix} 0 & x & -y & z \\ 0 & 0 & 0 & -y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad g = \begin{pmatrix} 1 & \eta & \xi & \theta \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & -\eta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we can find that

$$e^X = \begin{pmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & -y \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.9}$$

The left Haar measure on $G_{HW/\theta}$ $\mathrm{d}\mu = \mathrm{d}\eta\,\mathrm{d}\xi$ can be rewritten as $\mathrm{d}\mu = \mathrm{d}x\mathrm{d}y$ and the representation $U(\eta,\xi)$ as $U(x,-y)$:

$$(U(x,-y)\phi)(k) = e^{-iy(k-\frac{x}{2})}\phi(k-x) \tag{3.10}$$

Now ,the relation (3.8) takes the form:

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} \overline{< U_{HW}(x,-y)\phi_1\psi_1 >} < U_{HW}(x,-y)\phi_2|\psi_2 > \, \mathrm{d}\eta\mathrm{d}\xi$$
$$=< \phi_1|\phi_2 >< \psi_1|\psi_2 > \tag{3.11}$$

The Wigner function can be defined as a Fourier transform of an element $< U_{HW}(x,-y)\phi|\psi >\in L^2(\mathbb{R}^2)$

$$W(\phi,\psi|\vec{\gamma}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-i\gamma_1 x - i\gamma_2 y} < U_{HW}(x,-y)\phi|\psi > \mathrm{d}x\mathrm{d}y \tag{3.12}$$

After simple computation we obtain

$$W(\phi,\psi|\vec{\gamma}) = \frac{1}{2\pi}\int_{\mathbf{R}^2}\phi(\gamma_2-\frac{x}{2})e^{-i\gamma_1 x}\psi(\gamma_2+\frac{x}{2})\mathrm{d}x \tag{3.13}$$

which is exactly the Wigner Function (1) after putting $\gamma_1 = p$, $\gamma_2 = q$, $\hbar = 1$ and $\psi = \phi$.

Defining the Wigner function corresponding to two wave functions ϕ and ψ one can relate the Wigner formalism with the Schrödinger picture of Quantum Mechanics by the so called overlap condition:

$$\int_{\mathbf{R}^2}\mathrm{d}q\mathrm{d}pW(\phi,\psi|q,p)W(v,w|q,p) =< \phi|w >< v|\psi > \tag{3.14}$$

3.3 General Wigner function

The method introduced in [2] allows one to construct Wigner functions for a wide class of groups, possessing a square-integrable representation, as generalizations of (1). Denoting this representation by U and its carrier space by $\mathfrak{H}$, the generalized Wigner function for a Hilbert-Schmidt operator $\rho \in \mathcal{B}_2(\mathfrak{H})$ is defined by the following expression:

$$W(\rho\,|\vec{\gamma}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{N_0} e^{-i\vec{\gamma}^T\vec{x}}\mathrm{Tr}[U(e^{-X}\rho C^{-1}][\sigma(\vec{\gamma}^T)m_G(\vec{x})]^{\frac{1}{2}}\mathrm{d}\vec{x} \tag{3.15}$$

or equivalently

$$W(\phi,\psi\mid\vec{\gamma}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{N_0} e^{-i\vec{\gamma}^T\vec{x}} < C^{-1}\psi|U(e^{-X})\phi > [\sigma(\vec{\gamma}^T)m_G(\vec{x})]^{\frac{1}{2}}\mathrm{d}\vec{x} \tag{3.16}$$

This formula contains some symbols which we are presently defined: $\vec{\gamma}^T \in \mathbf{g}$, $\phi \in \mathfrak{H}$ and ψ is in the domain of the Duflo-Moore operator C related to the representation U. The density function m_G again expresses the Haar measure on G in terms of the Lebesgue measure $\mathrm{d}\vec{x}$ on $\mathbf{g}$ (see (2.19)):

$$\mathrm{d}\mu_G(e^X) = m_G(\vec{x})\mathrm{d}\vec{x} \tag{3.17}$$

where $X \in \mathbf{g}$ is expressed in terms of the components of the vector $\vec{x}$ in the basis $\{L^1, L^2, ...L^{2n}\}$ (i.e.,$X = \sum_{i=1}^{2n} x_i L^i$). We assume that there exist a symmetric subset N_0 of Lie algebra $\mathbf{g}$, such that the exponential map restricted to it is a bijection onto a dense set in G. The function σ is defined by expressing the Lebesgue measure $\mathrm{d}X^*$ in $\mathbf{g}^*$ in terms of invariant measures $\mathrm{d}\Omega_\lambda$ on the coadjoint orbits $\mathcal{O}^*_\lambda$ as follows:

$$\mathrm{d}X^* = \mathrm{d}\kappa(\lambda)\sigma_\lambda(X^*)\mathrm{d}\Omega_\lambda(X^*) \tag{3.18}$$

where the index λ parametrizes coadjoint orbit and $\mathrm{d}\kappa(\lambda)$ is a measure on the parameter space. This decomposability is not guaranteed in general and has to be assumed or proved for specific cases.

3.4 Basic Properties of general Wigner function

The appearance of σ_λ in the formula for the Wigner function is necessary in order to have the following important covariance property :

$$W(U(g)\rho U(g)^*|\vec{\gamma}^T) = W(\rho|Ad_{g^{-1}}{}^{\#}\vec{\gamma}^T) \tag{3.19}$$

which clearly can be regarded as a generalization of the covariance (3.4) for the original Wigner Function. The overlap condition (3.14) in this more general setting becomes:

$$\int_{\mathbf{g}^*} \overline{W(\rho_1|\vec{\gamma}^T)} W(\rho_2|\vec{\gamma}^T)[\sigma(\vec{\gamma})^T]^{-1}\mathrm{d}\vec{\gamma}^T = \mathrm{Tr}[\rho_1^*\rho_2] \tag{3.20}$$

3.5 Wavelet – general Wigner function relation

In order to establish this correspondence we need to introduce a (generalized) wavelet transform arising from a square-integrable group representation U. To any admissible vector η we can associate the map from the Hilbert space $\mathfrak{H}$ (carrier space of U) to the Hilbert space of square-integrable functions on G:

$$\phi \to f_{\eta,\phi}(g) = \frac{1}{[c(\eta)]^{\frac{1}{2}}} < U(g)\eta|\phi > \tag{3.21}$$

where $c(\eta)$ is as in(2.66). This map is an isometry as shown in [17]:

$$\int_G |f_{\eta,\phi}(g)|^2 \mathrm{d}\mu(g) = ||\phi||^2 \tag{3.22}$$

It is then clear that for any fixed admissible η an arbitrary vector $\phi \in \mathfrak{H}$ can be represented both by a wavelet transform $f_{\eta,\phi}(g)$ or by a Wigner function $W(\eta, \phi|\vec{\gamma})$. In slightly different notation using a Hilbert-Schmidt operator of the form:

$$\rho_{\eta,\phi} = \frac{1}{[c(\eta)^{\frac{1}{2}}]}|\phi >< \eta|C$$

the Wigner-wavelet relation is [2] :

$$W(\rho_{\eta,\phi}|\vec{\gamma}^T) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{N_0} e^{-i\vec{\gamma}^T\vec{x}} f_{\eta,\phi}(e^X)[\sigma(\vec{\gamma}^T)m(X)]^{\frac{1}{2}} \mathrm{d}X \tag{3.23}$$

3.6 Wigner function for Semidirect product groups

Before going to the case of semidirect product groups, let us make a few comments about the general construction.

We can see that the construction of the original Wigner function presented in section 3.1 is very similar to the general one, where both density functions $m(\vec{\mathbf{x}})$ and $\sigma(\vec{\gamma}^T)$ are equal to one and a Duflo-Moore operator is just an identity operator. The only difference is that we need to consider square integrability of Heisenberg-Weyl group representation with respect to the homogeneous space G_{HW}/θ instead of square integrability with respect to the whole group.

It should be stressed that this general procedure of constructing Wigner Functions rests on two requirements.

The first one is the decomposability (3.18) of the Lebesgue measure $dX^{\bullet}$. The second is that we consider only groups with square-integrable representations.

Semidirect product groups which we consider in this thesis satisfy both these conditions as we have seen in Chapter 2. Using their square-integrable representations as introduced in (2.51) we can rewrite the general Wigner function (2.51) as:

$$W(\widehat{\phi},\widehat{\psi}|\vec{\gamma}^T) = \frac{1}{(2\pi)^n}\int_{N_0} \mathrm{d}\vec{x}e^{-i\vec{\gamma}^T\vec{x}}\int_{\mathcal{O}^\bullet}\mathrm{d}\vec{\omega}^T\overline{C^{-1}\widehat{\psi}(\vec{\omega}^T)}$$
$$|\det\, e^{-X_q}|^{\frac{1}{2}}\exp\left(i\vec{\omega}^T e^{-\frac{X_q}{2}}\,\mathrm{sinch}\,\frac{X_q}{2}\vec{x}_p\right)\widehat{\phi}(\vec{\omega}^T e^{-X_q})[\sigma(\vec{\gamma}^T)m(\vec{x})]^{\frac{1}{2}} \quad (3.24)$$

Changing variables: $\omega'^T = \omega^T e^{-\frac{X_q}{2}}\,\mathrm{sinch}\,\frac{X_q}{2}$ and using the form for the density function $m(\vec{x})$ given in Eq.(2.24) we obtain:

$$W(\widehat{\phi},\widehat{\psi}|\vec{\gamma}^T) = \frac{1}{(2\pi)^n}\int_{N_{0q}}\int_{\mathbb{R}^2}\mathrm{d}\vec{x}_q\mathrm{d}\vec{x}_p e^{-i\vec{\gamma}_q^T\vec{x}_q}$$
$$\int_{\mathcal{O}^\bullet}\mathrm{d}\vec{\omega}'^T e^{i(\omega'^T-\vec{\gamma}_p^T)\vec{x}_p}\overline{C^{-1}\widehat{\psi}\left(\vec{\omega}'^T\frac{e^{\frac{X_q}{2}}}{\mathrm{sinch}\,\frac{X_q}{2}}\right)}\widehat{\phi}\left(\vec{\omega}'\frac{e^{-\frac{X_q}{2}}}{\mathrm{sinch}\,\frac{X_q}{2}}\right)\sigma(\vec{\gamma}^T)^{\frac{1}{2}}\left|\det\frac{e^{-\frac{X_q}{2}}}{\mathrm{sinch}\,\frac{X_q}{2}}\right|^{\frac{1}{2}}$$
$$\left|\det\left(e^{-ad\frac{X_q}{2}}\,\mathrm{sinch}\,ad\frac{X_q}{2}\right)\right|^{\frac{1}{2}} \quad (3.25)$$

We have shown in Theorem 2.3.2 that the Duflo-Moore operator in this case is related to the decomposition (3.18) of Lebesgue measure in $\mathfrak{g}$ and that we can use the structure constants of the Lie algebra $\mathfrak{g}$ to expressed them both. Applying the result (2.73) together with (2.69) and integrating over $\vec{x}_p$ we finally obtain:

$$W(\widehat{\phi},\widehat{\psi}|\vec{\gamma}^T) = \int_{N_{0q}}\mathrm{d}\vec{x}_q e^{-i\vec{\gamma}_q^T\vec{x}_q}\overline{\widehat{\psi}\left(\vec{\gamma}_p^T\frac{e^{\frac{X_q}{2}}}{\mathrm{sinch}\,\frac{X_q}{2}}\right)}\widehat{\phi}\left(\vec{\gamma}_p\frac{e^{-\frac{X_q}{2}}}{\mathrm{sinch}\,\frac{X_q}{2}}\right)$$
$$c\left(\vec{\gamma}_p^T\frac{1}{\mathrm{sinch}\,\frac{X_q}{2}}\right)^{-\frac{1}{2}} c(\vec{\gamma}_p^T)^{-\frac{1}{2}}\left|\frac{\det(\mathrm{sinch}\,ad\frac{X_q}{2})}{\det(\mathrm{sinch}\,\frac{X_q}{2})}\right|^{\frac{1}{2}} \quad (3.26)$$

Here we used the fact that the domain N_0 of the exponential map $\exp:\mathfrak{g}\to G$ in the case of semidirect product groups, is given by $N_{0q}\times\mathbb{R}^n$, where N_{0q} is the corresponding domain the exponential map $\exp:\mathfrak{h}\to H$. Again we assume that this map is a bijection onto a dense set in H such that its complement has measure zero.

3.7 Domain of the Wigner Function

As we have already mentioned, the advantage of using the original Wigner function as well as the Wigner function related to the wavelet group, is that it allows us to represent a signal as a function on a phase space (position-momentum or time-frequency). In other words we would like to be able to represent a signal as a function supported on coadjoint orbit, which together with a symplectic form Θ (discussed earlier in 2.43) can be considered as a phase space. We will investigate now under what condition this is the case for a Wigner function derived in the previous section. Recall first that the open free coadjoint orbits $\mathcal{O}_i^*$ in $\mathfrak{g}$ -for semidirect product groups - are in one to one correspondence with open free H-orbits $\widehat{\mathcal{O}}_i \subset \widehat{\mathbb{R}}^n$: indeed according to Thm 5.1, any $\mathcal{O}_i^*$ is a cotangent bundle of the form $\mathcal{O}_i^* = T^*\widehat{\mathcal{O}}_i = \widehat{\mathcal{O}}_i \times \mathbb{R}^n$.
Let $W_{\widehat{\mathcal{O}}_i}$ denote the Wigner function derived from a representation of G acting on $\mathfrak{H} = L^2(\widehat{\mathcal{O}}_i)$, which can be conveniently be thought of as the closed subspace of $L^2(\widehat{\mathbb{R}}^n)$ of functions which vanish almost everywhere outside $\widehat{\mathcal{O}}_i$. We are going to find sufficient conditions for a Wigner function $W_{\widehat{\mathcal{O}}_i}$ to have support concentrated on the corresponding coadjoint orbit $\mathcal{O}_i^* = \widehat{\mathcal{O}}_i \times \mathbb{R}^n$.
Let us start by recalling the Δ function (polynomial) introduced in section 1.1.

$$\Delta(\vec{\gamma}^T) = \det \begin{pmatrix} \vec{\gamma}^T L_1 \\ \vec{\gamma}^T L_2 \\ \dots \\ \vec{\gamma}^T L_n \end{pmatrix} \tag{3.27}$$

where $\{L_1, .., L_{n-1}, L_n\}$ is the basis in $\mathfrak{g}$. We have the following:

Proposition 3.7.1 *Let G be a semidirect product group $\mathbb{R}^n \rtimes H$ s.t. H acts on $\widehat{\mathbb{R}}^n$ with open, free orbits $\{\widehat{\mathcal{O}}_i\}_{i=1}^m$. If the orbits $\widehat{\mathcal{O}}_i$ are dihedral cones (i.e. if the zero level set of function Δ in (3.27) can be decomposed into hyperplanes) then the Wigner function $W_{\widehat{\mathcal{O}}_i}$ has support concentrated on the corresponding coadjoint orbit $\mathcal{O}_i^* = \mathbb{R}^n \times \widehat{\mathcal{O}}_i$.*

To prove it we will need the following lemma:

Lemma 3.7.2 *If a hyperplane* $\Pi(\vec{\gamma}^T) = 0$ *is a subset of* $\Delta(\vec{\gamma}^T) = 0$ *then it is invariant under* H.

Proof of Lemma 3.7.2

We will show first that we can always find $\vec{\gamma}_0^T \in \Pi^{-1}(0)$ s.t. there exists a neighborhood $U_{\vec{\gamma}_0^T}$ such that

$$U_{\vec{\gamma}_0^T} \cap \Pi^{-1}(0) = U_{\vec{\gamma}_0^T} \cap \Delta^{-1}(0) \tag{3.28}$$

Let us introduce a basis $\{Z_1, ..., Z_n\}$ in $\widehat{\mathbb{R}}^n$ such that the first $n-1$ elements constitute a basis in the hyperplane $\Pi(\vec{\gamma}^T) = 0$. In these coordinates, the the hyperplane can be written as $\Pi(\vec{\gamma}^T) = \gamma^n$ and the function Δ can be factored as:

$$\Delta(\vec{\gamma}^T) = (\gamma^n)^k P(\vec{\gamma}^T)$$

such that $P(\vec{\gamma})$ does not contain γ^n as a factor. Then $P(\gamma^1, ...\gamma^{n-1}, 0) \equiv 0$ iff $P(\vec{\gamma}) \equiv 0$ (which would imply $\Delta(\vec{\gamma}) \equiv 0$,a contradiction) or $P(\vec{\gamma})$ contains γ^n as a factor, which would contradict (3.7). Thus we can always choose $\vec{\gamma}_0 = (\gamma_1, ..., \gamma_{n-1}, 0)$ such that $P(\vec{\gamma}_0) = r \neq 0$ and $\Pi(\vec{\gamma}_0) = 0$. Since P is a polynomial, there exists an open neighborhood $U_{\vec{\gamma}_0^T}$ of γ_0^T such that $P(\vec{\gamma}_0^T) \in (r - \epsilon, r + \epsilon)$. Thus we have (3.28).

Thus, for any $\vec{\gamma} \in U_{\vec{\gamma}_0^T}$ the intersection of its orbit $\mathcal{O}_{\vec{\gamma}^T}$ with $U_{\vec{\gamma}_0^T}$ belongs to Π. This implies that for every $\vec{\gamma}^T \in U_{\vec{\gamma}_0^T}$ $\mathfrak{h}\vec{\gamma}^T \in \Pi$. We can choose a basis of Π formed by $N-1$ linearly independent elements $\{Z_1', ..., Z_{N-1}'\} \subset U_{\vec{\gamma}_0^T}$. Since $\mathfrak{h}Z_i' \subset \Pi$ is true for every basis element then also for every $\vec{\gamma}^T \in \Pi$, $\mathfrak{h}\vec{\gamma}^T \subset \Pi$, i.e. the hyperplane Π is stable under $\mathfrak{h}$ and hence also under H (by exponentiation). **QED**

Proof of Proposition 3.7.1

One sees from Eq.(3.26) that a sufficient condition for Wigner transform to preserve the decomposition into orbits $\mathcal{O}_i^s$ is that point $\vec{\gamma}^T \frac{e^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}}$ does not leave $\widehat{\mathcal{O}}_i$ as X_q varies in N_{0q}, or equivalently that the 'sinch' map preserves the orbits (which is not guaranteed because sinch (X) is not an element of the group H).

Let us take again a basis $\{Z_1, ..., Z_{n-1}, Z_n\}$ in $\widehat{\mathbb{R}}^n$ such as the first $n-1$ elements belong to the $(n-1)$-dim hyperplane as in Lemma (3.7.2) In the coordinates introduced above an element X of the *Lie* algebra $\mathfrak{h}$ of the group H is of the form:

$$X = \begin{pmatrix} X_{1,1} & \dots & X_{1,n-1} & 0 \\ \dots & & & \\ X_{n-1,1} & \dots & X_{n-1,n-1} & 0 \\ X_{n,1} & \dots & X_{n,n-1} & X_{n,n} \end{pmatrix}$$

because X must preserve the hyperplane $\gamma^n = 0$. Calculating the sinch of such element X we obtain:

$$S = \operatorname{sinch}(X) = \begin{pmatrix} S_{1,1} & \dots & S_{1,n-1} & 0 \\ \dots & & & \\ Sn-1,1 & \dots & S_{n-1,n-1} & 0 \\ S_{n,1} & \dots & S_{n,n-1} & \operatorname{sinch}(X_{n,n}) \end{pmatrix}$$

Notice that $\operatorname{sinch}(X_{n,n}) > 0$ from definition (2.15).

Applying $\operatorname{sinch}(X)$ to any vector $\vec{\gamma}^T$ in $\widehat{\mathbb{R}}^n$ written in the basis $\{Z_i\}_1^n$ we have :

$$\operatorname{sinch}(X)(\gamma^1, ..., \gamma^{n-1}, \gamma^n) = (\gamma^{1'}, ..., \gamma^{n-1'}, \operatorname{sinch}(X_{n,n}))\gamma^n)$$

Therefore the sign of γ^n remains unchanged, which also means that the hyperplane $\gamma^n = 0$ divides $\widehat{\mathbb{R}}^n$ into two halfspaces, invariant under the sinch map.

Since $\Delta^{-1}(0)$ is a union of hyperplanes $\Pi_1 \cup \Pi_2 ... \cup \Pi_r$ we can repeat the argument for each of them, proving that each open orbit is preserved. **QED**

In order to illustrate the problem of domain we give an example of a group for which the corresponding function Δ cannot be factored (over $\mathbb{R}$) into hyperplanes.

Example

Consider a group where *Lie* algebra $\mathfrak{h}$ has the following basis (equivalent to the case 1.b of the classification after change of basis):

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad F^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad F^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.29}$$

The orbit structure in $\widehat{\mathbb{R}}^n$ is given by the equation: $\Delta(\vec{\omega}) = -\frac{1}{3}x_3(-2x_3x_1 + x_2^2) = 0$ which clearly cannot be factored into hyperplanes.

We have the following open orbits in $\widehat{\mathbb{R}}^n$:

$\widehat{\mathcal{O}}_1$ - above the hyperplane $x_3 = 0$ and inside the cone $-2x_3x_1 + x_2^2 < 0$ $(\Delta > 0)$

$\widehat{\mathcal{O}}_2$ - above the hyperplane $x_3 = 0$ and outside the cone $-2x_3x_1 + x_2^2 > 0$ $(\Delta < 0)$

$\widehat{\mathcal{O}}_3$ - below the hyperplane $x_3 = 0$ and inside the cone $-2x_3x_1 + x_2^2 < 0$ $(\Delta > 0)$

$\widehat{\mathcal{O}}_4$ - below the hyperplane $x_3 = 0$ and outside the cone$(-2x_3x_1 + x_2^2 > 0$ $\Delta < 0)$

In order to see that the sinch map does not preserve orbits let us choose a point in $\widehat{\mathcal{O}}_2 : \vec{\omega}_0^T = (\omega_1, \omega_2, \omega_3)$ and apply to it $\operatorname{sinch}(tF^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{6}t^2 & 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$. Then one can compute $\Delta(\vec{\omega}_0^T \operatorname{sinch}(tF^1)) = \frac{1}{9}\omega_3 \left(6\,\omega_3\omega_1 + {\omega_3}^2t^2 - 3\,{\omega_2}^2\right)$. It is clear that, as a function of t, it changes sign whenever $2\,\omega_3\omega_1 - {\omega_2}^2 < 0$. This also means that the sinch map mixes two orbits $\widehat{\mathcal{O}}_1$ with $\widehat{\mathcal{O}}_2$ and also $\widehat{\mathcal{O}}_3$ with $\widehat{\mathcal{O}}_4$. By a continuity argument, this mixing property holds for a suitable open neighborhood of F^1 in the Lie algebra, i.e. a set of positive Lebesgue measure.

As a consequence, a Wigner function $W(\widehat{\phi}, \widehat{\psi}|X^*)$ corresponding to two functions supported in $\widehat{\mathcal{O}}_1$, $\widehat{\phi}, \widehat{\psi} \in L^2(\widehat{\mathcal{O}}_1)$ will have its support spread on *both* coadjoint orbits $\mathcal{O}_1^*$ and $\mathcal{O}_2^*$. To see that let us fix $\vec{\gamma}_p^T = \vec{\omega}_0^T \in \widehat{\mathcal{O}}_2$(recall that $X^* = (\vec{\gamma}_q^T, \vec{\gamma}_p^T) \in \mathcal{O}^*$). Then the Wigner function, as a function of $\vec{\gamma}_q^T \in \widehat{\mathbb{R}}^n$, is just the Fourier transform of a function $F(X_q)$

$$W_{\vec{\omega}_0^T}(\widehat{\phi}, \widehat{\psi}|\vec{\gamma}_q^T) = \int_{\mathbb{R}^n} d\vec{x}_q e^{-i\vec{\gamma}_q^T\vec{x}_q} F(X_q) \tag{3.30}$$

where:

$$F(X_q) = \overline{\widehat{\psi}\left(\vec{\omega}_0^T \frac{e^{\frac{X_q}{2}}}{\text{sinch } \frac{X_q}{2}}\right)} \widehat{\phi}\left(\vec{\omega}_0^T \frac{e^{-\frac{X_q}{2}}}{\text{sinch } \frac{X_q}{2}}\right)$$

$$c\left(\vec{\omega}_0^T \frac{1}{\text{sinch } \frac{X_q}{2}}\right)^{-\frac{1}{2}} c(\vec{\omega}_0^T)^{-\frac{1}{2}} \left|\frac{\det(\text{sinch } ad\frac{X_q}{2})}{\det(\text{sinch } \frac{X_q}{2})}\right|^{\frac{1}{2}} . \tag{3.31}$$

Since the map sinch (X_q) brings $\vec{\omega}_0^T$ from $\widehat{\mathcal{O}}_2$ to $\widehat{\mathcal{O}}_1$ (support of $\widehat{\phi}, \widehat{\psi}$) the function $F(X_q)$ is not identically zero, e.g. for X_q in a suitable open neighborhood of F^1. Then its Fourier transform $W_{\omega_0^T}(\widehat{\phi}, \widehat{\psi}|\vec{\gamma}_q^T)$ is also not identically zero. This means that the Wigner function $W(\widehat{\phi}, \widehat{\psi}|X^*)$ does not vanishes outside the orbit $\mathcal{O}_1^*$.

Chapter 4

Examples

It is very easy task to calculate explicitly Wigner function for each particular case from its general form for semidirect product groups derived earlier (3.26). Let us consider first examples of connected 4-dimensional semidirect product groups $G = \mathbb{R}^2 \rtimes H$ with open free H-orbits in $\widehat{\mathbb{R}}^2$. According to the theorem 1.2.1 there are only three conjugacy classes of them, i.e. when H is diagonal group, SIM(2) or one of the infinite family of H_c groups. We give Wigner functions for each of these cases.

Then we present an interesting example of an 8-dimensional group $G = H \rtimes H^*$, where H is a vector space of quaternions and H^* a group of invertible quaternions.

4.1 The diagonal group

Let H be the diagonal subgroup of $GL_2(\mathbb{R})$ that is $H = \{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}; a_1, a_2 \in \mathbb{R} - \{0\}\}$ The Wigner functions are defined on the coadjoint $G-$orbits $\mathcal{O}^*_{\vec{\gamma}^T_{ij}}$ of the element $\vec{\gamma}^T_0 = (0, 0, i, j), i = \pm 1, j = \pm 1$ the union of which is dense in $\widehat{\mathbb{R}}^4$.

$$W(\widehat{\phi}, \widehat{\psi}|\vec{\gamma}^T) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}x_1 \mathrm{d}x_2 e^{-i\gamma^1 x_1 - i\gamma^2 x_2} \overline{\widehat{\psi}(\gamma_1 \frac{e^{\frac{x_1}{2}}}{\operatorname{sinch} \frac{x_1}{2}})}$$
$$\widehat{\phi}(\gamma_2 \frac{e^{-\frac{x_2}{2}}}{\operatorname{sinch} \frac{x_2}{2}}) \frac{|\gamma^3 \gamma^4|}{\operatorname{sinch} \frac{x_1}{2} \operatorname{sinch} \frac{x_2}{2}} \tag{4.1}$$

Where we used the following relations:

$$X_q = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \quad e^{X_q} = \begin{pmatrix} e^{x_1} & 0 \\ 0 & e^{x_2} \end{pmatrix}$$

$$\operatorname{sinch}\frac{X_q}{2} = \begin{pmatrix} \operatorname{sinch}\frac{x_1}{2} & 0 \\ 0 & \operatorname{sinch}\frac{x_2}{2} \end{pmatrix}$$

$$c(\vec{\gamma}_p^T) = |\gamma^3\gamma^4|$$

4.2 The SIM(2) group

Let G denote the $SIM(2)$ i.e. the group of dilations rotations and translations in $\mathbb{R}^2$. $G = \mathbb{R}^2 \rtimes H$ where $H = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} : (a,b) \in \mathbb{R}^2 - \{0,0\}\}$. The Wigner function is defined on $\mathcal{O}^*_{\vec{\gamma}_0^T} = \{(\gamma^1,\gamma^2,\gamma^3,\gamma^4) : (\gamma_3,\gamma_4) \neq (0,0)\}$ (the coadjoint orbit of $\vec{\gamma}_0 = (0,0,1,0)$). This case was studied in [3]. The corresponding Wigner function is :

$$W(\widehat{\phi},\widehat{\psi}|\vec{\gamma}^T) = \frac{(\gamma^3)^2+(\gamma^4)^2}{2\pi}\int_{N_{0q}} e^{-i\gamma_1\lambda - i\gamma_2\theta}\overline{\widehat{\psi}(\vec{\gamma}_p\frac{e^{\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}})}\widehat{\phi}(\vec{\gamma}_p\frac{e^{-\frac{X_q}{2}}}{\operatorname{sinch}\frac{X_q}{2}})$$
$$\frac{\lambda^2+\theta^2}{2\cosh\lambda - 2\cos\theta}d\lambda d\theta \tag{4.2}$$

Where:

$$X_q = \begin{pmatrix} \lambda & -\theta \\ \theta & \lambda \end{pmatrix} \quad \theta \in (0,2\pi), \lambda >= 0$$

$$e^{X_q} = \begin{pmatrix} e^\lambda\cos\theta & -e^\lambda\sin\theta \\ e^\lambda\sin\theta & e^\lambda\cos\theta \end{pmatrix}$$

$$\det\ \operatorname{sinch}\frac{X_q}{2} = \frac{2\cosh\lambda - 2\cos\theta}{\lambda^2+\theta^2}$$

and

$$c(\vec{\gamma}_p^T) = |(\gamma^3)^2 + (\gamma^4)^2|^{-1}$$

Since H is abelian, $\det\,(\operatorname{sinch}\ ad\frac{X_q}{2}) = 1$

4.3 The one parameter family of groups H_c

Consider now the one parameter family of groups $H_c = \{\begin{pmatrix} a & 0 \\ b & a^c \end{pmatrix} : a,b \in \mathbb{R}, a > 0\}$ for $c \neq 0$. The Wigner functions are defined on coadjoint orbits $\mathcal{O}_+ = \{(\gamma^1,\gamma^2,\gamma^3,\gamma^4 : \gamma^4 > 0\}$ and $\mathcal{O}_- = \{(\gamma^1,\gamma^2,\gamma^3,\gamma^4 : \gamma^4 < 0\}$

In both cases the Wigner Function takes the form:

$$W(\widehat{\phi},\widehat{\psi}|\vec{\gamma}^T) = \frac{(|\gamma^4|^2)}{2\pi}\int_{N_{0q}} e^{-i\gamma^1 x_1 - i\gamma^2 x_2}\overline{\widehat{\psi}(\vec{\gamma}_p\frac{e^{\frac{X_q}{2}}}{\text{sinch }\frac{X_q}{2}})}\widehat{\phi}(\vec{\gamma}_p\frac{e^{-\frac{X_q}{2}}}{\text{sinch }\frac{X_q}{2}})$$
$$\frac{1}{\text{sinch }\frac{x_1c}{2}}\left(\frac{\text{sinch }\frac{x(c-1)}{2}}{\text{sinch }\frac{x_1}{2}\text{ sinch }\frac{x_1c}{2}}\right)^{\frac{1}{2}} \tag{4.3}$$

Where:

$$X_q = \begin{pmatrix} x_1 & 0 \\ x_2 & cx_1 \end{pmatrix}$$

$$e^{X_q} = \begin{pmatrix} e^{x_1} & 0 \\ \frac{x_2}{(c-1)x_1}(e^{x_1c} - e^{x_1}) & e^{x_1c} \end{pmatrix}$$

(in the case c=1 we should take the $\lim_{c\to 1}$)

$$\text{sinch }\frac{X_q}{2} = \begin{pmatrix} \text{sinch }\frac{x_1}{2} & 0 \\ \frac{1}{1-c}\frac{x_2}{x_1}(\text{sinch }\frac{x_1}{2} - \text{sinch }\frac{cx_1}{2}) & \text{sinch }\frac{cx_1}{2} \end{pmatrix}$$

$$\det(\text{sinch }ad\frac{X_q}{2}) = \text{sinch }\frac{(c-1)x_1}{2}$$

and $c(\vec{\gamma}_p) = |\gamma^4|^{-2}$

4.4 Quaternionic groups

Quaternions constitute a (nonabelian) field of numbers; they can be thought of as an extension of complex numbers in a similar way as complex numbers are an extension of the real ones. More specifically they are obtained by adding two more "imaginary units" customarily denoted by $\mathbf{j}, \mathbf{k}$ such that the following relations are fulfilled:

$$\mathbf{j}^2 = \mathbf{i}^2 = \mathbf{k}^2 = -1;\ \mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}.$$

The generic quaternion can be written as $x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ where x_0, x_1, x_2, x_3 are real numbers, or as $z_0 + z_1\mathbf{j}$, where $z_0 = x_0 + x_1\mathbf{i}$ and $z_1 = x_2 + x_3\mathbf{i}$ are complex numbers.

A very practical way of dealing with quaternions is to represent them as 2×2 matrices with complex entries

$$q := \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} .$$

By identifying the set of quaternions with $\mathbb{R}^4$ one can endow this latter space with a notion of multiplication.

It is also worthwhile to recall that any nonzero quaternion admits a (multiplicative) inverse which can be expressed by taking the inverse of the matrix representing it.

Let us consider now the semidirect product group $G = H \rtimes H^*$ where H denotes the vector space of quaternions and H^* the group of invertible quaternions. An element of the group can be written in the form:

$$g = \begin{pmatrix} h_q & h_p \\ 0 & 1 \end{pmatrix}$$

where $h_q \in H^*$ and $h_p \in H$. Then the element of the lie algebra $\mathbf{g} = Lie(G)$ is:

$$X = \begin{pmatrix} X_q & X_p \\ 0 & 0 \end{pmatrix}$$

Where X_q and X_p are both quaternions which can be written in coordinates as :

$$X_q = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} \quad X_p = \begin{pmatrix} x_4 + ix_5 & x_6 + ix_7 \\ -x_6 + ix_7 & x_4 - ix_5 \end{pmatrix}$$

The group can be equivalently written, in a manner more consistent with the rest of the thesis, as $\mathbb{R}^4 \rtimes M(h_q)$ where $M(h_q) \in GL(4, \mathbb{R})$ is of the form :

$$M(h_q) = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}$$

The quaternionic notation makes it easy to relate this group to the $G_1 = \mathbb{R} \rtimes \mathbb{R}^*$ and $G_2 = \mathbb{C} \rtimes \mathbb{C}^* = SIM(2)$, which are the wavelet groups in 1 and 2 dimensions respectively. It seems quite natural to use the field of quaternions to define a wavelet group in 4 dimensions. The concept of wavelet groups can be therefore extended (in rather straightforward way) to any Clifford algebra.

The Wigner function is defined on the single coadjoint orbit $\mathcal{O}^* = \mathbf{g}^* - \{0\}$.

$$W(\widehat{\phi}, \widehat{\psi}|\vec{X}^*) = \frac{|X_p^*|^4}{(2\pi)^4} \int_{\mathbf{R}^4} e^{-i<X_q^*, X_q>} \overline{\widehat{\psi}(X_p^* \frac{e^{\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}})} \widehat{\phi}(X_p^* \frac{e^{-\frac{X_q}{2}}}{\operatorname{sinch} \frac{X_q}{2}})$$
$$\frac{1}{16} \frac{|X_q|^4}{(\cosh^2 \frac{x_0}{2} - \cos^2 \frac{R}{2})^2} \frac{\sin R}{R} \mathrm{d}X_q \tag{4.4}$$

where $R = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ We also used :

$$c(X^*) = |X_p^*|^{-4}$$

$$\mathrm{d}\mu_G(\mathrm{e}^X) = \det(e^{-\frac{X_q}{2}} \operatorname{sinch} \frac{X_q}{2}) \frac{\sin R}{R} \mathrm{d}X$$

All those computation can be easily repeated for any Clifford algebra.

Chapter 5

Conclusions and outlook

In this thesis we have considered the particularly relevant case of Wigner functions associated to open and free actions of groups which generalize the well-known wavelet and similitude groups.
Within this framework, we have classified connected semidirect product groups $G = \mathbb{R}^n \rtimes H$ admitting open free H-orbits in $\widehat{\mathbb{R}}^n$ in dimensions $n = 3$ and $n = 4$, the latter with the further assumption that there exists a semisimple ideal. The starting point for this classification was the use of a particular quasi-invariant function Δ, which can be defined in any dimension. Our classification extends the results of H. Führ which treated the case $n = 2$, however, using different methods.

Subsequently we have undertaken a detailed study of the square-integrable representations of such groups. The geometry of their coadjoint orbits allows us to construct the corresponding Wigner functions explicitly. In attempting to interpret the support of such Wigner functions as phase spaces(symplectic manifolds) we have also proved that if the H-orbits $\widehat{\mathcal{O}}_i$ in $\widehat{\mathbb{R}}^n$ are dihedral cones then the Wigner transform maps each Hilbert-Schmidt operator $\rho \in L^2(\widehat{\mathcal{O}}_i)$ into a function supported on the corresponding coadjoint orbit $\mathcal{O}_i^*$. Each such orbit has the natural Kirillov symplectic structure and hence can be interpreted as a phase space.

Finally we have implemented the general framework outlined above for some relevant examples, the most interesting being the quaternionic group. Indeed, this group is the most natural candidate for the definition of a 4-dimensional non-abelian wavelet group; it extends the sequence of $\mathbb{R} \rtimes \mathbb{R}_+^\times$ (wavelets), $\mathbb{C} \rtimes \mathbb{C}^\times \simeq SIM(2)$ (similitude group) by $\mathbb{H} \rtimes \mathbb{H}^\times$.

As a possible continuation of this work we may consider generalized wavelet groups defined on Clifford algebras (as all of the three cases $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are), thus providing an infinite class of examples with (possibly) extra features coming from their algebraic structure.

A second generalization would consider the cases of groups with not necessarily open-free action: in this cases one should replace the notion of square integrability with respect to the whole group by a weaker square integrability on a suitable homogeneous space.

A third important direction in which one could move involves groups with non square-integrable representations most notably the kinematical groups, such as the Poincaré or the Galilei groups. Indeed, the construction of the generalized Wigner Function we were using in this thesis is based on the requirement that the group have a square-integrable representation and also that the image of the exponential map be a dense set in G with its complement having Haar measure zero.

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