SMOOTHNESS OF INVARIANT DENSITIES FOR CERTAIN CLASSES OF DYNAMICAL SYSTEMS

Abdusslam Osman

A Thesis

in

The Department

of

Mathematics and Statistics

Presented in Partial Fulfilment of the Requirements

for the Degree of Master of Science at

Concordia University

Montreal, Quebec, Canada

May 1996

SMOOTHNESS OF INVARIANT DENSITIES FOR CERTAIN CLASSES OF DYNAMICAL SYSTEMS

by

Abdusslam Osman

ABSTRACT

Under certain conditions a many-to-one transformation of the unit interval into itself possesses a finite invariant ergodic measure equivalent to the Lebesque measure. The purpose of this thesis is to investigate these conditions and to show how differentiable properties of the invariant density are inherited from the original transformation.

ACKNOWLEDGEMENT

I wish to express my deepest gratitude to my supervisor, Professor P. Góra, from whom I have learned the most about the field of dynamical systems. I would like to thank Dr. Góra for his careful guidance and his generous contributions of time and advice. The numerous discussions I had with him were of invaluable assistance in the completion of this thesis.

I am also grateful to Professor A. Boyarsky and Professor R. Stern for their careful reading and valuable suggestions.

I am grateful to my country Libya for their generous financial support.

Finally, my sincere thanks to my family for their patience.

TABLE OF CONTENTS

INTRODUCTION

In recent years there has been a surge of interest in research related to ergodic theory and dynamical systems. If a dynamical system τ has an invariant measure μ absolutely continuous with respect to Lebesgue measure μ (**acim**) corresponding to which it is mixing, then τ is chaotic. The field of dynamical systems and especially the study of chaotic systems has been hailed as one of the important breakthroughs in science in this century. While the field is still relatively young, there is no question that the field is becoming more and more important in a variety of scientific disciplines. The existence and properties of **acim** for a transformation of an interval have been studied by many authors, under different conditions, here we mention a few:

(1) Rényi's [**Ren**] in 1957 was first to define a class of transformations of the unit interval that satisfies a distortion condition (see Remark 2.1) and proved it has an **acim**.

(2) In 1973 Lasota and Yorke [**L-Y**] proved an important generalization of Renyi's result using the "bounded variation techniques". There, the authors proved a general sufficient condition for **acim** for expanding, piecewise C^2 transformations on the interval.

(3) In 1993 Góra [**Gór**] proved the existence of **acim** for C^1 expanding transformations of an interval, satisfying the Schmitt's [**Sch**] condition.

In 1972, Bowen [**Bow**] and Adler [**Adle**] defined the Markov map in a way that did not imply Rényi's distortion condition. Thus instead they introduced the so called second derivative condition, and proved existence of **acim**.

In chapter 1, we recall some basic results from real analysis and measure theory needed in this thesis.

In chapter 2, we will study the *admissible transformations* and their properties, and prove some results needed in the next chapter.

In chapter 3, which is the main focus of this thesis is to study the smoothness of invariant density of admissible transformations satisfy a distortion condition. Rényi [**Ren**] proved that the invariant density is bounded by the bounds of a distortion condition, if τ is an onto map of the unit interval onto itself and distortion condition is satisfied. For a transformation τ considered by Rényi with $\tau^{-1} \in C^r$, Halfant [**Hal**] proved that the invariant density $h \in C^{r-2}$. The inductive proof of the main result of [**Hal**] is proved only for $r = 2$, and $r = 3$. Although, [**Hal**] conjecture was correct, his final solution was unattainable due to certain obstacles. In this thesis, on the other hand, we give a details proof and the necessary calculations (see Theorem 3.2).

CHAPTER 1

PRELIMINARIES

In this chapter we include some basic definitions and results from real analysis and measure theory needed in the sequel (see [**Roy**],[**L-M**],[**Man**]).

Definition 1.1. A family $\mathfrak{B}$ of subsets of $I = [0,1]$ is called a *σ-algebra*, if and only if :

(1) $I \in \mathfrak{B}$;

(2) for any $B \in \mathfrak{B}$, $I \setminus B \in \mathfrak{B}$;

(3) if $B_n \in \mathfrak{B}$, for $n = 1, 2, \cdots$, then $\cup_{n=1}^{\infty} B_n \in \mathfrak{B}$.

Elements of $\mathfrak{B}$ are usually referred to as *measurable sets*.

Definition 1.2. A function $\mu : \mathfrak{B} \to R^+$ is called a *measure* on $\mathfrak{B}$, if and only if :

(1) $\mu(\emptyset) = 0$;

(2) for any sequence $\{B_n\}$ of disjoint measurable sets, $B_n \in \mathfrak{B}$,

$n = 1, 2, \ldots,$

$$\mu(\cup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n).$$

The triplet $(X, \mathfrak{B}, \mu)$ is called a *measure space*. If $\mu(X) = 1$, we say it is a *normalized measure space* or *probability space*.

Definition 1.3. The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called the *characteristic function of* E. A linear combination

$$s(x) = \sum_{i=0}^{n} a_i \chi_{E_i}(x),$$

is called a *simple function* if the sets E_i are measurable.

Throughout the thesis we denote by λ the *Lebesgue measure* on I.

Let $\tau : [0,1] \to [0,1]$ be a measurable, nonsingular transformation (i.e., $\lambda(\tau(A)) = 0$ implies $\lambda(A) = 0$, for any measurable A).

Definition 1.4. We define the *n-th iterated distribution* $\lambda_n([0,t])$ by

$$\lambda_n([0,t]) = \lambda(\{x \in I : \tau^n x \leq t\}).$$

We note that:

$$0 \leq \tau^n x \leq t \Rightarrow \tau^{-n}(0) \leq x \leq \tau^{-n}(t) \Rightarrow x \in (\tau^{-n}(0), \tau^{-n}(t))$$

$$\Rightarrow x \in \tau^{-n}[0,t].$$

Therefore, we have:

$$\lambda_n([0,t]) = \lambda(\tau^{-n}[0,t]), \quad t \in I.$$

In general, for every Borel set $A \in \mathcal{B}$, we have:

$$\lambda_n(A) = \lambda(\tau^{-n}(A)).$$

Definition 1.5. We say that a measure μ is τ-*invariant* if and only if

$$\mu(A) = \mu(\tau^{-1}A), \qquad \text{for any} \quad A \in \mathcal{B}.$$

Definition 1.6. We define the *invariant density* h by

$$\mu(A) = \int_A h d\lambda, \qquad \text{for any} \quad A \in \mathcal{B},$$

and so $h(t) = \frac{d\mu([0,t])}{dt}$, $t \in I$.

Definition 1.7. We denote by Δ_n the length of the largest interval of rank n:

$$\Delta_n = \max_i \{\tau_i^{-n}(1) - \tau_i^{-n}(0)\}.$$

$$\Delta_n = \max_{p_i \in \mathcal{P}_n} \lambda(P_i)$$

where $\mathcal{P}_n :=$ partition I under τ^{-n}.

Definition 1.8.

Suppose $(I, \mathcal{B}_1, \lambda)$, $(I, \mathcal{B}_2, \mu)$ are probability spaces.

(1) A transformation $\tau : I \to I$ is measurable if $\tau^{-1}(\mathcal{B}_2) \subset \mathcal{B}_1$ (i.e. $B_2 \in \mathcal{B}_2 \Rightarrow \tau^{-1}B_2 \in \mathcal{B}_2$);

(2) A transformation $\tau : I \to I$ is measure-preserving if τ is measurable and $\lambda(\tau^{-1}(B_2)) = \mu(B_2)$, $\forall B_2 \in \mathcal{B}_2$.

Definition 1.9. Let τ be a measure-preserving transformation of a probability space $(I, \mathcal{B}, \mu)$ then, τ is *(strongly) mixing* if $\forall A, B \in \mathcal{B}$

$$\lim_{n \to \infty} \mu(\tau^{-n} A \cap B) = \mu(A)\mu(B).$$

Remark 1.1. We note that

$$\int_{\tau^{-n} A} \chi_{B_i} d\mu = \mu(\tau^{-n} A \cap B_i).$$

Since

$$\chi_{B_i}(x) = \begin{cases} 1 & x \in B_i \\ 0 & x \notin B_i \end{cases}$$

and

$$\tau^{-n} A = (\tau^{-n} A \cap B_i) \cup (\tau^{-n} A \cap B_i^c),$$

we have

$$\begin{aligned} \int_{\tau^{-n} A} \chi_{B_i} d\mu &= \int_{\tau^{-n} A \cap B_i} \chi_{B_i} d\mu + \int_{\tau^{-n} A \cap B_i^c} \chi_{B_i} d\mu \\ &= \int_{\tau^{-n} A} 1 d\mu + \int_{\tau^{-n} A \cap B_i^c} 0 d\mu \\ &= \mu(\tau^{-n} A \cap B_i). \end{aligned}$$

Definition 1.10. If f is a bounded measurable function defined on a measurable set E with $\mu(E)$ finite, we have

$$\int_E f d\mu = \inf_{s \geq f} \int_E s d\mu,$$

where every s is a simple function.

Definition 1.11. The *conditional probability* of a set A with respect to another set B, is defined by

$$\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}.$$

Proposition 1.12. *Let f be defined and bounded on a measurable set with finite measure. Then f can be approximated uniformly by simple functions.*

Proposition 1.13. *Let $\{f_n\}$ be a non-negative, sequence of monotone continuous functions such that $f_n \to f$ as $n \to \infty$. Then the convergence is uniform.*

Definition 1.14. A family $\mathcal{F}$ of function continuous on $[0, 1]$ is said to be *equicontinuous* if for every $\epsilon > 0 \; \exists \; \delta$ such that

$$\mid x - y \mid < \delta \Rightarrow \mid f(x) - f(y) \mid < \epsilon$$

$\forall x, y \in [0, 1]$ and $\forall f \in \mathcal{F}$.

CHAPTER 2

ADMISSIBLE TRANSFORMATION

The main goal of this thesis is to study the smoothness of the invariant density of an *admissible transformation* which satisfy distortion condition. This transformation was defined for the first time in [**Ren**]. There the author establishes the existence of an **acim** for such transformations and proves that the invariant density is bounded by the bounds of a distortion condition.

Definition 2.1. The transformation $\tau : [0,1] \to [0,1]$ is called *admissible* if there exists a partition $0 = a_0 < a_1 < \cdots < a_q = 1$, of $[0,1]$ such that:

(1) $\tau(0) = 0$, $\tau(1) = 1$; for $0 < a_i < 1$, $\tau(a_i) =$ either 0 or 1;

(2) $\tau_i \in C^2[a_{i-1}, a_i]$, $i = 1, \cdots, q$, where $\tau_i = \tau_{|[a_{i-1},a_i]}$;

(3) $\tau_i' > 0$ on $[a_{i-1}, a_i]$, $i = 1, \cdots, q$.

We denote that $i \in \mathcal{I}_n$ and $\mathcal{I}_n = \{i : p_i \in \mathcal{P}_n\}$, where $\mathcal{P}_n :=$ partition I under τ^{-n}, then

$$\tau^{-n} = \tau_{i_1}^{-1} \circ \tau_{i_2}^{-1} \circ \tau_{i_3}^{-1} \circ \cdots \circ \tau_{i_n}^{-1}.$$

Remark 2.1 (Rényi's condition). Set

$$\frac{\sup_{t \in I} (\tau_i^{-n})'(t)}{\inf_{t \in I} (\tau_i^{-n})'(t)} = C_{n,i} \leq \infty,$$

and

$$\max_i C_{n,i} = C_n.$$

We say τ satisfies *Rényi's condition* if and only if

$$\sup_n C_n = C < \infty.$$

Remark 2.2. The above condition states that, for x, y restricted to the same interval in the Markov partition for τ^n, the quantity $|\frac{\tau^{n'}(x)}{\tau^{n'}(y)}|$ should be uniformly bounded, independent of n and the interval chosen.

Remark 2.3. The invariant density h which is defined in Definition 1.6 is shown by Rényi's [**Ren**] to satisfy everywhere on I the inequality

$$\frac{1}{C} \leq h \leq C.$$

We denote by $\mathfrak{R}$ the subset of admissible transformations which fulfill Rényi's condition.

Definition 2.2. Let $(X, \mathcal{B}, \mu)$ be a normalized measure space and let $\tau : X \to X$ be measure-preserving, such that $\tau(A) \in \mathcal{B}$ for each $A \in \mathcal{B}$. If

$$\lim_{n\to\infty} \mu(\tau^n A) = 1,$$

for every $A \in \mathcal{B}, \mu(A) > 0,$ then τ is called *exact*.

Lemma 2.1. *Let $\tau : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving transformation of a normalized measure space. Then, τ is exact, if and only if*

$$\mathcal{B}^T = \bigcap_{n=0}^{\infty} \tau^{-n}(\mathcal{B}),$$

*the tail σ-algebra consists of sets of μ-measure 0 or 1 (see [**L-M**]).*

Lemma 2.2. *If an admissible transformation τ satisfies the distortion condition, then it is exact.*

Proof. Let $\tau^{-n}(\mathcal{B})$ be the σ-algebra consisting of sets of the form $\tau^{-n}A,\ A$ being a Borel set in $\mathcal{B}$. Then $\tau^{-n}(\mathcal{B})$ form a nested decreasing sequence of σ-algebras, let

$$\mathcal{B}^T = \bigcap_{n=1}^{\infty} \tau^{-n}(\mathcal{B}).$$

We want to show that $\mathcal{B}^T$ consists only of sets of μ- measure 0 or 1. Let $[a, b]$ be a subinterval of I, and let

$$\Delta_n^i = [\tau_i^{-n}(0), \tau_i^{-n}(1)] = \tau_i^{-n}[0, 1],$$

denote the interval of rank n. Using the customary notation for conditional probability, we have

$$\begin{aligned} \tau_i^{-n}[a, b] \cap \Delta_n^i &= \tau_i^{-n}[a, b] \cap \tau_i^{-n}[0, 1] \\ &= \tau_i^{-n}[a, b], \quad \text{since} \quad [a, b] \subset [0, 1] \end{aligned}$$

$$\begin{aligned} \lambda(\tau^{-n}[a, b] | \Delta_n^i) &= \frac{\tau_i^{-n}(b) - \tau_i^{-n}(a)}{\tau_i^{-n}(1) - \tau_i^{-n}(0)} \\ &= \frac{(\tau_i^{-n})'(\theta_1)(b - a)}{(\tau_i^{-n})'(\theta_2)} \end{aligned}$$

where $\theta_1 \in (a, b)$ and, $\theta_2 \in (0, 1)$. By Rényi's condition,

$$\frac{1}{C} \le \frac{(\tau_i^{-n})'(\theta_1)}{(\tau_i^{-n})'(\theta_2)} \le C$$

$$\frac{b-a}{C} \le \frac{(\tau_i^{-n})'(\theta_1)(b-a)}{(\tau_i^{-n})'(\theta_2)} \le (b-a)C$$

$$\frac{b-a}{C} \le \lambda(\tau_i^{-n}[a,b]|\Delta_n^i) \le C(b-a).$$

It is clear that we may replace $[a,b]$ by an arbitrary Borel set $A \subset \mathcal{B}$ to obtain

$$\frac{\lambda(A)}{C} \le \lambda(\tau^{-n}A|\Delta_n^i) \le C\lambda(A).$$

Let

$$B = \tau^{-n}A,$$

then, we have

$$\lambda(B|\Delta_n^i) = \frac{\lambda(B \cap \Delta_n^i)}{\lambda(\Delta_n^i)} = \frac{\lambda(B)}{\lambda(\Delta_n^i)}$$

(1) $$\frac{\lambda(A)\lambda(\Delta_n^i)}{C} \le \lambda(B) \le C\lambda(A)\lambda(\Delta_n^i)$$

and by Definition 1.6 and Remark 2.3, it follows that

(2) $$\frac{1}{C}\lambda(A) \le \mu(A) \le C \cdot \lambda(A).$$

This is true for any $A \in \mathcal{B}$, thus it is true for $B \in \mathcal{B}$, thus by (1)

(3) $$\frac{\lambda(A)\lambda(\Delta_n^i)}{C^2} \le \frac{1}{C}\lambda(B) \le \mu(B) \le C\lambda(B) \le C^2\lambda(A)\lambda(\Delta_n^i)$$

Then from (2) we have the following equations

(4) $$\mu(A) \le C\lambda(A) \Rightarrow \frac{\mu(A)}{C} \le \lambda(A),$$

and

$$\mu(A) \geq \frac{1}{C}\lambda(A) \Rightarrow \lambda(A) \leq C\mu(A), \tag{5}$$

By (3), we have

$$\frac{\lambda(A)\lambda(\Delta_n^i)}{C^2} \leq \mu(B) \leq C^2\lambda(A)\lambda(\Delta_n^i). \tag{6}$$

Therefore by (4), (5) and (6) we get:

$$\frac{\mu(A)\lambda(\Delta_n^i)}{C^3} \leq \mu(B) \leq C^3\mu(A)\lambda(\Delta_n^i).$$

Similarly by Definition 1.6 and Remark 2.3, it follows that

$$\frac{1}{C}\lambda(\Delta_n^i) \leq \mu(\Delta_n^i) \leq C\lambda(\Delta_n^i)$$

and

$$\frac{\mu(\Delta_n^i)}{C} \leq \lambda(\Delta_n^i) \leq C\mu(\Delta_n^i)$$

therefore

$$\frac{\mu(A)\mu(\Delta_n^i)}{C^4} \leq \mu(B) \leq C^4\mu(A)\mu(\Delta_n^i).$$

Suppose now that A is a set in the tail σ-algebra $\mathcal{B}^T$. Then, for any n, there is a Borel set B such that $A = \tau^{-n}B$. Thus from(4), we obtain

$$\frac{\mu(A)}{C^4} = \frac{\mu(\tau^{-n}B)}{C^4} = \frac{\mu(B)}{C^4} \leq \mu(\tau^{-n}B|\Delta_n^i) = \mu(A|\Delta_n^i)$$

If $\mu(A) \geq 0$, then we may write

$$\frac{\mu(\Delta_n^i)\mu(A \cap \Delta_n^i)}{\mu(\Delta_n^i)} = \mu(A) \cdot \frac{\mu(\Delta_n^i \cap \mu(A)}{\mu(A)}$$

$$\mu(\Delta_n^i)\mu(A|\Delta_n^i) = \mu(A)\mu(\Delta_n^i|A)$$

$$\mu(\Delta_n^i) = \frac{\mu(A)\mu(\Delta_n^i|A)}{\mu(A|\Delta_n^i)} \leq C^4 \mu(\Delta_n^i|A). \tag{7}$$

The fundamental interval may be used to generate Borel sets. Therefore from (7), we deduce

$$\mu(A^c) \leq C^4 \mu(A^c|A),$$

for an arbitrary Borel set A^c (the complement of A). We then have

$$\mu(A^c) \leq C^4 \mu(A^c|A) = C^4 \cdot \frac{\mu(A^c \cap A)}{\mu(A)} = 0$$

and with $\mu(A^c) = 0$ follows $\mu(A) = 1$. This shows that $\mathcal{B}^T$ contains only sets of μ-measure 0 or 1. This completes the proof of exactness of τ. □

Lemma 2.3. *A transformation τ admissible and satisfies the distortion condition, if for any $n \geq 1$, we have*

$$\Delta_n \leq \beta^n,$$

where $\beta = \max_j \sup_{t \in I} \left(\tau_j^{-1}\right)'(t)$ and Δ_n as in Definition 1.7.

Proof. We observe that P_1: $\beta = \max_j \sup_{t \in I} (\tau_j^{-1})'(t)$, the length of an interval of rank 1 is given by

$$\Delta_1 = \tau_i^{-1}(1) - \tau_i^{-1}(0).$$

By the Mean Value Theorem $\tau_i^{-1}(1) - \tau_i^{-1}(0) = (\tau_i^{-1})'(\theta)(1-0)$, for some $\theta \in (0,1)$

$$\Delta_1 \leq \beta,$$

so P_1 is true. We now proceed to prove the result by induction.
Assume P_n, i.e.,

$$\Delta_n \leq \beta^n.$$

To prove P_{n+1}, we note that

$$\Delta_{n+1} = \tau_i^{-n-1}(1) - \tau_i^{-n-1}(0) = \tau_j^{-1}(\tau_k^{-n}(1)) - \tau_j^{-1}(\tau_k^{-n}(0)). \tag{8}$$

Once again, by the Mean Value Theorem, we have

$$\tau_j^{-1}(\tau_k^{-n}(1)) - \tau_j^{-1}(\tau_k^{-n}(0)) = (\tau_j^{-1})'(\alpha)(\tau_k^{-n}(1) - \tau_k^{-n}(0)), \tag{9}$$

for some $\alpha \in (\tau_k^{-n}(0), \tau_k^{-n}(1)) \subset [0,1]$.
Now note that

$$\tau_k^{-n}(1) - \tau_k^{-n}(0) = \Delta_n \leq \beta^n \tag{10}$$

(by P_k) implies

$$(\tau_j^{-1})'(\alpha) \leq \beta. \tag{11}$$

Using (10) and (11) in (9) and then replacing it in (8), we get

$$\Delta_{n+1} \leq \beta^{n+1},$$

which completes the proof. □

Definition 2.3. We define the quantities M and α as follows:

$$M = \max_j \sup_{t \in I} \mid (\tau_j^{-1})''(t) \mid < \infty$$

and

$$\alpha = \min_j \inf_{t \in I} (\tau_j^{-1})'(t) > 0.$$

Lemma 2.4. *A transformation τ admissible and satisfies the distortion condition, if*

$$C_{n+1} \leq C_n(1 + \frac{M}{\alpha}\Delta_n),$$

where Δ_n as in Definition 1.7, α and M as in Definition 2.3.

Proof. We use the functional relationship

$$\tau_i^{-n-1}(t) = \tau_j^{-1}(\tau_k^{-n}(t)).$$

Differentiating both side, by the chain rule, we get

$$(\tau_i^{-n-1})'(t) = \left(\tau_j^{-1}\right)'(\tau_k^{-n}(t))(\tau_k^{-n})'(t).$$

Taking the supremum on both sides, leads to

$$\sup_{t \in I}(\tau_i^{-n-1})'(t) = \sup_{t \in I}(\left(\tau_j^{-1}\right)'(\tau_k^{-n}(t))(\tau_k^{-n})'(t))$$

and

$$\sup_{t \in I}(\tau_i^{-n-1})'(t) \leq \sup_{t \in I}(\tau_k^{-n})'(t) \sup_{t \in I}\left(\tau_j^{-1}\right)'(\tau_k^{-n}(t)). \tag{12}$$

Similarly taking the infimum over the reciprocal of both sides, yields

$$\frac{1}{\inf_{t\in I}(\tau_i^{-n-1})'(t)} = \frac{1}{\inf_{t\in I}(\tau_k^{-n})'(t)\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)},$$

and

$$\frac{1}{\inf_{t\in I}(\tau_i^{-n-1})'(t)} \leq \frac{1}{\inf_{t\in I}(\tau_k^{-n})'(t)\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}. \tag{13}$$

Therefore we use (12) and (13), and arrive at

$$\frac{\sup_{t\in I}(\tau_i^{-n-1})'(t)}{\inf_{t\in I}(\tau_i^{-n-1})'(t)} \leq \frac{\sup_{t\in I}(\tau_k^{-n})'(t)}{\inf_{t\in I}(\tau_k^{-n})'(t)} \cdot \frac{\sup_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}{\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)},$$

which by use of

$$C_{n,i} = \frac{\sup_{t\in I}(\tau_i^{-n})'(t)}{\inf_{t\in I}(\tau_i^{-n})'(t)},$$

can be written as

$$C_{n+1,i} \leq C_{n,k} \cdot \frac{\sup_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}{\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}.$$

Thus, we have

$$\max_i C_{n+1,i} \leq \max_k C_{n,k} \cdot \max_j \frac{\sup_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau^{-n}(t)\big)}{\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}.$$

Therefore, by Rényi's distortion condition, we get

$$C_{n+1} \leq C_n \cdot \max_j \frac{\sup_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}{\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}.$$

To complete the proof, we need to show

$$\max_j \frac{\sup_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)}{\inf_{t\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(t)\big)} \leq 1 + \frac{M}{\alpha}\Delta_n.$$

Now,

$$\begin{aligned}
\frac{\sup_{\xi\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\xi)\big)}{\inf_{\eta\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\eta)\big)} &= 1 + \frac{\sup_{\xi\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\xi)\big) - \inf_{\eta\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\eta)\big)}{\inf_{\eta\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\eta)\big)} \\
&= 1 + \frac{\sup_{\theta\in I}\big(\tau_j^{-1}\big)''\big(\tau_k^{-n}(\theta)\big)\big(\tau_k^{-n}(\xi) - \tau_k^{-n}(\eta)\big)}{\inf_{\eta\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\eta)\big)} \\
&= 1 + \frac{\sup_{\theta\in I}\big(\tau_j^{-1}\big)''\big(\tau_k^{-n}(\theta)\big)}{\inf_{\eta\in I}\big(\tau_j^{-1}\big)'\big(\tau_k^{-n}(\eta)\big)} \cdot \Delta_n^k \\
&\leq 1 + \frac{\sup_{\theta\in I} \mid \big(\tau_j^{-1}\big)''(\theta) \mid}{\inf_{\eta\in I}\big(\tau_j^{-1}\big)'(\eta)} \cdot \Delta_n.
\end{aligned}$$

Taking the supremum on both sides completes the proof. □

Lemma 2.5. *A transformation τ admissible and satisfies the distortion condition, if for all k, we have:*

$$C_{k+1} \leq C_1 \prod_{n=1}^{k} \big(1 + \frac{M}{\alpha}\Delta_n\big),$$

where Δ_n as in Definition 1.7, α and M as in Definition 2.3.

Proof. We shall prove the Lemma by induction. For $k = 1$, by Lemma 2.4, we have

$$C_2 \leq C_1(1 + \frac{M}{\alpha}\Delta_1) = C_1 \prod_{n=1}^{1}(1 + \frac{M}{\alpha}\Delta_1).$$

We assume the inductional hypothesis for $k = i - 1$ and $i > 3$, i.e.,

$$C_i \leq C_1 \prod_{n=1}^{i-1}(1 + \frac{M}{\alpha}\Delta_n).$$

Now we perform the induction step. Let $k = i$. By Lemma 2.4, we have

$$\begin{aligned} C_{i+1} &\leq C_i(1 + \frac{M}{\alpha}\Delta_i) \\ &\leq C_1 \prod_{n=1}^{i-1}(1 + \frac{M}{\alpha}\Delta_n) \cdot (1 + \frac{M}{\alpha}\Delta_i) \\ &= C_1 \prod_{n=1}^{i}(1 + \frac{M}{\alpha}\Delta_n). \quad \square \end{aligned}$$

Remark 2.4. Taking the supremum over k on both sides in Lemma 2.5, leads to

$$\sup_k C_k \leq C_1 \prod_{n=1}^{\infty}(1 + \frac{M}{\alpha}\Delta_n);$$

thus

$$C \leq C_1 \prod_{n=1}^{\infty}(1 + \frac{M}{\alpha}\Delta_n).$$

Theorem 2.1. *If $\sum_{n=1}^{\infty} \Delta_n < \infty$, then the transformation τ is admissible and satisfying the distortion condition, i.e., $\tau \in \mathfrak{R}$.*

Proof. We have $\sum_{n=1}^{\infty} \Delta_n < \infty$ if and only if $\prod_{n=1}^{\infty}(1 + \frac{M}{\alpha}\Delta_n) < \infty$.

from Lemma 2.3 and Remark 2.4 we obtain

$$C_n \leq C_1 \prod_{n=1}^{\infty}\left(1 + \frac{M}{\alpha}\beta^n\right) < \infty \quad \text{for all} \quad n. \quad \square$$

Lemma 2.6. *If transformation τ is admissible and satisfying the distortion condition, i.e., $\tau \in \mathfrak{R}$, then $\Delta_n \to 0$ as $n \to \infty$*

Proof. Since the sequence Δ_n is monotonically decreasing, we can suppose there exists $c > 0$ such that $\Delta_n \to c$. Hence, there exists an interval $[a, b] \subset I$ such that $b - a = c$ with no fundamental end point in (a, b). We now examine the following 3 cases:

Case1: a is an endpoint, and b is not.

Case2: b is an endpoint, and a is not.

Case3: a and b are not fundamental end points.

We consider the most general case3. There are fundamental end points arbitrary close on either side of $[a, b]$. Now given $\epsilon > 0$, there exists an n such that

$$[a, b] \subset [\tau_i^{-n}(0), \tau_i^{-n}(1)]$$

where $| a - \tau_i^{-n}(0) | < \epsilon$ and $| \tau_i^{-n}(1) - b | < \epsilon$.

Now $\tau_i^{-n}(a) \in (\tau_i^{-n}(0), \tau_i^{-n}(1)) \subset [a - \epsilon, b + \epsilon]$; however, we cannot have $\tau_i^{-n}(a) \in (a, b)$, since that would imply the existence of m and j such that $\tau_j^{-m}(0)$ is arbitrary close to a. Thus we would have $\tau_i^{-n}(\tau_j^{-m}(0)) \in (a,b)$ for some m, j. But $\tau_i^{-n}(\tau_j^{-m}(0)) = \tau_k^{-n-m}(0)$ is a fundamental endpoint, contradicting the construction of $[a,b]$. Thus, we must have either

(3i) $\tau_i^{-n}(a) \in [a-\epsilon, a]$ or

(3ii) $\tau_i^{-n}(a) \in [b, b+\epsilon]$.

If (3i) takes place, we then have

$$\tau_i^{-n}(a) - \tau_i^{-n}(0) = (\tau_i^{-n})'(\xi)a \leq \epsilon$$

and

$$\tau_i^{-n}(1) - \tau_i^{-n}(a) = (\tau_i^{-n})'(\eta)(1-a) \geq c;$$

Consequently,

$$\frac{(\tau_i^{-n})'(\eta)}{(\tau_i^{-n})'(\xi)} \geq \frac{ac}{1-a} \cdot \frac{1}{\epsilon}$$

and since ϵ is arbitrary, Rényi's condition is violates. Likewise, if (3ii) holda, we have

$$\tau_i^{-n}(a) - \tau_i^{-n}(0) = (\tau_i^{-n})'(\xi)a \geq c$$

and

$$\tau_i^{-n}(1) - \tau_i^{-n}(a) = (\tau_i^{-n})'(\eta)(1-a) \leq \epsilon,$$

so

$$\frac{(\tau_i^{-n})'(\xi)}{(\tau_i^{-n})'(\eta)} \geq \frac{c(1-a)}{a} \cdot \frac{1}{\epsilon},$$

again contradicting Rényi's condition. In cases (1) or (2), the same kind of argument applies. □

Lemma 2.7. *If $\Delta_{2N} \leq C(\Delta_N)^2$, then*

$$\Delta_{2^n N} \leq C^{2^n} \frac{(\Delta_N)^{2^n}}{C}$$

for every integer N.

Proof. We prove the result by induction. For $n = 1$

$$\Delta_{2N} \leq C(\Delta_N)^2 \tag{14}$$

i.e.,

$$\Delta_{2^1 N} \leq C^{2^1} \frac{(\Delta_N)^{2^1}}{C}.$$

Assume the result is true for $n = k$, i.e.,

$$\Delta_{2^k N} \leq C^{2^k} \frac{(\Delta_N)^{2^k}}{C}.$$

For $n = k+1$, by (14), we have

$$\Delta_{2^{k+1} N} = \Delta_{2(2^k N)} \leq C \cdot (\Delta_{2^k N})^2.$$

Therefore we have

$$\begin{aligned}\Delta_{2^{k+1} N} &\leq C \cdot (C^{2^k} \frac{(\Delta_N)^{2^k}}{C})^2 \\ &= C \cdot (C^{2^{k+1}} \frac{(\Delta_N)^{2^{k+1}}}{C^2}) \\ &= C^{2^{k+1}} \frac{(\Delta_N)^{2^{k+1}}}{C}. \quad \square\end{aligned}$$

Theorem 2.2. *If a transformation τ is admissible and satisfying the distortion condition, i.e., $\tau \in \mathfrak{R}$, then $\sum_n \Delta_n < \infty$.*

Proof. For every n by the Mean Value Theorem there exists a θ_1, such that

$$\tau_i^{-n}(1) - \tau_i^{-n}(0) = (\tau_i^{-n})'(\theta_1)(1-0) = (\tau_i^{-n})'(\theta_1) = \Delta_n^i \leq \Delta_n. \tag{15}$$

Rényi's condition, implies

$$\sup_{t\in I}(\tau_i^{-n})'(t) \leq \inf_{t\in I}(\tau_i^{-n})'(t)C,$$

and so

$$\sup_{n\in I}(\tau_i^{-n})'(t) \leq (\tau_i^{-n})'(\theta_1)C$$

and

$$\sup_{t\in I}(\tau_i^{-n})'(t) \leq C\Delta_n. \tag{16}$$

Next, we note that, by the Mean Value Theorem, there exists an $\theta_2 \in (\tau_j^{-n}(0), \tau_j^{-n}(1))$ such that

(17)

$$\begin{aligned}\tau_j^{-n}(\tau_i^{-n}(1)) - \tau_j^{-n}(\tau_i^{-n}(0)) &= (\tau_j^{-n})'(\theta_2)(\tau_i^{-n}(1) - \tau_i^{-n}(0)) \\ &= (\tau_j^{-n})'(\theta_2)\Delta_n \\ &\leq C(\Delta_n)^2.\end{aligned}$$

where the last two inequalities are in consequence of (15) and (16). Now note that for some k

$$\tau_j^{-n}(\tau_i^{-n}(1)) = \tau_k^{-2n}(1) \quad \text{and} \quad \tau_j^{-n}(\tau_i^{-n}(0)) = \tau_k^{-2n}(0)$$

and thus

$$\tau_j^{-n}(\tau_i^{-n}(1)) - \tau_j^{-n}(\tau_i^{-n}(0)) = \tau_k^{-2n}(1) - \tau_k^{-2n}(0) = \Delta_{2n}^k. \tag{18}$$

From (17) and (18), we get

$$\Delta_{2n}^k \leq C(\Delta_n)^2.$$

Taking the supremum over k, we get

$$\Delta_{2n} \leq C(\Delta_n)^2.$$

By Lemma 2.6, we have $\Delta_n \to 0$ and thus we may choose N so large that $\Delta_N < \frac{1}{2C}$. We then have $C\Delta_N = \rho < \frac{1}{2}$, which yields

$$\Delta_{2N} \leq C(\Delta_N)^2.$$

By Lemma 2.7, we have

$$\Delta_{2^n N} \leq C^{2^n} \frac{(\Delta_N)^{2^n}}{C} = \frac{\rho^{2^n}}{C}, \tag{19}$$

and since $\Delta_k \geq \Delta_{k+1}$,

$$\begin{aligned} \sum_{k=2^n N}^{2^{n+1}N-1} \Delta_k &\leq (\#\text{of terms}) \cdot \Delta_{2^n N} \\ &= (2^{n+1}N - 1 - 2^n N + 1)\Delta_{2^n N} \\ &= 2^n N(2-1)\Delta_{2^n N} \\ &= 2^n N \Delta_{2^n N}. \end{aligned} \tag{20}$$

From (19) and (20) we have the following

$$\sum_{k=2^n N}^{2^{n+1}N-1} \Delta_k \leq 2^n N \cdot \frac{\rho^{2^n}}{C}, \tag{21}$$

$$\sum_{k=1}^{\infty} \Delta_k = \sum_{k=1}^{2N-1} \Delta_k + \sum_{k=2N}^{2^2 N-1} \Delta_k + \sum_{2^2 N}^{2^3 N-1} \Delta_k + \cdots$$

$$\sum_{k=1}^{\infty} \Delta_k = \sum_{k=1}^{2N-1} \Delta_k + \sum_{j=1}^{\infty} \sum_{k=2^j N}^{2^{j+1}N-1} \Delta_k. \tag{22}$$

Let $\sum_{k=1}^{2N-1} \Delta_k = \delta < \infty$. Then from (21) and (22) we get

$$\begin{aligned} \sum_{k=1}^{\infty} \Delta_k &\leq \delta + \sum_{j=1}^{\infty} 2^j N \cdot \frac{\rho^{2^j}}{C} \\ &= \delta + \frac{N}{C} \sum_{j=1}^{\infty} 2^j \rho^{2^j}. \end{aligned} \tag{23}$$

Furthermore, since $\rho < \frac{1}{2}$ and $\rho^2 < \rho$, it follows that

$$\begin{aligned} \sum_{j=1}^{\infty} 2^j \rho^{2^j} &< \sum_{j=1}^{\infty} 2^j \rho^j \\ &= \sum_{j=1}^{\infty} (2\rho)^j, \end{aligned}$$

which is a convergent geometric series with ratio equal to $2\rho < 1$, ans sum

$$\sum_{j=1}^{\infty}(2\rho)^j = \frac{2\rho}{1-2\rho}.$$

Therefore, finally we obtain

$$\sum_{k=1}^{\infty}\Delta_k < \delta + \frac{N}{C}\cdot\frac{2\rho}{1-2\rho} < \infty. \qquad \square$$

Theorem 2.3.

$$\inf_{\xi\in I}(\tau^n(\xi))' > 1, \quad \textit{for some } n \textit{ if and only if } \tau \in \mathfrak{R}.$$

Proof. $\Rightarrow$ It is sufficient to prove $\sum_n \Delta_n < \infty$. Recall that

$$\Delta_n = \tau_i^{-n}(1) - \tau_i^{-n}(0) = (\tau_i^{-n})'(\theta)(1-0)$$

we note

$$\frac{1}{(\tau_i^{-n})'(\theta)} = \tau_i^n(\tau_i^{-n}(\theta))'.$$

Let N be the smallest integer for which

$$\inf_{\xi\in I}(\tau^N(\xi))' = \beta > 1$$

and fix n such that $n \geq N$ and therefore

$$n = q\cdot N + r, \quad \text{where } 0 \leq r < N.$$

Now

$$\tau^n = \tau^r \circ \underbrace{\tau^N \circ \tau^N \circ \tau^N \circ \cdots \circ \tau^N}_{q\text{-times}}$$

and

$$(\tau^n)' = \left(\tau^r\right)'\left(\tau^{qN}\right) \cdot \left(\tau^N\right)'\left(\tau^{(q-1)N}\right) \cdots \left(\tau^N\right)' .$$

Thus

$$\frac{1}{\Delta_n} = \left(\tau^n\right)'(\xi) = \left(\tau^r\right)'\left(\tau^{qN}(\xi)\right) \cdot \left(\tau^N\right)'\left(\tau^{(q-1)N}(\xi)\right) \cdots \left(\tau^N\right)'(\xi)$$

and

$$\frac{1}{\Delta_n} \geq \inf_{\xi_0 \in I}\left(\tau^r\right)'(\xi_0) \cdot \inf_{\xi_q \in I}\left(\tau^{qN}\right)'(\xi_q) \cdots \inf_{\xi_1 \in I}\left(\tau^N\right)'(\xi_1).$$

Since $r < N$

$$0 < \inf_{\xi_0 \in I}(\tau^r)'(\xi_0) = \alpha \leq 1$$

and thus

$$\frac{1}{\Delta_n} \geq \alpha \cdot \beta^q,$$

where $q = [\frac{n}{N}]$ for $n \geq N$. We therefore have

$$\sum_{k=1}^{\infty} \Delta_k = \sum_{k=1}^{N-1} \Delta_n + \sum_{k=N}^{\infty} \Delta_k \leq N-1+\sum_{k=1}^{\infty} N \cdot \Delta_{kN} \leq N-1+\frac{N}{\alpha}\sum_{k=1}^{\infty} \beta^{-k}.$$

Since $\sum_{k=1}^{\infty} \beta^{-k}$ is a geometric series with ratio $\beta^{-1} < 1$, it converges. Therefore, $\sum_{k=1}^{\infty} \Delta_k < \infty$.

(The 'only if' part) By Theorem 2.1 we have

$$\tau \in \mathfrak{R} \Longleftrightarrow \sum_n \Delta_n < \infty,$$

so that $\Delta_n \to 0$ implies that there exists N and $\epsilon > 0$ such that $\Delta_n < \epsilon$. Then,

$$(\tau^{-n})'(\theta) < \epsilon, \quad \forall \quad \theta \in I$$

i.e.,

$$\frac{1}{(\tau^{-n})'(\theta)} > \frac{1}{\epsilon}, \quad \forall \quad \theta \in I.$$

Therefore

$$\inf_{\xi \in I} (\tau^n)'(\xi) > \frac{1}{\epsilon} > 1. \quad \square$$

Theorem 2.3 shows that there are certain transformations which do not belong to $\mathfrak{R}$. We present here two such transformations and using Theorem 2.3 we will show that one of these transformations is in $\mathfrak{R}$ and the other is not.

Example 2.1. *Let*

$$\tau : x \to 2x + \frac{3}{10}\sin(2\pi x) \quad (mod\ 1)$$

$$\tau(x) = 2x + \frac{3}{10}\sin(2\pi x)$$

$$\tau'(x) = 2 + \frac{3}{10}\cos(2\pi x) \cdot 2\pi$$

$$= 2 + \frac{6\pi}{10}\cos(2\pi x)$$

$$\min(\tau)' = 2 - \frac{3\pi}{5} \approx 0.115.$$

We find by numerical methods that $\inf_{\xi \in I}(\tau^5(\xi))' > 1$ *hence* $\tau \in \mathfrak{R}$.

Figure 1 illustrates the graph of $(\tau^5)'$.

Graph of $(\tau^5)'$.

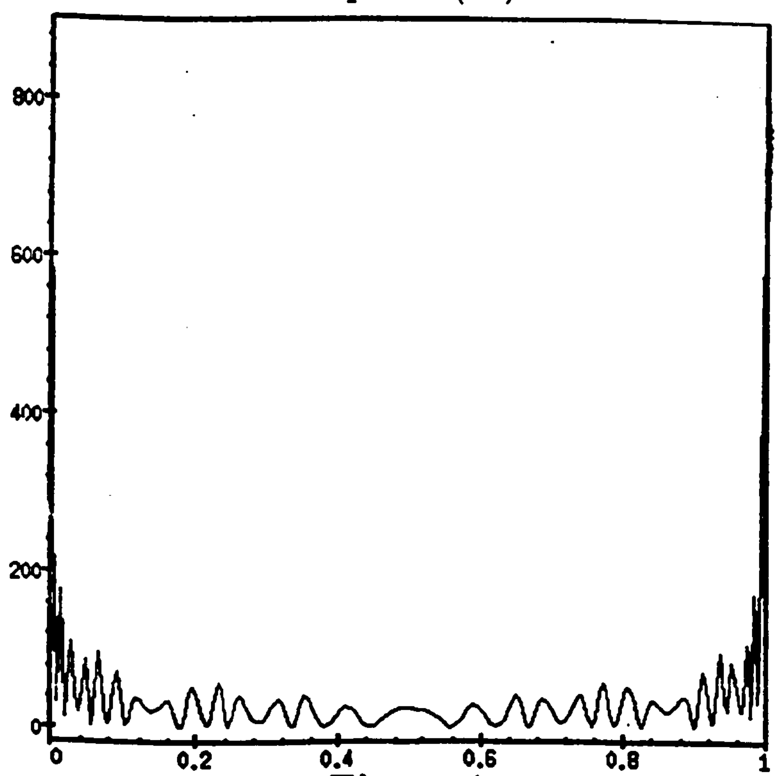

Figure 1.

In Figure 2 we take a closer look at $\inf_{\xi \in I}(\tau^5(\xi))'$.

Example 2.2. *Let*

$$\tau : x \to x + x^2 \ \ (mod\ 1).$$

We shall show that τ *is not in* $\mathfrak{R}$.

It is enough to show any iterate of τ will have derivative equal to 1 at $x = 0$.

Graph of $(\tau^5)'$.

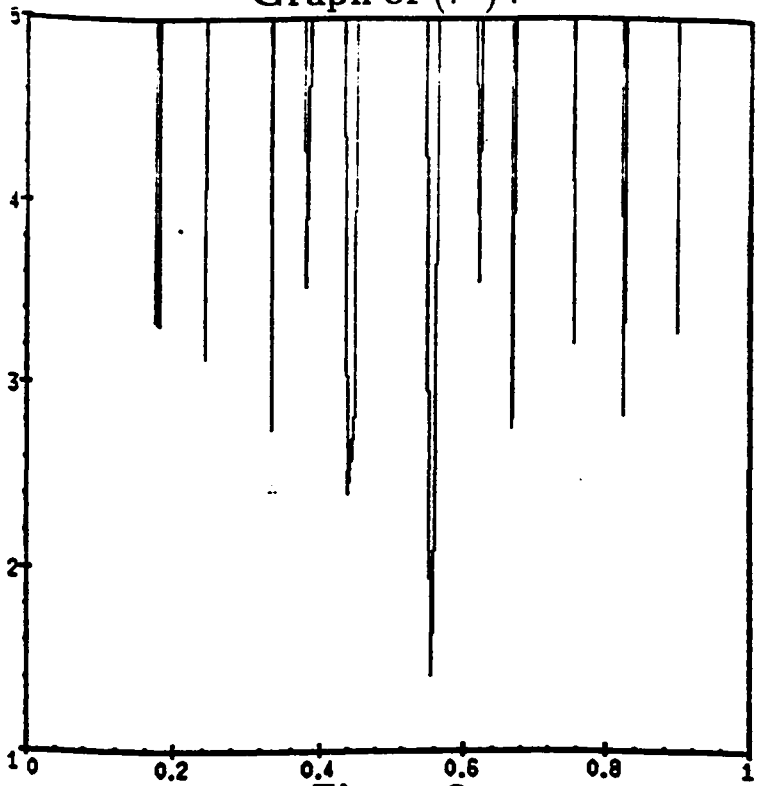

Figure 2.

Remark 2.5: We first show that for

$$\tau(x) = x + x^2$$

we have

$$\tau^n(x) = x + \sum_{i=2}^{2n} \alpha_i x^i \quad \text{for } \alpha_i \in \mathbb{N} \cup \{0\}. \tag{24}$$

For $n = 1$

$$\tau(x) = x + x^2 = x + \sum_{i=2}^{2} \gamma_i x^i,$$

where $\gamma_i = 1$.

Hence sssume (24) is true for $n = k$, i.e.,

$$\tau^k(x) = x + \sum_{i=2}^{2k} \beta_i x^i,$$

then for $n = k + 1$

$$\begin{aligned}
\tau^{k+1}(x) &= \tau^k(\tau(x)) \\
&= \tau(x) + \sum_{i=2}^{2k} \beta_i (\tau(x))^i \\
&= x + x^2 + \sum_{i=2}^{2k} \beta_i (x + x^2)^i \\
&= x + \sum_{i=2}^{2(k+1)} \alpha_i x^i. \quad \square
\end{aligned}$$

Remark 2.6: Let $f(x) = [P(x)]$, where $P(x)$ is a real polynomial. Then,

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[P(x + \Delta x)] - [P(x)]}{\Delta x}.
\end{aligned}$$

Suppose $k \leq P(x) < k + 1$, ($[P(x)] = k$) the continuity of $P(x)$ implies that given $\epsilon > 0$, $P(x + \Delta x) < P(x) + \epsilon < k + 1$ for Δx sufficentley small ($\mid P(x + \Delta x) - P(x) \mid < \epsilon$, provided $\mid x + \Delta x - x \mid = \mid \Delta x \mid < \delta$ for same suitable $\delta = \delta(a)$). Hence

$$k \leq P(x + \Delta x) < k + 1 \Rightarrow [P(x + \Delta x)] = k$$

and thus it follows that

$$f'(x) = 0.$$

Now let us consider,

$$\tau(x) = x + x^2 \pmod 1.$$

By Remark 2.5 we have

$$\tau^n(x) = x + \sum_{i=2}^{2n} \alpha_i x^i - [x + \sum_{i=2}^{2n} \alpha_i x^i]$$

by Remark 2.6 we have

$$(\tau^n(x))' = 1 + \sum_{i=2}^{2n} i\alpha_i \cdot x^{i-1} - 0.$$

Therefore, $(\tau(x))'_{|x=0} = 1$ and by Theorem 2.3 we conclude that $\tau \notin \mathfrak{R}$. □

CHAPTER 3

SMOOTHNESS OF INVARIANT DENSITIES FOR RÉNYI'S MAPS

In this chapter we prove the main results of this thesis. We will study the smoothness of invariant density of admissible transformations satisfying the distortion condition,. For a transformation τ considered by Rényi with $\tau^{-1} \in C^r$, Halfant [**Hal**] proved that the invariant density $h \in C^{r-2}$. The inductive proof of the main result of [**Hal**] is proved only for $r = 2$, and $r = 3$. Although the conjunction is correct , we believe the proof was not completed due to messy calculation.

Lemma 3.1. *If τ is mixing with respect to measure μ, then for any simple function $s : I \to I$*

$$\lim_{n\to\infty} \int_{\tau^{-n}A} s d\mu = \mu(A) \int_I s d\mu.$$

Proof. Let $s = \sum_{i=1}^{k} b_i \chi_{B_i}$. Remark 1.1 implies

$$\begin{aligned}
\lim_{n\to\infty} \int_{\tau^{-n}A} s d\mu &= \lim_{n\to\infty} \int_{\tau^{-n}A} \sum_{i=1}^{k} b_i \chi_{B_i} d\mu \\
&= \sum_{i=1}^{k} b_i \lim_{n\to\infty} \int_{\tau^{-n}A} \chi_{B_i} d\mu \\
&= \sum_{i=1}^{k} b_i \lim_{n\to\infty} \mu(\tau^{-n}A \cap B_i).
\end{aligned}$$

Since τ is mixing the following is valid:

$$\lim_{n\to\infty}\int_{\tau^{-n}A} s d\mu = \sum_{i=1}^{k} b_i \mu(A)\mu(B_i)$$

$$= \mu(A)\sum_{i=1}^{k} b_i \mu(B_i)$$

$$= \mu(A)\int_I s d\mu. \quad \square$$

Theorem 3.1. *If $\tau \in \mathfrak{R}$, then the sequence of iterated distributions $\{\lambda_n([0,t])\}$ converges uniformly to the invariant distribution $\mu([0,t])$ as $n \to \infty$.*

Proof. We have

$$\lambda(\tau^{-n}A) = \int_{\tau^{-n}A} d\lambda$$

and Definition 1.6 lat us write

$$\lambda(\tau^{-n}A) = \int_{\tau^{-n}A} \frac{1}{h} d\mu.$$

after noting that $\frac{1}{h}$ is measurable and bounded, Definition 1.11 leads to

$$\lim_{n\to\infty}\int_{\tau^{-n}A} \frac{1}{h} d\mu = \lim_{n\to\infty} \inf_{s\geq f}\int_{\tau^{-n}A} s d\mu,$$

where s is simple function. Thus by Lemma 3.1

$$\begin{aligned}\lim_{n\to\infty}\int_{\tau^{-n}A}\frac{1}{h}d\mu &= \inf_{s\geq f}\lim_{n\to\infty}\int_{\tau^{-n}A} sd\mu \\ &= \inf_{s\geq f}\mu(A)\int_I sd\mu \\ &= \mu(A)\inf_{s\geq f}\int_I sd\mu.\end{aligned}$$

Using Definition 1.11 once more, gives

$$\lim_{n\to\infty}\int_{\tau^{-n}A}\frac{1}{h}d\mu = \mu(A)\int_I\frac{1}{h}d\mu.$$

Since $\int_I \frac{1}{h}d\mu = 1$ we may write

(25) $$\lim_{n\to\infty}\int_{\tau^{-n}A}\frac{1}{h}d\mu = \mu(A).$$

Note that, from Definition 1.6, we have

(26)

$$\lim_{n\to\infty}\lambda_n(A) = \lim_{n\to\infty}\lambda(\tau^{-n}A) = \lim_{n\to\infty}\int_{\tau^{-n}A}d\lambda = \lim_{n\to\infty}\int_{\tau^{-n}A}\frac{1}{h}d\mu.$$

Thus (25) in (26) imply

$$\lim_{n\to\infty}\lambda_n(A) = \mu(A).$$

after letting $A = [0, t]$ in the previous equation

$$\lim_{n\to\infty} \lambda_n[0,t] = \mu[0,t].$$

Since $\lambda_n[0,t]$ and $\mu[0,t]$ are both continuous and monotone, the convergence is uniform (see Proposition 1.12). □

Definition 3.1. If τ is admissible then the Perron-Frobenius operator $\mathcal{P}_\tau : \mathcal{L}^1 \to \mathcal{L}^1$ is defined as

$$\mathcal{P}_\tau f(x) = \sum_{i\in I_1} \frac{f(\tau_i^{-1}(x))}{|\tau'(\tau_i^{-1}(x))|}, \quad \text{for } f \in \mathcal{L}^1.$$

It is well-known [**L-M**] that $h \in \mathcal{L}^1$ is a density of τ-invariant measure if and only if $\mathcal{P}_\tau h = h$ and we denote

$$\mathcal{P}_\tau^n f = \underbrace{\mathcal{P}_\tau(\mathcal{P}_\tau(\cdots(\mathcal{P}_\tau(f))\cdots))}_{n\text{-times}}.$$

Definition 3.2. We define the iterated densities as follows:

$$S_n(x) = \mathcal{P}_\tau^n 1(x) = \sum_{i\in\mathcal{I}_n} (\tau_i^{-n})'(x),$$

where $\mathcal{I}_n = \{i : p_i \in \mathcal{P}_n\}$, and $\mathcal{P}_n :=$ partition I under τ^{-n}.

Lemma 3.2. *If τ is an admissible transformation satisfying the distortion condition, then the sequence $\{S_n(t)\}$ of the iterated densities are bounded by the bounds of the invariant density (distortion constant C) -i.e.*

$$\frac{1}{C} \le S_n(t) \le C \quad \forall\, t \in [0,1] \quad \text{and any } n \in \mathcal{N}.$$

Proof. We consider

$$S_n(t) = \sum_{i \in \mathcal{I}_n} (\tau_i^{-n})'(t).$$

We have

$$\begin{aligned}
\frac{\sup_{t\in I} S_n(t)}{\inf_{t\in I} S_n(t)} &= \frac{\sup_{t\in I} \sum_{i\in\mathcal{I}_n} (\tau_i^{-n})'(t)}{\inf_{t\in I} \sum_{i\in\mathcal{I}_n} (\tau_i^{-n})'(t)} \\
&\leq \frac{\sum_{i\in\mathcal{I}_n} \sup_{t\in I} (\tau_i^{-n})'(t)}{\sum_{i\in\mathcal{I}_n} \inf_{t\in I} (\tau_i^{-n})'(t)} \\
&= \frac{|\mathcal{I}_n| \sup_{t\in I} (\tau_i^{-n})'(t)}{|\mathcal{I}_n| \inf_{t\in I} (\tau_i^{-n})'(t)} \\
&= \frac{\sup_{t\in I} (\tau_i^{-n})'(t)}{\inf_{t\in I} (\tau_i^{-n})'(t)} \leq C.
\end{aligned}$$

Thus we obtain

$$\sup_{t\in I} S_n(t) \leq C \inf_{t\in I} S_n(t).$$

Since $\int_0^1 S_n(t)dt = 1$, we must have $\inf S_n(t) \leq 1$. Therefore,

$$\sup_{t\in I} S_n(t) \leq C$$

and

$$S_n(t) \leq C \quad \text{for} \quad t \in I.$$

Furthermore,

$$\begin{aligned}\int_0^1 S_n(t)dt = 1 &\Rightarrow \sup_{t\in I} S_n(t) \geq 1 \\ &\Rightarrow 1 \leq C \inf_{t\in I} S_n(t) \\ &\Rightarrow \inf_{t\in I} S_n(t) \geq \frac{1}{C} \\ &\Rightarrow S_n(t) \geq \frac{1}{C}, \quad \text{for } t \in I.\end{aligned}$$

Thus

$$S_n(t) \geq \frac{1}{C}, \quad \text{on} \quad I. \quad \square$$

Lemma 3.3. *If τ is an admissible transformation satisfying the distortion condition, then there exists an N such that*

$$\sup_{t\in I} \sum_{i\in\mathcal{I}_N} ((\tau_i^{-N})'(t))^2 \leq 1.$$

Proof. For every N we have

$$\sum_{i\in\mathcal{I}_N} ((\tau_i^{-N})'(t))^2 \leq (\sup_i \sup_{t\in I} (\tau_i^{-N})'(t)) \sum_{i\in\mathcal{I}_N} (\tau_i^{-N})'(t).$$

By equation (14) and Definition 3.1 arrive at

$$\sum_{i\in\mathcal{I}_N} ((\tau_i^{-N})'(t))^2 \leq C\Delta_N S_N(t).$$

Thus Lemma 3.2 yields

$$\sum_{i\in\mathcal{I}_N} ((\tau_i^{-N})'(t))^2 \le C\Delta_N S_N(t) \le C^2\Delta_N.$$

By Lemma 2.6 we can choose N large enough so that $\Delta_N < \frac{1}{C^2}$. This completes the proof. □

Before proceeding with our next result, we derive an " iterative" expression for the iterated densities S_n in the following Remark.

Remark 3.1.

$$S_{K+N}(x) = \sum_{i\in\mathcal{I}_K} S_K(\tau_i^{-N}(x))(\tau_i^{-N})'(x).$$

Proof. By Definition 3.2 we have

$$S_K(\tau_i^{-N}(x)) = \sum_{i\in\mathcal{I}_K} (\tau_i^{-K})'(\tau_i^{-N}(x))$$

and thus

$$\sum_{i\in\mathcal{I}_N} S_K(\tau_i^{-N}(x))(\tau_i^{-N}(x))' = \sum_{i\in\mathcal{I}_N}\sum_{i\in\mathcal{I}_K} (\tau_i^{-K}(x))'(\tau_i^{-N}(x))(\tau_i^{-N}(x))'.$$

Note that

$$\begin{aligned}(\tau_i^{-(K+N)}(x))' &= (\tau_i^{-K}(\tau^{-N}(x)))'\\ &= (\tau_i^{-K}(x))'(\tau_i^{-N}(x))(\tau_i^{-N}(x))'\end{aligned}$$

in consequence of the chain rule and therefore

$$\sum_{i\in\mathcal{I}_N} S_K(\tau_i^{-N}(x))(\tau_i^{-N}(x))' = \sum_{i\in\mathcal{I}_N}\sum_{i\in\mathcal{I}_K}(\tau_i^{-(K+N)}(x))'$$

$$= \sum_{i\in\mathcal{I}_{N+K}}(\tau_i^{-(K+N)}(x))'.$$

Now, $i \in \mathcal{I}_{N+K}$ if and only if there exists $j \in \mathcal{I}_N$ and $k \in \mathcal{I}_K$ such that $P_i = P_j \cap P_k$. Thus $\sum_{i\in\mathcal{I}_N} = \sum_{i\in\mathcal{I}_N}\sum_{i\in\mathcal{I}_K}$ and this completes the proof. □

Lemma 3.4. *Let* $\{B_{K+iN}\}_{i=1}^{\infty}$ *be a sequence of real numbers. If* $B_{K+N} \le B_K\theta + M$, *where* $0 < \theta < 1$ *and* $M > 0$, *then* $\{B_{K+iN}\}_{i=1}^{\infty}$ *is bounded uniformly.*

Proof. First we prove by induction that

$$B_{K+jN} \le B_K\theta^j + M(1+\theta+\theta^2+\cdots+\theta^{j-1}),$$

which is true for $j = 2$ we have

$$B_{K+2N} = B_{K+N+N} \le B_{K+N}\theta + M$$

$$\le (B_K\theta + M)\theta + M$$

$$= B_K\theta^2 + M(1+\theta),$$

assume it is true for $j = i$ where $i > 2$ i.e.,

$$B_{K+iN} \le B_K\theta^i + M(1+\theta+\theta^2+\cdots+\theta^{i-1}).$$

Now, we will prove the hypothesis for $j = i + 1$ manely

$$B_{K+(i+1)N} = B_{K+iN+N} \le B_{K+iN}\theta + M$$
$$\le \left(B_k\theta^i + M(1+\theta+\theta^2+\cdots+\theta^{i-1})\right)\theta + M$$
$$= B_k\theta^{i+1} + M\big(1+\theta+\theta^2+\cdots+\theta^i\big).$$

This gives the following two facts

(1) $B_{K+jN} \le B_K\theta^j + M(1+\theta+\theta^2+\cdots+\theta^{j-1})$;

(2) $B_{K+jN} \le B_K\theta^j + \frac{M}{1-\theta}$ as $j \to \infty$.

The above two facts imply that

$$B_{K+jN} \le B_K + \frac{M}{1-\theta}, \quad \text{for all } j. \quad \square$$

Lemma 3.5. *Given*

$$S_{K+N}(t) = \sum_{i\in\mathcal{I}_N} S_K((\tau_i^{-N})(t))(\tau_i^{-N})'(t),$$

where $\mathcal{I}_N = \{i : p_i \in \mathcal{P}_N\}$, *and* $\mathcal{P}_N :=$ *partition* I *under* τ^{-N}.
Then, the r*-th derivative of* S_{K+N} *is given by*

$$S^{(r)}_{K+N}(x) = \sum_{i\in\mathcal{I}_N}\Bigg\{S^{(r)}_K(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{r+1}$$
$$+\sum_{j=0}^{r-1}\left(S^{(j)}_K(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\right)\Bigg\},$$

where each c_z is an integer, $a(t) \leq r$ and $b(t) < r$. For illustration see Appendix: A.

Proof. We prove this Lemma by induction. For $r = 1$

$$S'_{K+N}(x) = \sum_{i \in \mathcal{I}_N} \{S'_K((\tau_i^{-N}(x))((\tau_i^{-N})'(x))^2 + S_K(\tau_i^{-N}(x))(\tau_i^{-N})''(x)\}.$$

Thus with $z(j) = 1, c_z = 1, t(z) = 1, a(t) = 2$, and $b(t) = 1$ the Lemma is ture for $r = 1$. Now suppose the lemma is ture for $r = m$, i.e.,

$$S_{K+N}^{(m)}(x) = \sum_{i \in \mathcal{I}_N} \Bigg\{ S_K^{(m)}(\tau_i^{-N}(x)\big((\tau_i^{-N})'(x)\big)^{m+1}$$

$$+ \sum_{j=0}^{m-1} \Bigg(S_K^{(j)}(\tau_i^{-N}(x)) \sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)} \Bigg) \Bigg\},$$

where $a(t) \leq m + 1$ and $b(t) \leq m$. Next, we will prove the lemma for $r = m + 1$

$$S_{K+N}^{(m+1)}(x) = \frac{d}{dx}\{S_{K+N}^{(m)}(x)\}$$

$$= \frac{d}{dx} \Bigg\{ \sum_{i \in \mathcal{I}_N} S_K^{(m)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m+1}$$

$$+ \sum_{j=0}^{m-1} \Bigg(S_K^{(j)}(\tau_i^{-N}(x)) \sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)} \Bigg) \Bigg\},$$

differentiating over each summation we get:

$$S_{K+N}^{(m+1)}(x) = \frac{d}{dx}\left\{\sum_{i\in\mathcal{I}_N} S_K^{(m)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m+1}\right\}$$

$$+\frac{d}{dx}\left\{\sum_{j=0}^{m-1} S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\right\}$$

Differentiating prior to summation, we have:

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N}\frac{d}{dx}\left\{S_K^{(m)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m+1}\right\}$$

$$+\sum_{j=0}^{m-1}\frac{d}{dx}\left\{S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\right\}$$

thus we get

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N}\left\{\frac{d}{dx}\{S_K^{(m)}(\tau_i^{-N})(x)\}\big((\tau_i^{-N})'(x)\big)^{m+1}\right.$$

$$\left.+S_K^{(m)}(\tau_i^{-N}(x))\frac{d}{dx}\{(\tau_i^{-N})'(x)\}^{m+1}\right\}$$

$$+\sum_{j=0}^{m-1}\left\{\frac{d}{dx}\{S_K^{(j)}(\tau_i^{-N})(x)\}\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\right.$$

$$\left.+(S_K^{(j)}(\tau_i^{-N})(x))\frac{d}{dx}\{\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\}\right\}$$

which gives

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N}\Big\{S_K^{(m+1)}(\tau_i^{-N})(x)\big((\tau_i^{-N})'(x)\big)^{m+2}$$

$$+ (m+1)S_K^{m}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m}(\tau_i^{-N})''(x)$$

$$+ \sum_{j=0}^{m-1}\Big\{\big(S_K^{(j+1)}(\tau_i^{-N})(x)\big)(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\big)$$

$$+ (S_K^{(j)}(\tau_i^{-N})(x))\sum_{z=1}^{z(j)}\frac{d}{dx}\{c_z\prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\}\Big\}\Big\},$$

then

(27)

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N}\Big\{S_K^{(m+1)}(\tau_i^{-N})(x)\big((\tau_i^{-N})'(x)\big)^{m+2}$$

$$+ (m+1)S_K^{(m)}(\tau_i^{-N}(x))((\tau_i^{-N})'(x))^{m}(\tau_i^{-N})''(x)\Big\}$$

$$+ \sum_{j=0}^{m-1}\Big\{\Big(S_K^{(j+1)}(\tau_i^{-N}(x))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\Big)$$

$$+ (S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\Big\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\big((\tau_i^{-N})^{(a(\eta)+1)}(x)\big)^{b(\eta)-1}$$

$$\prod_{\substack{t=1\\t\neq\eta}}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\}\Big\}\Big\}.$$

Next, note that:

$$\sum_{j=0}^{m-1}\Bigg\{\Bigg(S_K^{(j+1)}(\tau_i^{-N}(x))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\Bigg)$$

$$+\,(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\times$$

$$\left\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1}\prod_{\substack{t=1\\t\neq\eta}}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\}\right\}\Bigg\}$$

$$=\sum_{j=0}^{m-1}(S_K^{(j+1)}(\tau_i^{-N}(x)))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}$$

$$+\sum_{j=0}^{m-1}(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\left\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1}\times\right.$$

$$\left.\prod_{\substack{t=1\\t\neq\eta}}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\}\right\}.$$

Changing the index of j for the second term on the right hand side we obtain:

(28)

$$\sum_{j=0}^{m-1}\Bigg\{\Bigg(S_K^{(j+1)}(\tau_i^{-N}(x))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\Bigg)$$

$$+\,(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\Bigg\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1}$$

$$\prod_{\substack{t=1\\t\neq\eta}}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\}\Bigg\}\Bigg\}$$

$$=\sum_{j=1}^{m}\Bigg(S_K^{(j)}(\tau_i^{-N}(x))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)}c_z\prod_{t=1}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\Bigg)$$

$$+\sum_{j=0}^{m-1}(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)}c_z\times$$

$$\Bigg\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1}\prod_{\substack{t=1\\t\neq\eta}}^{t(z)}\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)}\}\Bigg\}\Bigg\}.$$

Now, using equation (27)in (28) we obtain:

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N} \Big\{ S_K^{(m+1)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m+2}$$

$$+ (m+1)S_K^{(m)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m}(\tau_i^{-N})''(x)$$

$$+ \sum_{j=1}^{m}\Big(S_K^{(j)}(\tau_i^{-N}(x))(\tau_i^{-N})'(x)\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\Big)$$

$$+ \sum_{j=0}^{m-1}(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)} c_z \times$$

$$\Big\{\sum_{\eta=1}^{\eta=t(z)}\{b(\eta)\big((\tau_i^{-N})^{(a(\eta)+1)}(x)\big)^{b(\eta)-1}\prod_{\substack{t=1\\ t\neq\eta}}^{t(z)}\big((\tau_i^{-N})^{(a(t))}(x)\big)^{b(t)}\}\Big\}\Big\}.$$

Next, note that in

$$\sum_{j=1}^{m}(S_K^{(j)}(\tau_i^{-N}(x))\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)}((\tau_i^{-N})^{(a(t))})^{b(t)}),$$

the term where $j = m$ is

$$S_k^{(m)}(\tau_i^{-N}(x))\sum_{z=1}^{z(m)} c_z \prod_{t=1}^{t(z)}\big((\tau_i^{-N})^{(a(t))}\big)^{b(t)}, \tag{29}$$

on the other hand,

$$(m+1)S_k^{(m)}(\tau_i^{-N}(x))\big((\tau_i^{-N})'(x)\big)^{m}(\tau_i^{-N})''(x)$$

this term can be placed in (29) without changing its general format. Furthermore, every term in

$$\sum_{\eta=1}^{\eta=t(z)} b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1},$$

is of the form

$$\left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)},$$

times a constant where $a(t) \leq m+2$, and $b(t) \leq m+1$. Thus

$$\sum_{z=1}^{z(j)} c_z \left\{ \sum_{\eta=1}^{\eta=t(z)} b(\eta)\left((\tau_i^{-N})^{(a(\eta)+1)}(x)\right)^{b(\eta)-1} \prod_{\substack{t=1\\ t\neq\eta}}^{t(z)} \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)} \right\},$$

can be written as

$$\sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)},$$

where $a(t) \leq m+1$ and $b(t) \leq m$, with possibly new constants c_z and updating $z(j)$ and $t(z)$. Thus, we have:

$$S_{K+N}^{(m+1)}(x) = \sum_{i\in\mathcal{I}_N} \Bigg\{ S_K^{(m+1)}(\tau_i^{-N}(x))\left((\tau_i^{-N})'(x)\right)^{m+2}$$

$$+ \sum_{j=0}^{m} S_K^{(j)}(\tau_i^{-N}(x)) \sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)} \Bigg\},$$

this completes the inductive proof. $\square$

Theorem 3.2. *If $\tau \in \mathfrak{R}$ and $\tau_i^{-1} \in C^r[0,1]$, for all i and $r \geq 2$, then the invariant density $h \in C^{r-2}[0,1]$.*

Proof. We prove the theorem by induction. Let $r = 2$. By Remark 3.1 we have

$$S_{K+N}(t) = \sum_{i \in \mathcal{I}_N} S_K((\tau_i^{-N})(t))(\tau_i^{-N})'(t).$$

after differentiating the above, we obtain

$$S'_{K+N}(t) = \sum_{i \in \mathcal{I}_N} \left\{ S'_K((\tau_i^{-N})(t))((\tau_i^{-N})'(t))^2 + S_K(\tau_i^{-N}(t))(\tau_i^{-N})''(t) \right\}.$$

Taking the supremum on both sides yieds

$$\begin{aligned} \sup_{t \in I} S'_{K+N}(t) &= \sup_{t \in I} \sum_{i \in \mathcal{I}_N} \left\{ S'_K((\tau_i^{-N})(t))((\tau_i^{-N})'(t))^2 \right. \\ &\quad \left. + S_K(\tau_i^{-N}(t))(\tau_i^{-N})''(t) \right\} \\ &\leq \sup_{t \in I} \sum_{i \in \mathcal{I}_N} S'_K(\tau_i^{-N}(t))((\tau_i^{-N})'(t))^2 \\ &\quad + \sup_{t \in I} \sum_{i \in \mathcal{I}_N} S_K(\tau_i^{-N}(t)) \mid (\tau_i^{-N})''(t) \mid . \end{aligned}$$

Put

$$d = \sup_{t \in I} \sum_{i \in \mathcal{I}_N} \mid (\tau_i^{-N})''(t) \mid < \infty$$

$$B_n = \sup_{t \in I} \mid S'_n(t) \mid < \infty \quad (\text{for } n = 0, 1, 2, \cdots);$$

or otherwise, the distortion condition will not hold. Further,

(30)

$$\sup_{t\in I} B_{K+N}(t) \le \sup_{t\in I} \sum_{i\in\mathcal{I}_N} B_K((\tau_i^{-N})'(t))^2 + C \sup_{t\in I} \sum_{i\in\mathcal{I}_N} \mid (\tau_i^{-N})''(t) \mid .$$

Let $(\tau_i^{-N})'(t) = \theta < 1$ (see Lemma 3.3). From (30) we get

$$B_{K+N} \le B_K\theta + Cd.$$

By Lemma 3.4 are sequence

$$B_K, B_{K+N}, B_{K+2N}, \cdots$$

is uniformly bounded by some number $\hat{B}_K$. To find the bound for the entire sequence $\{B_n\}$, we let $K = 0, 1, \cdots, N-1$ and $B = \max\{\hat{B}_0, \hat{B}_1, \cdots, \hat{B}_{N-1}\}$. Then the sequence $\{B_n\} = \{\sup_{t\in I} \mid S_n'(t) \mid\}$ is uniformly bounded by B.

Therefore, the sequence $\{S_n\}$ of iterated densities is uniformly bounded and equicontinuous. By the Arzela-Ascoli Theorem $\{S_n\}$ possesses a uniformly convergent subsequence $\{\zeta_n\}$ such that

$$\lim_{n\to\infty} \zeta_n(t) = f(t).$$

Since the convergence is uniform on I

$$\lim_{n\to\infty} \int_0^1 \zeta_n(\tau)d\tau = F(t) \quad \text{for } t \in [0,1]$$

and because $\left\{\int_0^t \zeta(\tau)d\tau\right\}$ is a subsequence of $\lambda_n([0,t])$ and $F(t) = \mu([0,t])$, $F'(t) = f(t)$ must be equal to the invariant density h. Thus $h \in C^0$.

Next, we assume that the result is true for $\tau^{-1} \in C^{r-1}$, i.e., the sequence $\{S_n^{(r-2)}\}$ of iterated densities is uniformly bounded and if $\tau^{-1} \in C^{r-1}$ then $h \in C^{r-3}$. We next prove the Theorem for $\tau^{-1} \in C^r$. By Lemma 3.5 we have

$$S_{K+N}^{(r)}(x) = \sum_{i \in \mathcal{I}_N} \Bigg\{ S_K^{(r)}(\tau_i^{-N}(x))\left((\tau_i^{-N})'(x)\right)^{r+1}$$

$$+ \sum_{j=0}^{r-1} \left(S_K^{(j)}(\tau_i^{-N}(x)) \sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)} \right) \Bigg\},$$

where each c_z is an integer, $a(t) \le r$ and $b(t) < r$.

Now, we note that by induction we have for $j = 0, 1, 2, \ldots, r-1$ constants $B^{(j)}$ which are bounds for the sequences $\{\sup_{x \in I} S_n^{(j)}(x)\}$ respectively; furthermore,

$$\sup_{x \in I} \| \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)} \| < \infty.$$

Hence, for each z

$$\sup_{x \in I} c_z \prod_{t=1}^{t(z)} \left(\| \left((\tau_i^{-N})^{(a(t))}(x)\right)^{b(t)} \| \right) < \infty,$$

and thus for each j there exists a constant $\beta_j^{(z)}$ such that

$$\beta_j^{(z)} = \sup_{x \in I} \sum_{z=1}^{z(j)} c_z \prod_{t=1}^{t(z)} \left(\| \left((\tau_i^{-N})^{(a(t))}(x) \right)^{b(t)} \| \right) < \infty.$$

after setting

$$B_n^{(r)} = \sup_{x \in I} \|S_n^{(r)}\|(x),$$

we get

$$B_{K+N}^{(r)} \leq B_K^{(r)} \theta^r B^{(0)} + \sum_{j=0}^{r-1} B^{(j)} \beta_j^{(r)}.$$

For N large enough, we have $\theta^r B^{(0)} < 1$. Thus by Lemma 3.4

$$B_K^{(r)}, B_{K+N}^{(r)}, B_{K+2N}^{(r)}, \cdots$$

is uniformly bounded by some number $\hat{B}_K^{(r)}$. Therefore, the entire sequence $\{B_n^{(r)}\}$ is bounded by $B^{(r)} = \max\{\hat{B}_0^{(r)}, \hat{B}_1^{(r)}, \cdots \hat{B}_{N-1}^{(r)}\}$. Consequently, the sequence

$$\{B_n^{(r)}\} = \left\{ \sup_{t \in I} \mid S_n^{(r)} \mid \right\} \text{ is uniformly bounded by } B^{(r)}.$$

Therefore, the sequence $\{S_n^{(r-1)}\}$ is uniformly bounded and equicontinuous. By the Arzela-Ascoli Theorem $\{S_n^{(r-1)}\}$ possesses a uniformly convergent subsequence $\{\zeta_n^{(r-1)}\}$ such that

$$\lim_{n \to \infty} \zeta_n^{(r-1)}(t) = f^{(r-1)}(t).$$

Integrating r times leads to

$$\lim_{n\to\infty}\int_0^t \zeta_n(\tau)d\tau = F(t), \quad t\in[0,1].$$

Since $\left\{\int_0^t \zeta_n(\tau)d\tau\right\}$ is a subsequence of $\lambda(\tau^{-n}[0,t])$, $F(t) = \mu([0,t])$, where

$$F'(t) = f(t) = h(t)$$

and moreover $f^{(j)}(t) = h^{(j)}(t)$ for $0\le j\le r-2$, which completes the proof of the theorem . □

CONCLUSION

Our result has been generalized on the unit interval by **[Sze]** in the following two directions:

(1) He considered a class of transformation (Lasota - Yorke)which is a super class of transformation considered in this thesis.

(2) He also improved the degree of smoothness of invariant density.

Another generalization of our result has been done in [**Adl1**] and [**Adl3**]. There the author generalizes our result in two direction.

(1) He considers Markov maps which is a super class of our maps.

(2) It has been done in n-dimensions.

Next, we mention how these results can be improved. One problem of interest would be to establish the smoothness of invariant density for Lasota - Yorke maps in higher dimensions. For the Lasota-Yorke maps under general conditions [**Adl2**], it can be shown that τ has an **acim**. The result in [**Adl2**] is a generalization of results proved in [**Jab**],[**Can**],[**Kel**] and [**G-B**].

The dynamics of many physical systems are governed by a randomly changing environment and can thus described by a random map R whose evolution is represented by choosing a transformation from a given set of transformation and applying it with a given probability.

As an application of our result, we would like to mention that, the existence of an **acim** for random map composed of Lasota-Yorke maps on an interval has been establish in [**Pel**].

APPENDIX: A

In this appendix we present illustration of Lemma 3.5 using Maple package. We define a function f which represents the iterated density defined in Lemma 3.5. The function f will produce its subsequent derivatives f^r for $r = 1, 2, \cdots, 8$.

```
> T:=x->T(x);
```

$$T := T$$

```
> T(x);
```

$$\mathrm{T}(x)$$

```
> S:=x->S(x);
```

$$S := S$$

```
> S(x);
```

$$\mathrm{S}(x)$$

```
> S1:=x->S(T(x));
```

$$S1 := x \rightarrow \mathrm{S}(\mathrm{T}(x))$$

```
> S1(x);
```

$$\mathrm{S}(\mathrm{T}(x))$$

```
> f:=x->S1(x)*diff(T(x),x);
```

$$f := x \rightarrow \mathrm{S1}(x)\,\mathrm{diff}(\mathrm{T}(x), x)$$

```
> f(x);
```

$$\mathrm{S}(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)$$

```
> f1:=x->diff(f(x),x);
```

$$f1 := x \rightarrow \mathrm{diff}(\mathrm{f}(x), x)$$

```
> f1(x);
```

$$\mathrm{D}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 + \mathrm{S}(\mathrm{T}(x))\left(\frac{\partial^2}{\partial x^2}\mathrm{T}(x)\right)$$

```
> f2:=x->diff(f1(x),x);
```

$$f2 := x \rightarrow \mathrm{diff}(\mathrm{f1}(x), x)$$

```
> f2(x);
```

$$D^{(2)}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3$$

$$+ 3\,\mathrm{D}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^2}{\partial x^2}\mathrm{T}(x)\right)$$

$$+ \mathrm{S}(\mathrm{T}(x))\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

> f3:=x->diff(f2(x),x);

$$f3 := x \rightarrow \mathrm{diff}(\mathrm{f2}(x), x)$$

> f3(x);

$$D^{(3)}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^4 + 6\,D^{(2)}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \%1$$

$$+ 3\,\mathrm{D}(S)(\mathrm{T}(x))\,\%1^2$$

$$+ 4\,\mathrm{D}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ \mathrm{S}(\mathrm{T}(x))\left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$\%1 := \frac{\partial^2}{\partial x^2}\mathrm{T}(x)$$

> f4:=x->diff(f3(x),x);

$$f4 := x \rightarrow \mathrm{diff}(\mathrm{f3}(x), x)$$

> f4(x);

$$D^{(4)}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^5 + 10\,D^{(3)}(S)(\mathrm{T}(x))\left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \%1$$

$$+ 15\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \%1^2$$

$$+ 10\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^2 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)$$

$$+ 10\, \mathrm{D}(S)(\mathrm{T}(x))\, \%1 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)$$

$$+ 5\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$+ \mathrm{S}(\mathrm{T}(x)) \left(\frac{\partial^5}{\partial x^5} \mathrm{T}(x) \right)$$

$$\%1 := \frac{\partial^2}{\partial x^2} \mathrm{T}(x)$$

> f5:=x->diff(f4(x),x);

$$f5 := x \rightarrow \mathrm{diff}(\mathrm{f4}(x), x)$$

> f5(x);

$$D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^6 + 15\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^4 \%1$$

$$+ 45\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^2 \%1^2$$

$$+ 20\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^3 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)$$

$$+ 15\, D^{(2)}(S)(\mathrm{T}(x))\, \%1^3$$

$$+ 60\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \%1 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)$$

$$+ 15\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^2 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$+ 10\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)^2$$

$$+ 15\, \mathrm{D}(S)(\mathrm{T}(x))\, \%1 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$+ 6\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \left(\frac{\partial^5}{\partial x^5} \mathrm{T}(x) \right)$$

$$+ \mathrm{S}(\mathrm{T}(x)) \left(\frac{\partial^6}{\partial x^6} \mathrm{T}(x) \right)$$

$$\%1 := \frac{\partial^2}{\partial x^2} \mathrm{T}(x)$$

```
> f6:=x->diff(f5(x),x);
```

$$f6 := x \rightarrow \mathrm{diff}(\mathrm{f5}(x), x)$$

```
> f6(x);
```

$$\mathrm{S}(\mathrm{T}(x)) \left(\frac{\partial^7}{\partial x^7} \mathrm{T}(x) \right) + 21\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^5 \%1$$

$$+ D^{(6)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^7$$

$$+ 35\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^4 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 35\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$+ 105\, D^{(2)}(S)(\mathrm{T}(x))\ \%1^2 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 21\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \%2$$

$$+ 35\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)\left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$+ 21\, \mathrm{D}(S)(\mathrm{T}(x))\ \%1\ \%2$$

$$+ 7\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^6}{\partial x^6}\mathrm{T}(x)\right)$$

$$+ 105\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \%1^2$$

$$+ 105\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \%1^3$$

$$+ 210\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \%1 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 70\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)^2$$

$$+ 105\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \%1 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$\%1 := \frac{\partial^2}{\partial x^2} \mathrm{T}(x)$$

$$\%2 := \frac{\partial^5}{\partial x^5} \mathrm{T}(x)$$

> f7:=x->diff(f6(x),x);

$$f7 := x \to \mathrm{diff}(\mathrm{f6}(x), x)$$

> f7(x);

$$\mathrm{S}(\mathrm{T}(x)) \left(\frac{\partial^8}{\partial x^8} \mathrm{T}(x) \right) + 8\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right) \left(\frac{\partial^7}{\partial x^7} \mathrm{T}(x) \right)$$

$$+ 56\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^5 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x) \right)$$

$$+ D^{(7)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^8$$

$$+ 70\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^4 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$+ 56\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x) \right)^3 \%2$$

$$+ 210\, D^{(2)}(S)(\mathrm{T}(x))\, \%1^2 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x) \right)$$

$$+ 28\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right)^2 \left(\frac{\partial^6}{\partial x^6} \mathrm{T}(x)\right)$$

$$+ 35\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x)\right)^2$$

$$+ 56\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x)\right) \%2$$

$$+ 28\, \mathrm{D}(S)(\mathrm{T}(x))\, \%1 \left(\frac{\partial^6}{\partial x^6} \mathrm{T}(x)\right)$$

$$+ 105\, D^{(3)}(S)(\mathrm{T}(x))\, \%1^4$$

$$+ 210\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right)^4 \%1^2$$

$$+ 28\, D^{(6)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right)^6 \%1$$

$$+ 560\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right)^3 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x)\right) \%1$$

$$+ 420\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right)^2 \left(\frac{\partial^4}{\partial x^4} \mathrm{T}(x)\right) \%1$$

$$+ 280\, D^{(2)}(S)(\mathrm{T}(x))\, \%1 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x)\right)^2$$

$$+ 840\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x} \mathrm{T}(x)\right) \%1^2 \left(\frac{\partial^3}{\partial x^3} \mathrm{T}(x)\right)$$

$$+ 168\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \%2\ \%1$$

$$+ 280\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)\left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$+ 420\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \%1^3$$

$$+ 280\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)^2$$

$$\%1 := \frac{\partial^2}{\partial x^2}\mathrm{T}(x)$$

$$\%2 := \frac{\partial^5}{\partial x^5}\mathrm{T}(x)$$

```
> f8:=x->diff(f7(x),x);
```

$$f8 := x \rightarrow \mathrm{diff}(\mathrm{f7}(x), x)$$

```
> f8(x);
```

$$\mathrm{S}(\mathrm{T}(x)) \left(\frac{\partial^9}{\partial x^9}\mathrm{T}(x)\right) + 9\,\mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^8}{\partial x^8}\mathrm{T}(x)\right)$$

$$+ 36\,\mathrm{D}(S)(\mathrm{T}(x))\ \%1 \left(\frac{\partial^7}{\partial x^7}\mathrm{T}(x)\right)$$

$$+ 126\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^5 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$+ 36\, D^{(7)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^7 \%1$$

$$+ D^{(8)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^9$$

$$+ 126\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^4 \%2$$

$$+ 84\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \left(\frac{\partial^6}{\partial x^6}\mathrm{T}(x)\right)$$

$$+ 378\, D^{(2)}(S)(\mathrm{T}(x))\, \%1^2\, \%2$$

$$+ 126\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right) \%2$$

$$+ 315\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)^2$$

$$+ 84\, \mathrm{D}(S)(\mathrm{T}(x)) \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right) \left(\frac{\partial^6}{\partial x^6}\mathrm{T}(x)\right)$$

$$+ 1260\, D^{(3)}(S)(\mathrm{T}(x))\, \%1^3 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 945\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \%1^4$$

$$+ 36\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \left(\frac{\partial^7}{\partial x^7}\mathrm{T}(x)\right)$$

$$+ 1260\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^4 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right) \%1$$

$$+ 84\, D^{(6)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^6 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 1260\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right) \%1$$

$$+ 756\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \%2\ \%1$$

$$+ 1260\, D^{(2)}(S)(\mathrm{T}(x))\ \%1 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 1890\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \%1^2 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right)$$

$$+ 252\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^6}{\partial x^6}\mathrm{T}(x)\right) \%1$$

$$+ 504\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)\left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right) \%2$$

$$+ 280\, D^{(2)}(S)(\mathrm{T}(x)) \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)^3$$

$$+ 1260\, D^{(5)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \%1^3$$

$$+ 378\, D^{(6)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^5 \%1^2$$

$$+ 3780\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right) \%1^2$$

$$+ 840\, D^{(4)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^3 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)^2$$

$$+ 1260\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right)^2 \left(\frac{\partial^4}{\partial x^4}\mathrm{T}(x)\right) \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)$$

$$+ 2520\, D^{(3)}(S)(\mathrm{T}(x)) \left(\frac{\partial}{\partial x}\mathrm{T}(x)\right) \%1 \left(\frac{\partial^3}{\partial x^3}\mathrm{T}(x)\right)^2$$

$$\%1 := \frac{\partial^2}{\partial x^2}\mathrm{T}(x)$$

$$\%2 := \frac{\partial^5}{\partial x^5}\mathrm{T}(x)$$

References

[Adle] R. L. Adler, *F-expansions Revisited, in Recent Advances in Topological Dynamics (Lecture Notes in Mathematics)*, New York, Springer-Verlag **318** (1975), 1-5.

[Adl1] K. Adl-Zarabi, *Existence and Properties of ACIM For Higher Dimensional Chaotic Transformations*, PhD Thesis, Concordia University (1996).

[Adl2] ______, *Absolutely continuous invariant measures for piecewise expanding C^2 transformations in $\mathbb{R}^n$ on domains with cusps on the boundaries*, Ergodic Theory & Dynam. Syst. **15** (1996), 1-18.

[Adl3] ______, *Smoothness of invariant density for expanding Markov transformations in higher dimensions*, in press.

[Bow] R. Bowen, *Invariant Measures for Markov maps of the Interval*, Comm. Math.Phys **69** (1979), 1-17.

[Can] D. Candeloro, *Misure invariante per transformazioni in piu dimensionii*, Atti Sem. Mat. Fis. Univ. Modena **XXXV** (1987), 32-42.

[Gór] P. Góra, *Properties of invariant measures for piecewise expanding one dimensional transformations with summable oscillations of derivative*, Ergodic Theory & Dynam. Syst. **14** (1994), 475-492.

[G-B] P. Góra & A. Boyarsky, *Absolutely Continuous Invariant Measures for Piecewise Expanding C^2 Transformations in R^N*, J. Math **67(3)** (1989).

[Hal] M. Halfant, *Analytic Properties of Rényi's Invariant Density*, Israel J. Math **27(1)** (1977).

[Jab] M. Jabłoński, *On invariant measures for piecewise C^2- transformations of the n-dimensional cube*, Ann. Polon. Math **XLIII** (1983), 185-195.

[Kak] S. Kakeya, *On a Generalized Scale of Notations*, Japan J. Math **1** (1924), 95-108.

[Kuz] R. O. Kuz'min, *Sur un Probléme de Gauss*, Atti de Congresso Internazionale de Mathematici Bologna, **VI** (1928), 83-89.

[L-Y] A. Lasota & J. A. Yorke, *On the Existence of Invariant Measures for Piecewise Monotonic Transformation*, Trans. Amer. Math. Soc. **186** (1973), 481-488.

[L-M] A. Lasota & M. C. Mackey, *Probabilistic Properties of Deterministic Systems*, Cambridge Univ. Press, N.Y. (1985).

[Man] R. Mané, *Ergodic Theory and Differentiable Dynamics*, Springer, New York (1985).

[Rén] A. Rényi, *Representations for Real Numbers and their Ergodic Properties*, Acta. Math. Acad. Sci. Hunger. **8** (1957), 477-493.

[Roy] H. L. Royden, Macmillan Publishing Company, New York Third edition (1988), *Real Analysis*.

[Sch] B. Schmitt, *Contributions aleťude de Systeḿes Dynamiques Unidimensionnels en theórie Ergodique*, Theśe. Universiteḿ Bourgogne, Dijon, (1986).

[Sze] B. Szewc, *The Perron-Frobenius Operator in Spaces of Smooth Functions on an Interval*, Ergodic Th. & Dynam. Sys **4** (1984), 613-643.