Generalized Risk Processes and Lévy Modeling in Risk Theory

Manuel Morales

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ABSTRACT

Generalized Risk Processes and Lévy Modeling in Risk Theory

Manuel Morales, Ph.D.

Concordia University, 2003

A generalization to the classical risk model is presented. This generalization includes a Lévy process as the aggregate claims process. The compound Poisson process and the diffusion process are particular cases of this more general model. With this model we attempt to bridge two approaches often used in the literature to generalize the classical model.

We investigate applications of pure-jump Lévy processes to risk theory, in particular members of the family of generalized hyperbolic processes. We focus our interest in the normal inverse Gaussian process and in the generalized inverse Gaussian process. Both lead to purely discontinuous risk processes with infinite activity, i.e., these processes have an infinite number of small jumps and occasional larger movements.

We also present an approximation to the classical risk model when the claim severities belong to the domain of attraction of an extreme distribution, this allows for all kinds of heavy and medium tailed distributions. The model is based on a Lévy process with an underlying Lévy measure proportional to the generalized Pareto distribution.

Most of our results rely on properties that are not only valid for Lévy processes but for the larger class of semimartingales. As an illustration, we also introduce an even more general risk process with independent increments that would endow us with a periodic reserve process that can find applications in reinsurance or in the valuation of catastrophe insurance options. Although, this periodic risk process does

not belong to the Lévy family of processes, it does belong to the larger family of processes with independent increments.

The main contribution of this thesis takes the form of four independent chapters that illustrate the potential of Lévy modeling in risk theory. Each chapter constitutes a research paper in itself with different applications of the theory of processes with independent increments in risk theory. These applications range from new approximations to the aggregate claim process, to purely discontinuous risk models. Periodicity is also discussed in terms of a particular semimartingale for which a simulation approach is used. Together, they show the flexibility of pure-jump Lévy processes and their suitability as models in risk theory. Although, each of the applications represents a contribution to different aspects of the theory, all of them, as a whole, reflect the potential of Lévy modeling.

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Introduction

After a review of the literature, one can see that the use of Lévy processes in risk theory, other than the compound Poisson process, has been mainly restricted to Brownian motion and α-stable processes. There exist two different approaches when incorporating Lévy processes in risk theory: by replacing the classical aggregate claim process with it or by using it as perturbation to the classical model.

Consider the classical risk process [see for instance DeVylder (1996), Gerber (1979), Grandell (1991), Embrechts and Klüppelberg (1992) or Prabhu (1998)] given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \;, \qquad t \geqslant 0 \;, \tag{1}$$

where u is the initial surplus, c is a constant premium rate, $\{X_i\}$ are i.i.d. random variables (with c.d.f. F_X) denoting the claim amounts, $\mathbb{E}(X_i) = \mu$ and N is a Poisson process with intensity λ, denoting the claim occurrences (in a more general setting N can be a renewal process). We denote by $\theta = c(\lambda\mu)^{-1} - 1 > 0$ the safety loading factor.

Then a first approach is the one introduced by Iglehart (1969). He proposes a diffusion process as an approximation to the classical risk process (1). This approximation relies on the weak convergence of a sequence of processes as in (1), under certain conditions, to the diffusion process

$$U_D(t) = u + (c - \lambda\mu)t + \sigma\sqrt{\lambda}W(t) \;, \qquad t \geqslant 0 \;, \tag{2}$$

where λ^{-1} is the mean of the claims inter-occurrence times, $\mathbb{E}(X_i) = \mu$, $\sigma^2 = Var(X_i)$ and W is a one-dimensional standard Brownian motion.

Diffusion risk processes have been developed further since then. It turns out that for these processes there exist closed forms for functionals of (1) that are of concern in risk theory (ruin probabilities, hitting time distributions). In most of the useful cases there exist no closed forms for these functionals, like for the compound Poisson process. This mathematical tractability of (2) makes the use of such processes appealing in risk theory. Further work has been done in this direction such as Bohman (1972), Gluckman (1970) or Grandell (1972, 1977). Also generalizations have appeared such as Emmanuel, Harrison and Taylor (1975) or Garrido (1987) where processes that account for the interest earned on the risk reserve are considered. Developments in the theory of stochastic processes find their way into the field in papers like Ruohonen (1980) or Garrido (1989) where they studied risk processes satisfying a stochastic differential equation.

Another application of Lévy processes as risk models can be found in Dufresne, Gerber and Shiu (1991) where they propose a gamma process as a model for aggregate claims. They show that, despite its being composed by an infinite number of small claims, such a model can still be used in risk theory.

More recently, Furrer et. al. (1997) proposed a more general setting where the risk process is approximated by an α-stable Lévy process instead of a diffusion, this generalizes the diffusion approximation of Grandell (1977) since the Brownian motion belongs to the α-stable family of processes. Their model allows for greater variability than the diffusion approximation and hence, it performs better than the latter in the presence of heavy-tailed claims.

A second approach that has been taken to incorporate Lévy processes in risk theory is the one first introduced in Gerber (1970), where he enlarged the classical model by adding a diffusion component $\sigma W(t)$ to the classical risk process, yielding

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma W(t) , \qquad t \geqslant 0 , \tag{3}$$

where W is a standard one-dimensional Brownian motion. This new component adds extra uncertainty to the aggregate claims process that might account for fluctuations in the number of customers, in the premiums or return on the investment of the reserve. It is a white noise that is proposed to model all of these additional uncertainties.

The model (3) is known as a *perturbed risk process* [see Rolski *et al.* (1999)]. Renewal theory methods have been applied successfully for these new processes [Dufresne and Gerber (1991)] in the case of light-tailed claim sizes. Sub-exponential claim size distributions have been studied in this context by Veraverbeke (1993).

More general settings have been analyzed, as in Björk and Grandell (1988), Furrer and Schmidli (1994) and Schimidli (1995), where they considered the case when N is not necessarily a Poisson process. There exist even further generalizations where neither N is Poisson nor the perturbation is a Brownian motion as in Schlegel (1998) and Furrer (1998). Furrer (1998) generalizes the perturbed model to a process where the perturbation is given in terms of an α-stable process, this includes the perturbed model of Dufresne and Gerber (1991) as a particular case. A more general version of (2) is given in Paulsen (1993, 1998) where the risk reserve process and the perturbation process are Lévy processes.

In Gerber and Landry (1998) we find expressions for the discounted joint distribution function of the surplus prior to ruin and the deficit at the time of ruin for the perturbed model. Their study exploits further the concept of a discounted penalty function. This approach was first introduced in Gerber and Shiu (1998a) and has been further studied in the context of a perturbed model in Wang and Wu (2000), Wang (2001), Tsai (2001) and Tsai and Willmot (2002). Extensions to the case of a surplus with interest have been studied in Cai and Dickson (2002).

In the most recent to date article on Lévy processes in risk theory, Yang and Zhang (2001) propose a unifying approach using a spectrally negative Lévy process as the aggregate claim. They reproduce and formalize previous results.

The whole theory has been rebuilt and enlarged around these two approaches, this idea lies at the heart of our proposal. A unifying approach is proposed in terms of a general Lévy process.

Lévy processes are stochastic processes with independent and stationary increments, the Brownian motion, the compound Poisson process and the α-stable motion are a few examples. This wide family of processes contains many subclasses that have appealing features for financial and insurance risk modeling [see Barndorff-Nielsen, Mikosh and Resnick (2001) and Schoutens (2003)].

We plan to study a unifying model bridging the two approaches. We will consider a model following the dynamics:

$$dU(t) = cdt + bdZ(t) , \qquad t \geqslant 0 , \tag{4}$$

where Z is a Lévy process. Clearly, the classical model (1) is a solution if (4) since the compound Poisson process is a Lévy process.

It is known [Bertoin (1996)] that a Lévy process Z can be decomposed as the sum of three independent Lévy processes

$$Z(t) = Z^{(1)}(t) + Z^{(2)}(t) + Z^{(3)}(t) , \qquad t \geqslant 0 ,$$

where $Z^{(1)}$ is a linear transform of a Brownian motion, $Z^{(2)}$ is a compound Poisson process with jumps of at least size one, and $Z^{(3)}$ is a pure-jump martingale with jumps of at most size one. This brings the idea that, both, the perturbed models in the spirit of Dufresne and Gerber (1991), Furrer (1998) and Schmidli (2001)

$$U_\alpha(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \eta Z(t) , \qquad t \geqslant 0 , \tag{5}$$

with Z a Lévy process, and approximations in the spirit of Iglehart (1969), Grandell (1977) and Furrer, Michna and Weron (1997)

$$U_D(t) = u + (c - \lambda\mu)t + \eta Z(t) , \qquad t \geqslant 0 , \tag{6}$$

with Z a Lévy process, are models of the form (4).

A general Lévy process can include both diffusion and jump components as well as only a diffusion component. The two different approaches should become particular cases of (4) by turning on and off the compound Poisson component.

Results concerning the associated ruin probability for the approximation models as well as for perturbed risk processes are consequences of their being Lévy process. Therefore we should expect to find analogous results for a wider class of risk processes as in (4). For instance, the ladder height decomposition in the classical process is still valid for a subclass of processes of the form (4) as we discuss in Chapter 4.

In the classical risk model, where the surplus process is as in (1), it is well known [see for instance Asmussen (2000)] that the ultimate ruin probability

$$\psi(x) = P\left\{ \inf\left[t > 0 \mid x + ct - \sum_{i=1}^{N(t)} X_i < 0 \right] < \infty \right\}, \qquad x \geqslant 0 ,$$

satisfies

$$\psi(x) = P\{M > x\} , \qquad x \geqslant 0 , \tag{7}$$

where $M = \sup\limits_{0 \leqslant \tau}\{\sum_{i=1}^{N(t)} X_i - c\tau\}$ is the maximum aggregate loss. Moreover M is a compound geometric random variable [Kalashnikov (1997)] and its distribution is given by the so-called Beekman's convolution formula

$$1 - \psi(x) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n G^{*n}(x) , \tag{8}$$

where $\theta = c(\lambda\mu)^{-1} - 1$ is the safety loading factor and $G(x) = \mu^{-1} \int_0^x [1 - F_X(y)]\, dy$ is the so-called *ladder height* distribution.

The maximum aggregate loss M can be expressed as

$$M = \sum_{i=1}^{K} Y_i , \tag{9}$$

where $\{Y_i\}_{i=1\ldots K}$ are i.i.d. random variables with c.d.f. G and independent from a geometric random variable K with parameter $p = \frac{\theta}{1+\theta}$.

In the risk model perturbed by a diffusion of Dufresne and Gerber (1991) there exist similar results for the maximal aggregate loss

$$L = \max_{t \geqslant 0} \left\{ \sum_{i=1}^{N(t)} X_i - ct - W(t) \right\} .$$

They show that for the perturbed model there exist expressions analogous to (8) and (9).

Furrer (1998) presents a generalization of Beekman's formula when the risk process is perturbed by an α-stable Lévy process. Asmussen and Schmidt (1995) and Schmidli (2001) give a more general convolution formula for the maximal aggregate loss for a risk model described by an ergodic stationary marked point process perturbed by an α-stable Lévy process.

Yang and Zhang (2001) use a spectrally negative Lévy process to unify previous approaches. They show that the diffusion model and the perturbed (both by a Brownian motion and an α-stable Lévy motion) are particular cases of their model.

In the present work, we discuss how other Lévy processes can be used to drive a risk model. Special attention is paid to the applications of generalized hyperbolic Lévy motion in risk theory. These models were first introduced by Barndorff-Nielsen (1977) and Barndorff-Nielsen and Halgreen (1977) and have recently been applied in finance [see Eberlein and Keller (1995), Eberlein (2001) and Prause (1999)]. They show that the medium-tailed hyperbolic distribution fits well to stock returns and use it in a general option pricing model. A thorough account of more general Lévy processes in finance can be found in Schoutens (2003).

The family of generalized hyperbolic distributions is large, it contains either directly or as a limiting case the inverse Gaussian, normal, Student-t, Cauchy, exponential and gamma distributions. In financial applications it is often preferred to α-stable distributions since their density is known and all of the moments exist. It has been proven [Barndorff-Nielsen and Halgreen (1977)] that this family belongs to the infinitely divisible class of distributions, which allows us to define a generalized hyperbolic Lévy motion. But there are more appealing properties to the family of hyperbolic distributions. We find our motivation to propose in risk theory the application of this family in Chaubey, Garrido and Trudeau (1998). They showed that an inverse Gaussian distribution and particularly a mixture of an inverse Gaussian and a gamma provided a good approximation to the aggregate claims distribution in (1). The ability of these particular distributions to approximate the aggregate claims distribution suggest the implementation of a generalized hyperbolic motion in a risk model.

Motivated by the good fit found by Chaubey, Garrido and Trudeau (1998), we introduce a generalized inverse Gaussian Lévy risk process. Despite the fact of having an infinite number of small claims in any interval, it still accepts the ladder height decomposition for its ruin probability. This allows us to use existing results on bounds for its associated ruin probability. This model is an extension of the gamma process of Dufresne, Gerber and Shiu (1991) and is yet another example of the processes treated in Yang and Zhang (2001).

Rydberg (1997) uses the fact that pure-jump Lévy process is the limit of a sequence of compound Poisson processes to study the normal inverse Gaussian process. The normal inverse Gaussian is one of the members of the generalized hyperbolic family that is closed under convolutions. This makes it a more natural asset model. As another illustration of Lévy modeling in risk theory, we propose a normal inverse Gaussian risk process. The normal inverse Gaussian Lévy motion has medium-tailed finite-dimensional distributions and it has been applied recently in finance. We argue that this process is a good model for the aggregate claims distribution. The normal inverse Gaussian Lévy process exhibits a diffusion-like feature along with a jump-driven structure. This duality makes it another alternative to respond to the motivation of a perturbed model. This would yield a purely discontinuous risk process exhibiting, in one single object, small fluctuations and occasional larger jumps. A more general treatment of these kind of risk processes can be found in Morales and Schoutens (2003).

Implementing more general Lévy process in risk theory is of particular interest when bridging financial and insurance models. Examples of this interplay between financial and insurance mathematics can be found in Gerber and Shiu (1998b, 1999), Wang (2000) and Avram, Chan and Usabel (2002).

Approximations to the classical risk process can be divided into two cases, based on the claims distribution: approximations for light-tailed claims and for heavy-tailed claims. The diffusion approximation of Grandell (1977) deals with the first case and the α-stable approximation of Furrer, Michna and Weron (1997) deals with the second case. However, both approximations are linked by a general version of the central limit theorem and the concept of domain of attraction of an extreme distribution. In this context we introduce, as another illustration, a Lévy process that lies somewhere in-between these two approximations. Our approach is based on extreme value theory considerations, and in consequence, it adapts to the situation at hand to provide a better approximation regardless if the claim distribution is light or heavy tailed.

Most of the results concerning ruin probabilities for Lévy risk processes rely on properties that are common to the larger family of semimartingales with independent increments. Sørensen (1996) introduces a general risk model driven by a special class of semimartingales. In this framework, we deal with the case when the aggregate claims process is driven by a periodic non-homogeneous Poisson process, as in Garrido, Dimitrov and Chukova (1996). In the classical risk model (1) the claim counting process N is a Poisson process, but when working under periodic conditions, as for hurricane losses, this classical approach is no longer adequate. This is of particular interest when pricing catastrophe insurance options for example.

A generalized model when N is a non-homogeneous Poisson process (NPP) with a periodic intensity function has been discussed recently in Chukova, Dimitrov and Garrido (1993, 2000) Garrido, Dimitrov and Chukova (1996) and Morales (1999). Such a risk process is no longer a Lévy process but still belongs to the class of semimartingales with independent increments. Classical expressions for ruin probabilities can be extended to this general setting. We use the stationarity shown by the periodic process at complete cycles, to find a similar result for the periodic non-homogeneous Poisson risk reserve process.

Classical results on risk models with Lévy processes are presented in more detail in Chapter 1. A review of the theory for Lévy processes is presented in the Chapter 2. Finally, in the last chapters we present the already mentioned illustrations of Lévy modeling in risk theory. Chapter 3 discusses the potential of the normal inverse Gaussian process as a risk model. Chapter 4 generalizes the gamma process of Dufresne, Gerber and Shiu (1991) to a generalized inverse Gaussian process. Chapter 5 deals with the generalized Pareto-stable Lévy approximation. And Chapter 6 deals with the periodic risk reserve process in the context of processes with independent increments. These last chapters represent independent research articles by themselves but together shed new insight on the potential of Lévy modeling.

Chapter 1

A Review of Lévy Processes in Risk Theory

Applications in risk theory of Lévy processes have traditionally been in terms of compound Poisson processes and continuous Brownian motion. This is a consequence of the mathematical tractability of this process. It is not until recently that other Lévy processes, such as α-stable ones, have found applications in the theory.

Consider the classical risk process [see for instance DeVylder (1996), Gerber (1979), Grandell (1991), Embrechts and Klüppelberg (1992) or Prabhu (1998)] given by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i \, , \qquad t \geqslant 0 \, , \tag{1.1}$$

where u is the initial surplus, c is a constant premium rate, $\{X_i\}$ are i.i.d. random variables (with c.d.f. F_X) denoting the claim amounts, $\mathbb{E}(X_i) = \mu$ and N is a Poisson process with mean λ, denoting the claim occurrences (in a more general setting N can be a renewal process) and $\theta = c(\lambda\mu)^{-1} - 1 > 0$ is the safety loading factor. The process $\sum_{i=1}^{N(t)} X_i$ is refered to as the aggregate claims process.

In the following sections we present a brief account of the use of Lévy processes as generalizations of the classical process (1.1).

1.1 Diffusion Risk Processes

The first application of weak convergence in risk theory seems to be due to Iglehart (1969). He showed that a sequence of renewal risk reserve processes converges to a diffusion process.

Theorem 1.1 **[Iglehart (1969)]** *Consider the following sequence of renewal risk processes*

$$U_n(t) = u_n + c_n t - \sum_{i=1}^{N(t)} X_i^{(n)} , \qquad t \geqslant 0 , \tag{1.2}$$

where for each $\{n = 1, 2, \dots\}$ we have a process as the one defined in (1.1) plus the following conditions

i) $u_n = un^{1/2} + o(n^{1/2})$,

ii) $c_n = cn^{-1/2} + o(n^{-1/2})$,

iii) $\mathbb{E}[\tau_i] = \lambda^{-1}$, *where $\{\tau_i\}$ are the inter-occurrence times of $N(t)$,*

iv) $\mathbb{E}[X_i^{(n)}] = \mu n^{-1/2} + o(n^{-1/2})$,

v) $\mathbb{V}ar[X_i^{(n)}] = \sigma_n^2 \to \sigma^2 > 0$ *and*

vi) $\left(\mathbb{E}[X_i^{(n)}]\right)^{2+\epsilon}$ *is bounded for some ϵ.*

Then, for $t \geqslant 0$, $n^{-1/2}U_n(t)$ converges in distribution to $u + (c - \lambda\mu)t + \sigma\lambda^{1/2}W(t)$ where W is a standard Wiener process.

This limit theorem has been used to approximate the risk process (1.1) by a diffusion.

1.1.1 A First Diffusion Risk Process

Iglehart's result motivates the approximation proposed by Grandell (1977) for a case where the claims counting process N can be a more general renewal process. In the case of our concern, which is the classical model (1.1), both approaches yield the same approximation. We follow here the presentation of Grandell (1991).

Let us consider the aggregate claims in the process (1.1) and denote it by S as follows

$$S(t) = \sum_{i=1}^{N(t)} X_i \,, \qquad t \geqslant 0 \,.$$

$S(t) = 0$ if $N(t) = 0$. From sequence (1.2) in Theorem 1.1, we can construct the sequence

$$S_n(t) = \frac{S(nt) - \lambda\mu nt}{\sqrt{\lambda(\mu^2 + \sigma^2)n}} \,, \qquad n \geqslant 1 \,, \qquad t \geqslant 0 \,, \tag{1.3}$$

which converges in distribution to a standard Wiener process W.

Hence the limit risk process for (1.2) becomes

$$U_D(t) = u + \gamma\mu\lambda t - \sqrt{\lambda(\mu^2 + \sigma^2)}W(t) \,, \qquad t \geqslant 0 \,, \tag{1.4}$$

where

$$\gamma = \lim_{n\to\infty} \frac{c_n - \lambda\mu}{\lambda\mu}\sqrt{n} \,.$$

Notice that (1.4) is a Wiener process with drift.

Ruin Probabilities

Following Grandell (1977), from (1.3) we can construct the sequence

$$Y_n(t) = \frac{c_n nt - \lambda\mu nt}{\sqrt{n}} - \sqrt{\lambda(\mu^2 + \sigma^2)}S_n(t) \,. \tag{1.5}$$

This sequence converges in distribution to the two last terms in (1.4) that we denote by Y_D, i.e.

$$Y_D(t) = \gamma\mu\lambda t - \sqrt{\lambda(\mu^2 + \sigma^2)}W(t) \,, \qquad t \geqslant 0 \,.$$

This last equation implies that

$$\mathbb{P}\{\inf_{t\geqslant 0} Y_n(t) < -u\} \longrightarrow \mathbb{P}\{\inf_{t\geqslant 0} U_D(t) < -u\} \,, \qquad u \geqslant 0.$$

This yields the following approximation for the probability of ultimate ruin

$$\psi_n(u) = \mathbb{P}\{U_n(t) < 0 \mid \text{for some } t > 0\} \approx \mathbb{P}\{U_D(t) < 0 \mid \text{for some } t > 0\} = \psi_D(u) \,,$$

when n is large. Here $\psi_D(u)$ has the rather simple expression

$$\psi_D(u) = e^{-u\gamma\frac{2\mu}{\mu^2+\sigma^2}} , \qquad u \geqslant 0 .$$

Note that this formula is of the form $e^{-\mathcal{R}u}$ where $\mathcal{R}$ is the so-called adjustment coefficient [see Bowers *et al.* (1986)].

As for finite ruin probabilities, (1.4) also gives simple expressions. Denote by $\tau_n = \inf\{t > 0 \mid U_n(t) < 0\}$ the time to ruin for (1.5) and by $\tau = \inf\{t > 0 \mid U_D(t) < 0\}$ the time to ruin for (1.4). Then, by the same limiting process, we have

$$\begin{aligned} \psi_D(u,t) &= \mathbb{P}\{\tau \leqslant t\} = \lim_{n\to\infty} \mathbb{P}\{\tau_n \leqslant t\} \\ &= \Phi\left(\frac{-(c-\lambda\mu)t-u}{\sqrt{\sigma^2\lambda t}}\right) + e^{-\frac{2(c-\lambda\mu)u}{\sigma^2\lambda}}\Phi\left(\frac{(c-\lambda\mu)t-u}{\sqrt{\sigma^2\lambda t}}\right) , \end{aligned}$$

where Φ is the c.d.f. of a standard normal r.v. This gives the following approximation for n large

$$\psi_n(u,t) = \mathbb{P}\{\tau_n \leqslant t\} \approx \mathbb{P}\{\tau \leqslant t\} = \psi_D(u,t) , \qquad u,t \geqslant 0 .$$

For a recent discussion on a diffusion approximation in risk theory we also refer to Klugman, Panjer and Willmot (1998).

1.1.2 A Compounding Brownian Motion Model for the Risk Reserve Process

The simplicity of the limit risk process (1.4) encouraged further generalizations to models taking into account investment income. Emmanuel, Harrison and Taylor (1975) proposed an extension of (1.4) where the risk reserve earns interest. They based their analysis on the following risk reserve process

$$U_{CB}(t) = e^{\beta t}u + (c-\lambda\mu)\left(\frac{e^{\beta t}-1}{\beta}\right) + \left[\frac{\lambda(\sigma^2+\mu^2)}{2\beta}\right]^{1/2} W\left(e^{2\beta t}-1\right) , \qquad (1.6)$$

where the term $e^{\beta t}u$ represents the accumulated value of the initial reserve u at a constant interest force β. The second term is the accumulated value of the premiums paid c minus the expected claim cost and the third term is a re-scaled standard Wiener process.

They showed that (1.6) is a diffusion with mean $c-\lambda\mu+\beta u$ and variance $\lambda(\sigma^2+\mu^2)$. This process is related to the Ornstein-Uhlenbeck process [see Beekman (1975)] and is called a *compounding Brownian motion* by the authors.

The justification of (1.6) came from Harrison (1977) where he gives a weak convergence argument. Basically, the argument goes as follows. He constructs a sequence of processes that accounts for the interest earned

$$U_{CB}^{(n)}(t) = e^{\beta t}u + c_n\left(\frac{e^{\beta t}-1}{\beta}\right) - \sum_{i=1}^{N^{(n)}(t)} e^{\beta(t-T_i^{(n)})}X_i^{(n)} , \qquad t \geqslant 0 ,$$

where $N^{(n)}$ is a homogeneous Poisson process and $\{T_i^{(n)}\}$ are its inter-occurrence times. Then under certain asymptotic conditions for the parameters [see Harrison (1977)], the sequence $\left\{U_{CB}^{(n)}\right\}_{n=1,2,\ldots}$ converges in distribution to the compounding Brownian motion process (1.6).

Ruin Probabilities

For this process the ultimate ruin probability is given by [Emmanuel, Harrison and Taylor (1975)]

$$\psi_{CB}(u) = \frac{1-\Phi(a+b\mu)}{1-\Phi(a)} , \qquad u \geqslant 0 ,$$

where

$$a = \left[\frac{2\mu^2}{\beta\lambda(\sigma^2+\mu^2)}\right]^{1/2} \quad \text{and} \quad b = \left[\frac{2\beta}{\lambda(\sigma^2+\mu^2)}\right] .$$

These diffusion processes modeling risk reserves in the presence of compounding interest are no longer Lévy processes (this comes from the fact that their increments are not stationary), however, it is a natural extension once we have the diffusion approximation (1.4). These models are discussed here for completeness since similar generalizations can be worked out starting from a general Lévy risk process. In finance, Ornstein-Uhlenbeck type processes have been recently constructed from generalized hyperbolic Lévy processes [see Barndorff-Nielsen and Shephard (2001) and Schoutens (2003) for instance].

1.1.3 Reserve Processes Characterized by a Stochastic Differential Equation

As the theory of stochastic processes developed, more applications found their way into risk theory. Ruohonen (1980) derived analytical expressions for the probability of ultimate ruin for a wider class of risk processes than that defined in (1.4). He gave solutions for risk reserve processes satisfying a stochastic differential equation of the form

$$dU(t) = a\left[U(t)\right]dt + b\left[U(t)\right]dW(t)\ , \tag{1.7}$$

with $U(0) = u$ and a, b real functions. These results are given in the form of the following theorem:

Theorem 1.2 [Ruohonen (1980)] *If the risk process is a diffusion process satisfying (1.7), then the probability of ruin is*

$$\psi(u) = \frac{\int_u^\infty \exp\left(-2\int_0^s \frac{b(v)}{a^2(v)}dv\right)ds}{\int_0^\infty \exp\left(-2\int_0^s \frac{b(v)}{a^2(v)}dv\right)ds}\ , \qquad u \geqslant 0\ .$$

This expression is a generalization of previous results by Iglehart (1969) and Harrison (1977). If we let $a\left[U(t)\right] = c - \lambda\mu$ and $b\left[U(t)\right] = \sigma\sqrt{\lambda}$ then Iglehart's diffusion is a solution of (1.7). And if $a\left[U(t)\right] = c - \lambda\mu + \beta U(t)$ and $b\left[U(t)\right] = \sigma\sqrt{\lambda}$ then one solution to (1.7) is the compounding Brownian motion given by (1.6).

Another class of diffusion risk reserve processes was defined by Garrido (1989). This family is the set of solutions of a stochastic differential equation that takes into account the investment income earned and for the inflation on the claim amounts. The stochastic differential equation he considered is of the form

$$dU(t) = \left[\pi(t, U(t)) + \beta_t U(t) - \mu(t)\right]dt + \sigma(t)dW(t)\ , \tag{1.8}$$

where $U(0) = u > 0$, $\pi(t, U(t))$ is the rate at which premiums are collected and that might depend on the reserve level $U(t)$, β_t and $\mu(t)$ are the force of interest and the average aggregate claim rate at time t respectively and W is a standard Brownian motion.

All of these previous diffusion risk processes are members of this new class.

If we let $\mu(t) = \lambda\mu$, $\sigma(t) = \sqrt{\lambda(\mu^2+\sigma^2)}$, $\beta_t = 0$ and $\pi(t, U(t)) = c$, then a solution to (1.8) is the classical approximation to the compound Poisson ruin model with mean claim μ given in (1.4).

Letting $\mu(t) = \lambda\mu$, $\sigma(t) = \sigma\sqrt{\lambda}$, $\beta_t = 0$ and $\pi(t, U(t)) = c$, then the resulting process is Iglehart's diffusion reserve process constructed in Theorem 1.1.

Harrison's compounding Brownian motion process (1.6) is obtained by setting $\mu(t) = \mu$, $\sigma(t) = \sigma > 0$, $\beta_t = \beta > 0$ and $\pi(t, U(t)) = c$.

But there are also new processes in that family, namely, the *Inflated Compounding Brownian Motion* [see Garrido (1987)] that is obtained by letting $\mu(t) = \mu e^{A(t)}$, $\sigma(t) = \sigma e^{A(t)}$, $\pi(t, U(t)) = ce^{A(t)}$ where $A(t) = \int_0^t \alpha_s ds$ (α_t is the inflation force at t) and $B(t) = \int_0^t \beta_s ds$. Then the risk reserve process takes the form

$$U_{ICBM}(t) = e^{B(t)}u + (c-\mu)e^{A(t)} \int_0^t e^{D(t)} ds + \sigma e^{A(t)} \int_0^t e^{D(t)-D(s)} dW(s) , \qquad (1.9)$$

where $D(t) = B(t) - A(t) = \int_0^t \delta_s ds$.

Once again, these diffusion processes defined in terms of a stochastic differential equation, as in (1.8), are no longer Lévy processes. They are discussed here for completeness and as motivation for further research that goes beyond the scope of this thesis.

1.2 An α-stable Approximation

Furrer, Michna and Weron (1997) generalize the diffusion approximation of Grandell (1977). They propose a risk process of the form

$$U(t) = u + ct + \eta Z_\alpha(t) , \qquad t \geqslant 0 , \qquad (1.10)$$

where Z_α is an α-stable Lévy process. This process allows for greater flexibility than the Brownian motion.

α-stable processes are generalizations of Brownian motion, its one-dimensional distributions are α-stable. The tails of these distributions decrease like a power function. The rate of decay depends mainly on a parameter $\alpha \in (0, 2]$. The smaller the value of α, the slower the decay and the heavier the tails. In this family, distributions have four parameters, so they are denoted $S_\alpha(m, v, \beta)$, where m and v are location and scale parameters, respectively, so they are usually set to $m = 0$ and $v = 1$, leaving only α and β as free parameters. The p.d.f. for these distributions does not exist in close form, except for the cases $\alpha = 2$ (Gaussian distribution), $\alpha = 1$ (Cauchy distribution) and for $\alpha = 1/2$ (the one-sided stable distribution). However their characteristic functions have a explicit form (see Zolotarev (1986) or Janicki and Weron (1994) for a discussion on α-stable distributions). α-stable processes will be discussed further in Chapter 2. Meanwhile we give the following definition:

Definition 1.1 *A stochastic process Z_α is called a standard α-stable process if*

i) $Z_\alpha(0) = 0$, *a.s.,*

ii) Z_α *has independent increments,*

iii) $Z_\alpha(t) - Z_\alpha(s) \sim S_\alpha(0, (t-s)^{1/\alpha}, \beta)$ *for any* $0 \leqslant s < t < \infty$ *and for some* $0 < \alpha \leqslant 2$, $|\beta| \leqslant 1$.

Note that the standard Brownian motion is an α-stable Levy process with $\alpha = 2$ (see Samorodnitsky and Taqqu (1994) for a further discussion on α-stable processes).

α-stable distributions can be characterized as the limit of normalized sums of random variables. Let $S_n = \sum_{i=1}^n X_i$ where $\{X_i\}_{i=1,2,\dots}$ are i.i.d. copies of a random variable X. We say that X belongs to the domain of attraction of an α-stable random variable S_α, for $\alpha \in (0, 2]$, if there exist constants $a_n \in \mathbb{R}$, $b_n > 0$ such that

$$\frac{S_n - a_n}{b_n} \longrightarrow S_\alpha ,$$

in distribution as $n \to \infty$. Notice that this is a generalization of the central limit theorem since S_2 is a standard normal random variable. Iglehart (1969) used the central limit theorem to construct a sequence of compound Poisson processes converging to a Brownian motion. Likewise, Furrer, Michna and Weron (1997) define an analagous converging sequence.

In order to motivate their approximation, Furrer, Michna and Weron (1997) construct a sequence of risk reserve processes

$$U^{(n)}(t) = u^{(n)} + c^{(n)}t - \frac{1}{b_n}\sum_{i=1}^{N(t)}[Y_i - \mu] , \qquad t > 0 , \tag{1.11}$$

where $u^{(n)}$, $c^{(n)}$ are the corresponding sequences of initial reserves and premium rates such that $u^{(n)} \to u$ and $\frac{c^{(n)}-\mu}{b_n} \to c$, with constants $\mathbb{E}[Y] = \mu$ and b_n a slowly varying function. The counting process N is Poisson with mean λ and $\{Y_i\}_{i=1,2,\dots}$ are i.i.d. with d.f. given by F_Y with support on $\mathbb{R}^+$.

Depending on the tail of the claim distribution F_Y, the processes in (1.11) will converge weakly to an α-stable process with index $\alpha \in (1,2)$ or to a Brownian motion with drift. The first of these limiting processes is the α-stable approximation of Furrer, Michna and Weron (1997) and the second is the classical diffusion approximation of Grandell (1977). We refer to Alexander (1996) for a discussion on weak convergence in risk theory.

1.2.1 Ruin Probabilities

For the α-stable process in (1.10), Furrer, Michna and Weron (1997) worked out some results concerning finite ruin probabilities. They present the following asymptotic result for finite ruin probabilities of the process (1.10).

Theorem 1.3 *Let Z_α be an α-stable Lévy motion with skewness parameter $|\beta| < 1$. Then*

$$\mathbb{P}[u + cs - \eta^{1/\alpha} Z_\alpha(s) \leqslant t] \sim C_\alpha \frac{1+\beta}{2} \lambda t (u+ct)^{-\alpha} , \qquad u \longrightarrow \infty ,$$

where

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{1}{2}\pi\alpha)} .$$

The meaning of $f(x) \sim g(x)$ should be understood in the sense that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Theorem 1.3 leads to upper bounds for the finite ruin probability with the α-stable approximation.

The general risk process, presented in this section, incorporates Lévy processes that lead to approximations for ruin probabilities in the case of light and heavy tailed claim distributions. In the following section we present a first attempt to use non-standard Lévy motion as aggregate claims process.

1.3 A Gamma Risk Process

The first application to our knowledge of a Lévy process, other than a compound Poisson, as an aggregate claims model is due to Dufresne, Gerber and Shiu (1991). They presented a gamma process as a model for the aggregate claims process yielding the following risk model:

$$U(t) = u + ct - G(t) \qquad t \geqslant 0 \,, \tag{1.12}$$

where u is the initial reserve, G is a gamma process representing the claims and the loaded premium $c = (1+\theta)\mathbb{E}[G(1)]$.

They define a class of increasing processes G in terms of their Laplace transform

$$\mathbb{E}[e^{-zG(t)}] = \exp\left\{t \int_0^\infty \left[1 - e^{-zy}\right] dQ(y)\right\} \,, \qquad z > 0 \,, t \geqslant 0 \,, -1 < \alpha < 0 \,, \tag{1.13}$$

where $-dQ(x) = ax^{\alpha-1}e^{-bx}dx$ for positive constants a and b. All of its increments are gamma distributed, i.e., the random variable $G(t)$ has distribution with shape parameter at and scale parameter b. For $\alpha = 0$ we have that G is a gamma process. Another member of this family is the inverse Gaussian process that can be obtained by setting $\alpha = -1/2$.

Processes in this family are limits of a type of compound Poisson processes and are composed by an infinite number of small jumps. Dufresne, Gerber and Shiu (1991) discuss these and other properties in a risk theory context. Despite having an infinite number of jumps, these remain somehow small enough as to allow for a certain regularity of the process. For example, the jumps are always positive so that they can be used to model claims. The distribution of the aggregate claims process can be a gamma or an inverse Gaussian, both distributions have been succesfully used to fit aggregate claims [see Chaubey, Garrido and Trudeau (1998)]. Ruin probabilities find simple expressions that allow for the implementation of bounds as in Cai and Garrido (1998).

Dufresne, Gerber and Shiu (1991) showed that implementation of processes with infinite claims frequency has potential in risk theory. They started out a new approach by incorporating more general Lévy processes into the theory. Their treatment was for a general process with independent increments and with paths of finite variation, although they did not know the term at the time. Yang and Zhang (2001) formalized and extended this approach to include the more general class of spectrally negative Lévy processes. In this thesis we extend results in Dufresne, Gerber and Shiu (1991) to a wider class of processes giving yet another example of the models described in Yang and Zhang (2001).

In three previous section we present how Lévy processes have been used to approximate or to model a risk process. A second approach incorporates Lévy processes as a perturbation. Lévy processes, namely Brownian motion and α-stable processes, have been introduced to account for perturbations in the premium, as seen in the following section.

1.4 Perturbed Risk Processes

The first appearance of perturbed risk models in the literature seems to have been in Gerber (1970), where he proposed a model as in (3). Later, in Dufresne and Gerber (1991), they worked with

$$U_P(t) = u + ct - \sum_{i=1}^{N(t)} X_i + W(t) \ , \qquad t \geqslant 0 \ , \tag{1.14}$$

where W is a standard Wiener process independent of the compound sum $\sum_{i=1}^{N(t)} X_i$. This new model, is the same as in the classical case except for the inclusion of a Brownian motion with drift zero and infinitesimal variance $2V$. This intends to account for additional uncertainties in the premiums or/and in the claims. It is a white noise modeling the ever-changing economical environment. Theirs is also the idea of identifying two sources of ruin. Let us consider, as in the classical model, the probability of ultimate ruin

$$\psi(u) = \mathbb{P}\{U(t) < 0 \mid \text{for some } t > 0\} .$$

In this new model the ruin probability can be decomposed as

$$\psi(u) = \psi_d(u) + \psi_S(u) ,$$

where ψ_d is the probability of ruin induced by the diffusion, i.e. the surplus level at the time of ruin is zero, and ψ_S is the probability of ruin caused by a claim, i.e. the surplus level at the time of ruin is negative. Notice that because of the diffusion nature of the process we have that

$$\psi_S(0) = 0 \qquad \text{and} \qquad \psi(0) = \psi_d(0) = 1 .$$

1.4.1 A Renewal Approach to Ruin Probabilities

Dufresne and Gerber (1991) show that $1 - \psi(u)$ follows a defective renewal equation of the form

$$1 - \psi(u) = qH_1(u) + (1-q)\int_0^t [1 - \psi(s)]h_1 * h_2(u-s)ds , \qquad u \geqslant 0 , \tag{1.15}$$

where the operator $*$ is the convolution of functions on $\mathbb{R}^+$ defined as $f * g(x) = \int_0^x f(x-y)g(y)dy$, $q = \frac{c-\lambda\mu}{c}$ and h_1 and h_2 are p.d.f.'s (and therefore H_1 and H_2 are the corresponding c.d.f.'s) given by

$$\begin{aligned} h_1(x) &= \frac{c}{V}e^{-\frac{c}{V}x} , \qquad x > 0 , \\ h_2(x) &= \frac{1}{\mu}[1 - F_X(x)] , \qquad x > 0 . \end{aligned}$$

From (1.15), by using standard renewal theory techniques [see for example Cox (1967) or De Vylder (1996)], they arrived to a generalization of the classical Beekman convolution formula

$$1 - \psi(u) = \sum_{n=0}^{\infty} q(1-q)^n H_1^{*(n+1)} * H_2^{*n}(u) , \qquad u \geqslant 0 . \tag{1.16}$$

The solution (1.16) of the integral equation (1.15) is an infinite series of functions sometimes called a Neumann series.

Numerical solutions can be obtained from (1.16) using standard methods in risk theory [see for instance Dufresne and Gerber (1989)]. Similarly, they provided implicit solutions for $\psi(u)$, $\psi_d(u)$ and $\psi_S(u)$ since they satisfy defective renewal equations analogous to (1.15). Note that the Brownian motion W does not appear in the renewal equation and therefore, although it is a more complex model, the solution is still within the scope of renewal theory.

The renewal approach has brought new insight into the theory. More recent references to the perturbed model can be found in Tsai and Willmot (2002) and Cai and Dickson (2002).

Lundberg-type Bounds

If F_X is sufficiently regular in its right tail so that the moment generating function exists, then, by following a similar construction as in the classical case, from (1.14) Dufresne and Gerber (1991) define the adjustment coefficient R (when it exists) as the positive solution r of

$$\lambda \int_0^\infty e^{rx} dF_X(x) + Vr^2 = \lambda + cr \ ,$$

then the following Lundberg-type bound holds

$$e^{-Ru} > \psi_d(u) + \psi_S(u) = \psi(u) \ , \quad \text{for } u > 0 \ .$$

Moreover the following asymptotic expressions can be obtained by renewal arguments

$$\begin{aligned} \psi_d(u) &\sim C^d e^{-Ru} \ , \\ \psi_S(u) &\sim C^S e^{-Ru} \ , \end{aligned}$$

for large u and for certain constants C^d and C^S. The meaning of $f(x) \sim g(x)$ should be understood in the sense that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$. For further details and bounds see Cai and Garrido (2002).

1.4.2 Heavy-tailed Claim Size Distributions

Veraverbeke (1993) showed that when the claim-size distribution is sub-exponential [see Rolski et al. (1999)], then the ruin probability behaves asymptotically like the integrated tail of the claim-size distribution. His result also extends to exponential cases where the adjustment coefficient does not exist.

Let us consider the model proposed by Dufresne and Gerber (1991) given by (1.14), then just as in the classical case we have that

$$1 - \psi(u) = \mathbb{P}\{L \leqslant u\} , \qquad u \geqslant 0 ,$$

where $L = \max\{\sum_{i=1}^{N(t)} X_i - ct - W(t) | t \leqslant 0\}$ is the maximal aggregate loss. With the usual notation for the *ladder-height* (or equilibrium) distribution

$$F_X^L(x) = \frac{1}{\mathbb{E}[X]} \int_0^x [1 - F_X(s)]\, ds , \quad x > 0 ,$$

we have the following result.

Theorem 1.4 [Veraverbeke (1993)] *The following are equivalent*

i) F_X^L *belongs to the class of sub-exponential distributions,*

ii) $1 - \psi$ *belongs to the class of sub-exponential distributions,*

and either one of them imply

$$\psi(u) \sim \left[\frac{1}{c\lambda^{-1} - \mathbb{E}(X)}\right] \int_u^\infty [1 - F_X(s)]\, ds , \quad as \qquad u \to \infty .$$

The notation of $f(x) \sim g(x)$ should be understood in the sense that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

This gives an asymptotic approximation to the ruin probability in the case of sub-exponential claim-size distributions. It extends the theory so as to provide results for claim-size distributions such as Pareto or lognormal.

1.4.3 Risk Processes Perturbed by an α-stable Levy Process

A further generalization was presented in Furrer (1998) where he considered a risk process perturbed by a α-stable Levy process. He considered a risk process of the form

$$U_\alpha(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \eta Z_\alpha(t) , \qquad t \geqslant 0 , \tag{1.17}$$

where all the variables and parameters are as in the classical compound Poisson setting and Z_α is a α-stable Lévy process.

In the original perturbed model of Dufresne and Gerber (1991) the use of a Brownian motion does not allow for large fluctuations and it is not adequate to model large variations. By adding an α-stable process instead, Furrer achieved more flexibility through the insertion of the new parameter α. By changing α one can control the variability of the process, the smaller the α, the more dramatic the fluctuations; the closer to 2, the nicer it behaves.

For a perturbed process (1.17), Furrer (1998) proved the following convolution formula for the ultimate ruin probability:

Theorem 1.5 [Furrer (1998)] *Consider the perturbed risk process in (1.17) with $\alpha \in (0,2)$ and $\beta = -1$ (this allows only jumps downwards), then the probability of ruin ψ satisfies*

$$1 - \psi(u) = \left(1 - \frac{\lambda\mu}{c}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda\mu}{c}\right)^n F_X^{L*n} * Q^{*(n+1)}(u) , \qquad u \geqslant 0 , \tag{1.18}$$

where F_X^L is the ladder-height distribution

$$F_X^L(x) = \frac{1}{\mathbb{E}[X]} \int_0^x [1 - F_X(s)]\, ds , \qquad x \geqslant 0 , \tag{1.19}$$

and

$$Q(x) = 1 - \sum_{n=0}^{\infty} \frac{(-c/\eta^\alpha)^n}{\Gamma(1 + (\alpha - 1)n)} x^{(\alpha-1)n} , \qquad x \geqslant 0 . \tag{1.20}$$

This theorem is a generalization of the convolution formula given by Dufresne and Gerber (1991) since it includes, as a particular case, the risk process (1.14).

This result relies on the fact that β was set to -1 to avoid upwards jumps. This restriction makes the process (1.17) a spectrally negative Lévy process for which passage times have nicer expressions.

Furrer's results are derived from the following theorem in Zolotarev (1964)

Theorem 1.6 *Let Y be an α-stable Lévy process with no-positive jumps and $\gamma = \mathbb{E}[Y(1)] \geqslant 0$. Define $\psi(x) = \mathbb{P}[\inf_{t\geqslant 0} \{Y(t) < -x\}]$. Then ψ is implicitly determined by the characteristic exponent Ψ of Y in the relation*

$$s \int_0^\infty e^{-sx}\psi(x)dx = 1 - \frac{\gamma s}{\Psi(s)} , \qquad s > 0 .$$

The function Ψ is the exponent appearing in the characteristic function of Y and it will be formally defined later on.

If we translate this theorem into a ruin problem context, the Lévy process Y represents the perturbed aggregate claim and the premium components in (1.17), i.e.

$$Y(t) = \sum_{i=1}^{N(t)} X_i - \eta Z_\alpha(t) - ct , \qquad t \geqslant 0 .$$

Then the function $\psi(x)$ is the ruin probability for initial surplus x. One can solve this equation for ψ and obtain (1.18).

The convolution formulas (1.16) and (1.18) for the risk model perturbed by a diffusion and an α-stable Lévy motion are linked to a ladder-height decomposition that is common to Lévy processes, in particular to spectrally negative Lévy processes [see Yang and Zhang (2001)].

1.5 Ladder Height Distributions

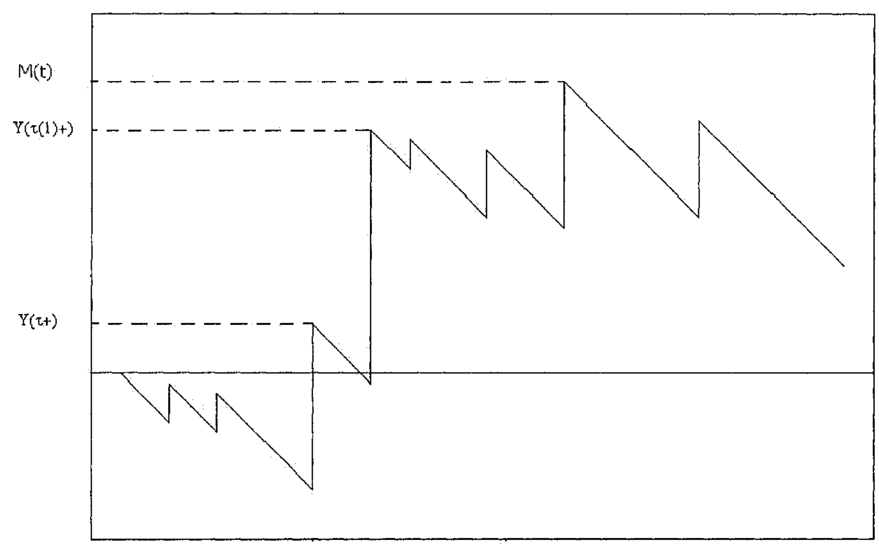
In the classical model (1.1) the ruin probability can be expressed in terms of the so-called ladder height distribution. This result, as seen in the previous sections, has been extended to models perturbed by diffusions or by α-stable processes. Here, we give a brief account of ruin probabilities and ladder heights distributions [see Asmussen (2000) for a thorough account on ruin probabilities].

We define the classical claim surplus process as

$$\Upsilon(t) = u - U(t) = \sum_{i=1}^{N(t)} X_i - ct \, , \qquad t > 0 \, , \tag{1.21}$$

($\Upsilon(0) \equiv 0$) where U is the risk process in (1.1) with initial reserve u. Then, consider the time to ruin $\tau(u) = \inf\{t > 0 \mid \Upsilon(t) > u\}$ in the particular case $u = 0$. We write $\tau_+ = \tau(0)$. Then we call *ladder height* the random variable $\Upsilon(\tau_+)$. The term ladder height comes from the ladder-like structure of the process of relative maxima [see Figure 1.1]. Let us define the *ladder epochs* $\tau_+^{n+1} = \inf\{t > \tau_+^n | \Upsilon_t > \Upsilon_{\tau_+^n}\}$, for $n = 0, 1, 2, \dots$ and $\tau_+^0 = \tau_+$. The ladder epochs $\{\tau_+^n\}_{n\in\mathbb{N}}$ are the moments at which the claim surplus process Υ attains a new maximum.

Figure 1.1: **The Surplus Process and the Ladder Heights**



In Figure 1.1 we have depicted the claim surplus and drawn the ladder steps, the first step of the ladder is precisely $\Upsilon(\tau_+)$, the second ladder point is $\Upsilon(\tau_+^1)$ and consequently the second ladder step is $\Upsilon(\tau_+^1) - \Upsilon(\tau_+)$. The process of relative maxima M is the total height of the ladder given by the sum

$$M(t) = \sum_{n=1}^{N(t)} \left[\Upsilon(\tau_+^n) - \Upsilon(\tau_+^{n-1})\right] \ , \qquad t > 0 \ .$$

Since this is a telescopic sum it can be written as

$$M(t) = \sup_{n \geqslant 0} \left\{\Upsilon(\tau_+^n)\right\} \ ,$$

which is the maximal aggregate loss.

It is known [see for instance Rolski *et al.* (1998) or Asmussen (2000)] that the ruin probability

$$\psi(u) = \mathbb{P}\left\{\inf\left[t > 0 \mid u + ct - \sum_{i=1}^{N(t)} X_i < 0\right] < \infty\right\} \ , \qquad u \geqslant 0 \ ,$$

satisfies

$$\psi(u) = \mathbb{P}\{M(\infty) > u\} \ , \qquad u \geqslant 0 \ .$$

Moreover $M(\infty)$ is a compound geometric random variable [Bowers *et al.* (1986), Kalashnikov (1997) or Asmussen (2000)] and its distribution is given by the so-called Beekman's convolution formula

$$1 - \psi(u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n F_X^{L*n}(u) \ ,$$

or

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{n=0}^{\infty} \left(\frac{1}{1+\theta}\right)^n (1 - F_X^L)^{*n}(u) \ , \tag{1.22}$$

where $\theta = c(\lambda\mu)^{-1} - 1$ is the safety loading factor and $F_X^L(x) = \mu^{-1}\int_0^x (1 - F_X(y))dy$ is the distribution of the *ladder heights* of the claims size distribution F_X. F_X^L is the so-called ladder height distribution, also known as the equilibrium distribution or the integrated tail distribution.

This implies that the ultimate ruin probability ψ is the tail of the distribution of a compound geometric random variable $M(\infty)$ that can be written as a sum of the form

$$M(\infty) = \sum_{i=1}^{M} L_i \ , \tag{1.23}$$

where M is a geometric random variable with parameter θ and $\{L_i\}_{i=0,1,\dots}$ are i.i.d. random variables with distribution F_X^L. M and L_i are independent where $\{L_i\}_{i=1,2,\dots}$ are the ladder heights of (1.21). This convolution series and its interpretation as a compound geometric distribution can be found, for instance, in Kendall (1957).

In renewal theory, the equation in (1.22) is known to be the solution of the following defective renewal equation [see De Vylder (1996) and Grandell (1991)]

$$\psi(u) = \frac{1 - F_X^L(u)}{1+\theta} + \frac{1}{1+\theta} \int_0^u \psi(u-y) dF_X^L(y) \ , \qquad u \geqslant 0 \ . \tag{1.24}$$

In turn, in the literature of integral equations, (1.24) is classified as a Volterra integral equation and its solution can be expressed as an infinite series, called a Neumann series, of the form (1.22) [see Pipkin (1991) for instance].

The distribution of the ladder height in the presence of a perturbation has been studied when the perturbing process is a Brownian motion [Dufresne and Gerber (1991)], when it is an α-stable distribution [Furrer (1998)] and when it is a spectrally negative Lévy process [Yang and Zhang (2001)]. In Prabhu (1998) we find a more general version of this result that is valid for increasing Lévy processes. Asmussen and Schmidt (1995) and Schmidli (2001) present a more general result for a risk process described by an ergodic stationary marked point process. Wang (2001) studies a perturbed model with return on investments. We will discuss in more detail some of these generalizations.

1.5.1 Maximal Aggregate Loss in the Presence of a Diffusion

In the presence of a perturbation we define the claim surplus process as

$$\Upsilon(t) = u - U(t) = \sum_{i=1}^{N(t)} X_i - ct - W(t) \ , \qquad t > 0 \ , \tag{1.25}$$

$(\Upsilon(0) = 0)$ where U is the perturbed risk process in (1.14) with initial reserve u.

Dufresne and Gerber (1991) find that, just as in the classical case, the non-ruin probability $1 - \psi$ satisfies

$$1 - \psi(u) = \mathbb{P}\{L \leqslant u\} ,$$

where $L = \max_{t>0}\{\sum_{i=1}^{N(t)} X_i - ct - W(t)\}$ is the maximal aggregate loss. They show that the convolution formula (1.16) is the tail of the distribution of the maximal aggregate loss L and is given by

$$\mathbb{P}\{L \leqslant u\} = \sum_{n=0}^{\infty} q(1-q)^n G^{*(n+1)} * F_X^{L*n}(u) , \qquad u \geqslant 0 ,$$

where $q = 1 - \frac{\lambda\mu}{c}$, G is an exponential distribution with parameter $\zeta = \frac{2c}{V}$, V is the variance of W) and F_X^L is the so-called *ladder* distribution. G and F_X^L are independent and, in this case, are given by

$$G(x) = 1 - e^{-\zeta x} , \qquad x > 0 ,$$

where

$$\zeta = \frac{c}{V} ,$$

and

$$F_X^L(x) = \frac{\int_0^x [1 - F_X(s)] ds}{\int_0^\infty [1 - F_X(x)]\, dx} , \qquad x > 0 . \tag{1.26}$$

This implies that the ultimate ruin probability ψ is the tail of the distribution of a compound geometric random variable L, i.e.

$$\psi(u) = \mathbb{P}(L > u) , \qquad u \geqslant 0 ,$$

where the random variable L is sum of the form

$$L = L_0^{(1)} + \sum_{i=1}^{M} \left[L_i^{(1)} + L_i^{(2)} \right] , \tag{1.27}$$

where M is a geometric random variable with parameter q, $\left\{L_i^{(1)}\right\}_{i=0,1,\ldots}$ are i.i.d. random variables with distribution G and $\left\{L_i^{(2)}\right\}_{i=1,2,\ldots}$ are i.i.d. random variables with distribution F_X^L. M, $L_i^{(1)}$ and $L_i^{(2)}$ are independent. $\left\{L_i^{(1)}\right\}_{i=0,1,\dots}$ are the parts of the ladder heights due to the perturbation in (1.25) and $\left\{L_i^{(2)}\right\}_{i=1,2,\dots}$ are the parts of the ladder heights due to a claim in (1.25).

Decompositions of the ruin probability as in (1.23) and (1.27) are valid for a wider class of risk processes as discussed in Furrer (1998).

1.5.2 Maximal Aggregate Loss in the Presence of an α-stable Perturbation

Furrer (1998) considers a model where the perturbation is an α-stable Lévy process. For this new perturbed model Furrer provides the convolution formula (1.18) that generalizes (1.16). However he does not show that the functions F_X^L and Q, in (1.18), are indeed the ladder heights in a decomposition similar to (1.23) and (1.27). Schmidli (2001) works with the perturbed process in the context of an ergodic stationary marked point process and shows that the ruin probability for the perturbed model of Furrer (1998) accepts a decomposition similar to (1.23) and (1.27). That is, the functions F_X^L and Q in (1.18) are the distributions functions of the ladder heights due to the perturbation and to the claims process respectively.

We have that the ultimate ruin probability ψ for the perturbed model of Furrer (1998) is the tail of the distribution of a compound geometric random variable L, i.e.

$$\psi(u) = \mathbb{P}(L > u) , \qquad u \geqslant 0 ,$$

where the random variable L is sum of the form

$$L = L_0^{(1)} + \sum_{i=1}^{M} \left[L_i^{(1)} + L_i^{(2)} \right] , \tag{1.28}$$

where M is a geometric random variable with parameter $q = 1 - \frac{\lambda\mu}{c}$, $\left\{L_i^{(1)}\right\}_{i=0,1,\dots}$ are i.i.d. random variables with distribution Q and $\left\{L_i^{(2)}\right\}_{i=1,2,\dots}$ are i.i.d. random variables with distribution F_X^L. Recall that the functions F_X^L and Q are given by (1.19) and (1.20). These are generalizations of the functions in decomposition (1.27). We can see that the part of the ladder height due to a claim ($L^{(2)}$ values) have the same distribution F_X^L regardless of the type of perturbation. Notice that the distribution Q [see (1.20)] of the part of the ladder height due to the perturbation ($L^{(1)}$) reduces to an exponential in the case $\alpha = 2$, which agrees with the result in Dufresne and Gerber (1991).

1.5.3 Maximal Aggregate Loss for the Gamma Risk Process

The gamma process of Dufresne, Gerber and Shiu (1991) exhibits the counterintuitive property of having an infinite number of small jumps in any interval, in spite of which, it keeps many of the nice features of the compond Poisson process. For instance, the non-ruin probability $1 - \psi(u)$ is the distribution function of the maximal aggregate loss for the gamma process (1.12). The maximal aggregate loss is defined as $L = \max_{t>0}\{G(t) - ct\}$.

Recall the construction of Dufresne, Gerber and Shiu (1991), they define the gamma process in terms of its Laplace transform (1.13)

$$\mathbb{E}[e^{-zG(t)}] = \exp\left\{t\int_0^\infty \left[1 - e^{-zy}\right] dQ(y)\right\}, \qquad z > 0\,,$$

with $-dQ(x) = ax^{-1}e^{-bx}dx$. Then, the distribution of L is still a compound geometric and accepts the ladder height decomposition

$$L = \sum_{i=1}^{M} L_i\,, \tag{1.29}$$

where M is a geometric random variable with parameter $\frac{1}{1+\theta}$ and $\{L_i\}_{i=0,1,\ldots}$ are i.i.d. random variables with distribution

$$F_X^L(x) = \frac{\int_x^\infty Q(y)dy}{\int_0^\infty ydQ(y)}\,, \qquad x > 0\,. \tag{1.30}$$

Moreover, M and L_i are independent.

We can see that decompositions like (1.23) and (1.27) are still valid for other processes. In Prabhu (1998) we find a more general version of this result that is valid for increasing Lévy processes.

This decomposition will be seen to be common to a large class of Lévy processes. In this thesis we generalize the gamma process to a generalized inverse Gaussian process. A generalized inverse Gaussian process family of processes contains the gamma and the inverse Gaussian processes as particular cases. It is, thus, a natural extension to the model of Dufresne, Gerber and Shiu (1991) and another example of the models in Yang and Zhang (2001). The gamma risk process finds its motivation on the decomposition (1.29). We show how, for a generalized inverse Gaussian process, decomposition (1.29) still holds. We also express its ladder height distribution in terms of the Lévy measure Q of the generalized inverse Gaussian processes. This leads to an expression similar to (1.30).

Chapter 2

Infinitely Divisible Distributions and Lévy Processes

This chapter presents a review of the theory of Lévy processes. Definitions and concepts used throughout the thesis are also discussed.

Although this review follows the modern presentation of Bertoin (1996), Itô (1969), Sato (1999) and Loève (1977), it is also interesting from the historical point of view, to refer to the original work of Lévy (1954).

For the section on stochastic integrals we follow the construction of Métivier and Pellaumail (1980).

2.1 Infinitely Divisible Distributions

Consider a r.v. X in a probability space $(\Omega, \mathcal{B}(\mathbb{R}), \mu)$ and its characteristic function given by

$$\phi_\mu(u) = \mathbb{E}_\mu\left[e^{iuX}\right] = \mathbb{E}_\mu\left[\cos uX\right] + i\mathbb{E}_\mu\left[\sin uX\right] \ , \qquad u \in \mathbb{R} \ ,$$

then we can state the following:

Definition 2.1 *The law μ is called infinitely divisible (ID) if for any integer $n > 0$,* *there exists a probability measure* μ_n *such that*

$$\phi_\mu(u) = [\phi_{\mu_n}(u)]^n .$$

In other words, μ can be expressed as the n^{th} convolution power of μ_n and X can be expressed as the sum

$$X(\omega) = \sum_{i=1}^{n} Y_i(\omega), \qquad \omega \in \Omega ,$$

where $\{Y_i\}_{i=1,\ldots,n}$ is a family of i.i.d. random variables having common law μ_n. Notice that the law μ_n does not have to be of the same family as μ.

One can easily verify the following property of infinitely divisible distributions:

Theorem 2.1 *The family of all infinitely divisible distributions is closed under linear transformations, convolutions and limits.*

There are two well-known distributions that are infinitely divisible: The Poisson and the normal distribution. One can easily verify that their characteristic functions, given respectively by

$$\phi_G(u) = \exp\left\{i\mu u - \frac{\sigma^2}{2}u^2\right\}, \qquad \mu \in \mathbb{R}\,, \sigma \in \mathbb{R}^+ \tag{2.1}$$

$$\phi_P(u) = \exp\left\{\lambda\left(e^{iu} - 1\right)\right\}, \qquad \lambda \in \mathbb{R}^+ , \tag{2.2}$$

satisfy Definition 2.1.

Applying Theorem 2.1 to (2.1) and (2.2) produces a bigger class of ID distribution, for example those characterized by

$$\phi(u) = \exp\left\{imu - \frac{\sigma^2}{2}u^2 + \sum_{i=1}^{K} \lambda_i\left(e^{iua_i} - 1\right)\right\} .$$

This idea of expanding the family of ID distributions with the aid of Theorem 2.1 leads to the Lévy characterization theorem for infinitely divisible distributions.

2.1.1 Lévy-Khintchine Characterization

The following result is due to Lévy and characterizes the family of ID distributions. For a proof we refer to Itô (1969).

Theorem 2.2 *Every ID distribution μ can be written in the form*

$$\phi_\mu(u) = e^{-\Psi_\mu(u)} , \qquad u \in \mathbb{R} ,$$

with

$$\Psi_\mu(u) = iau + \frac{b^2}{2}u^2 + \int_{\mathbb{R}} \left[1 - e^{iux} + iux\mathbb{I}_{(-1,1)}(x)\right] \nu(dx) , \tag{2.3}$$

where $a \in \mathbb{R}$, $b^2 > 0$ and ν is a measure on $\mathbb{R}_0 = \mathbb{R} - \{0\}$ satisfying

$$\int_{\mathbb{R}_0} (1 \wedge |x|^2)\nu(dx) < \infty .$$

The parameters a, b^2 and ν uniquely determine μ. We say that our infinitely divisible distribution has a triplet of Lévy characteristics (or Lévy triplet for short) $[a, b^2, \nu(dx)]$. The measure ν is called the Lévy measure and the exponent Ψ_μ is called the characteristic exponent of the distribution μ.

If the distribution μ has a finite mean then (2.3) can be alternatively written as

$$\Psi_\mu(u) = ia^*u + \frac{b^2}{2}u^2 + \int_{\mathbb{R}_0} \left[1 - e^{iux} + iux\right] \nu(dx) , \tag{2.4}$$

where the drift a^* is the mean of the distribution [see Sato (1999)]. This is, if the distribution μ has a finite mean the Lévy-Khintchine characterization takes on a simpler form where the mean of the distribution appears as the drift term. The new Lévy triplet is $[a^*, b^2, \nu(\mathrm{d}x)]$ with $a^* = a + \int_{\{|x|>1\}} x\nu(\mathrm{d}x)$.

Now we present some examples of ID distributions generated by Theorem 2.2.

2.1.2 Normal Distribution

We can see in (2.1) that the characteristic exponent of a normal distribution is given by

$$\Psi(u) = \frac{\sigma^2}{2}u^2 - imu , \qquad u \in \mathbb{R} .$$

It is easily verified that this exponent is of the form (2.3) with $a = -m$, $b^2 > 0$ and $\nu(A) = 0$, $\forall A \in \mathcal{B}(\mathbb{R}_0)$.

2.1.3 Poisson Distribution

Consider the Dirac measure $\delta_c : \mathcal{B}(\mathbb{R}) \longmapsto [0, \infty]$ where

$$\delta_c(A) = \mathbb{I}_{\{A\}}(c) .$$

The characteristic exponent of a Poisson distribution

$$\Psi(u) = \lambda \left(1 - e^{iu}\right) , \qquad u \in \mathbb{R} ,$$

is of the form (2.3) with $a = b^2 = 0$ and $\nu(A) = \lambda \delta_1(A)$.

2.1.4 Compound Poisson Distribution

The characteristic exponent of a compound Poisson distribution is given by

$$\Psi(u) = \int_{\mathbb{R}_0} \left(1 - e^{iux}\right) \lambda f(dx) ,$$

where λ is the parameter of the Poisson point process and f is the law of the jumps.

This characteristic exponent is also of the form (2.3) with $a = \int_{-1}^{1} x \lambda f(dx)$, $b^2 = 0$ and $\nu(dx) = \lambda f(dx)$.

2.1.5 α-Stable Distributions

These distributions are obtained as limits of normalized sums of i.i.d. random variables [see Zolotarev (1986)]. Their density function does not exist in closed form, but its characteristic exponent is given by

$$\Psi(u) = c|u|^\alpha \left[1 - i\beta \operatorname{sign}(u) \tan(\alpha\pi/2)\right] + imu , \qquad \alpha \in (0,1) \cup (1,2) .$$

For $\alpha = 1$ we have the Cauchy distribution and for $\alpha = 2$ we have the normal distribution.

This exponent is of the form (2.3) with $a = m$, $b^2 = 0$ and $\nu(dx) = c^+ x^{-\alpha-1} dx$ if $x > 0$ and $\nu(dx) = c^- |x|^{-\alpha-1} dx$ if $x < 0$ where c^+ and c^- are such that $\beta = \frac{c^+ - c^-}{c^+ + c^-}$.

The process has no positive (negative) jumps when $c^+ = 0$ ($c^- = 0$) or equivalently $\beta = -1$ ($\beta = 1$). It is symmetric when $\beta = 0$ ($c^+ = c^-$).

The appeal of this family of distributions lies in the parameter α. It controls the decay of the tail, going from the light-tailed Gaussian distribution to the heavy-tailed Cauchy distribution. Its Poisson component accounts for sudden and drastic changes in the, otherwise continuous, evolution of the system. These features render it a more flexible model to work with.

2.1.6 Generalized Hyperbolic Distribution

This family was first introduced by Barndorff-Nielsen (1977). Its density is given by

$$f(x; \lambda, \alpha, \beta, \delta, \mu) =$$
$$a(\lambda, \alpha, \beta, \delta)[\delta^2 + (x-\mu)^2]^{(\lambda-1/2)/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) \exp\left[\beta(x-\mu)\right] ,$$

for $x > 0$, where

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda} K_{\lambda}\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} ,$$

is the normalizing constant and K_λ denotes the modified Bessel function of the third kind with index λ

$$K_\lambda(z) = \frac{1}{2}\int_0^\infty y^{\lambda-1} \exp\left[-\frac{1}{2}z(y + y^{-1})\right] dy , \qquad z > 0 , \tag{2.5}$$

The parameter $\alpha > 0$ determines the shape, β with $0 \leqslant |\beta| < \alpha$ the skewness and $\mu \in \mathbb{R}$ the location. $\delta > 0$ is a scaling parameter. Finally $\lambda \in \mathbb{R}$ defines subclasses, it controls the heaviness of the tails. For $\lambda = 1$ we have the hyperbolic distribution, for $\lambda = -1/2$ we have the normal inverse Gaussian, which is one of the two sub-classes being closed under convolutions. The other subclass closed under convolution is the variance-gamma distribution, obtained when $\delta = 0$.

The name of this family comes from the fact that, for $\lambda = 1$, the logarithm of the density gives an hyperbola, unlike the case of a normal distribution which gives a parabola. This accounts for the slower decay of the tail with respect to the normal. By changing the axis of the hyperbola we get positively or negatively skewed densities.

Its characteristic exponent is of the form (2.4) and is given by

$$\Psi(u) = iu\mathbb{E}[GH] + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux\right) \nu(dx) , \tag{2.6}$$

where $\mathbb{E}[GH]$ is the mean of the density and the Lévy measure ν is given in terms of Bessel functions of the first and second kind as follows:

$$\nu(dx) = \begin{cases} \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp[-|x|\sqrt{2y+\alpha^2}]}{\pi^2 y[J_\lambda^2(\delta\sqrt{2y})+N_\lambda^2(\delta\sqrt{2y})]}dy + \lambda e^{-\alpha|x|}\right) dx\,, & \text{if} \quad \lambda \geqslant 0\,, \\ \\ \frac{e^{\beta x}}{|x|}\left(\int_0^\infty \frac{\exp[-|x|\sqrt{2y+\alpha^2}]}{\pi^2 y[J_{-\lambda}^2(\delta\sqrt{2y})+N_{-\lambda}^2(\delta\sqrt{2y})]}dy\right) dx\,, & \text{if} \quad \lambda < 0\,. \end{cases}$$

J and N are modified Bessel functions. We refer to Abramowitz and Stegun (1970) for further discussion on Bessel functions.

Notice that the Gaussian coefficient b^2 is zero. Another interesting feature of this family is that a random variable X having a generalized hyperbolic distribution can be written as a mean-variance mixture of a normal distribution. That is, X is conditionally distributed as a normal $N(\mu + \beta\sigma^2, \sigma^2)$ where, in turn, σ^2 has a generalized inverse Gaussian distribution.

2.1.7 Generalized Inverse Gaussian Distribution

This family is a generalization of the inverse Gaussian distribution and has been studied extensively in Jørgensen (1982). Barndorff-Nielsen and Halgreen (1977) showed that this family is infinitely divisible. Its density function is given by

$$f_{gig}(x) = \frac{(\gamma/\delta)^\lambda}{2K_\lambda(\delta\gamma)} x^{-\lambda-1} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}\,, \qquad x > 0\,,$$

where K_λ is the modified Bessel function of the third kind with index λ defined in (2.5). The parameter domain of the GIG distribution is

$$\begin{aligned} &\delta > 0\,, \quad \gamma \leqslant 0\,, \quad \text{if} \quad \lambda < 0\,, \\ &\delta > 0\,, \quad \gamma > 0\,, \quad \text{if} \quad \lambda = 0\,, \\ &\delta \leqslant 0\,, \quad \gamma > 0\,, \quad \text{if} \quad \lambda > 0\,. \end{aligned}$$

If $\lambda = -1/2$ it reduces to the inverse Gaussian distribution.

The generalized inverse Gaussian distribution has a characteristic exponent of the form (2.3) with Lévy measure

$$\nu(dx) = \frac{1}{x}\left[\delta^2 \int_0^\infty e^{-xt} g_\lambda(2\delta^2 t)dt + \max\{0,\lambda\}\right] e^{-\gamma^2 x/2}dx \ ,$$

where

$$g_\lambda(y) = \left\{\frac{\pi^2}{2} y \left[J^2_{|\lambda|}(\sqrt{y}) + N^2_{|\lambda|}(\sqrt{y})\right]\right\}^{-1} , \qquad y \geqslant 0 \ .$$

J and N are modified Bessel functions.

2.2 Lévy Processes

Following Karatzas and Shreve (1991) and Revuz and Yor (1994), consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$. We write $X = \{X(t)\}_{t\geqslant 0}$ to denote a $\mathbb{R}$-valued stochastic process on Ω. For a fixed sample point $\omega \in \Omega$, the mapping $t \longmapsto X_t(\omega)$; $t \geqslant 0$ is the sample path, or trajectory (realization) of the process X associated with ω. Now we can define some central concepts to this work.

Definition 2.2 *We say that a process X is càdlàg if for every $\omega \in \Omega$ the sample path $X_t(\omega)$ is right continuous and has a left limit in every $t \geqslant 0$ (càdlàg: from french* **continue à droite et admet une limite à gauche***).*

Definition 2.3 *We say a process X is an $(\mathcal{F}_t)$-martingale if*

i) X is $\mathcal{F}_t$-measurable for all $t \geqslant 0$.

ii) $\mathbb{E}_{\mathbb{P}}(X_t) < \infty$ for all $t \geqslant 0$.

iii) $\forall\ 0 < s < t$ we have that $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$.

Definition 2.4 *Let $0 = t_0 < t_1 < \cdots < t_m = T$ be a partition of $[0,t]$ then we define the p-variation of a process X over $[0,t]$ by*

$$V_t^{(p)} = \lim_{\Delta t_j \to 0} \sum_{t_j < t} |X_{t_{j+1}} - X_{t_j}|^p \ .$$

If the limit $V_t^{(1)} < \infty$ for all $t > 0$ we say that the process X is of finite variation. The limit $V_t^{(2)}$ can be proven to exist for all square integrable martingales. The limiting process is denoted by $< X >_t$ and it is the only adapted and increasing process for which $< X >_0 = 0$ and $X_t^2 - < X >_t$ is a martingale.

Definition 2.5 *A process X is a local martingale if there exists a strictly increasing sequence of $\mathcal{F}_t$-stopping times $\{T_n\}_{n>0}$ such that the stopped processes X^{T_n} for all $n > 0$ are uniformly integrable martingales.*

Any local martingale can be decomposed in its continuous and a purely discontinuous components as stated in the following theorem [Jacod and Shiryaev (1987)]

Theorem 2.3 *Any local martingale X admits a unique decomposition*

$$X = X_0 + X^c + X^d \, ,$$

where $X_0^c = X_0^d = 0$, X^c is a continuous local martingale and X^d is a purely discontinuous local martingale. X^c is called the continuous part of X and X^d its purely discontinuous part.

We also need to define the concept of semimartingale.

Definition 2.6 *A semimartingale is an adapted process X that can be decomposed as*

$$X_t = X_0 + M_t + A_t \, , \qquad t > 0 \, ,$$

where M is a local martingale and A is a process of finite variation and with $A_0 = 0$.

The continuous part of a semimartingale X^c is the continuos part of the local martingale M in Definition 2.6.

The concept of quadratic variation needs to be extended for semimartingales.

Definition 2.7 *The quadratic variation for a semimartingale X is defined by*

$$[X]_t =< X^c >_t + \sum_{s \leqslant t} (\Delta X_s)^2 \, ,$$

where X^c is the continous part of X and $\Delta X_s = X_s - X_{s-}$.

Note that if X is continuous then $[X]_t =< X >_t$ and if X is purely discontinuous then $[X]_t = \sum_{s \leqslant t}(\Delta X_s)^2$.

Now, we are in a position to state the key definition of this section.

Definition 2.8 *A càdlàg, adapted process X with $X_0 = 0$ is called a Lévy process (or process with stationary, independent increments) if the distribution of $X_t - X_s$ depends only on $t - s$ and if $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leqslant s < t$.*

We state in the form of a Lemma two important properties of Lévy processes.

Lemma 2.1

i) Lévy processes are semimartingales.

ii) If X is a Lévy process then X_t follows an ID law denoted f_t and its characteristic exponent $\Psi_t(u) = \ln \int_{\mathbb{R}} e^{iuy} f_t(dy)$ is such that $\Psi_t(u) = t\Psi_1(u)$, for $u \in \mathbb{R}$.

From *ii)* in Lemma 2.1 we can see that the law of a Lévy process is defined by the characteristic exponent of the ID distribution of the process at time one. We call Ψ_1 the characteristic exponent of the Lévy process X.

The main result of this section is the converse of *ii)* in Lemma 2.1 and is stated in the following theorem [see Bertoin (1996)].

Theorem 2.4 *Consider a function Ψ as in (2.3)*

$$\Psi(u) = iau + \frac{b^2}{2}u^2 + \int_{\mathbb{R}} \left[1 - e^{iux} + iux\mathbb{I}_{\{(-1,1)\}}(x)\right] \nu(dx) \ . \tag{2.7}$$

Then there exists a unique probability measure $\mathbb{P}$ on $\mathcal{F}$ under which X is a Lévy process with characteristic exponent Ψ.

The characteristic exponent Ψ uniquely characterizes the distribution of the Lévy process in terms of the parameters (a, b^2, ν). These parameters are refered to as the generating triplet of the Lévy process. The characterization (2.7) is the Lévy-Khintchine representation of a Lévy process.

It is useful to define the Laplace exponent Ψ_L. In an analogous way, it is the exponent appearing on the exponent of the Laplace transform of the law f_1. We can define it in terms of the characteristic exponent as follows $\Psi(u) = \Psi_L(-iu)$. Consequently, the Laplace transform of the process X is of the form $\mathbb{E}[e^{-uX(t)}] = e^{t\Psi_L(u)}$ for $u > 0$.

Notice that if $b^2 > 0$ in (2.7) there is a continuous Gaussian part in the Lévy process X, on the other hand, if $b^2 = 0$ then X is a pure jump process. Following Sato (1999) we can classify the class of pure jump Lévy processes into three types according to the behavior of its Lévy measure ν:

i) Type A: If $\int_{\mathbb{R}_0} \nu(dx) < \infty$ then the process is a compound Poisson process.

ii) Type B: If $\int_{\mathbb{R}_0} \nu(dx) = \infty$ and $\int_{\mathbb{R}_0} (1 \wedge |x|)\nu(dx) < \infty$, then the process has an infinite number of small jumps (infinite activity) but is of finite variation.

iii) Type C: If $\int_{\mathbb{R}_0} \nu(dx) = \infty$ and $\int_{\mathbb{R}_0} (1 \wedge |x|)\nu(dx) = \infty$, the process has infinite activity and is of unbounded variation.

The Lévy measure ν governs the occurrence and the size of the jumps of the process X. The jumps of size in $\Lambda \in \mathbb{R}$ (provided the closure $\bar{\Lambda}$ does not contain zero) form a compound Poisson process with rate $\int_\Lambda \nu(dx)$ and jump density $\frac{\nu(dx)}{\int_\Lambda \nu(dx)}$.

The following corollary to Theorem 2.4 will play an important role in our work.

Corollary 2.1 *A Lévy process X can be uniquely expressed as the sum of three independent Lévy processes*

$$X_t(\omega) = X_t^{(1)}(\omega) + X_t^{(2)}(\omega) + X_t^{(3)}(\omega) , \qquad \omega \in \Omega , \quad t > 0 ,$$

where $X^{(1)}$ is a linear transform of a Brownian motion with drift, $X^{(2)}$ is a compound Poisson process having jumps of size at least one and $X^{(3)}$ is a pure-jump martingale, having jumps of size less than one.

2.2.1 Lévy Processes with Finite First Moment

A subclass of interest is that class whose members are Lévy processes with finite first moment. For this subclass, we can rewrite the Lévy-Khintchine characterization in a simpler form. If the Lévy process X is such that $\mathbb{E}(X_1) < \infty$ then (2.7) can be alternatively written as

$$\Psi(u) = ia^*u + \frac{b^2}{2}u^2 + \int_{\mathbb{R}} \left[1 - e^{iux} + iux\right] \nu(dx) \;, \tag{2.8}$$

where the drift a^* is the mean of the process [see Sato (1999)]. This is, if the process has a finite mean the Lévy-Khintchine characterization takes on a simpler form where the mean of the process appears as the drift term. This form (2.8) might seem more natural for a risk model. The new Lévy triplet is $[a^*, b^2, \nu(\mathrm{d}x)]$ with $a^* = a + \int_{\{|x|>1\}} x\nu(\mathrm{d}x) = \mathbb{E}(X_1)$.

Processes of this type are known as *special semimartingales* and satisfy $\mathbb{E}_{\mathcal{P}}[|X_t|] < \infty$, for any $t > 0$, or equivalently $\mathbb{E}_{\mathcal{P}}[|X_1|] < \infty$, since the characteristic exponent at time one defines the distribution for the whole process.

For special semimartingales we have the following decomposition

Lemma 2.2 *If X is a special semimartingale it can be decomposed as the sum*

$$X_t = \sigma B_t + Z_t + \alpha t \;, \qquad t > 0 \;,$$

where B is a standard Brownian motion and Z a purely discontinuous martingale independent of B.

This class is easier to work with due to the simplification in the decomposition. Besides the fact that the assumption of a finite first moment is a rational one, we would like to have a heavy-tailed model, but not as heavy-tailed as not to allow for the existence of the first moment. This excludes the α-stable processes with $\alpha \leqslant 1$ but still includes most of the Lévy processes we are interested in. Especially the generalized hyperbolic Lévy motion, which is the subject of the next section.

2.2.2 Generalized Hyperbolic Lévy Processes

Since the generalized hyperbolic distribution belongs to the ID family of distributions we can define a generalized hyperbolic Lévy motion. This processes is characterized by its Lévy representation (2.6).

In the characteristic exponent we notice that the Gaussian coefficient b^2 is zero. The hyperbolic Lévy motion is a pure-jump process. The Lévy measure ν for the generalized hyperbolic process is such that it has infinite mass in every neighborhood of the origin. The behavior near zero of the Lévy measure ν of a generalized hyperbolic Lévy process has been proven to be [see Raible (2000)]:

$$x^2\nu(x) = \frac{\delta}{\pi} + \frac{\lambda + \frac{1}{2}}{2}|x| + \frac{\delta\beta}{\pi}x + o(x) ,$$

as $x \to 0$. In consequence, the process X has an infinite number of small jumps in every finite interval. In contrast to the continuous Brownian motion, the generalized hyperbolic Lévy motion lies at the other side of the spectrum.

Another feature of generalized hyperbolic Lévy processes is that, unlike the Brownian motion, their drift is a path property of the process. The Brownian motion carries along its paths information regarding the variance but not the drift. Also in Raible (2000) we find that, as $n \to \infty$,

$$\frac{1}{nt}\sum_{s \leqslant t} \mathbb{I}_{(\frac{1}{n},\infty)}(\Delta X_s) \longrightarrow \frac{\delta}{\pi} , \qquad a.s. ,$$

and

$$X_t - \sum_{s \leqslant t} \Delta X_s \mathbb{I}_{\{(-\infty,-\frac{1}{n})\cup(\frac{1}{n},\infty)\}}(\Delta X_s) \longrightarrow ct , \qquad a.s. .$$

This is, the parameters δ and μ are path properties of a generalized hyperbolic Lévy motion. In other words, we can recuperate these parameters from a sample path of the process. In an asymptotic sense: δ is the number of jumps larger than $1/n$, normalized by π/n and μ is the slope of the straight line that is left over after removing all jumps larger than $1/n$.

Despite its discontinuity at every point, these processes are mathematically tractable and have been used in finance to model stock prices. Their one-dimensional distributions are generalized hyperbolic and so they have moments of all orders. Besides, simulation and other numerical treatments are available for these processes. Generalized hyperbolic Lévy motion is the limiting process of a sequence of a type of compound Poisson processes. It suffices to notice that the Lévy measure ν of the hyperbolic Lévy motion is not bounded and that it can be approximated by a sequence of bounded Lévy measures ν_n, this is by Lévy measures of compound Poisson processes.

Another interesting feature is that increments of size one have a generalized hyperbolic distribution, but in general none of the increments of length different from one has a distribution from the same class within the large generalized hyperbolic family. This fact comes from the form of the characteristic function and *ii*) in Lemma 2.1. The only two members of this family that are closed under convolutions are the normal inverse Gaussian and the variance-gamma processes. This property makes them more natural models since all of their increments belong to the same class. For a further discussion on these process and their applications we refer to Eberlein (2001), Prause (1999) and Bibby and Sørensen (2001).

Lévy processes form a large class of processes arising from ID distributions, in the same way that Brownian motion does from the normal distribution. Many well-known processes belong to this family, among others we find Brownian motion (the only continuous Lévy process), compound Poisson and the α-stable processes. For an overview of the basic theory of Lévy process we refer to Barndorff-Nielsen (2001), Sato (2001) and Schoutens (2003).

We will pay special attention to the normal inverse Gaussian Lévy process as a risk model in Chapter 3.

2.3 Subordination

Let X be a Lévy process. The subclass of increasing Lévy process is that of subordinators as we can see from the following definition in Sato (1999):

Definition 2.9 *A Lévy process $X = \{X(t)\}_{t\geqslant 0}$ on $\mathbb{R}$ is a subordinator if it has increasing sample functions a.s. or, equivalently, if it has one-dimensional distributions with support on $[0, \infty)$.*

Subordinators form a special subclass of Lévy processes that play an important role in the theory. The following result found in Sato (1999) characterizes the subclass of subordinators:

Theorem 2.5 *Let $X = \{X(t)\}_{t \geqslant 0}$ be a Lévy process with Lévy measure ν. Then it is a subordinator if and only if it is of type A or B, ν is supported on $[0, \infty)$ and the drift μ is non-negative. In this case its characteristic function is of the form $\mathbb{E}[e^{izX(t)}] = e^{-t\Psi(-iz)}$ where Ψ is the Laplace exponent given by*

$$\Psi(z) = \mu z + \int_{0-}^{\infty} \left(1 - e^{-zx}\right) \nu(dx) \, , \qquad z > 0 \, . \tag{2.9}$$

We can see that for subordinators the Lévy-Khintchine representation (2.7) has a simpler form since the centering function h is not present. This implies that subordinators are of finite variation.

Alternatively, if we define the tail of the Lévy measure as $\Pi(x) = \int_x^\infty \nu(dw)$, we can rewrite (2.9) as [see Bertoin (1996)]

$$\frac{\Psi(z)}{z} = \mu + \int_0^\infty e^{-zx} \Pi(x) dx \, , \qquad z > 0 \, . \tag{2.10}$$

Note that

$$a = \lim_{s \to \infty} \frac{\Psi(s)}{s} \, ,$$

and that

$$\nu(dx) = -d\Pi(x) \, .$$

Subordination is a random time change of a stochastic process Y, through an independent subordinator τ, yielding the new process $Z(t) = Y[\tau(t)]$. The tranformation from Y to Z is called subordination by τ and Z is said to be subordinate to Y.

Recall that f_t are one-dimensional distributions of increments (of length t) and therefore f_1 is the law of the increments of length one, which defines Lévy processes according to Lemma 2.1. Then the following result form Sato (1999) characterizes a process obtained via subordination.

Theorem 2.6 *Let τ be a subordinator with Lévy measure ρ, drift β and one-dimensional law f_1. Let Y be a Lévy process on $\mathbb{R}$ with triplet (a, b, ν) and one-dimensional law g_1. If τ and Y are independent define the process*

$$Z(t) = Y[\tau(t)] \, .$$

Then Z is a Lévy process on $\mathbb{R}$ with generating triplet $(\widetilde{a}, \widetilde{\sigma}^2, \widetilde{\nu})$ where

$$\begin{aligned}
\widetilde{a} &= \beta a \int_{0-}^{\infty} \rho(ds) \int_{|x|\leqslant 1} x g_s(dx) , \\
\widetilde{\sigma}^2 &= \beta\sigma^2 , \\
\widetilde{\nu}(dx) &= \beta\nu(dx) + \int_{0-}^{\infty} g_s(dx)\rho(ds) .
\end{aligned}$$

Moreover, the characteristic function of the process Z is of the form

$$\mathbb{E}[e^{izZ(t)}] = e^{t\Psi_L[\ln \widehat{g}_1(z)]} , \qquad z \in \mathbb{R} ,$$

where Ψ_L is the Laplace exponent of the process τ and $\widehat{g}_1$ is the characteristics function of the law g_1.

Subordination will play an important role when we define a general risk process driven by a normal inverse Gaussian Lévy process. This is because of the connection between generalized hyperbolic processes and subordination.

2.3.1 Generalized Hyperbolic Lévy Processes via Subordination

Generalized Hyperbolic Lévy processes can be introduced via subordination [see for instance Prause (1999)]. Consider a generalized inverse Gaussian Lévy process τ. Because of (2.7) we can see that there exists a one-to-one correspondance between Lévy processes and infinitely divisible distributions. The generalized Inverse Gaussian distribution is infinitely divisible and has support on the positive axis. Therefore we can define an increasing Lévy process whose increments of length one are generalized inverse Gaussian distributed. By definition, such a process is a subordinator and can be used as a random time transformation.

Now let us consider a standard Brownian motion with drift $B(t) = \beta t + W(t)$ where W is a standard Brownian motion. Then we find the following result in Prause (1999) which is a consequence of Theorem 2.6:

Theorem 2.7 *Let B be a standard Brownian motion with drift β. Also let τ be a generalized inverse Gaussian Lévy process. By construction, its increments of length one are generalized inverse Gaussian distributed. Let us denote by $gig(\cdot;\delta,\gamma,\lambda)$ its density function, where $\gamma^2 = \alpha^2 - \beta^2$. Then, we can define a process Y via subordination as follows $Y(t) = \mu t + B[\tau(t)]$.*

The resulting Lévy process Y is a generalized hyperbolic Lévy motion with parameters $(\lambda, \alpha, \beta, \delta, \mu)$.

This construction of generalized hyperbolic Lévy motions will be used when we define a normal inverse Gaussian risk process.

2.4 Processes with Independent Increments

A larger class of processes that includes Lévy processes is that of processes with independent increments. Consider the *càdlàg* modification of an adapted process X. Then we have the following definition

Definition 2.10 *The process X with $X_0 = 0$ is called a process with independent increments if $X_t - X_s$ is independent of $\mathcal{F}_s$ for $0 \leqslant s < t$.*

Although these processes are non-stationary they still share a very important property with the Lévy family of processes.

Lemma 2.3 (Jacod and Shiryaev (1987)) *Let X be a process with independent increments, then X is a semimartingale. Moreover X_t has an infinitely divisible distribution and its characteristic function is of the form*

$$\phi_t(u) = \mathbb{E}[e^{iuX_t}] = e^{-\Psi_t(u)} \ ,$$

where

$$\Psi_t(u) = ia_t u + \frac{b_t^2}{2}u^2 + \int_{\mathbb{R}} \left[1 - e^{iux} + iux\mathbb{I}_{(-1,1)}(x)\right] \nu_t(dx) \ ,$$

with $a_t \in \mathbb{R}$, $b_t^2 > 0$ and ν_t is a measure on $\mathbb{R}_0 = \mathbb{R} - \{0\}$ satisfying

$$\int_{\mathbb{R}_0} (1 \wedge |x|^2)\nu_t(dx) < \infty \ , \qquad \forall\, t > 0 \ .$$

The parameters a_t, b_t^2 and ν_t now are functions of t and they are refered to as the characteristics of the process. This is a generalization of the Lévy-Khintchine representation for Lévy processes. The exponent Ψ_t is called the characteristic exponent of the process X.

2.4.1 Non-Homogeneous Poisson Process

A non-homogeneous Poisson process is an example of a process with independent but not stationary increments.

Definition 2.11 *An adapted point process N is said to be an non-homogeneous Poisson process (NPP) if*

i) $\mathbb{E}(N_t) < \infty$ *for each* $t > 0$,

ii) $N_t - N_s$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leqslant s < t$,

iii) The function $\mathbb{E}(N_t) = \Lambda(t)$ *is continuous. This function is called the integrated intensity of* N_t.

Its characteristic exponent is

$$\Psi_t(u) = \left(1 - e^{iux}\right) \Lambda(t) \ ,$$

where Λ is the integrated intensity function of the non-homogeneous Poisson process satisfying $\Lambda(t) = \int_0^t \lambda_s ds$.

2.4.2 Compound Non-Homogeneous Poisson Process

The compound non-homogeneous Poisson process is defined in terms of a non-homogeneous Poisson point process, just as the compound Poisson is derived from the homogeneous Poisson point process.

It is another example of processes with independent but not stationary increments. Its characteristic exponent is given by

$$\Psi_t(u) = \int_{\mathbb{R}_0} \left(1 - e^{iux}\right) \Lambda(t)\nu_t(dx) \ ,$$

where Λ is the integrated intensity of the non-homogeneous Poisson process and ν_t is the law of the jumps at t. A compound NPP will be a natural model when incorporating periodicity in the risk model as we will see in Chapter 6.

In the next section we see how we can use this larger family of processes with independent increments to model a stochastic system.

2.5 Stochastic Differential Equations

We start our discussion by motivating the construction of a stochastic integral [as in Øksendal (1992)].

Differential equations are widely used to model deterministic systems evolving through time. A stochastic analog of a classical differential equation for a process X is given by

$$\frac{dX_t}{dt} = a(t, X_t) + \sigma(t, X_t) W_t \,, \qquad t \geqslant 0 \,,$$

where W is a *reasonable* stochastic process. Since W may not have continuous paths and in some cases W_t may not even be $\mathcal{F}_t$-measurable. There exists then the need to replace W by a suitable stochastic process Z. In other words, we have to give a sense to the limit of

$$X_{t_k} = X_0 + \sum_{j=0}^{k-1} a(t_j, X_{t_j}) \Delta t_j + \sum_{j=0}^{k-1} \sigma(t_j, X_{t_j}) \Delta Z_{t_j} \,,$$

as $\Delta t_j \longrightarrow 0$ for some stochastic process Z. If this limit exists then we can apply the usual integral notation and write

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t \sigma(s, X_s) dZ_s \,. \tag{2.11}$$

The standard convention is that by

$$dX_t = a(t, X_t) dt + \sigma(t, X_t) dZ_t \,, \quad t > 0 \,,$$

it is meant that the stochastic process X satisfies (2.11).

The only Lévy process with continuous paths is the Brownian motion. Processes satisfying (2.11) have been largely studied when Z is a Brownian motion [see Karatzas and Shreve (1991) and Revuz and Yor (1994)]. Extensions of the definition of the integral (2.11) to a wider class of processes Z have been proposed in the last sixty years. After Itô (1942) first introduced the notion of stochastic integral, the theory has rapidly expanded. We find among others: Doob (1953) who pointed out the martingale character of the stochastic integral, later Kunita and Watanabe (1967) and Meyer (1976) who developed a unified martingale theory for stochastic integrals. This last extension of Meyer (1976) is for processes Z which are semimartingales. Later in Métivier and Pellaumail (1980) we find a stochastic integral defined for a wider class of processes Z that includes the class of semimartingales, and it turns out to be the widest class with respect to which the stochastic integral can be *nicely* defined. For an account on the theory of stochastic integrals see Itô and Watanabe (1976).

Since Lévy processes are semimartingales [i) in Lemma 2.1] we will focus our interest in the theory of integrals with respect to semimartingales. This choice is not restrictive since Lévy processes include a wide class of stochastic processes. In this case the integral in (2.11) is well defined and we can use a stochastic differential equation to model random systems. For the existence of solutions to these equations we refer to Itô and Watanabe (1976), Protter (1992) and Kallsen (1998).

2.5.1 A Chain Rule for Stochastic Integrals

We finish this section by presenting a generalization of the classical result by Itô. The key feature of the work of Itô (1942) does not lie particularly in the definition of the stochastic integral as much as it does in the derivation of a complete differential and integral calculus. The so-called Itô formula for Brownian motion is an analog to the chain rule in classical calculus.

In ordinary calculus we learn that if X and f are two real-valued functions then

$$df(X) = f'(X)dX \ ,$$

or more precisely

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s \ , \qquad t \geqslant 0 \ ,$$

where the integral is defined in the Riemann-Stieltjes sense. Itô introduced a similar rule for an integral with respect to Brownian motion. Here, we present an extension for this result for a continuous martingale [see for instance Karatzas and Shreve (1991)].

Theorem 2.8 *Let f be a real-valued function in $\mathcal{C}^2([0,\infty))$ and let M be a continuous square integrable martingale then*

$$f(M_t) - f(M_0) = \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d < X >_s , \qquad t \geqslant 0 .$$

In what follows we present the approach taken in Meyer (1976) for integrals with respect to semimartingales. We state the following

Theorem 2.9 *Let X be a semimartingale and let f be a real-valued function in $\mathcal{C}^2([0,\infty))$, then*

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_{s^-})dX_s + \frac{1}{2}\int_0^t f''(X_{s^-})d[X]_s \\ &+ \sum_{s\leqslant t}\left(f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s - \frac{1}{2}f''(X_{s^-})(\Delta X_s)^2\right) . \end{aligned}$$

Notice that this formula has an extra term that accounts for the jumps of the possibly discontinuous semimartingale. For a more detailed exposition on stochastic calculus for semimartingales we refer to Métivier (1982) and He, Wang and Yan (1992).

Notice that Itô's theorem is often written in terms of the quadratic variation for the continuous part of the semimartingale X. Recalling Definition 2.7 and the fact that $[X^c]_t =< X^c >_t$ we can rewrite the equation in Theorem 2.9 as follows [see Elliott (1982)]

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_{s^-})dX_s + \frac{1}{2}\int_0^t f''(X_{s^-})d < X^c >_s \\ &+ \sum_{s\leqslant t}(f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s) . \end{aligned}$$

We have then a complete theory on stochastic differential equations for Lévy processes and for processes with independent increments that can be used for general risk processes driven by a Lévy process. In the following chapters we explore different aspects of Lévy modeling in risk theory.

Chapter 3

On a Risk Model Driven by a Normal Inverse Gaussian Lévy Process

3.1 Introduction

Lévy processes have been traditionally applied in risk theory in the form of Brownian motion and α-stable Lévy motion. This approach can be extended to embrace more general Lévy processes. We believe that other Lévy processes have features that make them an object of interest in risk theory. Besides, the theory of stochastic calculus and Lévy processes that has been developed in the last fifty years has not, until recently, found its way in financial and insurance applications. For example applications of the hyperbolic and other Lévy processes in finance have just begun to be explored [see Schoutens (2003), Eberlein and Prause (2000) and references therein]. Unifying approaches using stochastic differential equations and semimartingales to model risk processes have just been initiated recently [Sørensen (1996) for instance].

Our goal is to extend the use of Lévy processes and processes with independent increments in risk theory. Implementing more general Lévy process in risk theory will be of interest when bridging financial and insurance models. The link between the

American option problem and the discounted penalty function has been discussed in Gerber and Shiu (1998a, 1998b, 1999) and Avram, Chan and Usabel (2002). Discussions on the duality of premium principle and financial risk measures can be found in Artzner *et al.* (1999), Reesor (2001) and Wang (2000).

Throughout this thesis we work with a general Lévy risk process. Consider

$$U(t) = u + ct + Z(t) , \qquad t > 0 , \tag{3.1}$$

where Z is a Lévy process that would bridge previous models. If Z is a compound Poisson process we have the classical model. The diffusion and the α-stable model are included in (3.1) by setting Z to be a Brownian or an α-stable motion respectively. Model (3.1) can also include diffusions with jumps, gamma process and other more general Lévy processes. We explore the model in (3.1) from different perspectives. In the present chapter, for instance, we let Z be a normal inverse Gaussian which shows some interesting features as a risk model. Chapter 4 generalizes previous work of Dufresne, Gerber and Shiu (1991) by letting Z be a generalized inverse Gaussian Lévy proces. In Chapter 5 we present an approximation to the classical risk process using extreme value theory, we construct a generalized Pareto-stable Lévy process and we use it as the aggregate claims process Z in (3.1). Finally, in Chapter 6 we go beyond the scope of Lévy process and we let Z be a non-homogeneous compound Poisson process which is a process with independent increments. This illustrates how some of the appealing properties of Lévy process are common to a wider class of processes.

In this chapter, we present a general risk model where the aggregate claims, as well as the premium function, evolve by jumps. This is achieved by incorporating a pure jump Lévy process into the model. This seeks to account for the discrete nature of claims and asset prices. We use a Normal Inverse Gaussian (NIG) Lévy process which allows us to incorporate aggregate claims and premium fluctuations in the same process. We discuss important features of such a process and their relevance to risk modeling. We also extend classical results on ruin probabilities to this model [see Morales and Schoutens (2003) for a more general setting of the discussion of this chapter].

In recent years, a case has been made for the use of pure jump Lévy processes in financial modeling, the purely discountinuous feature of such processes accounts for the discrete nature of the real world. Diffusions with jumps had been long favored when it came to asset price modeling, however such an approach is being abandoned in favor of pure-jump Lévy processes [see LeBlanc and Yor (1998) and Carr *et al.* (2002)]. In such processes the diffusion component is not present and an infinite number of small jumps drives its evolution. An important class of such processes is the generalized hyperbolic family. Eberlein and Keller (1995), Eberlein (2001) and Geman (2002) explore the potential of the generalized hyperbolic Lévy motion (GH) in finance. Barndorff-Nielsen (1998) and Rydberg (1997) work with the normal inverse Gaussian (NIG) process, which is a member of the generalized hyperbolic family with the property of being closed under convolutions. This property along with the simple expressions of its Laplace transform and Lévy measure are the motivation to our approach.

The NIG process is an extension of the Brownian motion that allows for finite dimensional distributions with semi-heavy tails. In a way, it can be seen as a purely discontinuous version of the latter. Within the wide spectrum of Lévy processes, it lies somewhere between the Brownian motion and the α-stable process. Grandell (1977) and Furrer, Michna and Weron (1997) work out risk models with a Brownian motion and an α-stable process respectively. A model based on the NIG Lévy motion allows for greater flexibility than the Brownian motion since its finite dimensional distributions decay at a slower rate than a normal distribution. Although, the NIG process does not account for heavy tails as the α-stable process does, it is mathematically tractable since it has analytical expressions for its one dimensional densities. We refer to Eberlein and Prause (2000), Voit (2001) and references therein for a discussion on empirical evidence of non-normality of assets returns and the adequacy of GH process as models in finance. Further research is needed to empirically asses the performance of the NIG distribution, and other GH distributions, in the insurance context.

NIG processes have recently become an object of interest in financial modeling because they enjoy both diffusion-like and jump properties at the same time. In finance, as well as in insurance, this has been achieved by adding extra components into the model. In finance, large fluctuations are incorporated via a jump process and in insurance small fluctuations are incorporated via a diffusion. NIG processes, and generalized hyperbolic processes in general, account for both types of structures.

Another interesting feature of NIG processes is their respresentation as a time-changed Brownian motion. Generalized hyperbolic Lévy processes, and NIG processes in particular, can be seen as a Brownian motion running in business time instead of calendar time. Such a business time can be understood as an alternative time unit, as reckoned by a different clock, which evolves according to a random process. In finance, such a clock can be the traded volume or the number of trades of a particular asset. In the insurance context it could be the aggregate claims process of our portfolio. Empirical studies [see Chaubey, Garrido and Trudeau (1998)] indicate that the inverse Gaussian distribution provides a good fit for aggregate claims. It turns out that an inverse Gaussian business time leads naturally to a NIG risk process.

We work with a general risk model along the lines of Sørensen (1996),

$$U(t) = u + ct + Z(t) - S(t) , \qquad t \geqslant 0 , \tag{3.2}$$

where u is the initial reserve, c is the constant loaded premium, Z is a pure jump Lévy process representing fluctuations in the risk premium, and S is the aggregate claims process. The fact that Z evolves only by jumps captures the fact that changes in the premium rate are of discrete nature. As for the loaded premium c, it is natural to define it as $c = (1+\theta)\mathbb{E}(Z-S)$, since it has to be greater than all random fluctuations to meet the net profit condition.

A very convenient choice for the processes Z and S should be such that $Z - S$ is a NIG Lévy process. This would lead to simple expressions for the ruin probabilities just as in the diffusion model of Grandell (1977).

In Section 3.2 we present a brief introduction to NIG distributions and Lévy processes, Section 3.3 represents the main body of our discussion where we incorporate appealing features of NIG processes into a general risk model. Section 3.4 discusses the NIG risk model as a time-changed diffusion model. Finally, Section 3.5 deals with a simulation approximation of the model.

3.2 Normal Inverse Gaussian Lévy Processes

The normal inverse Gaussian distribution is a member of the wider class of generalized hyperbolic distributions. This larger family was introduced in Barndorff-Nielsen and Halgreen (1977). The NIG is one of the only two subclasses being closed under convolutions (the other one being the variance-gamma distribution). Its density function is given by

$$nig(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta e^{\delta\gamma}}{\pi}\frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}e^{\beta(x-\mu)}\ , \qquad x\in\mathbb{R}\ , \tag{3.3}$$

where K_λ is the modified Bessel function of the third kind with index λ given by

$$K_\lambda(x) = \int_0^\infty u^{\lambda-1}e^{-\frac{1}{2}x(u^{-1}+u)}du\ , \qquad x>0\ ,$$

and $\gamma^2 = \alpha^2-\beta^2$. The parameter domain is $\delta>0$, $\alpha\geqslant 0$, $\alpha^2>\beta^2$ and $\mu\in\mathbb{R}$.

The mean and the variance are given respectively by

$$\begin{aligned}\mathbb{E}X &= \mu+\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\ ,\\ \mathbb{V}arX &= \frac{\delta\alpha^2}{(\alpha^2-\beta^2)^{\frac{3}{2}}}\ .\end{aligned}$$

Its Laplace transform is particularly simple:

$$\mathbb{L}(z) = e^{-\mu z+\delta(\gamma-\gamma_z)}\ , \qquad |\beta-z|<\alpha\ , \tag{3.4}$$

where $\gamma_z^2 = \alpha^2-(\beta-z)^2$.

From the form of the Laplace transform (3.4), we can see that the NIG distribution is closed under convolutions. If X_1 and X_2 are two independent random variables with NIG densities $nig(x;\alpha,\beta,\delta_1,\mu_1)$ and $nig(x;\alpha,\beta,\delta_2,\mu_2)$ respectively, then X_1+X_2 has density $nig(x;\alpha,\beta,\delta_1+\delta_2,\mu_1+\mu_2)$.

The NIG distribution was originally constructed in Barndorff-Nielsen (1977) as a normal variance-mean mixture where the mixing distribution is an inverse Gaussian. That is, if X is a NIG distributed random variable then, the conditional distribution given $W=w$ is $N(\mu+\beta w,w)$ where W is inverse Gaussian distributed $IG(\delta,\gamma)$ [see

Jørgensen (1982) for a reference on inverse Gaussian distributions]. This gives a simple way of simulating NIG r.v.'s.

Barndorff-Nielsen and Halgreen (1977) showed that the generalized hyperbolic family is infinitely divisible and its Laplace transform is of the form

$$\mathbb{L}(z) = e^{\Psi(z)} , \qquad z \in \mathbb{R} , \tag{3.5}$$

with

$$\Psi(z) = az + \int_{\mathbb{R}_0} \left[e^{zx} - 1 - zx\right] \nu(dx) , \qquad z \in \mathbb{R} , \tag{3.6}$$

where $a \in \mathbb{R}$ is the mean of the distribution and ν is a positive measure on $\mathbb{R}_0 = \mathbb{R} - \{0\}$ satisfying

$$\int_{\mathbb{R}_0} \left(1 \wedge |x|^2\right) \nu(dx) < \infty .$$

The exponent (3.6) corresponds to the Laplace exponent of an infinitely divisible distribution with finite mean [see equation (2.4)]. The measure ν is called the Lévy measure. The parameters a and ν uniquely determine any infinitely divisible distribution. Notice that the Gaussian coefficient is zero.

Because of the infinite divisibility of NIG distributions, we can construct a NIG Lévy process. Recall that this class of processes is in one-to-one correspondence with the class of infinitely divisible distributions. Every infinitely divisible distribution generates a Lévy process and the increments of every Lévy process are infinitely divisible distributed. We refer to Bertoin (1996) or Sato (1999) for comprehensive discussions on Lévy processes and to Barndorff-Nielsen, Mikosh and Resnick (2001) for recent applications.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space, then the NIG Lévy processes can be defined as follows:

Definition 3.1 *An adapted càdlàg $\mathbb{R}$-valued process $X = \{X(t)\}_{t\geqslant 0}$ with $X(0) = 0$ is a NIG Lévy process if $X(t)$ has independent and stationary increments distributed as $nig(\cdot; \alpha, \beta, \delta t, \mu t)$ and is continuous in probability.*

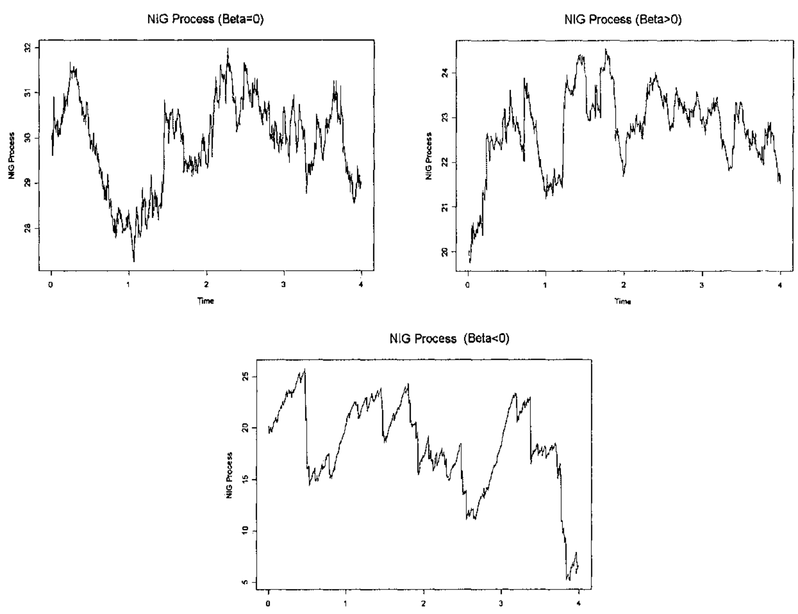
Its Laplace transform function $\phi_t(z) = \mathbb{E}\left(e^{-zX(t)}\right)$ is of the form $e^{t\Psi(z)}$ where Ψ is the Laplace exponent (3.6) in the Lévy-Khintchine representation of the NIG distribution. The Lévy triplet for the NIG Lévy process is especially simple as stated in the following alternative definition:

Definition 3.2 *An adapted càdlàg $\mathbb{R}$-valued process $X = \{X(t)\}_{t \geqslant 0}$ with $X(0) = 0$ is a NIG Lévy process if its Laplace transform is of the form $\mathbb{L}_t(z) = e^{t\Psi(z)}$ where $\Psi(z)$ is of the form (3.6) with $a = \mu + \frac{\beta\delta}{\gamma}$ and Lévy measure ν_{nig} given by*

$$\nu_{nig}(dx) = \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|) e^{\beta x} dx \,, \qquad x \in \mathbb{R} \,. \tag{3.7}$$

We can see that since there is no Gaussian constant in (3.7), the NIG process is a pure jump process plus a drift term. The drift term is nothing but the expected value of $X(1)$. In Figure 3.1 we can see different paths of NIG processes. Despite the apparent continuity, these paths are composed by an infinite number of small jumps.

Figure 3.1: **Simulated paths of a NIG Lévy processes for different values of** β



The Lévy measure governs the occurrence and the size of the jumps of the process X. The jumps of size in $\Lambda \in \mathbb{R}$ (provided that the closure $\bar{\Lambda}$ does not contain zero) form a compound Poisson process with rate $\int_\Lambda \nu_{nig}(dx)$ and jump density $\frac{\nu_{nig}(dx)}{\int_\Lambda \nu_{nig}(dx)}$.

At this point we make some remarks about the Lévy measure of a NIG process that are important to our risk model. First, notice that ν_{nig} has infinite mass around the origin. This implies that the NIG process is composed of an infinite number of small jumps. This pure discontinuity will account for discrete changes in the constant premium. Second, the jump size distribution has medium or exponentially decaying tails, i.e. the jump size density behaves as $C|x|^{\rho}e^{-\lambda|x|}$ for $x \to \pm\infty$ and for some real constants $C > 0$, ρ and λ. This follows directly from the form of its Lévy measure (3.7) and the well-known [see Abramowitz and Stegun (1970)] asymptotic relation for the Bessel function K_ξ

$$K_\xi(x) \sim \sqrt{\frac{\pi}{2}} x^{-\frac{1}{2}} e^{-x} \,, \qquad \text{as } x \to \infty \,. \tag{3.8}$$

This would make a risk process driven by a NIG motion a good model for medium tailed claims since the jump size distribution will account for claims over a certain threshold.

Figure 3.1 also illustrates the role played by the parameter β in (3.7). By changing the sign of β we control the weight of positive versus negative jumps, this is because the only element in (3.7) affected by its sign is the exponential term. A parameter $\beta = 0$ implies a symmetric NIG distribution and hence an equilibrium between positive and negative jumps in the corresponding process. A value of $\beta < 0$ would induce larger and more frequent positive jumps. And not only that, the disparity between positive and negative jumps grows exponentially, i.e. positive jumps would be exponentially larger and more frequent than negative ones, which would be exponentially smaller and less frequent. The opposite applies for $\beta > 0$.

This will be an important feature of our model; thanks to the role of β we can incorporate aggregate claims and premium perturbations in the same process. By changing β we can obtain larger jumps downwards that represent the claims along with small jumps accounting for premium fluctuations. We are interested in paths like the one shown for $\beta < 0$.

Also notice in Figure 3.1 that other possible paths can be obtained. We can see why this process is appealing in finance modeling, it incorporates fine diffusion-like and jump-driven structures into one single process.

Finally, we discuss briefly a second way to construct a NIG process that might give a new interesting perspective to our model. A NIG Lévy process can also be constructed via subordination. Consider a standard Brownian motion W. If we add a drift term β we have the new Brownian motion Y

$$Y_t = \beta t + W_t \ , \qquad t > 0 \ . \tag{3.9}$$

Now, recall that the inverse Gaussian distribution $IG(\cdot;\alpha,\delta)$ is infinitely divisible and therefore we can define an inverse Gaussian Lévy process τ. Since the increments of τ are inverse Gaussian distributed and can only be positive, the process τ is increasing. Such processes are called subordinators and can be interpreted as a random time transform. If we apply this random transformation to (3.9) we get a new process Y

$$Y_t = \beta\tau(t) + W_{\tau(t)} \ , \qquad t > 0 \ . \tag{3.10}$$

The transformed process (3.10) is a NIG Lévy process with finite dimensional distribution $nig(\cdot;\alpha,\beta,\delta t,\mu = 0)$. If we add a drift term $\mu t + Y_t$ we introduce the parameter μ.

This construction allows us to see a NIG as a transformed Brownian motion, where the subordinator τ represents a business time reflecting a varying market activity. In terms of our model this means that the NIG risk model is the diffusion model of Grandell (1977) distorted by a randomly changing business time. We elaborate further on this in Section 3.4.

3.3 A Purely Discontinuous Risk Model

We consider a general risk model as in (3.2)

$$U(t) = u + ct + Z(t) - S(t) \ , \qquad t \geqslant 0 \ . \tag{3.11}$$

We would like the aggregate claims process S to be a compound Poisson process with medium tailed claim size distribution. We would also like the perturbation Z to account for ups and downs in the premium, this is usually achieved by means of a diffusion [Dufresne and Gerber (1991)] or of an α-stable process [Furrer (1998)]. Instead we aim somewhere in between these two approaches, we would like Z to account for more drastical fluctuations than those allowed by a diffusion but not as dramatic as those implied by an α-stable perturbation.

We can achieve all of these desired properties by letting Z and S be pure jump Lévy processes defined by appropriate Lévy measures ν_Z and ν_S. For instance, if $Z-S$ forms a NIG Lévy process then ν_S should be the Lévy measure of a compound Poisson process with medium tailed jump distribution of the form $C|x|^{-\rho}e^{-\lambda x}$. By contrast, ν_Z would represent small positive and negative discrete fluctuations. This reasoning leads to the following characteristic exponent for the process $U-u$:

$$\begin{aligned}\Psi(z) &= \left(c+\frac{\beta\delta}{\gamma}\right)z \\ &+ \int_{-1}^{\infty}\left[e^{zx}-1-zx\mathbb{I}_{(-1,1)}(x)\right]\nu_Z(dx)+\int_{-\infty}^{-1}\left(e^{zx}-1\right)\nu_S(dx)\ ,\end{aligned} \tag{3.12}$$

where

$$\nu_S(dx)=\nu_Z(dx)=\frac{\delta\alpha}{\pi|x|}K_1(\alpha|x|)e^{\beta x}dx\ , \qquad x\in\mathbb{R}\ ,\ \beta<0\ . \tag{3.13}$$

The condition on β is needed to assure larger negative jumps.

The loaded premium c plays the role of the parameter μ and it has to satisfy $c>-\frac{\beta\delta}{\gamma}$ to meet the net profit condition. The loaded premium is then of the form $c=(1+\theta)\frac{-\beta\delta}{\gamma}$ for some $\theta>0$.

In Figure 3.2 we can see a NIG risk process. Notice the diffusion-like structure perturbed by large negative jumps. This would allow us to have the same features as the model of Dufresne and Gerber (1991) in one single object instead of considering two different processes.

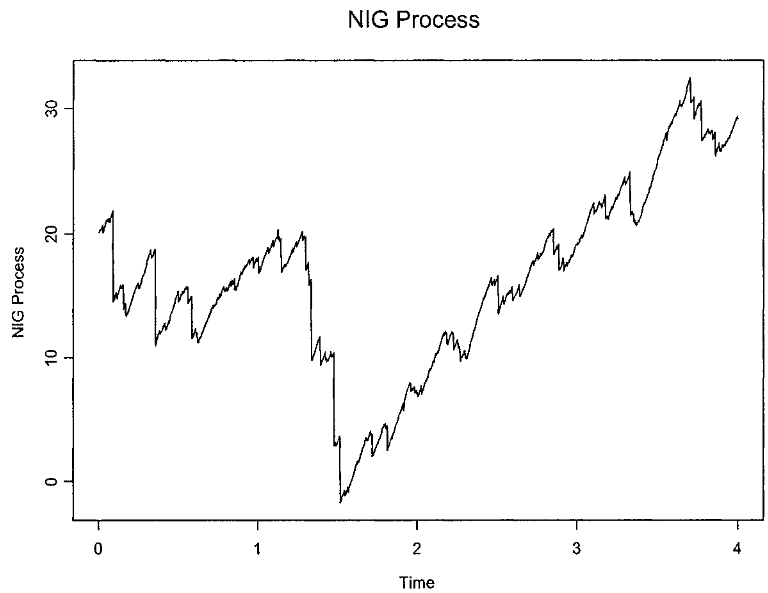
The jump sizes, ΔX_s of a NIG Lévy process X with characteristic exponent (3.12), are such that [see Raible (2000)], as $n\to\infty$

$$\frac{1}{nt}\sum_{s\leqslant t}\mathbb{I}_{(\frac{1}{n},\infty)}(\Delta X_s)\longrightarrow\frac{\delta}{\pi}\ , \qquad a.s.\ ,$$

and

$$X_t-\sum_{s\leqslant t}\Delta X_s\mathbb{I}_{\{(-\infty,-\frac{1}{n})\cup(\frac{1}{n},\infty)\}}(\Delta X_s)\longrightarrow ct\ , \qquad a.s.\ .$$

Figure 3.2: **A simulated path of a NIG risk process.** $\beta = -4$, $\delta = 20$, $c = 30$ **and** $\alpha = \sqrt{13}$



That is, the process carries along its path some information from the distribution. The parameters δ and μ are imprinted in the paths of the NIG process. We can draw some analogies with the classical risk model: in a way δ plays the role of the claim rate λ, it is the limit of the normalized number of jumps larger than $\frac{1}{n}$. As for the drift, c is the constant premium collected that is visible if we disregard all large fluctuations and claims.

The martingale approach of Schmidli (1995) and Sørensen (1996) can be applied to a NIG risk model as in (3.11) to work out expressions for the ruin probability. We refer to Grandell (1991) for a review of martingale methods in risk theory.

From the general theory of processes with independent increments [see Jacod and Shiryaev (1987)] we have that, if X is a Lévy process with Laplace exponent Ψ, then the process

$$M^r(t) = \frac{e^{-rX(t)}}{e^{t\Psi(r)}}, \qquad r \in \mathbb{R}, \tag{3.14}$$

is a local martingale for r within the right domain. The finite dimensional distributions of M^r are the Esscher transforms of the finite dimensional distributions of X. [The Esscher transform has been extensively used throughout the actuarial literature, for instance Gerber and Shiu (1994a, 1994b, 1996) and references therein]. We can define a new measure $\mathbb{Q}$ in the following way:

Definition 3.3 *Let $X = \{X(t)\}_{t\geqslant 0}$ be a Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. We define an Esscher transform as any change of $\mathbb{P}$ to a locally equivalent measure $\mathbb{Q}$ with a density process $M(t) = \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}$ of the form (3.14) for some r in the domain of definition of the Laplace exponent Ψ.*

In other words, this means that, for any $A \in \mathcal{F}_t$, the new measure $\mathbb{Q}$ is given by

$$\mathbb{Q}_t^r[A] = \mathbb{E}_{\mathbb{P}}[M^r(t)\mathbb{I}_{\{A\}}] \, .$$

Now, recall that the ruin probability is defined as

$$\psi(u) = \mathbb{P}\{\tau < \infty\} \;, \qquad u \geqslant 0 \;, \tag{3.15}$$

where $\tau = \inf[t > 0 : U(t) < 0]$ is the first time the process falls below zero. Since $U - u$ is a Lévy process we can write its associated ruin probability ψ in terms of the new measure as follows:

$$\psi(u) = \mathbb{E}_{\mathbb{P}}[\mathbb{I}_{\{\tau<\infty\}}] = \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{M^r(\tau)}\mathbb{I}_{\{\tau<\infty\}}\right] = \mathbb{E}_{\mathbb{Q}}[e^{\{r(U(\tau)-u)+\tau\Psi(r)\}}\mathbb{I}_{\{\tau<\infty\}}] \, .$$

If we can find a value $r = R$ in the domain of definition of the Ψ such that $\Psi(R) = 0$ and $\tau < \infty$ a.s. we could simply write

$$\psi(u) = \mathbb{E}_{\mathbb{Q}}[e^{RU(\tau)}]e^{-Ru} \;, \qquad u \geqslant 0 \, . \tag{3.16}$$

This approach to the ruin problem can be found in Asmussen (1987) and he has attributed the idea to von Bahr (1974) and Siegmund (1975).

Equation (3.16) defines a straight forward simulation scheme to compute the ruin probability for our model. A first concern is how an Esscher transformation affects the NIG Lévy process. It turns out that a NIG Lévy process stays a NIG process under an Esscher transform. In fact, all Lévy processes are transformed into Lévy processes under an Esscher change of measure [see Jacod and Shiryaev (1987), Raible (2000) or Morales and Schoutens (2003)]. Yong Yao, then a graduate student at Iowa University was the first to mention this property of Lévy processes in the actuarial literature [see discussion of Gerber and Shiu (1994a) in page 168 and appendix in Gerber and Shiu (1994b)]. For the NIG process the parameter β plays the role of the Esscher parameter r as stated in the following Lemma:

Lemma 3.1 *Let $X = \{X(t)\}_{t\geqslant 0}$ be a NIG Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. And also let $\mathbb{Q}$ be an equivalent probability measure defined through the Esscher density process of Definition 3.3. Then, the process X under $\mathbb{Q}$ is again a NIG Lévy process with parameter $\tilde{\beta} = \beta - r$.*

Proof. An Esscher transformation defined through the density process (3.14) as in Definition 3.3 implies an Esscher transformation in the one-dimensional distributions. Let $nig(dx)$ be the density function of $X(1)$ and $\mathbb{L}$ its Laplace transform. The Esscher transform $nig_r(dx)$ of $X(1)$ is

$$nig_r(dx) = \frac{e^{-rx} nig(dx)}{\mathbb{L}(r)}\,, \qquad |\beta - r| < \alpha\,.$$

Clearly, the Laplace transform of $nig_r(dx)$ is

$$\mathbb{L}_r(z) = \frac{\mathbb{L}(z+r)}{\mathbb{L}(r)}\,, \qquad |\beta - (r+z)| < \alpha\,.$$

Since $X(1)$ is a NIG random variable, $\mathbb{L}$ is the Laplace transform given in (3.4) and we can write

$$\mathbb{L}_r(z) = \exp\left\{\Psi(z+r) - \Psi(r)\right\}\,,$$

where Ψ is the Laplace exponent in (3.4). This exponent becomes

$$\begin{aligned}
\Psi(z+r) - \Psi(r) &= -\mu(z+r) + \delta\left\{\gamma - \sqrt{\alpha^2 - [\beta - (z+r)]^2}\right\} \\
&+ \mu r - \delta\left\{\gamma - \sqrt{\alpha^2 - [\beta - r]^2}\right\} \\
&= -\mu z + \delta\left[\sqrt{\alpha^2 - [\beta - r]^2} - \sqrt{\alpha^2 - [(\beta - r) - z]^2}\right]\,,
\end{aligned}$$

which is the Laplace exponent of a NIG random variable with parameters $(\alpha, \tilde{\beta} = \beta - r, \delta, \mu)$. ■

Now, since a NIG process stays a NIG under an Esscher change of measure with the parameter β acting exactly as the Esscher parameter r, we can write the ruin probability ψ as in (3.16) as stated in the following result:

Proposition 3.1 *Let U be a risk reserve process driven by a NIG Lévy process as in (3.11). Its associated ultimate ruin probability ψ satisfies:*

i) If $-\frac{\beta\delta}{\gamma} \leqslant c \leqslant \frac{\delta\gamma}{\alpha+\beta}$ then

$$\psi(u) = \mathbb{E}_{\mathbb{Q}}[e^{RU(\tau)}]e^{-Ru} , \qquad u \geqslant 0 ,$$

for $R = \frac{2(\beta+\gamma\frac{c}{\delta})}{1+(\frac{c}{\delta})^2}$ and where $\mathbb{Q}$ is an equivalent measure induced by an Esscher transform with parameter R.

ii) If $c > \frac{\delta\gamma}{\alpha+\beta}$ then

$$\psi(u) \leqslant \frac{e^{-r^*u}}{\mathbb{E}_{\mathbb{P}}[e^{-\tau\Psi(r^*)}|\tau < \infty]} , \qquad u \geqslant 0 ,$$

for some r^* in such that $\beta - \alpha < r^* < \alpha + \beta$, where $\mathbb{P}$ is the original measure.

Proof. *i)* We had already stated that, for $|\beta - r| < \alpha$,

$$\psi(u) = \mathbb{E}_{\mathbb{Q}}[e^{\{r(U(\tau)-u)+\tau\Psi(r)\}}\mathbb{I}_{\{\tau<\infty\}}] , \qquad u \geqslant 0 . \tag{3.17}$$

Now, if there exist a value $r = R$ in the domain of definition of Ψ such that $\Psi(R) = 0$ we would have half of the proof. Notice that, for a NIG distribution, the Laplace exponent Ψ is finite for all r in the domain of definition, including the endpoints. This implies a restriction in the possible values for c to insure that such a number R exists. This can be seen if we solve the equation $\Psi(r) = 0$, which implies the following relation

$$cr = \delta(\gamma - \gamma_r) , \qquad -\alpha - \beta \leqslant r \leqslant \alpha + \beta . \tag{3.18}$$

The function in the right-hand side takes the value $\delta\gamma$ at both endpoints. If a positive solution R is to exist, the line cr should intersect the curve $\delta(\gamma-\gamma_r)$ at a lower point than $\delta\gamma$. This is, $c(\alpha+\beta)\leqslant\delta\gamma$ which implies the upper bound for c in the proposition.

Now, if we take derivatives we can see that the function Ψ is a convex function and that $\Psi'(0)=-(c+\frac{\beta\delta}{\gamma})$. Since we assume that $c+\frac{\beta\delta}{\gamma}>0$ to meet the net profit condition, we have that a positive solution R would exist as long as $-\frac{\beta\delta}{\gamma}\leqslant c\leqslant\frac{\delta\gamma}{\alpha+\beta}$. Solving (3.18) yields $R=\frac{2\left(\beta+\gamma\frac{c}{\delta}\right)}{1+\left(\frac{c}{\delta}\right)^2}$ which is only positive and well defined in the specified range.

In order to complete the proof we have to show that, under $\mathbb{Q}$, $\tau<\infty$ almost surely. Since the process $U-u$ is still a NIG Lévy process as stated in Lemma 3.1, if we compute $\mathbb{E}_{\mathbb{Q}}[U(1)-u]$ we find

$$\mathbb{E}_{\mathbb{Q}}[U(1)-u]=c+\frac{(\beta-R)\delta}{\sqrt{\alpha^2-(\beta-R)^2}}=-\Psi'(R)\ .$$

Now, since Ψ is convex and strictly increasing on the positive axis, we have that $-\Psi'(R)<0$ which implies that, under the measure $\mathbb{Q}$, the process $U-u$ drifts away to $-\infty$ and $\tau<\infty$ $\mathbb{Q}$-almost surely and $\mathbb{I}_{\{\tau<\infty\}}=1$.

ii) If $c>\frac{\delta\gamma}{\alpha+\beta}$ we have that there is no solution R in the domain of definition of the Laplace exponent such that $\Psi(R)=0$. However, if we use the fact that M^r is a martingale, and therefore supermartingale, we have for $t>0$,

$$1\geqslant\mathbb{E}_{\mathbb{P}}[M^r(t\wedge\tau)]\geqslant\mathbb{E}_{\mathbb{P}}[M^r(t)|\tau<t]\mathbb{P}(\tau<t)\ ,\qquad|\beta-r|<\alpha\ .$$

This last inequality follows from the optional stopping theorem and the fact that $M^r(0)=1$. If we substitute the expression for M^r and we get

$$\mathbb{P}(\tau<t)\leqslant\frac{e^{-ru}}{\mathbb{E}_{\mathbb{P}}[e^{-rU(\tau)-\tau\Psi(r)}|\tau<t]}\leqslant\frac{e^{-ru}}{\mathbb{E}_{\mathbb{P}}[e^{-\tau\Psi(r)}|\tau<t]}\ ,\qquad u\geqslant 0\ .$$

This last inequality comes from the fact that $U(\tau)<0$ conditioned on the set $\tau<\infty$. Now, via Jensen's inequality and the previoulsy established fact that $\Psi'(0)<0$ because of the net profit condition, we have that there exists a point r^* in the domain of the function Ψ such that $\frac{e^{-ru}}{\mathbb{E}_{\mathbb{P}}[e^{-\tau\Psi(r)}|\tau<t]}$ attains its minimum. If we let $t\to\infty$, this leads to the form that has become standard in the actuarial literature [Grandell

(1991)]:

$$\psi(u) \leqslant \frac{e^{-r^* u}}{\mathbb{E}_{\mathbb{P}}[e^{-\tau\Psi(r^*)}|\tau < \infty]}\,, \qquad u \geqslant 0\,.$$

■

Notice that from Proposition 3.1 we can recover Lundberg's inequality for the NIG risk process

$$\psi(u) \leqslant e^{-\mathcal{R}u}\,,$$

where $\mathcal{R} = \sup\{r|\Psi(r) \leqslant 0\}$. In the first case of Propostion 3.1 this yields the exact adjusment coefficient R for which $\Psi(R) = 0$. In the second case this yields r^*.

Proposition 3.1 defines a straight forward simulation scheme to evaluate ruin probabilities in the case R exists. This imposes a certain condition on the loaded premium. Although it might seem restrictive at first, such condition depends on other parameters that can be adjusted to allow for greater flexibility.

Altogether, we have shown that the NIG Lévy process can be the basis of a risk reserve process for which classical ruin theory results still hold. Its counterintuitive but appealing features make it a model that accounts for the discrete infinite activity of the real world. We say discrete infinite activity because the evolution of real processes seem to be governed by discrete jumps that, as we zoom into smaller time intervals, appear to be infinitely many.

3.4 On the NIG Risk Process as a Transformed Diffusion Approximation

Grandell (1977) constructed a sequence of risk reserve processes $\{U_n\}_{n=1,2,...}$ of the form

$$U_n(t) = u_n + c_n t - \sigma_n S_n(t)\,, \qquad t \geqslant 0\,,$$

where u_n, $c_n = \beta_n \rho_n$ are the corresponding sequences of initial reserves and premium rates such that $u_n \to u$, $\sigma_n^2 \to \sigma^2$ and $c_n \to c = \beta\rho$. The aggregate claim processes S_n is compound Poisson with mean β_n and variance σ_n^2. He showed that for claim sizes in the domain of attraction of the normal distribution, the sequence $\{U_n\}_{n=1,2,...}$

converges weakly in the Skorokhod topology to the diffusion process

$$U_D(t) = u + \beta\rho t - \sigma W(t) \,, \qquad t > 0 \,, \tag{3.19}$$

as $n \to \infty$, where W is a standard Brownian motion.

Compare this last equation (3.19) to the subordination construction of the NIG process (3.10). Notice that the diffusion approximation is of the form (3.10) with a variance factor. If we are to incorporate the parameter σ we have to set $\rho = \sigma$ and then we can consider the following generalization of (3.19) via subordination through the inverse Gaussian Lévy process τ:

$$\widetilde{U}_D(t) = u + \beta\sigma\tau(t) - \sigma W[\tau(t)] \,, \qquad t > 0 \,. \tag{3.20}$$

τ is a subordinator with one-dimensional distribution given by

$$IG(x; \delta t, \gamma_\sigma) = \frac{\delta t}{\sqrt{2\pi x^3}} \exp\left\{ -\frac{\gamma_\sigma (x - \delta t/\gamma_\sigma)^2}{2x} \right\} , \qquad x > 0 \,,$$

where $\gamma_\sigma^2 = \alpha^2 - (\beta\sigma)^2$.

The process $\widetilde{U}_D(t)$ in (3.20) is a NIG Lévy process with one-dimensional distribution $nig(\cdot; \alpha, \beta\sigma, \delta t, \mu = 0)$. This can be seen if we apply the following result of Sato (1999):

Theorem 3.1 *Let Z be a subordinator with Laplace exponent Ψ and let Y be a Lévy process with characteristic function $\widehat{\mu}$. If Z and Y are independent then the characteristic function of the process $X(t) = Y(Z(t))$ obtained through subordination is given by*

$$\mathbb{E}[e^{izX(t)}] = e^{t\Psi[-\ln \widehat{\mu}(z)]} \,, \qquad z \in \mathbb{R} \,.$$

In this case the subordinator is an inverse Gaussian process and

$$\Psi(z) = \delta\gamma_\sigma \left[1 - \sqrt{1 + \frac{2}{\gamma_\sigma^2} z} \right] , \qquad z \in \mathbb{R} \,,$$

and the transformed process is the Brownian motion with drift (3.19) having characteristic function

$$\widehat{\mu}(z) = \exp\left(i\beta\sigma z - \frac{\sigma^2 z^2}{2} \right) , \qquad z \in \mathbb{R} \,.$$

If we apply Theorem 3.1 we get that the subordinated process in (3.20) has characteristic function given by

$$\mathbb{E}[e^{iz\widetilde{U}_D(t)}] = e^{t\Psi[-\ln\widehat{\mu}(z)]} = e^{\delta[\gamma_\sigma - \gamma_\sigma(z)]} , \qquad z \in \mathbb{R} , \tag{3.21}$$

where $\gamma_\sigma^2(z) = \alpha^2 - (\beta\sigma - iz)^2$.

Equation (3.21) is the characteristic function of a NIG Lévy process with parameters $(\alpha, \beta\sigma, \delta, \mu = 0)$.

Notice that in order to introduce the variance parameter σ into the construction via subordination of the NIG process (3.10) we need to force $\rho = \sigma$. This is, we need to incorporate the variance into the collected premium. This is not such an unusual choice since the risk due to the variability of the aggregate claim process should be reflected in the premium.

If we wish to incorporate the parameter μ we get a step further in the generalization yielding

$$U_{NIG}(t) = \mu t + \widetilde{U}_D(t) , \qquad t > 0 . \tag{3.22}$$

This last process is the NIG risk process that we introduce in Section 3 with parameters $(\alpha, \beta\sigma, \delta, \mu)$.

The NIG risk process (3.22) is a transformation of the diffusion approximation of Grandell (1977). Our risk process is a still a diffusion but operating in *business time.* The subordinator τ is a random time transformation that accounts for different speeds at which the market evolves. In a way, the business time does not flow continuously, but by an infinite number of jumps of different lengths which are represented by the inverse Gaussian process.

The NIG risk process (3.22) is a generalization of (3.19). If we set $\tau(t) = t$, the business time flows just like regular time and we recover the diffusion risk process. In fact, the NIG risk process (3.22) is embedded in a larger class of risk processes defined via subordination. Consider the following model

$$U_\tau(t) = u + \mu t + \sigma W[\tau(t)] , \qquad t > 0 , \tag{3.23}$$

where W is a Brownian motion with drift β and τ is any subordinator representing a randomly changing business time. If τ is an inverse Gaussian subordinator we obtain the NIG model (3.22) which is of the form (3.23) as we had stated. The diffusion model is also of the form (3.23) by setting $\tau(t) = t$. We can also recuperate the α-stable model, if τ is an $\alpha/2$-stable subordinator and the drift term $\beta = 0$, then (3.23) is the α-stable model of Furrer, Michna and Weron (1997) [see Sato (1999) and Cherny and Shiryaev (2002) for a reference on semimartingales as a time-changed diffusions].

In finance they often refer to the time change τ as the clock of the process [see Ané and Geman (2000)]. For instance, if $\tau(t) = t$ then the process runs in calendar clock. Other subordinators τ are used to model different clocks, for example τ might be the traded volume which reflects the business activity. In insurance, when talking about a clock for the process (3.23), the natural analogy for traded volume will be the aggregate claims process. This is the measure of insurance business activity, it represents the total claims filed up to time t.

The inverse Gaussian distribution is traditionally used to model aggregate claims [see Chaubey, Garrido and Trudeau (1998)]. This gives another interpretation to the NIG risk process (3.22). The NIG risk process models risk reserves with a Brownian motion. However, this Brownian motion does not run in calendar time but in a business clock defined in terms of the aggregate claims process of the company.

3.5 Simulation for the NIG Risk Model

Proposition 3.1 endows us with a straight forward simulation approximation to the associated ruin probability for the NIG risk process. If the adjustment coefficient exists then we can simulate the risk process U under the Esscher-induced change of measure. Under this measure the stopping rule for the simulated paths

$$\tau = \inf\{t > 0 | U(t) \leqslant 0\}$$

is well defined since τ is finite $\mathbb{Q}$-almost surely. Then, for each path we evaluate the expression $e^{R[U(\tau)-u]}$ and we average over all simulated paths.

This scheme requires the simulation of a NIG Lévy process. As we had previously pointed out this is simple because all the one-dimensional densities f_t are NIG distributed. If we want a NIG process with parameters $(\alpha, \beta, \delta, \mu)$ we simulate a process skeleton by generating NIG random variables at time increments of length Δ. That is, we simulate NIG variates with parameters $(\alpha, \beta, \delta\Delta, \mu\Delta)$.

If we use the mean-variance representation of the NIG distribution we have that, if W and Y are independent random variates distributed as standard normal and $\mathrm{IG}(\delta, \sqrt{\alpha^2 - \beta^2})$ respectively, then

$$X = \mu + \beta Y + \sqrt{Y} W \ ,$$

is a NIG random variate with parameters $(\alpha, \beta, \delta, \mu)$. There exist now standard techniques to simulate normal and inverse Gaussian variates [see for instance Devroye (1986)].

3.6 Conclusions

We present a risk model based on a normal inverse Gaussian Lévy process. We show how the infinite activity feature of such family of processes can be used to account for discrete premium fluctuations as well as for semi-heavy tailed claims. Despite its counterintuitive properties, the NIG risk process still can be incorporated into standard risk theory results as shown in Proposition 3.1.

The subordination construction of the NIG process implies that our risk model is a generalization of the diffusion risk model of Grandell (1977). The fact that the NIG is still a diffusion but operating in business time allows for larger fluctuations making it a better and more flexible model to fit risk reserves with exponentially decaying claims. The concept of business time for a transformed risk reserve process is used to generalize the diffusion model. In such a generalized process, time evolves by an infinite number of small jumps with occasional larger time jumps. The random time increments can be seen as a randomly varying market activity.

The mean-variance representation of NIG random variables allows us to easily simulate a skeleton for the NIG risk process and hence to evaluate its associated ruin probabiities using Proposition 3.1.

Our discussion is mainly introductory. We present the NIG Lévy process as well as some of its features within a risk theory context. Further research is needed to asses the performance of such processes compared to other risk models. Other directions to be explored for the NIG risk process are those bridging financial and insurance mathematics: risk measures and option pricing. In conclusion, risk processes driven by NIG, or other Lévy processes, have merits to be considered an object of further research in risk theory.

Chapter 4

Risk Theory with the Generalized Inverse Gaussian Lévy Process

4.1 Introduction

Dufresne, Gerber and Shiu (1991) introduced a general risk model defined as the limit of compound Poisson processes. They work with a model

$$U(t) = u + ct - Z(t)\ , \qquad t > 0\ , \tag{4.1}$$

where Z is an increasing Lévy process. Such model is either a compound Poisson process itself or a Lévy process with infinitely many small jumps. Their construction is based on a non-negative non-increasing function Q that governs the jumps of the process. This function, it turns out, is the tail of the Lévy measure of the process. They show that the gamma process is one of the processes that can be generated this way and use it as a model for the aggregate claims. We enlarge their model to a Generalized Inverse Gaussian (GIG) Lévy process. Although mathematically more complex, such a process keeps some of the nice properties of the simpler gamma process.

Dufresne, Gerber and Shiu (1991) constructed a general aggregate claim process S with independent and stationary increments. They define it in terms of a non-negative and non-increasing function Q defined as

$$Q(x) = \int_x^\infty q(s)ds \ , \quad x > 0 \ .$$

Moreover, the function Q should be such that

$$\int_0^\infty x dQ(x) < \infty \ . \tag{4.2}$$

The process S is uniquely defined by its Laplace transform

$$\mathbb{L}_t(z) = \mathbb{E}\left[e^{-zS(t)}\right] = e^{-t\Psi(z)} \ , \qquad z > 0 \ , \tag{4.3}$$

where the exponent Ψ is given by

$$\Psi(z) = \int_0^\infty \left[e^{-zx} - 1\right] dQ(x) \ , \qquad z > 0 \ . \tag{4.4}$$

This last equation can be recognized as the Laplace exponent of a Lévy process with sample paths of finite variation [see Bertoin (1996) or Sato (1999) for an account on Lévy processes]. The measure $q(dx) = -dQ(x)$ is the Lévy measure of the process S.

From the theory of Lévy processes we have that S is either a compound Poisson process (if $Q(0) < \infty$) or a process with an infinite number of small jumps (if $Q(0) = \infty$). In both cases, the process S can be seen as the limit of a sequence of compound Poisson processes $\{S_\epsilon\}_{\epsilon > 0}$ described by their Laplace transform

$$\mathbb{L}_t^{(\epsilon)}(z) = e^{-t\int_0^\infty \left[e^{-zx} - 1\right] dQ_\epsilon(x)} \ , \qquad z > 0 \ , \tag{4.5}$$

where the measure Q_ϵ is the restriction to the interval $[\epsilon, \infty)$ of the tail of Lévy measure Q, i.e.

$$dQ_\epsilon(x) = dQ(x)\mathbb{I}_{[\epsilon,\infty)}(x) \ , \qquad x > 0 \ .$$

We can see that

$$\lim_{\epsilon \to 0} dQ_\epsilon(x) = dQ(x) \ , \qquad x > 0 \ ,$$

which implies that the sequence of compound Poisson processes $\{S_\epsilon\}_{\epsilon > 0}$ converges weakly in the Skorokhod topology to the process S defined by (4.3) [see Jacod and Shiryaev (1987)].

Notice that (4.4) implies that S is a Lévy process with Lévy measure $-dQ(x)$, moreover, condition (4.2) implies that S is of finite variation.

Dufresne, Gerber and Shiu (1991) explore the process S for

$$q(dx) = -dQ(x) = x^{-1}e^{-x}dx \ , \qquad x > 0 \ ,$$

and

$$q(dx) = -dQ(x) = x^{-3/2}e^{-x}dx \ , \qquad x > 0 \ .$$

The first choice of Q leads to a gamma process while the second leads to an inverse Gaussian process. These processes are such that their one-dimensional distributions are gamma and inverse Gaussian respectively.

Here we extend this aggregate claims process S to a generalized inverse Gaussian Lévy process (GIG). Such a process is a non-decreasing (subordinator) Lévy process exhibiting the intriguing property of having an infinite number of small jumps. Moreover its intervals of length one follow a generalized inverse Gaussian (GIG) distribution. A standard reference on GIG distribution is Jørgensen (1982). The inverse Gaussian process and the gamma process of Dufresne, Gerber and Shiu (1991) are particular or limiting cases of the GIG Lévy process, which in turn is another example of the spectrally negative Lévy processes discussed in Yang and Zhang (2001).

Section 4.2 introduces some basic facts about the GIG distribution. In Section 4.3 we construct the GIG Lévy process and describe some of the properties that make it appealing to ruin modeling. In Section 4.4 we extend ruin theory results to the GIG Lévy process.

4.2 Generalized Inverse Gaussian Distribution

The class of generalized inverse Gaussian distribution is described by three parameters and it has support on the positive axis as we discussed in Chapter 2. It has been extensively studied by Jørgensen (1982). Recall that its density function is given by

$$f_{gig}(x) = \frac{(\gamma/\delta)^{\lambda}}{2K_{\lambda}(\delta\gamma)} x^{-\lambda-1} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\} \ , \qquad x > 0 \ , \tag{4.6}$$

where K_λ is the modified Bessel function of the third kind with index λ given by

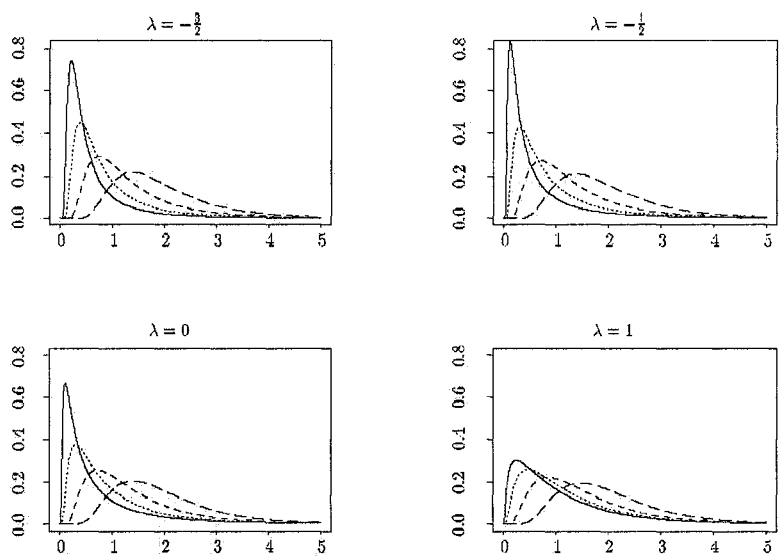
$$K_\lambda(x) = \int_0^\infty u^{\lambda-1} e^{-\frac{1}{2}x(u^{-1}+u)} du \,, \qquad x > 0 \,.$$

The parameter domain of the GIG distribution is

$$\begin{aligned} \delta > 0\,, \quad & \gamma \leqslant 0\,, \quad && \text{if} \quad \lambda < 0\,, \\ \delta > 0\,, \quad & \gamma > 0\,, \quad && \text{if} \quad \lambda = 0\,, \\ \delta \leqslant 0\,, \quad & \gamma > 0\,, \quad && \text{if} \quad \lambda > 0\,. \end{aligned}$$

If $\lambda = -1/2$ the density (4.6) reduces to that of the inverse Gaussian distribution. The gamma distribution is a limiting case of the GIG distribution (4.6) for $\lambda > 0$ and $\gamma > 0$ and $\delta \to 0$. These make the GIG Lévy processes a natural extention to the gamma processes.

Figure 4.1: **Some GIG densities for different values of λ and $\omega = \delta\gamma$**



The Laplace transform of the GIG is

$$\mathbb{L}_{GIG}(z) = \frac{K_\lambda\left(\delta\gamma\sqrt{1-\frac{2z}{\gamma^2}}\right)}{K_\lambda(\delta\gamma)\left(1-\frac{2z}{\gamma^2}\right)^{\lambda/2}}\,,$$

for $\delta > 0$ and $\gamma > 0$. Its domain is $z < \gamma^2/2$ when $\lambda \leqslant 0$ and $z \leqslant \gamma^2/2$ when $\lambda < 0$. Figure 4.1 shows some GIG densities for different values of the parameters.

Barndorff-Nielsen and Halgreen (1977) showed that the GIG distribution is infinitely divisible. We use this property to construct a Lévy process in the following section.

4.3 Generalized Inverse Gaussian Lévy Process

Let us recall some basic facts about Lévy process. Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space.

Definition 4.1 *An adapted càdlàg $\mathbb{R}$-valued process $X = \{X(t)\}_{t\geqslant 0}$ with $X(0) = 0$ is a Lévy process if and only if its characteristic function is of the form $\phi_t(s) = e^{-t\Psi(s)}$ where*

$$\Psi(s) = ias + \frac{b^2}{2}s^2 + \int_{\mathbb{R}_0} \left[1 - e^{isx} + isx\mathbb{I}_{(-1,1)}(x)\right] \nu(dx) , \qquad s \in \mathbb{R} , \tag{4.7}$$

while $a, b \in \mathbb{R}$ and ν is a positive measure on $\mathbb{R}_0 = \mathbb{R} - \{0\}$ as discussed in Chapter 2. The parameters a, b^2 and ν uniquely determine X. The measure ν is called the Lévy measure and the exponent Ψ is called the characteristic exponent of the process X.

This class of processes is in a one-to-one correspondence with the class of infinitely divisible distributions. Every infinite divisible distribution generates a Lévy process and the increments of every Lévy process are infinite divisible distributed.

The Lévy measure ν governs the occurrence of the jumps of the process X. If $b^2 > 0$ and the Lévy measure is identically zero then the process is a Brownian motion (the only continuous Lévy process). If the Guassian coefficient $b^2 = 0$ the process is entirely composed by jumps, if in addition $\int_{\mathbb{R}_0} \nu(dx) < \infty$ then the process is a compound Poisson process where the distribution of the jumps is $\frac{\nu(dx)}{\int_{\mathbb{R}} \nu(dx)}$ and the jumps epochs occur at rate $\int_{\mathbb{R}} \nu(dx)$. On the other hand, if $\int_{\mathbb{R}_0} \nu(dx) = \infty$ and $\int_{\mathbb{R}_0} 1 \wedge |x|\nu(dx) < \infty$, then the process has an infinite number of small jumps but is of finite variation. Finally, if $\int_{\mathbb{R}_0} \nu(dx) = \infty$ and $\int_{\mathbb{R}_0} 1 \wedge |x| \nu(dx) = \infty$, the process has infinitely many jumps and is of unbounded variation.

If the Gaussian coefficient $b^2 = 0$ and if the Lévy measure ν is defined on $(0, \infty)$ such that $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$ then the corresponding Lévy process is called a subordinator. Its increments are always positive.

If $b^2 = 0$ and the Lévy measure ν satisfies

$$\int_{\mathbb{R}_0} (1 \wedge |x|)\nu(dx) < \infty \,, \tag{4.8}$$

then the process X is of finite variation and we can discard the centering function in (4.7) and simply write

$$\Psi(s) = ias + \int_{\mathbb{R}_0} \left[1 - e^{isx}\right] \nu(dx) \,, \qquad s \in \mathbb{R} \,. \tag{4.9}$$

Particularly, increasing Lévy processes, also called subordinators, satisfy (4.8). For an account of the theory we refer to Sato (1999) or Bertoin (1996) and for recent applications to Barndorff-Nielsen, Mikosh and Resnick (2001).

Notice that the general aggregate claim process S defined by (4.4) has a characteristic exponent of the form (4.9) with $Q(x) = \int_x^\infty q(s)ds$ where $q(ds) = -dQ(s)$ is the Lévy measure. This is because the measure $q(ds)$ is non-negative and the increments of the process can only be positive making S a subordinator. Also notice that the jumps of S larger than ϵ form a compound Poisson process with jump rate $Q(\epsilon)$ and jump density $\frac{q(dx)}{Q(\epsilon)}$.

In the spirit of Dufresne, Gerber and Shiu (1991) we define the generalized inverse Gaussian Lévy process in terms of a non-negative and non-increasing function Q, which, as we have seen, is the Lévy measure of the process. Barndorff-Nielsen and Shephard (2001) show that the Lévy measure in the Lévy-Khintchine representation of the GIG$(\lambda, \delta, \gamma)$ distribution is

$$q(dx) = \frac{1}{x}\left[\delta^2 \int_0^\infty e^{-xt} g_\lambda(2\delta^2 t)dt + \max\{0, \lambda\}\right] e^{-\gamma^2 x/2} dx \,, \qquad x > 0 \,, \tag{4.10}$$

where

$$g_\lambda(y) = \left\{\frac{\pi^2}{2} y \left[J^2_{|\lambda|}(\sqrt{y}) + N^2_{|\lambda|}(\sqrt{y})\right]\right\}^{-1} \,, \qquad y > 0 \,.$$

J and N are modified Bessel functions.

Since the GIG distribution is infinite divisible and with support on the positive axis we can define a positive Lévy process S_{GIG} described by its characteristic function $\phi_t(s) = e^{t\Psi_{GIG}(s)}$ where Ψ_{GIG} is the characteristic exponent of the GIG distribution, i.e.,

$$\Psi_{GIG}(s) = \int_{\mathbb{R}_0} \left[e^{isx} - 1\right] q(dx) , \qquad s \in \mathbb{R} . \tag{4.11}$$

This last equation (4.11) is of the form (4.4) with $Q(x) = \int_x^\infty q(dt)$. Notice that since S_{GIG} is a subordinator, its Lévy measure satisfies $\int_0^\infty xq(dx) < \infty$. Moreover, $Q(0) = \int_0^\infty q(dx) = \infty$ and the process S_{GIG} is composed of an infinite number of small jumps.

Such a process is a generalization of the gamma and inverse Gaussian process. From the form of its Laplace transform we can see that the GIG distributions are not closed under convolutions. It follows that all increments of length one follow a GIG distribution. However, increments of other lengths follow an infinite divisible distribution that does not belong to the GIG class. This is because, by construction, $\mathbb{E}[e^{izS_{GIG}(t)}] = \phi_t(z) = [\phi_1(z)]^t$.

Nonetheless, we can compute this class of infinite divisible densities using the Fourier inversion formula

$$f_t(x) = \frac{1}{\pi} \int_0^\infty \cos(ux)\phi_t(u)du . \tag{4.12}$$

4.4 Ruin Theory for the GIG Lévy Process

A general risk model based on a GIG aggregate claim process would be

$$U(t) = u + ct - S_{GIG}(t) , \qquad t \geqslant 0 , \tag{4.13}$$

where S_{GIG} is a generalized inverse Gaussian Lévy process, u is the initial surplus, c is a constant premium rate defined as $c = (1+\theta)\mathbb{E}\left[S_{GIG}(1)\right]$ where θ is the security loading factor. For an account on the classical risk model we refer to Grandell (1991).

Chaubey, Garrido and Trudeau (1998) show that the inverse Gaussian distribution provides a good fit for aggregate claims for a wide choice of claim size distributions. The extra parameter of the GIG distribution might make it a more flexible distribution to model aggregate claims. This is yet to be explored. But if this turns out to be the case and aggregate claims are well described by a GIG distribution, then a risk model based on a GIG Lévy motion like (4.13) would be a natural model.

Despite the fact that such an aggregate claim process has an infinite number of small claims in any interval, the fact that it is of finite variation, implies that the claims remain small enough as to assure that the *maximum aggregate loss* is still a compound geometric random variable.

For the process (4.13), the maximum aggregate loss is the random variable defined by

$$L = \sup_{t>0}\{S_{GIG}(t) - ct\} , \tag{4.14}$$

and the ruin probability is

$$\psi(u) = \mathbb{P}\{\inf[t > 0 \mid u + ct - S_{GIG}(t) < 0] < \infty\} , \qquad u \geqslant 0 . \tag{4.15}$$

In the classical compound Poisson model [see Grandell (1991)], L is related to the ruin probability ψ as indicated by the following relation:

$$\psi(u) = \mathbb{P}(L > u) , \qquad u \geqslant 0 , \tag{4.16}$$

i.e., the ruin probability ψ is the tail of the distribution of the maximum aggregate loss. Moreover, L is a compound geometric random variable with parameter $\frac{\theta}{1+\theta}$ and jump distribution given by

$$F(x) = \frac{\int_0^x [1 - G(y)]dy}{\int_0^\infty [1 - G(y)]dy} , \qquad x > 0 ,$$

where G is the claim distribution of the original compound Poisson risk process. This last fact, along with (4.16) implies the following equation for the ruin probability in the classical case

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n [1 - F^{*n}(u)] , \qquad u \geqslant 0 , \tag{4.17}$$

where F^{*n} indicates the n-fold convolution of F [see equation (1.22)].

For the GIG process we cannot talk about a claim size distribution equivalent to G in (4.17). However, all of these relations are preserved if the aggregate claim process is a GIG Lévy process as stated in the following result.

Theorem 4.1 *Let U be the process in (4.13). Also let L and ψ be as in (4.14) and (4.15) respectively. Then, the ruin probability ψ is related to L as in (4.16). Moreover, L is a compound geometric random variable of the form*

$$L = \sum_{i=0}^{M} L_i ,$$

where M is a geometric random variable with parameter $\frac{\theta}{1+\theta}$ and $\{L_i\}_{i=1,2,\ldots}$ are i.i.d. random variables with density

$$m(x) = \frac{Q(x)}{\int_0^\infty sq(s)ds} , \qquad x > 0 .$$

Recall that Q is the tail of the Lévy measure $q(dx)$ of the GIG Lévy process given by (4.10).

Proof. This is a direct consequence of the fact that S_{GIG} is a limit of a sequence of compound Poisson processes as in (4.5).

The converging sequence $\{S_\epsilon\}_{\epsilon>0}$ is defined by its characteristic function

$$\phi_t^{(\epsilon)}(z) = e^{t\int_0^\infty \left[e^{izx}-1\right]dQ_\epsilon(x)} , \qquad z > 0 ,$$

where $q_\epsilon(dx) = -dQ_\epsilon(x) = -\mathbb{I}_{[\epsilon,\infty)}(x)dQ(x)$. Therefore, the claim rate is $\int_0^\infty q_\epsilon(x)dx$ and the claim size density is $\frac{dQ_\epsilon(x)}{\int_0^\infty dQ_\epsilon(s)}$.

For such a compound Poisson process the maximum aggregate loss L_ϵ is a compound geometric r.v. of the form

$$L_\epsilon = \sum_{i=0}^{M} L_i^\epsilon ,$$

and it has the following characteristic function [see Asmussen (2000)]:

$$\phi_\epsilon(z) = \frac{\frac{1}{1+\theta}}{1 - \frac{\theta}{1+\theta}\phi_{L_i^\epsilon}(z)} , \qquad z > 0 , \tag{4.18}$$

where $\phi_{L_i^\epsilon}$ is the characteristic function of the random variable L_i^ϵ with density

$$m_\epsilon(x) = \frac{Q_\epsilon(x)}{\int_0^\infty s q_\epsilon(s) ds} , \qquad x > 0 . \tag{4.19}$$

This is the classical ladder-height decomposition for the classical risk process. Moreover, we have that the sequence of ruin probabilities ψ_ϵ is such that

$$\psi_\epsilon(u) = \mathbb{P}(L_\epsilon > u) , \qquad u > 0 . \tag{4.20}$$

Now, notice that the density (4.19) converges to

$$m(x) = \frac{Q(x)}{\int_0^\infty s q(s) ds} , \qquad x > 0 , \tag{4.21}$$

as $\epsilon \longrightarrow 0$. This is because S_{GIG} is a subordinator and $\int_0^\infty x q(x) dx < \infty$. In consequence, its characteristic function $\phi_{L_i^\epsilon}$ converges to the characteristic function ϕ_{L_i} of the density m. In light of this we have that the characteristic function (4.18) converges to

$$\phi(z) = \frac{\frac{1}{1+\theta}}{1 - \frac{\theta}{1+\theta}\phi_{L_i}(z)} , \qquad z > 0 , \tag{4.22}$$

as $\epsilon \longrightarrow 0$. This last equation is the characteristic function of a compound geometric random variable with rate $\frac{\theta}{1+\theta}$ and with jump distribution given by (4.21). Further, because of equation (4.20), we have that the ruin probability ψ is the tail of the compound geometric distribution described by (4.22). This completes the proof. ∎

Remark 4.1 *The result in Theorem 4.1 is valid for a wider class of Lévy processes. In fact, it is true for any risk model with a subordinator as the aggregate claims process.*

This result allows us to do risk theory with the GIG Lévy process even though is composed by an infinite number of claims. In practice, the fact that the ladder height distribution (4.21) is given in terms of the integral (4.10) might be seen as a setback. However this integral might be computed numerically. Or as Dufresne, Gerber and Shiu (1991) had already pointed out for the gamma process, this property leads to upper and lower bounds for the ruin probability.

4.5 Ruin Probabilities for the GIG Risk Process

Theorem 4.1 shows that the ladder height decomposition is still valid for the GIG risk process. This allows us to derive equations analogous to those in the classical case. For instance, the ruin probability (4.15) satisfies

$$\psi(u) = \frac{\theta}{1+\theta} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta} \right)^n [1 - M^{*(n)}(u)] , \qquad u \geqslant 0 , \tag{4.23}$$

where M is the distribution function of the density m in (4.21). This last equation is the analogous to equation (4.17).

Furthermore, equation (4.23) is of the form (1.22) and therefore ψ satisfies the following renewal equation analogous to (1.24)

$$\psi(u) = \frac{1 - M(u)}{1+\theta} + \frac{1}{1+\theta} \int_0^u \psi(u-y) dM(y) , \qquad u \geqslant 0 . \tag{4.24}$$

Just as in the classical case, the computation of (4.23) and (4.24) is complicated. This situation is aggravated by the form of the distribution M. Recall that

$$m(x) = \frac{Q(x)}{\int_0^\infty t[-dQ(t)]} = \frac{\int_x^\infty q(dt)}{\int_0^\infty tq(t)dt} , \qquad x \geqslant 0 ,$$

where $q(dx)$ is the Lévy measure of the GIG process given by (4.10) and that involves all three modified Bessel functions.

However, we can compute bounds for the ruin probability (4.15) using existing results for compound geometric tails. For instance, Cai and Garrido (1998) give lower and upper bounds for ruin probabilities satisfying equations of the form (4.23). These bounds are given by

$$\frac{\bar{M}(u)}{\theta + \bar{M}(u)} \leqslant \psi(u) \leqslant \frac{\bar{M}(u) + \mathbb{E}(L)M(u)/u}{1 + \theta + \mathbb{E}(L)M(u)/u} , \qquad u \geqslant 0 , \tag{4.25}$$

where $\bar{M}(x) = 1 - M(x)$ and $\mathbb{E}(L)$ is the expected value of the maximum aggregate loss random variable.

We have gone around the problem of computing convolutions of M, but we still have to deal with $\bar{M}(x)$ and $\mathbb{E}(L)$ in (4.25). These two functions are given in terms of the Lévy measure (4.10) as follows:

$$\bar{M}(x) = \frac{\int_x^\infty \int_y^\infty \frac{1}{\xi}\left[\delta^2 \int_0^\infty e^{-\xi t} g_\lambda(2\delta^2 t)dt + \max\{0,\lambda\}\right] e^{-\gamma^2\xi/2} d\xi dy}{\int_0^\infty \left[\delta^2 \int_0^\infty e^{-xt} g_\lambda(2\delta^2 t)dt + \max\{0,\lambda\}\right] e^{-\gamma^2 x/2} dx}\ , \ , \qquad x \geqslant 0 \tag{4.26}$$

and

$$\mathbb{E}(L) = \frac{1}{\theta}\int_0^\infty \bar{M}(x)dx\ , \tag{4.27}$$

where

$$g_\lambda(y) = \left\{\frac{\pi^2}{2} y \left[J^2_{|\lambda|}(\sqrt{y}) + N^2_{|\lambda|}(\sqrt{y})\right]\right\}^{-1}\ , \qquad y > 0\ .$$

Integrals involving modified Bessel functions can be computed numerically. Here we illustrate the particular case $\lambda = 1/2$ and $\lambda = -1/2$. For this choice of the parameter λ, the function g_λ takes a much simpler form thanks to the following relations [see Abramowitz and Stegun (1970)]

$$J_{\frac{1}{2}}(y) = \sqrt{\frac{2}{\pi y}}\sin(y) \qquad \text{and} \qquad N_{\frac{1}{2}}(y) = -\sqrt{\frac{2}{\pi y}}\cos(y)\ , \qquad y > 0\ .$$

This yields

$$g_{\frac{1}{2}}(y) = g_{-\frac{1}{2}}(y) = \left\{\frac{\pi^2}{2} y \left[J^2_{\frac{1}{2}}(\sqrt{y}) + N^2_{\frac{1}{2}}(\sqrt{y})\right]\right\}^{-1} = \frac{1}{\pi\sqrt{y}}\ , \qquad y > 0\ . \tag{4.28}$$

Using (4.28) we get that, for $\lambda = 1/2$, equation (4.26) becomes

$$\begin{aligned}
\bar{M}(x) &= \frac{\int_x^\infty \int_y^\infty \frac{1}{\xi}\left[\delta^2 \int_0^\infty \frac{e^{-\xi t}}{\pi\delta\sqrt{2t}} dt + \frac{1}{2}\right] e^{-\gamma^2\xi/2} d\xi dy}{\int_0^\infty \left[\delta^2 \int_0^\infty \frac{e^{-xt}}{\pi\delta\sqrt{2t}} dt + \frac{1}{2}\right] e^{-\gamma^2 x/2} dx}\ , \qquad x > 0\ , \\
&= \frac{\int_x^\infty \int_y^\infty \frac{1}{\xi}\left[\delta^2 \frac{\Gamma(1/2)}{\pi\delta\sqrt{2\xi}} + \frac{1}{2}\right] e^{-\gamma^2\xi/2} d\xi dy}{\int_0^\infty \left[\delta^2 \frac{\Gamma(1/2)}{\pi\delta\sqrt{2x}} + \frac{1}{2}\right] e^{-\gamma^2 x/2} dx} \\
&= \frac{\int_x^\infty \left[\frac{\delta\Gamma(1/2)}{\pi\sqrt{2}} \int_y^\infty \xi^{-3/2} e^{-\gamma^2\xi/2} d\xi + \frac{1}{2}\int_y^\infty \xi^{-1} e^{-\gamma^2\xi/2} d\xi\right] dy}{\frac{\delta\Gamma(1/2)}{\pi\sqrt{2}} \int_0^\infty x^{-1/2} e^{-\gamma^2 x/2} dx + \frac{1}{2}\int_0^\infty e^{-\gamma^2 x/2} dx} \\
&= \frac{\gamma^2}{\delta\gamma + 1}\int_x^\infty \left[\frac{\delta\Gamma(\frac{1}{2})}{\pi\sqrt{2}} \int_y^\infty \xi^{-\frac{3}{2}} e^{-\frac{\gamma^2}{2}\xi} d\xi + \frac{1}{2}\int_y^\infty \xi^{-1} e^{-\frac{\gamma^2}{2}\xi} d\xi\right] dy\ .
\end{aligned}$$

This last equation comes from the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ which reduces the denominator to the constant $\frac{\delta\gamma+1}{\gamma^2}$. As for the numerator, if we integrate by parts, we can rewrite

$$\begin{aligned} \bar{M}(x) &= \frac{\gamma^2}{\delta\gamma+1}\left\{\int_x^\infty \frac{\delta\Gamma(\frac{1}{2})}{\pi\sqrt{2}}\left[2y^{-\frac{1}{2}}e^{-\frac{\gamma^2}{2}y} - \gamma^2\int_y^\infty \xi^{-\frac{1}{2}}e^{-\frac{\gamma^2}{2}\xi}d\xi\right]dy \right. \\ &\left. + \int_x^\infty \frac{1}{2}\int_y^\infty \xi^{-1}e^{-\frac{\gamma^2}{2}\xi}d\xi dy\right\}, \qquad x \geqslant 0 . \end{aligned} \tag{4.29}$$

The function inside the integral is the ladder height density m. If we change the order of integration in (4.29), it simplifies even further yielding

$$\begin{aligned} \bar{M}(x) &= \frac{\gamma^2}{\delta\gamma+1}\left\{\frac{2\delta\Gamma(\frac{1}{2})}{\pi\gamma}\bar{\Gamma}(x;\frac{1}{2}) \right. \\ &-\frac{\delta\Gamma(\frac{1}{2})}{\pi}\left[\frac{2}{\gamma}\bar{\Gamma}(x;\frac{3}{2}) - \gamma x\bar{\Gamma}(x;\frac{1}{2})\right] \\ &\left. +\frac{1}{2}\left[\frac{2}{\gamma^2}e^{-\frac{\gamma^2}{2}x} - x\Gamma(x;0)\right]\right\}, \end{aligned} \tag{4.30}$$

where $\bar{\Gamma}(u;\alpha) = \int_u^\infty x^{\alpha-1}e^{-x}dx$ is the incomplete gamma function [see Abramowitz and Stegun (1970)].

Consequently, $\mathbb{E}(L)$ is given by

$$\mathbb{E}(L) = \frac{1}{\theta}\int_0^\infty \bar{M}(x)dx .$$

If we substitute $\bar{M}$ by (4.29) and compute the double integrals by changing the order of integration we have that

$$\mathbb{E}(L) = \frac{1}{\theta}\frac{\gamma^2}{\delta\gamma+1}\left\{\frac{2\delta}{\gamma^3} - \frac{3}{8}\frac{\delta\gamma^2}{\sqrt{2}} + \frac{1}{4}\left(\frac{2}{\gamma^2}\right)^2\right\} . \tag{4.31}$$

In this last equality we use the fact that $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$, $\Gamma(\frac{5}{2}) = \frac{3}{4}\Gamma(\frac{1}{2})$ and $\Gamma(2) = 1$.

Substituting (4.30) and (4.31) into the expressions (4.25) yields lower and upper bounds for the ruin probability of the GIG risk process with $\lambda = \frac{1}{2}$.

Bounds for the ruin probability for the inverse Gaussian case ($\lambda = -\frac{1}{2}$) can be derived in the same manner. The middle integral in the numerator and denominator of (4.26) simplifies since $\max\{0, -\frac{1}{2}\} = 0$ implying the following forms for $\bar{M}$ and $\mathbb{E}(L)$:

$$\bar{M}(x) = \frac{\gamma}{\delta}\left\{\frac{2\delta\Gamma(\frac{1}{2})}{\pi\gamma}\bar{\Gamma}(x;\frac{1}{2}) - \frac{\delta\Gamma(\frac{1}{2})}{\pi}\left[\frac{2}{\gamma}\bar{\Gamma}(x;\frac{3}{2}) - \gamma x\bar{\Gamma}(x;\frac{1}{2})\right]\right\}, \quad (4.32)$$

and

$$\mathbb{E}(L) = \frac{1}{\theta}\frac{\gamma}{\delta}\left[\frac{2\delta}{\gamma^3} - \frac{3}{8}\frac{\delta\gamma^2}{\sqrt{2}}\right].$$

Equations (4.30) and (4.32) generalize similar expressions for the gamma and the inverse Gaussian processes in Dufresne, Gerber and Shiu (1991).

4.6 Numerical Results

We carry out the numerical evaluation of the bounds defined by (4.25), (4.30) and (4.31) for the case $\lambda = \frac{1}{2}$. The incomplete gamma functions were computed using well-known analytical expantions [see Abramowitz and Stegun (1970)]. In Table 4.1 we present bounds for the ruin probability of a GIG risk process with parameters $\lambda = \frac{1}{2}$, $\gamma = \frac{1}{10}$ and $\delta = 10\sqrt{2}$ which correspond to an expected aggregate claim value of 158.58. Different values for the safety loading θ are shown.

4.7 Conclusions

We have extended the gamma risk process of Dufresne, Gerber and Shiu (1991) to a wider class of Lévy processes generated by the GIG distributions. We showed that they share the same counterintuitive property of having infinitely many claims, and, in spite of which, they accept a ladder height-like decomposition.

This ladder-height decomposition has been used in the literature [Cai and Garrido (1998)] to produce bounds for the corresponding ruin probability. We implemented these bounds and show that they are given in terms of Bessel functions. These take on a simple form in the case $\lambda = \frac{1}{2}$. Numerical results for this case are provided.

The fact that increments of length one follow a GIG distribution makes it interesting for applications since aggregate claims are better fitted by an inverse Gaussian

Table 4.1: **Upper and lower bounds for ruin probabilities of the GIG risk process for different values of θ. The parameters of the underlying GIG distribution are $\lambda = \frac{1}{2}$, $\gamma = \frac{1}{10}$ and $\delta = 10\sqrt{2}$.**

	θ			
	0.2		0.5	
u	LOWER	UPPER	LOWER	UPPER
10	0.66326	0.98363	0.44067	0.94662
15	0.65770	0.97583	0.43457	0.92315
50	0.61729	0.92653	0.39216	0.79689
100	0.55677	0.86706	0.33443	0.67772
150	0.49452	0.81577	0.28126	0.59248
200	0.43243	0.77028	0.23358	0.52618
250	0.37240	0.72934	0.19182	0.47239
300	0.31606	0.69223	0.15601	0.42769
350	0.26465	0.65842	0.12584	0.38998
400	0.21893	0.62752	0.10081	0.35780
450	0.17918	0.59920	0.08030	0.33012
500	0.14530	0.57319	0.06367	0.30612
550	0.11692	0.54924	0.05030	0.28518

distribution. The extra parameter of the GIG would make a more flexible model for aggregate claims.

These larger class of processes contain the particular cases of the gamma and inverse Gaussian processes explored in Dufresne, Gerber and Shiu (1991). This implies that the bounds we provide here can be used in those particular cases. This is yet a further improvement since, the bounds we implement, have been shown to be tighter than those they use in their paper.

A topic of future research remains the study of the GIG risk process in the context discussed in Gerber and Shiu (1998a) where they studied the discounted penalty function.

Chapter 5

Approximating the Risk Reserve Process Using Extreme Value Theory: With Applications in Reinsurance

5.1 Introduction

A general insurance portfolio consists of several independent contracts issued for a limited time period (usually one year). During this period the company faces claims from policyholders, multiple claims from the same portfolio are possible. If we assume that the risk characteristics of such a portfolio are preserved through different periods then a homogeneous Poisson process describes, in a natural way, the occurrence of claims in this portfolio.

In the classical risk model, the aggregate claims process for such a portfolio is given by

$$S(t) = \sum_{i=1}^{N(t)} Y_i \,, \qquad t \geqslant 0 \,, \tag{5.1}$$

where $\{Y_i\}$ are i.i.d. random variables (with c.d.f. F_Y) representing the claim amounts, $E(Y_i) = \mu$ and N is a homogeneous Poisson process with parameter λ.

The corresponding risk reserve process is

$$U(t) = u + ct - S(t) , \qquad t \geqslant 0 , \tag{5.2}$$

where S is the aggregate claims process defined in (5.1), u is the initial surplus, c is a constant premium rate defined as $c = \lambda\mu(1+\theta)$ and θ is the security loading factor. For an account on the theory we refer to Grandell (1991) or Rolski *et al.* (1999).

Risk theory is concerned with functionals of the reserve process, one that is of particular interest is the associated ultimate ruin probability ψ. This functional is often used as a measure of the riskiness of the portfolio and it can be used as a risk index for reinsurance purposes. The ruin problem in risk theory involves the, possibly defective, random variable

$$T(u) = \inf\{t > 0 : U(t) = u + ct - S(t) < 0\} , \qquad u \geqslant 0 .$$

The main interest lies in evaluating the probability of ruin over a finite or an infinite horizon, this is

$$\psi_t(u) = P\{T(u) \leqslant t\}, \quad 0 < t < \infty; \quad \text{and} \quad \psi(u) = P\{T(u) < \infty\} , \tag{5.3}$$

respectively, where the functions ψ and ψ_t are functions of the initial level u. Formulas for functionals of the probability of ruin have been worked out yielding complicated expressions that are not always easy to evaluate [see Asmussen (2000) for a thorough discussion].

Simulation is one approach used to estimate ruin probabilities [see Vazquez-Abad (2000) and references therein]. Implementing a simulation scheme to estimate the probability (5.3) is not straight-forward. One of the problems is that there is no stopping rule if we simulate the risk process (5.2) as the time-to-ruin random variable may be defective. Some of the simulated paths will never fall below zero because of the net profit condition $\theta > 0$, making a straight-forward simulation inviable. A change of measure or the ladder-height decomposition can be used to define simulation schemes that deal with this problem, however their implementation still depends heavily on the claims distribution F_Y.

Another approach has been to use Lévy processes to approximate the original risk process (5.2) as we discussed in Chapter 1.

In this chapter we use extreme value theory to construct a Lévy motion that approximates the classical risk theory under different underlying claim distributions. The present approach leads naturally to a simulation scheme that allows us to approximate the sequence of classical risk processes converging to either a Brownian or to an α-stable process at any step of the convergence.

We start by decomposing the aggregate claims process (5.1) into two independent sums, one containing the small claims and the other one the large claims as follows

$$\sum_{i=1}^{N(t)} Y_i = \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(0,\epsilon)}(Y_i) + \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(\epsilon,\infty)}(Y_i) , \qquad t \geqslant 0 , \tag{5.4}$$

for any threshold $\epsilon > 0$. Then we approximate the large claims by a generalized Pareto-stable Lévy process (using extreme value theory techniques) and the small claims by a Brownian motion (using the classical approach of Grandell (1977)). The approximating process is then of the form

$$\sum_{i=1}^{N(t)} Y_i \approx \kappa + \eta W(t) + J(t) , \qquad t \geqslant 0 , \tag{5.5}$$

where W is a standard Brownian motion and J is a generalized Pareto-stable Lévy process independent of W. η and κ are constants.

What we call a generalized Pareto-stable Lévy process is a particular type of compound Poisson process. It is a compound sum of the form

$$\sum_{i=1}^{N(t)} Z_i , \qquad t \geqslant 0 ,$$

where N is a Poisson process and $\{Z_i\}_{i=1,2,\ldots}$ are i.i.d. random variables following a generalized Pareto distribution. Approximation (5.5) circumvents some of the difficulties posed by an aggregate process of the form (5.1). We will see that a generalized Pareto distribution is either a Pareto or an exponential distribution. This reduces to some extent the ruin problem. Instead of dealing with a general sum (5.1) where the random variables $\{Y_i\}_{i=1,2,...}$ follow an arbitrary distribution F_Y, we now have to deal with a perturbed model (5.5) where the process J is always the compound sum of Pareto or exponential random variables. For processes of this form we show that there is always a straight-forward simulation approximation that can be used to compute ultimate ruin probabilities, which is not the case for a general process of the form (5.1). Our approximation is then a simulation scheme that allows us to estimate ruin probabilities in the classical case for all kinds of claims distributions. The approximating process (5.5) can be recognized as a perturbed risk model, such models have been extensively studied in the literature: Bounds for the ruin probability can be found in Cai and Garrido (2002), some expressions for the joint distribution prior and at ruin can be found in Wang and Wu (2000).

The decomposition (5.4) and its approximation (5.5) are of particular interest in a reinsurance context, since it gives a way to define a risk measure for excesses-of-loss using the concept of distortion and its relation to relative entropy [see Reesor (2001)]. These concepts have appeared recently in the actuarial literature: Wang (1996), Wang, Young and Panjer (1997), Kamps (1998) and Artzner (1999) follow an axiomatic approach to insurance prices and embed premium principles into more general risk measures. We will present an application of our generalized Pareto-stable process to the pricing of reinsurance layers in this context. We will consider the second sum in (5.4) and its corresponding approximation term in (5.5). We exploit the fact that the generalized Pareto compound Poisson process is a Lévy process and define a distorted risk measure via an Esscher transform. Other distortions are also considered.

Let us briefly recall some facts about Lévy processes discussed in Chapter 2. Lévy processes are in a one-to-one correspondence with the class of infinitely divisible distributions. Their characteristic function $\phi_t(s) = \mathbb{E}\left(e^{isX(t)}\right)$ is of the form $e^{-t\Psi(s)}$ where Ψ is the so-called characteristic exponent in the Lévy-Khintchine characterization.

The compound Poisson process, the Brownian motion and the α-stable motion are well known examples of Lévy processes. If the Lévy measure ν is the null measure then we obtain the Brownian motion defined by its characteristic exponent

$$\Psi(s) = ias + \frac{b^2}{2}s^2 \,, \qquad s \in \mathbb{R} \,.$$

The compound Poisson is obtained by setting $a = \int_{-1}^{1} x\lambda\nu(dx)$ and with $\nu(dx) = \lambda dF(x)$ where the jumps occur at rate λ with law dF, yielding

$$\Psi(s) = \int_{\mathbb{R}_0} \left(1 - e^{isx}\right) \lambda\nu(dx) \,, \qquad s \in \mathbb{R} \,.$$

The α-stable Lévy process has the following characteristic exponent

$$\Psi(s) = \int_{\mathbb{R}_0} \left[1 - e^{isx} + isx\mathbb{I}_{(-1,1)}(x)\right] \nu(dx) \,, \qquad s \in \mathbb{R} \,,$$

with Lévy measure given by $\nu(dx) = c^+ x^{-\alpha-1}dx$ if $x > 0$ and $\nu(dx) = c^-|x|^{-\alpha-1}dx$ if $x < 0$, where c^+ and c^- are such that $\beta = \frac{c^+ - c^-}{c^+ + c^-}$. The process has no positive (negative) jumps when $c^+ = 0$ ($c^- = 0$) or equivalently $\beta = -1$ ($\beta = 1$). It is symmetric when $\beta = 0$ ($c^+ = c^-$). The parameter α is restricted to the interval $(0, 2)$.

These processes lie at different latitudes in the wide spectrum of Lévy processes. For an account of the theory we refer to Sato (1999) or Bertoin (1996) and for recent applications to Barndorff-Nielsen, Mikosh and Resnick (2001).

In Section 5.2 we present some basic notions of extreme value distributions. In Section 5.3 we present the generalized Pareto-stable Lévy motion as a sequence of compound Poisson processes. The problem of simulating the ultimate ruin probability is explored in Section 5.4. Numerical illustrations for the simulation of ruin probabilities are presented in Section 5.5. Finally, in Section 5.6, we briefly discuss basic concepts concerning distorted measures and relative entropy and hint some applications to reinsurance of our approximation.

5.2 Extreme Value Theory

In the following we attempt to give a brief account of extreme value theory (EVT). EVT has been developed in connection with applications in hydrology and climatology and it is only until recently that it has found its way into insurance loss modeling [Beirlant and Teugels (1992) and McNeil (1997)].

Just as the normal distribution arises as a limit of sums of sample averages, the family of extreme value distributions arises as the limit of normalized sums of sample extrema. The family of extreme value distributions is given in the following definition.

Definition 5.1 *The standard generalized extreme value distribution (GEV) is given by*

$$H_\xi(x) = \begin{cases} e^{-(1+\xi x)^{-\frac{1}{\xi}}} & \text{for } \xi \neq 0 \ , \\ \\ e^{-e^{-x}} & \text{for } \xi = 0 \ . \end{cases} \tag{5.6}$$

with $1 + \xi x > 0$.

The parameter ξ is a shape parameter that defines three families. If $\xi > 0$ we have the Fréchet distribution, if $\xi = 0$ we have the Gumbel distribution and if $\xi < 0$ we obtain the Weibull distribution. The generalized extreme value distribution is obtained from the standard GEV distribution by introducing a location and a scale parameter μ and $\sigma > 0$. The GEV distribution is defined as $H_{\xi,\mu,\sigma}(x) = H_\xi(\frac{x-\mu}{\sigma})$.

The following theorem is the basic result in EVT.

Theorem 5.1 (Fisher-Tippet Theorem) *Let $X_1, X_2, \ldots$ be i.i.d. random variables with distribution function F and let M_n be the maximum of the first n observations $M_n = \max\{X_1, \ldots, X_n\}$. If there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$ and a random variable H such that*

$$\frac{M_n - d_n}{c_n} \longrightarrow H \ ,$$

in distribution as $n \to \infty$, then H has distribution function of the form H_ξ for some ξ.

If this condition holds we say that F is in the maximum domain of attraction (MDA) of H_ξ.

Characterizations of the family of distributions that fall in the domain of attraction of GEV distributions have been studied. We can find thorough accounts of the theory in Embrechts, Klüppelberg and Mikosh (1997) and Reiss and Thomas (2001).

Distributions in the MDA of the Fréchet distribution ($\xi > 0$) have heavy tails, among others we find the Pareto, Cauchy, Burr and α-stable distributions.

Distributions in the MDA of the Gumbel distribution ($\xi = 0$) have medium tails. The normal, lognormal, exponential and gamma distributions are some examples.

Distributions in the MDA of the Weibull ($\xi < 0$) are short tailed such as the uniform and the beta distributions.

Of all these families of distributions we focus our concern on the Fréchet and the Gumbel distributions since many of the most commonly used distributions in loss modeling fall in their MDA.

Another distribution that plays an important role in EVT is the generalized Pareto distribution (GPD).

Definition 5.2 *The standard generalized Pareto distribution is given by*

$$G_\xi(x) = \begin{cases} 1-(1+\xi x)^{-\frac{1}{\xi}}\,, & \text{if} \quad \xi \neq 0\,, \\ \\ 1-e^{-x}\,, & \text{if} \quad \xi = 0\,. \end{cases} \tag{5.7}$$

for $x \geqslant 0$ if $\xi \geqslant 0$ and $0 \leqslant x \leqslant -1/\xi$ if $\xi < 0$. By introducing location and scale parameters μ and σ we get the generalized Pareto distribution $G_{\xi,\mu,\sigma}(x) = G_\xi(\frac{x-\mu}{\sigma})$.

Notice that if $\xi > 0$ we have a reparameterized Pareto distribution, a shape parameter of $\xi = 0$ yields the exponential distribution and if $\xi < 0$ we obtain a type II Pareto distribution. We will focus only on the case $\xi \geqslant 0$ since this is the more relevant for insurance loss modeling.

The GPD proves to be important in loss modeling as implied by the following theorem.

Theorem 5.2 (Pickands-Balkema-deHaan Theorem) *Let F be a distribution function with right endpoint x_F and let $F^{[x_0]}$ be its excess distribution function over the threshold x_0 defined as*

$$F^{[x_0]}(x) = P(X - x_0 \leqslant x | X > x_0)\,, \qquad x \geqslant 0\,.$$

Then, for $\xi \in \mathbb{R}$ the following are equivalent:

i) $F \in MDA(H_\xi)$.

ii) There exists a positive, measurable function β *such that*

$$\lim_{x_0 \to x_F} F^{[x_0]}(x_0 + x\beta(x_0)) = G_\xi(x) .$$

This last theorem suggests that the exceedance over a threshold of a certain d.f. $F \in \text{MDA}(H_\xi)$ can be approximated by a GPD with shape parameter ξ. This notion lies at the heart of our approach.

Another concept of importance is that of the domain of attraction (DA) of an α-stable distribution and arises in connection with a general version of the central limit theorem.

Definition 5.3 *Let* $S_n = \sum_{i=1}^{n} X_i$ *where* $\{X_i\}_{i=1,2,\dots}$ *are i.i.d. copies of a random variable* X. *We say that* X *belongs to the domain of attraction of an* α*-stable distribution* S_α*, for* $\alpha \in (0, 2]$*, if there exist constants* $a_n \in \mathbb{R}$*,* $b_n > 0$ *such that*

$$\frac{S_n - a_n}{b_n} \longrightarrow S_\alpha ,$$

in distribution as $n \to \infty$. *We write* $X \in DA(\alpha)$ *and say that* X *satisfies the general central limit theorem with limit* S_α.

Notice that the case $\alpha = 2$ yields the classic central limit theorem since S_2 is the standard normal distribution. For a comprehensive reference on α-stable distributions see Janicki and Weron (1994).

5.3 Generalized Pareto-Stable Lévy Approximation

The general central limit theorem implied in Definition 5.3 lies behind the two existing limiting approximations in risk theory. Consider the following sequence of risk reserve processes

$$U^{(n)}(t) = u^{(n)} + c^{(n)}t - \frac{1}{b_n} \sum_{i=1}^{N(t)} Y_i , \qquad t > 0 , \tag{5.8}$$

where $u^{(n)}$, $c^{(n)}$ are the corresponding sequences of initial reserves and premium rates such that $u^{(n)} \to u$ and $\frac{c^{(n)}-a_n}{b_n} \to c$ where the constants a_n and b_n are those of Definition 5.3. The counting processes $N(t)$ and the random variables $\{Y_i\}_{i=1,2,...}$ are as in (5.1).

Depending on the tail of the claim distribution F_Y, the processes in (5.8) will converge weakly to an α-stable process with index $\alpha \in (1,2)$ or to a Brownian motion with drift. The first of these limiting processes is the α-stable approximation of Furrer, Michna and Weron (1997) and the second is the classic diffusion approximation of Grandell (1977).

Proposition 5.1 *Let $U^{(n)}$ be a sequence of processes as in (5.8).*

i) If F_Y is in the MDA of the Fréchet distribution ($F_Y \in MDA(H_\xi)$ for $1/\xi \in (1,2)$) then (5.8) converges in the Skorokhod topology to

$$U_{\{\alpha\}}(t) = u + ct - \lambda^{\frac{1}{\alpha}} Z_\alpha(t) , \qquad t > 0 , \tag{5.9}$$

as $n \to \infty$, where Z_α is a α-stable Lévy motion.

ii) If F_Y is in the MDA of the Gumbel distribution ($F_Y \in MDA(H_\xi)$ for $\xi = 0$) then (5.8) converges in the Skorokhod topology to

$$U_{\{2\}}(t) = u + ct - \sqrt{\lambda(\mu + \sigma^2)} W(t) , \qquad t > 0 , \tag{5.10}$$

as $n \to \infty$, where $E(Y) = \mu$, $Var(Y) = \sigma^2$ and W is a standard Brownian motion.

Proof.

i) If F_Y is in MDA of the Fréchet distribution then its tail decays as a power function [see Embrechts, Klüppelberg and Mikosh (1997)], i.e.

$$1 - F_Y(y) = y^{-\alpha} L(y) , \qquad y > 0 ,$$

for some slowly variate function L and some $\alpha \in (1,2)$. This characterizes the DA of an α-stable distribution with index $\alpha \in (1,2)$. Therefore $F_Y \in DA(\alpha)$

and we have

$$\frac{\sum_{i=1}^{n}(Y_i) - a_n}{b_n} \longrightarrow S_\alpha \,, \tag{5.11}$$

for centering constants $a_n = nE(Y)$ and $b_n = n^{\frac{1}{\alpha}}L(n)$. Notice that $E(Y) < \infty$ since $\alpha \in (1,2)$.

Condition (5.11) applied to the sequence (5.8) yields the risk reserve processes in Furrer, Michna and Weron (1997). They show that such a process converges to (5.9).

ii) If F_Y is in MDA of the Gumbel distribution then $E(X^k) < \infty$ for all $k > 0$ [see Embrechts, Klüppelberg and Mikosh (1997)]. In particular $Var(Y) = \sigma^2 < \infty$. This implies that $F_Y \in DA(2)$, i.e. is in the DA of the normal distribution and

$$\frac{\sum_{i=1}^{n}(Y_i) - a_n}{b_n} \longrightarrow S_2 \,, \tag{5.12}$$

for centering constants $a_n = nE(Y)$ and $b_n = \sigma\sqrt{n}$.

Condition (5.12) applied to the sequence (5.8) yields the risk reserve processes in Grandell (1977). He shows that such a process converges to (5.10).

■

Both approximations rely on the convergence of a sequence of risk reserves. The restriction of the parameter ξ is needed to ensure the existence of the α-stable Lévy process. As we have seen this limit can be a Brownian motion or an α-stable Lévy process. We will construct an approximating sequence of risk reserve process that can be as close as needed to the limiting process but before we present a brief motivation to our approach.

Let us consider the α-stable Lévy process of Furrer, Michna and Weron (1997) with Lévy measure given by

$$\nu_\alpha(dy) = \lambda \frac{\alpha}{y^{\alpha+1}} \mathbb{I}_{(0,\infty)}(y)dy \,, \qquad y > 0 \,, \alpha \in (1,2) \,. \tag{5.13}$$

Such a model was proven to be a good approximation for a classical risk reserve with heavy-tailed claims distribution. We analyze further the approximation of Furrer, Michna and Weron (1997) by decomposing it into two sums containing the small and large jumps.

Decompositions of a Lévy process in terms of a compound sum of large jumps and a Brownian motion has been suggested recently in Asmussen (2001). Following his model, consider an α-stable Lévy motion with Lévy measure (5.13), which can be approximated by the following process:

$$X_\epsilon(t) = \mu_\epsilon t + \sigma_\epsilon W(t) + N_\epsilon(t) \ , \qquad t \geqslant 0 \ , \tag{5.14}$$

where W is a standard Brownian motion and N_ϵ is the compound Poisson sum of the jumps larger than ϵ

$$N_\epsilon(t) = \sum_{s \leqslant t} \Delta X(s) \mathbb{I}(|\Delta X(s)| \geqslant \epsilon) \ .$$

The constants μ_ϵ and σ_ϵ^2 are given by

$$\mu_\epsilon = -\int_{\epsilon \leqslant |x| \leqslant 1} x \nu_\alpha(dx) \ ,$$

$$\sigma_\epsilon^2 = \int_{|x| \leqslant \epsilon} x^2 \nu_\alpha(dx) \ .$$

Our motivation is found in the process of large jumps N_ϵ. Its Lévy measure is the restriction to a finite interval of the Lévy measure of an α-stable Lévy process and is given by:

$$\nu_\alpha^{(\epsilon)}(dy) = \lambda \frac{\alpha}{y^{\alpha+1}} \mathbb{I}_{(\epsilon,\infty)}(y) dy \ . \tag{5.15}$$

The process N_ϵ defined by such a Lévy measure is nothing but the large jumps of an α-stable process defined by (5.13). The Lévy measure (5.15) defines a compound Poisson process with jump distribution given by

$$F_Y^{(\epsilon,\alpha)}(y) = 1 - \left(\frac{\epsilon}{y}\right)^\alpha \ , \qquad y \geqslant \epsilon \ , \qquad \alpha \in (1,2) \ , \tag{5.16}$$

and Poisson rate given by

$$\lambda_\epsilon^{(\alpha)} = \lambda \left(\frac{1}{\epsilon}\right)^\alpha \ .$$

Notice how the Poisson rate goes to infinity as ϵ approaches zero. This gives an infinite number of small jumps.

We can see that, in a way, the α-stable motion of Furrer, Michna and Weron (1997) always approximates the large claims with the Pareto distribution (5.16), regardless of the underlying claim distribution. Our approach aims to improve this in the sense that the large claims will be approximated by a generalized Pareto distribution instead of the Pareto distribution (5.16). Extreme value theory models exceedances over a threshold ϵ using a generalized Pareto distribution. This yields a generalized Pareto distribution that approximates better the claim distribution for large claims.

Now we can construct our approximation. Let $J = \{J(t)\}_{t\geqslant 0}$ be a Lévy process with characteristics exponent given by

$$\Psi_J(s) = \int_{\mathbb{R}_0} \left[1 - e^{isy} + isy\mathbb{I}_{(-1,1)}(y)\right] \nu_\alpha(dy) \ , \qquad s \in \mathbb{R} \ , \tag{5.17}$$

where the Lévy measure in (5.17) is proportional to the density function of the GPD with location parameter $\beta \geqslant 0$ and shape parameter $\xi = 1/\alpha$, i.e.

$$\nu_\alpha(dy) = \begin{cases} \lambda \frac{\alpha}{(\beta+y)^{\alpha+1}} \mathbb{I}_{(0,\infty)}(y) dy \ , & \text{if} \quad \frac{1}{\xi} = \alpha \in (1,2) \ , \\ \\ \frac{\lambda}{\beta} e^{-\frac{y}{\beta}} \mathbb{I}_{(0,\infty)}(y) dy \ , & \text{if} \quad \xi = 0 \ , \end{cases} \tag{5.18}$$

for some $\lambda > 0$. The process J is a compound Poisson process if $\beta > 0$. If $\beta = 0$ and $\xi = 1/\alpha > 0$ the process J is an α-stable Lévy motion with non-positive jumps as in Furrer, Michna and Weron (1997). If $\alpha = \infty$ the process J is a compound Poisson process with Poisson parameter λ and an exponential jump distribution with mean β.

Notice that (5.17) is the limit, as $\epsilon \to 0$, of the sequence of characteristic exponents

$$\Psi_{J_\epsilon}^{(\alpha)}(s) = \int_{\mathbb{R}_0} \left[1 - e^{isy} + isy\mathbb{I}_{(-1,1)}(y)\right] \nu_\alpha^{(\epsilon)}(dy) \ , \qquad s \in \mathbb{R} \ , \tag{5.19}$$

where the Lévy measure is the restriction of the measure (5.18) to a finite set away from zero i.e.

$$\nu_\alpha^{(\epsilon)}(dy) = \begin{cases} \lambda \frac{\alpha}{(\alpha\beta_1(\epsilon)+y-\epsilon)^{\alpha+1}} \mathbb{I}_{(\epsilon,\infty)}(y) dy \ , & \text{if} \quad \frac{1}{\xi} = \alpha \in (1,2) \ , \\ \\ \frac{\lambda}{\beta_2(\epsilon)} e^{-\frac{y}{\beta_2(\epsilon)}} \mathbb{I}_{(\epsilon,\infty)}(y) dy \ , & \text{if} \quad \xi = 0 \ , \end{cases} \tag{5.20}$$

where β_1 and β_2 are functions such that

$$\beta_1(\epsilon) \longrightarrow \frac{\beta}{\alpha} \, ,$$

$$\beta_2(\epsilon) \longrightarrow \beta \, ,$$

as $\epsilon \to 0$, and

$$\frac{\beta_1(\epsilon)}{\epsilon} \longrightarrow \alpha^{-1} \, ,$$

as $\epsilon \to \infty$. Such functions will be seen to be the scale function β of Theorem 5.2, β_1 will be the scale function needed for a distribution in the MDA of the Fréchet and β_2 the one needed for a distribution in the MDA of the Gumbel distribution.

This sequence of characteristic functions defines, in turn, a sequence of Lévy processes $J_\epsilon^{(\alpha)}$ converging weakly in the Skorokhod topology to J (see Jacod and Shiryaev (1987) for a reference on limits of stochastic processes). This is because the sequence of Lévy measures (5.20) converges to the measure (5.18).

Because of the decomposability property of Lévy processes we have that $\left\{J_\epsilon^{(\alpha)}\right\}_{\epsilon>0}$ is a sequence of compound Poisson processes of the form

$$J_\epsilon^{(\alpha)}(t) = \sum_{i=1}^{N^{(\epsilon,\alpha)}(t)} Y_i^{(\epsilon,\alpha)} \, , \qquad t > 0 \, , \tag{5.21}$$

where $\left\{Y_i^{(\epsilon,\alpha)}\right\}_{i=1,2,\ldots}$ are i.i.d. following a generalized Pareto distribution with location parameter ϵ given by

$$F_Y^{(\epsilon,\alpha)}(y) = \begin{cases} 1 - \left(\frac{\alpha\beta_1(\epsilon)}{\alpha\beta_1(\epsilon)+y-\epsilon}\right)^\alpha , & y \geqslant \epsilon \, , \quad \text{if} \quad \alpha \in (1,2) \, , \\ \\ 1 - e^{-\frac{(y-\epsilon)}{\beta_2(\epsilon)}} \, , & y \geqslant \epsilon \, , \quad \text{if} \quad \alpha = \infty \, . \end{cases} \tag{5.22}$$

The Poisson process $N^{(\epsilon,\alpha)}$ has rate

$$\lambda_\epsilon^{(\alpha)} = \int_{\mathbb{R}_0} \nu_\alpha^{(\epsilon)}(dy) \, . \tag{5.23}$$

If $\alpha = \infty$, the rate $\lambda_\epsilon^{(\alpha)}$ of the sequence of compound Poisson processes (5.21) becomes

$$\lambda_\epsilon^{(\alpha)} = \lambda e^{-\frac{\epsilon}{\beta_2(\epsilon)}} \, . \tag{5.24}$$

On the other hand, if $\alpha \in (1,2)$ equation (5.23) becomes

$$\lambda_\epsilon^{(\alpha)} = \frac{\lambda}{[\alpha\beta_1(\epsilon)]^\alpha} \ . \tag{5.25}$$

Now we are in a position to present our approach. The sequence (5.19), not the limiting process, is the basis for our approximation. Theorem 5.2 implies that, for a compound Poisson sum of r.v. following a distribution F, the sum of excesses over a high enough threshold ϵ is, approximately, a compound Poisson sum of i.i.d. generalized Pareto random variables. We can decompose the aggregate claims process (5.1) into the sum of small and big jumps as follows

$$S(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(0,\epsilon)}(Y_i) + \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(\epsilon,\infty)}(Y_i) \ , \qquad t \geqslant 0 \ ,$$

for any threshold $\epsilon > 0$.

Under considerations from extreme value theory, the second term can be approximated by a compound sum of generalized Pareto r.v. as suggested by the following result:

Proposition 5.2 *Let S be an aggregate claims process as in (5.1). Its aggregate process of claims over a threshold ϵ is defined by*

$$S_\epsilon(t) = \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{\{(\epsilon,\infty)\}}(Y_i) \ , \qquad t \geqslant 0 \ ,$$

where $\{Y_i\}$ are i.i.d. random variables with c.d.f. F_Y such that $\mathbb{E}(Y_i) = \mu$ and N is a homogeneous Poisson process with parameter λ. And let $J_\epsilon^{(\alpha)}$ be the sequence of processes defined in (5.21) and (5.22). If $F_Y \in MDA(H_{\frac{1}{\alpha}})$ then

$$\lim_{\epsilon \to \infty} \left|\mathbb{P}[S_\epsilon(t) \leqslant y] - \mathbb{P}[J_\epsilon^{(\alpha)}(t) \leqslant y]\right| = 0 \ , \qquad t > 0 \ , \tag{5.26}$$

The positive functions β_1 and β_2 in the definition of $J_\epsilon^{(\alpha)}$ are those of Theorem 5.2.

Proof. Notice that the characteristic exponent of the process S_ϵ is of the form

$$\Psi_{S_\epsilon}(s) = \int_{\mathbb{R}_0} \left[1 - e^{isy}\right] \lambda^{[\epsilon]} dF_{Y|Y>\epsilon}(dy) \ , \qquad s \in \mathbb{R} \ , \tag{5.27}$$

where $\lambda^{[\epsilon]}$ and $F_{Y|Y>\epsilon}$ are the claim rate and the conditional claim size distribution given respectively by

$$\lambda^{[\epsilon]} = \lambda\left[1 - F_Y(\epsilon)\right] , \tag{5.28}$$

$$F_{Y|Y>\epsilon}(y) = P\left(Y \leqslant y | Y > \epsilon\right) . \tag{5.29}$$

Since the generalized Pareto-stable process $J_\epsilon^{(\alpha)}$ is the compound Poisson process described in (5.21), its characteristic exponent can be written as

$$\Psi_{J_\epsilon}^{(\alpha)}(s) = \int_{\mathbb{R}_0} \left[1 - e^{isy}\right] \lambda_\epsilon^{(\alpha)} dF_Y^{(\epsilon,\alpha)}(dy) , \qquad s \in \mathbb{R} . \tag{5.30}$$

Now, since $F_Y \in MDA(H_{\frac{1}{\alpha}})$ we have [see Embrechts, Klüppelberg and Mikosh (1997)]

$$\lim_{\epsilon \to \infty} |F_{Y|Y>\epsilon}(y) - F_Y^{(\epsilon,\alpha)}(y)| = 0 . \tag{5.31}$$

This is, the generalized Pareto distribution function $F_Y^{(\epsilon,\alpha)}$, defined in (5.22), converges to the original conditional distribution function of claim sizes $F_{Y|Y>\epsilon}$ as the threshold ϵ becomes large.

Recall that the rate $\lambda^{[\epsilon]}$ in (5.28) is proportional to the tail of F_Y and that $\lambda_\epsilon^{(\alpha)}$, defined in (5.23), is given in terms of the functions β_1 and β_2. Since these functions (β_1 and β_2) are those of Theorem 5.2 we have that

$$\lim_{\epsilon \to \infty} |\lambda^{[\epsilon]} - \lambda_\epsilon^{(\alpha)}| = 0 . \tag{5.32}$$

This is because F_Y belongs to $H_{\frac{1}{\alpha}}$. If it belongs to the domain of attraction of the Fréchet then the tail of F_Y decays as a power function and $\lambda^{[\epsilon]}$ is close enough to (5.25) for a large enough ϵ. If it belongs to the domain of attraction of the Gumbel then the tail of F_Y decays as an exponential function and $\lambda^{[\epsilon]}$ is close enough to (5.24) for a large enough ϵ.

Equations (5.31) and (5.32) along with dominated convergence imply that

$$\lim_{\epsilon \to \infty} |\Psi_{S_\epsilon}(s) - \Psi_{J_\epsilon}^{(\alpha)}(s)| = 0 , \qquad \text{for all } s \in \mathbb{R} .$$

This is, the difference of the characteristic exponents of both processes goes to zero as the threshold ϵ increases. This implies weak convergence of the random variables

$S_\epsilon(1)$ and $J_\epsilon^{(\alpha)}(1)$ which for Lévy processes is equivalent to weak convergence of the processes S_ϵ and $J_\epsilon^{(\alpha)}$ [see Jacod and Shiryaev (1987)]. This implies the result (5.26). ■

Proposition 5.2 means that the one-dimensional distributions of the processes S_ϵ and $J_\epsilon^{(\alpha)}$ are equal for a large enough threshold ϵ. This is what allows us to approximate the original sum of excesses S_ϵ with the sum $J_\epsilon^{(\alpha)}$ of generalized Pareto distributed random variables. In this way we reduce, to some extent, the difficulties associated with the claims size distribution F_Y when computing ruin probabilities.

We have that the sum of claims over a high enough threshold ϵ can be approximated by one of the processes $J_\epsilon^{(\alpha)}$ defined in (5.21) for some high enough ϵ. Consider the aggregate claims process S as defined in (5.1), because is a Lévy process it accepts the following decomposition into two independent Lévy processes:

$$S(t) = \sum_{i=1}^{N(t)} Y_i = \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(0,\epsilon)}(Y_i) + \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(\epsilon,\infty)}(Y_i) , \qquad t \geqslant 0 .$$

By Proposition 5.2, if ϵ is a high enough threshold, the second term can be approximated by $J_\epsilon^{(\alpha)}$ yielding

$$S(t) \approx \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{\{(0,\epsilon)\}}(Y_i) + J_\epsilon^{(\alpha)}(t) , \qquad t \geqslant 0 .$$

We are approximating the large jumps by a generalized Pareto-stable Lévy process thanks to approximations used in extreme value theory. Now we focus on the compound Poisson sum of small jumps of the original aggregate claims process

$$X_\epsilon(t) = \sum_{i=1}^{N(t)} Y_i \mathbb{I}_{(0,\epsilon)}(Y_i) , \qquad t \geqslant 0 . \tag{5.33}$$

the Lévy process X_ϵ has characteristic exponent

$$\Psi_{X_\epsilon}(s) = \int_{\mathbb{R}_0} \left[1 - e^{isy}\right] Q_\epsilon(dy) , \qquad s \in \mathbb{R} ,$$

where the Lévy measure Q_ϵ is given by

$$Q_\epsilon(dy) = \lambda \mathbb{I}_{(0,\epsilon)}(y) dF_Y(y) .$$

Consequently we have that

$$\mathbb{E}[X_\epsilon(t)] = t \int_{\mathbb{R}_0} y Q_\epsilon(dy) ,$$

and

$$\mathbb{V}ar[X_\epsilon(t)] = t \int_{\mathbb{R}_0} y^2 Q_\epsilon(dy) .$$

The sum of small jumps (5.33) is a compound Poisson process with claims following a distribution with finite support. This distribution is given by

$$dF_{Y|Y<\epsilon}(y) = \frac{Q_\epsilon(dy)}{\int_{\mathbb{R}} Q_\epsilon(dx)} = \frac{\mathbb{I}_{\{(0,\epsilon)\}}(y)}{F_Y(\epsilon)} dF_Y(y) , \qquad y > 0 ,$$

which is the conditional distribution $F_{Y|Y<\epsilon}$ given that $Y < \epsilon$ and the claim epochs occur at rate $\lambda_\epsilon = \int_{\mathbb{R}} Q_\epsilon(dx) = \lambda F_Y(\epsilon)$. The variance of X_ϵ is then given by $\lambda_\epsilon(\mu_\epsilon^2 + \sigma_\epsilon^2)t$ where μ_ϵ and σ_ϵ^2 are, respectively, the mean and variance of the conditional distribution $F_{Y|Y<\epsilon}$.

This compound Poisson sum can be approximated via the classical diffusion approximation of Grandell (1977). This is possible because we have separated the large claims. We have the following Lévy approximation of the aggregate claims processes where the sum of small claims has been replaced by a Brownian motion with drift.

$$S(t) \approx GPS(t) = \lambda_\epsilon \mu_\epsilon t + \sqrt{\lambda_\epsilon(\mu_\epsilon^2 + \sigma_\epsilon^2)} W(t) + J_\epsilon^{(\alpha)}(t) , \qquad t \geqslant 0 , \tag{5.34}$$

where $J_\epsilon^{(\alpha)}$ is the EVT compound Poisson approximation of the exceedances and W is a standard Brownian motion.

The accuracy of the approximation (5.34) depends on the choice of ϵ and on the function β_i. One of the goals of EVT is to approximate the tails of distributions in terms of the generalized Pareto distribution. The *mean excess function* defined as

$$e(u) = \mathbb{E}(X - u|X > u) , \qquad u > 0 ,$$

plays a crucial role in optimally choosing the threshold ϵ and the scaling functions β_i. The choice of $\beta(\epsilon)$ and ϵ depends on whether F_Y is in the MDA of the Fréchet distribution or in that of the Gumbel distribution [see Embrechts, Klüppelberg and Mikosh (1997)]. In both cases, the functions β_i have been shown to be proportional to the *mean excess function.* Besides, the mean excess function can also be used to determine the value ϵ. A standard technique in extreme value theory is to plot the mean excess function of the distribution we want to approximate. A good choice for a threshold value is then the point at which this mean excess function becomes linear. Recall from the literature [see Embrechts, Klüppelberg and Mikosh (1997) for instance] that the mean excess function of a generalized Pareto distribution is linear in u.

Notice that such an approximation based on a generalized Pareto-stable distribution is sensitive to the tails of the underlying claim distribution. If F_Y is in the MDA of the Fréchet ($\alpha \in (1,2)$) then the process $J_\epsilon^{(\alpha)}$ behaves like the large jumps of the α-stable approximation of Furrer, Michna and Weron (1997). Our approximation differs from theirs in the small jumps X_ϵ, in Furrer, Michna and Weron (1997) the small jumps are approximated by an α-stable Lévy process whereas in our approach we deal separately with the small jumps using a Brownian motion.

If F_Y belongs to the MDA of the Gumbel distribution then the process X_ϵ is the diffusion approximation of Grandell (1977) for the claims under the threshold ϵ. The difference lies in how large claims are handled. In Grandell (1977) the influence of large claims is also modeled by a Brownian motion. Our approach deals with the large claims separately using a compound Poisson process $J_\epsilon^{(\infty)}$.

5.4 Ruin Probabilities for the GPS Lévy Process

The generalized Pareto-stable approximation (5.34) can be recognized to be a perturbed aggregate claims process in the spirit of Dufresne and Gerber (1991). For such a model, the ultimate ruin follows a convolution formula which leads naturally to a simulation scheme. Such simulation scheme can always be implemented regardless of the original claims distribution F_Y. This is because through the GPS approximation we substitute the original compound sum by a compound sum of generalized Pareto random variables. Generalized Pareto random variables are either Pareto or exponential, for which our simulation scheme is easy to implement.

Consider a risk reserve process where the aggregate claims process has been replaced by the GPS approximation defined in (5.34) yielding

$$U_\epsilon(t) = u + ct - \left[\lambda_\epsilon\mu_\epsilon t + \sqrt{\lambda_\epsilon(\mu_\epsilon^2 + \sigma_\epsilon^2)}W(t) + J_\epsilon^{(\alpha)}(t)\right] , \quad t \geqslant 0 . \tag{5.35}$$

Recall that u is the initial surplus, c is a constant premium rate defined as $c = \lambda\mu(1+\theta)$ where θ is the security loading factor, λ is the rate of occurrences of the claims and μ is the mean of claims severities as defined in (5.2). Now, the drift $\lambda_\epsilon\mu_\epsilon$ is the expectation of the compound sum of small claims thus if we decompose c in terms of $\lambda_\epsilon\mu_\epsilon$ as follows:

$$\begin{aligned} c = \lambda\mu(1+\theta) &= \left[\lambda\int_0^\epsilon y dF_Y(y) + \lambda\int_\epsilon^\infty y dF_Y(y)\right](1+\theta) \\ &= \left[\lambda_\epsilon\mu_\epsilon + \lambda\int_\epsilon^\infty y dF_Y(y)\right](1+\theta) . \end{aligned} \tag{5.36}$$

Substituting equation (5.36) into the risk reserve process (5.35) we have

$$U_\epsilon(t) = u + c^* t - \left[\sqrt{\lambda_\epsilon(\mu_\epsilon^2 + \sigma_\epsilon^2)}W(t) + J_\epsilon^{(\alpha)}(t)\right] , \quad t \geqslant 0 ,$$

where

$$c^* = \lambda\int_\epsilon^\infty y dF_Y(y)(1+\theta) + \lambda_\epsilon\mu_\epsilon\theta .$$

Notice that if we choose the threshold ϵ high enough then $\lambda\int_\epsilon^\infty y dF_Y(y)$ is approximately the expectation of the process $J_\epsilon^{(\alpha)}$ and c^* can be approximated by

$$c^{(\alpha)} = \lambda_\epsilon^{(\alpha)}\mu_\epsilon^{(\alpha)}(1+\theta_2) ,$$

where

$$\theta_2 = \theta\left(1 + \frac{\lambda_\epsilon\mu_\epsilon}{\lambda_\epsilon^{(\alpha)}\mu_\epsilon^{(\alpha)}}\right) . \tag{5.37}$$

$\lambda_\epsilon^{(\alpha)}$ is the rate of the jumps of $J_\epsilon^{(\alpha)}$ as defined in (5.23) and $\mu_\epsilon^{(\alpha)}$ is the mean of the distribution of the jumps of $J_\epsilon^{(\alpha)}$ (this distribution is the generalized Pareto-stable distribution given in (5.22)).

Thus we have the following generalized Pareto-stable approximation to the classical risk reserve (5.2)

$$U_{GPS}(t) = u + c^{(\alpha)}t - \left[\sqrt{\lambda_\epsilon(\mu_\epsilon^2 + \sigma_\epsilon^2)}W(t) + J_\epsilon^{(\alpha)}(t)\right] , \quad t \geqslant 0 . \tag{5.38}$$

where u is the initial surplus.

Since $J_\epsilon^{(\alpha)}$ is a compound Poisson process, the approximation can be seen as a classic risk reserve process with $J_\epsilon^{(\alpha)}$ as aggregate claims process and perturbed by a Brownian motion $\sqrt{\lambda_\epsilon(\mu_\epsilon^2+\sigma_\epsilon^2)}W$. Recall that, because of the way we have defined $J_\epsilon^{(\alpha)}$, we have that $\lambda_\epsilon^{(\alpha)}\mu_\epsilon^{(\alpha)}$ is the mean of the process $J_\epsilon^{(\alpha)}$.

Consequently, we have the following approximation for the corresponding ruin probability

$$\begin{aligned}\psi(u) &= \mathbb{P}\left[\inf\{t>0 \mid u+ct-S(t)<0\}<\infty\right] \\ &\approx \mathbb{P}\left[\inf\left\{t>0 \mid u+c^{(\alpha)}t-\sqrt{\lambda_\epsilon(\mu_\epsilon^2+\sigma_\epsilon^2)}W(t)-J_\epsilon^{(\alpha)}(t)<0\right\}<\infty\right] \\ &= \psi_{GPS}(u)\,.\end{aligned}$$

Dufresne and Gerber (1991) found an expression for the ultimate ruin probability for a perturbed model of the form (5.38). The ultimate ruin probability is given by (1.16) which in this context yields

$$1-\psi_{GPS}(u)=\sum_{k=0}^{\infty}\theta_2(1-\theta_2)^k H_1^{*(k+1)} * H_2^{*k}(u)\,, \qquad u\geqslant 0\,, \tag{5.39}$$

where θ_2 is the security loading of the approximating process as defined in (5.37), H_1 is an exponential distribution with parameter $\zeta=\frac{2c^{(\alpha)}}{D}$ ($c^{(\alpha)}$ is the premium rate and D is the infinitiesimal variance of W) and H_2 is the so-called *ladder* distribution. H_1 and H_2 are the distributions of two independent random variables. In this case they are given by

$$H_1(x)=1-e^{-\zeta x}\,, \qquad x>0\,,$$

where

$$\zeta=\frac{2c^{(\alpha)}}{\lambda_\epsilon(\mu_\epsilon^2+\sigma_\epsilon^2)}\,,$$

and

$$H_2(x)=\frac{1-F_Y^{(\epsilon,\alpha)}(x)}{\int_\epsilon^\infty\left[1-F_Y^{(\epsilon,\alpha)}(s)\right]ds}\,, \qquad x>0\,. \tag{5.40}$$

This implies that the ultimate ruin probability $\psi_{GPS}(u)$ is the tail of the distribution of a compound geometric random variable L, i.e.

$$\psi_{GPS}(u)=\mathbb{P}(L>u)\,, \qquad u\geqslant 0\,,$$

where the random variable L is sum of the form

$$L = L_0^{(1)} + \sum_{i=1}^{M} \left[L_i^{(1)} + L_i^{(2)} \right] \ , \tag{5.41}$$

where M is a geometric random variable with parameter θ, $\left\{ L_i^{(1)} \right\}_{i=0,1,\ldots}$ are i.i.d. random variables with distribution H_1 and $\left\{ L_i^{(2)} \right\}_{i=1,2,\ldots}$ are i.i.d. random variables with distribution H_2. M, $L_i^{(1)}$ and $L_i^{(2)}$ are independent.

This allows a straight forward simulation algorithm for the ultimate ruin probability of our approximating GPS risk reserve. It is sufficient to simulate K copies of the random variable L as defined in (5.41) and then compute the estimate for the tail of its distribution

$$\psi_{GPS}(u) = \mathbb{P}(L > u) = \mathbb{E}\left[\mathbb{I}_{\{L>u\}} \right] \approx \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}_{\{L^k > u\}} \ , \tag{5.42}$$

where L^k are the simulated copies of L.

Notice that each simulated random variable L^k is a function h of the random variables in the decomposition (5.41)

$$L^k = h \left(M_k, L_0^{(1)k}, \left\{ L_i^{(1)k}, L_i^{(2)k} \right\}_{i=1,2,\ldots,M_k} \right) \ .$$

In order to simulate L^k we need to simulate these random variables first. This poses no problem since M and $L_i^{(1)}$ are geometric and exponential random variables, respectively. As for the copies of $L_i^{(2)}$, the distribution $F_Y^{(\epsilon,\alpha)}$ is a generalized Pareto distribution, therefore it is either a Pareto or an exponential distribution, as given in (5.22). The transformation defined in (5.40) for a generalized Pareto distribution stays in the generalized Pareto family. This is, H_2 is also an exponential or a Pareto, which are easily simulated. Implementing a simulation for the original process is not always plausible, it depends on the underlying claim size distribution F_Y. Our approximation circumvents this inconvenient since a simulation algorithm is always available.

The particular form (5.41) of the ruin probability for the GPS allows for the implementation of well-known bounds for compound geometric sums. Cai and Garrido

(2002) have established bounds for the ruin probability of a perturbed ruin model with heavy tailed claims distribution as in (5.38).

In the last section we present some numerical results for some particular cases of claim distributions.

5.5 Numerical Results

We present numerical results for two risk reserve processes, one in each domain of attraction. These two examples serve only as illustrations since, under extreme value theory considerations, they are approximated by distributions in their same class. However, the same exercise can be done with any other claims size distribution belonging to the domain of attraction of the Gumbel or the Fréchet.

Table 5.1: **Comparison of $\Psi(u)$ for a classical risk process (λ=1, $\theta = 0.05$ and Pareto claims) with $\Psi_{GPS}(u)$, its corresponding GPS approximation (100, 000 simulations).**

u	α	Ψ(u)		ΨGPS(u)		ε
		lower	upper			
50	1.2	0.888	0.899	0.9	(+-0.002)	30
100	1.2	0.874	0.885	0.888	(+-0.002)	30
250	1.2	0.852	0.866	0.869	(+-0.002)	30
1000	1.2	0.814	0.830	0.835	(+-0.002)	30
50	1.5	0.666	0.752	0.725	(+-0.003)	30
100	1.5	0.585	0.682	0.66	(+-0.003)	30
250	1.5	0.472	0.576	0.553	(+-0.003)	30
1000	1.5	0.309	0.392	0.374	(+-0.003)	30
50	1.8	0.333	0.534	0.476	(+-0.003)	20
100	1.8	0.223	0.393	0.345	(+-0.003)	20
250	1.8	0.122	0.219	0.188	(+-0.002)	20
1000	1.8	0.044	0.066	0.055	(+-0.001)	20

The first risk process has Pareto claims with distribution function

$$F_Y(y) = 1 - \left(\frac{\beta}{\beta + y}\right)^{\alpha} , \qquad y > 0 ,$$

with $\beta = 0.5$, a safety loading factor $\theta = 0.05$ and claims occurrence rate $\lambda = 1$. We chose the scaling function to be

$$\beta_1(\epsilon) = \frac{\beta + \epsilon}{\alpha} ,$$

which is proportional to the mean excess function of the Pareto distribution.

Table 5.1 shows results for different parameter values of β and α. We compare the ultimate ruin probability of the classical risk reserve as given by (5.2) with our estimate of ultimate ruin probability given by (5.34). Since there is no closed form for this process in the Pareto case, we present bounds for the ruin probability. Such bounds have been calculated following Cai and Garrido (2002). We also include the threshold ϵ used for the GPS approximation. We performed 100,000 independent replications.

Table 5.2: **Comparison of $\Psi(u)$ for a classical risk process (λ=1, $\theta = 0.5$ and exponential claims) with $\Psi_{GPS}(u)$, its corresponding GPS approximation (100, 000 simulations).**

u	β	Ψ(u)	ΨGPS(u)		ε	Relative error
10	2	0.1259	0.1227	(+-0.002)	4	-2.56%
15	2	0.0547	0.0541	(+-0.002)	4	-1.13%
20	2	0.0238	0.0234	(+-0.001)	4	-1.68%
10	3	0.2195	0.2238	(+-0.003)	6	1.98%
15	3	0.1259	0.1227	(+-0.002)	6	-2.56%
20	3	0.0723	0.0715	(+-0.002)	6	-1.04%
10	4	0.2897	0.2786	(+-0.003)	8	-3.84%
15	4	0.1910	0.1931	(+-0.002)	8	1.10%
20	4	0.1259	0.1227	(+-0.002)	8	-2.56%

The second risk reserve process to be compared has exponential claims with distribution function given by

$$F_Y(y) = 1 - e^{-\frac{y}{\beta}} , \qquad y > 0 ,$$

a safety loading factor $\theta = 0.5$ and claim occurrence rate $\lambda = 1$. We chose the scaling function to be

$$\beta_2(\epsilon) = \beta \ ,$$

which is the mean excess function of the exponential distribution. The true value reported is the theoretical value which in the exponential case is known to be

$$\psi(u) = \frac{1}{1+\theta} e^{-\frac{\theta u}{(1+\theta)\beta}} \ , \qquad u \geqslant 0 \ .$$

Table 5.2 shows results for different parameter values of β. We compare the ultimate ruin probability of the classical risk reserve as given by (5.2) with our estimate of ultimate ruin probability given by (5.34). We also include the relative error of our approximation with respect to the true value and the threshold ϵ. We performed 100,000 independent replications.

These two distributions are representative of the families defined by the domain of attraction of the Fréchet and the Gumbel distribution. The approximation lies within an acceptable 5% of the true value or within the theoretical bounds. Further testing reveals that the accuracy of the approximation strongly depends on the threshold ϵ. For a heavy tailed distribution the threshold has to be quite high to yield a decent approximation whereas for the medium tailed distributions a somehow lower value does it.

5.6 Pricing of Reinsurance Premiums

The problem of calculating insurance premiums is an important topic in the actuarial literature [see for instance Wang (2000) and references therein]. Given a random loss X we want to define a premium principle ρ that, in turn, would define a suitable loaded premium $\rho(X)$ for the random loss X. Among other properties, such principle should meet the net profit condition $\rho(X) > \mathbb{E}(X)$. Wang (1996) discusses insurance premiums following an axiomatic approach. He defines suitable properties that a premium principle should posses and then search for insurance premiums satisfying the stated requirements. This leads to the concept of *coherent insurance premium* or coherent risk measure in a broader sense [see Artzner (1999) and Artzner *et al.* (1999)]. We use the following definition from Reesor (2001):

Definition 5.4 *An insurance premium ρ is said to be coherent if for any two random losses X and Y we have*

i) Monotonicity: If $\mathbb{P}(X \leqslant Y) = 1$, then $\rho(X) \leqslant \rho(Y)$.

ii) Positive Homogeneity: For all $\kappa \geqslant 0$, $\rho(\kappa X) = \kappa\rho(X)$.

iii) Translation Invariance: For all $\kappa \in \mathbb{R}$, $\rho(X + \kappa) = \rho(X) + \kappa$.

iv) Subadditivity: $\rho(X + Y) \leqslant \rho(X) + \rho(Y)$.

At this point, we would like to acknowledge that there is not consensus yet in the actuarial literature on this axiomatic construction of risk measures. Subaditivity, for instance, is highly contested by some researchers [see discusion to Artzner (1999) for instance]. Without any further discussion on the ongoing debate regarding coherent measures, we adopt, for illustration purposes, the axioms as stated in Definition 5.4.

Notice that if ρ is coherent then it satisfies the net profit condition $\rho(X) \geqslant \mathbb{E}(X)$. A natural way of loading a net premium is to transform the underlying probability measure of the random loss X so that the new expectation under the new measure would act as a coherent premium for the risk. It turns out that if we search for the *closest* (in some sense) probability measure $\mathbb{P}^{\vartheta}$ to the original measure $\mathbb{P}$, we obtain a very tractable transformation that serves our purposes. Such a measure $\mathbb{P}^{\vartheta}$ is the so-called *minimum relative entropy* measure and its closedness to the original measure $\mathbb{P}$ is given in terms of the relative entropy distance [see Reesor (2001) for a thorough discussion].

Definition 5.5 *A minimum relative entropy probability measure is the solution to the following optimization problem:*

$$\min_{\mathbb{P}^*} \mathbb{H}(\mathbb{P}^*, \mathbb{P}) \,, \tag{5.43}$$

subject to the constraints

$$\mathbb{E}[G_i(X)] = c_i \ , \qquad i = 1, 2, \ldots, M \ ,$$

$$\int d\mathbb{P}^* = 1 \ ,$$

and $\mathbb{P}^*$ *is absolutly continuous with respect to* $\mathbb{P}$ *(*$\mathbb{P}^* \ll \mathbb{P}$*). The distance* $\mathbb{H}(\cdot,\cdot)$ *is the relative entropy distance [see Reesor (2001)].*

Without further discussion on relative entropy we state the following result that gives the form of this kind of measures.

Theorem 5.3 *If there is a measure which is the solution of (5.43), then it is defined by its Radon-Nikodym derivative with respect to the original measure* $\mathbb{P}$ *as follows:*

$$\frac{d\mathbb{P}^{\vartheta}}{d\mathbb{P}} = \exp\left[\sum_{i=1}^{M} \vartheta_i G_i(X) - \phi(\vartheta)\right] \ , \tag{5.44}$$

where $\{G_i\}_{i=1,2,\ldots,M}$ *are the constraints in (5.43),* ϑ *is a vector of parameters and* ϕ *is defined as*

$$e^{\phi(\vartheta)} = \mathbb{E}_{\mathbb{P}}\left\{\exp\left[\sum_{i=1}^{M} \vartheta_i G_i(X)\right]\right\} \ ,$$

provided that $\mathbb{E}_{\mathbb{P}}\left\{\exp\left[\sum_{i=1}^{M} \vartheta G_i(X)\right]\right\} < \infty$.

Reesor (2001) establishes the relation between relative entropy and the distortion approach of Wang, Young and Panjer (1997). In either one of these contexts, there exist conditions that assure the existence of a coherent insurance premium in terms of a Choquet integral. We will not elaborate further on the Choquet integral and its role in definig coherent insurance premiums. Instead we state a result that links relative entropy, distortion and coherent risk measures and that will be the basis for our application. But before we just define the concept of distortion as found in Wang (1996).

Definition 5.6 *A distortion function* g *is any non-decreasing function on* $[0, 1]$ *such that* $g(0) = 0$ *and* $g(1) = 1$.

In terms of such a distortion function we can distort a probability distribution for premium calculation purposes as follows: Let F be the distribution function of the random loss X. In terms of the distortion function g we can define a new distribution function $F^*(x) = 1-g[1-F(x)]$. As we have mentioned, a probability transformation via a distortion function g and a probability transformation via relative entropy are equivalent.

Theorem 5.4 *A minimum relative entropy measure $\mathbb{P}^{\vartheta}$ defines a distortion function g (and vice versa) in the following way:*

$$g'(x) = \exp\left[\sum_{i=1}^{M} \vartheta_i G_i(x) - \phi(\vartheta)\right] ,$$

where G_i and ϕ are as in Definition 5.5. Moreover, if g is a concave function then $\mathbb{E}_{\mathbb{P}^{\vartheta}}$ defines a coherent insurance premium.

Now we can use the generalized Pareto-stable Lévy approximation to price a reinsurance layer above a retention level ϵ. Because of Proposition 5.2, the GPS process $J_\epsilon^{(\alpha)}$ can be used to model claims in excess of a threshold ϵ. Since $J_\epsilon^{(\alpha)}$ is a Lévy process we can focus on the random variable $J_\epsilon^{(\alpha)}(1)$. We would like to find an insurance premium for this risk using a probability transformation that satisfies an optimization problem like in (5.43). By doing so, we circumvent the problem of working an insurance premium for a process based on the original claims. We use extreme value theory to approximate the exceedances over a retention ϵ by a generalized Pareto-stable Lévy process eliminating the case-by-case dependance of the classical process.

We choose the following setting as an illustration:

$$\min_{\mathbb{P}^*} \mathbb{H}(\mathbb{P}^*, \mathbb{P}) ,$$

subject to the constraint

$$\mathbb{E}_{\mathbb{P}}[J_\epsilon^{(\alpha)}(1)] = C ,$$
$$\int d\mathbb{P}^* = 1 ,$$

and $\mathbb{P}^*$ is absoluty continuous with respect to $\mathbb{P}$ ($\mathbb{P}^* \ll \mathbb{P}$). Notice that we are forcing the mean $E_{\mathbb{P}}[J_\epsilon^{(\alpha)}(1)]$ to equal a certain value C. In this setting the function G is the identity function.

By Theorem 5.3 we have that such a transformation exists and is given by its Radon-Nikodym derivative with respect to the original measure $\mathbb{P}$ as follows:

$$\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}} = \exp\left[\vartheta J_\epsilon^{(\alpha)}(1) - \phi(\vartheta)\right] \ , \qquad \vartheta \in \mathbb{R} \, , \tag{5.45}$$

where ϕ is defined by $e^{\phi(\vartheta)} = \mathbb{E}\left[\exp\left(\vartheta J_\epsilon^{(\alpha)}(1)\right)\right]$. The measure transformation defined by (5.45) can be recognized to be an Esscher transform of the original measure [see Raible (2000) for a discussion on Esscher transforms for Lévy processes].

Since the GPS Lévy process is the compound Poisson process defined in (5.21), we have that the density process $\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}$ in (5.45) is given by

$$\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}} = \exp\left[\vartheta J_\epsilon^{(\alpha)}(1) - \int_0^\infty \left[e^{\vartheta y} - 1\right] \lambda_\epsilon^{(\alpha)} dF_Y^{(\epsilon,\alpha)}(y)\right] \ , \qquad \vartheta \in \mathcal{D} \, , \tag{5.46}$$

where $\mathcal{D}$ is a neighborhood of $\mathbb{R}$ containing zero.

Now, this density process is well defined if $\mathbb{E}_{\mathbb{P}}[\exp(\vartheta J_\epsilon^{(\alpha)}(1))] < \infty$. Recall that the jump size distribution $F_Y^{(\epsilon,\alpha)}$ for a generalized Pareto process can be either an exponential or a Pareto distribution. If it is a Pareto distribution, then the moment generating function $\mathbb{E}_{\mathbb{P}}[\exp(\vartheta J_\epsilon^{(\alpha)}(1))]$ does not exist for a neighborhood $\mathcal{D}$ of zero and the density process $\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}$ is not defined. In light of this, we focus first on the case where the original claims size distribution belongs to the domain of attraction of the Gumbel distribution so that $F_Y^{(\epsilon,\alpha)}$ is an exponential. The case of the Fréchet domain of attraction is dealt with in a different fashion that avoids the use of the Esscher transform.

If $F_Y^{(\epsilon,\alpha)}$ is an exponential distribution the density process in (5.46) becomes

$$\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}} = \exp\left\{\vartheta J_\epsilon^{(\alpha)}(1) - \lambda_\epsilon^{(\alpha)}\left[\frac{1}{1-\beta_2(\epsilon)\vartheta} - 1\right]\right\} \ , \qquad \vartheta < \frac{1}{\beta_2(\epsilon)} \, .$$

By Theorem 5.4 we have that if $\vartheta < 0$ then, the new measure constructed via $\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}$ defines a coherent insurance premium for the reinsurance layer modeled by $J_\epsilon^{(\alpha)}(1)$ in the following way: $\rho(J_\epsilon^{(\alpha)}(1)) = \mathbb{E}_{\mathbb{P}^\vartheta}[J_\epsilon^{(\alpha)}(1)]$. Recall that such transformation can be seen as induced by a distortion function g defined through Theorem 5.4.

This is only one distortion or probability tranformation that leads to a coherent insurance premium for the reinsurance layer $J_\epsilon^{(\alpha)}(1)$. Wang (1996) discusses other choices for the distortion function g that lead to suitable coherent risk measures.

In order to deal with the case when the original claims distribution belongs to the Fréchet domain of attraction, we can use, for instance, the proportional hazards distortion of Wang (1995). If we let $g(x) = x^a$, for $a \leqslant 1$, we have a concave distortion function that implies a probability measure transformation. Because of Theorem 5.4 we have that

$$\begin{aligned} g'(X) &= aX^{a-1} \\ &= \exp\left[(a-1)\ln(X) + \ln(a)\right] \\ &= \exp\left[\vartheta G(X) - \phi(\vartheta)\right] \ , \end{aligned}$$

where $G(x) = -\ln(x)$, $\vartheta = 1 - a$ and $\phi(\vartheta) = \ln\left(\frac{1}{1-\vartheta}\right)$.

We have that the last equation is the density process $\frac{d\mathbb{P}^\vartheta}{d\mathbb{P}}$ leading to a transform measure $\mathbb{P}^\vartheta$. Such a probability measure is a minimum relative entropy measure as in (5.43) subject to the constraint $\mathbb{E}_{\mathbb{P}}[-\ln(J_\epsilon^{(\alpha)}(1))] = C$. This distortion leads to a proportional transformation of the hazard rate of the random loss, this is where the name proportional hazards come from. The effect of such transformation is to inflate the fatness of the tail of the loss distribution. From Theorem 5.4, we have that the expectation $\mathbb{E}_{\mathbb{P}^\vartheta}$ under this transformed measure defines a coherent insurance premium for the reinsurance layer $J_\epsilon^{(\alpha)}(1)$ when the claims distribution belongs to the Fréchet domain of attraction.

In this section we have described how we can define loaded insurance premiums for a reinsurance layer with retention limit ϵ. The use of the generalized Pareto-stable approximation simplifies the case-by-case dependance that using the classical model would bring otherwise. The distortions we presented are only illustrations, varying the constraints in the optimization problem (5.43) or using different distortion functions g bring about a wide range of suitable insurance premiums.

5.7 Conclusions

Motivated by the adequacy of α-stable Lévy motion to approximate risk reserve with heavy tailed claims we construct a new approximation in terms of a generalized Pareto-stable Lévy process. We discuss how the α-stable Lévy reserve of Furrer, Michna and Weron (1997) approximates the large claims with a Pareto distribution. This leads to the idea of splitting the claims of the aggregate claim process into large and small claims and treat them separately.

We use the EVT classification of distributions in terms of domains of attraction to approximate the large claims. The recent work on approximations of small jumps of Lévy processes suggest the use of a Brownian motion to model the small claims. Our approach is then the Lévy process resulting from the independent sum of a compound Poisson process and a Brownian motion. This leads naturally to a simulation scheme that allows us to estimate the ultimate ruin probability of the approximating process.

This represents a universal simulation approximation for the ultimate ruin probability in the classical model. The GPS approach can be implemented for a classical risk process with any claims distribution belonging to either the Fréchet or the Gumbel domain of attraction. This is because we have reduced the problem of simulating a risk process with an arbitrary claims distribution F_Y to simulating a risk process with a Pareto or an exponential claims distribution.

The generalized Pareto-stable approximation can be used to define a loaded premium for a reinsurance layer above a retention level ϵ. We discuss how using the concept of coherent risk measure and its links to relative entropy we can transform the original probability measure to obtain a coherent premium principle. By using the generalized Pareto-stable Lévy process instead of the original process, we can define a premium principle that holds for all kinds of light and heavy tailed claims distributions.

In the numerical results we present empirical confirmation of the accuracy and applicability of our approach. Figures 5.1 and 5.2 show that, once a high enough threshold ϵ is chosen, the proposed approximation yields estimates for the ultimate ruin probability that lie within 5% of the true value or, at least, within well-established theoretical bounds.

Further analysis should be made on the role of the threshold ϵ. The applicability to the GPS approximation of bounds for the ruin probability in the perturbed model should be studied further. The theoretical bounds presented in Cai and Garrido (2002) are of a very general form. And therefore, they can be applied to, both, a classical risk model as well as to a perturbed model. This might provide grounds to compare the performance of the GPS approximation with respect to the classical model and the influence of the threshold ϵ.

Chapter 6

On a Periodic Risk Reserve Process: A Simulation Approach

6.1 Introduction

The problem of modeling claims occurring in periodic random environments is discussed in this chapter. In the classical approach of risk theory, the occurrence of claims is modeled by counting processes which do not account for claims following a periodic pattern. We discuss how the use of the classical approach to model a periodic portfolio might lead to the miscalculation of important risk indices, namely the associated ruin probability.

In previous chapters we worked with a general risk model

$$U(t) = u + ct - Z(t) , \qquad t > 0 , \tag{6.1}$$

where Z is a Lévy process. In order to deal with periodicity we have to allow Z to belong to a larger family. We let Z be a non-homogeneous compound Poisson process. Such processes belong to the family of processes with independent increments.

From the theory of semimartingales with independent increments we have that the process

$$\frac{e^{\alpha[Z(t)-ct]}}{e^{\Psi_t(\alpha)}} ,$$

is a martingale, where Ψ_t is the Laplace exponent (or Lévy-Khintchine representation) of $Z(t) - ct$. This leads to extensions of well-known results to the periodic model [see Asmussen and Rolski (1994)] and, in our approach, allows us to implement a straight forward simulation scheme.

We present a periodic model, in terms of non-homogeneous Poisson processes, that has potential practical applications. We base our discussion on some properties of the proposed periodic intensities. We adapt existing simulation techniques to this periodic model, which provides a practical way to evaluate ruin probabilities.

A general insurance portfolio consists of several independent contracts issued for a limited time period (usually one year). During this period the company faces claims from policyholders, multiple claims from the same portfolio are possible. For some portfolios, these claims are caused by periodic phenomena. This, for instance, is the case of property insurance issued in geographical areas where hurricanes or floods are of concern. Assuming that the risk characteristics of such a portfolio are preserved through different periods, then a non-homogeneous Poisson process with a periodic intensity describes, in a natural way, the occurrence of claims in this portfolio. This periodic case has been discussed in Asmussen and Rolski (1994) and Chukova, Dimitrov and Garrido (2000). The discussion in Asmussen and Rolski (1994) relies on martingale properties of the non-homogeneous Poisson process, whereas the approach in Chukova, Dimitrov and Garrido (2000) comes from reliability theory. Here, we propose a practical simulation approach that relies on the periodic nature of certain intensities.

As in the classical risk model, the aggregate claims process for such a portfolio is given by

$$Z(t) = \sum_{i=1}^{N(t)} Y_i , \qquad t \geqslant 0 , \tag{6.2}$$

where $\{Y_i\}$ are i.i.d. random variables [with c.d.f. F_Y, $\mathbb{E}(Y_i) = \mu$ and with m.g.f. $M_Y(\alpha)$] representing the claim amounts. N is a non-homogeneous Poisson process (NPP) with periodic intensity $\lambda(t)$. This intensity is a function of time and drives the seasonality at which claims occur.

This compound process is fundamentally different from the classical compound Poisson process as implied by the following definitions.

Following Çinlar (1974), we first give a formal definition of a non-homogeneous Poisson process.

Definition 6.1 *A non-homogeneous Poisson process (NPP) N_t is a counting process such that*

i) $E(N_t) < \infty$ *for* $t > 0$*;*

ii) $N_t - N_s$ *is independent of* N_s*, for* $0 \leqslant s < t$ *(i.e. it has independent increments).*

iii) $N_t - N_s$ *is Poisson distributed with mean* $\Lambda(t) - \Lambda(s)$*. Where the function* $\Lambda(t) = E(N_t)$ *is continuous.*

The function $\Lambda(t)$ *is called the integrated intensity of* N_t *and it has the following representation*

$$\Lambda(t) = \int_0^t \lambda(s)ds \; , \qquad t \geqslant 0 \; ,$$

for some positive function λ_t*. If* $\Lambda(t) = \lambda t$ *for some constant* λ*,* N_t *is a homogeneous Poisson process (PP).*

Chukova, Dimitrov and Garrido (2000) introduced the following notion

Definition 6.2 *A non-homogeneous Poisson process* N_t *is said to be periodic with period* ς *if its intensity function* λ_t *is periodic, i.e. it satisfies*

$$\lambda(n\varsigma + t) = \lambda(t) \; , \qquad n = 0, 1, 2, \dots \; , \qquad t \geqslant 0 \; ,$$

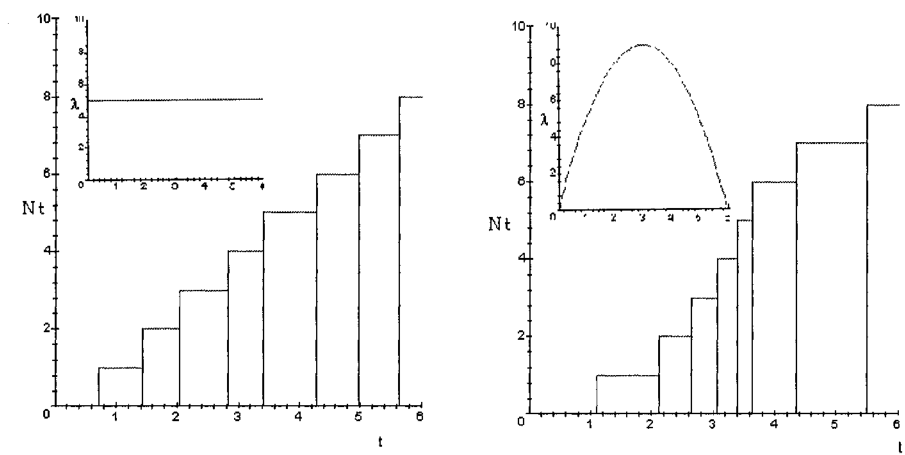
or in terms of the integrated intensity

$$\Lambda(n\varsigma + t) = n\Lambda(\varsigma) + \Lambda(t) \; , \qquad n = 0, 1, 2, \dots \; , \qquad t \geqslant 0 \; .$$

This definition of periodicity is given in connnection to random variables exhibiting the so-called *almost lack of memory property.* These random variables were first introduced by Chukova and Dimitrov (1992). For a review of the results on almost lack of memory see Morales (1999).

In Figure 6.1 we show realizations of two different counting processes. The one on the left is a sample path of a homogeneous Poisson process with mean $\lambda = 5/6$, and the one on the right is for a periodic non-homogeneous Poisson process with period $\varsigma = 6$ and with an intensity such that $\Lambda(6) = 5$. Both processes have the same expectation over the period $\varsigma = 6$. However the way in which claims occur in time is different, as described by their corresponding intensity functions depicted in the left upper corner of each graph. We can see that while the one on the left is always constant the one on the right shows a high season in the middle of the period.

Figure 6.1: **Homogeneous and non-Homogeneous Poisson Processes**



Another difference between an aggregate claims process driven by a PP and one driven by a NPP is that the latter depends on the initial season s. The initial season s is the point at which we start our observation or measurement of the process. This is not a problem in the homogeneous PP since its intensity is constant in time, but the NPP depends on the level of the intensity function λ, which will affect the seasonality of the NPP.

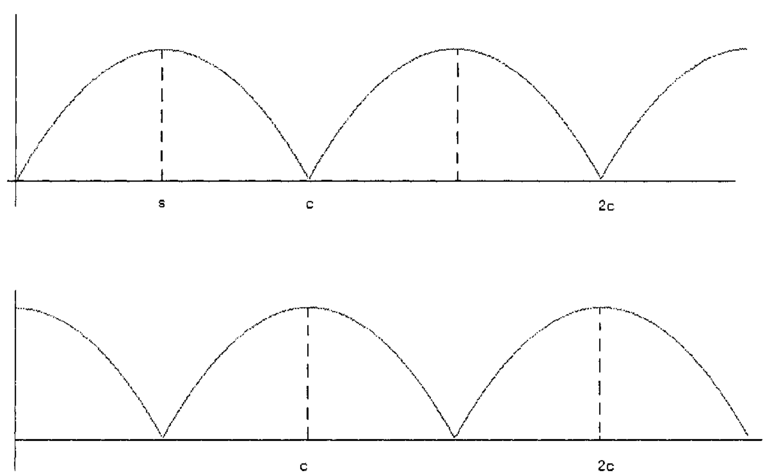
Non-homogeneous Poisson processes are not Lévy processes but they belong to the wider class of processes with independent increments. From the theory of semi-martingales with independent increments [see Jacod and Shiryaev (1987)] we have that the characteristic exponent of a periodic non-homogeneous compound Poisson process is

$$\Psi_t(u) = \int_{\mathbb{R}_0} \left(1 - e^{iux}\right) \Lambda(t)\nu(dx) , \qquad t \in [0, \varsigma] , \tag{6.3}$$

where Λ_t is the integrated intensity of the Poisson point process and ν is the law of the jumps. This uniquely characterizes a periodic non-homogeneous compound Poisson process. Notice that at point $t = \varsigma, 2\varsigma, 3\varsigma, \dots, n\varsigma, \dots$ this characterization does not depend on t and it reduces to the characteristic exponent of a compound Poisson with Lévy measure $\Lambda(\varsigma)\nu(dx)$.

Figure 6.2 shows the same intensity function λ under different initial season s. In the upper graph we have a periodic intensity with a high season in the middle of the period, while the second graph shows the same intensity but starting at a different initial season s. The resulting seasonality is different, now with two high seasons, one at the beginning and another at towards the end of the period. These two intensities will produce different NPP and, as a consequence, they will produce different risk processes with different ruin probabilities.

Figure 6.2: **Effect of the of the initial season** s



Consider the model used for risk reserves under periodic environments [Asmussen and Rolski (1994)]

$$U^{(s)}(t) = u + \varrho t - Z^{(s)}(t) , \qquad t \geqslant 0 , \tag{6.4}$$

where $Z^{(s)}$ is the aggregate claims process defined in (6.2) with initial season s. Here u is the initial surplus, ϱ is a constant premium rate defined as $\varrho = \lambda^* \mu(1+\theta)$ where θ is the security loading factor and $\lambda^* = \frac{\Lambda(\varsigma)}{\varsigma}$ is the average intensity of the NPP over the period ς. The net profit condition for this model is $\varrho > \lambda^* \mu$ [Asmussen and Rolski (1994)].

Risk theory is concerned with functionals of the reserve process. One of particular interest is the associated ultimate ruin probability ψ. This functional measures the riskiness of the portfolio and can be used as a risk index for reinsurance purposes [Bowers *et al.* (1986), Asmussen (2000)]. The ruin problem in the periodic model is based on the random variable

$$T^{(s)}(u) = \inf\left\{t > 0 : U^{(s)}(t) = u + \varrho t - Z^{(s)}(t) < 0\right\} .$$

The main interest lies in evaluating the probability of ruin over a finite or an infinite horizon:

$$\psi_t^{(s)}(u) = P\{T^{(s)}(u) \leqslant t\}, \quad 0 < t < \infty; \quad \text{and} \quad \psi^{(s)}(u) = P\{T^{(s)}(u) = \infty\} ,$$

where the functions $\psi^{(s)}$ and $\psi_t^{(s)}$ are functions of the initial level u as well as of the initial season s.

It is well known [Grandell (1991)] that if the premium ϱ is a function of time such that $\varrho(t) = \mu(1+\theta)\Lambda(t)$, then the reserve process (6.4) is equivalent to the transformed process

$$\bar{U}(t) = U^{(s)}\left[\Lambda^{-1}(t)\right] = u + \mu(1+\theta)t - \bar{Z}(t) , \qquad t \geqslant 0 , \tag{6.5}$$

where $\bar{Z}_t$ is a compound Poisson process driven by a PP with mean one. The time scale $\Lambda^{-1}(t)$ is called the *operational time scale* of a NPP. Since the process in (6.5) is a classical risk reserve process driven by a PP with mean one, the periodic problem has been reduced to the non-periodical case.

The problem arises when we work in a more realistic setting where the premium charged for a contract is constant over a year (or period ς). In this case ϱ is not a function of time but is proportional to the average of the intensity over the period ς, i.e. $\varrho = \lambda^* \mu(1+\theta)$ where $\lambda^* = \frac{\Lambda(\varsigma)}{\varsigma}$.

In Asmussen and Rolski (1994) they derive Lundberg-type bounds for the ultimate ruin probability $\psi^{(s)}$. Their analysis is an extention of an approach introduced in Gerber (1973). It relies on the fact that the NPP is still an infinitely divisible process and therefore, from semimartingale theory of processes with independent increments [Jacod and Shiryaev (1987)], we have that the process

$$\frac{e^{\alpha[Z^{(s)}(t)-\varrho t]}}{e^{\kappa_t(\alpha)}} ,$$

is a martingale, where κ_t is the cummulant function (or Lévy-Khintchine representation) of $S(t) = Z^{(s)}(t) - \varrho t$. This is a rather general property that holds for a wide class of risk reserves processes and yields Lundberg-type bounds for the associated ruin probabilities [see Sørensen (1995) and Morales and Schoutens (2003) for some examples]. Note that since $Z^{(s)}$ is a compound NPP then

$$\kappa_t(\alpha) = \Lambda^{(s)}(t)[M_Y(\alpha) - 1] - \mu\lambda^* t(1+\theta)\alpha , \qquad t, \alpha \geqslant 0 .$$

Notice that κ_t is periodic with period ς and that at the end of each period it reduces to

$$\kappa(\alpha) = \lambda^*\varsigma[M_Y(\alpha) - 1] - \mu\lambda^*\varsigma(1+\theta)\alpha , \qquad \alpha \geqslant 0 ,$$

since $\Lambda^{(s)}(\varsigma) = \lambda^*\varsigma$. This is the cumulant function for the classical (homogeneous) case [see Asmussen and Rolski (1994)]. Recall from the literature [Bowers *et al.* (1986)] that the adjustment coefficient $\mathcal{R}$ is defined as the solution of $\kappa(\mathcal{R}) = 0$.

Our approach relies on path-wise properties of the chosen intensity Λ to approximate, via simulation, the ultimate ruin probability in the periodic case.

The main problem that arises when simulating ruin probabilities is that, for certain simulated paths, the event ruin does not occur and there is no stopping rule for the simulation. We deal with that problem by changing the measure to one, under which the ruin event occurs with certainty. Then, switching back to the original measure, we obtain an estimator of the ruin probability. This application of importance sampling in risk theory can be found in Asmussen (1985) and Vazquez-Abad (2000).

6.2 The Model

Consider the periodic risk reserve process presented in (6.4)

$$U^{(s)}(t) = u + \varrho t - Z^{(s)}(t)\ , \qquad t \geqslant 0\ , \tag{6.6}$$

where $Z^{(s)}$ is a periodic compound Poisson process driven by the integrated intensity $\Lambda^{(s)}$. Henceforth, we make the assumption that $\varsigma = 1$. This is in no way restrictive since the length of a period can be arbitrarily set by the company.

Chukova, Dimitrov and Garrido (2000) considers intensities having a beta shape and argue that the beta function is flexible enough to account for different seasonal patterns. However, working with this function requires approximations of the incomplete beta function and, in addition, might yield discontinuous intensity functions λ. Here, we use a function proportional to the normal density to model the claims arrival intensity. This has the advantage of being continuous and requiring only values of a standard normal distribution for numerical computations.

6.2.1 Bell-Shaped Intensities

We consider bell-shaped intensities of the form

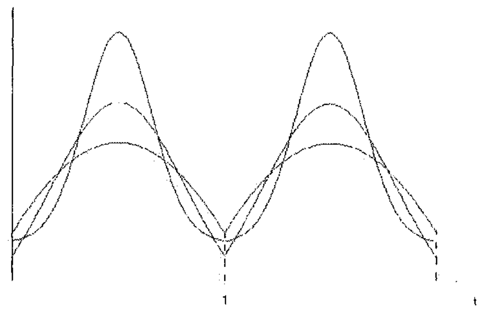
$$\begin{aligned} \lambda(t) &= \frac{\lambda}{\sigma\sqrt{2\pi}} e^{-\frac{(t-1/2)^2}{2\sigma^2}}\ , \qquad \text{for } t \in [0,1)\ , \\ \lambda(n+t) &= \lambda(t)\ , \qquad \text{for}\ , \qquad n = 0, 1, 2, \dots\ , \end{aligned} \tag{6.7}$$

where

$$\lambda = \frac{\lambda^*}{\Phi(\frac{1}{2\sigma}) - \Phi(-\frac{1}{2\sigma})} = \frac{\lambda^*}{2\Phi(\frac{1}{2\sigma}) - 1}\ ,$$

and Φ is the standard normal distribution function. In Figure 6.3 we show different shapes of this family of intensities parameterized by σ and λ^*. The value σ drives the variability of the seasons while the factor λ^* is the area under the intensity over a period, i.e. $\Lambda^{(s)}(1) = \lambda^*$. These intensities show a high season in the middle of the period, its amplitude depends on the factor σ, the smaller it is the more extreme the seasons are. This choice for the intensities is in no way restrictive. Given an initial

Figure 6.3: **Different Bell-shaped Intensities**

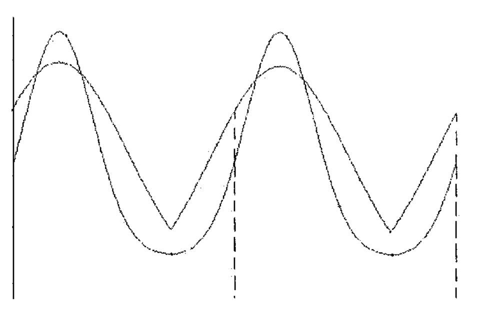


season $s > 0$ we can replicate almost any seasonal pattern in a rather smooth fashion, as can be seen in Figure 6.4. By changing s we go from a pattern with a high season in the middle (Figure 6.3), to one with a high season at the begining and a low season towards the end of the period (Figure 6.4). This three parameter family is defined through

$$\lambda^{(s)}(t) = \lambda(t+s) \ , \qquad s > 0 \ .$$

where the additional parameter s is the initial season.

Figure 6.4: **Bell-shaped Intensities with Different Initial Seasons**



The difference between this family and the beta family proposed in Chukova, Dimitrov and Garrido (2000) is that for the bell-shaped intensities the change in the seasons is always smooth whereas in the beta intensities this change can be sometimes abrupt. Another friendly feature of this family of intensities is that its integrated intensity and its inverse it is given in terms of the cumulative function of a standard normal distribution. This makes this family easier to implement.

The integrated intensity for an initial season $s = 0$ is given by

$$\Lambda(t) = \lfloor t \rfloor \lambda^* + \lambda \left\{ \Phi\left(\frac{t - \lfloor t \rfloor - 1/2}{\sigma} \right) - \Phi\left(-\frac{1}{2\sigma} \right) \right\} , \qquad t > 0 , \tag{6.8}$$

where $\lfloor \cdot \rfloor$ is the maximum integer function and Φ is the cumulative function of a standard normal distribution. For other initial season s the integrated intensity can be written as

$$\Lambda^{(s)}(t) = \Lambda(t+s) - \Lambda(s) , \qquad t \geqslant 0 .$$

The inverse Λ^{-1} is needed in simulations and its expression is also simple. For an initial season of 0, it is given by

$$\Lambda^{-1}(a) = \lfloor \frac{a}{\lambda^*} \rfloor + \sigma \Phi^{-1} \left\{ \frac{a - \lambda^* \lfloor a/\lambda^* \rfloor}{\lambda} + \Phi\left(-\frac{1}{2\sigma} \right) \right\} + \frac{1}{2} , \tag{6.9}$$

where Φ^{-1} is the inverse function or quantile function of a standard normal distribution and $\lfloor \cdot \rfloor$ the maximum integer function. For any initial season $s > 0$ we can write

$$\Lambda^{(s)^{-1}} = \Lambda^{-1}\left(a + \Lambda(s)\right) - s .$$

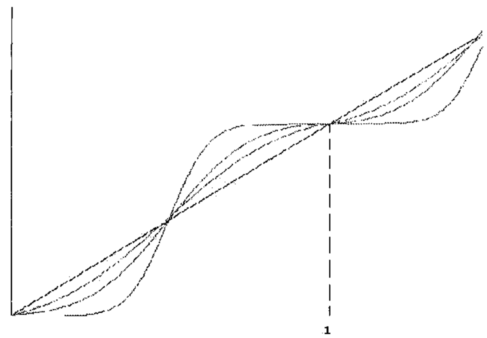
Another function related to the intensity family that is of interest to our study is the function

$$\eta^{(s)}(t) = \frac{\lambda^* t}{\Lambda^{(s)}(t)} , \qquad t > 0 . \tag{6.10}$$

The function (6.10) measures at every instant the difference between the periodic integrated intensity $\Lambda^{(s)}$ and a constant integrated intensity $\lambda^* t$. We show in Figure 6.5 both functions so that we can observe how this difference varies. Different functions Λ_t values of σ are plotted over the straight line with slope λ^*. Notice that the function (6.10) is bounded for all $t > 0$ by

$$\eta^+ = \max_{0 \leqslant t < 1} \left\{ \frac{\lambda^* t}{\Lambda^{(s)}(t)} \right\} ,$$

Figure 6.5: **Integrated Intensities**



and

$$\eta^- = \min_{0 \leqslant t < 1} \left\{ \frac{\lambda^* t}{\Lambda^{(s)}(t)} \right\} .$$

Neither one of them depend on the initial season s because of the periodic nature of $\Lambda^{(s)}$. The following bounds for the integrated intensity are immediate:

$$\eta^- \Lambda^{(s)}(t) \leqslant \lambda^* t \leqslant \eta^+ \Lambda^{(s)}(t) .$$

We can compute the time points at which these maxima occur for the bell-shaped intensities. These are given by

$$t_0^+ = \frac{1}{2} - \sqrt{-2\sigma^2 \ln\left(\frac{\lambda^*}{\lambda}\sigma\sqrt{2\pi}\right)} ,$$

and

$$t_0^- = \frac{1}{2} + \sqrt{-2\sigma^2 \ln\left(\frac{\lambda^*}{\lambda}\sigma\sqrt{2\pi}\right)} .$$

Estimators for the parameters for these intensities can be obtained from raw data using standard techniques for the normal distribution.

6.3 The Simulation Model

Consider the embedded discrete risk reserve model

$$U^{(s)}(T_n) = u + \varrho T_n - \sum_{i=1}^{n} Y_i , \qquad n = 0, 1, 2, \dots , \tag{6.11}$$

where T_n is the n^{th} arrival time of a NPP with intensity $\lambda^{(s)}$, $\varrho = \mu\lambda^*(1+\theta)$ as described in (6.4) and Y_i's are i.i.d. positive random variables with distribution function F_Y.

The aim is to simulate the stopping time adapted to this reserve process given by

$$\tau = \min\left\{n = 1, 2, \cdots : u + \varrho T_n \leqslant \sum_{i=1}^{n} Y_i\right\}. \tag{6.12}$$

Notice that τ is a function of u and represents the number of claims observed before ruin . The time to ruin $T^{(s)}(u)$ is the τ^{th} arrival time, i.e. $T^{(s)}(u) = T_\tau$.

In order to simulate the arrivals T_n we use the fact that a NPP under the *operational time scale* is a Poisson process with rate one. This result is presented in the following lemma.

Lemma 6.1 (Çinlar (1974)) $\{T_1, T_2, ...\}$ *are the arrival times of a non-homogeneous Poisson process with integrated intensity* Λ *if and only if* $\{\Lambda(T_1), \Lambda(T_2)...\}$ *are the arrival times in an homogeneous Poisson process with mean one.*

Lemma 6.1 implies that $E_i = \Lambda(T_i) - \Lambda(T_{i-1})$ is exponentially distributed with mean one, for all $i = 1, 2, \ldots$. As a consequence

$$T_n = \Lambda^{-1}(E_1 + \cdots + E_n)\,, \tag{6.13}$$

where the E_i's are exponentially distributed with mean one.

In order to simulate the n^{th} arrival T_n of a NPP with integrated intensity Λ_t we need to simulate n exponential variates with mean one and then evaluate (6.13). This poses no problem as long as the function Λ^{-1} is invertible, as is the case for the particular bell-shaped intensities chosen in this study.

Our main concern at this point is the fact that the time to ruin (6.12) is not always finite and there will be paths for which the stopping rule will not exist. This is an impediment to any straight-forward simulation analysis.

The way to go around this problem is to use importance sampling [Asmussen (1985)]. The idea is to simulate (6.12) under a measure where ruin is certain (and therefore all the paths admit a stopping rule) and then to switch back to the original measure to define an estimator of the ruin probability. This technique, commonly used as a variance reduction technique in simulation applications [L'Ecuyer (1994)], relies on the Radon-Nikodym theorem.

Theorem 6.1 *Let $\mathbb{P}$, $\mathbb{Q}$ be two probability measures in $(\Omega, \mathcal{F})$ and $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$, then there exists a non-negative r.v. Z such that $E_{\mathbb{P}}[Z] = 1$ and*

$$\mathbb{P}(A) = \int_A Z(\omega) d\mathbb{Q}(\omega) \, , \qquad \forall A \in \mathcal{F} \, .$$

In particular

$$E_{\mathbb{P}}[h(X)] = \int_\Omega h(X(\omega)) d\mathbb{P}(\omega) = \int_\Omega h(X(\omega)) Z(\omega) d\mathbb{Q}(\omega) = E_{\mathbb{Q}}[h(X)Z] \, .$$

$Z(\omega) = \frac{d\mathbb{P}(\omega)}{d\mathbb{Q}}$ *is the Radon-Nikodym derivative.*

This allows us to write the expectation under one measure of a random variable in terms of an expectation under a different measure.

We can write the ultimate ruin probability $\psi^{(s)}$ as an expectation of a function of (6.12) as follows

$$\psi^{(s)}(u) = \mathbb{E}\left[\mathbb{I}_{\{\tau < \infty\}}\right] \, ,$$

where $\mathbb{I}_{\{A\}}$ is the indicator function over the set A. In our simulation model, this last expression is, in turn, a function of the independent random variables $\{(E_i, Y_i) | i = 1, 2, \ldots, \tau\}$ where E_i is exponentially distributed with mean one, and Y_i is distributed as F_Y for all $i = 1, 2, \ldots, \tau$. Recall that the random variables $\{E_i\}_{i=1,2,\ldots}$ are not the interarrival times but the independent random variables used to define the interarrivals. We would like to find a measure $\mathbb{Q}$ under which $\mathbb{I}_{\{\tau<\infty\}} = 1$ a.s. and $\frac{d\mathbb{P}(\omega)}{d\mathbb{Q}} < 1$ [see Vazquez-Abad (2000)].

Now, consider the process (6.11) simulated under a new measure, under which the claims severity distribution is the R-Esscher transform of Y and the underlying variates E_i's are exponentially distributed with parameter $\alpha = (\delta\eta^- + 1)$ where

$$\eta^- = \min_{0 \leqslant t < 1} \left\{ \frac{\lambda^* t}{\Lambda^{(s)}(t)} \right\} \, , \tag{6.14}$$

and for some $\delta > 0$. Notice that the periodic nature of $\Lambda^{(s)}$ assures the existence of $\eta^- > 0$ which does not depend on the initial season s.

The transformed severity density function is

$$f_{\tilde{Y}}(y) = \frac{e^{Ry}}{M_Y(R)} f_Y(y) ,$$

where M_Y is the moment generating function of the claim severity distribution and the T_n are now the arrival times of NPP with integrated intensity $\widetilde{\Lambda}^{(s)}(t) = (\delta\eta^- + 1)\Lambda^{(s)}(t)$. This last fact comes from the form of our particular choice for the intensity. If $\tilde{E}_i$'s are exponential with parameter α we have that $\tilde{T}_n = \Lambda^{-1}(\frac{1}{\alpha}\{E_1 + \cdots + E_n\})$ where E_i's are exponential with parameter one. Using (6.9) for Λ^{-1} we see that this implies $\tilde{T}_n = \alpha\Lambda^{-1}(E_1 + \cdots + E_n)$ and hence the arrival times $\tilde{T}_n$ are the arrival times of a NPP with integrated intensity $\widetilde{\Lambda}^{(s)}$, which is proportional to the original one. We set $R = \frac{\delta}{\mu(1+\theta)}$.

In a general semimartingale setting we know that ruin is certain if [see Sørensen (1995)]

$$\varrho(t) < \mathbb{E}[Z(t)] , \qquad \text{for all } t > 0 ,$$

where ϱ is the premium function, which in this case is the linear function ϱt, and $\mathbb{E}[Z(t)]$ is the expectation of the aggregate claims process. Then for the periodic case it reduces to [Asmussen and Rolski (1994)]

$$\varrho < \mathbb{E}[Z(1)] . \tag{6.15}$$

Under the new measure the expected claim amount per unit time $\widetilde{\mathbb{E}}[Z(1)]$ is

$$\mathbb{E}[\widetilde{N}(1)]\mathbb{E}[\widetilde{Y}] = (\delta\eta^- + 1)\lambda^* \left[\frac{M_Y'(R)}{M_Y(R)}\right] .$$

Let us set $R = R^*$ where R^* is the solution to

$$M_Y(R) = \delta\eta^- + 1 = R\mu(1+\theta)\eta^- + 1 . \tag{6.16}$$

Notice that this is analogous to the definition of the adjustment coefficient $\mathcal{R}$ and it implies that $R^* < \mathcal{R}$.

Then the expected claim amount per unit time becomes

$$\mathbb{E}[\widetilde{N}(1)]\mathbb{E}[\widetilde{Y}] = \lambda^* M_Y'(R^*) .$$

This implies that if we want ruin to be certain under the new measure [condition (6.15)] we need to set $\lambda^* M'_Y(R^*) > \varrho = \lambda^* \mu(1+\theta)$.

Following Vazquez-Abad (2000), we have that under this new measure the ruin probability can be expressed as

$$\begin{aligned}\psi^{(s)}(u) &= \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) = \sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{I}_{\{\tau=n\}}\right] \\ &= \sum_{n=0}^{\infty} \widetilde{\mathbb{E}}\left[\left\{\prod_{i=1}^{n} \frac{e^{-E_i} f_Y(Y_i)}{(\delta\eta^- + 1)e^{-(\delta\eta^-+1)E_i}\left[\frac{e^{R^* Y_i} f_Y(Y_i)}{M_Y(R^*)}\right]} \mathbb{I}_{\{\tau=n\}}\right\}\right] ,\end{aligned}$$

where the expectation $\widetilde{\mathbb{E}}$ is taken under the new measure. The ratio in this formula is the likelihood ratio between the two measures and it has this form since the $\{E_i\}_{i=1,2,\ldots}$ and the $\{Y_i\}_{i=1,2,\ldots}$ are independent and, moreover, we know that they follow an exponential and a R^*-Esscher transformed distribution respectively. Notice that the $\{E_i\}_{i=1,2,\ldots}$ are not the interarrival times but the independent random variables in Lemma 6.1.

Under this new measure we have that $\mathbb{I}_{\{\tau<\infty\}} = 1$ a.s. and we can simply write

$$\psi^{(s)}(u) = \widetilde{\mathbb{E}}\left[\left(\frac{M_Y(R^*)}{\delta\eta^- + 1}\right)^{\tau} e^{\left\{\delta\eta^- \sum_{i=1}^{\tau} E_i - R^* \sum_{i=1}^{\tau} Y_i\right\}}\right] .$$

Since R^* is such that $M_Y(R^*) = \delta\eta^- + 1$ and recalling that $\Lambda^{(s)}(T_n) = \sum_{i=1}^{n} E_i$ we have

$$\psi^{(s)}(u) = \widetilde{\mathbb{E}}\left[e^{-R^*\left\{\sum_{i=1}^{\tau} Y_i - \mu(1+\theta)\eta^- \Lambda^{(s)}(T_\tau)\right\}}\right] . \tag{6.17}$$

The expression inside the expectation is the Radon-Nikodym derivative of the original measure, restricted to the set $\{\tau < \infty\}$, with respect to the original measure. If this derivative is bounded by one then the variance of the simulation estimator for the expectation is bounded [L'Ecuyer (1994)] and (6.17) can be used to obtain reliable estimators via simulation. Our concern is the function in the exponent in (6.17). Since R^* is always positive, we need to verify that

$$\sum_{i=1}^{\tau} Y_i - \mu(1+\theta)\eta^- \Lambda^{(s)}(T_\tau) > 0 ,$$

so that the exponential part of the derivative is less than one.

From (6.12) we know that

$$\sum_{i=1}^{\tau} Y_i - \mu(1+\theta)\lambda^* T_\tau \geqslant u \ .$$

This justifies our initial choice of α as a function of η^-. Since η^- is the minimum of all the proportions between $\lambda^* t$ and $\Lambda^{(s)}(t)$ we have that

$$\eta^- \Lambda^{(s)}(t) \leqslant \lambda^* t \ , \qquad t > 0 \ .$$

This implies

$$\sum_{i=1}^{\tau} Y_i - \mu(1+\theta)\eta^- \Lambda^{(s)}(T_\tau) \geqslant \sum_{i=1}^{\tau} Y_i - \mu(1+\theta)\lambda^* T_\tau \geqslant u \ ,$$

and the exponent is always positive.

Notice that the final simulation estimator (6.17) has the same form as the estimators found in Asmussen and Rolski (1994), specifically of the form defined in equation (3.12) of their article. We can translate their equation into our notation as follows:

$$\psi^{(s)}(u) = \widetilde{\mathbb{E}}\left[\frac{h(s;\alpha)}{h(T_u^{(s)}+s;\alpha)} e^{-\alpha S(T_u^{(s)})}\right] \ , \qquad \alpha \geqslant R_0 \ , \tag{6.18}$$

where $S(t) = Z(t) - \lambda^* \mu(1+\theta)t$, $\mathbb{E}[e^{\alpha S_t}] = \frac{h(s;\alpha)}{h(t+s;\alpha)}$ and R_0 is the solution to

$$0 = \kappa'(R_0) = \lambda^* M_Y'(R_0) - \lambda^* \mu(1+\theta) \ ,$$

i.e. R_0 is the point at which κ attains its minimum. Because of convexity we have that $R_0 < \mathcal{R}$.

If we calculate $\frac{h(s;\alpha)}{h(T_u^{(s)}+s;\alpha)}$ for our model we have

$$\mathbb{E}[e^{\alpha S(T_u^{(s)})}] = \exp\left\{[M_Y(\alpha)-1][\Lambda(T_u^{(s)}+s) - \Lambda(s)] - \alpha\lambda^*\mu(1+\theta)T_u^{(s)}\right\} \ .$$

Because of periodicity we have that $\Lambda^{(s)}(t) = \Lambda(t+s) - \Lambda(s)$ yielding

$$\frac{h(s;\alpha)}{h(T_u^{(s)}+s;\alpha)} = \exp\left\{[M_Y(\alpha)-1]\Lambda^{(s)}(T_u^{(s)}) - \alpha\lambda^*\mu(1+\theta)T_u^{(s)}\right\} \ .$$

Now, if we substitute this last equation into (6.18) we get

$$\psi^{(s)}(u) = \widetilde{\mathbb{E}}\left[e^{-\alpha Z(T_u^{(s)}) - [M_Y(\alpha)-1]\Lambda^{(s)}(T_\tau)}\right] , \qquad \alpha \geqslant R_0 ,$$

If we set $\alpha = R^*$ then $M_Y(R^*) - 1 = \mu(1+\theta)\eta^-$ and we finally have

$$\psi^{(s)}(u) = \widetilde{\mathbb{E}}\left[e^{-R^*\left\{Z(T_u^{(s)}) - \mu(1+\theta)\eta^- \Lambda^{(s)}(T_\tau)\right\}}\right] , \qquad u \geqslant 0 . \tag{6.19}$$

Now we can easily see this equivalence if we recall that the ruin time $T_u^{(s)}$ and that the aggregate claim process $Z(s)$ are given in terms of τ and Y_i by $T_u^{(s)} = T_\tau$ and $Z(T_u^{(s)}) = \sum_{i=1}^{\tau} Y_i$. Substituiting these in (6.17) we get (6.19).

All we have to check is $R_0 \leqslant R^*$. If we look at the total claim amount per unit time for the new measure $\widetilde{\mathbb{E}}[Z(1)]$ we have

$$\widetilde{\mathbb{E}}[Z(1)] = \lambda^* M_Y'(R^*) = \kappa'(R^*) + \lambda^*\mu(1+\theta) .$$

Recall that since ruin is certain under the new measure we have that $\widetilde{\mathbb{E}}[Z(1)] > \lambda^*\mu(1+\theta)$. It follows that $\kappa'(R^*) > 0$ and by convexity we have that $R^* > R_0$.

In order to obtain estimates for the ruin probability in the periodic case we simulate E_i's (exponential mean $1/\alpha$) and claims Y_i's (having as distribution the Esscher transform $\widetilde{F}_Y$ of the original claims severity distribution). We simulate as many as are needed to observe a negative reserve in (6.11), this always happens because under this measure ruin is certain. Then we evaluate the term inside the expectation in (6.19). We repeat this procedure N times and we average over all the obtained values. This average is an estimator of the expectation in (6.17) and therefore of the ruin probability.

6.4 A Simulation Study

Consider two risk reserves processes. One is the classical model

$$U_{HP}(t) = u + \lambda^*\mu(1+\theta)t - \sum_{i=1}^{N(t)} Y_i , \qquad t \geqslant 0 , \tag{6.20}$$

where the claims are assumed to occur following a homogeneous Poisson process $N(t)$ with mean per unit time λ^*. The second process is the periodic reserve process defined in Section 2 and given by

$$U_{NPP}^{(s)}(t) = u + \lambda^* \mu(1+\theta)t - \sum_{i=1}^{N^{(s)}(t)} Y_i\,, \qquad t \geqslant 0\,, \tag{6.21}$$

where the claims occur following a NPP with periodic intensity.

The ruin probabilities for both models with different values of u and s are presented in Table 6.1.

Table 6.1: **Ruin Probabilities for the Periodic and non-Periodic Case**

	Non-Periodic	Periodic Reserve Process						
		s						
u		0	0.2	0.4	0.6	0.8	1	1.2
0	0.526	0.349	0.61	0.7	0.605	0.373	0.349	0.61
0.5	0.415	0.278	0.513	0.595	0.485	0.274	0.278	0.513
1	0.328	0.223	0.432	0.505	0.387	0.203	0.223	0.432
1.5	0.259	0.181	0.364	0.425	0.306	0.151	0.181	0.364
2	0.204	0.147	0.305	0.358	0.243	0.114	0.147	0.305
2.5	0.161	0.12	0.256	0.299	0.191	0.087	0.12	0.256
3	0.127	0.098	0.215	0.25	0.149	0.066	0.098	0.215

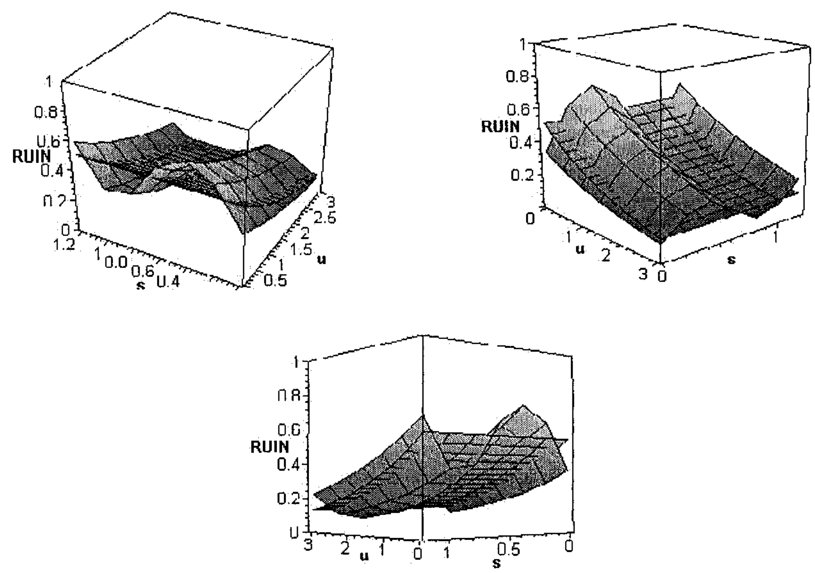
The simulation study was carried out with 100000 paths for the periodic process (6.21), the claims are assumed to be exponentially distributed with mean $\mu = 1$, the average number of claims over a year is estimated to be $\lambda^* = 10$ and the safety loading factor is $\theta = 0.9$. The intensities are bell-shaped with moderate seasons given by a value of $\sigma = 0.25$. The simulation estimators have a precision of (± 0.001).

The values for the ruin probability of the homogeneous process (6.20) are computed using the exact expression, which in the exponential case is known to be

$\psi_{HP}(u) = \frac{1}{1+\theta} e^{-\frac{\theta u}{(1+\theta)\mu}}$. Notice that this ruin probability (shown in the first darker column in Table 6.1) does not depend on the initial season s.

Figure 6.6 shows both surfaces (from different angles) from the simulated estimations. We can observe a periodic fluctuation of the ruin probabilities in the periodic

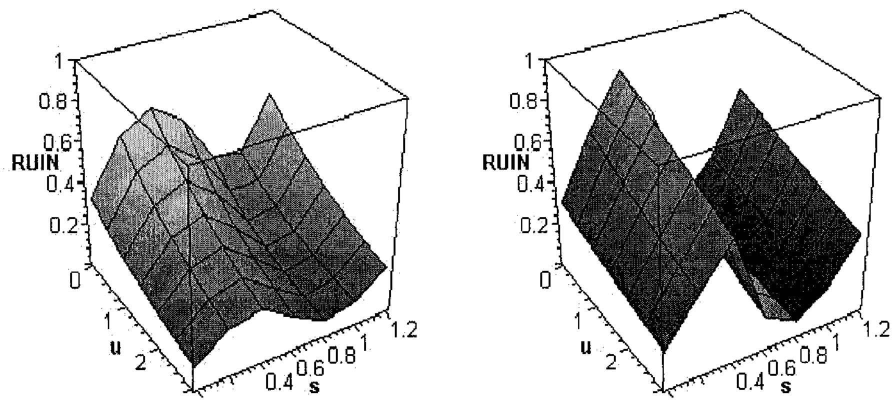
Figure 6.6: **Ruin Probabilities: the Homogeneous and the Periodic Case**



process (6.21) depending on the initial season s. It fluctuates above and below the surface of the homogeneous process (6.20) but as function of u it decreases exponentially, just like the ruin function in the non-periodic case. This difference is the error we can incur in when we use the homogeneous Poisson process to model periodic claims. This error depends on the initials season s and sometimes can be quite large depending on how extreme the seasons are during the year (parameter σ). This empirically verifies the approximation of Asmussen and Rolski (1994). His Lundberg-type approximation is an exponential function of u and a periodic function of the initial season s.

Figure 6.7 presents plots of the ruin probability surfaces for two values of σ. The first graph is for a model with moderate seasons $\sigma = 0.5$ and the second one is for a model with more extreme seasons $\sigma = 0.1$. We can see that the fluctuations are larger when the underlying intensity experiences extreme seasons.

Figure 6.7: **Ruin Probabilities: Periodic Case**



σ=0.5 σ=0.1

6.5 Conclusions

There exist insurance portfolios which are subject to periodic random environments, as it is the case of property and car insurance. The classical risk reserve process of risk theory uses an homogeneous Poisson process to model the claims occurrences. This process has a constant intensity over time, this is, its occurrences are homogeneously distributed over time, which makes it a crude model for insurance portfolios under periodic environments.

Modeling risk reserves under periodicity can be done using a periodic non-homogeneous Poisson process for the claims occurrences. The resulting periodic risk reserve process is fundamentally different from the classical one and yields different ruin probabilities. This is why the use of an homogeneous risk reserve process under a periodic setting might lead to miscalculations, as shown in our simulation analysis.

The periodic intensities of the NPP can be fitted by bell-shaped intensities. This family of intensities can model many types of periodic patterns in a smooth fashion. This feature is important since most season changes in nature occur in a rather continuous way. Hot seasons occur in between mild seasons that then turn into cold seasons, winter never follows immediately after summer. Besides, since this bell-shaped intensities are defined in terms of the standard normal distribution, it is suitable for statistical estimation. In the same spirit we can use gamma or lognormal versions of this family by simply changing the underlying distribution. This might provide a better fit for different data sets.

In this chapter we use the fact that a non-homogeneous Poisson process is a process with independent increments to implement a simualtion scheme that allows us to evaluate ruin probabilities in this context. We approach the periodic model of Asmussen and Rolski (1994) via simulation. Although they hint at a simulation estimator implied by their analysis, they do not go as far as to implement it. Here, we use the periodic properties of the intensities to derive a stopping rule for the simulated paths which endows us with an estimator for the ruin probability of the periodic risk reserve process. This estimator assumes the existence of the moment generating function of the claims distribution which rules out heavy tailed distributions. Further research should be done in this direction to implement aproximations for these cases.

Results of these simulations for the periodic case are presented and compared to the non-periodic one. The difference between the two is then more obvious in view of the empirical analysis. We also verify that the difference between the two models depends on the measure of variability σ. The more extreme the seasons are throughout the year, the larger the difference is.

A further general case of study would be a model where the distribution of the claims is also time-dependent Asmussen and Rolski (1994). This is, the claims sizes are distributed as $F_Y^{(t)}$ where the superscript t indicates this dependence over time. This would be the case in settings where, in addition to the periodicity of the occurrences, the severity distribution also changes over time. This allows for environments where the size of the claims is higher or lower in some seasons. Our simulation estimator might be flexible enough to include variations in the distributions of the risks, this is a subject of future research.

Conclusions

A generalized risk process in terms of Lévy processes is considered in this work. A first goal is to unify different approaches used in risk theory to introduce a Lévy process in the model. From this new perspective, we extend the theory to allow for more general Lévy processes than those previously used.

Brownian and α-stable processes have been applied in risk theory to generalize the classical model, our approach allows for more general Lévy processes, namely a normal inverse Gaussian Lévy motion. This model has been successfully applied in finance and we believe will perform just as well in the insurance context. This process is a purely-discontinuous semimartingale and cannot be treated with the tools introduced so far in risk theory. A topic of future research is to explore further the consequences of introducing a normal inverse Gaussian risk reserve, for instance modeling reserves in the presence of compound interest. It also remains yet to explore models bridging financial and insurance mathematics under an NIG risk model, premium principles and risk measures for instance.

We address several aspects of risk theory and present a contribution in the context of Lévy modeling. Chapter 3 and 4 represent generalizations of Dufresne, Gerber and Shiu (1991) where we show that purely discontinuous risk processes are still mathematically tractable. In Chapter 5 we construct an approximation that uses extreme value theory to adapt to the situation at hand and provide a better estimate for ruin probabilities under different types of claim distributions.

In Chapter 6 we go beyond the scope of Lévy process to be able to include periodicity into our model. For processes with independent increments there still exist results that allow us to evaluate ruin probabilities in this case. We implement a simulation scheme that allows us to compare the classical risk process and the periodic risk process in terms of their associated ruin probabilities.

These four chapters can be viewed as individual contributions in their own. They have been put together here to illustrate the potential of Lévy modeling as a tool yet to be fully explored in risk theory. We hope to have established grounds for future research in this direction that will lead to new insight in the bridging of finance and insurance modeling.

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