Identification and Fostering of Mathematically Gifted Children at the Elementary School

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Abstract

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In the modern dynamic mathematics classroom environment teachers often meet with a variety of abilities. Finding a balance between helping students with difficulties in mathematics and keeping gifted children on the right track of development is not an easy task. Our study aims to develop an operational model for the identification and fostering of mathematically gifted students at the elementary school. It reflects seven years of experience in the teaching of challenging mathematics curriculum to Kindergarten through Grade 6 children having various levels of abilities.

In our analysis of lessons we are going to discuss two questions:

1. How to interpret children's mathematical work in terms of mathematical giftedness ?
2. What kind of didactical situations are favourable in order to foster mathematical abilities ?

In order to develop efficient teaching strategy for the gifted students, we developed a theoretical framework based on the "challenging situations" approach. On the basis of classroom observations and examples of students' solutions of challenging mathematical tasks, we discovered several characteristics of mathematical giftedness, such as mathematical precocity, thinking in concepts and relationships, high motivation and perseverance, systematic and reflective mind.

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Table of Contents

List of Figures

List of Tables

INTRODUCTION

The biographers of famous mathematicians often refer to the evidence of a particular nature of their talent which can be detected already at a very young age. One can ask where this deep insight in mathematics comes from. How can teachers discover their talent and nurture it? And, as a result of this discovery, what kind of a classroom environment would be advantageous for these children? What can be done in order to help these children to realise their potential?

Our teaching and research practice shows the importance of a closer look at this educational issue. In our study, we will focus on gifted children in mathematics, their identification and fostering. Active and curious in their learning, persistent and innovative in their efforts, flexible and fast in grasping complex and abstract mathematical concepts, they do represent a unique human intellectual resource for our society, which we have no right to waste or to loose.

Two major questions have been studied by a number of educators and psychologists:

- ***Who are gifted students and how to recognise (identify) them?***
- ***What can be done in order to foster and to nurture mathematically gifted students?***

Our study aims to design and reflect on various classroom situations which help the teacher to discriminate mathematical giftedness in young children and to create an appropriate challenging environment.

In Chapter I, we will overview and analyse research on mathematical giftedness. There are various models of identification of mathematically gifted children. In our analysis we focus on those of them which help to understand mathematically specific characteristics of abilities of gifted children and their particular way to think mathematically.

In Chapter II, we look closely at the theoretical thinking as characteristics of mathematical giftedness. We will also discuss various problems and questions related to the very nature of mathematical thinking and search for methods of fostering it in gifted children.

In Chapter III, we are going to describe paradoxes of teaching and learning of didactical, epistemological and methodological nature and discuss different approaches to overcome these paradoxes.

In Chapter IV, we will describe a theoretical framework of our study and to construct our definition of a challenging situation and an operational model of analysis of children's mathematical behaviour corresponding to this definition and analyse classroom situations which are favourable for the nurturing and fostering of mathematical abilities in gifted children.

Our teaching experiment is based on 7 years of teaching challenging mathematics to Kindergarten - Grade 6 children. On the basis of classroom observations and examples of student's solutions of challenging mathematical tasks, we discover several characteristics of mathematical giftedness such as spontaneous use of mathematics, mathematical precocity, thinking in concepts and relationships, high motivation and perseverance, systematic and reflective mind. Having many of these characteristics, gifted children demonstrate different styles of work organisation and communication within a challenging situation.

We conclude with various recommendations for the use of challenging courses, teachers preparation, relationships between giftedness and high achievement as well as open questions for further research.

CHAPTER I. LITERATURE SURVEY ON THE PROBLEM OF IDENTIFICATION AND FOSTERING OF MATHEMATICAL GIFTEDNESS

The problem of identification and nurturing young people with special inclination towards mathematics is probably as old as mathematics. From ancient schools led by philosophers and famous mathematicians to today's school system with its qualified mathematics teachers, didacticians and psychologists, the mathematical giftedness has always been an attractive field of study and practice. In this chapter, we are going to analyse different approaches to identification and fostering of mathematical giftedness.

Whatever the point of view on mathematical giftedness, practical or theoretical, psychological or didactical, historical or epistemological, there are two questions that must be asked: how to recognise mathematical talent and how to nurture it.

We start this chapter by looking at the different ways of defining mathematical giftedness. We continue by discussing different existing models of a mathematically gifted student as well as different methods of identification of giftedness. Finally, we will analyse the problem of the fostering of mathematical giftedness within and beyond the existing school system and curriculum.

I.1 Defining mathematical giftedness

The terminology used in the context of identification of children with special abilities operates with many words, such as "promising", "advanced", "talented", "high ability", "extraordinary", "above average", "gifted", etc., whose sense may differ from one author to another.

There are several approaches to study mathematical giftedness. One of them is related to theories of general giftedness in which mathematics is seen as one particular giftedness along with particular talents in music, arts, science, etc. Researchers working on problems of gifted education see mathematical abilities as part of the so-called "specific abilities" which are distinguished from "general abilities", or "general intelligence".

There exist various models representing a large spectrum of abilities. Among them, one finds Guilford's model of 120 + elementary abilities (Guilford,1967), Vernon's hierarchy of abilities - general, verbal-educational, practical-mechanical, minor and specific (Vernon, 1971).

Sternberg (1979) used an information-processing model to identify five ways of thinking or processing information which includes metacomponents for higher-order planning, problem solving and decision taking, as well as performance, acquisition, retention and transfer components.

Gardner (1983) proposed six abilities as a base for a so-called 'multiple intelligence' model: linguistic, logical-mathematical, spatial, musical, bodily-kinaesthetic and personal abilities.

Some researchers focused on the specificity of mathematical abilities.

More than a century ago, in order to understand the nature of mathematical abilities, Calkins (1894) was trying to evaluate the extent of the main mental operations used by children in doing mathematics. She mentioned the power of thought, identification, comparison and reasoning as essential characteristics of the student of mathematics. Among abilities that are more likely to be developed in mathematically inclined individuals comparatively to those who do not like mathematics, Calkins mentioned the ability to notice similarity or dissimilarity between objects or relations as well as the ability to classify and to reason.

Rosenbloom's (1960) observations about his experience of teaching gifted children mathematics also gives some initial clues about the difference between mathematically able students and the average ones: "The bright youngster has a capacity to make abstractions and generalisations, and so can go deeper as well as faster. He can discover for himself what others have to be told".

As our study of mathematical giftedness is focused on "mathematical thinking" and "mathematical ability", we have been led to paying special attention to the work of Krutetskii (1976). According to Krutetskii, mathematical giftedness is a unique combination of mathematical abilities that opens up the possibility of successful performance in mathematical activity and/or the possibility of a creative mastery of the subject. Krutetskii (1962) understands ability to learn about mathematics as individual psychological qualities of the human being that help him or her acquire mathematical knowledge and mathematical skills faster, easier and deeper.

With time, several researchers developed various models of mathematical giftedness related to the ability to learn mathematics.

According to Krutetskii (1962) abilities to do mathematics are composed of:

General abilities (hard working, persistence, productivity, active memory, concentration, motivation)

General mathematical abilities (flexibility, dynamic thinking)

Mathematics specific abilities

Many researchers relate mathematical ability to intelligence.

Young, Tyre (1992) give some practical characteristics of intelligence:

- The ability to deal with new situations
- The ability to see relationships, including complex and abstract ones
- The ability to learn and to apply what has been learnt to new situations
- The capacity to inhibit instinctive behaviour
- The ability to handle complex stimuli
- The ability to respond quickly to information
- A group of mental processes involving perception, association, memory, reasoning, imagination

Many of these characteristics can be detected already at an early age. Krutetskii (1976, p.302) observed highly precocious children who demonstrated a clear interest in mathematics, worked with mathematics with pleasure and without compulsion, mastered different mathematical skills and habits faster and attained of a comparatively (by age) high level of mathematical development.

Several models are based on different lists of characteristics of mathematically able students (see Hlavaty (1959), Krutetskii (1976), Livne & Milgram (2000), Lupkowski & Assouline (1997), Miller (1990), Tempest (1974), Tuttle & Becker (1980)).

We can summarise them in the following combined list.

Mathematically gifted are children who:

<u>Love mathematics:</u>

- Are ready to spend time doing mathematical activities

- Are able to see beauty in mathematical symbols and relationships: numbers, formulas, shapes, graphs, etc.
- Enjoy doing mathematics

Like to learn more about math, which could be seen as:

- motivation
- curiosity
- persistence
- discovery-orientation
- initiative
- wide range of interests

Think mathematically in various school situations that allow them to:

- collect and organise information
- formalise situations
- analyse facts, patterns and relationships
- generalise
- reason abstractly
- count and calculate
- interpret data
- explain and prove logically

Demonstrate behaviour that increases chances to succeed in mathematical activity:

- are hard workers
- have long attention span
- have good memory
- are flexible
- plan, control and verify they actions (work efficiency)
- are fast in their thinking
- think deeply
- think critically
- are able to focus/concentrate

- are able to complete their work
- are able to communicate their results orally and in writing
- pay attention to details
- see a whole structure
- apply intuitive thinking efficiently
- are able to compete

Discover the world by means of:

- "mathematical eyes"
- "mathematical creativity"
- "mathematical logic"

Some researchers consider giftedness as an intersection of different factors. Mingus & Grassl (1999) focus their study on students who display a combination of willingness to work hard, natural mathematical ability and / or creativity.

The authors consider **natural mathematical ability,** which might be represented by several characteristics discovered by Krutetskii (see above) as well as non-mathematical ones such as **willingness to work hard** (that means being focused, committed, energetic, persistent, confident, and able to withstand stress and distraction) or **high creativity** (i.e. capacity of divergent thinking and of combining the experience and skills from seemingly disparate domains to synthesise new products or ideas). The authors labelled students possessing a high degree of mathematical ability, creativity, and willingness as "truly gifted".

Ridge and Renzulli (1981, Fig.1) define giftedness as an interaction among three basic clusters of human traits: above average general abilities, high levels of task commitment, and high levels of creativity. Upon their definition, gifted and talented children are those possessing or capable of developing this composite set of traits and applying them to any potentially valuable area of human performance.

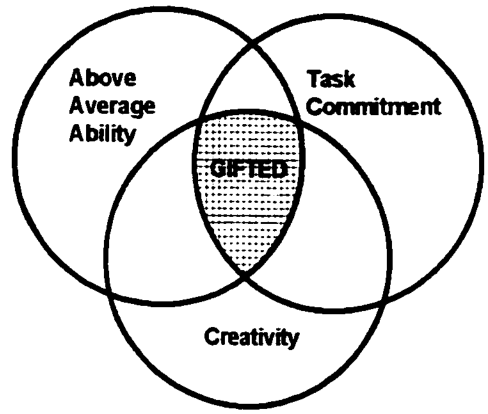


Fig. 1 Renzulli's model of giftedness

Greenes (1981) analyses **various strategies** used by children working on mathematical problems. He points out seven attributes that characterise the gifted student in mathematics:

- Spontaneous formulation of problems
- Flexibility in handling data
- Data organisation ability
- Mental agility or fluency of ideas
- Originality of interpretation
- Ability to transfer ideas
- Ability to generalise

I.2 IDENTIFYING MATHEMATICALLY GIFTED CHILDREN

Having dressed a profile of the mathematically able student in the previous section, we can now move to the next point of our discussion: **What tools could help to identify**

mathematical giftedness in young children? Analysis of various studies in mathematics education shows a large spectrum of theories and practices.

Baroody & Ginsburg (1990) remark that even among children just beginning school, there is a wide range of individual differences. Kindergarteners and first-graders are far from uniform in their informal mathematical knowledge and readiness to master formal mathematics. With each grade, individual differences increase.

Although there seems to be no agreement on terminology nor on procedures of identification of the gifted, all researchers agree that school marks do not reflect mathematical abilities. School success in mathematics does not imply the presence of mathematical ability. Conversely, also, children who do not succeed in school mathematics are not necessarily mathematically unable.

We now look at the different ways of identification mathematical giftedness through intelligence tests, achievement tests and ability tests.

I.2.1 Tools for the identification of mathematical giftedness

There exist many different tools of identifications of mathematical giftedness. But there is no agreement between researchers regarding their efficiency. Moreover, there is still a large disparity among them. The aim of our analysis is to take the most objective look at them and focus not on **how** they help to identify, but mostly **what** exactly they tend to identify and what is the **purpose** of their use.

I.2.1.1 Intelligence testing

Historically, the first scientific methods of identifying gifted children were related to the measurement of intelligence by mean of various tests and their results. Applied since the beginning of the 20th century, the intelligence test produce a three-digit IQ coefficient which is sometimes considered as an indicator of the level of mathematical giftedness.

This way of identification of mathematically gifted students has been criticized by many researchers working in psychology and/or in mathematics education.

Fox (1981) argues that measures of global intelligence do not provide sufficient information about the pattern of specific abilities and levels of achievement for program placement. By mentioning that the intelligence test fails to measure many important cognitive abilities, such as divergent thinking skills, he opts for batteries of aptitude and achievement tests for diagnostic purposes.

Stanley & Benbow (1986) found it illogical and inefficient to group students for instruction in mathematics mainly on the basis of overall mental age or IQ, recognising however a value of IQ as perhaps the best single index of general learning rate.

I.2.1.2 Reasoning test (SAT)

An alternative to the IQ tests was a test of reasoning abilities, which, it was hoped, would predict further success in mathematics far better than items measuring learned concepts, learned algorithms, and computational speed and accuracy. As an addition to this major requirement, the authors of the test proposed the principle of the appropriate difficulty level. They also wanted a professionally prepared, carefully standardised, reliable test for which several well-guarded forms existed and well-known, meaningful interpretations of scores were available.

This approach to the search of mathematical talents was used in Stanley's "Study of Mathematically Precocious Youth" (SMPY), which started in 1969 and was oriented to helping mathematically talented children use their abilities more effectively in the various academic areas.

The choice was a Standard Achievement Test (SAT) which is normally used (in the US) by The College Board for the evaluation of achievement of 12-th Grade college-bound students.

The researchers were trying this multiple-choice test with 12-13 year old students in order to discover early mathematical talents. They estimated that SAT-M test would be excellent for identifying the level of mathematical reasoning ability and SAT-V could provide additional assessment of overall mental age as well as a level of verbal reasoning ability.

Pendarvis, Howly & Howly (1990) argue, nevertheless, that the mathematics portion of SAT for the identification of mathematical *brilliance* (another interesting term, V.F.), is of limited use. The lack of a comprehensive individually-administered test of mathematical ability makes it impossible to identify mathematical talent on the basis of mathematical ability. As an alternative, authors propose to base identification on mathematical achievement. Ideally, achievement tests for this purpose would sample the skills that most reflect the characteristics of mathematical ability: (1) apprehension, (2) logic, (3) generalisation, (4) curtailment, (5) flexibility, (6) elegance, (7) reversibility of thought, and (8) retained insights. These would not be influenced strongly by (1) speed, (2) computation, (3) rote memory, or (4) spatial skills and concepts.

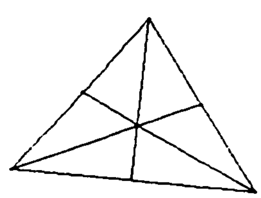
I.2.1.3 Use of special tests

Paek, Holland and Suppes (1999) found that the above mentioned ways to assess gifted children are not satisfactory, because:

- standardised achievement tests rely on a cluster of high scores for gifted students (and thus leaves gifted underachievers or slow thinkers unidentified and unnurtured)
- intelligence tests assess only basic arithmetic skills
- tests designed for older students measure students' exposure to mathematics, not their ability

Therefore, they concluded, there was a need to design a test specifically aimed for the gifted population. Such test, called The Stanford EPGY Mathematics Aptitude Test (SEMAT) for 9 to 13 year old students, has been developed within the Educational Program for Gifted Youth (EPGY). 32 items of this test have been classified in three categories: arithmetic (series, probability, word problems), geometry (geometric shapes and spatial rotation problems) and logic (deductive logic problems written as word problems). Following example, however, raises questions about the relevance of a multiple-choice form in such tests.

What is the total number of triangles in the figure below?



The choices are

*a) 6 _b) 7_ c) 12_ d) 13_*

But, it seems that none of the given answers is the right one:

We have 3 big triangles with the common vertex in the middle, 6 triangles ("halves" of a big one), 6 little triangles with the common vertex in the middle, plus the biggest one. So, it looks like we have 16 triangles in all !?

This example, along with the next two, taken from different multiple-choice tests (see p. 16), allows us to suggest that the same problems could provide more information about children's mathematical abilities if they were given in an open-ended form.

Some researchers express the quite popular modern view that we have far more gifted children than the classical 3 % discovered using IQ's, or SAT's tests.

For example, Sternberg (1997) mentioned that the ability tests we currently use, whether to measure intelligence or achievement or to determine college admissions, value only memory and analytical abilities because they emphasise the same abilities that are emphasised in the classroom. Sternberg's concern is that through grades and test scores, we may be rewarding only a fraction of the students who should be rewarded. Following this theory, Sternberg adds creative and practical abilities as important and somewhat crucial components of the talent identification. He claims that his Triarchic Abilities Test helps us discover and recognise these abilities in mathematics. The examples given in his article raise, however, more questions than confirmations.

Let us look at the following one:

There is a new mathematical operation called "graph". It is defined as follows:
x graph y=x+y, if x<y but
x graph y=x-y, if otherwise.
How much is 4 graph 7?

A.-3 B.3 C.11 D.-11

The ability of this problem to identify mathematically creative minds is rather questionable. A mathematically weak student will definitely get confused by the strange name of the operation and its symbolism and his/her maximum (creative?) effort would probably consist in circling any answer. An average student (one used to follow given instructions), would simply follow the instruction 4 < 7; therefore, 4 graph 7 = 4+7, the answer is 11. A talented student will probably grasp the essence of this problem, that is operation + or - depending on the routine "if - else" condition. For him/her, this represents a simple (and more analytical than creative) technique.

In a certain way, some typical SAT-M problems like

"If 16*16*16=8*8*P, then P=___

A.4 B.8 C.32 D.48 E.64 (10 SATs, 1990:22)

would give more 'creative' opportunity for showing creativity than the above cited example. Some mathematically talented students might actually enjoy the discovery that 16=2*8, and use it to solve the problem.

But this problem allows for a simple (although time consuming) way to calculate the answer simply by doing all operations arithmetically, which a good "average" student can do quite fast. We could raise another question regarding the use of a multiple-choice tests. What if one student simply estimates that on the left side we have something like 20*20*15=6000, on the right side we have about 100, so the answer is around 60. Isn't this student smart (or/and creative) ?

Yet, another test for mathematical giftedness has been created as an addition to the standard ones - Hamburger Test fur Mathematische Begabung (German word 'Begabung' means 'ability'). It contains following complex mathematical activities:

(1) organising material
(2) recognising patterns or rules
(3) changing representation of the problem and recognising patterns and rules in this new area
(4) comprehending very complex structures and working with structures
(5) reversing processes
(6) finding (constructing) related problems

The idea of the test is based on the assumption that if one scores high in almost all of these activities, there is a "high probability of successive creative work later on in the mathematical field or related areas." (Wagner & Zimmermann, 1986)

I.2.1.3 Competitions

Besides the regular or experimental methods of searching for mathematical talents, there is a powerful set of out-of-regular-classroom activities such as mathematical clubs, mathematical camps, or mathematical competitions (Olympiads).

Over the years, the major objectives of the Olympiads have stayed the same: to stimulate student's interest in solving non-typical mathematical problems and to assist in finding those students who have a superior interest and ability in mathematics (Skvortsov, 1978). A guiding principle in problem selection is that solving the problem should not require knowledge beyond the general school curriculum.

Skvortsov stresses that besides mathematical ability, a number of purely competitive qualities are demanded by the Olympiad; for example, the power of high concentration during a limited time period - a requirement not necessarily reflecting the conditions of real creative scientific work.

While some are sceptical about mathematical competitions as a tool in the selection of the best students because of their elitist flavour; Kahane (1999) affirms that large popular competitions, like the *Australian Mathematics Competition* or *the French Mathematics competition Kangourou (see p.139)*, can reveal hidden aptitudes and talents, and they stimulate large numbers of children and young people. When a mathematical activity looks like a game and when children like the game, the parents cannot consider mathematics as an inhuman tyranny anymore.

However, competition as a tool of identification of giftedness has its limits related to time constraints (we might lose some gifted children who are slow but deep thinkers) and stress (some children would get anxious facing the necessity to win), factors which seem to be somewhat outside of the characteristics of mathematical giftedness.

I.2.1.5 Alternative approaches

The few examples given above show that the selection of problems for the identification of gifted children as well as the interpretation of the results is not an easy task. Many researchers indicate that a process of identification of gifted children requires many different steps and techniques including classroom observation, surveys, interviews and a variety of tests.

We would agree with Kulm (1990) that assessment should be aimed at revealing the extent, complexity, and functional characteristics of mathematical thinking rather than focusing simply on final, well formed ability or performance.

Young & Tyre (1992) mentioned that "instead of assessing children's intelligence as measured by intelligence tests, we would be better employed finding out what they can do and how well and quickly they can learn to do more".

Following Johnson's (1983) remark that many achievement tests (primarily computational) allow assumptions to be made about giftedness from the number of items answered correctly (quantitative data), one should give more attention to *how student actually reasons in mathematics* (qualitative data).

Johnson also mentioned that, in his opinion, the characteristics that separates a gifted from non-gifted child in mathematics is the **quality of the child's thinking**.

I.2.1.6 Investigative, open-ended problem solving alternative

One of the often-used categories, that accompanies multiple choice testing is "time". Kulm (1990) remarks that since so much of school mathematics in the past has been focused on practising skills, the completion of a large number of exercises in a limited time period has been accepted not only as a measure of mastery, but as an indication of giftedness and potential for doing advanced work. On the other hand, higher order thinking in mathematics is by its very nature complex and multifaceted, requiring reflection, planning, and consideration of alternative strategies. Only the broadest limits on time for completion make sense on a test purposing to assess this type of thinking.

We would agree with Burjan's (1991) recommendations to use:

- Open-ended investigations and open-response problems rather than multiple-choice short questions
- Problems allowing several different approaches

- Non-standard tasks rather than standard ones
- Tasks focusing on high-order-abilities rather than lower-level-skills
- Complex tasks requiring the use of several "pieces of mathematical knowledge" from different topics) rather than specific ones (based on one particular fact or technique)
- Knowledge-independent tasks rather than knowledge-based ones.

Unfortunately, as mentioned by Greenes (1981), the bulk of our mathematics program is devoted to the development of computational skills and we tend to assess students' ability or capability based on successful performance of these computational algorithms (so called "good exercise doers") and have little opportunity to observe students' high order reasoning skills.

Sometimes, even a very banal math problem might deliver a clear message about distinguishing the gifted student from the good student. Greenes analyses a very simple word problem (given to 5th Grade children):

Mrs. Johnson travelled 360 km in 6 hours. How many kilometres did she travel each hour?

One bright student surprised the teacher by having difficulty to solve this easy problem. Finally, the teacher realised that the student had discovered that nothing was said about the same number of kilometres travelled each hour. This example demonstrates the child's ability to detect ambiguities in the problem, which may be a sign of mathematical giftedness.

Another aspect of choosing appropriate problems for the identification mathematically bright children has been analysed by Greenes (1997) who underlines the importance of presenting situations in which students can demonstrate their talents: "One vehicle for both challenging students and encouraging them to reveal their talents is the use of rich problems and projects". Greenes mentions that such problems accomplish the following:

- Integrate the disciplines (application of concepts, skills, and strategies from the various sub-discipline of mathematics or from other content areas (including non-academic ones)
- Are open to interpretation or solution (open-beginning and open-ended problems)
- Require the formation of generalisations (recognition of common structures as basic to analogue reasoning)
- Demand the use of multiple reasoning methods (inductive, deductive, spatial, proportional, probabilistic, and analogue)
- Stimulate the formulation of extension questions
- Offer opportunities for firsthand inquiry (explore real-word problems, perform experiments and conduct investigations and surveys)
- Have social impact (well-being or safety of members of the community)
- Necessitate interaction with others

I.2.1.7 Challenging tasks

Many authors analyse a teacher's role in the process of identification of mathematically able children. Kennard (1998) affirms that the nature of the teacher's role is critical in terms of facilitating pupils exploration of challenging material. Hence, the identification of very able pupils becomes inextricably linked with both the provision of challenging material and forms of teacher-pupil interaction capable of revealing key mathematical abilities. The author votes for interactive and continuous model for providing identification through challenge which integrates the following strands:

- The interpretative framework employed by classroom teachers to identify mathematically able pupils

- The selection of appropriately challenging mathematical material
- The forms of interaction between teachers and pupils which provide opportunities for mathematical characteristics to be recognised and promoted
- The continuous provision of opportunities for mathematically able children to respond to challenging material

The above characteristics are very different from what we had been discussing in the previous sections. They refer to a research approach, not to the practice of identification. Author stresses the need to search different ways of interacting with pupils "which maximise the opportunities for *simultaneously* recognising and promoting mathematical abilities."

In Kennard's case study based on this model the identification was conducted by the so-called teacher-researcher in the classroom environment where the pupils are being taught as well as observed. The questioning approach was used in order to reveal aspects of pupils' mathematical approaches and understanding.

In his paper, Kennard (1998) analyzes an example of a situation in which a teacher-researcher worked with a group of four year 9 students (13-14 years old) to whom he gave so-called "extension activity" (as extension of the topic "angles in polygons"). The students were asked to *devise a test on whether or not a shape tessellates*. The work was organised in groups. The teacher was constantly questioning students' reasoning which was revealed in different stages of the investigation:

Stage 1. The teacher asks students what they mean by tessellation and invite them to explore what they mean by 'fit together'. Students conduct an experiment stating *whether or not the shapes 'fit together'*. The result of this stage revealed that students grasped the essence using a practical way to test their conjecture.

Stage 2. The teacher *directs students work toward more theoretical work* asking for the search of a way to test the tessellation of shapes that are not available for manipulations. Students were engaged in conjecturing and checking initial hypotheses (if two angles form 180 degrees, shapes would fit together). The result was a refutation of the conjecture by using a counter-example (quadrants of a circle) and a demonstration of flexibility of thought to try a new approach.

Stage 3. The teacher *encourages students to continue their explorations* with regular polygons (without any concrete manipulations). Students look for links between 360 degrees and an interior angle of the polygon thus putting together a chain of reasoning which allows them to formulate a generalisation.

Stage 4. The teacher *asks students to test their theory*: how could you check it is true? Students propose an idea for predicting for a shape they didn't try. Their way of testing is evaluated by the researcher as based on conviction rooted in reasoning, not in practice.

Stage 5. The teacher *asks students to think* at home about how to express their finding by means of a formula. One student comes up with a formula 360/ [180(n-2)/n] which students succeed to simplify to 2n/(n-2). The author evaluates this work as an ability to use algebraic skills (of transformation of algebraic expressions) and thus moving away from the geometric context of the problem to an algebraic one, more generalised and simplified.

As a conclusion from the case study, the author suggests the use of Krutetskii's categories of analysis of students abilities, teacher-students interactions as stimulus of investigations along with the use of the appropriate challenging material.

I.3 KRUTETSKII'S METHODS OF STUDY MATHEMATICAL ABILITIES THROUGH THINKING

One of the most respected, by mathematics educators, research studies of the differences in mathematical thinking has been conducted in Russia in 50-60ss, within a fundamental 15-year long Krutetskii's work on mathematical abilities. An English translation of Krutetskii's findings published in 1976 helped it to remain a valuable model for the identification of mathematically gifted children in Western countries.

In the next few paragraphs, we will analyze Krutetskii's method of testing mathematical abilities as well as his most important result regarding a particular way of mathematical thinking in mathematically gifted children.

In his critique of IQ-testing for the selection of pupils, Krutetskii (1976) states that IQ-tests measure only the ability to perform the task required by the test situation, and nothing more. In his study Krutetskii offers a different way to study mathematical abilities within appropriate mathematical activity, which, taken in school instruction, consists of solving various kind of problems in the broad sense of the word, including proof, calculation, transformation, and construction.

I.3.1 Choice of mathematical problems

Krutetskii (1976: 89) analyzes seven principles of choosing mathematical problems suitable for the discovery of a mathematically able student:

Principle 1. Correspondence with the contents of a pupil's mathematical activity

> Our experimental problems, especially chosen or composed, represent about equally the different provinces of school mathematics - arithmetic, algebra, and geometry - comprehending rather fully the essence of a pupil's mathematical activity and shaping it in certain sense. (Krutetskii, 1976: 89)

He claims that an analysis of solutions to these "chiefly mathematical" problems would unable him to understand how the distinctiveness of the capable pupils' mental activity differs from that of the less capable pupils and to reveal individual features of pupils' mental activity in problem-solving.

In the last paragraph, the word "capable", used in the English translation, is one of possible interpretations of the Russian word "sposobni". Another version of it is the word "able" used in the chapter that contains an analysis of exceptional cases of the high mathematical ability. Krutetskii uses also the word "mathematicheskaya odarennost" that in English translation would be equivalent to "mathematical giftedness" or "mathematical talent".

Among various problems used in the study, one can find some arithmetical ones. Advocating his use of this kind of problems, Krutetskii argues that

> they interest us not so much in themselves but as a means of revealing traits of thinking, characteristics of the reasoning process. Solving arithmetical problems clearly reveals the process of reflection or reasoning and provides an opportunity to penetrate to the 'laboratory of thought'. (Krutetskii, 1976:90)

Principle 2. Various degrees of difficulty

A set of problems should contain items at various levels of difficulty (low, average, and high), including nonstandard problems requiring elements of mathematical creativity. Problems should thus vary from very simple questions, accessible even to the pupil of ordinary abilities, up to the most complex, not accessible even to every able pupil. Krutetskii bases his choice on either objective criteria (like appraisal of the complexity of relations given in the problem), or subjective ones (like comparing problems solved by 75 % of children with problems solved by 25 % of children).

Principle 3. Potential for revealing the structure of abilities in the pupil

The solution of the problems should reveal those features of mental activity that are specific of mathematical activity.

Principle 4. Process-oriented tasks

The problems should aim to establish not only and not so much the ulimate result of the examinees' performance on a task but primarily the process used in that performance.

Principle 5. Reducing the influence of the past experience (the aggregate of knowledge, habits, and skills) as much as possible.

Krutetskii achieved this reduction through the so-called equalization: choosing problems that do not require any particular knowledge, skills or habits; are new to the pupils and are based on unfamiliar material; are given on recently learnt material (here the experimenter himself organized the instruction in a whole series of cases).

Principle 6. Use of instructive methods for diagnostic purposes

This principle was aimed at helping the researcher to determine "how rapidly a pupil progressed in solving problems of a certain type, how well he achieved skill in solving these problems, what were his maximum possibilities in this regard" (Krutetskii, 1976: 95). Referring to Vygotskii who expressed the important notion that the condition of development is not determined only by maturation but also by learning potential (the so-called Zone of Proximal Development, which is a measure of not only what the child has already learnt but also what he is capable of learning), Krutetskii continues with two indices needed to evaluate children' abilities:

> How independently they solve the problems given to them, and how they solve the same problems with the aid of adults. The discrepancy between these two indices

> would be the index of the child's so-called Zone of Proximal Development - an important component in the total evaluation of his mental potential. (Krutetskii, 1976:96)

Principle 7. Use of quantitative descriptions when necessary

In evaluating the 'progress rate' Krutetskii proposed to look at the following quantitative data: the number of exercises necessary for mastering something new, the number of problems solved independently, solved with the experimenter's help, or unsolved, the number of exercises needed for mastering the solution of a very complex problem on a test, and so on. In evaluating the curtailment of the reasoning process he looked at the number of links that were skipped. In evaluating the flexibility of the mental process he counted the number of different methods used to solve one problem.

Following these principles, Krutetskii constructed a system of experimental problems that included 26 series, containing 79 tests (including 22 arithmetical, 17 algebraic, 25 geometric, and 15 others).

I.3.2 Choice of students

According to his assumption that "all able students possess certain mental traits that are lacking in all incapable ones, [and that] these qualities must play an important role in the structure of mathematical abilities", Krutetskii postulates that these individual differences can be studied only by investigating individual differences in a suitable activity. He states thus that "the most natural way to study abilities is by comparing those who perform a certain activity successfully or creatively (who are called capable) with those who do not (who consequently are regarded as incapable or less able) (Krutetskii, 1976:175).

Therefore, for the purpose of his study, Krutetskii has chosen several groups of students that were tentatively placed in the categories of mathematically capable, average, and relatively incapable pupils. "Examinees were chosen with the help of their teachers according to their success or lack of success in mathematics (these categories were interpreted in a broad framework as success in the independent and creative mastery of mathematics) for the purpose of studying how this was determined" (Krutetskii, 1976:176).

Krutetskii counts among the 'capable' those pupils who quickly and easily mastered mathematical material and acquired skills in performing mathematical operations, who thought independently and somewhat creatively while studying new material, and who came up with original solutions of nonstandard problems.

In a group of pupils with an "average ability", he places those ones for whom successful work in mathematics required a considerable - in comparison with the "capable" pupils - expenditure of time and effort. Their teachers have noted their great difficulties in transferring to solving problems of a new type. But having mastered the methods of solving them, they subsequently cope with similar assignments rather well.

In his comments on placing pupils in the "incapable" group, Krutetskii underlines a relative character of such labeling. According to him, an absolute inability to study mathematics doesn't exist. So, every normal, mentally healthy pupil is capable of mastering the school mathematics course more or less successfully with proper instruction. With teachers' advice, Krutetskii singled out pupils who tried, who worked assiduously, but who still achieved no special distinction in mathematics although they might have done well in their other subjects. He thus put in this group those children who had trouble understanding teacher's explanation (extra lessons were often needed) and those who could not work out problems that went beyond the limits of the standard they had mastered. Their mathematical habits were formed with difficulty, required a large number of exercises, and were shaky; they disintegrated easily in the absence of practice.

I.3.3 Discrimination between mathematically capable and other students

In his profound analysis of the experimental data, Krutetskii makes clear a discrimination between mathematically able and other students. He groups them around various characteristics and features of mathematical thinking; among them we retain for our study:

- Information gathering
- Information processing
- Information retention
- Mathematical cast of mind
- Inspiration and insight
- Indefatigability

According to the goals of our study, we leave aside differences related to age and gender.

I.3.3.1 Information gathering features of mathematical ability

The way a mathematically able student perceives a mathematical problem and grasps its structure is an important clue for the teacher: capable students systematize the mathematical quantities given in the problem through a clear differentiation or isolation of three qualitatively different elements in its structure:

- Complexes and indices essential for the given type of problem: those relations in a problem which have basic mathematical meaning
- Quantities not essential for the given type of problem but essential for the given concrete variant
- Superfluous, inessential quantities, unnecessary for solving the specific problem

In other words, capable pupils perceive the mathematical material of a problem *analytically* (they isolate different elements in its structure, assess them differently, systematize them, determine their "hierarchy") and *synthetically* (they combine them into complexes, they seek out mathematical relationships and functional dependencies). The able student perceives each such complex as a *composite whole:*

> First, he perceives individual elements in this complex, each element as a part of the whole, and second, he perceives these elements as interrelated and forming an integral structure, as well as the role of each element in this structure. Thus, the able pupil creates a clear, integrally dismembered image of the problem. Apparently, this also underlies the ability, which distinguishes capable pupils, to 'grasp' a problem as a whole without losing sight of all its data. (Krutetskii, 1976:228)

Compared to these pupils, children with average, ordinary abilities in mathematics, when confronted with a new type of problem, usually see only its separate mathematical elements. "Going outside" the limits of the perception of one element often means "losing" it. The average pupil thus must face the special task of connecting the mathematical elements of the problem, and in the process of analysis and synthesis he finds this connection. As for the incapable pupils, such connections and relations between the elements of a problem, even with outside help, are established with great difficulty.

The analysis of differences in children' perception of a mathematical problem gives another clue for the understanding of the mechanism of curtailing in mathematically able children: in their initial orientation to a problem (if the problem was not very complicated), they curtailed the process so much that it was frequently almost "knit together" with the first instant of perception, since any noticeable elements of reasoning were absent. This special kind of analytic-synthetic vision in the mathematically able contrasts very much with the analytic-synthetic process of perceiving a problem's structure in ordinary children. Krutetskii sees a difference in the speed of mental orientation as a factor of discrimination between the gifted and the average pupils:

> A single-act, single-feature analytic-synthetic 'vision' of mathematical material, an immediate judgment of the relations given in the material, seeing the 'skeleton' of the problem at once, purging all concrete values and as if visible through the specific data (Krutetskii, 1976:231).

Generalizing his data about information gathering by children of different mathematical abilities, Krutetskii summarize that "under identical conditions for the perception of mathematical material, pupils with different mathematical abilities obtained (or more precisely, actively procured) different information." (Krutetskii, 1976:233). In able pupils, the volume of the information procured is always greater than it is in average or, even more, in incapable pupils. In this sense, Krutetskii speaks about an ability to "actively extract from the given terms of a problem the information maximally useful for its solution" (ibid.: 233). Giving a psychological interpretation of this phenomenon, Krutetskii mentions a factor of a "formalized perception of mathematical material". The word "formalization" is used here as a rapid "grasping", in a specific problem or in a mathematical expression, of the formal structure, as if all of the content (numerical facts, specific material) had fallen out, leaving pure relations between indices that determine whether the problem or mathematical expression belongs to a definite type.

Yet, another observation made by Krutetskii (p.235) is worth mentioning: in capable students, the process of analysis - synthesis has an "instrumental" character; they isolate problem' features for the purpose of determining appropriate operations for solving the problem. Ordinary students would also be able to distinguish differences between the problems or their features but would rarely turn their discoveries into an active search for solutions.

I.3.3.2 Information processing features

Krutetskii groups various characteristics of information processing during problem-solving be mathematically capable pupils in several categories of abilities:

- *Ability to generalize mathematical objects, relations and operations*

His results can be summarized as follows:

1. They recognize every specific problem at once as a representative of a class of problems of a single type and solve it in a general form.
2. They very easily find the essential and the general in the particular, the hidden generality in what are, apparently, different mathematical expressions and problems.
3. They generalize methods of solution and the principles of an approach to problem-solving and therefore are better prepared to use effective strategies in solving non-standard mathematical problems.

One of Krutetskii's examples presented by Johnson (1983) illustrates the investigation of the ability to generalise and the ability for "curtailment". Following group of problems was constructed on the base of one algebraic property identity $(a+b)^2 = a^2+2ab+b^2$ given in various expressions placed in the order of the gradually increasing level of difficulty. The student starts with problem # 1. And he/she is then confronted straight away with the last one. If he/she is not able to solve # 8 the item # 2 is given, then another trial with # 8 and so on.

1. $(a+b)^2$
2. $(1 + 1/2\, a^3b^2)^2$
3. $(-5x + 0.6xy^2)^2$
4. $(3x - 6y)^2$
5. $(m + x + b)^2$
6. $(4x + y^3 - a)^2$
7. 51^2
8. $(C + D + E)(E + C + D)$

According to Krutetskii, the level of ability depends on how fast (it means after how many iterations) the student will pass from the first to the last problem in applying the formula of the square of the sum of two numbers.

In some cases researchers observed that the structure of reasoning in mathematically able students was so curtailed that it started to look as a direct connection between the problem and the result. Actually, it could be even interpreted as the absence of any reasoning or thinking. (Krutetskii, 1976:270)

Krutetskii stressed that the presence of curtailed mathematical thinking in the able pupils he studied, does not contradict the assumption that mathematical thought is sequential and logical. He argues that "in a curtailed deduction or argument the necessary links are not missing altogether; they are shortened, are not articulated and are not realized, but they are 'invisibly' present" (ibid.: 271)

- *Flexibility of mental processes*

The study showed that mathematically able pupils are distinguished by great flexibility, by mobility of their mental processes in solving mathematical problems.

1. They switch freely and easily from one mental operation to qualitatively different one.

2. They demonstrate the diversity of aspects in the approach to problem solving, the freedom from binding influence of stereotyped methods of solution.
3. They easily reconstruct established thought patterns and system of operations (Ibid.: 282)

- *Striving for clarity, simplicity, and economy ("elegance") in a solution*

The following Krutetskii's remarks illustrate his interpretation of this characteristic of mathematically able students:

1. A striving for simplest, most rational solution begins to appear at a very early age
2. All the capable pupils, in finding the solution to a problem, continue to search for the best variant, even though not required to do so.

- *Reversibility of mental processes in mathematical reasoning*

The most interesting conclusion from Krutetskii's study at this point, is that capable pupils appear to solve reverse problems without any special instruction, identifying them quickly as the opposites to the ones just solved. Another important observation is that reversed problems were solved more rapidly and easily if they were given immediately after solving direct ones. This suggests that, in able students, the process of establishing or forming direct associations goes hand in hand with forming the reverse ones. This process would be extremely difficult for less capable pupils. (Ibid.: 288)

- *Differences in the use of trial-and-error method between mathematically capable and average students*

In a polemic with Western psychologists, Krutetskii insists on the validity of his results regarding a particular mechanism of regulation of the process of trial and error in solving mathematical problems. His results show that unlike inept pupils who do their trials by blind, unmotivated manipulations, chaotic and unsystematic attempts at guessing, capable students are marked by an organized system of searching, subordinated to a definite program or plan. Their trials are purposeful, systematized attempts, directed toward verifying the assumptions they had made. In making a trial, capable pupils usually know why it was being made, what was expected, and what was to come next. (ibid.: 292)

We found a very interesting idea expressed by Lee (1982) in her analysis of student's mathematical behavior on solving challenging mathematical tasks (like the famous problem of the 'parking lot' - *How many cars and motorcycles are on a parking lot if we count 100 wheels*). She found that some of student's sudden 'illuminations' that appear as the result of the leap-frog thinking (trial and error) might be in fact examples of a so-called 'parallel' processing when students are pursuing two, or more, lines of thought simultaneously.

I.3.3.3 Information Retention

During his study, Krutetskii made very intriguing observations regarding the memory of mathematically able students. Giving them a problem of a definite type and repeating the same type two or three months later (same type, not the same problem), he observed the same repeating phenomenon: capable pupils believed that they were doing exactly the same problem as before, they would even indicate approximately in what experimental lesson it had been done. The explanation given by Krutetskii is as follows: "the pupil was retaining the problem type and generalized scheme for solving it; therefore he perceived another problem of the same type and with the same solution pattern as familiar to him" (ibid.: 296). While remembering the type and the general character of the operations of a problem they have solved, they did not remember the problem's specific data or numbers.

Thus, Krutetskii sees in this generalized recollection of typical schemas of reasoning and of operations the very essence of a mathematical memory. As for memory for specific data, for numerical parameters, he considers it "neutral" with respect to mathematical ability.

I.3.3.4 Mathematical cast of mind

According to Krutetskii, this trait representing a unique organization of the mind begins to show up in elementary forms by the age of 7 or 8 and later acquires a very broad character. It is expressed in the children's striving to make the phenomena of the environment mathematical, to notice spatial and quantitative relationships, bonds, and functional dependencies.

They see the world "through mathematical eyes". Their "journey" through the real world is marked thus by many mathematical questions they ask while walking, reading, watching movies, during lessons and at home. They would be willing to

- Estimate the volume of a huge building
- Calculate the speed of the bus they travel in
- Compute the area of a stadium,

and so on.

This particular inclination to interpret reality mathematically Krutetskii defines as a *particular synthetic expression of mathematical giftedness (ibid.: 305).*

I.3.3.5 Sudden solution (inspiration, insight)

Without a deep look at the nature of insights, Krutetskii shows that "many incidents of sudden and seemingly inexplicable 'inspiration' in the solution of problems by capable pupils can be explained by the unconscious influence of past experience; underlying these incidents is also an ability to generalize in the realm of mathematical objects, relations, and operations and an ability to think in curtailed structures" (ibid.: 309).

I.3.3.6 Indefatigability

Krutetskii noticed much evidence of the reduced fatigue of capable pupils as they process mathematical information. His capable students could spend 3 to 4 hours working on complicated mathematics problems requiring great mental effort, and not get tired. What is more intriguing, this trait can be observed only during the mathematical activity and not during other activities which can also be interesting for these children. As a psychological explanation of this phenomenon, Krutetskii speaks of a kind of "partiality of properties of a person's nervous process in conformity with the nature of one or another of his activities" (ibid.: 311).

When we look globally at Krutetskii's result, we understand better how powerful tool it represents for teachers. In his study, Krutetskii

- constructs a complete system of tasks that can be used in studying students' mathematical ability
- describes in depth children's genuine thinking and reasoning
- points at features discriminating mathematical behaviour of gifted vs ordinary students

- demonstrates the efficiency of methods of identification of giftedness through-teaching and learning, not just through testing (qualitative vs quantitative methods).

I.4 DIFFERENT LEVELS AND TYPES OF GIFTEDNESS

Many studies of mathematical giftedness indicate that beside the differences between more and less mathematically able students, there are differences between mathematically talented children as well.

I.4.1 Binet

A very interesting typology of intellectual and moral types was created by Binet (1922) regarding reading abilities:

- Descriptor type
- Observer type
- Emotional type
- Erudite type

It seems that one could discover these types in mathematically gifted children as well.

I.4.2 Hlavaty

J.Hlavaty (1959) mentions 4 groups of academically talented students:

Group A consists of students with high general intelligence, outstanding accomplishment in all courses throughout their school careers, and deep interest in and satisfaction derived from the study of mathematics.

Group B includes students with high measurable ability, but whose performance in mathematics has not been commensurate with their ability.

Group C is formed by students with innate ability but deprived of opportunity to develop it to the extent that can be measured by intelligence tests.

In the last *Group D* the author puts children with high general intelligence but low school results in many school subjects, except mathematics where they excel due to their motivation and special talents.

I.4.3 Krutestskii

Krutetskii's study of mathematical abilities (1976) permitted him to isolate three types of mathematically able students: analytic (mathematically abstract) cast of mind, geometric (mathematically pictorial) cast of mind and harmonic (combines both, analytical and geometrical types) cast of mind.

I.4.4 Burjan

Burjan (1991) characterises two main "archetypes" among wide variation in the forms and manifestations of giftedness:

The problem-solver type:

- Extremely skilful (successful) in solving both standard and non-standard (challenging, Olympiad-type) problems
- Is able to perform optimally in timed tests and competitions
- His/her main interest is in searching for solutions to problems that have been posed by somebody else

The investigative type:

- Is interested (and successful) in detailed investigation of open-ended problems
- May be slow and not able to perform well in a definite time and under time pressure, prefers long-term investigation of a problem
- His/her main interest is in asking (and answering) own questions, including own concepts, own methods or algorithms

I.5 EDUCATION OF THE GIFTED : PEDAGOGICAL APPROACHES FOR GIFTED, ABLE AND PROMISING

I.5.1 Different options for gifted students

An identification of mathematically gifted children is only the first step in a long term work with them which is usually followed by various methods of **fostering and nurturing**.

At the elementary school level, there exist several program options for gifted mathematics students according to the culture and policy adapted in a country. Among them, Sovchik (1989) mentions:

- **Acceleration**, which results in an early admission into kindergarten, advanced placement or credit (skipping grades)
- **Enrichment**, which allows to go beyond the standard grade-level work without skipping grades
- **Special schools** for the gifted
- **Private schools** which often offer a gifted program of their own
- **Special classes** in which mathematically gifted students within one school can be brought together in order to study a more advanced program in mathematics
- **"Pull-out"** programs in which gifted students are pulled out of the regular classroom to study special topics in mathematics 2-3 times a week
- **Regular classroom** in which teacher forms meaningful, instructionally valid grouping schema (ability grouping)

I.5.2 Curriculum for mathematically gifted children

Regardless of the pro and contra of each form of work with mathematically able students, we would agree with Kiefer who concludes that "no matter what grouping practice we choose, grouping methods alone cannot solve all teaching-learning problems. What remains most important is the nature and quality of instructional program." (Kiefer, 1995).

Such instructional program, according to Sheffield (1999), helps students develop their mathematical abilities to the fullest. It may thus allow them to move faster than others in the class to avoid deadly repetition of material that they have already mastered. We also agree with the author that

> Such a program may also introduce them to topics that others might not study but, most important, it introduces students to the joys and frustrations of thinking deeply about a wide range of original, open-ended, or complex problems that encourage them to respond creatively in ways that are original, fluent, flexible and elegant. (Sheffield, 1999:46)

In the Enrichment-Triad Model developed by Renzulli one can find the following three types of activities which are important for nurturing mathematical talents (Ridge & Renzulli, 1981: 218):

- General exploratory activities to stimulate interest in specific subject areas: experiences that would demonstrate various procedures in the professional or scientific world (through children's museums and science centres) in which students would get an opportunity to choose, explore, and experiment without the threat of having to prepare a report or provide any sort of formal recapitulation.
- Group training activities to develop processes related to the areas of interest developed through general activities. The aim of these activities is to enable students to deal more effectively with content through the power of mind. Typical for these thinking and feeling processes are critical thinking, problem solving, reflective thinking, inquiry training, divergent thinking, sensitivity training, awareness development, and creative or productive thinking. Problem solving applies to

1. The application of mathematics to the solution of problems in other fields
2. The solution of puzzles or logically oriented problems

3. The solution of problems requiring specific mathematical content and processes.

- Individual and small-group investigation of real problems. As giftedness becomes manifest as a result of student's willingness to engage with more complex, self-initiated investigative activities, the essence of this type of activities is that students become problem finders as well as problem solvers and that they investigate a real problem using methods of inquiry appropriate to the nature of the problem (p.231).

I.5.3 School policy and needs of gifted children: the existence of a gap

As we saw in the previous section, finding appropriate ways of fostering mathematically gifted students is an issue in mathematics education. A number of researches are devoted to the creation of resources for gifted students (list of some available resources is given in the references (pp. 138-139).

Here is an important question: what else can be done for these students beyond the special programs and particular projects? What are the didactical issues of the development of appropriate teaching methods that take into consideration the needs of gifted students in a regular classroom? Does the existing mathematics curriculum (even in its reform efforts) meet the needs of gifted children?

During the past 10 years, school systems of many countries have been looking for changes in their curriculum that would help each student to meet challenges of the modern complex world. The school reform undertaken in Quebec aims to follow these tendencies. As mentioned in the new Program (Programme de formation de l'école québécoise, 2001), the school has to allow each child to receive the best possible education and to reach the highest possible level of self-realisation.

According to this, the learning shall be differentiated in order to meet the educational needs on the basis of respect of individual differences. Thus, a particular attention will be brought to each student helping him/her to fully use his/her personal resources and taking into consideration his/her knowledge and interests.

However, these good intentions can be misunderstood and misinterpreted in the classroom practice. They may lead to oversimplifying mathematical learning in order to make it fun, taking out all abstract, theoretical, difficult, "unnecessary" material of the "good old times" and replacing it by the colourful new "supermarket curriculum" oriented to the so-called consumer needs. As it is phrased by Berry (1996), "If you can't use it in the supermarket, let's not teach it" .

This tendency is at odds with the needs of gifted students.

The intensive use of so-called new methods of teaching: project based learning, group work, social oriented teaching based on the student's real life experience, was supposed to help students to understand mathematics better.

However, it remains an open question, to what extent these innovations provide a harmonious passage to more abstract levels of mental development and are sufficiently helpful to form a student's theoretical thinking. The approach might create a gap between school and higher levels of education, such as college and university. Some students would be having difficulties in following advanced mathematics and science courses. The most able students would lose their interest in mathematics and diminish their high intellectual potential.

At the same time, the new curriculum gives schools and teachers a real opportunity to develop classroom situations that meet the needs and interests of all students. That is why it is important to create effective didactical tools that would help teachers to develop each student's full potential.

According to Tempest (1974 :9), "it is sometimes urged that there is no need to make any special provision for gifted pupils in class. They will do well, anyway, it is said, and the teacher should concentrate her attention on those at the other end of the scale who find difficulty with their work". However he stresses that a gifted child

> can easily and quickly do what is required of him and, in so doing gain the approval of teacher and parents, why should he do more? If he finds the task given him easy to accomplish, he may well become bored and his parents may have difficulty in sending him to school. Alternatively, having completed his work easily and quickly, he may look around for more challenging problems and become a ringleader of all sorts of mischievous exploits. (Ibid.)

Another aspect of this issue is related to the cognitive structures that have to be taken into account by educators. Often, the organisation of the classroom activities gives too much freedom to children in order to help them to develop their own knowledge.

We would agree with Gelman's (1991:319) observation that learners who have not developed structures to guide their search for and correct interpretation of nurturing inputs, can no longer be counted on to find their own food for thought. Left to their own devices, students have every reason to have misconceptions. We have to be prepared to probe their understandings and make special efforts to draw children's attention to relevant characteristics of data. Finally, she underlines that "pupils are more likely to encounter structured data, to practice at using what they are learning, and to come to understand how to interpret the terms and symbols of the domain if given a fair chance to become fluent both in the structure and the language of the domain" (Ibid.).

Finally, the curriculum that meets the interests of gifted students (and not only the gifted), shall encourage teachers to create situations of a real learning.

Thus, the Piagetian notion of cognitive conflict is useful for us in so far as it is a conflict really experienced by the learner, who "feels it", "lives it internally" as his/her own problem. Only then can this conflict lead to changes in knowledge (Schleifer, 1989).

From these considerations, we assume that all the children would benefit from more challenging curriculum that would give them better chances to confront rich mathematical content. We think that the content of learning itself is not as important for the students as the way **how** it is taught and **what** is the *learning environment* in which it is taught.

To make learning and teaching intellectually enriching and developing, some authors suggest the use the method of content and genetic logic and its powerful tool, reflection (Shchedrovitskii, 1969, Rosin, 1967).

By bringing reflective ideas to the classroom, we bring in a methodology, a structure and a dynamic. Through the development of classroom situations that are challenging and contain a rupture with child's previous knowledge and beliefs, we would disturb the child's state of comfort (fun), make him search for problems, and look anew at old stereotypes. The rebuilding of problem solving process according to new means and rules of play would increase child's motivation to learn, develop his/her critical thinking ability, capacity to ask questions, to investigate.

CHAPTER II. MATHEMATICAL THINKING AND GIFTEDNESS: PROBLEMS AND QUESTIONS

II.1. INTRODUCTION

The literature survey shows a variety of models and methods of identification and fostering mathematical giftedness. In our study, we focus on those which can be applied to the everyday teaching practice. We base our study on the assumption that mathematical giftedness appears already in the early school age. Therefore, we as primary school teachers cannot wait till mathematically gifted children are detected in the middle grades and transferred into special programs.

We have to take care of these children as early as we can and create favourable conditions for their development on the everyday basis. Questions of teaching and learning organization become thus crucial for identification and fostering mathematically gifted children.

In this chapter, we analyse examples that demonstrate a shortage of efficient methods of working with mathematically gifted children. In order to develop such methods, we shall look deeper at the phenomenon of giftedness with regard to its purely mathematical part: mathematical thinking.

We thus relate the notion of mathematical ability to a particular way of thinking theoretically that appears early in gifted children and determines their mathematical behaviour in various learning situations. The following didactical question arises then: how to create an appropriate environment to make theoretical thinking work.

Following are examples that show, how certain classroom situations might stimulate children's mathematical thinking.

II.2 Example from the history: Gauss

The story of Gauss solving a routine problem of calculating the sum of the first hundred natural numbers is one of the well known examples of this kind. While all other children were desperately trying to add terms one by one, Gauss impressed the teacher by finding a quick and easy way to do it regrouping the terms in a special way (see, for example, *Dunham, 1990, pp. 236-237*).

But one can ask: what were the characteristics of the classroom situation, which allowed the gifted student to demonstrate his talent in mathematics?

The same story says that the teacher had chosen the task for its accessibility to all students (the task is routine) and the probably very long time that it would take the students to solve; he hoped to thus keep them all quiet and busy for a good while. What he didn't expect that one of the students would turn the routine task into a challenging one of finding a quick way of solving an otherwise tedious and long computational exercise. The situation was not planned to reveal a mathematical talent, yet it did so 'spontaneously'. The situation became a challenging one by chance.

In many cases, mathematical giftedness would not be identified. We could say that using routine drill tasks involving numerous standard algorithms are not, in general, offering a good opportunity to identify and nurture mathematical talents.

Sheffield (1999) calls such routine tasks 'one dimensional'. As an example, she cites a class of three-four graders reviewing addition of two-digit numbers with regrouping . Children are asked to complete a page of exercises such as: 57+45, 48+68, 59+37. As it usually happens with brighter and faster students, they finish all exercises before their classmates. So the teacher would 'challenge' them with 3- or 4-digit additions.

additions. Although the calculations become longer and time consuming, the tasks themselves are not more complex or more mathematically interesting.

As a better didactical solution for these children, Sheffield suggests the use of meaningful tasks as *finding three consecutive integers with a sum of 162*:

> Students would continue to get the practice of adding two-digit numbers with regrouping, but they also would have the opportunity to make interesting discoveries. Students who are challenged to find the answer in as many ways as possible, to pose related questions, to investigate interesting patterns, to make and evaluate hypotheses about their observations, and to communicate their findings to their peers, teachers, and others will get plenty of practice adding two digit numbers, but they will also have the chance to do some real mathematics. (Sheffield, 1999: 47)

She claims that following an open-ended heuristic model, we would contribute into a student's creative development of mathematical abilities (Ibid.:46, Fig.2)

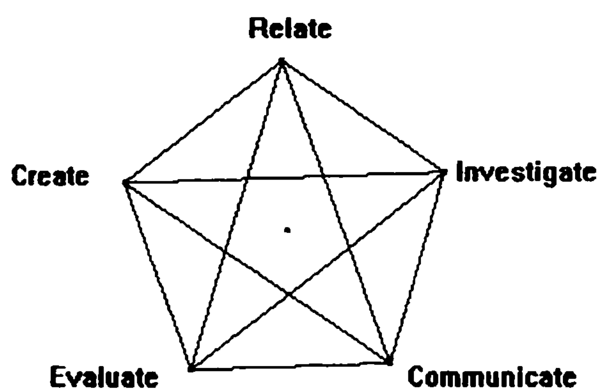


Fig. 2 Open-ended heuristic model

II.3 ELEMENTARY SCHOOL CHILDREN'S STRATEGIES ON CHALLENGING TASKS IN THE MATHEMATICS CLASSROOM

By giving children a challenging task we would expect them to make efforts in understanding a problem, to search for an efficient strategy of solving it, to find appropriate solutions and to make necessary generalizations.

In following examples we present three very different approaches to the same problem of *finding a number of handshakes that we obtain when n people shake hands of each other.*

Marc-Etienne (10) organized an experience with his classmates, considering systematically the cases n=2, n=3 , etc. Then he made necessary generalizations. Here is a transcript of his report (see Appendix A1, p. 141):

Picture 1: two circles connected with an arrow representing two people - one handshake. He wrote beside the picture '=1'.

Picture 2: three circles forming a triangle connected with three arrows representing three people - three handshakes. He wrote beside the picture '=3'

Picture 3: four circles forming a square connected with six arrows representing four people - six handshakes. He wrote aside: '3+2+1=6' commenting:

'1 à un ils se retirent'

Picture 4: Five circle forming a 'domino-5-dots' disposition connected with only six arrows (some arrows are missing). However, he wrote '4+3+2+1=10' continuing the same pattern.

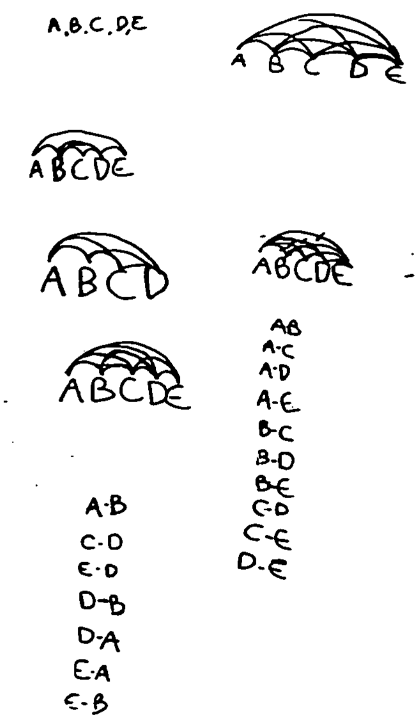
Picture 5: Six circles disposed in two rows (by three), no arrows. He wrote '5+4+3+2+1=15'

Picture 6: Seven circles disposed in two rows (three+four), no arrows. He wrote '6+5+4+3+2+1=21'

Picture 7: Eight circles disposed in two rows (by four), no arrows. He wrote '7+6+5+4+3+2+1=28'

He wrote than sentence that he called 'Formule': 'On calcule toujours de la manière que le prochain chiffre soit -1 '2+1' et que chaque chiffre qui précède soit +1'

Charlotte (10) made diagrams showing her systematic search for all possible combinations.



Christopher (10) wrote just one single sentence : 1+2+3+4+5+7+8=36

He provided a comment that if we have a group of people, each person has to shake hands to all people who came before, so with 2 people we would have 1 handshake, with 3 people - 2 more handshakes (1+2), and so on.

So, with a simple question what would be the number of handshakes with 100 people, the class arrived to the same problem that Gauss had to deal with but posed in different, challenging way. Thus, a generalization might be provoked here in a more natural way.

And, from the point of view of practicing teacher, we can ask:

How to organize children's mathematical activities so that they were motivated to act this way?

So far, the problems of identification and nurturing mathematical giftedness has been analyzed separately but the examples given above show the pertinence of the search for combined didactical methods of identification and nurturing through the everyday teaching and learning.

The crucial element that allows to link both processes is a particular way of mathematical thinking that we provoke in children in order to reveal their above average mathematical ability and at the same time ensure its constant development.

Thus we link mathematical giftedness to a particular way of thinking mathematically. But first, we have to look closer at the notions of "thinking" and "mathematical thinking".

II.4 Analysis of the Nature of Mathematical Thinking

Many authors point at the **ability to recognise patterns and to see relationships** as a *key element* in mathematical thinking.

Fisher argues (1990) that since mathematics is a highly structured network of ideas, to **think mathematically** is to *form connections* in this network: the task of a teacher then is to help children to see the structure inherent in mathematics, not just rules and facts learned in isolation. He states that in encouraging children to think mathematically we need to engage all aspects of a child's intelligence. There are different ways of processing mathematics according to the following scheme (Fig.3):

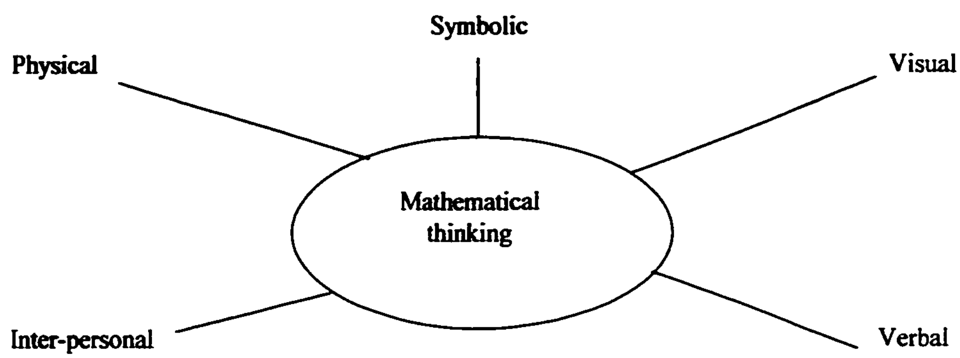


Fig.3 Different ways of mathematical thinking

We see that, according to this model, mathematical problems can be modelled or represented in a variety of ways:

- *Verbally:* through inner speech and talking things through, using linguistic intelligence, putting planning procedures and process into words, making sense and meaning for oneself
- *Inter-personally:* learning through collaboration observing others, working together to achieve a shared goal, exchanging and comparing ideas, asking questions, discussing problems
- *Physically:* using physical objects in performing mathematical tasks, working with practical apparatus, equipment and mathematical tools, modelling a problem or process, having hands-on experience, using bodily-kinaesthetic skills, practical applications into the physical world
- *Visually:* putting processes into pictorial form, making drawings or diagrams visualising patterns and shapes in the mind's eye, thinking in spatial terms, graphical communication, geometric designs, manipulating mental images
- *Symbolically:* using written words and abstract symbols to interpret, record and work on mathematical problems, using different recording systems, logically exact languages, translating into mathematical codes

According to Baroody (1987), genuine learning involves **changing in thinking patterns**. More specifically,

> Making a connection can change the way knowledge is organised, thus changing *how* a child thinks about something. Put differently, insights can provide fresh and more powerful perspectives. Take the child who does not know the basic subtraction combinations and must use finger counting to calculate differences. Given the series to problems 2 - 1 =_, 4 - 2 =_, 6 - 3 =_, 8 - 4 =_, 10 - 5 = _, the child laboriously calculates each answer. Suddenly the child has an insight: The subtraction combinations are a mirror image of the well-known addition doubles (1 + 1 = 2, 2 +2 = 4, 3 + 3 = 6, 4+ 4 = 8, and 5 + 5 = 10). There is a relationship between subtraction combinations and the familiar addition facts! Afterward, she views subtraction in a different light. Given a problem like 5 - 3 = _, the girl now thinks to herself; "Three plus what makes five? Oh, yeah, two." Her new perspective now enables her to solve subtraction combinations efficiently without laborious calculation. Mathematical development, then, entails qualitative changes in thinking as well as quantitative changes in the amount of information stored. Essential to the development of understanding are changes in thinking patterns. (Baroody, 1987: 11)

According to Schrag (1988), mental activity is *purposeful* thinking only if it is experienced as *directed* to a problem or task one has set oneself. This, admittedly normative, conception is meant to include cases in which we may suddenly see a solution without any awareness of "wrestling" with a problem. But even in such cases, an idea does not appear as a *solution* unless it is experienced in relation to some difficulty one has been worrying about. Thus thinking is evoked in **situations** where one is not quite sure how to go on. Schrag calls such situations *problems*.

In order to solve a problem, a person has to use various strategies. Here is a list compiled by Ernst (1998):

- re-reading the problem
- simplifying the problem
- understanding the problem (by group discussion)
- trying to solve a simpler problem

- holding one variable constant (at a small value) and exploring values of another variable
- drawing a diagram
- representing the problem by symbolisation
- generating examples
- obtaining data
- making a table of results
- putting the results in a table in a suggestive order
- making conjectures
- thinking up different approaches
- trying them out
- justifying answers.

Referring to works of Polya (1957) and Schoenfeld (1979), Ernst points at two types of activities: cognitive and metacognitive. **Cognitive** activities include using and applying facts, skills, concepts and all forms of mathematical knowledge. They also include applying general and topic specific mathematical strategies, and carrying out problem-solving plans. **Meta-cognitive** activities, involve planning, monitoring progress, making effort calculations (e.g. 'Is this approach too hard or too slow?'), decision making, checking work, choosing strategies, and son on. Metacognition (literally: "above cognition") is about the management of thinking.

Sierpinska (2002) characterizes mathematical thinking as "a good balance between **theoretical** and **practical** thinking". The following table shows the differences between these two ways of thinking with respect to their aims, objects, main concerns and results (Tab.1).

Tab. 1 Comparative analysis of theoretical and practical thinking

Aspect of thinking	Theoretical thinking	Practical thinking
Aim	• Is thinking for the sake of thinking	• Thinking for the sake of getting things done or making things happen
Object	• Systems of concepts	• Thinking about particular 'objects' (things, matters, events, people, phenomena)
Main concerns	• Meanings of concepts	• Significance of actions
Nature of connections	• Conceptual connections: establishing and studying relations between concepts as they are characterized within a system of other concepts	• Factual connections: contingency in time and space, analogy between observed circumstances across time, focus on particular examples and personal experience
Sources of validity	• Epistemological validity: Concerns about conceptual coherence and internal consistency of systems of symbolic representations • Makes no claims to stating 'the truth' about experience - cares more about producing conditional or hypothetical statements (propositions)	• Factual validity: the proof of a plan of action is in the results of the action and in the agreement of the assumptions of the plan with experience
Methodology	• Operates on two levels: reasons about concepts and reasons about this reasoning • Is concerned with the symbolic notation and forms of graphical representation, and with the rules and principles of reasoning and validation that it uses.	• Operates always on a single level of its relation with its aims and objects • Uses only some ad hoc symbols, different in solving each particular problem.
Results	• Changes in the objects of this thinking	• Theories and specialized notations

In their study of relationships between theoretical thinking and high achievement in linear algebra, Sierpinska, Nnadozie & Oktaç (2002) assume that theoretical thinking "is not a continuation but a reversal of the practical thought." (p.11). They thus view practical thinking as an "epistemological obstacle" that cannot and should not be avoided. However, they claim that teaching abstract mathematical concepts that puts too much emphasis on the "concrete" experience based on so-called "geometric" or "numerical" approaches might leave students with representations irrelevant from the point of view of the concepts and lead them to contradictions. (Ibid.: p.19).

Defining theoretical thinking as reflective, systemic (definitional, proof-based and hypothetical) and analytic (linguistic sensitive and meta-linguistic sensitive), the authors argue for the necessity of theoretical thinking in understanding linear algebra as following:

- The undergraduate learner of linear algebra must be even more theoretically inclined than the inventors of the theory
- Meanings of concepts must be sought in their relations with other concepts
- The learner must engage in proving activity and therefore use systemic approaches to meanings and validity
- The learner has to accept that his or her ontological questions will remain unanswered
- The learner must engage in hypothetical thinking
- The learner must become mathematically "multilingual" (Ibid.:33-35)

If we project these ideas on the elementary school level, we could see that today's tendencies (discussed in previous sections) do not favor the education of a "theoretical thinker" although our practice shows that mathematically gifted students, even at an early age hold certain epistemological views about mathematics that are close to the theoretical thinking.

When we think about the interpretation of these aspects of thinking in terms of giftedness, then we may suppose that a mathematically gifted child who is a high achiever would probably demonstrate a balanced ability to think mathematically (theoretically and practically). A mathematically able child who is not a high achiever would be rather more "theoretically inclined". The question is whether a mathematically able child can be only "practical"? Another question is to what point we can identify a child as a theoretical thinker?

Another question arises: what kind of classroom situations would enhance the fostering the development of mathematical thinking in young children ?

II.5 How to foster the development of mathematical thinking?

In order to *foster* the development of mathematical thinking, Baroody (1993) stresses a use of a problem-solving approach which focuses on the processes of mathematical inquiry: problem solving, reasoning, and communicating. It is a teacher-guided approach in which a student plays an active role.

Ernst (1998) makes a comparative analysis of different teaching approaches related to the mathematical thinking. It shows the didactical transition of mathematical process which "progresses from the application of facts, skills and concepts, to a limited repertoire of problem-solving strategies including generalisation and the induction of pattern, to the full range of problem-solving strategies, and finally adding problem-posing processes as well"(p.132) happens when the classroom teaching becomes more open and challenging (Tab.2):

Tab. 2 Ernst's comparative analysis of different teaching approaches

Teaching approach	*Role of the teacher*	*Role of the learner*	*Process involved*
Direct instruction	To state an item of knowledge explicitly. To provide exercises for application	To apply the given knowledge to exercises	The application of facts, skills and concepts
Guided discovery	To present a rule or other form of mathematical knowledge implicitly in a sequence of examples	To infer the rule or knowledge implicit in the given examples	Generalisation. Induction from pattern
Problem-solving	To present a problem to the student, leaving the solution method open	To attempt to solve the problem using own method(s)	Problem-solving strategies
Investigatory mathematics	To present an initial area of investigation, or to vet a student's own choice	To choose questions for investigation within the topic given. To explore the topic freely	Problem-posing and problem-solving strategies

Fishbein (1990) defines a teacher's task as to "create an environment that would require a mathematical attitude, mathematical concepts, and mathematical solutions". He argues that facing a challenging task children might not be able to find solutions spontaneously. He might get engaged in a constructional process combining various conditions. He has then to produce a method to work on the problem systematically. Fishbein sees this aspect of finding a method, an algorithm used consciously as fundamental for the development of mathematical reasoning.

According to Fishbein (1990, p.8), the question is whether the teacher should wait until children find the method by themselves without any help. In his opinion

> Formal reasoning doesn't develop spontaneously as a main way of thinking. This conclusion doesn't imply that the teacher should simply offer the solution. What the teacher should do is to direct the student's efforts to a solution by asking adequate questions. The student builds the answers as a reaction to a certain environment. This environment should be programmed as a problematic one in order to inspire student's solution endeavors. (Fishbein, 1990: 8)

These theoretical guidelines interfere with Driscoll's (1999) remarks that through

- consistent modeling of algebraic thinking
- giving well timed pointers to students that help them shift or expand their thinking, or that help them to pay attention to what is important.
- making it a habit to ask a variety of questions aimed at helping students organize their thinking and respond to algebraic prompts.

teachers would promote those habits of mind that are specific to the algebraic thinking and should be developed in children as following:

- Reversibility as a capacity not only to use a process to get to the goal, but also to understand the process well enough to *work backward* from the answer to the starting point.
- Building rules as a capacity to *recognize patterns* and *organize data.*
- *Abstracting from calculations* as a capacity to think about computations independently of particular numbers that are used. (Driscoll, 1999:3)

Drawing from the above mentioned theoretical considerations, we will move now towards more practice-oriented questions: what are mathematical activities that would help to foster the emergence of a theoretical thinking in mathematically gifted elementary school students allowing them to progress in the mixed-ability classroom?

Many studies point at mathematically rich tasks as an engine of such fostering. Peressini & Knuth (2000) mention that mathematically rich are tasks that fit following criteria:

- encourage a range of solution approaches,
- address significant mathematical concepts,
- require students to justify their explanations,
- are open ended

We saw examples of these tasks in previous sections ('tessellations', p. 22 ,'parking lot', p. 35, 'handshakes', p. 49).

The use of such tasks requires a rethinking of the role of both the teacher and the student. Burton (1984) stresses that the teacher's role shifts from that of providing information, to question-asker and resource-provider. The teacher would challenge pupils to justify or falsify arguments and to reflect on what has been done. The tone of the teacher's interventions is also important. It has to emphasise enquiry rather than instruction.

As for the student, the author underlines that along with sustaining interest, motivation and success in problem-solving, the following aspects are also to be mentioned:

- Choosing and using representations that would enhance the ability to model a problem
- Resolving instead of solving which helps to develop an attitude to build a network of new questions, new resolutions, and further questions out of an initial problem following a spiral development instead of a linear one (problem - solution)
- Communicating using different tools of communication

At the same time, new questions arise:

- how to design appropriate classroom situations in which challenging tasks could be posed in a natural meaningful way ?
- how to make challenging situations not an isolated episode of teaching but a permanent component of teaching and learning ?

CHAPTER III. THEORIES UNDERLYING THE DEVELOPMENT OF A CHALLENGING SITUATIONS APPROACH TO IDENTIFYING AND NURTURING MATHEMATICALLY GIFTED CHILDREN

III.1 INTRODUCTION

Based on studies of mathematical giftedness on the one hand and on theories of mathematical thinking on the other hand, we are going to construct a theoretical framework for our study of identification and nurturing mathematically gifted children at a young age.

We claim that within today's system of identification and nurturing, many mathematically gifted children are left unrecognised (either not being detected by test or not having possibilities to be admitted to special program) and unnurtured (bored with too simple or too mechanical tasks). Actually, every teacher might discover mathematical talents in her classroom if equipped with an appropriate approach to teaching. Our study presents a model of such an approach as well as examples of its applications in practice.

In the second chapter, we have linked mathematical thinking and mathematical giftedness by the notion of theoretical thinking as a particular way of thinking in mathematically gifted children. Presently, we will look at didactical tools that could help teachers create a meaningful learning environment in which theoretical thinking would be given a chance to work.

Although some authors point to the fact that solving mathematically demanding problems requires rich knowledge about numbers and number relationships, normally not available to elementary school students (see, for example, Lorenz, 1994), others (e.g. Krutetskii,1976) affirm that already at the age of 7 or 8, gifted children "mathematize" their environment, giving particular attention to the mathematical aspects of the phenomena they perceive. They realise spatial and quantitative relationships and functional dependencies, in a variety of situations, that is, they see the world through mathematical eyes.

These children are eager to learn mathematics, they enjoy it, and teachers should use every opportunity to nurture their young fresh minds. Thus, a special environment has to be created in order to maintain their genuine interest. We shall call this environment *challenging,* as it is composed of a variety of situations that provoke mathematical questioning, investigations, use of different strategies, reasoning about problems and reasoning about reasoning. These situations also require special abilities to organise and re-organise mathematical knowledge for solving new problems by developing and using different strategies. Altogether, these situations help reveal mathematical giftedness while at the same time they foster further progress in children.

Our framework draws on the work of Krutetskii (1976), who postulated that a mathematically able child can, at a young age,

- **formalize** a problem situation by linking logically related data,
- **generalize** particular cases by combining separate data into more general structures,
- **curtail** mathematical operations keeping in mind all intermediate steps,
- demonstrate a **flexible way of thinking** switching easily from one idea to another, and **rationalize** their thinking, by critically evaluating different ways to solve a problem.

Krutetskii claimed that these components of mathematical activity are integrated into a specific mental structure called the ***"mathematical cast of mind"***.

Moreover, we assume that mathematically gifted children are inclined to theoretical thinking in their approach to reality and mathematical problem solving, that is, they are likely

- to engage in thinking for the sake of thinking and not only for the sake of getting things done (reflective thinking)
- to be concerned with the structure of relations between concepts, and not only with concrete objects or actions on them; to be critical with respect to the validity of their own and other's claims; and to view mathematical statements as conditional and hypothetical (systemic thinking)
- to be aware of the arbitrary and conventional character of representations of concepts (analytic thinking)

We are aware that very young learners may not exhibit all these features of theoretical thinking. However, in specially constructed learning and teaching situation, children's behaviour hints at the potential for the development of these features in further advanced mathematical learning.

We now go on to make explicit those pedagogical conditions, under which mathematically gifted children demonstrate their superior mathematical abilities.

III.2 Paradoxes of teaching

In his theory of didactical situations, Brousseau (1997, p. 41) describes the so-called paradox of devolution of situations. He states that in the situation where the teacher "is induced to tell the student how to solve the given problem or what answer to give, the student, having had neither to make a choice nor to try out any methods nor to modify her own knowledge or beliefs, will not give the expected evidence of the desired acquisition." But at the same time, the teacher has a social obligation to "teach everything that is necessary about the knowledge. The student - especially when she has failed - asks her for it." This situation is obviously paradoxical: "the more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for."(ibid.)

Brousseau thus claims that everything the teacher undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the ***necessary conditions for the understanding and the learning of the target notion***.

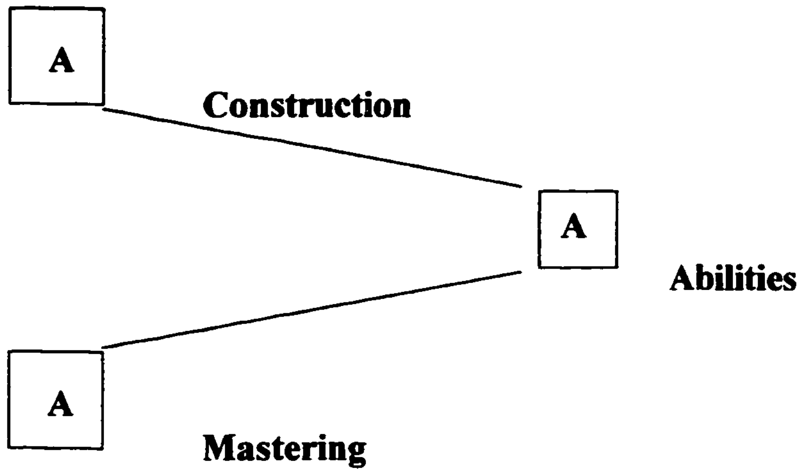
Questioning further the nature of the ability to reflect on a mathematical task, we found interesting links between Brousseau's theory and the works of the Russian methodologist G. Shchedrovitskii which are probably less known to Western educators.

Shchedrovitskii (1968) illustrates his logical analysis of teaching and learning with a similar paradoxical phenomenon. He remarks that when we as educators want our children to master some kind of action, we often tend to teach it directly by giving children tasks which are identical with this action. But classroom practice shows that the children not only do not learn actions that go beyond the tasks, they do not even learn the actions that we teach them within the tasks.

The following schema (Fig.4) shows that in a direct instruction the "inputs" (- what do we teach) and the "outputs" (- what children learn), are the same identical actions.

Schema 1.

Actions - products



Actions - contents

Fig.4 Relations between actions and abilities in direct instruction

This schema demonstrates that since all elements are identical (A), there is no need for the development of an "ability", that is, of possibility of constructing a similar action in different circumstances. For example, teachers may pose the task of constructing a square out of the given four identical cubic blocs. This activity presents no real challenge for children. It doesn't require any construction of a process of learning as a movement from the known to the unknown. It is, instead, a move from the known (a shape of the square as image in child's mind) to the known (reproduction of the shape by means of cubes), but in a different form. In this situation, all normal children succeed easily. When the same children are asked to produce a square with no given number of cubes, they might puzzle a teacher by making rectangles instead of squares. They might reveal a lack of understanding of notion of square (or, an inability to control the condition of the 'squareness'), and thus the need to learn something new would arise.

Shchedrovitskii's logical analysis of the learning of some action is based on the principle shared by many psychologists and educators: the subjective conditions of activity, or "abilities" are just "copies" of actions mastered by the individual and then appearing in specific new situations. So, the abilities are the same actions but in potentio. It seems that the use of the term "compétence" in the recent Quebec's curriculum reform is based on a similar understanding of the notion of "ability".

The third paradox is related to the idea that, in order to access a higher level of knowledge or understanding, a person has to be able to proceed at once with an integration and reorganisation (of previous knowledge). This idea can be found in Piaget's theory of equilibration of cognitive structures, as well as in Bachelard's notion of epistemological obstacle (see, e.g. Sierpinska,1994). For example, a passage from arithmetic to algebra via whole numbers requires a perception that natural numbers are no more collections of objects (pizzas, cakes, apples) but a structure with operations that can be a base for further generalisations.

Sierpinska (1994) sees the need of "reorganisations" as one of the most serious problems in education. In teaching, we do not follow the students' "natural development" but rather precede it, trying, of course, as far as possible, to find ourselves within our students' "zones of proximal development". But we can not just tell the students to "how reorganise" their previous understanding, we can not tell them what to change and how to make shifts in focus or generality because we would have to do this in terms of a knowledge they have not acquired yet. So, we must involve students in new problem situations and expect all kind of difficulties, misunderstandings and obstacles to emerge and it is our main task as teachers to help the students in overcoming of those, in becoming aware of differences, in the hope that then the students would be able to make the necessary reorganisation.

All three researchers, Brousseau, Shchedrovitskii and Sierpinska, construct models of teaching which give the teacher efficient tools of minimization of negative effects of this paradox: meaningful didactic situations (Brousseau, 1997), construction of new means by mean of reflective actions (Shchedrovitskii, 1968) and stimulating of "good understanding" (Sierpinska, 1994). For the purpose of our study, we shall look more closely at these models.

III.3 APPROACHES TO OVERCOMING THE PARADOXES OF TEACHING

In this section we will look at three approaches to overcoming the paradoxes of teaching: didactic approach (Brousseau, 1997); a logical approach (Shchedrovitskii, 1968); and an epistemological approach (Sierpinska, 1994).

III.3.1 Constructing of teaching based on problem situations

In his theory of **didactic situations,** Brousseau (1997) states that

> The construction of meaning, as we understand it, implies a constant interaction between the student and problem-situations, a dialectical interaction (because the subject anticipates and directs her actions) in which she engages her previous knowing, submits them to revision, modifies them, completes them or rejects them to form new conceptions.
> The main object of didactique is precisely to study the conditions that the situations or the problems put to the student must fulfil in order to foster the appearance, the working and the rejection of these successive conceptions. (Brousseau, 1997:82-83)

In his fundamental work, Brousseau brings a new perspective to a vision of a problem situation based on the conception of learning which relies on the study of the development of knowledge in terms of overcoming obstacles. This conception differs thus appreciably from the classical conception especially concerning the role and organisation of problem situation.

The posing of a problem consists, according to Brousseau, "of finding situation with which the student will undertake a sequence of exchanges concerning a question which creates an "obstacle" for her, and from which she will derive support for her acquisition or construction of a new piece of knowledge."(Ibid.:87)

Brousseau underlines that "the conditions under which this sequence of exchanges is displayed are initially chosen by the teacher, but the process must quickly move under the partial control of the subject, who will, in her turn, 'question' the situation. Motivation is generated by this investment and maintains itself by it. Instead of being a simple external motor, in balancing frustration, it builds up both the subject (her word) and her knowledge". (Ibid.)

This last remark seems to be very fruitful for the organisation of work with mathematically gifted because, according to Brousseau (1997), under such conditions, the resolution of a problem will be for the student a kind of "experimental path".

According to Brousseau (Ibid.), the process of overcoming an obstacle has a dialectical character: dialectic of a priori and a posteriori, of knowledge and action, of self and others, etc (p.88). He continues:

> Organising the overcoming of an obstacle will consist of offering a situation which is likely to evolve and to make the student evolve according to a suitable dialectic. It will be a question not of communicating a piece information that we wish to teach but of finding a situation in which it is the only satisfactory or optimal one-among those with which it is competing-for obtaining a result in which the students is investing.
>
> That is not sufficient: the situation must immediately allow the construction of an initial solution or of an attempt in which the student invests her current knowledge. If the attempt fails or is inappropriate, the situation must nonetheless produce a new situation, modified by this failure in an intelligible but intrinsic way; that is to

say, not depending arbitrarily on the aims of the teacher. The situation must allow the voluntary repetition of the testing of all student's resources... (Brousseau, 1997:88)

In the student functioning, Brousseau considers different types of dialectics:

- Questions of validation
- Questions of formulations
- Questions of action.

None of these dialectics is independent of the others: formulation is often facilitated if an implicit model of action exists: the subject knows better how to formulate a problem if she has been able to solve it, action at its turn, is facilitated by a suitable formulation. But at the same moment, each domain can be an obstacle to progress within the others. Some things are better done than said. Implicit models are better able to take a larger number of facts at the same time, and are more versatile and easier to restructure. Conditions that are too favourable to action make explanation useless.(Brousseau, 1997:89)

III.3.2 Shchedrovitskii's "reflective" schemas for the fostering of abilities

Shchedrovitskii (1968) puts an emphasis on the methodological component of didactical study. He develops various schemas that represent the reflective character of teaching and learning.

In order to overcome the paradox of teaching stated in III.2, Shchedrovitskii (1968) proposes to include the contents of the learning and products of the learning into a new, more complex schema (Fig.5) of relationships which becomes an object of didactical study.

Schema 2.

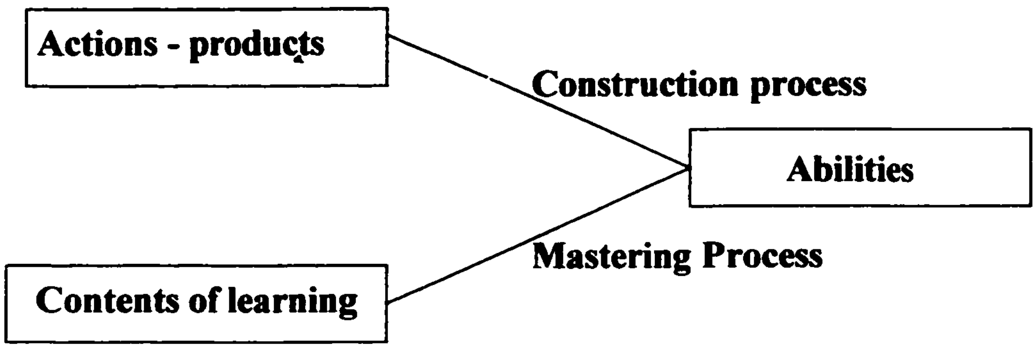


Fig.5 Relations between actions and abilities in the challenging approach

Let us come back to our example of the task of constructing squares with cubic blocs (see p. 66). Here is how the problem could be posed: giving a certain number of little cubes to the students, we ask them to construct as many squares as they can. Asking a problem in this way, we help students to develop an ability because they have to:

- Choose an appropriate number of cubes
- Construct the right shape (without confusing squares with rectangles)
- Search for different solutions

Instead of a simple reproduction of already acquired knowledge and skills, a child is engaged in a process of construction of "new" (to her) actions making thus an active use of her abilities.

III.3.2.1 Objective and psychological planes of learning

In his analysis, Shchedrovitskii puts the learner's action in the centre of the process of learning, considering two planes: **objective**, or logical, which considers abstraction from the psychological sphere of individual learning and **psychological**, or logical-psychological, in which the attention is brought upon the description of subjective mental processes. He underlines, however, that these two levels are linked together.

Schema 3.

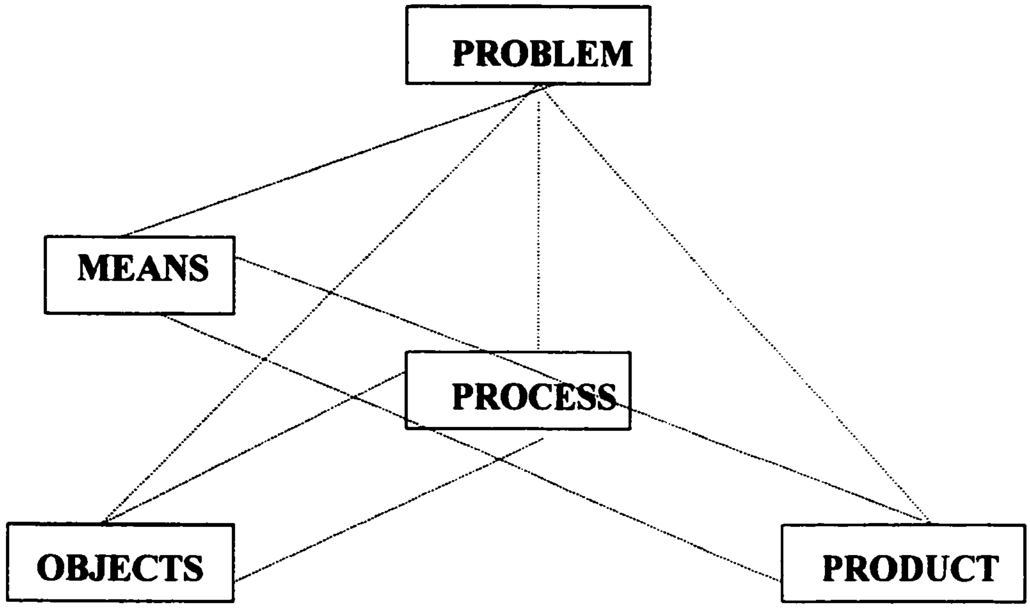


Fig.6 Objective analysis of a teaching environment

The **objective analysis of a teaching environment or a didactic situation** includes (schema 3, Fig. 6) the following analyses:

- an analysis of the nature of problems and forms of their objective existence, of their relationships to the means and processes of action
- an analysis of the structure of the processes themselves, e.g. the number and the character of operations required to solve the problems, the nature of objects and symbols (signs) to which these operations are applied, links between operations in the context of processes of action
- an analysis of means used for the construction of processes (often they have the form of symbols (signs))
- an analysis of products of action (main and secondary), their properties, relationships to the problems themselves and the means used.

For example, when we give a problem of finding a sum 2+3= ...to a five year-old child, we expect her to count two objects and to continue counting adding three more objects. If we ask this problem in a different way, like ... + ... =5, we would activate completely different processes with different objects, different means and different products.

In order to understand how the individual constructs his/her action and what is the mechanism of child's development and thus to be able to analyse problem solving at a **subjective psychological level**, one would need a different schema.

Shchedrovitskii (1968) suggested that we need to include abilities in a bloc of means of construction of the process, or even to call abilities the whole bloc of means considering that an individual appropriates and internalises these symbolic means (schema 4, Fig.7).

Schema 4.

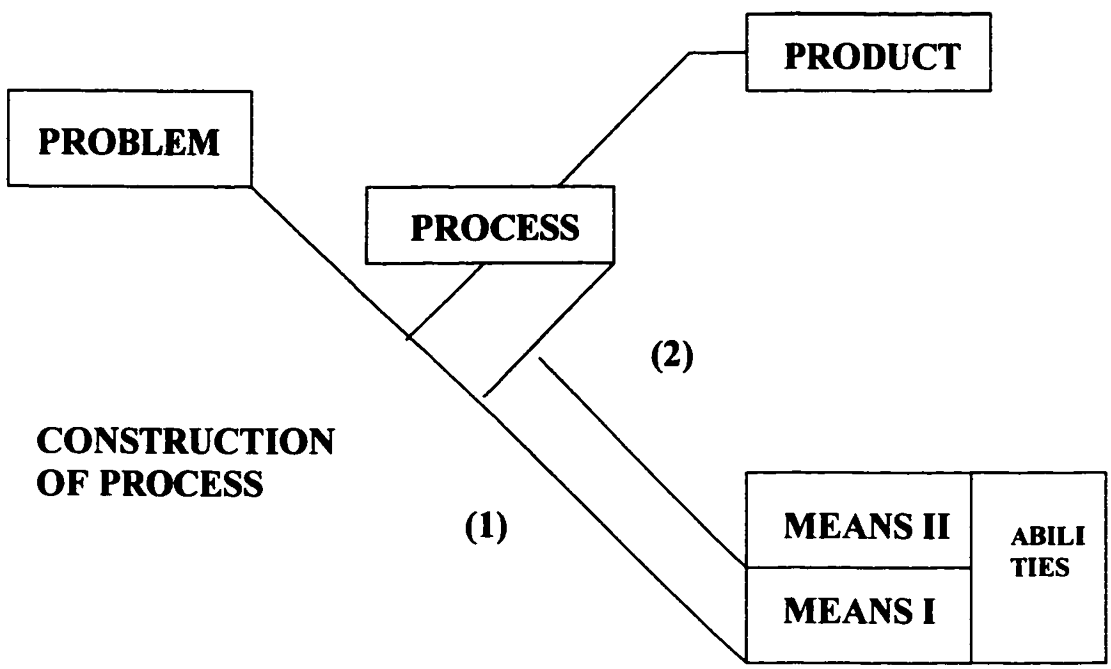


Fig.7 Individual approach to the construction of the problem solving process

For example. the problem 2+3 = ... would require the use of counting abilities only. The problem ... + ... =5, instead, would require not only a meaningful use of old means but also the construction of new ones. Therefore, various new abilities can be developed (*for example*: decomposing, taking commutativity into account, looking at number as a concept, search for different solutions, (systemic) search for all solutions) Therefore, one could observe two components of construction of action:

- mechanisms of correspondence of the problem with the available means taking into account the fact that a child already has choice of certain means to solve the problem (we call this an "understanding" of the problem; see arrow 1 on the schema);
- mechanisms of using specific regulations (rules or models of previous activities) in order to construct a new process (arrow 2 of the schema).

III.3.2.2 The role of reflective action in learning

The next schema, developed by Shchedrovitskii (Schema 5, Fig.8), illustrates that, on the one hand, using abilities, the child can build the links between the new and old processes, and, on the other hand, abilities themselves could provide the individual with the links between the old and the new processes and, therefore, they have to be developed.

To develop these abilities, the child has to construct various processes of actions using appropriate means (signs, symbols, formulas, sequences, structures) and master the methods of activity in which these means are organised.

Shchedrovitskii hypothesized that for the creation of new objective means and of new objective methods of action along with them, it is necessary that the action itself becomes an object of transformation, and a new secondary, reflective, action would influence it assuring the development of appropriate mental functions.

III.3.2.3 Example of creating new means by young children

In this section, we will illustrate the act of creation of new means on the basis of reflection. Grade 3 children were working on the problem of calculating the number of weeks in one year.

They were familiar with the fact that one year has 365 days and a week has 7 days but they could not use division to directly calculate the number of weeks. Children were aware that, by grouping days by 7, they would reach 365 somehow but their means were not powerful enough to solve the problem. Amelie started searching for a different way of counting. She divided the year into 12 periods (months). In each month, she counted

Schema 5.

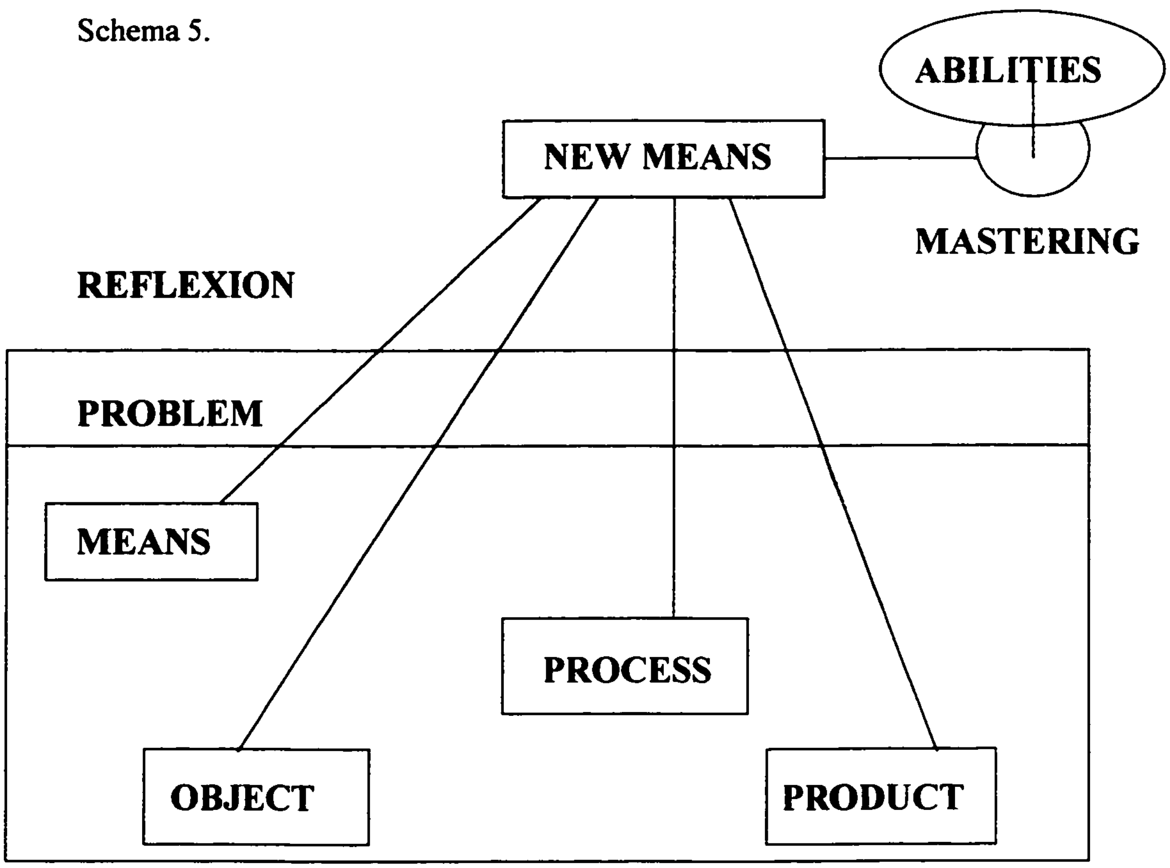


Fig.8 Development of abilities through reflection

weeks (grouping by 7). She also created new symbols (weeks and days), and added all results to get a total of 52 weeks.

She has thus developed a new mechanism which included old means (counting by 7), old objects (days, weeks, year), old process (grouping), and old products (number of ...) as well as a reflection on the problem as a whole (what does not work in my direct approach), that pushed her towards the creation of new means(symbols and procedures). But beside all this we can see the influence of the combination of abilities and meta-abilities (to work on a problem and to work on the work) which is an important characteristic of a mathematically gifted child (see Fig.9, next page).

III.3.2.4 Teaching reflective action to young children: rupture situations

This reflective action is a complex and difficult subject of study for children. In order to help children to develop their abilities, we might use a **methodology** which helps to build up a *science* as specific action of selection of new tools and means from a set of all human actions. This set of actions should include a *learning action* that helps the individual to master and to use these new tools and means and a particular *system of teaching* with its **particular methods** of *representation* of these means as appropriate semiotic and subjective operative systems and *construction* of specific processes and models (as problems) of action.

The question that arises from these considerations is, how to transpose a system made of content knowledge and skills that has been empirically and randomly created over a long period of time into an appropriate teaching and learning sequence, respecting such principles of children's cognitive development as the ability to construct the processes of mental action and to master the method of construction as a specific content.

Every process of a child's mental action exists in a system of certain external conditions, or in the so called "situation". The same process of action, is, on the one hand, confronted with the elements of the situation itself, and, on the other, it creates the situation as a whole, that is, it puts some structure on it.

Sometimes, the elements "fit" the action and can be easily organised in the process, sometimes, they do not. The latter is a special kind of situation where the conditions do not correspond to the process, the objects can not be easily included in the process, or there is a certain incoherence between conditions and process as result of interaction of many individuals, or there are some external factors that destabilise the system. Shchedrovitskii calls "*rupture*" this kind of situation.

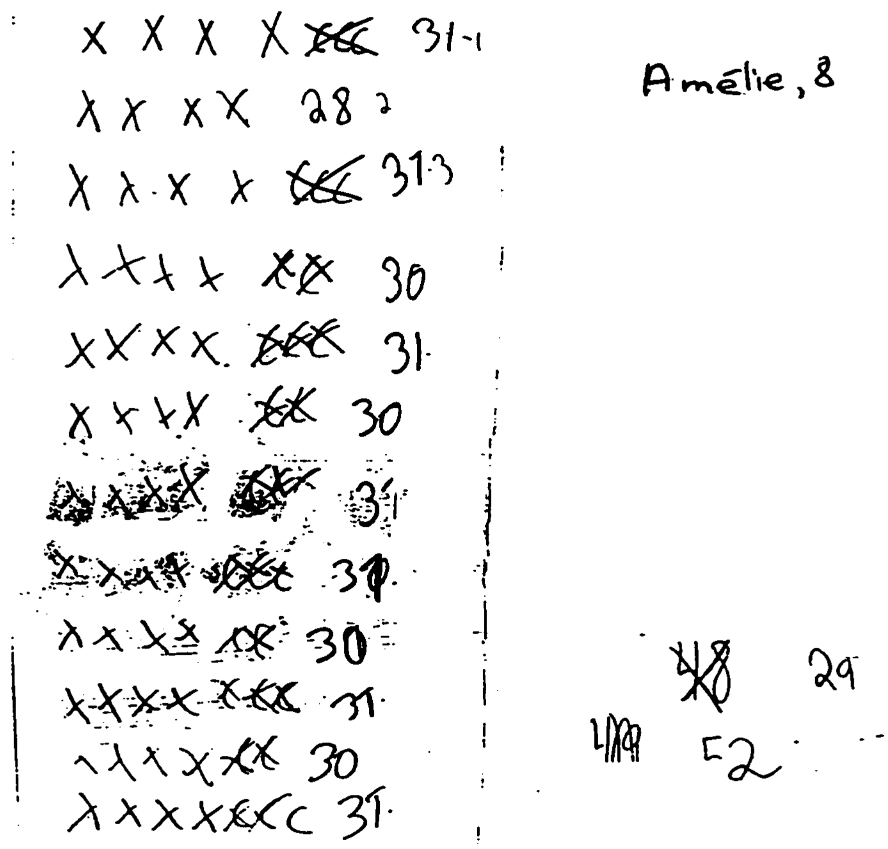


Fig. 9 Example of construction of new means by a student

In the situation of *rupture*, the child must construct a new process of action in order to stabilise the situation (he must, actually, construct a new situation). It is exactly in this kind of situation that favours real learning. Such situations may not occur spontaneously; they have to be arranged by purposeful educational intervention. The intervention is meant to:

- purposefully create a *rupture* in children's actions (this can be achieved by including or eliminating certain significant new objects, signs, people, etc.)
- help learners understand the nature of the rupture (what is wrong?) and formulate new tasks
- create situations that stimulate children to create new means and to develop new methods of action required to overcome the *rupture*.

The following example illustrate a possible way to create a situation of *rupture*.

III.3.2.5 Example of a situation of rupture

In our experiment with elementary school children (see Chapter IV), we created several situations of rupture with the students' previous knowledge. In this section, we are going to show how Shchedrovitskii's model can be helpful in analyzing such classroom situations.

One day, Grade 3 class started to read a text on the history of mathematical computations, which might have led to the constructions of computing devices (abaci). Let us focus our attention on the following paragraph:

These new commercial exchanges required more and more computations. The bookkeepers from Asia had to find means to do more complex arithmetical operations, like 3561x14. (Lyons, Lyons, 2001:44, English translation)

We can easily imagine a classroom where the reading would simply go on. It is quite clear that for 8 year-old children the operation is complex. But in the challenging classroom, this sentence is just the beginning of an exploration. The initiative could come from one student who might just exclaim: '*it is not difficult, I know how to do it*' or, from the teacher who might provoke children with a question like '*I don't think that it is difficult to find an answer, each of you would be able to do it*'. After such an 'optimistic' introduction few children would refuse to try it out. In our class, everyone tried, as a result, we had 30 reports on children's efforts featuring a variety of strategies and approaches that we have grouped into 8 categories.

No strategy, work not completed.

One child gave up, just writing 'impossible' in his report. Two children left too few traces to judge about their thinking. The fact that they didn't succeed can be due to other factors (two of them are hyperactive, one demonstrates a very unstable attitude toward classroom routine, the other is just struggling with her impatience.). The third student left few marks, but they pointed to being on the right track: he started to add 3561 up consequentially but stopped at the 5th step and erased his results. He is an extremely slow working student. Lack of time might be an explanation. All three students showed, however, several characteristics of giftedness working with some other problems.

Messy work, some ideas can be seen (see Appendix A2, p.142).

Another mathematically able student (+ a very high achiever) left us with a real puzzle (one would need to conduct a separate study just to decode his ideas).

Use of pictograms to represent numbers

Three students were trying to use pictograms (that we use to present three-digit numbers). They reinvested recently learnt methods in a new situation. The result of their work shows their abilities to represent numbers and to do various grouping (two of three students).

The solution given by one of these students (who has a very original, sometimes far from conventional thinking in mathematics) is rather messy and incomplete. The second student did his calculation almost correctly but didn't take 'thousands' into account, in fact, he did 561x14 (instead of 3561 x 14) (see Appendix A3, p.143). The third student in this group used a triangle to represent thousands and computed his result almost correctly, obtaining 47820 instead of 49854 (see Fig. 10a).

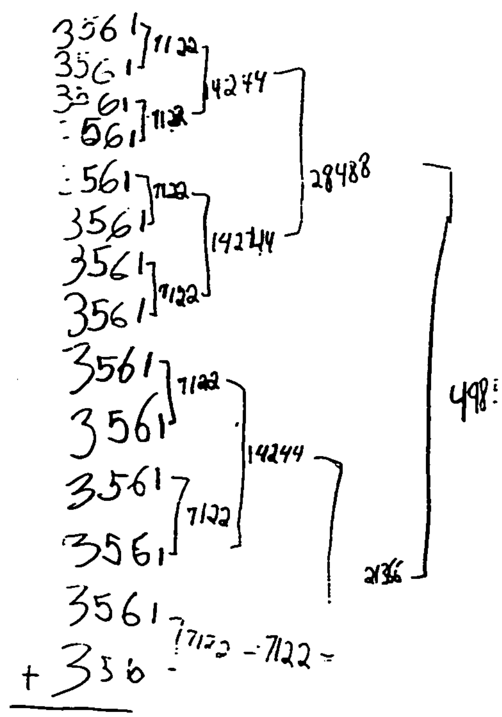


Fig. 10 a Example of use of pictograms to represent numbers

<u>Decomposing (kind of distributive property) (Appendix A4-A5, pp. 144-146).</u>

Two mathematically gifted students identified as such during our teaching experiment, were using a decomposition of 3561 as 3000+500+60+1 (one did it in a curtailed way - mentally, the other presented a very elaborated report). Neither came up with decomposing 14 as 10+4 (they did it as 1+4) - an amazing fact.

Conventional way applied to the two-digit number (Appendix A6, p.147).

Seven students posed the multiplication directly in a conventional way.

One of them obtained the right answer: he is a very high achiever and a mathematically gifted child. He considered this exercise as a technical routine. Two other gifted students got lost in the algorithm they definitely saw but did not master (both are high achievers). Four other students (good achievers but rather average abilities) were trying to adjust the algorithm of multiplication by a single digit but without success.

Multiplication as repeated addition (Appendix A7, p.148).

Ten students simply added 3561 fourteen times. One of them (gifted + very high achiever) did it successfully. Five of them (very good 'calculators' but rather limited in reasoning) made some computational mistakes that they would not normally do in simpler calculations but they were too confident in their skills to think of different ways of computing.

Addition with grouping.

Two students (with average abilities) got lost in the structure of numbers. Two other children (gifted ones + high achievers) were trying interesting groupings of numbers (by 2). One of them did it successfully constructing an elegant structure with a consecutive doubling (see next page, Fig. 10b), the other came close to a correct result (see Appendix A8, p.149).

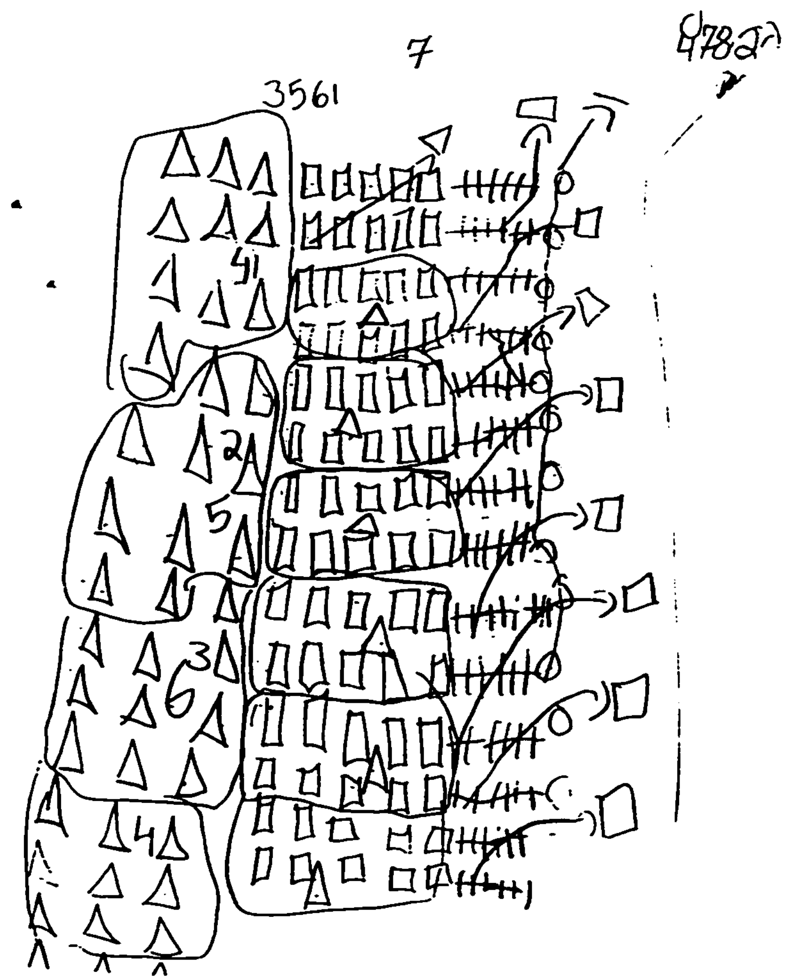


Fig 10 b. Example of addition with grouping

Decomposition (associative property) (Fig. 10 c).

Only one student falls in this category. His answer was correct. He used a curtailed and ingenious reasoning (for his age of 8). He did a decomposition: 3561x14=(3561+3561)x7 and got almost mentally the right answer.

His report (Fig. 10c) contained a 'strange' line with '14x3=42+7=49'. This was his way to calculate 7x7. Instead of memorising his multiplication facts and getting good marks, he is inventing repetition strategies (here: 7x7= 7x2x3+7).

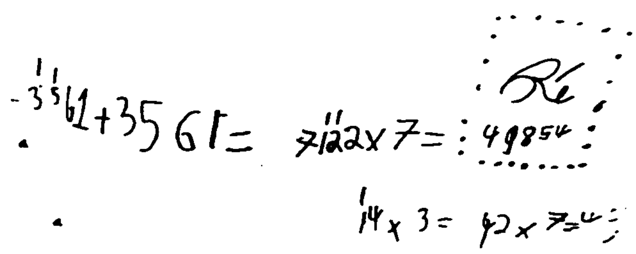


Fig 10 c. Example of decomposition (associative property)

This example shows a generally good involvement of students in a reflective action in a challenging situation. It shows also that in such open situations, children can produce very different means. The role of the teacher is to organise a common discussion in which the new knowledge would become 'institutionalised' (in Brousseau's terminology) and thus mastered as a 'new ability' (using Shchedrovitskii's terminology).

Shchedrovitskii concludes that both contents and pedagogical guidance are important for the child's development. The character of these two elements determines the way of development of the whole system of learning. This theory would help teachers to enhance the active role of students as (co-) organisers of meaningful process of learning in accordance with their abilities.

If we want to construct our teaching on the principles of creating classroom situations in which theoretical thinking can be detected and nurtured, another aspect has to be taken into consideration: mathematics itself with its complex system of interconnected and interrelated concepts and structures. Thus, the meaningful learning of mathematics can be seen not as a smooth and static accumulation of knowledge but as a complex and dynamic process of overcoming difficulties and developing a deeper understanding of mathematical concepts and theories. We deal with this problem in the next section.

III.3.3 Epistemological foundations for the development of "good understanding" in mathematics

Sierpinska (1994) has developed a theoretical framework for the notion of understanding in mathematics, based on three fundamental models: Historical empirical genesis of knowledge (intra-, inter-, tras- triad of Garcia and Piaget), Vygotsky's theory of thinking (complexes and concepts) and Bachelard's notion of "epistemological obstacle" (a term imported and adapted in the Theory of Didactic Situations by Brousseau). We focus here on the idea of developing understanding through overcoming obstacles.

According to Sierpinska, one of the important aims of the didactical analysis of a subject of teaching is to clarify, "what it is that we want our students to understand when they study mathematics, and what exactly it is that they don't understand" (p. 41). She thus links the problem of objects of understanding with the contents of teaching along with the goals of this teaching. Taking into consideration the fact that a school as an institution has not been chosen by children's free will, we can suppose that from teacher to the student, the object of understanding can easily change its identity:

> What is, for the teacher, an 'algebraic method of solving problems' may become, for the student, a mechanical procedure, a school activity that is done in order to comply with the requirements of the teacher and the school institution. It may have nothing to do with 'methodology' and certainly nothing with answering interesting questions. The student's activity does not always have a cognitive character; very often it is a strategic activity aiming at going through the school and graduating with as little intellectual investment as possible."(Sierpinska, 1994:42)

From this remark, we can make an important conclusion: If we want to foster the development in mathematically gifted children, we have to look for pedagogical instruments of organisation of meaningful learning. Our didactical efforts should be directed to the development of a basis for understanding.

III.3.3.1 Mental models as obstacles to new knowledge

We often base our understanding of mathematics on certain **mental models**; their "concreteness" helps us to become familiar with various abstract mathematical "objects". However, such models have the tendency to acquire, in our minds, the status of the "whole truth" about the conceptual domain we are exploring with their help. They become thus obstacles to further explorations.

Sierpinska's remark that "it may even be so that the more we make a mental model function and the better it works, the bigger the obstacle we thus create for ourselves" (p.55) is followed by an example of "number sense", proposed to be developed in schoolchildren, which is the knowledge of the logistics rather the arithmetic, the knowledge of the artisan rather than the knowledge of the architect: numbers look very much like wooden blocks of various lengths, and operations are almost physical operations on the blocs. Problems with this model arise when we teach children to multiply integers as a structure that extends the structure of natural numbers in such a way that it preserves the properties of operations in it.

Our practice shows the existence of a very interesting link between giftedness and mental models (in the above sense). Weak students tend to ignore "structural contradictions" within mental models, artificially memorising new rules with new objects (such as integers). Able students, on the other hand, who see the structure beyond a model might experience dramatic frustration in their need to overcome obstacles and equilibrate the new knowledge with their old mental models. For example, they become very used to sharply distinguish between the 'squareness' and the 'rectangleness' of a shape. Later, they struggle to accept that, by definition, the square counts as a rectangle. Our practice shows that these students could be able to overcome more easily this obstacle if they confronted it in their early stages of development and did it in appropriate classroom situations.

In a way, certain abstract notions are better concluded on the base of a certain logical pattern than through search for an intuitive meaning. (Sierpinska, 1994:55).

Children who really succeed in overcoming this obstacle, proceed purely logically: *I know it is a square; why would it be a rectangle? - Well, it is a quadrilateral and it has 4 right angles, so it **must** be called a rectangle.*

III.3.3.2 'Single vision' as an obstacle

Sierpinska considers the process of the overcoming of an obstacle (developmental or epistemological) as the most important learning act that allows someone to access to the advanced scientific knowledge. Overcoming an obstacle thus is not an acceptance of a new system of beliefs or another (universal) schema of thinking but more a modification of our conception of the object by means of "a different point of view", "one of the possible attitudes", "one of possible methods to handle a problem". This would be an important personal quality that we observe in children we identify as gifted. For example, the problem *'how many shoe laces are in 5 pairs of shoes'* requires a *flexible vision* of shoes as one single object (one 'shoe' -one 'shoe lace') or as another single object (one 'pair of shoes - 'two shoes'). Our practice show that already at a very young age, some children are struggling with problems that require this *flexible vision.* In the contrary, gifted children demonstrate a particular facility for this kind of task.

III.3.3.3 A systemic thinking as an obstacle

Another important piece of Sierpinska's model of understanding is based on Vygodsky's theory of stages of development from thinking in syncretic heaps through thinking in complexes to thinking in concepts, which Vygotsky understands as meanings of scientific terms. The main characteristics of concepts is that they form a system. These stages can be distinguished by observing how children use logically coherent criteria while ordering and classifying objects. In gifted children, the passage to thinking in concepts takes place earlier. They pass very early to generalizations in the form of pseudo-concepts and become ready to work with abstract concepts using "correct" definitions.

III.4 CONCLUSION

The Krutetskii's model of identification of mathematically gifted students, analysed in the first chapter, and the problems and questions of the fostering of theoretical thinking at an early age, raised in the second chapter, led us to the necessity of looking in depth at didactical, methodological and epistemological aspects of learning and teaching mathematics in order to design classroom situations appropriate to the need of gifted children.

We analysed thus three paradoxes of learning and teaching that can be seen as obstacles to the elaboration of challenging mathematics curriculum related to the necessity of provoking re-organisations in previous knowledge in order to construct a new one. We also studied three approaches of overcoming this paradoxes through the didactical design of challenging teaching based on the Brousseau's theory of Didactical Situations in Mathematics, Schedrovtskii's methodological analysis of learning and teaching by means of reflective actions and Sierpinska's epistemological analysis of aspects of re-organisations of mathematical thinking based on good understanding.

Moving now to the description of our challenging situation approach for identifying and fostering of mathematical giftedness in young children, we put emphasis on finding a balance between children's autonomy in construction of a new mathematical knowledge and a pedagogical necessity of didactical provision for a meaningful mathematical teaching an learning.

We thus choose a position of an active constructor of a challenging environment that would be beneficial for all children (aiming however at a highest level of mathematical thinking for each child) rather then an observer of difficulties that children meet if they are left to working on their own.

<u>CHAPTER IV. CHALLENGING SITUATION APPROACH TO MATHEMATICAL GIFTEDNESS</u>

IV.1 THE NOTION OF CHALLENGING SITUATION

Krutetskii's characterization of mathematically gifted and the three approaches to overcoming the paradoxes of teaching outlined in previous section form the theoretical background for a study of mathematical giftedness.

First of all, following Krutetskii, we assume that one can not study giftedness separately of the mathematical content in which it reveals itself. Thus any study of giftedness must start with an epistemological analysis of the mathematical content of the problems or tasks proposed to students. This analysis should be aimed at answering questions such as, what does it mean to understand this content? What level of theoretical thinking is necessary? What ruptures with old knowledge must take place, what obstacles have to be overcome?

Brousseau's didactical principles and Shchedrovitskii's method of task analysis may then help the educator to design special, "challenging situations", in which the desired understanding could take place and new knowledge/abilities could be developed.

In order to link the different 'pieces' of our theoretical framework into a system of teaching and learning, we establish a **developmental 'recursive chain' of identification and fostering of mathematical giftedness: the challenging situation requires rupture with old knowledge and construction of (new) abilities, thus at the same time revealing the obstacles and giving the teacher an opportunity to address them in all students (not only the gifted ones); reflection on the situation and the shortcomings**

of thinking about it leads to new questions and indeed creates a new challenging situation.

We postulate that the use of teaching approaches based on challenging situations would help to engage all students into meaningful learning through:

- *Early beginning* of work on challenging mathematical tasks : 3-5 year old (fostering precocious mind)
- *Stimulating questioning* (fostering critical / reflective mind) :
  - Why ?
  - What if not ?
  - Is there a different way ?
  - Does it always hold ? etc.
- *Encouraging search for new original ideas* by means of open-ended tasks (fostering creative / investigative mind
- *Promote full and correct explanations* (fostering logical / systematic mind)
- *Introduce children to the complexity and variety of mathematical concepts and methods* (fostering looking at the world with mathematical eyes)
- *Provide children with tasks that require complex data organisations and reorganisations* (fostering selective / reversible / analytical / structural mind)

Any textbook problem can be turned into a challenging learning situation or, on the contrary, into a dull exercise.

Challenging situations cannot be used only on exceptional occasions in a teaching approach. Some of them must, of course, be carefully prepared, but, for the approach to work, it must become a style pervading all teaching all the time at all levels of education. The teacher must be ready to use any opportunity that presents itself in class (e.g. a puzzling question posed by a students, an interesting error or unusual solution) to interrupt the routine and engage in reflective and investigative activities on the spot, or suggest that students think about the problem at home. Thus, in fact, what is needed are not occasional challenging situations, but a "challenging learning environment".

In the challenging situations used in our own teaching, we favour **open-ended problems** which are situated in a conceptual domain familiar enough to the child who appropriates the situation as his/her own and engages in an interplay of trials and conjectures, examples and counter-examples, organisations and reorganisations (as stated by Arsac, Germain, Mante (1988, p.7)).

In each situation, we observe various elements of the child's mathematical behaviour:

- How the child enters into the situation (introductory stage, pre-organisation) and how different ways of presenting the problem affects children's work (*Brousseau, 1997*);
- How the child constructs his/her process of problem solving (choice of strategy, use of manipulative, systematic search, autonomy, self-control, mathematical components);
- How the child acts in case of an error (destroys his/her previous work and starts form scratch or tries to modify/correct certain actions);
- How the child modifies his/her strategy when the conditions are slightly / completely changed;
- How the child presents his/her results (orally or in writing, clearly or not, communicating or not with other participants (children or adults), symbolism used by the child, organisation of results (on paper)).

Based on our observations of children's mathematical behavior in challenging situations, we describe gifted children's mathematical thinking as:

1. *Precocious*: performing on a higher level than the one theoretically established for child's age normal level (e.g., in Piagetian terms of intra-, inter-, trans)
2. *Systematic and/or systemic* : systematic and logical work with complex data
3. *Selective* : focus of attention on appropriate (sometimes implicit) data, links and relations
4. *Structural* : consideration of many parameters/links/data/conditions at a time
5. *Logical* : giving full and correct explanations
6. *"Optomathematical"* : looking for mathematically essential details (sees the world with mathematical eyes)
7. *Critical* : acceptance of a certain level of rigor and concern with validation; constant 'state of alert', looking for ambiguities and for ways of proving or disproving using logical inferences
8. *Reflective* : reinvestment of known methods in a new situation
9. *Reversible* : making 'two-ways' links between mathematical operations and data
10. *Analytical* : operating with symbols / formulas / graphs
11. *Creative / investigative* : asking questions / investigating / making conjectures / creating new problems
12. *Constructive* : acting mathematically in non-mathematical situations
13. *Conceptual* : fluency in operations with abstract mental objects
14. *Intuitive* : sudden "spontaneous" insights about the problem
15. *Abstract* : looking for general categories and relationships in concrete situations

The context of challenging situation allows children to go faster beyond the level established by a regular curriculum without losing their interest and motivation of learning more.

In fact, our practice demonstrates the evidence that within a challenging environment very young children actively demonstrate their willingness to learn more advanced and abstract topics like:

- big numbers, zero, infinity
- negative numbers
- fractions and proportional reasoning
- logical inference
- variables and functions
- shapes and their properties (definitions and proofs)
- geometric transformations
- equations with missing terms

In order to test our challenging situation model in a real classroom, we have organized a teaching experiment during which we have developed a challenging curriculum based on challenging situations and used it with elementary grade students in a mixed-ability classroom. We give selected information about this experiment in the next section.

IV.2 The Experiment

Our experiment reflects 7 years of classroom activities and observations with Grades K-6 children while teaching challenging mathematics courses. In this section, we will analyse a few classroom situations in which the discrimination between mathematically gifted and ordinary students became possible. We will discuss how a challenging curriculum might help teachers to develop a higher order of reasoning and thinking in all their students. We will conclude with striking particular examples of mathematical thinking in gifted children.

IV.2.1 The context [a]

Our teaching experiment has been conducted at Académie Marie-Claire, a private bilingual elementary school with French and English both taught as a first language. Despite a short history of 7 years of its existence, the academy has built a good reputation as a school with solid and enriched academic curriculum.

Along with a strong linguistic program (with a third language, Spanish or Italian), the school insists on offering enriched programs in all subjects including mathematics to all its students independently of their abilities and academic performance.

The school thus promotes education as a fundamental value by instilling the will to learn while developing the following intellectual aptitudes:

- being able to analyse and synthesize
- critical thinking
- art of learning

The mathematics curriculum is composed of a solid basic course whose level is almost a year ahead in comparison to the program of the Quebec's Ministry of Education (Programme de formation de l'école québécoise, 2001) and an enrichment (deeper exploration of difficult concepts and topics: logic, fractions, geometry, numbers as well as a strong emphasis on problem solving strategies). The active and intensive use of "Challenging mathematics" text-books (Lyons, Lyons) along with carefully chosen additional materials helps us create a learning environment in which the students participate in decisions about their learning in order to grow and progress at their own pace. Each child competes with himself (herself) and is encouraged to surpass himself (herself).

Since the school doesn't do any selection of students for the enriched mathematics courses, all children of Académie Marie-Claire participated in the experiment. With some of them, this author started to work at their age of 3-5, as a computer teacher. Following table shows a number of participated students within the years of our study and school Grades.

Tab. 3 Students involved in our experiment

Year of experiment	Grades involved	Number of students
1995-96	Grade 1	12
1996-97	Grade 1	17
	Grade 2	11
1997-98	Grade 1	17
	Grade 2	15
	Grade 3	11
1998-99	Grade 1	23
	Grade 2	19
	Grade 3	15
	Grade 4	9
1999-2000	Grade 1	32
	Grade 2	23
	Grade 3	19
	Grade 4	15
	Grade 5	9
2000-2001	Kindergarten	33
	Grade 1	31
	Grade 2	34
	Grade 4	21
	Grade 5	15
	Grade 6	7
2001-2002	Kindergarten	35
	Grade 1 (last term-Mai, June)	39
	Grade 3	29
	Grade 4	25
	Grade 5	21
2002-2003 (fall)	Kindergarten	34
	Grade 3	33
	Grade 4	32
	Grade 6	23

This table shows that there were certain groups that we could observe during a long period of time (for example Grade 6 children in 2002-2003 were our students since Grade 1, some of them since the age of 3-5). During this period, some children had to leave the school, some of them joined the class later (in the same Grade 6, there were 2 students who started in our school in Grade 6).

In terms of abilities, we can characterise our classroom as a mixed ability classroom with a significant variation in the level of achievement.

The enriched course aims to foster children's logical reasoning and problem solving skills in all children. It is based on *challenging situations* presented in the 'Challenging Mathematics' textbook collection (Défi mathématique (Lyons, Lyons)) along with other different computer and printed resources (LOGO, Cabri, Game of Life, Internet, and so on) as well as situations created by the author.

The course is difficult for many students. It includes several topics earlier than in the regular curriculum; some topics are presented in more depth than in the regular curriculum; various topics which are not included in the regular curriculum. Such curriculum thus requires a mobilising of all the inner resources of the child: her motivation, hard mental work, curiosity, perseverance, thinking ability. Since all our students are exposed to this enriched curriculum, the differences between them become more evident.

In our examples, we will focus first on a global analysis of situation describing all children's mathematical behaviour and then on particular episodes with some children who demonstrate characteristics of giftedness described in our theoretical model.

We will also make explicit the role of the challenging situation itself showing that without the context of challenging situation, such opportunity for students and teachers would be lost.

IV.2.2 Design of a challenging curriculum: example of an enriched course for the Kindergarten

There are two basic approaches to design a mathematics curriculum for 5-6 year old children; one can be labelled as traditional and the other as innovative. The former is based on counting, ordering, classifying, introduces basic numbers, operations (addition and subtraction), relations (more, less, bigger, smaller, greater) and shapes. The latter puts more emphasis on learning while allowing children to play using manipulatives, colouring, arts and crafts, games with numbers and shapes. During the past decade, many creative teachers have been trying to use the best ideas from each of the two approaches also adding reasoning activities to the mathematics curriculum.

Many studies has been conducted on the child's mathematical development within different psychological theories (see, for example, Piagetian studies (Rosskopf, 1975), Gelman's study of child's understanding of number (Gelman & Gallistel, 1978), Resnick's developmental theory of number understanding (Resnick, 1983)).

In our school, we use a traditional approach based on Quebec's *Passeport Mathématique* Grade 1 textbook along with a new French collection *Spirale (Maths CP2)* which represents the second, modern approach. However, even this combination doesn't provide our children with the material necessary for their mathematical development. There is still a gap between their level of ability and the requirements of the challenging curriculum that we use starting from Grade 1 (collection "Défi mathématique") and which is based on discovery, reasoning and understanding.

In order to fill the gap, we developed an enriched course offered to all kindergarten students (we have 30-35 children every year). The course is given on a weekly basis (1 hour a week). We base our teaching on the challenging situations approach, developing activities that stimulate mathematical questioning and investigations along with reflective thinking.

Each class starts with such questions as *What did we do last time? What problem did we have to solve? What was our way to deal with the problem? What strategies did we use?*, etc. This questioning aims to provoke reflection on the problems that children solved as well as on methods that they used. Without this reflection, rupture situation (in Shchedrovitskii's sense, see section III.3.2.5) would never arise, because a rupture is a break with previous knowledge, which needs to be brought to mind.

At the same time, we would ask questions that would indicate children's understanding of underlying mathematical concepts or methods that we aim to introduce (using appropriate vocabulary and/or symbolism).

During this initial discussion we usually try to bring in a new aspect which provides children with an opportunity to ask new questions, to look at the problem in a different way. Sometimes, we might ask them, simply, *what do they think we should do today?*

Thus we can pass to the new situation/new problem/new aspect of the old problem. We may do it by means of provoking questions, of interesting stories or introductory games. Following Shchedrovitskii and Brousseau, we try to avoid the teaching paradox by not providing children with direct description of the tasks or methods of solutions. We try also to keep their attention and motivate them.

After this introductory stage, children begin investigating a problem using different manipulatives: cubes, geometrical blocs, counters, etc. They work alone or in groups. During the phase of investigation, the role of the teacher becomes more modest: we give children certain autonomy to get familiar with the problem, to choose a necessary material, organise their work environment, choose an appropriate strategy.

However, some work has to be done by the teacher to guide children through their actions. We have to make sure that the child understands the problem, the conditions that are given (rules of the game), the goal of the activity. As the child moves ahead, we shall verify his control of the situation: what she is doing now and what is the purpose of the action (activating reflective action). We have to keep in mind that the exploration is used not only as a way to make the child do some actions but also and foremost as an introduction to mathematical concepts or methods.

Therefore, the teacher needs to be prepared to introduce the necessary mathematical vocabulary along with its mathematical meaning as well as mathematical methods of reasoning about the concepts and about the reasoning. In our experiment, we try to choose those mathematical aspects that are considered as difficult and are not normally included in the Kindergarten curriculum.

For example, when we want to introduce an activity with patterns, we would organise a game. We would start to make a line 'boy, girl, boy, girl,..' children find it easy and are happy to discover a pattern. Then we would start a new 'pattern' : 'boy, girl, boy, girl, boy, boy'. Many children would protest, saying that the pattern is wrong. But perhaps, some of them would try to look for different pattern, like "glasses, no glasses, glasses, no glasses, ...".

As the game goes on, children get used to looking for familiar patterns. This is the time to challenge them more. For example, we may ask them, how many children would be in the line with the pattern 'boy, girl, boy, girl,...'. Since there were only 8 boys in the classroom, one child could make a hypothesis that it gives 8+8 children in the line. After such a line had been completed, teacher's silence could be broken by a child's voice - 'we can add one more child to the line – a girl in the beginning'.

The course is built of various challenging situations that we create in order to give children an opportunity to take a different look at mathematical activities that they usually do, to question their knowledge about mathematics trying to discover hidden links between different objects, to discover structures and relationships between data, learn to reason mathematically based on logical inference and at the same leave some space to children's mathematical creativity. We use different didactical variables in order to create obstacles making children re-organise their knowledge and create new means in order to overcome the obstacle.

We often ask our children to report on their investigations inviting them to communicate their discoveries by developing appropriate tools: diagrams, schemas, symbols, signs.

We will now analyse an example of a challenging situation and show how different aspects of our theoretical model help to identify and nurture mathematically gifted children.

IV.2.2 Analysis of a challenging situation given to Kindergarten (5 years old) children in an enriched mathematics course

The problem given to the students (as it is formulated in the book 'Challenging mathematics' (Lyons, 2001)) was

Find all ways of dividing five counters into two groups.

Mathematical content of the problem

As shown in many studies in mathematics education and psychology, children have difficulties learning numbers because of their limited conceptual understanding. The source of these limitations might be traced back to the very beginning of mathematics instruction which introduces number as a result of a counting.

Not only the number itself, but also basic operations on numbers are also introduced by means of counting with elements of different sets. More abstract and theoretical aspects of numbers related to their structural characteristics and operational properties are often put aside.

Therefore, many children see addition as a 'counting on' with the necessity of getting a final result - the sum. Therefore, they do not understand that the expression 2+3=1+4 makes sense. They think that 2+3 can be only equal to 5. Thus, the operational structure of number might be inaccessible to young learners.

That is why, it is important to bring structural aspects of number in mathematics teaching to very young children. By asking them to find different ways of decomposing 5 into a sum of two numbers we aim to introduce different properties of numbers (like 'fiveness') and operations (like the commutative law and the neutral element) and thus creating 'multifaceted' picture of mathematical action.

The situation was as follows: The teacher takes 5 counters in his hands and asks children what one could do with them (all the counters are identical: they have the same shape (circle), colour (blue) and size (around 3 cm in diameter). The students make their guesses. The teacher does not make any comments starting to play with counters, putting some of them in one hand and others in the second. The students give comments on the significance of these actions. Finally, they get familiar with the situation of partitioning counters into two groups and the problem of finding all the ways to do it. There were no words about number 0 as an option or about inverse combinations such as 2+3 and 3+2, or what one can call two different partitions.

The students worked on the problem in the following setting: children were sitting around 6 tables (5-6 children around one table). They had to communicate their solutions by drawing them in the Défi-1 workbook. The experimenter and two teachers moved from one table to another checking on children's work and giving them some hints (like *'regarde, si tu as déjà cette solution, tu dois essayer d'en trouver d'autres'*)

Here is how we analysed the situation:

Trying to make guesses about the task during the introduction stage, children knew that since it is a math class, the activity must be mathematical. So, they proposed counting counters, ordering them, using them to draw circles (Sic!). We could observe here an emergence of spontaneous brain-storming thinking process: they tried to predict the possible nature of a mathematical activity yet unknown to them. They could use their previous experience in order to build links between different activities. Thus, they got an opportunity to ask questions, to make hypotheses, to learn about the rules of the 'game' (instructions to follow, conditions to respect). Finally, they approach the formulation of problem. In this very beginning of activity, we could already identify children who manifested their interest, understanding and creativity **more explicitly** than others.

During the teacher's demonstration with hands, we introduce a new **'variable didactique'** trying to direct children's search towards a pre-planned activity. Some children look very confused as they saw that no counting, no drawing happened. They had to modify their 'guesses' adjusting it to the new situation. At one moment, one child pronounces: "you are always changing the number of counters in each hand", thus a word 'partition' has been pronounced by the teacher. Now we could talk about different ways to divide counters. At the same time, we do not tell children how to do it or how to validate a solution - thus we let the child organise the process of solution, construct necessary tools, try different strategies. This openness and autonomy left to children is very important for the fostering of mathematical giftedness in young children and for any learning to occur, for that matter, according to Brousseau and Shchedrovitskii.

Our observation shows that children organise their work differently. Some children pose all the counters on the table in a line, or form different configurations: (squares, circles, towers). We interpret this spontaneous organisation of material as an important indicator of mathematical ability (thinking in terms of structures). This is also a sign of a high level of thinking discipline. We can also suggest that in the child's head the partition is being made at this moment.

Other children start immediately to move the counters from one hand to the other (imitating the teacher's demonstration). We can give two kinds of explanation of this phenomenon: they are trying to understand the problem or they have the need to simply touch the counters.

However, we could already evaluate children's readiness to solve the problem: some of them seemed to know how to proceed, others showed signs of confusion.

Looking for solutions, some children formed two groups of counters on the table, others kept the counters in their hands. In their work organisation, we see a very different approach: very systematic and orderly in some children and chaotic and messy in others.

While several children understood the need of verifying the solutions (respecting conditions of five counters and different partitions) and were able to do it, some children could not do it even with help of the teachers.

If we look at children's drawing in the exercise book (some examples of these drawing are given in the Appendix A9, pp.150-152), we find that of the 31 children participating in the activity, 6 have obtained all 6 solutions: 0-5, 5-0, 1-4, 4-1, 2-3, 3-2. Other children made different kinds of mistakes:

- Losing one or two solutions (0-5 or 0-5, 2-3)
- Not considering 0
- Not considering commutative partitions like 2+3 and 3+2 as different ones
- Not finding all partitions (0-5, 1-4, 2-3)
- Presenting same partitions several times
- Losing 'fiveness' in partition

Eleven drawings suggest some systematic search, 20 - non-systematic

It is interesting to note that we did not ask children to use an ordered disposition of counters on their drawings. However, we could observe some particular dispositions of circles ('domino', 'rows', 'circle', 'triangle'); it happens often within challenging situations that some children give themselves new tasks, new interpretations of task moving thus beyond pre-planed activity.

There were 4 children who used numbers in their drawings (2 of them solved the problem successfully). We do not know, however, why they used numbers (aesthetic or mathematical reason).

One more observation - while some children could keep all the process under control, others lost the control passing from the manipulation to the communication (on the paper). They considered it as a different task: focusing on particular aspects of drawing instead of on mathematical 'parameters' of the task. They could even draw a completely different partition from the one found with counters.

Our attention was drawn to the few students who proceeded systematically in their search of all the different ways (e.g. 5 and 0, 4 and 1, 3 and 2), considered zero as a significant element in their partition (neutral element property), made a distinction between a+b and b+a cases (commutative property), kept the total number of counters constant (addition as operation). These students not only succeeded in this partition problem but also were able to solve a similar next problem (3 counters) without any reference to the counters.

Unlike other children who started to solve the problem of partitioning the set of 3 counters in the same way as they approached the previous one: placing counters in their hands, trial and error partitions, etc., our "candidates" for gifted children just took the pencil and drew all the partitions in two seconds. It was as if they took the whole structure of the first situation and transferred it into the new one.

Let us see now how these children's behavior can be interpreted in terms of our operational model of mathematical abilities.

When these children were shown a group of objects (in this case: counters), they almost instantly (even before the problem was presented) started to describe it mathematically using their usual routine knowledge. They said, "We see counters, they are all the same: they are round (circles), they are blue, they are big (small), we can count them, there are five of them". This behavior would be an instance of what we have called "*seeing the world with mathematical eyes*".

They were then shown a textbook page featuring 6 boxes in two columns with a pair of hands drawn on each of them. The number 5 was written in each box and beneath the number there was a two-way arrow (⇔) meant to suggest that the number five had to be split into two parts, one for each hand. To the question of the teacher, "What does this have to do with our five counters?" the children responded, "We have to put counters in these hands". These children were able to establish a relationship between the counters and the hands. For us, this behavior represents the "*constructive mind*" feature of our operational model.

The next question of the teacher was: "Why do we repeat this operation several times?" The answers of the children were along the lines, "because there are several ways of putting counters in two hands" ("*creative mind*").

. In this introductory stage, we could already see that these children have grasped the essential structure of the problem: abstracting from the context of putting counters into two hands, they saw it as a problem of partition ("*selective mind*").

We noticed that these children were using manipulations with the counters in a very special, orderly and systematic way. Even before starting to do any manipulations with the counters, they would put them first on their desks in some (appropriate) order, for example, in a row, and then they would just take a quick look at their constructions. It seemed that they already played with the essential elements of the problem ("*intuitive mind*"). Later, they used manipulations to verify their mental solutions ("*critical mind*"). While doing it, they would take care simultaneously of all essential conditions: keep 5 counters constant, do a partition, find all the different ways ("*structured mind*"). They organized and coordinated their actions ("*systematic mind*").

Moreover, they could always explain what they were doing and why: "I'll put 2 counters in the right hand because we have 5 counters in all and I put 3 of them in my left hand" ("*logical mind*").

When these children passed from the first example to another and reinvested their experience in a new situation, they were transferring the method of dealing with a partition problem, not the experience of playing with counters ("*reflective mind*").

They demonstrated their early ability to transfer a whole structure ("*precocious mind*"), representing the problem in a symbolic analytical form ("analytical mind"). They worked fluently with data, easily incorporating the commutative and zero properties in their solution ("*reversible and conceptual mind*") and making generalizations: "We can do like this with any number" ("*abstract mind*").

The example shows that the identification of giftedness occurs when children work on a real mathematical problem. Teacher's task therefore is to organize and to guide this work, to interpret the results of children's work adjusting the teaching process along the way.

The example also demonstrates that our model provides us with the links of the idea of 'challenging situation' to our notion of mathematical ability. The able child is not only able to deal with challenging situations, but also looks for them and sometimes turns dull exercises into challenging mathematical problems. Thus, he or she is better equipped for the study of mathematics.

In our example, an uninspired teacher could simply solve one example for the students on the board, make them copy the solution in their exercise books and give them a similar one to solve in class and four more to solve as homework. This way, the chance would be lost not only to identify mathematically able children but also to bring other children to a higher level in mathematics.

IV.3 Challenging situations approach and development of mathematical reasoning

As it was shown in our theoretical part, mathematical reasoning is an important part of fostering theoretical thinking in all children and especially in those who are identified as mathematically gifted.

Piaget established five stages in the development of intelligence in young children (Piaget, 1950):

- Sensori-Motor Intellegence (Birth-2 years)
- Pre-Conceptual Thought (2-4 years)
- Intuitive Thought (4-7 years)
- Concrete Operations (7-11 years)
- Formal Operations (11-12 years)

A number of studies with mathematically gifted students show that these children easily 'break' this "natural" sequence of development. For example, in Shields' analysis of logical thinking in gifted children, we can find examples that demonstrate that the performance of the gifted junior-school child in logical thinking is equal to the performance of the average pupil who is 3-5 years older (Shields, 1968).

Since logical thinking is a foundation of mathematical reasoning, we can expect that presenting our students with challenging situations that stimulate the growth of logical thinking, we help them reason at a higher theoretical level.

That is why logic is an important part of our challenging elementary curriculum. On the one hand, our children learn to solve different logical puzzles, use logical operations (negation, implication, class inclusion, etc.), play different strategy games (like chess). On the other hand, they are constantly invited to think logically in various mathematical situations (like working with definitions, looking for logical explanations, proofs, using examples, non-examples and counter examples). We challenge them constantly with 'little questions' provoking logically grounded mental actions.

Following are examples of situations that feature different aspects of logical thinking in our students.

IV.3.1 Example of the use of logical inference by mathematically gifted children: fluency, control, rigour

Our curriculum provides an early exposure of children to different task that require the use of logical inference.

In Grades 1-3, our children learn to solve logical puzzles on the 3x3 board. They have to place different objects (cards, shapes, letters) following various instructions, like *'There is no red card on the top", "Each row has to contain shapes of different colour", "Letter A is in the right column".* In order to solve these problems, children have to use logical reasoning, like *"The letter B is under A, therefore, it is not on the top".*

Sometimes, they have to make conclusions combining several conditions.

In Grade 2, we add another type of problems, in which children are asked to construct a cross-table following several instructions. For example, the next problem is taken from a Grade 3 Challenging Mathematics Book (Lyons, Lyons, 1989, p.10): *"Find the favourite subject and the favourite drink of each student. Mathematics student likes juice. Carole likes arts. Serge likes coffee. Who likes science? Does Victor like milk?"*

Starting from Grade 2, we include chess game in our curriculum. In Grade 4, children learn to use logical inferences based on inclusion classes (all the cars are red, I have a car, therefore my car is red).

Facing logical problems, many children would use trial and error method (in enigmas) or 'childish' intuitions (in logical inferences). However, there is an evidence that some students demonstrate, in fact, a maturity unusual for their age in their reasoning. They form a logical sequence of conclusions based on the given rules, keep control of given instructions, use schemas and linking words (because, therefore, etc.).

For example, in Caroll-Ann's work (7 years old) we see a complete analysis of data (Fig.11, next page), ability to make logical inference ("La soupe n'est pas dans la grande casserole et les carottes non-plus." → "Il y a juste le bouilli qu'on peut mettre dans la grande casserole").

Here is a text of the problem (Défi mathématique-2):

Le souper mijote dans trois casseroles.

- *La soupe n'est pas dans la grande casserole, et les carottes non plus.*
- *Le bouilli n'est pas dans la petite casserole et la soupe non plus.*

Trouve ce qu'il y a dans chaque casserole.

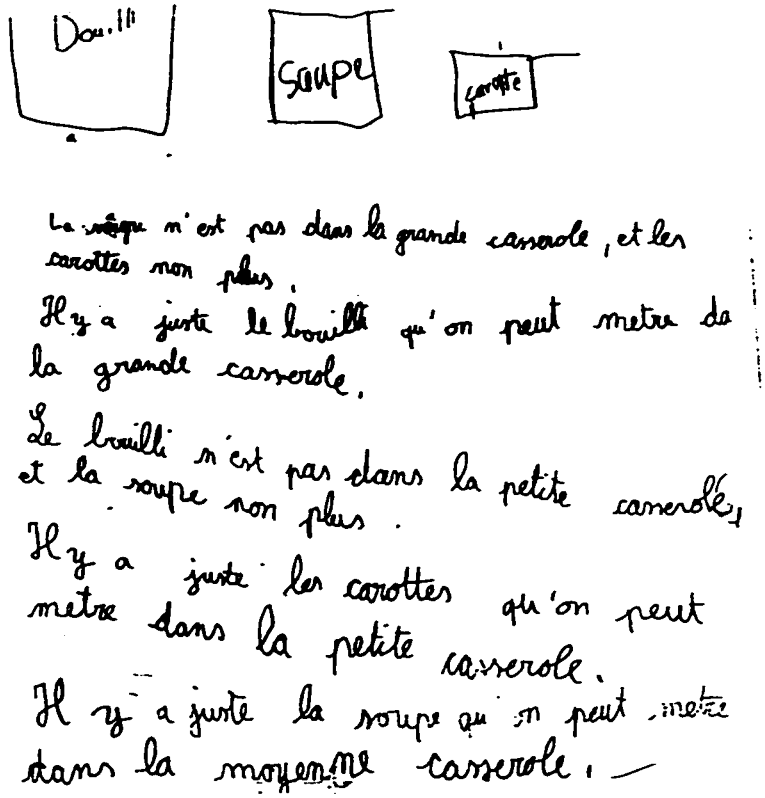


Fig. 11 Example of logical thinking in Grade 2 children

The following situation is very difficult for an 8 year old child:

When it's sunny, Tim always puts on his white hat. When he puts his hat he never puts on his blue shoes. Yesterday, he was wearing his blue shoes during a whole day. What was the weather that day?

Therefore, Sarah's solution is outstanding in many ways : she uses two important key words : proof and inverse, constructs schemas, gives a short explanation. (Fig.12)

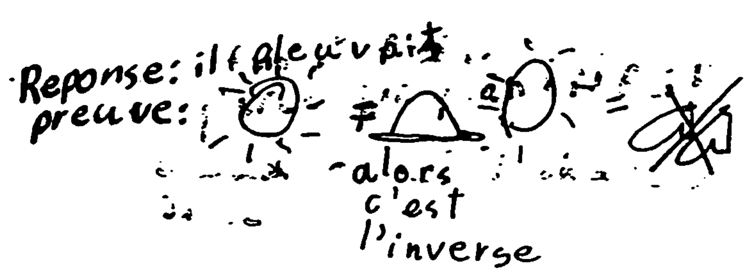


Fig. 12 Example of children's schemas of logical inference

With the time, problems become more and more difficult, like, for example, the following one from The Challenging mathematics-4 book:

Text of the problem: Défi mathématique-4, (Lyons, Lyons, 1989) -in French, p. 214 # 18)
"Rebecca la sorcière est furieuse.
(Rebecca) - Quelqu'un a volé mon balai. Je suis certaine que c'est l'un de mes lutins qui a fait le coup. Lutins, venez ici!
Aussitôt accourent Flip, Flap et Flop, les trois lutins espiègles. Rebecca les interroge.
(Flip) - C'est Flop qui a volé ton balai.
(Flap) - Ce n'est pas moi, belle Rebecca, qui ai pris ton balai.
(Flop) - C'est moi le voleur.
Mais Rebecca connaît bien ses lutins. Il y en a toujours au moins un qui dit la vérité et au moins un qui ment lorsqu'ils a'adressent à elle en m^mem temps. Pourtant, elle sait maintenant qui a volé son balai. Qui est-ce et comment a-t-elle pu le découvrir?"

Gifted children handle these problems with more rigour and complexity in reasoning. Mark-Alexandre's (9 year old) work shows an appropriate use of "connectors" (parce que, alors), look for contradiction as a logical foundation of 'proof' (see also Appendix A10, p.153)

"Si Flip a menti d'abord Flop a menti aussi, alors ça fera
(Flip) Ce n'est pas Flop qui a volé ton balai (contraire de ce qu'il a dit)
(Flap) Ce n'est pas moi, belle Rebecca, qui ai pris ton balai (ce qu'il a dit)
(Flop) Ce n'est pas moi le voleur (le contraire de ce qu'il a dit) "

They make coherent conclusions ("*si ce n'est pas Flap ni Flop alors c'est Flip*")

IV.3.2. Example of use of logic in problem solving

It comes as no surprise that, facing challenging tasks, gifted children use their abilities to the full in making logical inferences and thus demonstrate their potential of proving. In their 'proofs' they use schemas, symbols (letters), relationships between data and generalisations.

For example, Alice (10 years old, Fig.13) shows abilities of

- Construction of proof (*"Si la somme des ages de Peter et d'Elisabeth = 20, alors l'âge d'Elisabeth = 5 et celui de Melissa = 7, car la somme de son âge et celui d'Alex est égale à celle de Peter et d'Elisabeth et d'âge d'Alex est 13 ans."*)

- Use of a generalising remark as a justification of relationships between data ("*ce nombre (20) peut toujours changer, mais la différence d'âge entre celui d'Elisabeth et celui de Mélissa sera toujours 2 et Mélissa sera toujours plus âgée*")

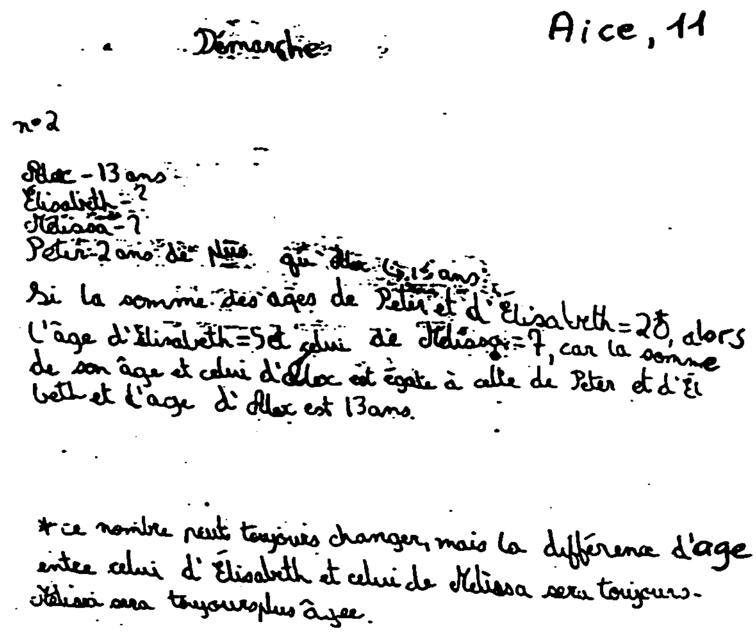


Fig.13 Example of logical explanation given by a mathematically gifted student

IV.3.3. Example of construction of a 'proof' by mathematically gifted students.

In the following situation, the answer found within a few seconds (actually a wrong one) has initiated a whole chain of reasoning accompanied by schemas and logical inferences in a mathematically gifted child.

The Grade 6 class was working on standard tasks with fractions (addition, multiplication , and so on). Some students finished this routine work early. John was among them. The teacher decided to add a bonus question based on a non-standard task :

What is a minimum number of steps needed to determine which of 12 equal-sizes identical pieces of money is false (that means its weight is different). One can use only a balance without weights.

John, who is often struggling with explanation of his (curtailed) reasoning and thus sometimes lowing his achievement ('lack' of communication skills), suddenly came up with a very detailed schema. While an attentive look at it reveals a misinterpretation of the task (John is an attentive reader, he does what is written, thus his interpretation of words 'minimum steps' was 'minimum possible with a good outcome' is correct, but we meant, of course, a minimum which would **guarantee** a good outcome), John's solution presents an interesting example of mathematical reasoning.

Actually, he could limit himself with much simpler explanation, like following one:

We can not do it in one step, because even if by chance, our first weighting of two pieces will be successful (one piece weights more than the other), another check is important to see which one is false.

But John took his time to create a very detailed schema with all logical steps of his reasoning (Fig.14).

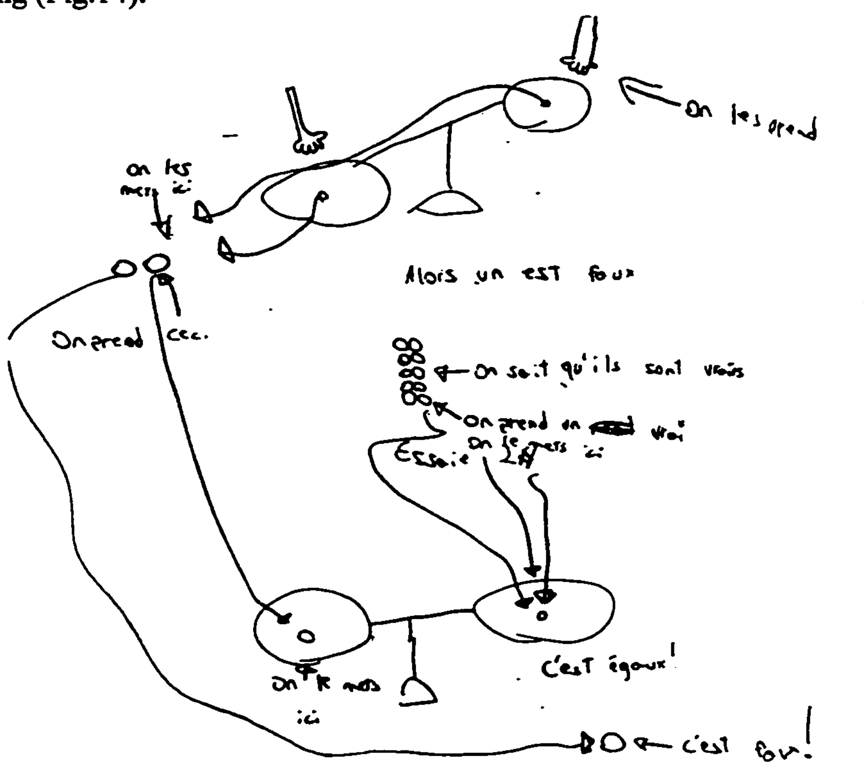


Fig.14 Example of a schema of proof invented by a gifted student

We see that John's schema combines linguistic and symbolic ways of explaining .
He constructs his proof in two steps that he calls "Essaie #1" and "Essaie # 2".
He provides us with a detailed instruction what to do: "prendre", "mettre".
He uses sometimes words, sometimes arrows to make conclusions, "alors".
Arrows are also used do indicate an action.

The question remains: why did he decide to give such a detailed communication of his reasoning?

One of the possible explanations is that the test in its main part was boring for him. He thus saw the bonus problem as a 'reward' or 'dessert' for this regular 'meal'. He enjoyed fully the opportunity to do 'real mathematics'. Thus the challenging character of the situation might have affected his mathematical behaviour.

The results of our study thus confirm the importance of the fostering of reasoning abilities in young children. As it is mentioned by Flores (2002), many students show self-reliance and clear understanding of the facts they have learned.

> Teachers can build on this understanding and methods to help students develop more efficient and systematic ways to provide evidence and to include mathematical reasoning in their arguments and proofs, at an appropriate level ... As children progress through more complex mathematics such as operations with fractions and negative numbers, their ability to explain why the facts that they learn are true seems to fall behind. As teachers, we need to help students make the transition from the empirical methods that work well with small numbers to methods based on relationships between numbers and between operations. To gain insight into children's thinking, we must ask them to explain or to justify what they are learning (Flores, 2002: 274)

As our examples show, the challenging situations approach allows creation a large variety of classroom situations featuring different didactical, methodological and epistemological aspects of mathematics. In the next section, we will summarise our findings giving more examples of challenging situations and analysing teacher's and student's role in them.

IV.4 GUIDELINES FOR THE DESIGN OF CHALLENGING SITUATIONS

We consider three kinds of challenging situations:

- open-ended problems and investigations
- routine work turned into a challenge by the teacher
- routine work turned into a challenge by a student

Let us consider these options in details:

IV.4.1 Open-ended problems and investigations

As we look at the video protocol of interviews with 4-6 year old children conducted by Bednarz and Poirier (1987) within their study of number acquisition by young children, we see how the evidence of differences in organisation of mathematical work by very young children becomes explicit with the open character of given tasks.

The video presents children's work on different tasks related to the concept of number: counting, formation of collections, order, conservation, comparison. Each task that in a regular classroom might be seen as ordinary, was given by authors in a very original challenging, dynamic, and open-ended way.

The child was constantly invited to think about the process of her work (how did you do it?), to develop an efficient strategy, to re-organise, if necessary, her process, to co-ordinate her actions. Thus, the routine tasks became open-ended and a child was given an opportunity to become an organiser of her mathematical work.

In our experiment, we also tried to make problems more open than they were usually presented to the students.

For example, we can take a problem from one mathematical competition :

--->	**1**	**2**	**3**	
	4	**5**	**6**	
	7	**8**	**9**	--->

In this table, we enter by 1 and exit by 9.
One can only move horizontally or vertically, and it is impossible to step twice on one box. For example, moving through boxes 1-2-5-8-9, one gets a sum of 25. But not all the trajectories lead us to the number 25.
Give all others 9 numbers.

This problem was given to participants of the regional final of the Championnat International des Jeux Mathématiques et Logiques in 2000 for Grade 4-5 children (10-11 year old) **http://www.cijm.org/cijm.html**.

We found that this problem would become more challenging for children if posed in a different way (open-ended) :

Someone is going to visit a museum, which has 9 exhibition halls, arranged in a square 3x3. The number of paintings in each hall is written in the box. What are all the possible numbers of paintings that could be seen by this visitor who does not like to be in one hall twice ?

Not only do we hide the number of different ways, which makes this problem open, we give it to our Grade 1 students (6-7 year old). Every student has a task at his/her level (They will all be able to find at least a couple of solutions). And we will be able to check for characteristics of mathematically gifted students.

Among open-ended problems, we distinguish a class of situations where the openness is being pushed to its extreme limit of mathematical *investigation*.

In section 3.2, we mentioned a way to transform a problem of constructing squares by means of cubes into a more challenging one. We can also use it to engage children in a real investigation making them to discover square numbers. For example, we can ask them, how many bricks would be needed to construct a square-shaped wall. We did this activity with grade 3 children (8 year old) and we could observe many characteristics of giftedness appearing.

IV.4.2 Routine work turned into a challenge by the teacher

The recent Quebec's school curriculum puts emphasis on the learning of basic number facts (like multiplication tables). This routine task can become challenging by many different ways. For example, one day we wrote on the board the 9-table operations:

1 x 9 =

2 x 9 =

3 x 9 = , and so on

Grade 3 children said immediately that it is a very easy table, because there is a well known regularity (writing down first digits of the product in order from 0 to 9 and the second ones down from 9 to 0, we obtain all the multiples of 9 : 09, 18, 27, and so on). Among the answers one could find that 6 x 9 = 54.

So, the teacher comes to the board and writes 6 x 9 = 56 telling the story that when he was young, he had to memorise all answers, not just 'tricks', and he is sure that 6 x 9 = 56. The students are confused, but many of them started to think how to prove that their result (54 was the correct one).

Many of them went to the board to share their ideas as well as other ways to obtain a 9-table. As a result of the lesson, the 9-table has appeared a couple of times on the board, children said it many times aloud, so they could memorise it and at the same time do it in a meaningful way questioning and proving their methods and ways of reasoning.

IV.4.3 Routine work turned into a challenge by a student

When Grade 4 children are asked to represent 1/8 of a rectangle, they find it an easy and routine task. That's why we were surprised by Christopher's way to divide a rectangle in 64 boxes (8 rows x 8 columns) and to colour 8 boxes randomly. He found that the task was not challenging enough and he wanted to make it more complicated.

IV.4.4 Transformation of challenge within one situation

All three ways of creating of challenging situations are not isolated from one another. They can also be transformed one into another.

For example, a kindergarten class (5-6 year old) is working on an **open-ended** problem:

Amelie needs to build new houses for her farm animals. When one looks at the house from the sky, she sees that all of them have a roof in shape of a 'digit'. She has to build now a new house for her cows. What 'digit' you suggest to use for the roof of this new house.

Children used blocs in form of different solids. The activity aimed to make them to explore different solids, to make different constructions with them. There are, basically, two ways of making constructions: three-dimensional or two-dimensional establishing thus different spatial relationships. For example, we may teach children to verify which shapes fit together recovering certain surface. The activity that we gave to our students didn't aim to teach any particular way of making constructions: many textbooks contain a lot of exercises asking children to reproduce one construction or another. Our situation was designed in order to help children get a certain 'spatial feeling' trying different way to layout blocs. The main challenge in it was to organise a mathematically meaningful investigation within an 'ill-defined' problem.

Some of them chose to imitate shapes of digits in the way we write them, others looked for different ways to create more 'economic' constructions taking care of geometric properties (like seeing if the blocs fit one to another) . Finally, there was a group of children who moved from the initially given situation of building a new 'house' and started to construct many digits 'writing' numbers (up to the "1000") (Appendix A11, p.154).

Soon, we could see that originally challenging and creative, the task became routine for many children. So, we decided to put some restrictions (new 'variables') that were sought as means to engage children in the investigation of a different problem in which we would be willing to construct house that has a "5"- shape and do this with a minimum of blocs. Thus, with the intervention of the teacher, a routine problem became a challenging one once again.

This method of a 'sudden' change of didactic variable (Brousseau, 1997) is important in our study of relationship between child's organisation of the problem-solving and mathematical giftedness because it provokes a reflection (what is new?) and re-organisation of the whole process of thinking and acting (what do I need to modify?) and thus gives students a chance to show their full potential.

In our experimental work with young children, we obtained a constant confirmation of the fruitfulness of such an approach, especially if one wants to identify and nurture gifted children. The picture (Appendix A12, p.155), shows David (5 years old) working on the 'minimum' task (see above); he looked happy with his solution (4 blocks) but still in what looked like a 'state of alert'. At this moment, we began to discuss children's solutions. One group of children has presented a three-blocs solution. Suddenly, David started to change something in his configuration. As we see, the number of blocs is still the same (4) but the blocs are smaller. But what is the most intriguing, is the rapid reactions of this child to the changing conditions (someone has found a better solution). This constant state of 'alert' is an important characteristics of giftedness which could be better activated in challenging situation that in the ordinary one.

This state of 'alert' leads them to constantly verify all the conditions going back and forth through the situation. Here is one more observation. Grade 4 students worked on their test. Answering a question of 'Is it true that if the sum a+c=8 then a and c are two different numbers?', Christopher hesitated a lot, saying however, that the numbers have to be different. As his work on the test went on, he had to solve a system of two equations with two variables: ab=16, a+b=8. He found easily a=4, b=4 as a solution then went back to his previous task and corrected his answer.

We could also observe another interesting phenomenon: challenging situation created by the teacher may initiate its further explorations by gifted students.

For example, doing the same activity with Grade 1 children (6-7 year old), we could state that it was seen as a routine problem by many of them and some of students lost completely their interest in it. Yet, we could still observe one girl looking for many different ways of building "5" using 4 blocs.

Not only she kept herself working on this problem, she came out with a new one : she started to look for possibilities to built a digit "4" with a minimum of blocks. Here, the problem was turned into a challenging one by the student.

IV.5 THE ROLE OF THE TEACHER

In a challenging environment the **role of the teacher** becomes crucial in all the stages:

- choice of a problem
- way of presenting it to the students
- organisation of student's work
- interpretation of results
- follow-up

One of the very important conditions of success of the challenging situations approach is the teacher's attitude. How should we, as teachers, control the student's work? Related to the learning paradox (described in the previous chapter), it is far from being obvious how to find a solution to this problem. On the one hand, every word and every gesture said by teacher can affect the whole challenge of situation in either a positive or a negative sense. On the other hand, the teacher has to have a full didactical control of the situation (otherwise a mathematical learning activity might become a sort of 'arts and crafts in mathematical wrapping').

Our experiment didn't provide us with clear recipes but rather with examples that can be open to further questioning and investigations. These examples allowed us to formulate teacher's approaches favourable for the challenging situation:

- Give a child an opportunity to think: being a flexible teacher
- Support of children's willingness to learn more about math
- Challenge students in informal situations: sense of humour
- Support children their desire to go beyond pre-planed situations
- Giving hints without telling solutions

- Management of particular cases of mathematical giftedness
- Use of 'little tricks' as follows :

- While distributing manipulative material (blocs, cubes, etc.), we would give children time to touch it, to play with it, to get a feeling of it; sometimes it gives us important clues of children's organisations (how they put material, arrange it, order, classify, build different forms, etc.)
- When children finish their manipulation, we ask them to write a report. Sometimes it makes sense to give them time to break up their constructions. This opens the door to a variety of presentations (will the child reproduce his construction, add new details, draw a completely different pictures)
- When children are asked to communicate their results, it is important to motivate them to give detailed explanations. We often ask them to be 'mini-teachers' - to explain to somebody who doesn't understand the problem
- Children often ask us to teach them complicated things. Sometimes, a pedagogical effect can be bigger if the teacher make them wait. Then, starting to teach it, children might become more motivated: *finally, we got it*!

IV.6 THE ROLE OF THE STUDENT

The role of the students in a challenging situation differs significantly from those in the regular learning activity. They have to adapt to a new, open environment. They have no precise algorithm of actions, no clear instruction what to do. Therefore, they have an opportunity to:

- **demonstrate different approaches to the problem**
- **act differently in different situations**

- **overcome obstacles, construct various means, discover new relationships**
- **work on mathematical problems based on structures and systems using properties and definitions, conjectures and proofs**
- **use of logical inference with fluency, control, rigour**
- **combine logic and creativity in problem solving**
- **invent new symbols and signs, use schemas and abstract drawings**
- **use reflective thinking**
- **ask mathematical questions, create new problems, investigate, use mathematics in non-mathematical situations, look around with 'mathematical eyes'**

IV.7 CONCLUSION

Based on Krutetskii's characterization of mathematically gifted children, Brousseau's theory of didactical situations and Shchedrovitskii's model of reflective thinking, we established a developmental 'recursive chain' of identification and fostering of mathematical giftedness.

In the enriched mathematics offered to all students, we use a "challenging situations" approach which helps develop a higher level of theoretical thinking in young children by means of open-ended tasks, mathematical investigations, and solving challenging tasks that require mathematical reasoning based on logical inference.

The experiment conducted at Académie Marie-Claire, a private elementary school with enriched K-6 curriculum in all subjects offered to all students, demonstrated that this approach helps teachers create a learning environment favourable for the intellectual development of not only gifted students, but also other children. The atmosphere created in the mathematics classroom allowed them to ask interesting mathematical questions, to search for new original ideas, to give full and logical explanations, to organise and re-organise data, to develop different strategies in problem solving and share them with their peers.

The results of our experiment show the importance of epistemological, methodological and didactical analysis of mathematical content in order to find the best way for creating meaningful learning and teaching situations that provoke theoretical mathematical thinking in young children and foster its reflective, systemic and analytic components.

CHAPTER V. CONCLUSIONS AND RECOMMENDATIONS

There exists a number of educational studies of mathematical giftedness. Various models of giftedness based on different characteristics of mathematically gifted students have been developed and implemented. Different programs of support provide gifted students with advanced curriculum and guidance of highly qualified professionals. Several mathematical contests, Olympiads, and competitions help in searching for mathematically gifted children and taking care of their development.

Yet, the problems of identification and nurturing of mathematical giftedness are far from being solved. Many children become bored, at a very early age, with the simplified curriculum, lose their interest in mathematics and waste their intellectual potential. Despite the ingenious testing system, some children never get admitted to special programs for gifted students. The regular school system is not equipped to help these children.

Our study aims to contribute to filling this gap and providing elementary school (Grades K-6) teachers with methods of identification and fostering mathematically gifted children in the mixed ability classroom.

We have called our approach, the "challenging situations approach". The approach is theoretically grounded in Krutetskii's notion of mathematical ability, Shchedrovitskii's developmental model of reflective action, Bachelard's notion of epistemological obstacle, Sierpinska's distinction between theoretical and practical thinking in mathematics, and Brousseau's theory of didactic situations.

Following Krutetskii, we have defined mathematical ability as a 'mathematical cast of mind', which represents a unique combination of psychological traits that enable young children to think in structures, to formalise, to generalise, to grasp relations between different concepts, structures, data and models and thus solve different mathematical problems more successfully than children of average or low ability.

At a very early age, these children demonstrate high thinking potential in reasoning about mathematical concepts and systems of concepts along with the capacity to reason about their reasoning. From the outset, they are better prepared than other children for theoretical thinking, which is the foundation of pure mathematical thinking.

The critical point of our study was an understanding that a discovery and nurturing of theoretical thinking is not possible if children are working with routine arithmetical tasks, merely applying algorithms that had been provided by the teacher, telling her students what to do and how to do it.

The paradoxes of such classroom situations have been described by Brousseau in his Theory of Didactical Situations. Following Brousseau's theory, we bring a notion of challenging situation into our model of mathematical giftedness postulating that a gifted child will show her talent in mathematics only in specific situations when a real question has been asked and a real problem has been posed.

"Challenging situations" use open-ended problems and mathematical investigations. A challenging situation initiates the student's action of structuring a problem, and of searching for links between data and with her previous experience. Since a real challenge is possible only when the situation is new for the learner, the challenging situation must contain a rupture with what the student has previously learned, provoking the student to reflect on the insufficiency of the past knowledge and construct new means, new mechanisms of action adapted to the new conditions, activating her full intellectual potential. A challenging situation could also provide the student with an opportunity to face an obstacle of a pure mathematical nature, the so called epistemological obstacle. In order to overcome it, the student will have to re-organise her mathematical knowledge, create new links, new structures following laws of logical inference. We claim that situations satisfying these conditions allow the teacher to identify and nurture mathematical giftedness among her students.

A challenging situation often presents the child with a problem, which goes above or beyond the average level of difficulty. The child is encouraged to surpass what is normally expected of children of her age, thus demonstrating her precocity, which is a sign of mathematical giftedness.

A challenging situation cannot be created as an isolated learning task. It full developmental potential can be realised only within a system of teaching based on a challenging curriculum as a whole. This would allow to create a learning environment in which every child would be able to demonstrate her highest level of ability.

This is why, using a challenging situation model we are not only able to get gifted children involved in genuine mathematical activity but also help all children to increase their intellectual potential.

A challenging situation helps to create a friendly environment in which a child compete with herself sharing her discoveries with other children and learning from others. Thus it gives mathematically gifted children who are not high achievers to participate actively in class and to succeed.

Finally, a challenging situation has another opening for gifted children: they can always go further, go beyond situations, ask new questions, initiate their own investigations, be more creative in their mathematical work. This spontaneous mathematical reaction feeds back into the learning environment in a positive way and further enhances its potential for all children. We consider this feature of the approach as crucial from the point of view of mathematics education for all children.

Our study prompts different teaching approaches in mathematics. The teacher is no more re-translator of knowledge or instructor of methods of problem solving. In a challenging situation her role becomes more moderator of discussions, listeners of student's ideas, student's guide through the discovery.

In helping students go through various obstacles, we shall encourage them to:

- **Organise** his/her mathematical work
- **Reason** mathematically
- **Control** several conditions (verification, adjustment, modification, reorganisation, awareness of contradictions, validation)
- **Choose/develop** efficient strategies/tools of problem solving
- **Reflect** on methods of mathematical work
- **Communicate** his/her results in a "mathematical" way (oral/written form, use of symbols, giving valid explanations)

Thus, we will be able to **identify** gifted children who:

- ask spontaneously questions beyond given mathematical task
- look for patterns and relationships
- build links and mathematical structures
- search for a key (essential) of the problem
- produce original and deep ideas
- keep a problem situation under control

- pay attention to the details
- develop efficient strategies
- switch easily from one strategy to another, from one structure to another
- think critically
- persist in achieving goals

At the same time, we will nurture their curiosity, willingness to learn more about mathematics, provide them with an opportunity to go further in their mathematical learning, to create new structures, to pose new problems and thus **foster** the development of their mathematical abilities.

This approach is very demanding to the teaching. The teacher has to think constantly about challenging the students, look for different ways to stimulate children's work, demonstrate a high flexibility, ability to react spontaneously on changing conditions of the classroom situation, be ready to provoke students and to get provoked by students asking question which the teacher can not answer immediately.

The understanding of how highly talented children think mathematically would lead to elaboration of efficient didactical approaches for all students. We shall agree with following general remark made by Young &Tyre (1992): "If we examine more closely what it is that makes prodigies, geniuses, gifted people, high achievers, champions and medallists, we may be better able to increase their number dramatically".

REFERENCES

Arsac, G., Germain, G., & Mante, M. (1988). Problème ouvert et situation-problème. Lyon: Université Claude Benaud.

Bachelard, G. (1938). La formation de l'esprit scientifique. Paris: Presses Universitaires de France.

Baroody, A. (1987). Children's mathematical thinking: a developmental framework for pre-school, primary and special education teachers. Columbia: Teachers College, Columbia University.

Baroody, A. (1993). Problem solving, reasoning, and communication, K-8: helping children think mathematically. Macmillan Publishing Company.

Baroody, A., & Ginsburg, H. (1990). Children's Learning: A Cognitive View. In: R.Davis, C.Maher, & N. Noddings (Eds.), Constructivist Views on the teaching and Learning of Mathematics (pp 51-64). Reston, Va: NCTM..

Bednarz, N.(1987). Les mathématiques et l'enfant (Le concept du nombre)-bande vidéo. Montreal: UQAM.

Berry, J., "Do Not Shirk a Testing Subject", *Daily Mail*, December 3, 1996.

Binet, A. (1922). Étude expérimentale de l'intellegence. Paris: A. Costes.

Brousseau, G.(1997). Theory of didactical situations in mathematics. Dordrecht: Kluwer Academic Publishers.

Burjan, V.(1991). Mathematical Giftedness - Some Questions To Be Answered. In: F. Moenks, M. Katzko, & H. Van Roxtel (Eds.), Education of the Gifted in Europe: Theoretical and Research Issues: report of the educational research workshop held in

Nijmegen (The Netherlands) 23-26 July 1991 (pp.165-170). Amsterdam / Lisse: Swetz & Zeitlingen Pub. Service.

Burton, L. (1984). Thinking Things Through. Oxford: Basil Blackwell Limited.

Calkins, M.W. (1894). A Study of the Mathematical Consciousness. Educational Review, New York, Vol. VII,.

Denton, C. (1985). Able children, Identifying them in the Classroom. NFER-NELSON.

Driscool, M. (1999) Fostering Algebraic Thinking: A Guide for Teachers, Grades 6-10. Portsmouth, NH: Heinemann.

Dunham ,W.(1990). Journey Through Genius. New York: Penguin Books.

Ernst, P.(1998). Recent development in mathematical thinking. In: R. Burden, & M. Williams (Eds.), Thinking through the curriculum (pp.113-134). London, New York: Routledge.

Fishbein, E. (1990). Introduction. In: P.Nesher, & J.Kilpatrick (Eds.), Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education. Cambridge: University Press.

Fisher, R.(1990). Teaching children to think, Oxford: Basil Blackwell.

Flores, A. (2002, January).How do children know that what they learn in mathematics is true? Teaching children mathematics, 269-274.

Fox, L. Identification of Academically Gifted, American Psychologist, Vol. 36, No. 10, 1103-1111.

Gardner, H.(1983). Frames of Mind: The Theory of Multiple Intelligences. New York: Basic Books.

Gelman, R.(1991). Epigenetic Foundations of Knowledge Structures: Initial and Transcendent Constructions. In: S. Carey, & R. Gelman (Eds.), The Epigenesis of Mind: Essays on Biology and Cognition, Nillsdale (pp.293-322). NJ: L. Erlbaum Associates.

Gelman, R., & Gallistel, C. (1978). The Child's Understanding of Number. Cambridge: Harward University Press.

Greenes, C. (1981, February). Identifying the Gifted Student in Mathematics. Arithmetic Teacher, 14-17.

Greenes, C. (1997, October). Honing the abilities of the mathematically promising. Mathematics Teacher.

Guilford, J.A.(1967). The Nature of Human Intelligence,. New York: McGraw-Hill.

Johnson, M. (1983, January). Identifying and Teaching Mathematically Gifted Elementary School Children. Arithmetic Teacher, 25-26; 55-56.

Kahane, J-P. (1999). Mathematics Competitions. ICMI Bulletin No. 47.

Kennard, R. (1998). Providing for mathematically able children in ordinary classrooms", Gifted Education International, Vol. 13 , No 1, 28-33.

Kiefer, E. (1995). Ability Grouping in Mathematics and Testing for Mathematical Aptitude: A Project in the Department of Mathematics and Statistics. Montreal: Concordia University.

Krutetskii, V. (1962). K voprosou o mathematitcheskih sposobnostyah u shkolnikov (Problems of mathematical abilities in school children). In N. Levitov, & V.Krutetskii (Eds.), Sposobnosti i interesy (Abilities and Interests), in Russian.

Krutetskii V.A.(1976). The psychology of mathematical abilities in school children. Chicago: The University of Chicago Press.

Krutetskii, V.A. (1969. An investigation of mathematical abilities in schoolchildren. In: J.Kilpatrick, & I.Wirszup (Eds.), The structure of mathematical abilities. Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. 2. Stanford, Calif.: School Mathematics study Group.

Kulm, G. (1990). New Directions for Mathematics Assessment. In: G. Kulm, Assessing higher order thinking in mathematics. Washington, DC: American Association for the Advancement of Science.

Lee, L. (1982). A Study of the problem solving behaviour of twenty-four students. Montreal: Concordia University.

Livne N., & Milgram R.(2000, January). Assessing Four Levels of Creative Mathematical Ability in Israely Adolescents Utilising Out-of School Activities: A Circular Three-Stage Technique. Roeper Review, 111-116.

Lorenz, J.H. (1994). Mathematically retarded and gifted students. In: R.Biehler R.Scholz, R. Straesser, & B. Winkelman (Eds.), Didactics of mathematics as a scientific discipline (pp. 291-302). Dordrecht, Kluwer: Academis Publishers.

Lupkowski, A., & Assouline, S.(1997). Jane and Johnny love math: recognising and encouraging mathematical talent in elementary students: a guide book for educators and parents. Unionville. NY: Royal Fireworks Press.

Lyons, M., & Lyons, R. (2001-2002). Défi mathématique. Cahier de l'élève. 1-2-3-4. Montreal: Chenelière McGraw-Hill.

Lyons, M., & Lyons, R. (1989). Défi mathématique. Manuel de l'élève. 3-4-5-6. Laval: Mondia Editeurs Inc.

Hlavaty, J.H. (Ed.). (1959). Mathematics for the Academically Talented Student in the Secondary School: report of a conference sponsored jointly by National Educational Association on the academically talent students and the NCTM. Washington: National Educational Association on the academically talent students and the NCTM.

Miller, R.(1990). Discovering Mathematical Talent. ERIC Digest #E482.

Mingus, T., & Grassl, R. What Constitutes a Nurturing Environment for the Growth of Mathematically Gifted Students? School Science and Mathematics, 99 (6), 286-293.

Paek, P., Holland, P., Suppes, P. Development and Analysis of Mathematics Aptitude Test for Gifted Elementary School Students. School Science and Mathematics, 99 (6), 338 – 347.

Pendarvis, Howly, & Howly (1990). The abilities of gifted children, Engelwood Cliffs, NJ: Prentice Hall.

Programme de formation de l'école québecoise: 2001, Québec, 2001.

Peressini D., & Knuth, E. (2000, January). The Role of Tasks in Developing Communities of Mathematical Inquiry. Teaching Children Mathematics,391-396.

Piaget, J.(1950). The Psychology of Intelligence. London: Routledge & Kegan Paul.

Piaget, J. & Garcia, R.(1989). Psychogenesis and the history of science. New York : Columbia University Press.

Polya, G. (1957). How To Solve It. Princeton: Princeton University Press.

Resnick, L.B. (1983). A Developmental Theory of Number Understanding. In: H.P. Ginsburg (Ed.), The development of mathematical thinking (pp. 110-152). Academy Press Inc.

Ridge, L., & Renzulli, J. (1981). Teaching Mathematics to the Talented and Gifted. In: V.Glennon (Ed.), The Mathematical Education of Exceptional Children and Youth, An Interdisciplinary Approach (pp. 191-266). NCTM.

Rosenbloom, P.C. (19600 Teaching gifted children mathematics. In E.Torrance (Ed.), Talent and Education: present status and future directions: papers presented at the 1958 Institute on Gifted Children (pp. 351-370). Minneapolis: University of Minnesota.

Rosin, V. (1967). Semiotic analysis of signs in mathematics. Semiotica i vostochnye yasiki, in Russian.

Rosskopf, M.F.(1975. Children's mathematical concepts: six Piagetian studies. Teachers College, Columbia University.

Rotigel J., & Lupkowski-Shoplik A.(1999, October). Using talent searches to identify and meet the educational needs of mathematically talented youngsters. School Science and Mathematics,.

10 SATs plus advice from the college board on how to prepare for them, 1990.

Shchedrovitskii, G. (1968) Pedagogika i logika. Unedited version (in Russian).

Sheffield, L.(1999) Serving the Needs of the Mathematically Promising. In: L. Sheffield (Ed.), Developing mathematically promising students (pp. 43-56). NCTM.

Shields, J.B. (1968). The Gifted Child. National Foundation for Educational Research in England and Wales.

Schleifer, M. (1989). Le conflict cognitif chez Piaget: Une interprétation. In: N. Bednarz, & C. Garnier (Eds.), Construction des savoirs : Obsatacles et conflicts (pp. 156-161). Ottawa: Agence d'ARC inc..

Schoenfeld, A. H. (1979). Explicit heuristic training as a variable in problem-solving performance. Journal for Research in Mathematics Education, 1979, 10, 173-187.

Schrag, F. (1988). Thinking in school and Society. NY & London: Routledge.

Sierpinska, A.(1994). Understanding in mathematics, London: The Falmer Press.

Sierpinska, A., & Freiman, V.(2002). Mathematically gifted children and high achievers. A workshop of NTCM Canadian regional conference, Montreal, August 15-16, 2002.

Sierpinska, A., Nnadozie, A., & Octaç, A.(2002). A study of relationships between theoretical thinking and high achievement in linear algebra. Montreal: Concordia University.

Skvortsov V. (1978). Mathematical Olympiads. In F.Swetz (Ed.), Socialist mathematics education (pp. 351-370). Southampton, PA: Burgundy Press.

Sovchik, R.(1989). Teaching mathematics to children. Harper & Row, Publishers.

Stanley, J., & Benbow, C. (1986).Youth who reason exceptionally well mathematically. In: R.Sternberg, & J.Davidson (Eds.), Conception of giftedness. (pp. 361-367). Cambridge, NY: Cambridge University Press.

Sternberg, R., "What Does It Mean to Be Smart?", Educational leadership, March 1997, pp.20-24.

Tempest, N.(1974). Teaching clever children, 7 - 11. London: Routledge & Kegan Paul.

Tuttle F., Jr., & Becker L.(1980). Characteristics and Identification of Gifted and Talented Students. Washington: National Educational Association.

Vernon, P.E. (1971). The Structure of Human Abilities. London: Methuen.

Vygotsky, L.S.(1987). The Collected Works of L.S. Vygotsky. Volume 1. Problems of general psychology including Thinking and Speech. New York and London : Plenum Press.

Wagner, H., & Zimmermann, B. (1986). Identification and fostering of mathematically gifted students. In: A. Cropley, K.Urban, H.Wagner, & W.Wieczerkowski (Eds.), Giftedness: A continuing world-wide challenge (pp.273-287). New York: Trillium Press.

Young P., & Tyre C.(1992). Gifted or able?: realising children's potential. Open University Press.

Resources for teachers: open door into education of mathematically gifted children.

Books:

Developing mathematically promising students /Ed. by Linda Jensen Sheffild, NTCM, 1999

Krutetski V.A. **The psychology of mathematical abilities in school children**, 1976

Silver E.A., Kilpatrick J, Schlesinger B. **Fostering Inquiry And Communication In Mathematics Classrooms**, 1990

Baroody A. **Problem Solving, Reasoning and Communicating: Helping children think mathematically**, 1993

Articles:

Greenes, C. (1981). **Identifying the gifted student in mathematics.** Arithmetic Teacher, February 1981, 14-17.

Miller, R. (1990). **Discovering mathematical talent.** ERIC Digest #E482.

Mingus, T.T.Y. & Grassl, R.M. (1999): **What constitutes a nurturing environment for the growth of mathematically able students?** School Science and Mathematics 99(6), 286-293.

Electronic resources:

What makes giftedness?

Renzully Model
(Defining giftedness, common terms)

http://www.misd.net/Gifted/Renzullimodel.html

Fostering mathematical giftedness: NRICH-project (mathematics enrichment: problems, solutions, articles, inspirations, games, LOGOland, Interactivities)

http://nrich.maths.org/mathsf/about.htm

Interactive Mathematics Miscellany and Puzzles

http://www.cut-the-knot.org/front.shtml

National Association for Gifted Children

http://www.nagc.org/

Resources for Parents, Teachers

And Gifted, Talented, Creative

And Promising Mathematics Students

http://www.nku.edu/~mathed/gifted.html

Math Competitions:

Canadian Competitions

http://www.cms.math.ca/Olympiads/

International Competitions

In French: Comité International des Jeux Mathématiques (International Math Competitions)

http://www.cijm.org/cijm.html

In French: Kangourou mathématique

www.mathkang.org

Australian Mathematics Competition

www.amt.canberra.edu.au/

APPENDIX A

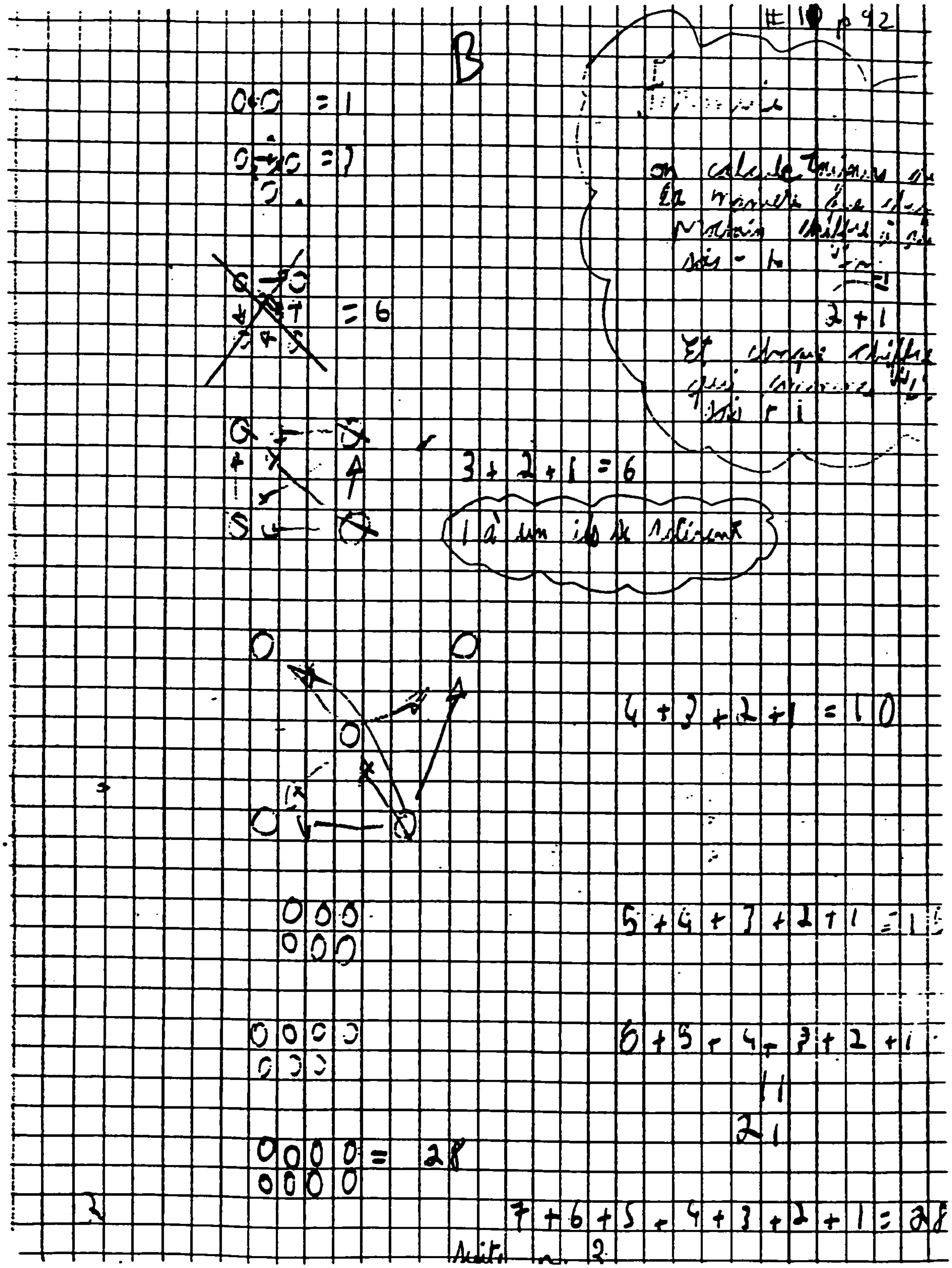


Fig. A1 Marc-Etienne's (10) report on the problem of handshakes.

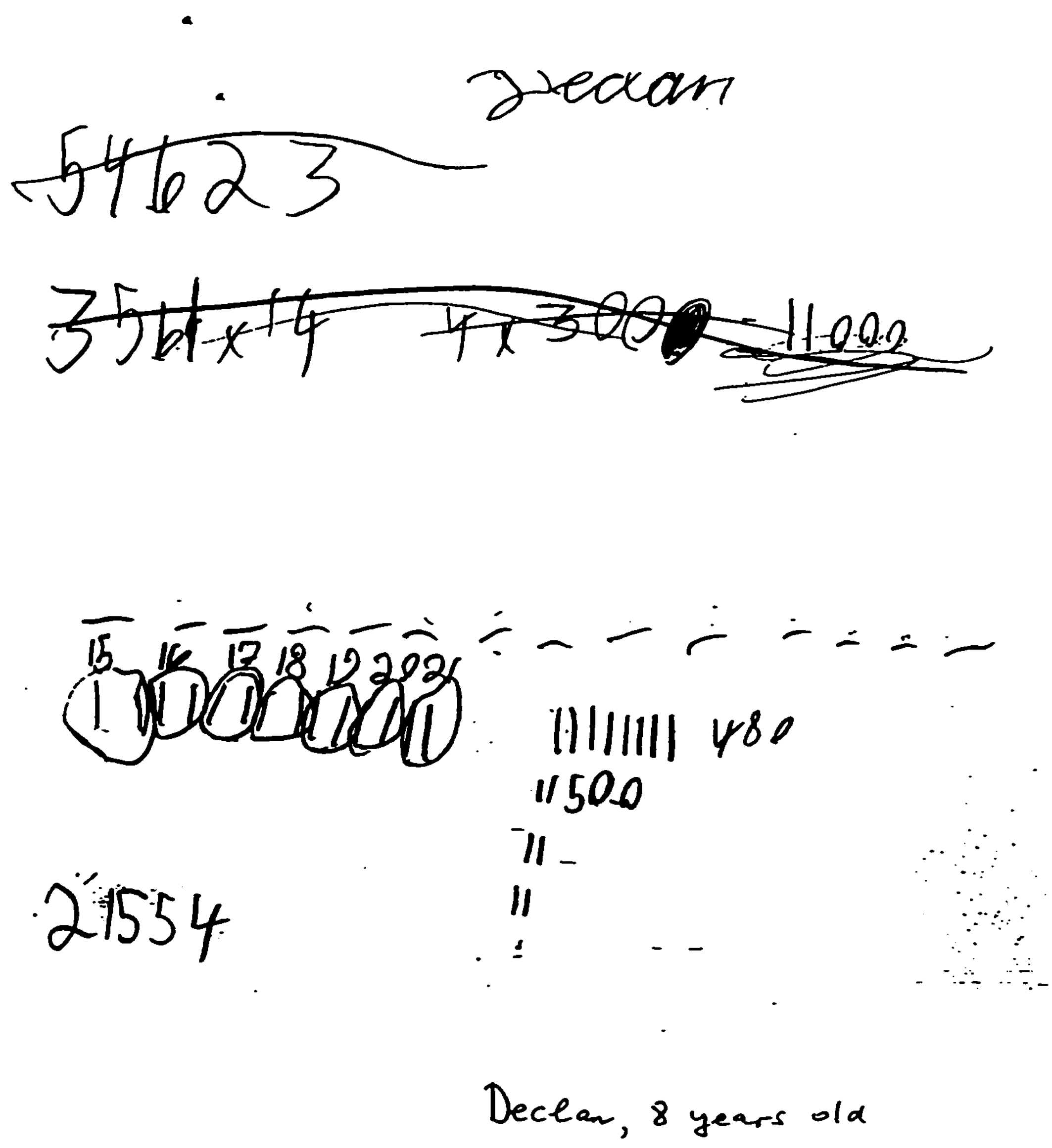


Fig. A2 Declan's (8) report on the calculation 3561 x 14

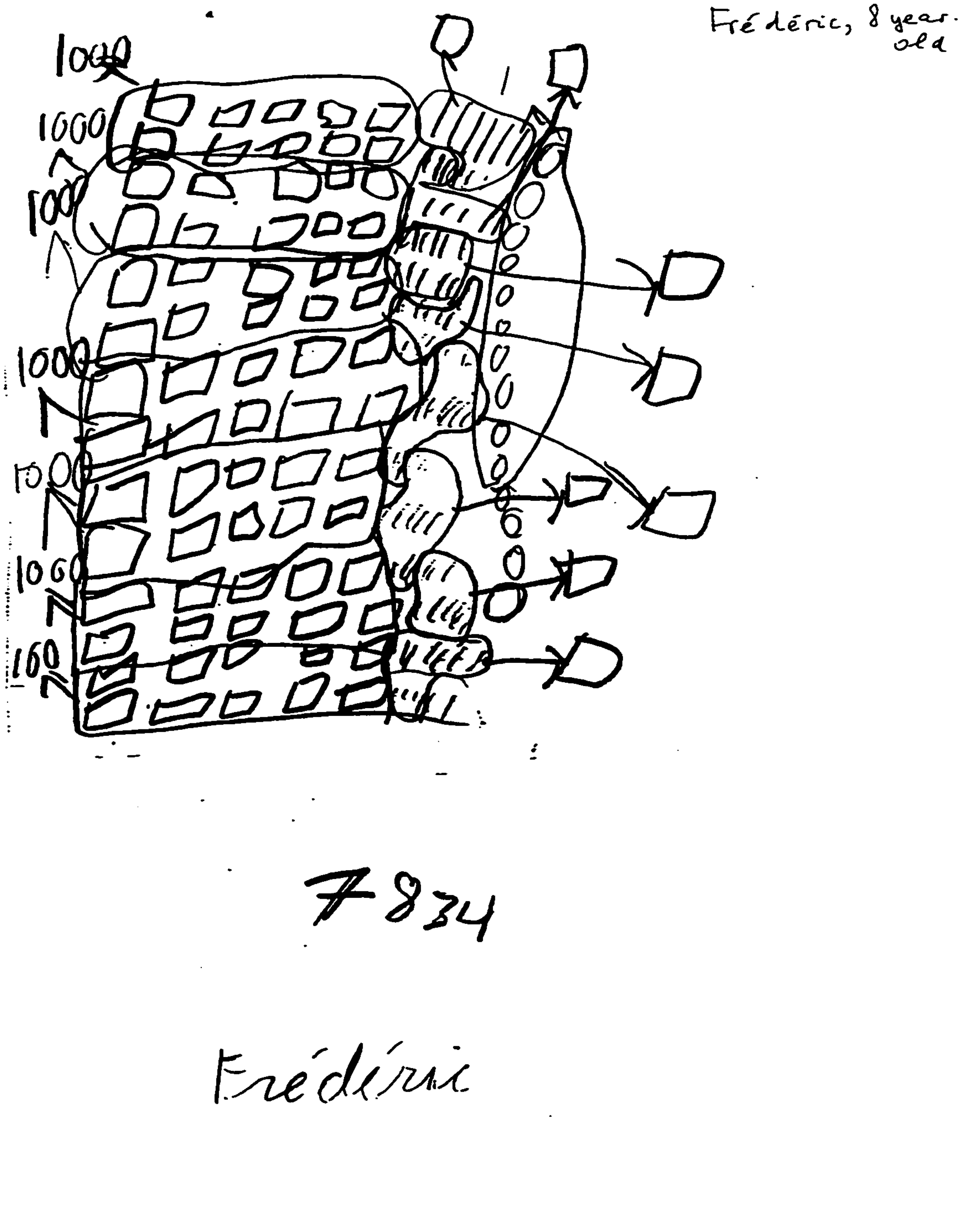


Fig. A3 Frédéric's (8) use of pictograms to calculate 3561 x 14

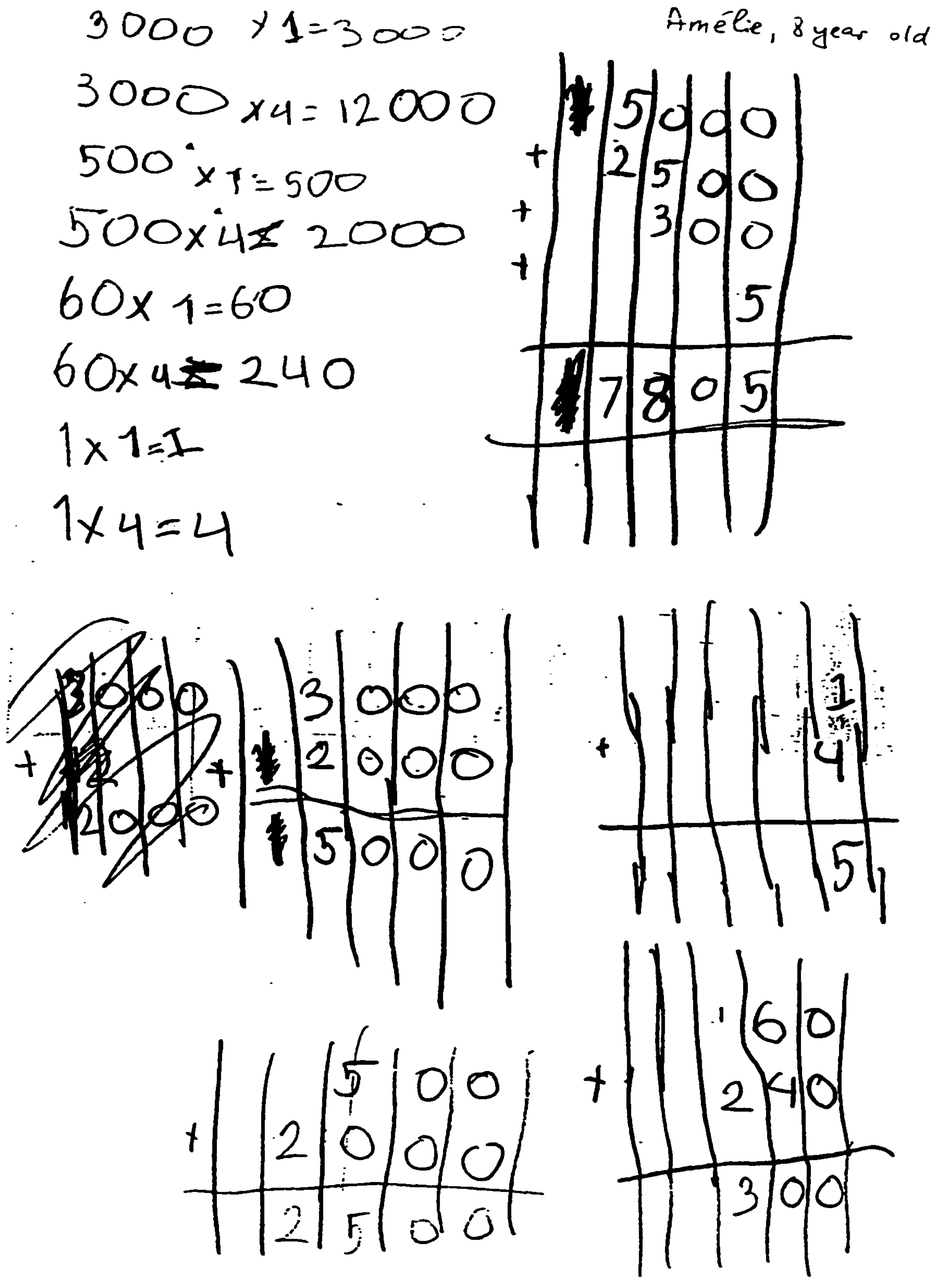


Fig. A4a Amélie's (8) method of decomposing 3561 x 14

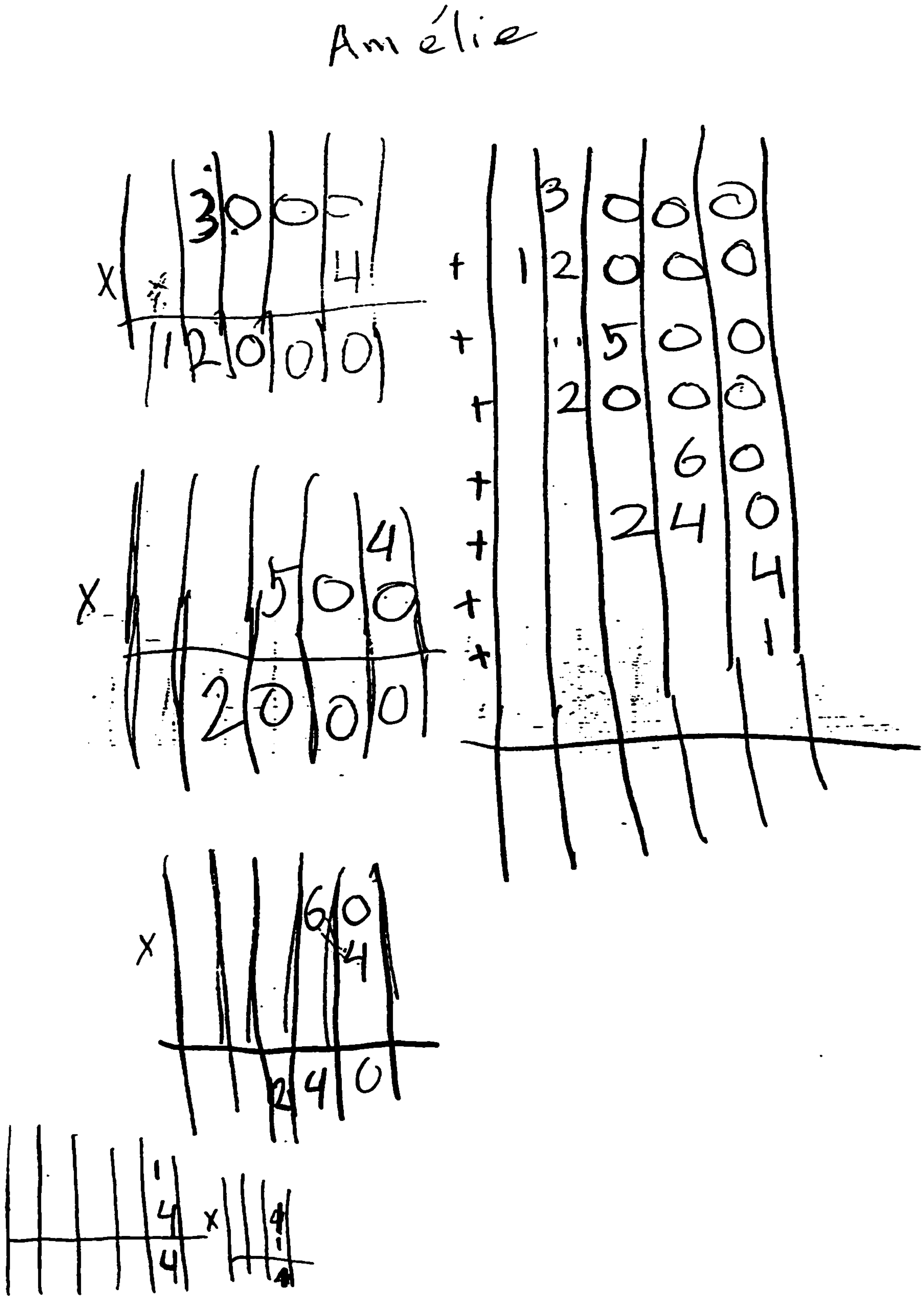


Fig. A4b Amélie's (8) method of decomposing 3561 x 14 (continued)

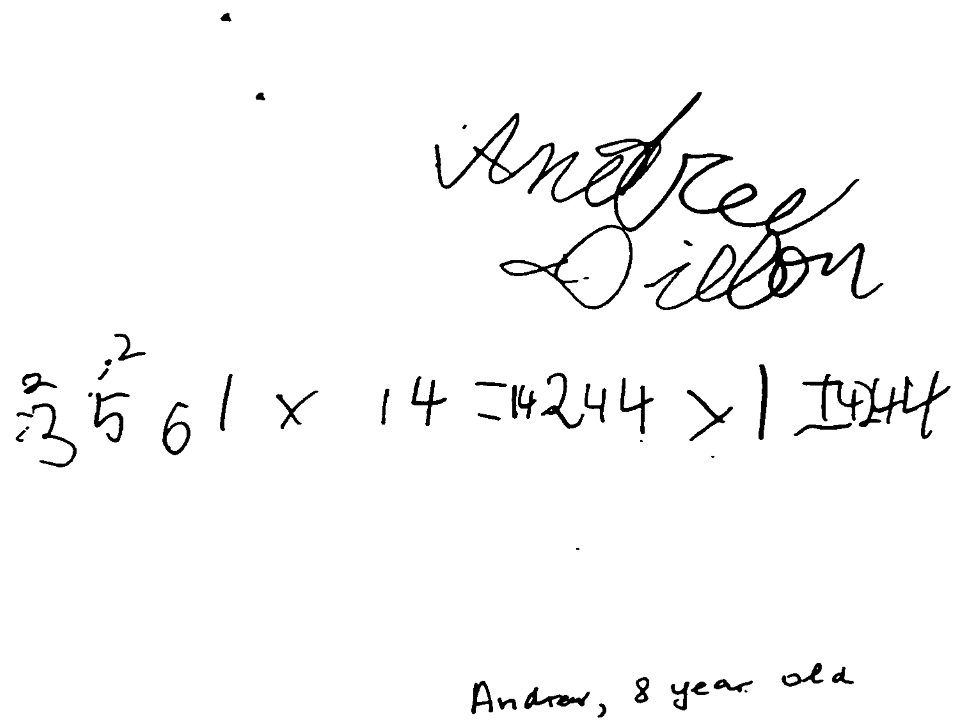


Fig. A5 Andrew's (8) solution to the problem 3561 x 14

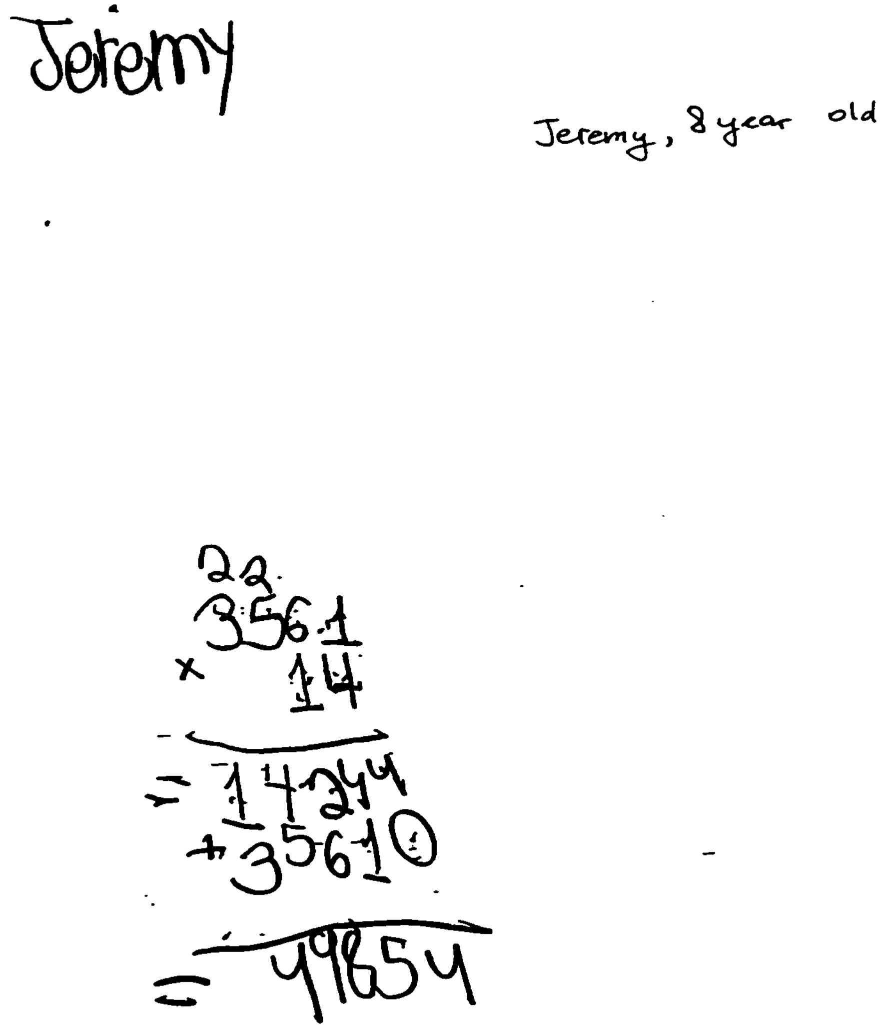


Fig. A6 Jeremy's (8) way of calculations 3561 x 14

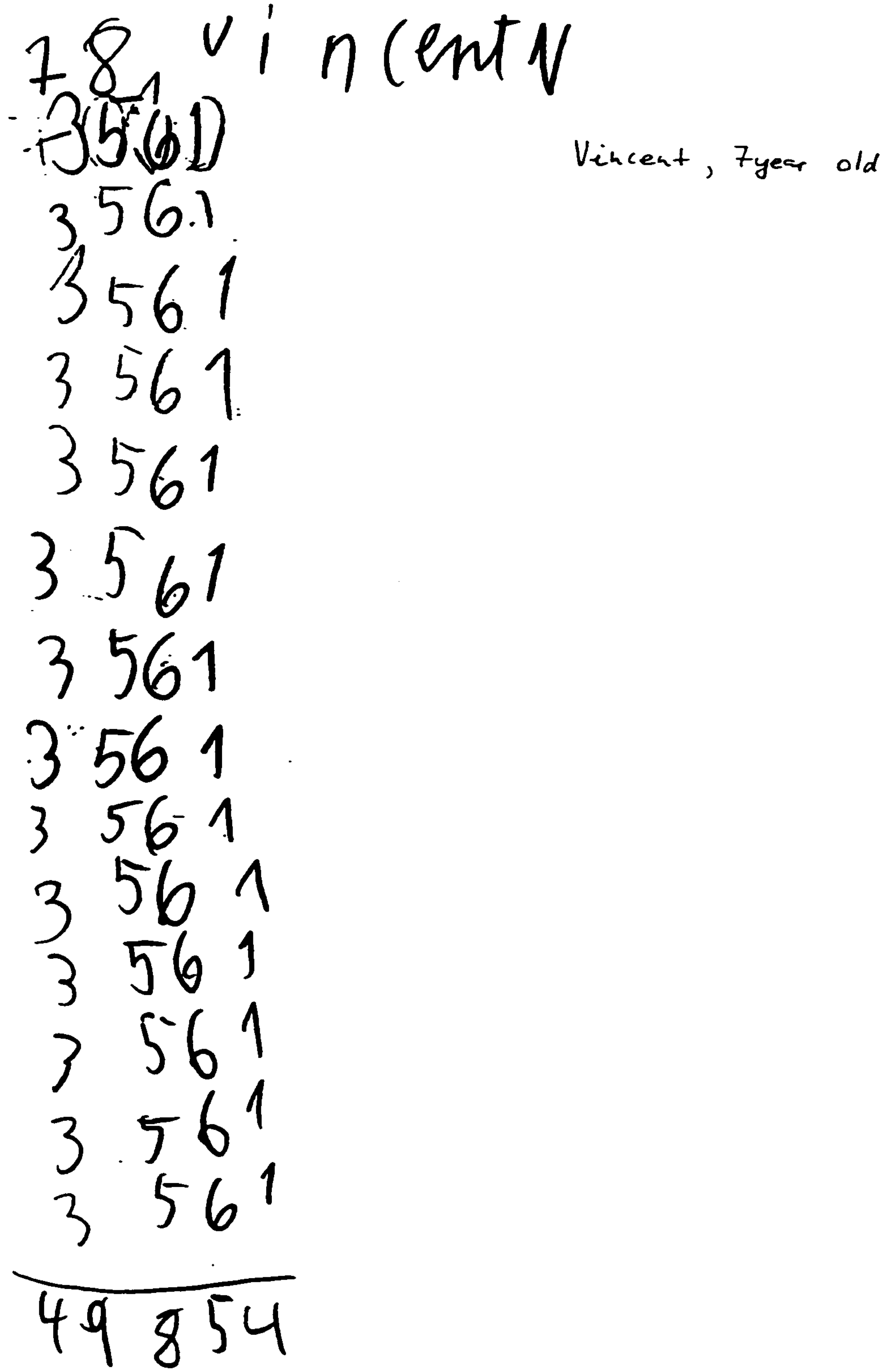


Fig. A7 Vincent's (8) method of repeated addition

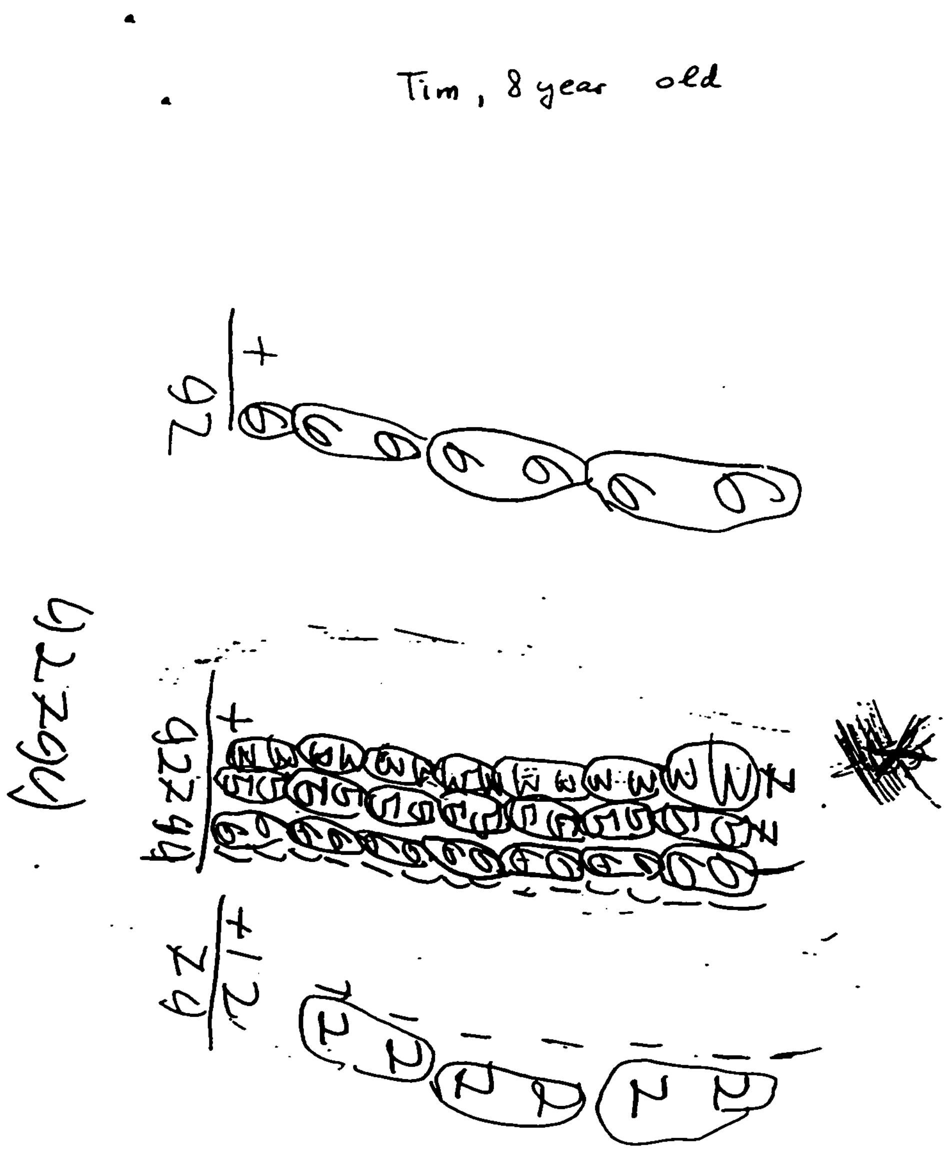


Fig. A8 Tim's (8) way of grouping numbers

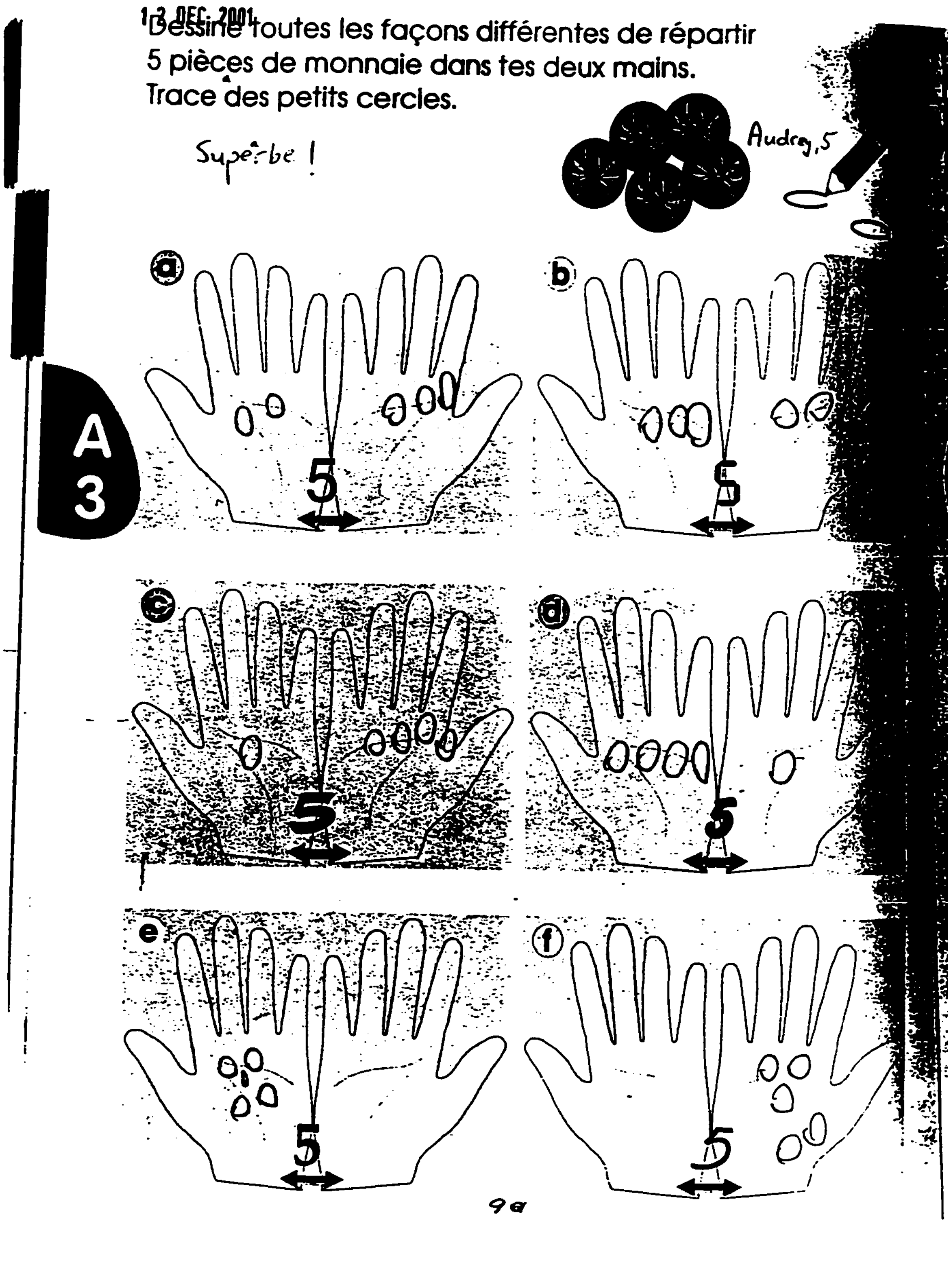


Fig. A9a Audrey's (5) report on the partition problem

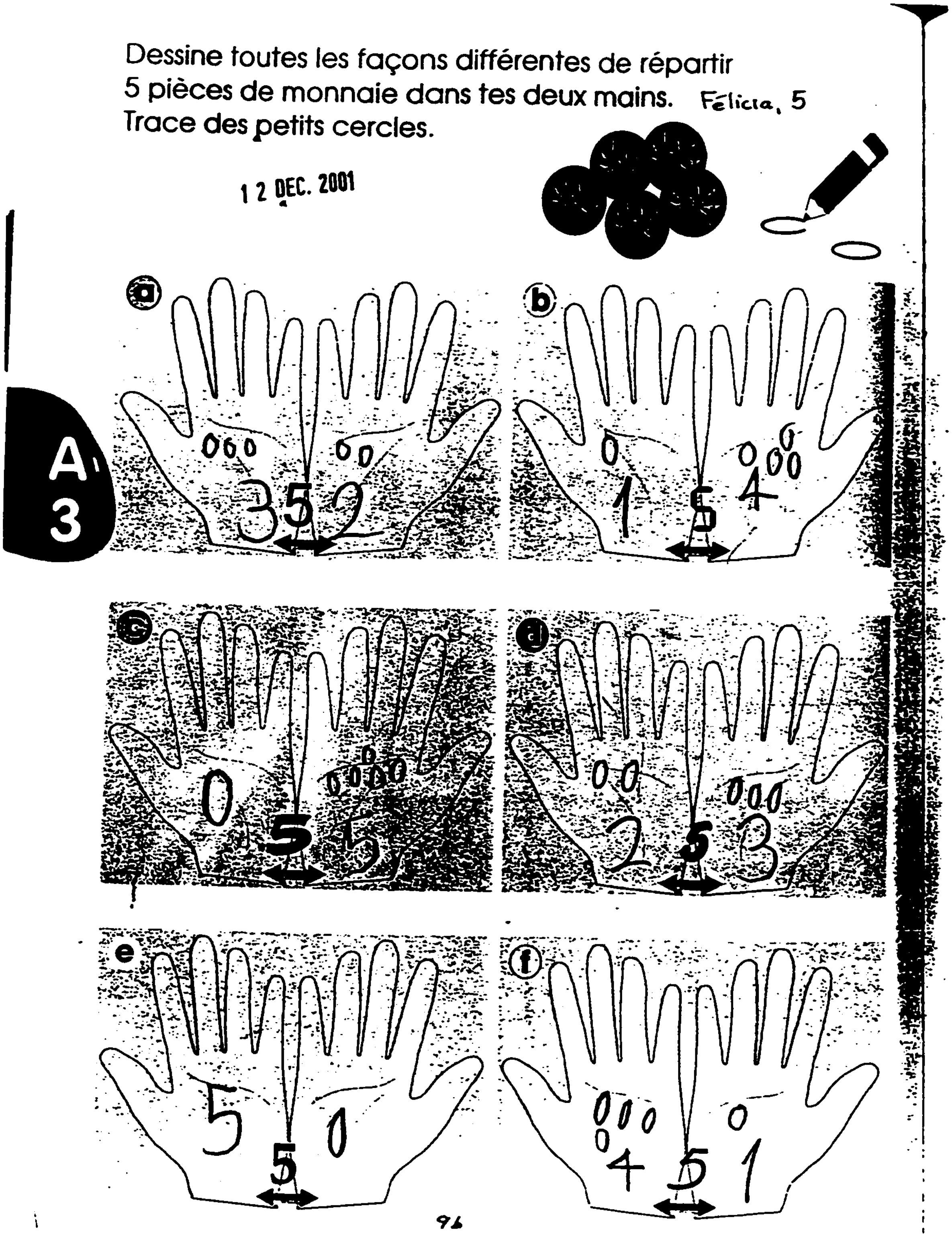


Fig. A9b Félicia's (5) report on the partition problem

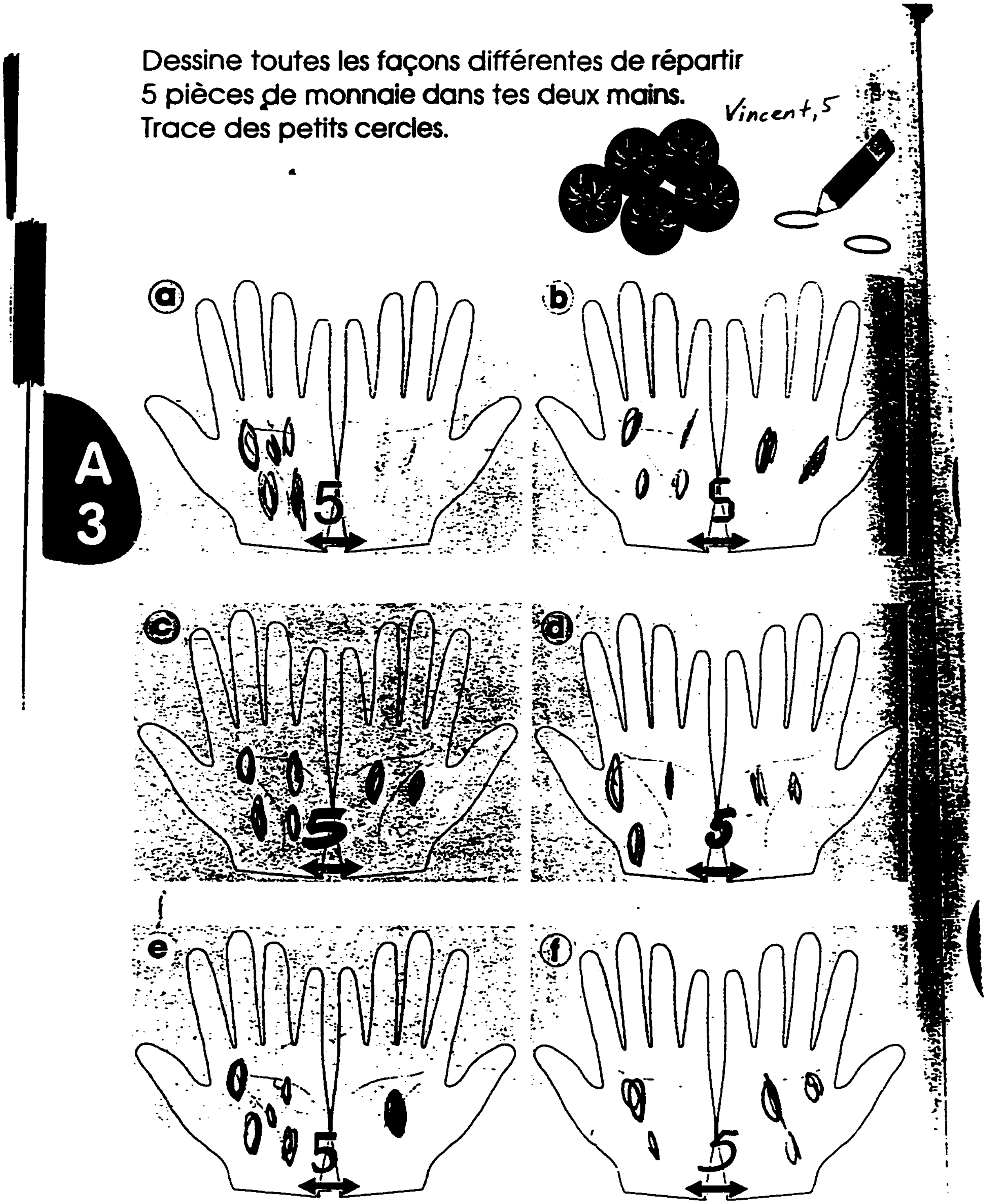
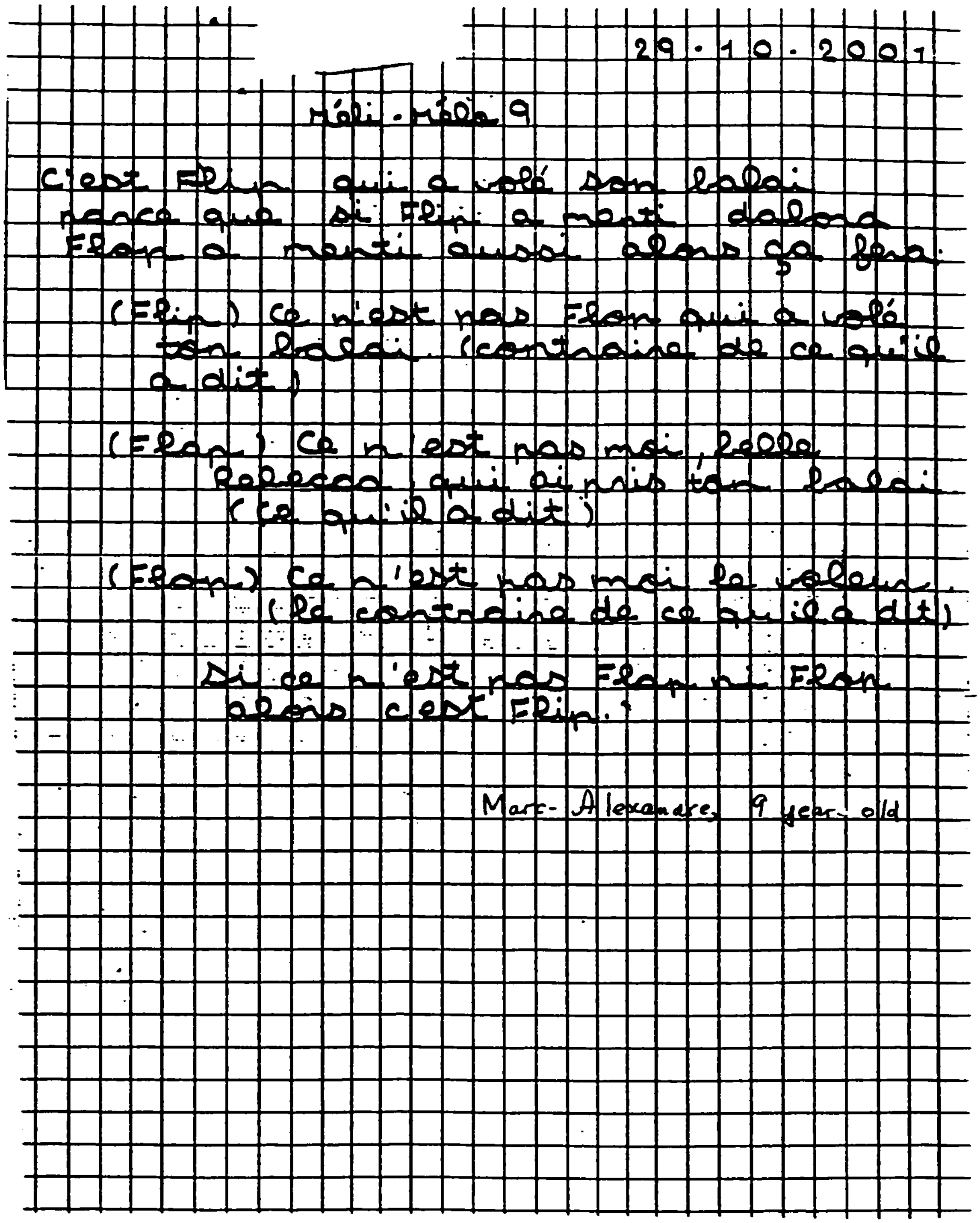


Fig. A9c Vincent's (5) report on the partition problem



29.10.2001

Méli-mélo 9

C'est Flip qui a volé son balai parce que si Flip a menti alors Flop a menti aussi alors ça fera:

(Flip) ce n'est pas Flop qui a volé ton balai (contraire de ce qu'il a dit)

(Flop) Ce n'est pas moi, c'est Rebecca qui a pris ton balai (ce qu'il a dit)

(Flop) Ce n'est pas moi le voleur (le contraire de ce qu'il a dit)

Si ce n'est pas Flop ni Flop alors c'est Flip.

Marc-Alexandre, 9 year old

Fig. A10 Example of Marc-Alexander's (9) logical reasoning

Matthew, 5 year old

Fig. A11 Picture of Matthew (5) "writing" number "1000"

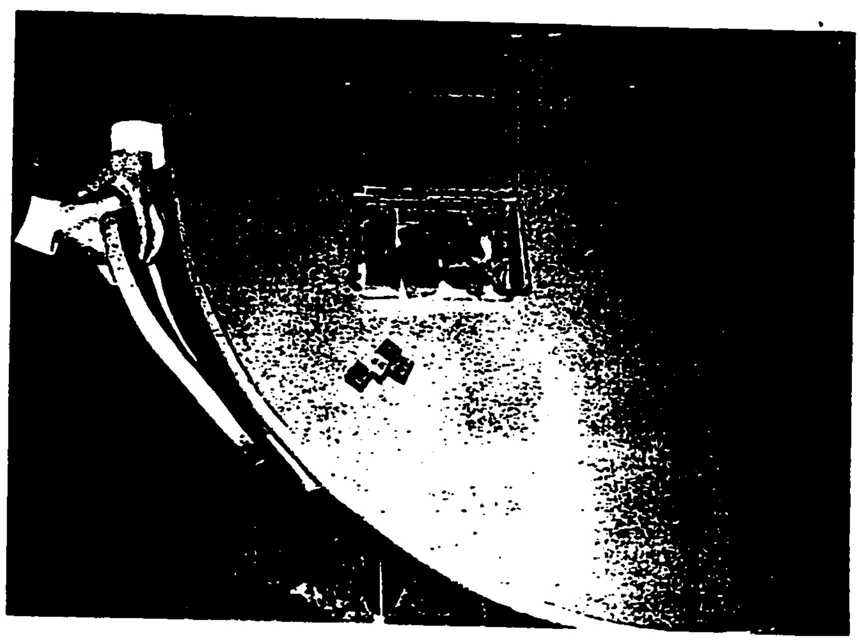


Fig. A12 Picture of David (5) puzzled with a minimization problem