BEHAVIOR OF THE POSTERIOR DISTRIBUTION OF COMMON MEAN IN CASE OF TWO NORMAL POPULATIONS

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Abstract

Behavior of the Posterior Distribution of Common Mean in Case of two Normal Populations

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In this thesis we consider the estimation of common mean of two normal populations and investigate the theoretical behavior of the posterior density, its mean and variance when the scale parameter of the prior is extremely large or small. We further investigate the behaviour of the posterior and its mean and variance when the distance between the mean of the normalized likelihood and the mean of prior becomes large. The choice of prior distribution considered is from normal and t-densities.

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Chapter 1

Introduction

1.1 Introduction

Let $(X_1, \ldots, X_m)$ be a random sample from $N(\theta, \sigma_1^2)$, and $(Y_1, \ldots, Y_n)$ be that from $N(\theta, \sigma_2^2)$. The problem of statistical inference on the parameter θ is known as the common mean problem and arises usually in combining information from two independent sources. This problem was first considered by Fisher (1935) and was later generalized by Cochran (1937) for $k \geq 2$ normal populations and it subsequently attracted enormous attention of researchers. The reader is referred to Ahmad *et al.* (1991) for a short review concerning estimation of θ. The reader may further be referred to the paper by Chaubey and Gabor (1981) for providing another look at Fisher's solution and related references. Box and Tiao (1973) have discussed Bayesian inference concerning θ, using a non-informative prior. In this thesis we concentrate on the Bayesian inference about the common mean θ for two normal populations. The motivation for the approach pursued here comes from the paper of Fan and Berger (1991) which considers the case of single normal population.

1.2 Form of the Likelihood Function

Denoting the sample observations by $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ from the two normal populations respectively, we may easily write the general likelihood function for θ given (σ_1^2 and σ_2^2) as

$$l(\theta|\sigma_1, \sigma_2, \tilde{x}, \tilde{y}) = l_x(\theta)l_y(\theta), \tag{1.2.1}$$

where $l_x(\theta) = (2\pi)^{-\frac{m}{2}}\sigma_1^{-m}exp\{-\frac{\sum_{i=1}^{m}(x_i-\theta)^2}{2\sigma_1^2}\}$ is the likelihood function of θ with respect to sample of X-observation and $l_y(\theta) = (2\pi)^{-\frac{n}{2}}\sigma_2^{-n}exp\{-\frac{\sum_{j=1}^{n}(y_j-\theta)^2}{2\sigma_2^2}\}$ is that for the sample of Y-observation.

However, when σ_1 and σ_2 are unknown, the likelihood function of θ can be defined by integrating the above likelihood function with respect to the noninformative prior $\sigma_1^{-2}\sigma_2^{-2}d\sigma_1^2\sigma_2^2$ for (σ_1^2, σ_2^2) giving:

$$
\begin{aligned}
l(\theta|\tilde{x},\tilde{y}) &= \int_0^\infty (2\pi)^{-\frac{m+n}{2}}\sigma_1^{-m}\sigma_2^{-n} exp\{-\frac{\sum_{i=1}^m (x_i-\theta)^2}{2\sigma_1^2} - \frac{\sum_{j=1}^n (y_j-\theta)^2}{2\sigma_2^2}\}\sigma_1^{-2}\sigma_2^{-2} d\sigma_1^2\sigma_2^2 \\
&= \int_0^\infty (2\pi)^{-\frac{m}{2}}\sigma_1^{-m-2} exp\{-\frac{\sum_{i=1}^m (x_i-\theta)^2}{2\sigma_1^2}\} d\sigma_1^2 \\
&\qquad \times \int_0^\infty (2\pi)^{-\frac{n}{2}}\sigma_1^{-n-2} exp\{-\frac{\sum_{j=1}^n (y_j-\theta)^2}{2\sigma_2^2}\} d\sigma_2^2 \\
&= \frac{\sqrt{m}K_{m-1}}{s_1}(1+\frac{m(\theta-\bar{x})^2}{(m-1)s_1^2})^{-m/2} \times \frac{\sqrt{n}K_{n-1}}{s_2}(1+\frac{n(\theta-\bar{y})^2}{(n-1)s_2^2})^{-n/2} \\
&= f_{m-1}(\frac{\bar{x}-\theta}{s_1}\sqrt{m}) f_{n-1}(\frac{\bar{y}-\theta}{s_2}\sqrt{n}),
\end{aligned}
\tag{1.2.2}
$$

where $\bar{x} = \frac{1}{m}\sum_{i=1}^m x_i$, and $\bar{y} = \frac{1}{n}\sum_{j=1}^n y_j$, and $s_1^2 = (m-1)^{-1}\sum_{i=1}^m (x_i-\bar{x})^2$ and $s_2^2 = (n-1)^{-1}\sum_{i=1}^n (y_i-\bar{y})^2$ and $f_\nu(t) = K_{\nu-1}(1+\frac{t^2}{\nu})^{-\frac{\nu}{2}}$ represents t−density function with degree of freedom ν, the constant $K_{\nu-1}$ being given by

$$
K_{\nu-1} = \frac{\Gamma(\frac{l}{2})}{\Gamma(\frac{l-1}{2})\sqrt{(\nu-1)\pi}}. \tag{1.2.3}
$$

1.3 Form of the Posterior Distribution

To be able to perform Bayesian inference on θ, we need to specify a prior distribution for it. In what follows, we consider the choice between two priors, namely, the normal prior and the t-prior and analyze the resulting posterior.

Let $p(\theta)$ denote the prior density for θ. Then the general form of posterior distribution of θ is given by the following probability density function:

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{l_x(\theta)l_y(\theta)p(\theta)}{\int_\Theta l_x(\theta)l_y(\theta)p(\theta)d\theta}, \quad \text{for } \sigma_1 \text{ and } \sigma_2 \text{ known,} \tag{1.3.1}$$

and

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{l(\theta|\tilde{x},\tilde{y})p(\theta)}{\int_\Theta l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta}, \quad \text{for } \sigma_1 \text{ and } \sigma_2 \text{ unknown,} \tag{1.3.2}$$

We will assume $p(\theta)$ to have one of the following two forms:

$< 1 >$ Normal prior

$$p(\theta) = (2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}. \tag{1.3.3}$$

$< 2 >$ t-prior

$$p(\theta) = \frac{K_{l-1}}{\tau}(1+\frac{(\theta-\mu)^2}{(l-1)\tau^2})^{-l/2}. \tag{1.3.4}$$

where the parameter μ represents the best prior guess for θ and the scale parameter τ measures the accuracy of μ.

1.3.1 Normal Prior

a) if σ_1, σ_2 known:

When the prior is normal density as given (1.3.3), we can easily get the posterior distribution function corresponding the nature of normal distribution:

$$\pi(\theta|\tilde{x},\tilde{y}) \propto N(\theta_{(\mu,\bar{x},\bar{y})}, \sigma^2_{(\mu,\bar{x},\bar{y})}) \tag{1.3.5}$$

where

$$\theta_{(\mu,\bar{x},\bar{y})} = \frac{\frac{1}{\tau^2}\mu + \frac{m}{\sigma_1^2}\bar{x} + \frac{n}{\sigma_2^2}\bar{y}}{\frac{1}{\tau^2} + \frac{m}{\sigma_1^2} + \frac{n}{\sigma_2^2}}, \quad \sigma^2_{(\mu,\bar{x},\bar{y})} = [\frac{1}{\tau^2} + \frac{m}{\sigma_1^2} + \frac{n}{\sigma_2^2}]^{-1}.$$

b) if σ_1, σ_2 unknown:

From (1.3.2), we get posterior density:

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}(1+\frac{m(\theta-\bar{x})^2}{(m-1)s_1^2})^{-m/2}(1+\frac{n(\theta-\bar{y})^2}{(n-1)s_2^2})^{-n/2}}{\int_\Theta exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}(1+\frac{m(\theta-\bar{x})^2}{(m-1)s_1^2})^{-m/2}(1+\frac{n(\theta-\bar{y})^2}{(n-1)s_2^2})^{-n/2}d\theta}. \tag{1.3.6}$$

1.3.2 *t*-Prior

a) if σ_1, σ_2 known:

when the prior is given by a t-distribution as in (1.3.4), the form of posterior is given as:

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{[1+\frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2}exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}}{\int_\Theta [1+\frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2}exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}d\theta}, \tag{1.3.7}$$

where

$$\theta_{(\bar{x},\bar{y})} = \frac{m\sigma_2^2\bar{x} + n\sigma_1^2\bar{y}}{m\sigma_2^2 + n\sigma_1^2}, \quad \sigma^2_{(\bar{x},\bar{y})} = \frac{\sigma_1^2\sigma_2^2}{n\sigma_1^2 + m\sigma_2^2}.$$

Obviously, $\theta_{(\bar{x},\bar{y})}$ is between $\bar{x}$ and $\bar{y}$.

b) if σ_1, σ_2 unknown:

In this case, the posterior does not have simple form. The nature and behavior of these complex forms are discussed in the chapter IV of the thesis.

The posterior has following form:

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{(1+\frac{m(\theta-\bar{x})^2}{(m-1)s_1^2})^{-m/2}(1+\frac{n(\theta-\bar{y})^2}{(n-1)s_2^2})^{-n/2}(1+\frac{(\theta-\mu)^2}{(l-1)\tau^2})^{-l/2}}{\int_\Theta(1+\frac{m(\theta-\bar{x})^2}{(m-1)s_1^2})^{-m/2}(1+\frac{n(\theta-\bar{y})^2}{(n-1)s_2^2})^{-n/2}(1+\frac{(\theta-\mu)^2}{(l-1)\tau^2})^{-l/2}d\theta}. \tag{1.3.8}$$

Comparatively, this formula is much more complex than the other formulas above. We will do more analyzing on it.

1.4 Some Notations

We will discuss behavior of the posterior distribution along with its mean and variance in following chapters. We introduce some notations in the following section which will be used throughout the thesis.

Define $E^l(\theta^i)$ for $i \geq 0$ as follows. When σ_1 and σ_2 are known,

$$E^l(\theta^i) = \frac{\int_\Theta \theta^i l(\theta|\sigma_1,\sigma_2,\tilde{x},\tilde{y})d\theta}{\int_\Theta l(\theta|\sigma_1,\sigma_2,\tilde{x},\tilde{y})d\theta}, \tag{1.4.1}$$

and when σ_1 and σ_2 are unknown,

$$E^l(\theta^i) = \frac{\int_\Theta \theta^i l(\theta|\tilde{x},\tilde{y})d\theta}{\int_\Theta l(\theta|\tilde{x},\tilde{y})d\theta}, \tag{1.4.2}$$

Thus the mean and the variance of the normalized likelihood, denoted by $\delta_{(x,y)}$ and $V_{(x,y)}$ respectively are given by;

$$\delta_{(x,y)} = E^l(\theta), \tag{1.4.3}$$

and

$$V_{(x,y)} = E^l(\theta^2) - [E^l(\theta)]^2. \tag{1.4.4}$$

Further define the mean $\delta^\pi_{(x,y)}$ and the variance $V^\pi_{(x,y)}$ of the posterior distribution as follows;

$$E^\pi(\theta^i) = \int_\Theta \theta^i \pi(\theta|\tilde{x},\tilde{y})d\theta; \tag{1.4.5}$$

and

$$\delta^\pi_{(x,y)} = E^\pi(\theta), \tag{1.4.6}$$

$$V^\pi_{(x,y)} = E^\pi(\theta^2) - [E^\pi(\theta)]^2. \tag{1.4.7}$$

1.5 Summary of the Thesis

For the case of known variances of the two normal populations, the posterior distribution for a given normal prior is still normal (see equation 1.3.5). Hence, it is easy to study the properties and characteristics of the the posterior distribution in this case. For example, it is unimodal; and when τ converges to infinity, $\theta_{(\mu,\bar{x},\bar{y})}$ converges to $\theta_{(\bar{x},\bar{y})}$, etc.. We may be interested to investigate how far such properties hold in other cases when the posterior is not as simple as in the above case. This needs a careful consideration and analysis.

In Chapter 2, we consider the behavior of posterior with normal prior when σ_1 and σ_2 are unknown. The shape of posterior is not always unimodal. Some numerical examples are provided to demonstrate this property. The behavior of posterior mean and variance is investigated when the parameter τ of normal prior tends to zero or infinity. Similar analysis is carried out in Chapters 3 and 4 for the case of t-prior with known and unknown variances respectively. Theoretical conditions of unimodality of the posterior are also obtained. Fan and Berger (1991) have given similar conclusions for single sample.

Chapter 2

Analysis of Posterior with Unknown Variances under Normal Prior

In this chapter, we consider the case of a normal prior when variances σ_1 and σ_2, are unknown. The behavior of the posterior is discussed when the variance of prior distribution is either extremely large or small respectively.

Intuitively, the larger the variance of normal prior is, the smaller the influence to posterior is. From the Theorem 2.1, we can see this characteristic; *i.e.*, the posterior density tends to the normalized likelihood function and as a consequence of this result, the mean and variance of the posterior tend respectively to $\delta_{(x,y)}$ and $V_{(x,y)}$.

We introduce the normalized likelihood $m_{(x,y)}$ given by

$$m_{(x,y)} = C_{(x,y)}^{-1} \int_{\Theta} l(\theta|\tilde{x}, \tilde{y}) p(\theta) d\theta.$$

where

$$C_{(x,y)} = \int_{\Theta} l(\theta|\tilde{x}, \tilde{y}) d\theta.$$

Theorem 2.1

(i)

$$\lim_{\tau \to \infty} \frac{m_{(x,y)}}{p(\delta_{(x,y)})} = 1.$$

(ii)

$$\lim_{\tau \to \infty} \pi(\theta|\tilde{x}, \tilde{y}) = C_{(x,y)}^{-1} l(\theta|\tilde{x}, \tilde{y}).$$

(iii)

$$\lim_{\tau \to \infty} \delta_{(x,y)}^{\pi} = \delta_{(x,y)}.$$

(iv)

$$\lim_{\tau \to \infty} V_{(x,y)}^{\pi} = V_{(x,y)}.$$

First we prove the following lemmas which are used in the proof of the above as well as the theorem 2.2.

lemma 2.1

$$\int_{\Theta} |\theta|^{i} l(\theta|\tilde{x}, \tilde{y}) d\theta < \infty. \quad \text{for } i = 0, 1, 2.$$

Proof:

set

$$c_m = \frac{m}{(m-1)s_1^2}$$

$$c_n = \frac{n}{(n-1)s_2^2}$$

$$c_{mn} = \frac{\sqrt{m}K_{m-1}\sqrt{n}K_{n-1}}{s_1 s_2}$$

Without loss of generality, let (a,b) $\subseteq$ R such that

$$(1+\frac{m\eta^2}{(m-1)s_1^2})^{-m/2} \leq (1+\frac{n(|\eta|+\bar{x}-\bar{y})^2}{(n-1)s_2^2})^{-n/2} \quad \text{for } \eta \in (a,b);$$

$$(1+\frac{m\eta^2}{(m-1)s_1^2})^{-m/2} > (1+\frac{n(|\eta|+\bar{x}-\bar{y})^2}{(n-1)s_2^2})^{-n/2} \quad \text{otherwise.}$$

where constant a and b can be infinite.

We have known the fact that when $\nu > k$,

$$E_{t_\nu}(|\eta|^k) = \int |\eta|^k (1+\frac{\eta^2}{\nu})^{-(\nu+1)/2} d\eta < \infty \tag{2.1}$$

Set $\quad \eta = \theta - \bar{x}$

For $\quad$ i=0, 1, 2,

$$\begin{aligned}
\int_\Theta |\theta|^i l(\theta|\bar{x},\bar{y})d\theta &= \int_\eta |\eta+\bar{x}|^i f_{m-1}(\tfrac{\eta}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta-d}{s_2}\sqrt{n})d\eta \\
&\leq \int_\eta (|\eta|+|\bar{x}|)^i f_{m-1}(\tfrac{\eta}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta-d}{s_2}\sqrt{n})d\eta \\
&= \textstyle\sum_{j=0}^{i} \binom{i}{j} |\bar{x}|^{i-j} [\int_{(a,b)} + \int_{(a,b)^c}] |\eta|^j f_{m-1}(\tfrac{\eta}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta-d}{s_2}\sqrt{n})d\eta
\end{aligned} \tag{2.2}$$

The first part of (2.2) is finite, that is,

$\int_{(a,b)} |\eta|^j f_{m-1}(\frac{\eta}{s_1}\sqrt{m}) f_{n-1}(\frac{\eta-d}{s_2}\sqrt{n}) d\eta$

$$= c_{mn} \int_{(a,b)^c} |\eta|^j (1 + c_m\eta^2)^{-m/2} (1 + c_n(\eta - d)^2)^{-n/2} d\eta$$

$$\leq c_{mn} \int_a^b [|\eta - d| + d]^j (1 + c_n(\eta - d)^2)^{-n} d\eta$$

$$\overset{\zeta=\eta-d}{=\!=\!=\!=\!=} c_{mn} \sum_{k=0}^{j} \binom{j}{k} d^{j-k} \int_{a_1}^{b_1} |\zeta|^k (1 + c_n\zeta^2)^{-n} d\zeta$$

$$\overset{\xi=\sqrt{(2n-1)c_n}\zeta}{=\!=\!=\!=\!=\!=\!=} c_{mn} \sum_{k=0}^{j} \binom{j}{k} d^{j-k} [(2n-1)c_n]^{-\frac{k+1}{2}} \int_{a_2}^{b_2} |\xi|^k (1 + \frac{\xi^2}{2n-1})^{-\frac{(2n-1)+1}{2}} d\xi$$

$$\leq c_{mn} \sum_{k=0}^{j} \binom{j}{k} d^{j-k} [(2n-1)c_n]^{-\frac{k+1}{2}} \int_{\Theta} |\xi|^k (1 + \frac{\xi^2}{2n-1})^{-\frac{(2n-1)+1}{2}} d\xi \tag{2.3}$$

by (2.1), since $k \leq j \leq i \leq 2 < 2n - 1$, (2.3) is finite.

Similarly,

$$\int_{(a,b)^c} |\eta|^j f_{m-1}(\frac{\eta}{s_1}\sqrt{m}) f_{n-1}(\frac{\eta - d}{s_2}\sqrt{n}) d\eta < \infty \tag{2.4}$$

Combining (2.3) and (2.4), we conclude that the expression in (2.2) is finite.

lemma 2.2. Recall (1.3.2), at point $\theta = \delta_{(x,y)}$,

$$p(\delta_{(x,y)}) = (2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\},$$

then

$$\lim_{\tau\to\infty}\frac{p(\theta)}{p(\delta_{(x,y)})} = 1.$$

Proof:

$$\begin{aligned}
\lim_{\tau\to\infty}\frac{p(\theta)}{p(\delta_{(x,y)})} &= \lim_{\tau\to\infty}\frac{(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}}{(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\}} \\
&= \lim_{\tau\to\infty} exp\{-\frac{(\theta-\mu)^2-(\delta_{(x,y)}-\mu)^2}{2\tau^2}\} \\
&= 1.
\end{aligned}$$

$$\begin{aligned}
\lim_{\tau\to\infty}\frac{m_{(x,y)}}{p(\delta_{(x,y)})} &= \lim_{\tau\to\infty}\frac{\int_\Theta l(\theta|\tilde{x},\tilde{y})(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}d\theta}{C_{(x,y)}(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\}} \\
&= C^{-1}_{(x,y)}\int_\Theta l(\theta|\tilde{x},\tilde{y})\lim_{\tau\to\infty}\frac{exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}}{exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\}}d\theta \\
&= C^{-1}_{(x,y)}\int_\Theta l(\theta|\tilde{x},\tilde{y})d\theta \\
&= 1.
\end{aligned}$$

Proof of theorem 2.1:

(i)

$$\lim_{\tau\to\infty}\frac{m_{(x,y)}}{p(\delta_{(x,y)})} = \lim_{\tau\to\infty}\frac{\int_\Theta l(\theta|\tilde{x},\tilde{y})(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}d\theta}{C_{(x,y)}(2\pi)^{-1/2}\tau^{-1}exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\}}$$

$$= C^{-1}_{(x,y)}\int_\Theta l(\theta|\tilde{x},\tilde{y})\lim_{\tau\to\infty}\frac{exp\{-\frac{(\theta-\mu)^2}{2\tau^2}\}}{exp\{-\frac{(\delta_{(x,y)}-\mu)^2}{2\tau^2}\}}d\theta$$

$$= C^{-1}_{(x,y)}\int_\Theta l(\theta|\tilde{x},\tilde{y})d\theta$$

$$= 1.$$

(ii):

$$\begin{aligned}\lim_{\tau\to\infty}\pi(\theta|\tilde{x},\tilde{y}) &= C^{-1}_{(x,y)}l(\theta|\tilde{x},\tilde{y})\lim_{\tau\to\infty}\frac{p(\delta_{(x,y)})}{m_{(x,y)}}\lim_{\tau\to\infty}\frac{p(\theta)}{p(\delta_{(x,y)})}\\ &= C^{-1}_{(x,y)}l(\theta|\tilde{x},\tilde{y}).\end{aligned}$$

(iii):

$$\begin{aligned}\lim_{\tau\to\infty}\delta^{\pi}_{(x,y)} &= \int_\Theta \lim_{\tau\to\infty}\theta\pi(\theta|\tilde{x},\tilde{y})d\theta\\ &= C^{-1}_{(x,y)}\int_\Theta \theta l(\theta|\tilde{x},\tilde{y})d\theta\\ &= \delta_{(x,y)}.\end{aligned}$$

(iv):

$$\begin{aligned}\lim_{\tau\to\infty}V^{\pi}_{(x,y)} &= \lim_{\tau\to\infty}E^{\pi}(\theta^2)-\lim_{\tau\to\infty}[E^{\pi}(\theta)]^2\\ &= [V_{(x,y)}+\delta^2_{(x,y)}]-\delta^2_{(x,y)}\\ &= V_{(x,y)},\end{aligned}$$

where

$$
\begin{aligned}
\lim_{\tau\to\infty} E^{\pi}(\theta^2) &= \lim_{\tau\to\infty} \int \theta^2 \pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= C^{-1}_{(x,y)} \int \theta^2 l(\theta|\tilde{x},\tilde{y})d\theta \\
&= V_{(x,y)} + \delta^2_{(x,y)}.
\end{aligned}
$$

In above theorem gives the behavior of the posterior when the variance of the prior is extremely large, however, if the variance of prior is very small, the posterior will concentrate on point μ. So the mean of posterior is close to the mean of prior and the variance of posterior is close to zero. These are explained in the following theorem.

Theorem 2.2

(i)

$$\lim_{\tau\to 0} m_{(x,y)} = C^{-1}_{(x,y)} f_{m-1}\left(\frac{\bar{x}-\mu}{s_1}\sqrt{m}\right) f_{n-1}\left(\frac{\bar{y}-\mu}{s_2}\sqrt{n}\right).$$

(ii)

$$\lim_{\tau\to 0} \pi(\theta|\tilde{x},\tilde{y}) = \delta(\mu), \text{where } \delta(\mu) \text{ denotes a point mass at } \mu.$$

(iii)

$$\lim_{\tau\to 0} \delta^{\pi}_{(x,y)} = \mu.$$

(iv)

$$\lim_{\tau\to 0} V^{\pi}_{(x,y)} = 0.$$

Proof:

Set $\eta = \frac{\theta-\mu}{\tau}$

(i):

$$\begin{aligned}
\lim_{\tau\to 0} m_{(x,y)} &= \lim_{\tau\to 0} \int_\eta C^{-1}_{(x,y)} l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta \\
&= C^{-1}_{(x,y)} \int_\eta \lim_{\tau\to 0} l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta \\
&= C^{-1}_{(x,y)} f_{m-1}(\tfrac{\bar{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\bar{y}-\mu}{s_2}\sqrt{n}) \int_\eta (2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta \\
&= C^{-1}_{(x,y)} f_{m-1}(\tfrac{\bar{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\bar{y}-\mu}{s_2}\sqrt{n})
\end{aligned}$$

(ii):

$$\begin{aligned}
P(|\theta_\tau-\mu|>\varepsilon) &= \int_{|\theta-\mu|>\varepsilon} C^{-1}_{(x,y)} l(\theta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{(\theta-\mu)^2}{2}\}d\theta \\
&= C^{-1}_{(x,y)} \int_{(|\eta|>\varepsilon/\tau)} l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta \\
&= C^{-1}_{(x,y)} \int_\eta I_{(|\eta|>\varepsilon/\tau)} l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta
\end{aligned}$$

so, $\lim_{\tau\to 0} P(|\theta_\tau-\mu|>\varepsilon)=0$,

where $|I_{(|\eta|>\varepsilon/\tau)} l(\mu+\tau\eta|\tilde{x},\tilde{y}) exp\{-\frac{\eta^2}{2}\}| \le exp\{-\frac{\eta^2}{2}\} I_{(|\eta|>\varepsilon/\tau)}$.

so $\lim_{\tau\to 0} P(|\Theta_\tau-\mu|=0)=1$,

which equivalent to $\lim_{\tau\to 0} \pi(\theta|\tilde{x},\tilde{y})=\delta(\mu)$.

(iii):

$\lim_{\tau\to 0} \delta^\pi_{(x,y)} = lim_{\tau\to 0} m^{-1}_{(x,y)} \int_\Theta \theta C^{-1}_{(x,y)} l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$

$$= lim_{\tau\to 0} C^{-1}_{(x,y)} m^{-1}_{(x,y)} \int_\eta (\mu+\tau\eta) l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\tfrac{\eta^2}{2}\}d\eta$$

$$= \frac{\int_\eta lim_{\tau\to 0}(\mu+\tau\eta) l(\mu+\tau\eta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\frac{\eta^2}{2}\}d\eta}{f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})}$$

$$= \frac{\int_\eta \mu f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})(2\pi)^{-1/2} exp\{-\frac{\eta^2}{2}\}d\eta}{f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})}$$

$$= \mu$$

(iv):

By Lemma 2.1

$$\begin{aligned}
\lim_{\tau\to 0} E^\pi(\theta^2) &= \lim_{\tau\to 0} m^{-1}_{(x,y)} \int_\Theta \theta^2 C^{-1}_{(x,y)} l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta \\
&= \lim_{\tau\to 0} C^{-1}_{(x,y)} m^{-1}_{(x,y)} \int_\eta (\mu+\tau\eta)^2 l(\mu+\tau\delta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\frac{\eta^2}{2}\}d\eta \\
&= \frac{\int_\eta lim_{\tau\to 0}(\mu+\tau\eta)^2 l(\mu+\tau\delta|\tilde{x},\tilde{y})(2\pi)^{-1/2} exp\{-\frac{\eta^2}{2}\}d\eta}{f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})} \\
&= \frac{\int_\eta \mu^2 f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})(2\pi)^{-1/2} exp\{-\frac{\eta^2}{2}\}d\eta}{f_{m-1}(\frac{\tilde{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\tilde{y}-\mu}{s_2}\sqrt{n})} \\
&= \mu^2
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{\tau\to 0} V^\pi_{(x,y)} &= \lim_{\tau\to 0}\{E^\pi(\theta)^2 - [E^\pi(\theta)]^2\} \\
&= 0
\end{aligned}$$

At the end of chapter2, we give figure 1 and figure 2 to show that the shape of posterior can have one or two peaks.

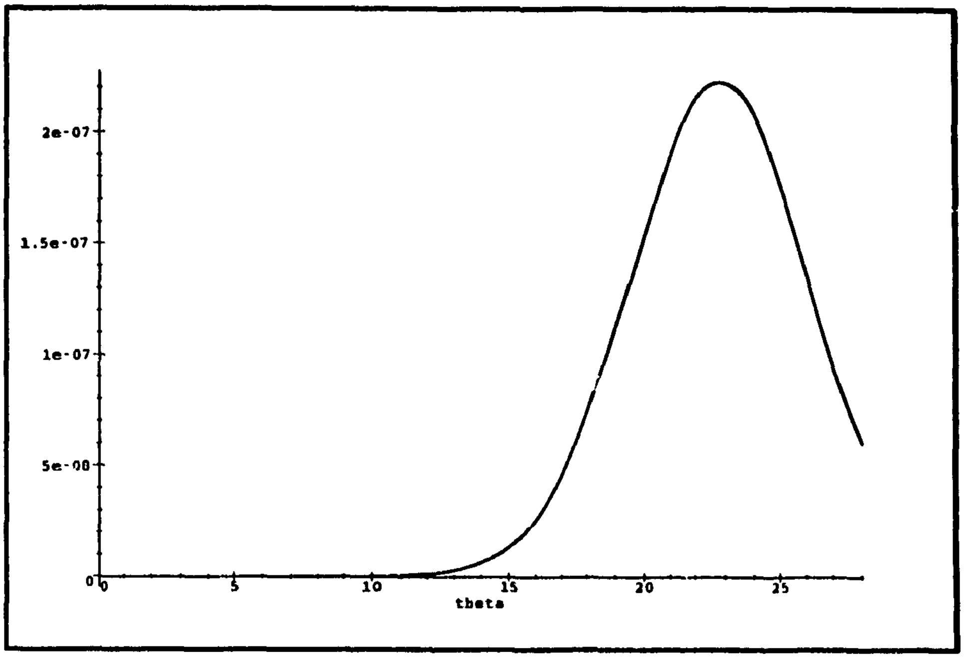

Figure 1: $\bar{x} = 0, m = 2, s_1 = 3.2, \bar{y} = 1, n = 6, s_2 = 4.5, \tau = 3, \mu = 26$

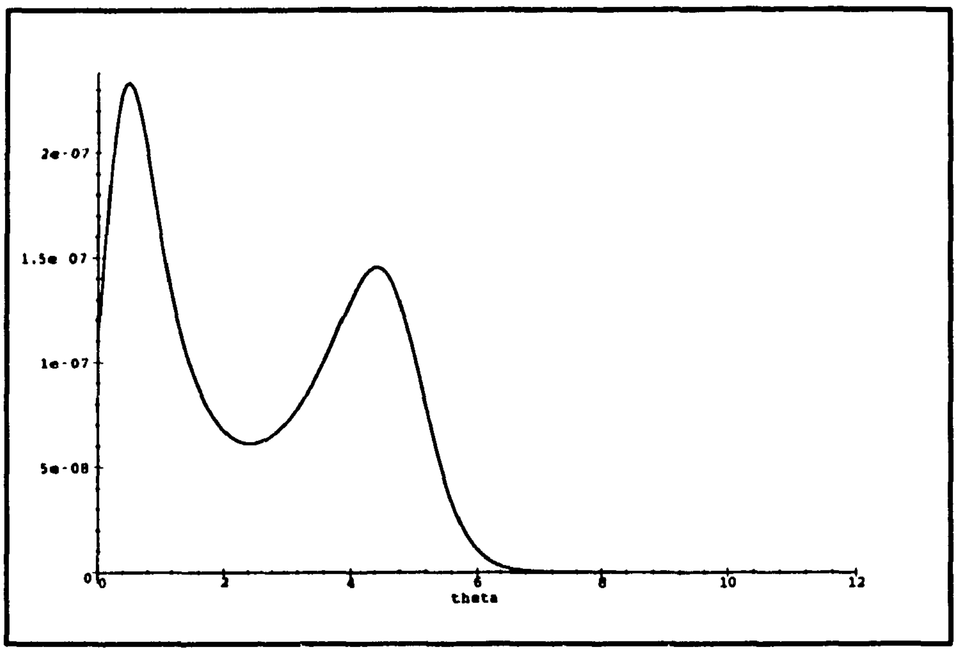

Figure 2: $\bar{x} = 0, m = 10, s_1 = 1.2, \bar{y} = 5, n = 8, s_2 = 1.9, \tau = 2.1, \mu = 8$

Chapter 3

Analysis of Posterior with t-Prior and σ_1^2, σ_2^2 Known

In this chapter, first, we discuss about the shape of the posterior and the condition of unimodality. Then we discuss about the natures of mean and variance of the posterior when τ tends to infinity and zero.

Recall from (1.3.7)

$$\pi(\theta|\bar{x},\bar{y}) \propto [1+\frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2} exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\} \stackrel{\text{def}}{=} L(\theta|x,y)$$

Without loss of generality, assume $\bar{x} \leq \bar{y} \leq \mu$, which implies that

$$\bar{x} \leq \theta_{(\bar{x},\bar{y})} \leq \bar{y} \leq \mu$$

3.1 The Unimodality Condition of Posterior Distribution

Intuitively, the shape of posterior could be unimodal and bimodal. Figure 3 and figure 4 show these natures respectively. Comparing these two figures, we can see changing μ from 10 to 20 changes bimodal to unimodal. Now we analyze the property theoretically.

Without loss of generality, set $\theta_{(\tilde{x},\tilde{y})} = 0$. Do first derivative on log of $L(\theta|\tilde{x},\tilde{y})$ with respect to θ:

$$\begin{aligned} \frac{\partial}{\partial\theta} logL(\theta|\tilde{x},\tilde{y}) &= \frac{\partial}{\partial\theta}[-\frac{l}{2}log(1+\frac{(\theta-\mu)^2}{(l-1)\tau^2}) - \frac{\theta^2}{2\sigma^2_{(\tilde{x},\tilde{y})}}] \\ &= -[\frac{l(\theta-\mu)}{(l-1)\tau^2+(\theta-\mu)^2} + \frac{\theta}{\sigma^2_{(\tilde{x},\tilde{y})}}] \end{aligned} \tag{3.1.1}$$

It is clear that $\pi(\theta|\tilde{x},\tilde{y})$ is monotone increasing when $\theta \le \theta_{(\tilde{x},\tilde{y})}$, and it is monotone decreasing when $\theta \ge \mu$. This mean the peaks of the shape of posterior can only appear between μ and $\theta_{(\tilde{x},\tilde{y})}$. Second, for any fixed $\theta_0 > 0$, $\frac{\partial}{\partial\theta} logL(\theta|\tilde{x},\tilde{y})$ tends to negative when μ tends to infinity, which implies the posterior is monotone decreasing at the point θ_0. Now let $\frac{\partial}{\partial\theta} logL(\theta|\tilde{x},\tilde{y}) = 0$, we obtain the following equation,

$$\theta^3 - 2\mu\theta^2 + [l\sigma^2 + (l-1)\tau^2 + \mu^2]\theta - l\sigma^2\mu = 0.$$

Set $r = -2\mu$, $s = l\sigma^2 + (l-1)\tau^2 + \mu^2$, and $t = -l\sigma^2\mu$, let $p = \frac{3s-r^2}{3}$ and $q = \frac{2r^3}{27} - \frac{rs}{3} + t$, $D = (\frac{p}{3})^3 + (\frac{q}{2})^2$, then we obtain

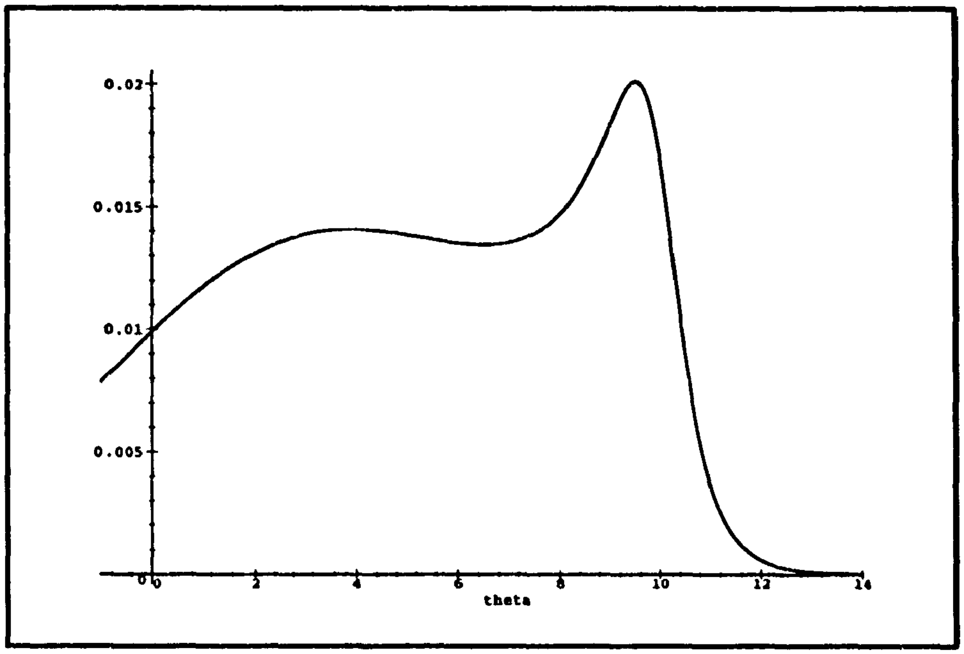

Figure 3: $\sigma_{(\bar{x},\bar{y})} = 3.5, \tau = 1, \mu = 10, \theta_{(\bar{x},\bar{y})} = 0, l = 2$

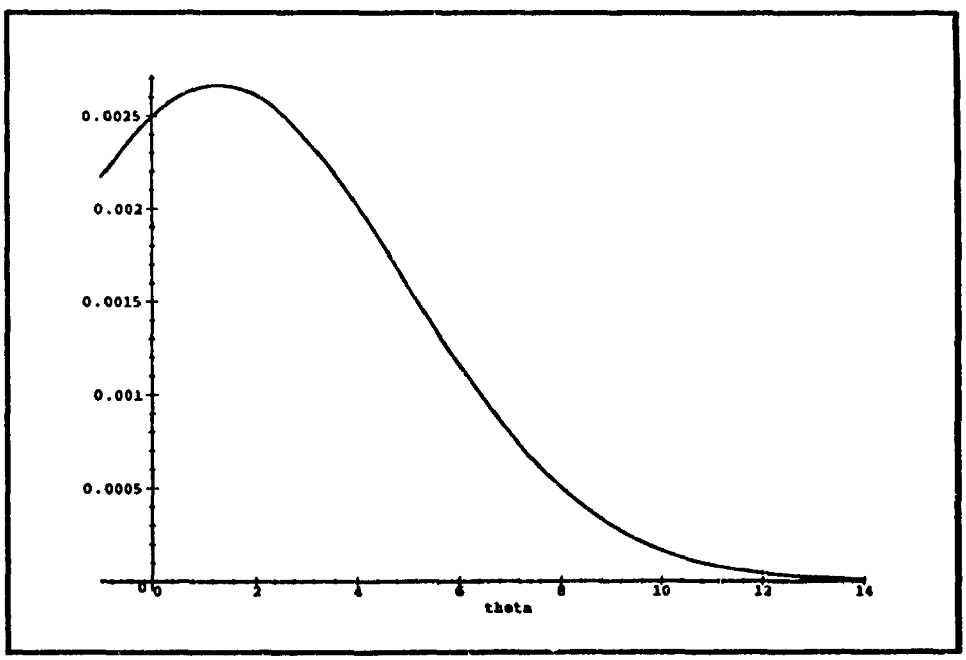

Figure 4: $\sigma_{(\bar{x},\bar{y})} = 3.5, \tau = 1, \mu = 20, \theta_{(\bar{x},\bar{y})} = 0, l = 2$

$$D = \frac{1}{3^6}[\Delta_1\mu^4 + \Delta_2\mu^2 + \Delta_3]$$

where

$$\Delta_1 = 27(l-1)\tau^2 > 0,$$

$$\Delta_2 = 54(l-1)^2\tau^4 - 135(l-1)l\tau^2\sigma^2 - \frac{27}{4}l^2\sigma^4$$

and

$$\Delta_3 = 27l^3\sigma^6 + 81(l-1)l^2\tau^2\sigma^4 + 81(l-1)^2l\tau^4\sigma^2 + 27(l-1)^3\tau^6$$

The necessary and sufficient condition of unimodality for posterior is that D is great than zero. From the results above, D will be always positive for any value of θ when μ tends to infinity, because D is dominated by the coefficient of μ^4. So the posterior is unimodal and the peak is close to $\theta_{(\bar{x},\bar{y})}$ when the μ is far from the $\theta_{(\bar{x},\bar{y})}$. Also, the degree of freedom influence the shape of posterior. Figure 3 and figure 5 show these influence when l is changed from 2 to 3.

3.2 Behavior of Posterior Mean and Variance as τ Tends to Infinity and Zero.

Before we obtained the prior density, we may known nothing about the distribution of parameter θ. We may made the prior density by some way, for example, by expert

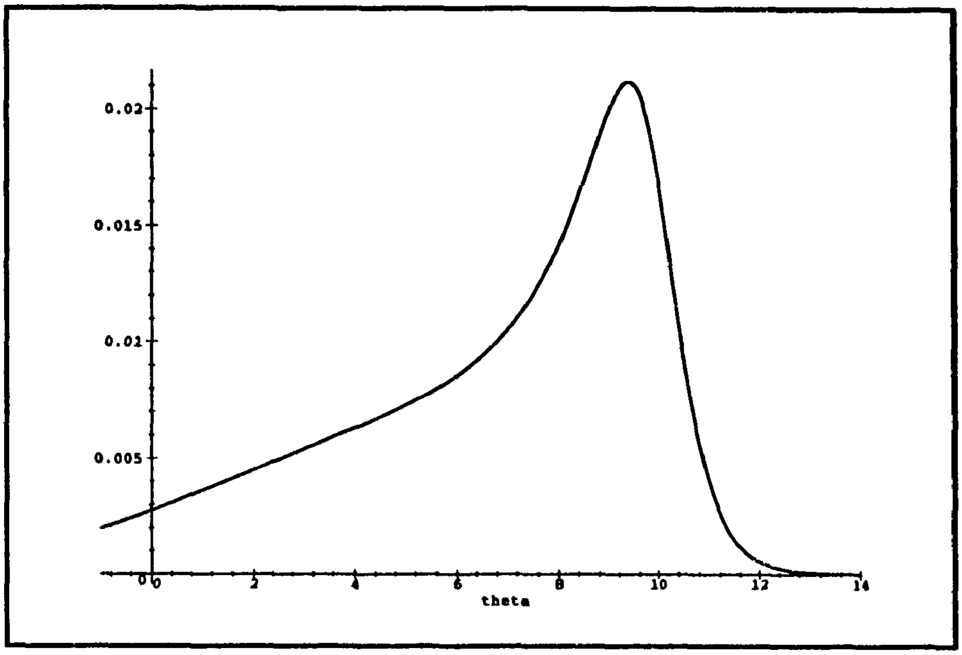

Figure 5: $\sigma_{(\bar{x},\bar{y})} = 3.5, \tau = 1, \mu = 10, \theta_{(\bar{x},\bar{y})} = 0, l = 3$

prediction. However after we obtain two observations from two different samples, we may have more information about the distribution of θ. So that the posterior is affected more by likelihood function is more reasonable and understandable. On the other hand, if we knew much about prior distribution, the posterior will be affected more by the prior distribution. From this point of view, we discuss about the case when the τ is extremely small. As we will see from theorem 3.2.2, the behavior of posterior will be fewer influenced by the likelihood function.

We have seen that τ can influence the shape of posterior density in section 3.1. From figure 6 and figure 7, we can see posterior distribution is close to likelihood distribution when τ tends to infinity, and to a point mass at μ when τ tends to zero.

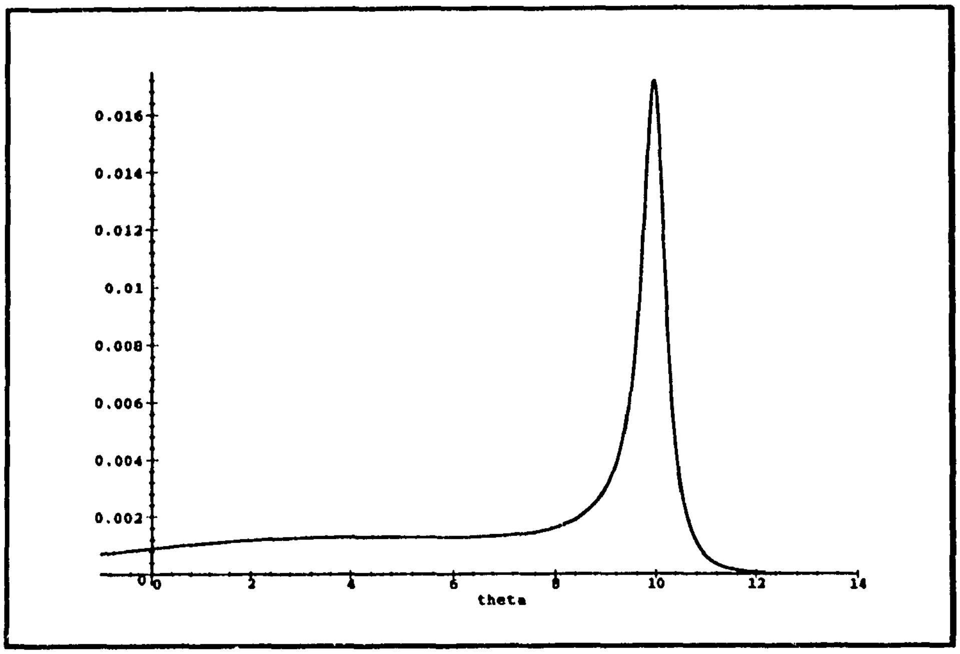

Figure 6: $\sigma_{(\bar{x},\bar{y})} = 3.5, \tau = 0.3, \mu = 10, \theta_{(\bar{x},\bar{y})} = 0, l = 3$

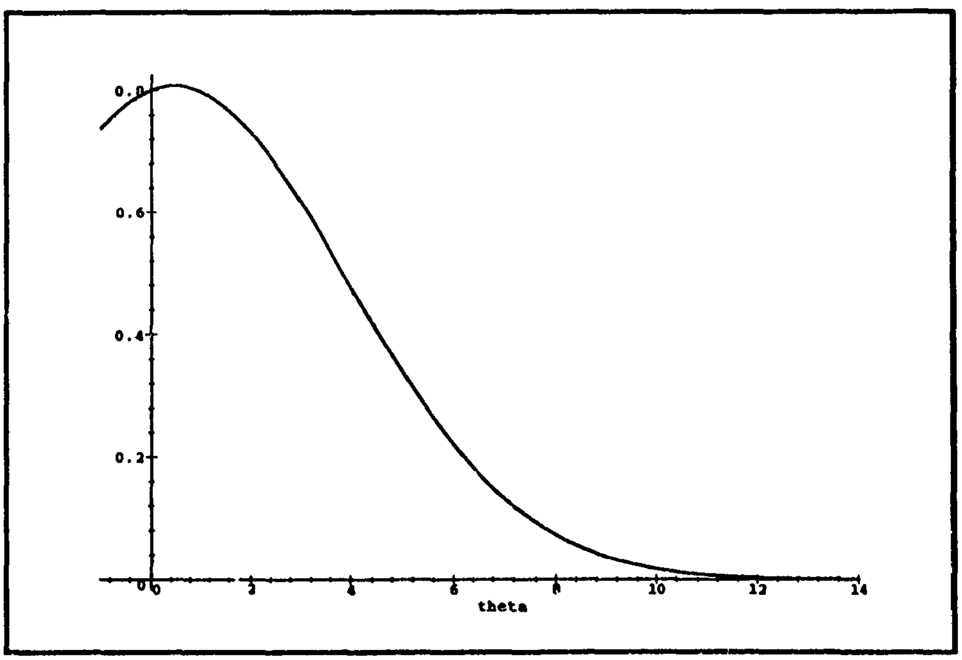

Figure 7: $\sigma_{(\bar{x},\bar{y})} = 3.5, \tau = 20, \mu = 10, \theta_{(\bar{x},\bar{y})} = 0, l = 3$

Under this view, we obtain the similar result as in chapter 2 with same idea. Some proof here are omitted.

lemma 3.2

$$\lim_{\tau\to\infty}\frac{p(\theta)}{p(\delta_{(x,y)})}=1$$

Proof:

$$\begin{aligned}
\lim_{\tau\to\infty}\frac{p(\theta)}{p(\delta_{(x,y)})} &= \lim_{\tau\to\infty}\frac{[1+\frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2}}{[1+\frac{(\delta_{(x,y)}-\mu)^2}{(l-1)\tau^2}]^{-l/2}} \\
&= \frac{[1+\lim_{\tau\to\infty}\frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2}}{[1+\lim_{\tau\to\infty}\frac{(\delta_{(x,y)}-\mu)^2}{(l-1)\tau^2}]^{-l/2}} \\
&= 1.
\end{aligned}$$

Theorem 3.2.1

(i)

$$\lim_{\tau\to\infty}\frac{\int_\Theta(2\pi)^{-1/2}\sigma^{-1}_{(\bar{x},\bar{y})}exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}p(\theta)d\theta}{p(\delta_{(x,y)})}=1.$$

(ii)

$$\lim_{\tau\to\infty}\pi(\theta|\tilde{x},\tilde{y})=(2\pi)^{-1/2}\sigma^{-1}_{(\bar{x},\bar{y})}exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}.$$

(iii)

$$\lim_{\tau\to\infty}\delta^{\pi}_{(x,y)}=\theta_{(\bar{x},\bar{y})}.$$

(iv)

$$\lim_{\tau\to\infty}V^{\pi}_{(x,y)}=\sigma^2_{(\bar{x},\bar{y})}.$$

Proof of (iv):

Similar as Theorem 2.1 d), only to find the tendency of expectation of θ^2.

$$\begin{aligned}
\lim_{\tau\to\infty} E^{\pi}(\theta^2) &= \lim_{\tau\to\infty} \int_{\Theta} \theta^2 \pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta} \theta^2 \lim_{\tau\to\infty} \pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta} \theta^2 (2\pi)^{-1/2}\sigma^{-1}_{(\tilde{x},\tilde{y})} exp\{-\tfrac{(\theta-\theta_{(\tilde{x},\tilde{y})})^2}{2\sigma^2_{(\tilde{x},\tilde{y})}}\}d\theta \\
&= \sigma^2_{(\tilde{x},\tilde{y})} + \theta^2_{(\tilde{x},\tilde{y})}.
\end{aligned}$$

Theorem 3.2.2

(i)

$$\lim_{\tau\to 0}\int_{\Theta}(2\pi)^{-1/2}\sigma^{-1}_{(\tilde{x},\tilde{y})} exp\{-\frac{(\theta-\theta_{(\tilde{x},\tilde{y})})^2}{2\sigma^2_{(\tilde{x},\tilde{y})}}\}p(\theta)d\theta$$

$$= (2\pi)^{-1/2}\sigma^{-1}_{(\tilde{x},\tilde{y})} exp\{-\frac{(\mu-\theta_{(\tilde{x},\tilde{y})})^2}{2\sigma^2_{(\tilde{x},\tilde{y})}}\}.$$

(ii)

$$\lim_{\tau\to 0}\pi(\theta|\tilde{x},\tilde{y}) = \delta(\mu), \quad \text{where } \delta(\mu) \text{ denotes a point mass at } \mu.$$

(iii)

$$\lim_{\tau\to 0}\delta^{\pi}_{(x,y)} = \mu.$$

(iv)

$$\lim_{\tau \to 0} V^{\pi}_{(x,y)} = 0.$$

Proof:

(i):

Similar as the proof of Theorem 2.2, set $\eta = \frac{\theta-\mu}{\tau}$, by (1.3.7)

$$\lim_{\tau\to 0} \int_{\Theta} (2\pi)^{-1/2} \sigma^{-1}_{(\bar{x},\bar{y})} exp\{-\frac{(\theta-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\} p(\theta) d\theta$$

$$= \int_{\eta} \lim_{\tau\to 0} (2\pi)^{-1/2} \sigma^{-1}_{(\bar{x},\bar{y})} exp\{-\frac{(\tau\eta+\mu-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\} K_{l-1}(1+\frac{\eta^2}{l-1})^{-l/2} d\eta$$

$$= (2\pi)^{-1/2} \sigma^{-1}_{(\bar{x},\bar{y})} exp\{-\frac{(\mu-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\} \lim_{\tau\to 0} \int_{\eta} K_{l-1}(1+\frac{\eta^2}{l-1})^{-l/2} d\eta$$

$$= (2\pi)^{-1/2} \sigma^{-1}_{(\bar{x},\bar{y})} exp\{-\frac{(\mu-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}$$

(ii):

Similar as theorem 2.2 (ii) and by (1.3.7) and in the proof of (i),

$$P(|\theta-\mu| > \varepsilon) = \frac{K_{l-1}}{m_{(x,y)}} \int_{\Theta} I_{(|\eta|>\varepsilon/\tau)} (2\pi)^{-1/2} \sigma^{-1}_{(\bar{x},\bar{y})} exp\{-\frac{(\tau\eta+\mu-\theta_{(\bar{x},\bar{y})})^2}{2\sigma^2_{(\bar{x},\bar{y})}}\}(1+\frac{\eta^2}{l-1})^{-l/2} d\eta$$

where $|I_{(|\eta|>\varepsilon/\tau)}(1+\frac{\eta^2}{l-1})^{-l/2}| \leq (1+\frac{\eta^2}{l-1})^{-l/2}$, and the right hand side is integrable, therefore $\lim_{\tau\to 0} P(|\theta-\mu| = 0) = 1$.

(iii):

$$\begin{aligned}
\lim_{\tau\to 0}\delta^{\pi}_{(x,y)} &= lim_{\tau\to 0}\int_{\Theta}\theta\pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta} lim_{\tau\to 0}\pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta}\theta I_{(\theta=\delta_{(\mu)})}d\theta \\
&= \mu
\end{aligned}$$

where $I_{(\theta=\delta_{(\mu)})} = \begin{cases} 1 & \text{if } \theta = \delta_{(\mu)} \\ 0 & \text{if } \theta \neq \delta_{(\mu)} \end{cases}$

(iv) :

$$\begin{aligned}
\lim_{\tau\to 0} E^{\pi}(\theta^2) &= \lim_{\tau\to 0}\int_{\Theta}\theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta}\lim_{\tau\to 0}\theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta \\
&= \int_{\Theta}\theta^2 I_{(\theta=\delta_{(\mu)})}d\theta \\
&= \mu^2,
\end{aligned}$$

So

$$\begin{aligned}
\lim_{\tau\to 0} V^{\pi}_{(x,y)} &= \lim_{\tau\to 0}\{E^{\pi}(\theta)^2 - [E^{\pi}(\theta)]^2\} \\
&= \lim_{\tau\to 0} E^{\pi}(\theta)^2 - \lim_{\tau\to 0}[E^{\pi}(\theta)]^2 \\
&= 0.
\end{aligned}$$

Chapter 4

Analysis of Posterior with Unknown Variances under t-Prior

4.1 Introduction

In this chapter, we consider the case of unknown variances for the two normal populations along with a t-prior. The likelihood function to be considered in this case is thus taken to be as in the equation (1.2.2). The posterior therefore consists of a multiple of three t-density functions. Section 4.2 provides analyses similar to those given in previous chapter where the prior is taken to be normal. In section 4.3, we discuss the behavior of posterior mean and variance under the condition $L \rightarrow \infty$ where, $L = min(|\mu - \bar{x}|, |\mu - \bar{y}|)$.

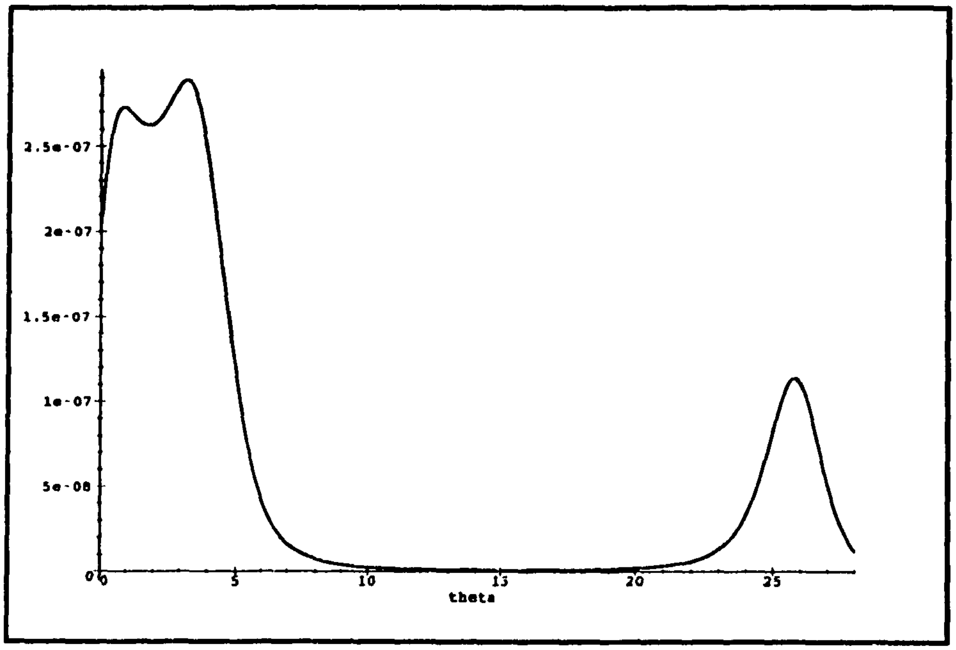

Figure 8: $s_1 = 1.8, m = 3, \bar{x} = 0, s_2 = 2.3, n = 3, \bar{y} = 4, \tau = 1.0, \mu = 26, l = 5$

4.2 Analysis of Posterior Distribution

In this section we provide extensions of the analysis given in chapter 3. First, we investigate the shape of the posterior distribution with respect to its unimodality and then we provide its limiting behaviour as the scale parameter of the prior distribution tends to zero or infinity.

Intuitively, since the posterior is a multiple of three unimodal densities it could have one, two or three peaks between $\bar{x}$ and μ, figure 8, figure 9 and figure 10 demonstrate these facts.

The following theorems, namely, theorem 4.2.1 and theorem 4.2.2 provide the behavior of the posterior density function as $\tau \rightarrow \infty$ and $\tau \rightarrow 0$ respectively. We note

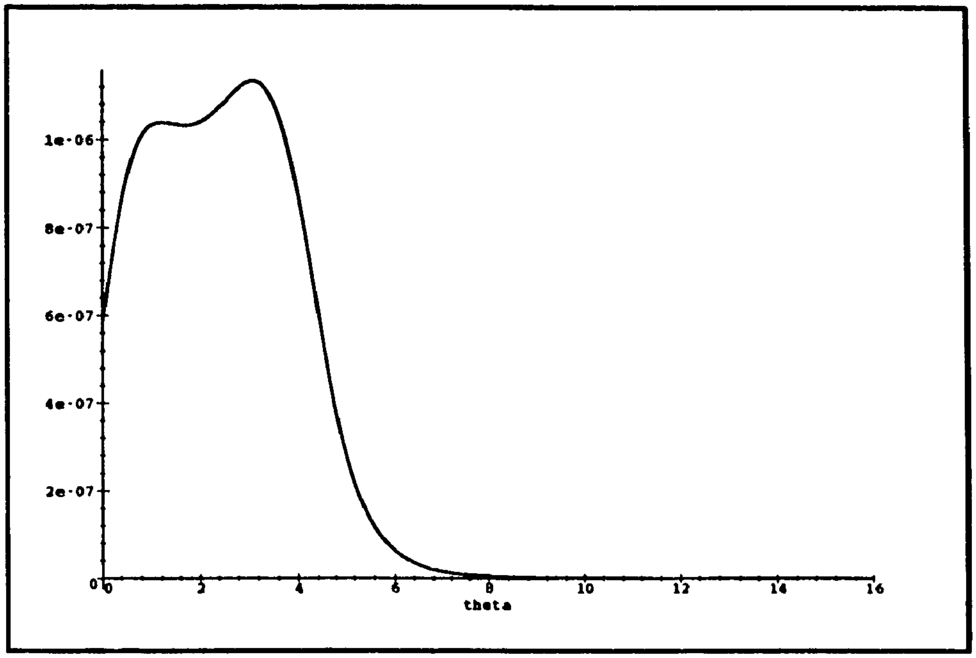

Figure 9: $s_1 = 1.8, m = 5, \bar{x} = 0, s_2 = 2.3, n = 5, \bar{y} = 4, \tau = 1.0, \mu = 16, l = 5$

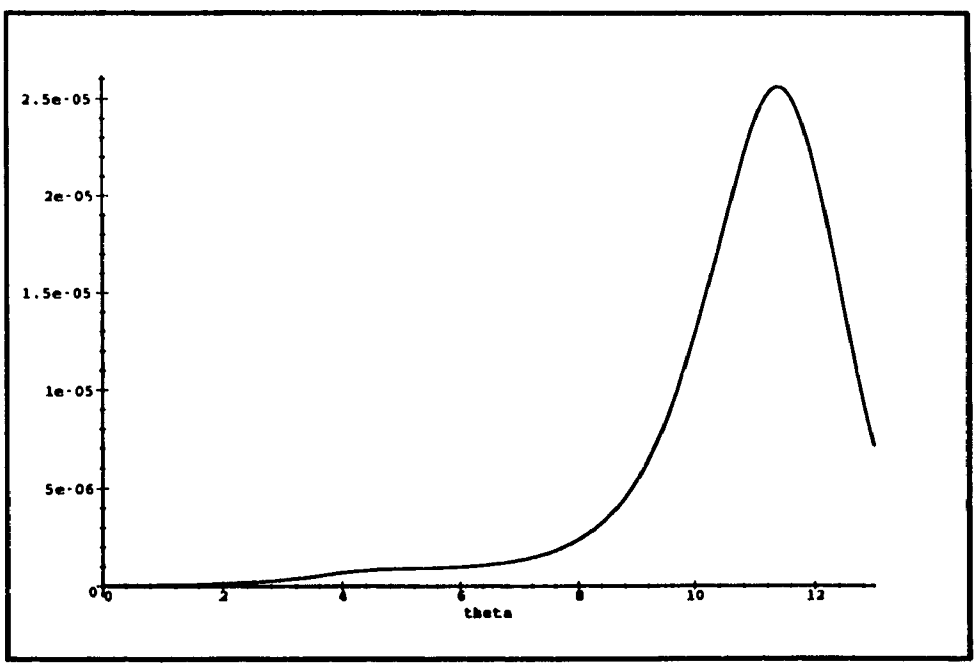

Figure 10: $s_1 = 1.8, m = 3, \bar{x} = 0, s_2 = 2.3, n = 3, \bar{y} = 4, \tau = 1.0, \mu = 12, l = 11$

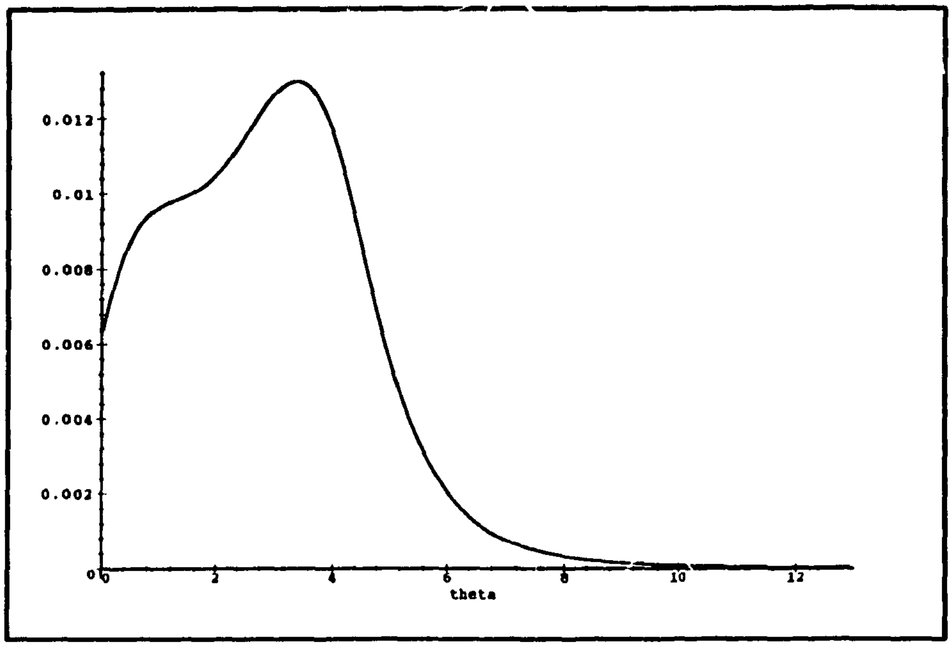

Figure 11: $s_1 = 1.8, m = 3, x = 0, s_2 = 2.3, n = 3, \bar{y} = 4, \tau = 5.0, \mu = 12, l = 11$

that this is parallel to the conclusions obtained under the normal prior.

Theorem 4.2.1

(i)

$$\lim_{\tau \to \infty} \frac{m_{(x,y)}}{p(\delta_{(x,y)})} = 1.$$

(ii)

$$\lim_{\tau \to \infty} \pi(\theta|\tilde{x}, \tilde{y}) = C_{(x,y)}^{-1} l(\theta|\tilde{x}, \tilde{y}).$$

(iii)

$$\lim_{\tau \to \infty} \delta_{(x,y)}^{\pi} = \delta_{(x,y)}.$$

(iv)

$$\lim_{\tau \to \infty} V^{\pi}_{(x,y)} = V_{(x,y)}.$$

Theorem 4.2.2

(i)

$$\lim_{\tau \to 0} m_{(x,y)} = C^{-1}_{(x,y)} f_{m-1}(\frac{\bar{x}-\mu}{s_1}\sqrt{m}) f_{n-1}(\frac{\bar{y}-\mu}{s_2}\sqrt{n}).$$

(ii)

$$\lim_{\tau \to 0} \pi(\theta|\tilde{x},\tilde{y}) = \delta(\mu), \quad \text{where } \delta(\mu) \text{ denotes a point mass at } \mu.$$

(iii)

$$\lim_{\tau \to 0} \delta^{\pi}_{(x,y)} = \mu$$

(iv)

$$\lim_{\tau \to 0} V^{\pi}_{(x,y)} = 0$$

The proof of these theorems follows along the same lines as in chapter 2 after noting the fact that $[1 + \frac{(\theta-\mu)^2}{(l-1)\tau^2}]^{-l/2}$ increases monotonically to 1 as $\tau \to \infty$. We may further compare figure 12 and figure 9 demonstrating that the posterior distribution tends to prior distribution when the variance of prior tends to zero.

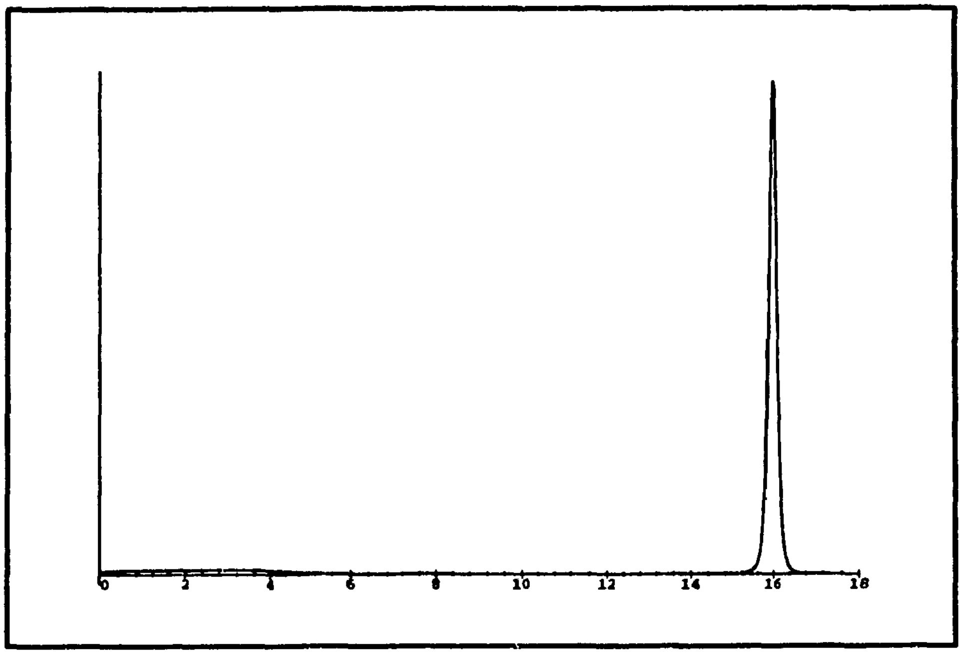

Figure 12: $s_1 = 1.8, m = 3, \bar{x} = 0, s_2 = 2.3, n = 3, \bar{y} = 4, \tau = 0.1, \mu = 16, l = 5$

4.3 Behavior of Posterior Distribution as $L \to \infty$

This is another interesting case to be analyzed as this portrays the case prior guess is extremely divergent from the evidence obtained by the sample means. In the analysis which follows we may assume without loss of generality that $\bar{x} \leq \bar{y} \leq \mu$.

Define:

(i) $\phi(x) = O^*(x)$ if $\exists c < \infty$ such that $\lim_{x\to\infty} \frac{\phi(x)}{x} = c$;

(ii) $\phi(x) = O^*(1)$ if $\exists 0 < c < \infty$ such that $\lim_{x\to\infty} \phi(x) = c$;

(iii) $\phi(x) = O(x)$ if $\exists c \leq \infty$ such that $\lim_{x\to\infty} \frac{\phi(x)}{x} \leq c$.

(iv) $\phi(x) = o(1)$ if $\lim_{x\to\infty} \frac{\phi(x)}{x} = 0$.

4.3.1 Behaviour of the Posterior PDF

We obtain in this case that the posterior density tends to the prior density if the degrees of freedom of the prior is larger than the total sample size, otherwise it tends to the normalized likelihood function. This observation is given in the following theorem.

Theorem 4.3.1

(i) If $l < m + n$, then

$$\lim_{L\to\infty} \pi(\theta|\tilde{x}, \tilde{y}) = C^{-1}_{(x,y)} l(\theta|\tilde{x}, \tilde{y}) \qquad a.s$$

(ii) If $l > m + n$,then

$$\lim_{L\to\infty} \pi(\theta|\tilde{x}, \tilde{y}) = p(\theta) \qquad a.s$$

Proof:

(i):

$\forall$ Borel set $A \subset R$

$$\int_{A\cap R_1} l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta = p(\delta_{(x,y)})O^*(1)\int_{A\cap R_1} l(\theta|\tilde{x},\tilde{y})d\theta \tag{4.3.1}$$

$$\int_{A\cap R_2} l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta = f_{m-1}(\frac{L}{s_1}\sqrt{m})f_{n-1}(\frac{L}{s_2}\sqrt{n})O^*(1)\int_{A\cap R_2} p(\theta)d\theta \tag{4.3.2}$$

$$\begin{aligned} \int_{A\cap R_1^c\cap R_2^c} l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta &\leq \int_{R_1^c\cap R_2^c} l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta \\ &= C_0 + D_0 + E_0 \leq (A_0 + B_0)o(1) \end{aligned} \tag{4.3.3}$$

then from (4.4.13) and above three equations,

$$\begin{aligned} \int_A \pi(\theta|\tilde{x},\tilde{y})d\theta &= \int_{A\cap R_1} \pi(\theta|\tilde{x},\tilde{y})d\theta + \int_{A\cap R_2} \pi(\theta|\tilde{x},\tilde{y})d\theta + \int_{A\cap R_1^c\cap R_2^c} \pi(\theta|\tilde{x},\tilde{y})d\theta \\ &= \frac{[p(\delta_{(x,y)})\int_{A\cap R_1} C_{(x,y)}^{-1} l(\theta|\tilde{x},\tilde{y})d\theta + C_{(x,y)}^{-1} f_{m-1}(\frac{L}{s_1}\sqrt{m})f_{n-1}(\frac{L}{s_2}\sqrt{n})\int_{A\cap R_2} p(\theta)d\theta]O^*(1)}{[C_{(x,y)}^{-1} f_{m-1}(\frac{L}{s_1}\sqrt{m})f_{n-1}(\frac{L}{s_2}\sqrt{n}) + p(\delta_{(x,y)})]O^*(1)} \end{aligned}$$

Define.

$$w_L = \frac{p(\delta_{(x,y)})}{C_{(x,y)}^{-1} f_{m-1}(\frac{L}{s_1}\sqrt{m})f_{n-1}(\frac{L}{s_2}\sqrt{n}) + p(\delta_{(x,y)})}$$

by the definition and the lemma 4.4.7,

$$\begin{aligned} \text{RHS} &= [w_L \textstyle\int_{A\cap R_1} C^{-1}_{(x,y)} l(\theta|\tilde{x},\tilde{y})d\theta + (1-w_L)\textstyle\int_{A\cap R_2} p(\delta_{(x,y)})d\theta]O^*(1) \\ &\to \textstyle\int_A C^{-1}_{(x,y)} l(\theta|\tilde{x},\tilde{y})d\theta \quad \text{when} L \to \infty \end{aligned}$$

(ii):

Similarly to prove (ii), notice if $l > m+n$, $w_L \to 0$, and $A \cap R_2 \to A$ when $L \to \infty$ and $\int_{A\cap R_1} l(\theta|\tilde{x},\tilde{y})d\theta < 1$.

4.3.2 The Behavior of the Posterior Mean

It may be difficult to calculate the posterior mean because of computational difficulties. Here we provide an approximation and study its behaviour for large L. It is interesting to note that the given approximation is between $\delta_{(x,y)}$ and μ, hence it may be close neither to $\delta_{(x,y)}$ nor to μ. We further investigate the question of "how close the posterior mean is to Fisher's solution?".

Theorem 4.3.2

Recall from (1.4.5),

$$\delta^{\pi}_{(x,y)} = E^{\pi}(\theta) = \int_{-\infty}^{\infty} \theta\pi(\theta|\tilde{x},\tilde{y})d\theta$$

Let

$$\hat{\delta}^{\pi}_{(x,y)} = \delta_{(x,y)}w_L + \mu(1-w_L)$$

then

$$|\delta^{\pi}_{(x,y)} - \hat{\delta}^{\pi}_{(x,y)}| = o(1) \qquad (4.3.4)$$

Proof: (See Appendix <A-2>)

Now we study the behaviour of $\hat{\delta}_{(x,y)}$ for both m and n extremely large. It is easy to see that in this case $\hat{\delta}^{\pi}_{(x,y)}$ tends to $\delta_{(x,y)}$ which lies between $\bar{x}$ and $\bar{y}$ (see lemma 4.3.1 (iv)). It is further observed that the Fisher's solution as investigated in Chaubey and Gabor(1981) also lies between $\bar{x}$ and $\bar{y}$, hence we study its closeness to $\delta_{(x,y)}$. Chaubey and Gabor (1981) got the MLE estimator from the likelihood function $l(\theta|\tilde{x}, \tilde{y})$ which is written in form

$$\tilde{\theta} = \tilde{w}\bar{x} + (1 - \tilde{w})\bar{y}$$

where

$$\tilde{w} = \frac{m^2/\Sigma(x_i - \tilde{\theta})^2}{m^2/\Sigma(x_i - \tilde{\theta})^2 + n^2/\Sigma(y_j - \tilde{\theta})^2}$$

Theorem 4.3.3

For large m and n

$$\delta_{(x,y)} - \tilde{\theta} \doteq \tilde{w}d$$

Proof:

To prove the above theorem, we first establish the following lemma giving some basic properties of $\delta_{(x,y)}$.

lemma 4.3.1

(i) Obviously, $\delta_{(x,y)} = \bar{x}$ if $\bar{x} = \bar{y}$.

(ii) From the discussion of Professor Y.P.Chaubey and G.Gabor (1981),if $n = m$, $s_1^2 = s_2^2$, the function $l(\theta|\tilde{x}, \tilde{y})$ is symmetric, i.e.

$$l(\theta|\tilde{x}, \tilde{y}) = f_{m-1}(\frac{\eta + d/2}{s_1}\sqrt{m})f_{n-1}(\frac{\eta - d/2}{s_2}\sqrt{n}),$$

so we get $\quad \delta_{(x,y)} = \frac{\bar{x}+\bar{y}}{2}$

(iii) if $\frac{m}{(m-1)s_1^2} = \frac{n}{(n-1)s_2^2}$, we also have $\delta_{(x,y)} = \frac{\bar{x}+\bar{y}}{2}$

(iv) $\delta_{(x,y)}$ lies between $\bar{x}$ and $\bar{y}$.

Proof:

Set $\eta = \theta - \bar{y}$, $\xi = \theta - \bar{x}$, then

$$l(\theta|\tilde{x}, \tilde{y}) = f_{m-1}(\frac{\xi}{s_1}\sqrt{m})f_{n-1}(\frac{\xi - d}{s_1}\sqrt{n}) = f_{m-1}(\frac{\eta + d}{s_1}\sqrt{m})f_{n-1}(\frac{\eta}{s_2})$$

so

$$\begin{aligned}\delta_{(x,y)} &= C_{(x,y)}^{-1} \int_\eta \eta f_{m-1}(\tfrac{\eta+d}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta}{s_2}) d\eta + \bar{y} \\ &= C_{(x,y)}^{-1} \int_\xi \xi f_{m-1}(\tfrac{\xi}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\xi-d}{s_2}\sqrt{n}) d\xi + \bar{x}\end{aligned}$$

Now

$$\int_\eta \eta f_{m-1}(\tfrac{\eta+d}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta}{s_2}\sqrt{n}) d\eta$$

$$= [\int_0^\infty + \int_{-\infty}^0] \eta f_{m-1}(\tfrac{\eta+d}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta}{s_2}\sqrt{n}) d\eta$$

$$= \int_0^\infty \eta f_{m-1}(\tfrac{\eta+d}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta}{s_2}\sqrt{n}) d\eta - \int_0^\infty \eta f_{m-1}(\tfrac{\eta-d}{s_1}\sqrt{m}) f_{n-1}(\tfrac{\eta}{s_2}\sqrt{n}) d\eta$$

$$= \int_0^\infty \eta f_{n-1}(\tfrac{\eta}{s_2}\sqrt{n}) [f_{m-1}(\tfrac{\eta+d}{s_1}\sqrt{m}) - f_{m-1}(\tfrac{\eta-d}{s_1}\sqrt{m})] d\eta \leq 0$$

because of $f_{m-1}(\frac{\eta\pm d}{s_1}\sqrt{m})$, $f_{n-1}(\frac{\eta}{s_2}\sqrt{n})$, η are non-negative when $\eta \in (0, \ \infty)$ and $f_{m-1}(\frac{\eta+d}{s_1}\sqrt{m}) \leq f_{m-1}(\frac{\eta-d}{s_1}\sqrt{m})$, so $\delta_{(x,y)} \leq \bar{y}$.

Similarly $\delta_{(x,y)} \geq \bar{x}$.

End of proof of lemma.

Without loss of generality, set $C_{(x,y)} = 1$. Note $\delta_{(x,y)} - \bar{x} = (1 - \tilde{w})d$; $\tilde{\theta} - \bar{y} = -\tilde{w}d$

$$
\begin{aligned}
\hat{\theta} - \tilde{\theta} &= \int_{\Theta} (\theta - \tilde{\theta}) l(\theta | \tilde{x}, \tilde{y}) d\theta \\
&\overset{\xi = \theta - \tilde{\theta}}{=\!=\!=\!=} \int_{\xi} \xi f_{m-1}\left(\tfrac{\xi + (1-\tilde{w})d}{s_1}\sqrt{m}\right) f_{n-1}\left(\tfrac{\xi - \tilde{w}d}{s_2}\sqrt{n}\right) d\xi \\
&\overset{\eta = \xi - \tilde{w}d}{=\!=\!=\!=} \int_{\eta} \eta f_{m-1}\left(\tfrac{\eta + d}{s_1}\sqrt{m}\right) f_{n-1}\left(\tfrac{\eta}{s_2}\sqrt{n}\right) d\eta + \tilde{w}d
\end{aligned}
$$

set

$$
\begin{aligned}
\Delta &\overset{\text{def}}{=\!=} \int_{\eta} \eta f_{m-1}\left(\tfrac{\eta + d}{s_1}\sqrt{m}\right) f_{n-1}\left(\tfrac{\eta}{s_2}\sqrt{n}\right) d\eta \\
&= c_{mn}\left[\int_{-\infty}^{0} + \int_{0}^{\infty}\right] \eta [1 + c_m(\eta + d)^2]^{-m/2} [1 + c_n \eta^2]^{-n/2} d\eta
\end{aligned}
$$

and

$$
\Delta_1 = \int_0^{\infty} \eta (1 + c_n \eta^2)^{-n/2} (1 + c_m(\eta + d)^2)^{-m/2} d\eta
$$

$$
\Delta_2 = \int_0^{\infty} \eta (1 + c_n \eta^2)^{-n/2} (1 + c_m(\eta - d)^2)^{-m/2} d\eta
$$

then

$$
\Delta = \Delta_1 - \Delta_2
$$

when m and n are large enough,

$$
[1 + c_m(\eta \pm d)^2]^{-m/2} \doteq e^{-\frac{1}{2} m c_m (\eta \pm d)^2}
$$

$$
[1 + c_n \eta^2]^{-n/2} \doteq e^{-\frac{1}{2} n c_n \eta^2}
$$

then

$$
\begin{aligned}
\Delta_i &\doteq \int_0^\infty \eta e^{-\frac{1}{2}[mc_m(\eta^2 \pm 2\eta d + d^2) + nc_n\eta^2]} d\eta \\
&= c_1 \int_0^\infty \eta e^{-\frac{1}{2}(mc_m+nc_n)(\eta \pm \frac{mc_m d}{mc_m+nc_n})^2} d\eta \\
&= c_1 [\int_0^\infty (\eta \pm \frac{mc_m d}{mc_m+nc_n}) e^{-\frac{1}{2}(mc_m+nc_n)(\eta \pm \frac{mc_m d}{mc_m+nc_n})^2} d\eta \qquad (4.3.5) \\
&\mp \frac{mc_m d}{mc_m+nc_n}) \int_0^\infty e^{-\frac{1}{2}(mc_m+nc_n)(\eta \pm \frac{mc_m d}{mc_m+nc_n})^2} d\eta] \\
&\triangleq c_1 q_1 \mp c_1 \frac{mc_m d}{mc_m+nc_n} q_2
\end{aligned}
$$

where

$$
c_1 = e^{-\frac{mc_m nc_n d^2}{2(mc_m+nc_n)}}
$$

After making a transformation $\eta \rightarrow \xi \rightarrow z$ where $\xi = \eta \pm \frac{mc_m d}{mc_m+nc_n}$ and $z = \sqrt{mc_m + nc_n}\xi$, we can define q_1 and q_2 of equation (4.3.5) as following:

$$
q_1 = (mc_m + nc_n)^{-1} e^{-\frac{1}{2}(mc_m+nc_n)^{-1}(mc_m d)^2}
$$

$$
\begin{aligned}
q_2 &= \int_{\pm\frac{mc_m d}{mc_m+nc_n}}^\infty e^{-\frac{1}{2}(mc_m+nc_n)\xi^2} d\xi \\
&= (\sqrt{mc_m + nc_n})^{-1} \int_{\pm\frac{mc_m d}{\sqrt{mc_m+nc_n}}}^\infty e^{-\frac{z^2}{2}} dz
\end{aligned}
$$

Hence

$$\begin{aligned} \Delta_i &\doteq c_1[(mc_m+nc_n)^{-1}e^{-\frac{1}{2}(mc_m+nc_n)^{-1}(mc_md)^2} \\ &\mp \tfrac{mc_md}{(mc_m+nc_n)^{3/2}}\int_{\pm\frac{mc_md}{\sqrt{mc_m+nc_n}}}^{\infty} e^{-\frac{z^2}{2}}dz] \end{aligned}$$

where i=1,2.

so

$$\begin{aligned} |\Delta| &= c_{mn}|\Delta_1-\Delta_2| \\ &\doteq c_{mn}c_1\tfrac{mc_md}{(mc_m+nc_n)^{3/2}}[\int_{-\frac{mc_md}{\sqrt{mc_m+nc_n}}}^{\frac{mc_md}{\sqrt{mc_m+nc_n}}} e^{-\frac{z^2}{2}}dz + 2\int_{\frac{mc_md}{\sqrt{mc_m+nc_n}}}^{\infty} e^{-\frac{z^2}{2}}dz] \\ &= c_{mn}c_1\tfrac{mc_md}{(mc_m+nc_n)^{3/2}}\int_{-\infty}^{\infty} e^{-z^2/2}dz \\ &= c_{mn}c_1\tfrac{mc_md}{(mc_m+nc_n)^{3/2}}\sqrt{2\pi} \end{aligned}$$

note

$$\begin{aligned} \tfrac{c_{mn}c_1mc_md}{(mc_m+nc_n)^{3/2}} &= \tfrac{\sqrt{n}\sqrt{m}\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})m^2dc_1(n-1)s_2^2}{\Gamma(\frac{m-1}{2})\Gamma(\frac{n-1}{2})\pi[m^2(n-1)s_2^2+n^2(m-1)s_1^2]^{3/2}} \\ &\triangleq \varepsilon(m,n)\tfrac{ds_2^2}{\pi} \end{aligned}$$

where

$$\frac{\Gamma(\frac{t}{2})}{\Gamma(\frac{t-1}{2})}\to 1 \quad \text{as } t\to\infty.$$

Since $\sigma_1^2<\infty$ and $\sigma_2^2<\infty$, we have $s_1^2<\infty$ and $s_2^2<\infty$.

$$\varepsilon(m,n) = \frac{\sqrt{n}\sqrt{m}m^2(n-1)}{[m^2(n-1)s_2^2 + n^2(m-1)s_1^2]^{3/2}} \to 0 \quad \text{and} \quad c_1 \to 0$$

as $m \to \infty$ and $n \to \infty$

so

$$\begin{aligned} \Delta &\doteq \varepsilon(m,n)\tfrac{\sqrt{2}s_2^2}{\sqrt{\pi}}d \\ &\Rightarrow \delta_{(x,y)} - \tilde{\theta} = \tilde{w}d \end{aligned}$$

By the way, when the m or n are large,

$$l(\theta|\tilde{x}, y) \doteq c_{mn}e^{-\frac{mc_m(\theta-\bar{x})^2}{2}}e^{-\frac{nc_n(\theta-\bar{y})^2}{2}}$$

using MLE method, we get the approximation estimator:

$$\tilde{\theta} \doteq \frac{mc_m}{mc_m + nc_n}\bar{x} + \frac{nc_n}{mc_m + nc_n}\bar{y}$$

where

$$\frac{mc_m}{mc_m+nc_n} \doteq \frac{m^2(n-1)s_2^2}{m^2(n-1)s_2^2+n^2(m-1)s_1^2}$$

$$\to \begin{cases} 1 & \text{if the speed of m} \to \infty \text{ is faster than of n} \to \infty \\ \frac{s_2^2}{s_2^2+s_1^2} & \text{if the speed of m} \to \infty \text{ is equal to of n} \to \infty \\ 0 & \text{if the speed of m} \to \infty \text{ is slower than of n} \to \infty \end{cases}$$

4.3.3 The Behavior of Posterior Variance

The posterior variance should tend to infinity intuitively when L tends to infinity. Theorem 4.3.4 shows this fact and give an approximate estimator of the posterior variance.

Theorem 4.3.4

Set

$$\Psi(L) = w_L a_2 + (1 - w_L) b_2 + w_L (1 - w_L) L^2$$

where a_2 and b_2 are defined in lemma 4.4.6.

then

$$|V^{\pi}_{(x,y)} - O^{*}(1)\Psi(L)| = \begin{cases} o(1) & \text{if } l \neq m + n \\ O^{*}(1) & \text{if } l = m + n \end{cases} \tag{4.3.6}$$

Proof: (See Appendix <A-3>)

The last theorem 4.2.5 show the variance of posterior is portion to L^2.

Theorem 4.3.5

$$lim_{L \to \infty} Sup_{\tau} \frac{V^{\pi}_{(x,y)}}{L^2} = \frac{1}{4}$$

Proof: (See Appendix <A-4>)

4.4 Some Lemmas

This section presents some lemmas which are used at various places in the proofs.

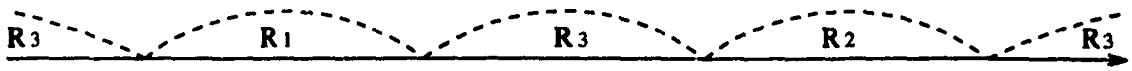

Figure 13: Intervals R_j Integrations

Set $d = |\bar{x} - \bar{y}|$, which is constant after obtaining two sample observations from X- and Y-distribution respectively.

set

$$1 > r > \frac{2 + min(m+n, l)}{m+n+l-1}$$

where r is a fixed constant. No doubt when L is large enough, it makes following inequality existed:

$$L^r > d \qquad L > 2^{\frac{1}{1-r}}.$$

All succeeding discussion are based on above assumption.

Let

$$R_1 \stackrel{\text{def}}{=} \{\theta : |\theta - \delta_{(x,y)}| < L^r\}$$

$$R_2 \stackrel{\text{def}}{=} \{\theta : |\theta - \mu| < L^r\}$$

$$R_3 \stackrel{\text{def}}{=} R - (R_1 \cup R_2)$$

Assume $\bar{x} \leq \bar{y} \leq \mu$. Here we divide the real line into three parts, the length of R_1 and R_2 are less than double L^r.

Before proving some theorems, we need following lemmas provided.

Set

$$\gamma_{(t)} = \sqrt{t}K_{t-1}(\frac{t-1}{t})^{t/2}S_t^{t-1}$$

lemma 4.4.1

For $\eta \in [0,\infty)$, and constant ν greater than zero,

$$f_{t-1}(\frac{\eta+\nu}{S_t}\sqrt{t}) = \gamma_{(t)}O^*(1)(\eta+\nu)^{-t}. \tag{4.4.1}$$

Proof:

$$\begin{aligned} f_{m-1}(\tfrac{\eta+\nu}{S_t}\sqrt{t}) &= \sqrt{t}K_{t-1}(t-1)^{t/2}S_t^{t-1}[(t-1)S_t^2 + t(\eta+\nu)^2]^{-t/2} \\ &= \gamma_{(t)}[1+\tfrac{(t-1)S_t^2}{t(\eta-\nu)^2}]^{-\frac{t}{2}}(\eta+\nu)^{-t} \end{aligned}$$

Note

$$[1+\frac{(t-1)S_t^2}{t\nu^2}]^{-\frac{t}{2}} \leq [1+\frac{(t-1)S_t^2}{t(\eta+\nu)^2}]^{-\frac{t}{2}} \leq 1$$

then (4.4.1) is existed.

lemma 4.4.2

(i) If $m+n > i+1$ then

$$\int_{L^r}^{\infty} \eta^i l(\eta + \delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(L^{-r(m+n-1-i)}) \tag{4.4.2}$$

$$\int_{L^r}^{\infty} \eta^i l(-\eta + \delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(L^{-r(m+n-1-i)}) \tag{4.4.3}$$

where $i \geq 0$.

(ii) If $m+n=i+1$ and $i=3$ or $i=4$, then

$$\int_0^{L^r} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(lnL^r) \tag{4.4.4}$$

$$\int_0^{L^r} \eta^i l(-\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(lnL^r)$$

(iii) if $m+n=i=4$. then

$$\int_0^{L^r} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(L^r) \tag{4.4.5}$$

$$\int_0^{L^r} \eta^i l(-\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta = O^*(L^r)$$

Proof:

(i) proof of (4.4.2):

From one side,

$$\int_{L^r}^{\infty} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \geq \int_{L^r}^{\infty} \eta^i f_{m-1}(\frac{\eta+\delta_{(x,y)}-\bar{x}}{s_1}\sqrt{m})f_{n-1}(\frac{\eta+\delta_{(x,y)}-\bar{x}}{s_2}\sqrt{n})d\eta$$

using (4.4.1)

$$\begin{aligned} \text{RHS} &= O^*(1)\textstyle\int_{L^r}^{\infty} \eta^i(\eta+\delta_{(x,y)}-\bar{x})^{-m-n}d\eta \\ &= O^*(L^{-r(m+n-1-i)}) \end{aligned} \tag{4.4.6}$$

From the other side,

$$\int_{L^r}^{\infty} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \leq \int_{L^r}^{\infty} \eta^i f_{m-1}(\frac{\eta+\delta_{(x,y)}-\bar{y}}{s_1}\sqrt{m})f_{n-1}(\frac{\eta+\delta_{(x,y)}-\bar{y}}{s_2}\sqrt{n})d\eta$$

set $\zeta = \eta - \bar{y}$, then

$$\begin{aligned} \text{RHS} &= \int_{L^r-\bar{y}}^{\infty}(\zeta+\bar{y})^i f_{m-1}\left(\frac{\zeta+\delta_{(x,y)}}{s_1}\sqrt{m}\right) f_{n-1}\left(\frac{\zeta+\delta_{(x,y)}}{s_2}\sqrt{n}\right) d\zeta \\ &= O^\star(1)\int_{L^r-\bar{y}}^{\infty}(\zeta+\bar{y})^i(\zeta+\delta_{(x,y)})^{-m-n}d\eta \\ &\leq O^\star(1)\int_{L^r-\bar{y}}^{\infty}(\zeta+\delta_{(x,y)})^{i-m-n}d\eta \\ &= O^\star(L^{-r(m+n-1-i)}) \end{aligned} \tag{4.4.7}$$

Combine (4.4.6) and (4.4.7), we get (4.4.2). Similarly to prove (4.4.3)

(ii) proof of (4.4.4):

$$\begin{aligned} \int_0^{L^r}\eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta &= \left(\int_0^{\bar{y}}+\int_{\bar{y}}^{L^r}\right)\eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \\ &\leq O^\star(1)\int_{\bar{y}}^{L^r}\eta^i(\eta+\delta_{(x,y)}-\bar{y})^{-m-n}d\eta+O^\star(1) \\ &\leq \int_{\bar{y}}^{L^r}(\eta+\delta_{(x,y)}-\bar{y})^{-1}d\eta+O^\star(1)=O^\star(lnL^r) \end{aligned} \tag{4.4.8}$$

converse, set $\eta = \zeta + \bar{y}$

$$\begin{aligned} &\int_0^{L^r}\eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \\ &= \left(\int_{-\bar{y}}^{0}+\int_0^{L^r-\bar{y}}\right)[(\zeta+\bar{y})^i f_{m-1}\left(\frac{\zeta+\delta_{(x,y)}+d}{s_1}\sqrt{m}\right) f_{n-1}\left(\frac{\zeta+\delta_{(x,y)}}{s_2}\sqrt{n}\right)d\zeta+O^\star(1) \\ &= O^\star(1)\int_0^{L^r-\bar{y}}(\zeta+\bar{y})^i(\zeta+\delta_{(x,y)}+d)^{-m}(\zeta+\delta_{(x,y)})^{-n}d\zeta+O^\star(1) \\ &\geq O^\star(1)\int_0^{L^r-\bar{y}}(\zeta+\delta_{(x,y)}+d)^{i-m-n}d\zeta=O^\star(lnL^r) \end{aligned} \tag{4.4.9}$$

(4.4.8) and (4.4.9) implies(4.4.4).

proof of (4.4.5) is similar.

lemma 4.4.3

As in lemma 4.4.2, we set $\theta = \eta + \delta_{(x,y)}$, get following conclusions:

(i)

$$\begin{aligned}\int_{R_1} l(\theta|\tilde{x},\tilde{y})d\theta &= C_{(x,y)} - \int_{-\infty}^{-L^r} l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta - \int_{L^r}^{\infty} l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \\ &= C_{(x,y)} - O^*(L^{-r(m+n-1)})\end{aligned}$$

(ii)

$$\int_{R_1}(\theta-\delta_{(x,y)})l(\theta|\tilde{x},\tilde{y})d\theta = O^*(L^{-r(m+n-2)})$$

(iii)

$$\int_{R_1}(\theta-\delta_{(x,y)})^2 l(\theta|\tilde{x},\tilde{y})d\theta = C_{(x,y)}V_{(x,y)} - O^*(L^{-r(m+n-3)})$$

(iv)

if $m+n \geq 5$ and $i=3$, or $m+n>5$ and $i=4$, by 1) of lemma 4.4.2, then

$$\begin{aligned}&\int_{R_1}(\theta-\delta_{(x,y)})^i l(\theta|\tilde{x},\tilde{y})d\theta \\ &= C_{(x,y)}E(\theta-\delta_{(x,y)})^i - [\int_{-\infty}^{\delta_{(x,y)}-L^r} - \int_{\delta_{(x,y)}+L^r}^{\infty}](\theta-\delta_{(x,y)})^i l(\theta|\tilde{x},\tilde{y})d\theta \\ &= C_{(x,y)}E(\theta-\delta_{(x,y)})^3 - O^*(L^{-r(m+n-1-i)})\end{aligned}$$

if $m+n=4$ and $i=3$, by 2) of lemma 4.4.2, then

$$\begin{aligned}&\int_{R_1}(\theta-\delta_{(x,y)})^3 l(\theta|\tilde{x},\tilde{y})d\theta \\ &= \int_{-L^r}^{0}\eta^3 l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta + \int_0^{L^r}\eta^3 l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})d\eta \\ &= O^*(lnL^r)\end{aligned}$$

if $m+n=i=4$, then

$$\int_{R_1}(\theta-\delta_{(x,y)})^4 l(\theta|\tilde{x},\tilde{y})d\theta = O^*(L^r)$$

if $m+n=5$ and $i=4$, then

$$\int_{R_1}(\theta-\delta_{(x,y)})^4 l(\theta|\tilde{x},\tilde{y})d\theta = O^*(lnL^r).$$

lemma 4.4.4

For $\theta \in R_1$

$$p(\theta)=p(\delta_{(x,y)})O^*(1)[1-(\theta-\delta_{(x,y)})O^*(L^{-1})+(\theta-\delta_{(x,y)})O^*(L^{-2})]$$

Proof:

step 1) Do derivative to $p(\theta)$ w.r.t. θ.

$\frac{\partial p(\theta)}{\partial\theta}=p(\theta)[-\frac{l(\theta-\mu)}{(l-1)\tau^2+(\theta-\mu)^2}]$

$\frac{\partial^2 p(\theta)}{\partial\theta^2}=p(\theta)[\frac{l(\theta-\mu)}{(l-1)\tau^2+(\theta-\mu)^2}]^2-p(\theta)\frac{(l-1)l\tau^2+l(\theta-\mu)^2-2l(\theta-\mu)^2}{[(l-1)\tau^2+(\theta-\mu)^2]^2}$

$=p(\theta)\frac{l(l+1)(\theta-\mu)^2-(l-1)l\tau^2}{[(l-1)\tau^2+(\theta-\mu)^2]^2}$

step 2)

$$\frac{p(\delta_{(x,y)})}{p(\delta_{(x,y)})} = [\frac{(l-1)\tau^2+L^2}{(l-1)\tau^2+(\mu-\delta_{(x,y)})^2}]^{l/2} = O^\star((\frac{L}{\mu-\delta_{(x,y)}})^l) = O^\star(1)$$

similarly, for any $\theta_0 \in R_1$, $\frac{p(\theta_0)}{p(\delta_{(x,y)})} = O^\star(1)$

step 3) Expanding $p(\theta)$, by the Taylor theorem, $\exists\theta_0 \in R_1$ such that

$$p(\theta) = p(\delta_{(x,y)}) + (\theta - \delta_{(x,y)})p'(\theta)|_{\theta=\delta_{(x,y)}} + \frac{1}{2!}(\theta - \delta_{(x,y)})^2 p''(\theta)|_{\theta=\theta_0}$$

$$= p(\delta_{(x,y)})\{1+(\theta-\delta_{(x,y)})[-\frac{l(\delta_{(x,y)}-\mu)}{(l-1)\tau^2+(\delta_{(x,y)}-\mu)^2}]+\frac{1}{2}(\theta-\delta_{(x,y)})^2\frac{h(\theta_0)}{p(\delta_{(x,y)})}\cdot\frac{l(l+1)(\theta_0-\mu)^2-(l-1)l\tau^2}{[(l-1)\tau^2+(\theta_0-\mu)^2]^2}\}$$

$$= p(\delta_{(x,y)})O^\star(1)[1 - (\theta - \delta_{(x,y)})O^\star(L^{-1}) + (\theta - \delta_{(x,y)})^2 O^\star(L^{-2})].$$

lemma 4.4.5.

For $\theta \in R_2$

$$f_{m-1}(\frac{\bar{x}-\theta}{s_1}\sqrt{m}) = f_{m-1}(\frac{L}{s_1}\sqrt{m})(1-(\theta-\mu)O^\star(L^{-1}))O^\star(1)$$

$$f_{n-1}(\frac{\bar{y}-\theta}{s_2}\sqrt{n}) = f_{n-1}(\frac{L}{s_2}\sqrt{n})(1-(\theta-\mu)O^\star(L^{-1}))O^\star(1)$$

Proof: Similar as proof of lemma 4.4.4.

lemma 4.4.6

Set

$$\gamma_0 = C_{(x,y)}^{-1}\sqrt{m}K_{m-1}(\frac{m-1}{m})^{m/2}s_1^{m-1}\sqrt{n}K_{n-1}(\frac{n-1}{n})^{n/2}s_2^{n-1}(l-1)^{l/2}K_{l-1}\tau^{l-1},$$

Let

$$A_i \stackrel{\text{def}}{=} C_{(x,y)}^{-1}\int_{R_1}(\theta-\delta_{(x,y)})^i l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta$$

$$B_i \stackrel{\text{def}}{=} C_{(x,y)}^{-1}\int_{R_2}(\theta-\mu)^i l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta$$

$$C_i \stackrel{\text{def}}{=} C_{(x,y)}^{-1}\int_{-\infty}^{\delta_{(x,y)}-L^r}(\theta-\delta_{(x,y)})^i l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta$$

$$D_i \stackrel{\text{def}}{=} C_{(x,y)}^{-1}\int_{\delta_{(x,y)}+L^r}^{\mu-L^r}(\theta-\delta_{(x,y)})^i l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta$$

$$E_i \stackrel{\text{def}}{=} C_{(x,y)}^{-1}\int_{\mu+L^r}^{\infty}(\theta-\mu)^i l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta$$

where $i \geq 0$.

For the fixed d and large L, $L^r > d$, i=0,1,2, we have following conclusions.

(i)

$$A_i = p(\delta_{(x,y)})O^*(1)a_i. \tag{4.4.10}$$

where

$$a_i = C_{(x,y)}^{-1} \int_{R_1} (\theta - \delta_{(x,y)})^i l(\theta|\tilde{x}, \tilde{y})$$
$$\times [1 - (\theta - \delta_{(x,y)})O^*(L^{-1}) + (\theta - \delta_{(x,y)})^2 O^*(L^{-2})]d\theta$$

$$a_0 = 1 + V_{(x,y)}O^*(L^{-2}) - C_{(x,y)}^{-1}O^*(L^{-r(m+n-1)}).$$

$$a_1 = C_{(x,y)}^{-1}O^*(L^{-r(m+n-2)}) - O^*(L^{-1})V_{(x,y)}.$$

$$a_2 = \begin{cases} V_{(x,y)} - C_{(x,y)}^{-1}O^*(L^{-r(m+n-3)}) - E(\theta - \delta_{(x,y)})^3 O^*(L^{-1}) & \text{if } m+n > 5 \\ V_{(x,y)} - C_{(x,y)}^{-1}O^*(L^{-r(m+n-3)}) - E(\theta - \delta_{(x,y)})^3 O^*(L^{-1}) & \text{if } m+n = 5 \\ V_{(x,y)} - C_{(x,y)}^{-1}O^*(L^{-r(m+n-3)}) - O^*(L^{-1} ln L^r) & \text{if } m+n = 4 \end{cases}$$

(ii)

$$B_i = C_{(x,y)}^{-1} f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n}) O^*(1) b_i \tag{4.4.11}$$

where

$$b_i = \int_{R_2} (\theta - \mu)^i [1 - (\theta - \mu)O^*(L^{-1}) + (\theta - \mu)^2 O^*(L^{-2})] p(\theta) d\theta$$

$$b_0 = \begin{cases} 1 + O(L^{-r(l-1)}) + O(L^{-2}) & \text{if } l > 3 \\ 1 + O(L^{-r(l-1)}) & \text{if } l \leq 3 \end{cases}$$

$$b_1 = \begin{cases} O(L^{-1}) & \text{if } l > 3 \\ O(L^{-1} ln(L)) & \text{if } l = 3 \\ O(L^{r-1}) & \text{if } l = 2 \end{cases}$$

$$b_2 = \begin{cases} V_h + O(L^{-r(l-3)}) & \text{if } l > 3 \\ O(ln(L)) & \text{if } l = 3 \\ O(L^{r}) & \text{if } l = 2 \end{cases}$$

(iii)

$$|C_i| \leq O^{\star}(L^{-r[(m+n+l-1)-i]})$$

(iv)

$$|D_i| \leq O^{\star}(L^{-r[(m+n+l-1)-i]})$$

(v)

$$|E_i| \leq O^{\star}(L^{-r[(m+n+l-1)-i]})$$

Proof:(See Appendix <A-1>)

From the lemma 4.4.6,

$$A_0 = p(\delta_{(x,y)})O^{\star}(1)$$

$$B_0 = C^{-1}_{(x,y)} f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n}) O^*(1)$$

$$C_0 = O(L^{-r(m+n+l-1)})$$

$$D_0 = O(L^{-r(m+n+l-1)})$$

$$E_0 = O(L^{-r(m+n+l-1)})$$

lemma 4.4.7

As $L \to \infty$ for i, j, k≥ 0, then

$w_L^i [1 - w_L]^j L^k$

$$= \begin{cases} \gamma_1^j L^{k-j(m+n-l)}(1 + O^*(L^{-(m+n-l)}) + jO^*(L^{-2})) & \text{if } l < m+m \\ \gamma_1^j (1+j)^{-(i+j)} L^k (1 + O^*(L^{-2})) & \text{if } l = m+m \\ \gamma_1^{-i} L^{k-i[l-(m+n)]}(1 + O^*(L^{-[l-(m+n)]}) + iO^*(L^{-2})) & \text{if } l > m+m \end{cases} \tag{4.4.12}$$

where

$$\gamma_1 = \frac{K_{m-1}(\frac{s_1}{m})^{m/2-1}(m-1)^{m/2} K_{n-1}(\frac{s_2}{n})^{n/2-1}(n-1)^{n/2}}{k_{l-1}(l-1)^{l/2}\tau^{l-1}}$$

Proof:

By defined w_L and notice that $f_\nu(L) = K_{\nu-1}\nu^{\frac{\nu}{2}}(1 + \nu t^2)^{-\frac{\nu}{2}} t^{-\nu}$ we have

$$\begin{aligned} w_L^i (1 - w_L)^j L^k &= (\frac{f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n})}{C_{(x,y)} p(\delta_{(x,y)})})^j [1 + \frac{f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n})}{C_{(x,y)} p(\delta_{(x,y)})}]^{-(i+j)} L^k \\ &= \frac{\gamma_1^j L^{k-j(m+n-l)}(1 + jO^*(L^{-2}))}{[1 + \gamma_1 L^{-(m+n-l)}(1 + O^*(L^{-2}))]^{i+j}} \end{aligned}$$

if $l = m + n$ then

$$
\begin{aligned}
w_L^i(1-w_L)^j L^k &= \frac{\gamma_1^j L^k(1+jO^\star(L^{-2}))}{[1+\gamma_1(1+O^\star(L^{-2}))]^{i+j}} \\
&= \frac{\gamma_1^j(1+\gamma_1)^{-(i+j)}L^k(1+jO^\star(L^{-2}))}{[1+\frac{\gamma_1}{1+\gamma_1}O^\star(L^{-2}))]^{i+j}} \\
&= \gamma_1^j(1+\gamma_1)^{-(i+j)}L^k[1+jO^\star(L^{-2})]
\end{aligned}
$$

if $l < m + n$, then

$$
\begin{aligned}
w_L^i(1-w_L)^j L^k &= \frac{\gamma_1^j L^{k-j(m+n-l)}(1+jO^\star(L^{-2}))}{[1+\gamma_1 L^{-(m+n-l)}+O^\star(L^{-2-(m+n-l)}))]^{i+j}} \\
&= \gamma_1^j L^{k-j(m+n-l)}(1+jO^\star(L^{-2}))[1+O^\star(L^{-(m+n-l)})] \\
&= \gamma_1^j L^{k-j(m+n-l)}[1+jO^\star(L^{-2})+O^\star(L^{-(m+n-l)})]
\end{aligned}
$$

if $l > m + n$, then

$$
\begin{aligned}
w_L^i(1-w_L)^j L^k &= \frac{\gamma_1^j L^{k-j(m+n-l)}(1+jO^\star(L^{-2}))}{\gamma_1^{i+j} L^{-(m+n-l)(i+j)}[1+O^\star(L^{-2})+L^{-(l-m-n)})]^{i+j}} \\
&= \gamma_1^{-i} L^{k-i(l-m-n)}[1+iO^\star(L^{-2})+O^\star(L^{-(l-m-n)})].
\end{aligned}
$$

lemma 4.4.8

For fixed $d > 0$, when $L \to \infty$,

$$
m_{(x,y)} = [C_{(x,y)}^{-1} f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n}) + p(\delta_{(x,y)})]O^\star(1) \tag{4.4.13}
$$

Proof:

By the definition of Intervals R_j Integration,

$$m_{(x,y)} = A_0 + B_0 + C_0 + D_0 + E_0.$$

It is easy to see

$$f_{m-1}(\frac{L}{s_1}\sqrt{m}) = O^*(L^{-m}),$$
$$f_{n-1}(\frac{L}{s_2}\sqrt{n}) = O^*(L^{-n}),$$

and

$$p(\delta_{(x,y)}) = O^*(L^{-l}),$$

then

$$\begin{aligned} \frac{C_0+D_0+E_0}{A_0+B_0} &\le \frac{O(L^{-r(m+n+l-1)})}{[C^{-1}_{(x,y)} f_{m-1}(\frac{L}{s_1}\sqrt{m}) f_{n-1}(\frac{L}{s_2}\sqrt{n}) + p(\delta_{(x,y)})] O^*(1)} \\ &= \frac{O(L^{-r(m+n+l-1)})}{O^*(L^{-(m+n)}) + O^*(L^{-l})} \\ &= o(1) \end{aligned}$$

So

$$\begin{aligned} m_{(x,y)} = (A_0+B_0)(1+o(1)) &= [C^{-1}_{(x,y)} f_{m-1}(\tfrac{L}{s_1}\sqrt{m}) f_{n-1}(\tfrac{L}{s_2}\sqrt{n}) + p(\delta_{(x,y)})] O^*(1) \\ &= O^*(L^{-(m+n)}) + O^*(L^{-l}) \end{aligned} \tag{4.4.14}$$

Appendix

<A-1>: Proof of Lemma 4.4.6

Proof of (i)

1) By the Lemma 4.2.1.4, for i=0, 1, 2

$$
\begin{aligned}
A_i &= C^{-1}_{(x,y)} p(\delta_{(x,y)}) O^*(1) [\textstyle\int_{R_1} (\theta - \delta_{(x,y)})^i l(\theta|\tilde{x}, \tilde{y}) d\theta \\
&- O^*(L^{-1}) \textstyle\int_{R_1} (\theta - \delta_{(x,y)})^{i+1} l(\theta|\tilde{x}, \tilde{y}) d\theta \\
&+ O^*(L^{-2}) \textstyle\int_{R_1} (\theta - \delta_{(x,y)})^{i+2} l(\theta|\tilde{x}, \tilde{y}) d\theta]
\end{aligned}
$$

By the lemma 4.4.3

$$
\begin{aligned}
a_0 &= C^{-1}_{(x,y)} [C_{(x,y)} - O^*(L^{-r(m+n-1)}) - O^*(L^{-1}) O^*(L^{-r(m+n-2)}) \\
&+ O^*(L^{-2}) (C_{(x,y)} V_{(x,y)} - O^*(L^{-r(m+n-3)}))] \\
&= 1 - C^{-1}_{(x,y)} O^*(L^{-r(m+n-1)}) - C^{-1}_{(x,y)} O^*(L^{-1-r(m+n-2)}) + O^*(L^{-2}) V_{(x,y)} \\
&- O^*(L^{-2-r(m+n-3)}) \\
&= 1 - C^{-1}_{(x,y)} O^*(L^{-r(m+n-1)}) + O^*(L^{-2}) V_{(x,y)}
\end{aligned}
$$

$$
\begin{aligned}
a_1 &= C_{(x,y)}^{-1} O^{\star}(L^{-r(m+n-2)}) - C_{(x,y)}^{-1} O^{\star}(L^{-1})(C_{(x,y)} V_{(x,y)} - O^{\star}(L^{-r(m+n-3)})) \\
&+ C_{(x,y)}^{-1} O^{\star}(L^{-2}) \begin{cases} O^{\star}(lnL^{r}) & \text{if } m+n=4 \\ C_{(x,y)} E(\theta - \delta_{(x,y)})^3 - O(L^{-r(m+n-4)}) & \text{if } m+n \geq 5 \end{cases} \\
&= C_{(x,y)}^{-1} O^{\star}(L^{-r(m+n-2)}) - O^{\star}(L^{-1}) V_{(x,y)}
\end{aligned}
$$

$$
\begin{aligned}
a_2 &= V_{(x,y)} - C_{(x,y)}^{-1} O^{\star}(L^{-r(m+n-3)}) \\
&- C_{(x,y)}^{-1} O^{\star}(L^{-1}) \begin{cases} E(\theta - \delta_{(x,y)})^3 C_{(x,y)} - O^{\star}(L^{-r(m+n-4)}) & \text{if } m+n \geq 5 \\ O^{\star}(lnL^{r}) & \text{if } m+n=4 \end{cases} \\
&+ C_{(x,y)}^{-1} O^{\star}(L^{-2}) \begin{cases} O^{\star}(L^{r}) & \text{if } m+n=4 \\ O^{\star}(lnL^{r}) & \text{if } m+n=5 \\ C_{(x,y)} E(\theta - \delta_{(x,y)})^4 - O^{\star}(L^{-r(m+n-5)}) & \text{if } m+n>5 \end{cases}
\end{aligned}
$$

proof of (ii):

By the lemma 4.4.5 for i=0, 1, 2,

$$\begin{aligned} B_i &= C^{-1}_{(x,y)} \int_{R_2} (\theta-\mu)^i f_{m-1}(\tfrac{L}{s_1}\sqrt{m}) f_{n-1}(\tfrac{L}{s_1}\sqrt{n}) \\ &\times (1-(\theta-\mu)O^*(L^{-1})+(\theta-\mu)^2 O^*(L^{-2}))O^*(1)p(\theta)d\theta \end{aligned}$$

set

$$b_i = \int_{R_2} (\theta-\mu)^i [1-(\theta-\mu)O^*(L^{-1})+(\theta-\mu)^2 O^*(L^{-2})]p(\theta)d\theta$$

then by Fan and Berger(1991), (4.4.11) is obtained.

proof of (iii)

Set

$$\eta = \theta - \delta_{(x,y)}$$

then

$$\begin{aligned} |C_i| &= |C^{-1}_{(x,y)} \int_{-\infty}^{-L^r} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})p(\eta+\delta_{(x,y)})d\eta| \\ &\leq C^{-1}_{(x,y)} \int_{L^r}^{\infty} \eta^i l(-\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})p(-\eta+\delta_{(x,y)})d\eta \end{aligned}$$

by lemma 4.4.1, definition γ_0 and $i-n-l<0$,

$$\begin{aligned} \text{rightmost} &\leq \gamma_0 \int_{L^r}^{\infty} \eta^i (\eta-d)^{-m} \eta^{-n-l} d\eta \\ &\leq \gamma_0 \int_{L^r}^{\infty} (\eta-d)^{i-m-n-l} d\eta \\ &= \gamma_0 \frac{(L^r-d)^{i-m-n-l+1}}{m+n+l-i-1} \\ &= O^*(L^{-r[(m+n+l-1)-i]}) \end{aligned}$$

proof of (iv)

Set $\eta = \theta - \delta_{(x,y)}$. Before proving d), we first see two facts:

fact (1)

$$[1 + \frac{(\eta + \delta_{(x,y)} - \mu)^2}{(l-1)\tau^2}]^{-l/2} \le O^*(L^{-rl}) \quad \text{for } \forall \eta \in (L^r, \mu - L^r - \delta_{(x,y)}).$$

It is easy to see

$$\eta - (\mu - \delta_{(x,y)}) \in (L^r - \mu + \delta_{(x,y)}, -L^r)$$

which implies $1 + \frac{(\eta-\mu+\delta_{(x,y)})^2}{(l-1)\tau^2} > 1 + \frac{L^{2r}}{(l-1)\tau^2}$

then

$$(1 + \frac{(\eta - \mu + \delta_{(x,y)})^2}{(l-1)\tau^2})^{-l/2} < (1 + \frac{L^{2r}}{(l-1)\tau^2})^{-l/2} = O^*(L^{-rl})$$

fact (2)

$$\frac{(\mu - L^r - \delta_{(x,y)} - d)^{i-(m+n)+1} - (L^r - d)^{i-(m+n)+1}}{L^{-r(m+n-1-i)}} = O^*(1)$$

indeed,

$$\begin{aligned} \frac{(\mu-L^r-\delta_{(x,y)}-d)^{i-(m+n)+1}}{L^{-r(m+n-1-i)}} &= \frac{L^{i-(m+n)+1}}{L^{-r(m+n-1-i)}}O^*(1) \\ &= O^*(L^{(r-1)(m+n-1-i)}) \\ &< O^*(1) \end{aligned}$$

and

$$\frac{(L^r - d)^{i-(m+n)+1}}{L^{-r(m+n-1-i)}} = (\frac{L^r - d}{L^r})^{1-(m+n)+i} = O^*(1)$$

Now

$$
\begin{aligned}
|D_i| &= |C^{-1}_{(x,y)} \int_{L^r}^{\mu-L^r-\delta_{(x,y)}} \eta^i l(\eta+\delta_{(x,y)}|\tilde{x},\tilde{y})p(\eta+\delta_{(x,y)})d\eta| \\
&\leq \gamma_0 \int_{L^r}^{\mu-L^r-\delta_{(x,y)}} \eta^{i-m}(\eta-d)^{-n} O^\star(L^{-rl})d\eta \\
&\leq \tfrac{\gamma_0 O^\star(L^{-rl})}{i-(m+n)+1}(\eta-d)^{i-(m+n)+1}\Big|_{L^r}^{\mu-L^r-\delta_{(x,y)}} \\
&= \tfrac{\gamma_0 O^\star(L^{-rl})}{i-(m+n)+1}[(\mu-L^r-\delta_{(x,y)}-d)^{i-(m+n)+1}-(L^r-d)^{i-(m+n)+1}] \\
&= O^\star(L^{-r(m+n+l-1-i)})
\end{aligned}
$$

proof of (v)

Set $\eta=\theta-\mu$, then it can be proved as the same way as of c). Note that

$$m+n+l-1 \geq 2+2+2-1=5>i, \quad i=0,1,2.$$

<A-2>: Proof of theorem 4.3.2

Without loss of generality, set $C_{(x,y)}=1$.

set

$$A^* \stackrel{\text{def}}{=} m_{(x,y)}^{-1} \int_{R_1} (\theta - \delta_{(x,y)} w_L - \mu + \mu w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

$$B^* \stackrel{\text{def}}{=} m_{(x,y)}^{-1} \int_{R_2} (\theta - \delta_{(x,y)} w_L - \mu + \mu w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

$$C^* \stackrel{\text{def}}{=} m_{(x,y)}^{-1} \int_{R_3} (\theta - \delta_{(x,y)} w_L - \mu + \mu w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

then

$$|\delta^{\pi}_{(x,y)} - \hat{\delta}^{\pi}_{(x,y)}| = A^* + B^* + C^*$$

step 1) consider C^* first:

$$\int_{-\infty}^{\delta_{(x,y)}-L^r} (\theta - \delta_{(x,y)} w_L + \mu + w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

$$= C_1 + (\delta_{(x,y)} - \mu)(1 - w_L) C_0$$

and

$$\int_{\delta_{(x,y)}+L^r}^{\mu-L^r} (\theta - \delta_{(x,y)} w_L + \mu + \mu w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

$$= D_1 + (\delta_{(x,y)} - \mu)(1 - w_L) D_0$$

and

$$\int_{\mu+L^r}^{\infty} (\theta - \delta_{(x,y)} w_L + \mu + \mu w_L) l(\theta|\tilde{x},\tilde{y}) p(\theta) d\theta$$

$$= E_1 - (\delta_{(x,y)} - \mu) w_L E_0$$

then we get

$$\begin{aligned} m_{(x,y)}|C^*| &= |C_1 + D_1 + E_1 + (\delta_{(x,y)} - \mu)[(1 - w_L)(C_0 + D_0) - w_L E_0]| \\ &\leq |C_1| + |D_1| + |E_1| + (\mu - \delta_{(x,y)})[(1 - w_L)(|C_0| + |D_0|) + w_L |E_0|] \end{aligned}$$

Using lemma 4.4.6 and lemma 4.4.7,

- if $m+n=l$

$$\begin{aligned} m_{(x,y)}|C^*| &\leq O^*(L^{-r(m+n+l-2)}) + O^*(L)O^*(L^{-r(m+n+l-1)}) \\ &= O^*(L^{1-r(m+n+l-1)}) \end{aligned} \qquad (A-2.1)$$

- if $m+n>l$

$$\begin{aligned} m_{(x,y)}|C^*| &\leq O^*(L^{-r(m+n+l-2)}) + O^*(L)O^*(L^{-r(m+n+l-1)})(O^*(L^{-r(m+n-l)} + O^*(1)) \\ &= O^*(L^{1-r(m+n+l-1)}) \end{aligned} \qquad (A-2.2)$$

- if $m+n<l$

$$m_{(x,y)}|C^*| \leq O^*(L^{1-r(m+n+l-1)}) \qquad (A-2.3)$$

By (A-2.1), (A-2.2) and (A-2.3), and lemma 4.4.8,

$$\begin{aligned} |C^*| &\leq m_{(x,y)}^{-1} O^*(L^{1-r(m+n+l-1)}) \\ &\leq O^*(L^{1+min(m+n,l)-r(m+n+l-1)}) \\ &= o(1) \end{aligned}$$

step 2) consider A^* and B^*,

$$\begin{aligned}
|A^* + B^*| &= m_{(x,y)}^{-1}|A_1 + (\delta_{(x,y)} - \mu)(1 - w_L)A_0 \\
&+ B_1 - (\delta_{(x,y)} - \mu)w_L B_0| \\
&= O^*(1)|a_1 w_L + a_0 w_L(\delta_{(x,y)} - \mu)(1 - w_L) \\
&+ (1 - w_L)b_1 - (1 - w_L)(\delta_{(x,y)} - \mu)w_L b_0| \\
&= O^*(1)|a_1 w_L + (1 - w_L)b_1 \\
&+ (1 - w_L)(\delta_{(x,y)} - \mu)w_L(a_0 - b_0)|
\end{aligned}$$

- if $l = m + n$, this implies $l > 3$,

$$a_0 - b_0 = O^*(L^{-2}) - O^*(L^{-r(l-1)})$$

then using (4.4.12), note $l \geq 4$,

$$\begin{aligned}
|A^* + B^*| &= O^*(L^{-r(m+n-2)}) + O^*(L^{-1}) - O^*(L^{1-r(l-1)}) \\
&= o(1)
\end{aligned} \qquad (A-2.4)$$

- if $l > m + n$, similarly

$$|A^* + B^*| = o(1) \qquad (A-2.5)$$

• if $l < m+n$, then $a_0 - b_0 = o(1)$ and

$$
\begin{aligned}
|A^* + B^*| &= O^*(L^{-r(m+n-2)}) - O^*(L^{-1}) + O^*(L^{-(m+n-l)})O^*(L) \\
&+ O^*(L^{-(m+n-l)}) \begin{cases} O(L^{-1}) & \text{if } l > 3 \\ O(L^{-1} ln(L)) & \text{if } l = 3 \\ O(L^{r-1}) & \text{if } l = 2 \end{cases} \qquad (A-2.6) \\
&= o(1)
\end{aligned}
$$

By (A-2.4), (A-2.5) and (A-2.6),

$$A^* + B^* = o(1).$$

step 3) By step 1) and 2), theorem 4.3.2 is proved.

<A-3>: Proof of theorem 4.3.4

Without lose generality, set $C_{(x,y)} = 1$, $\delta_{(x,y)} = 0$.

step 1) To prove

$$
\Psi(L) = \begin{cases} O^*(1) + O^*(L^{2-(m+n-l)}) & \text{if} l < m+n \\ O^*(1) + O^*(L^2) & \text{if } l = m+n \\ O^*(1) + O^*(L^{2-(l-m-n)}) & \text{if } l > m+n \end{cases}
$$

By (4.4.12)

• if $l = m + n$,

$$\Psi(L) = O^\star(1)[a_2 + b_2 + L^2] = O^\star(L^2) + O^\star(1)$$

• if $l < m + n$, then

$$w_L = O^\star(1), \quad 1 - w_L = O^\star(L^{-(m+n-l)})$$

and

$$\Psi(L) = O^\star(1) + O^\star(L^{2-(m+n-l)})$$

• if $l > m + n$, which implies $l > 3$, then

$$w_L = L^{-(l-m-n)}, \quad 1 - w_L = O^\star(1)$$

and

$$\Psi(L) = O^\star(1) + O^\star(L^{2-(l-m-n)})$$

step 2) By theorem 4.3.2,

$$V^\pi_{(x,y)} = E^\pi(\theta - \hat{\delta}^\pi_{(x,y)})^2 - o(1)$$

and

$$E^\pi(\theta - \hat{\delta}^\pi_{(x,y)})^2 = m^{-1}_{(x,y)}[\int_{R_1} + \int_{R_2} + \int_{R_3}](\theta - \mu(1 - w_L))^2 l(\theta|\tilde{x}, \tilde{y})p(\theta)d\theta$$

step 3) Similar as C^* obtained in <A-2>

$$
\begin{aligned}
|C_*| &\triangleq m_{(x,y)}^{-1}|[C_2 - 2(1-w_L)\mu C_1 + (1-w_L)^2\mu^2 C_0 \\
&+ D_2 - 2(1-w_L)\mu D_1 + (1-w_L)^2\mu^2 D_0 \\
&+ E_2 + 2w_L\mu E_1 + w_L^2\mu^2 E_0]| \\
&= m_{(x,y)}^{-1}|C_2 + D_2 + E_2 + O^*(L)[-(1-w_L)(C_1+D_1) + w_L E_1] \\
&+ O^*(L^2)[(1-w_L)^2(C_0+D_0) + w_L^2 E_0]|
\end{aligned}
$$

By the lemma 4.4.8 and lemma 4.4.6, we have

$$
C_* \le O^*(L^{min(m+n,l)})|O^*(L^{-r[(m+n+l-1)-2]}) + O^*(L)[(1-w_L)O^*(L^{-r[(m+n+l-1)-1]})
$$
$$
+w_L O^*(L^{-r[(m+n+l-1)-1]})] + O^*(L^2)[(1-w_L)^2 O^*(L^{-r(m+n+l-1)}) + w_L^2 O^*(L^{-r(m+n+l-1)})]|
$$

- if $l = m+n$ then by definition of r and w_L we get

$$
\begin{aligned}
|C_*| &\le O^*(L^{min(m+n,l)})[O^*(L^{-r[(m+n+l-1)-2]}) + O^*(L^{1-r[(m+n+l-1)-1]}) + O^*(L^{2-r(m+n+l-1)})] \\
&= O^*(L^{min(m+n,l)+2-r(m+n+l-1)}) \\
&= o(1)
\end{aligned}
$$

- for $l < m+n$ and $l > m+n$ we similarly get

$$
|C_*| \le o(1)
$$

step 4) Similarly, we get

$$
\begin{aligned}
A_* + B_* &\triangleq m^{-1}_{(x,y)}[\textstyle\int_{R_1} + \int_{R_2}](\theta - \mu(1-w_L))^2 l(\theta|\tilde{x},\tilde{y})p(\theta)d\theta \\
&= m^{-1}_{(x,y)}[A_2 - 2(1-w_L)\mu A_1 + (1-w_L)^2\mu^2 A_0 + B_2 \\
&+ 2w_L\mu B_1 + w_L^2\mu^2 B_0] \\
&= O^\star(1)\{w_L a_2 + (1-w_L)b_2 + w_L(1-w_L)\mu^2 + \mu(1-w_L)w_L(b_1 - a_1) \\
&+ \mu^2[(1-w_L)^2 w_L a_0 + w_L^2(1-w_L)b_0] - w_L(1-w_L)\mu^2\} \\
&= O^\star(1)\{\Psi(L) + O^\star(L)(1-w_L)w_L(b_1 - a_1) \\
&+ O^\star(L^2)(1-w_L)w_L[(1-w_L)a_0 + w_L b_0 - 1]\}
\end{aligned}
$$

Let

$$\Delta \stackrel{\text{def}}{=} O^\star(L)(1-w_L)w_L(b_1 - a_1) + O^\star(L^2)(1-w_L)w_L[(1-w_L)a_0 + w_L b_0 - 1]$$

• if $l > m+n$, by (4.4.10) and (4.4.11)

$$
\begin{aligned}
(1-w_L)a_0 + w_L b_0 - 1 &= (1-w_L)O^\star(L^{-r(m+n-1)}) + (1-w_L)O^\star(L^{-2}) \\
&+ w_L O^\star(L^{-r(l-1)}) + w_L O^\star(L^{-2}) \\
&= O^\star(L^{-r(m+n-1)}) + O^\star(L^{-2}) + O^\star(L^{-(l-m-n)-r(l-1)})
\end{aligned}
$$

then

$$
\begin{aligned}
\Delta &= O^\star(L^{1-(l-m-n)})[O^\star(L^{-1}) - O^\star(L^{-r(m+n-2)})] \\
&+ O^\star(L^{2-(l-m-n)})[O^\star(L^{-2}) - O^\star(L^{-r(m+n-1)}) + O^\star(L^{-(l-m-n)-r(l-1)})] \\
&= o(1)
\end{aligned}
$$

• if $l = m + n$

$$\begin{aligned}(1-w_L)a_0 + w_L b_0 - 1 &= (1-w_L)O^\star(L^{-r(m+n-1)}) + (1-w_L)O^\star(L^{-2}) \\ &+ w_L O^\star(L^{-r(l-1)}) + w_L O^\star(L^{-2}) \\ &= O^\star(L^{-r(l-1)}) + O^\star(L^{-2})\end{aligned}$$

then

$$\begin{aligned}\Delta &= O^\star(L)[O^\star(L^{-1}) - O^\star(L^{-r(m+n-2)})] + O^\star(L^2)[O^\star(L^{-r(l-1)}) + O^\star(L^2)] \\ &= O^\star(1)\end{aligned}$$

• if $l < m + n$, the discussion is similar as of $l > m + n$, we get $\Delta = o(1)$.

So we get

$$A_* + B_* = \begin{cases} O^\star(1)\{\Psi + O^\star(1)\} & \text{if } i = m + n \\ O^\star(1)\{\Psi + o(1)\} & \text{if } i \neq m + n \end{cases}$$

that means

$$A_* + B_* + C_* = O^\star(1)\Psi(L)\begin{cases} O^\star(1) & \text{if } l = m + n \\ o(1) & \text{if } l \neq m + n \end{cases}$$

So Theorem(4.3.4) is proved.

<A-4>: Proof of theorem 4.3.5

Without loss of generality, set $\delta_{(x,y)} = 0$, and $C_{(x,y)} = 1$

The proof is divided into two parts, the first part is to prove the limit of supremum of $V^{\pi}_{(x,y)}$ great than $\frac{L^2}{4}$; the second part is to prove the inverse.

Part i) To prove

$$lim_{L\to\infty} Sup_{\tau} \frac{V^{\pi}_{(x,y)}}{L^2} \geq \frac{1}{4}$$

Choose

$$\tau_* = K_0 \mu^{\frac{m+n-l}{l-l}}$$

Define

$$p_{\tau_*}(\theta) = \frac{K_{l-1}}{\tau_*}(1 + \frac{(\theta-\mu)^2}{(l-1)\tau_*^2})^{-l/2}$$

$$K_0 = [\frac{K_{l-1}(l-1)^{l/2}}{K_{m-1}K_{n-1}(m-1)^{m/2}(n-1)^{n/2}}(\frac{\sqrt{m}}{s_1})^{m-1}(\frac{\sqrt{n}}{s_1})^{n-1}]^{\frac{1}{1-l}}$$

$$k_0 = (m-1)^{m/2}(n-1)^{n/2}(\frac{s_1^2}{m})^{(m-1)/2}(\frac{s_1^2}{n})^{(n-1)/2}$$

then

$$K_0^{l-1} = k_0 \frac{K_{m-1}K_{n-1}}{K_{l-1}(l-1)^{l/2}}$$

$$K_{l-1}(l-1)^{l/2} K_0 = k_0 K_{m-1} K_{n-1}$$

Set

$$\epsilon = \begin{cases} \frac{1}{3} & \text{if } l-(m+n) \leq 2 \\ \frac{1-(m+n)}{1-l} & \text{if } l-(m+n) > 2 \end{cases}$$

if $\quad l-(m+n)\leq 0, \qquad$ then

$$\frac{m+n-l}{1-l}\leq 0\leq \frac{2}{3}, \text{ and } \tau_*\leq K_0\mu^{\frac{2}{3}}=K_0\mu^{1-\epsilon};$$

if $\quad 0<l-(m+n)\leq 2, \quad$ then

$$0<\frac{m+n-l}{1-l}<\frac{2}{3}, \text{ and } \tau_*\leq K_0\mu^{1-\epsilon};$$

if $\quad l-(m+n)>2, \qquad$ then

$$1-\epsilon=\frac{m+n-l}{1-l}>0, \text{ and} \tau_*=K_0\mu^{1-\epsilon}$$

So,

$$\frac{\tau_*}{\mu^{1-\epsilon}}=O(1)$$

let

$$A_i^* \stackrel{\text{def}}{=} \int_{|\theta|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta$$

$$B_i^* \stackrel{\text{def}}{=} \int_{|\theta-\mu|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta$$

$$C_i^* \stackrel{\text{def}}{=} \int_{[\{|\theta|\leq\mu^{1-\frac{1}{2}\epsilon}\}\cup\{|\theta-\mu|\leq\mu^{1-\frac{1}{2}\epsilon}\}]^c} \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta$$

then

$$\int_\Theta \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta = A_i^*+B_i^*+C_i^*$$

step 1) To prove

$$A_i^* = K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}(E\theta^i + o(1))$$
$$= k_0 K_{m-1}K_{n-1}\mu^{-(m+n)}(E\theta^i + o(1)) \qquad (A-4.1)$$

Obviously, for $i = 0, 1, 2$

$$0 \leq |\int_{|\theta|>\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x}, \tilde{y})d\theta| = o(1) \qquad (A-4.2)$$

we just prove case $i = 2$.

$$\int_{\theta>\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x}, \tilde{y})d\theta \leq c_{mn}c_m^{-m/2}c_n^{-n/2}\int_{\theta>\mu^{1-\frac{1}{2}\epsilon}}[(\theta-\bar{y})^2 + 2\bar{y}(\theta-\bar{y}) + \bar{y}^2](\theta-\bar{y})^{-m-n}d\theta$$
$$\leq o(1) + c_{mn}c_m^{-m/2}c_n^{-n/2}\int_{\theta>\mu^{1-\frac{1}{2}\epsilon}}(\theta-\bar{y})^{2-m-n}d\theta$$
$$= O^*(\mu^{(1-\epsilon/2)(3-m-n)})$$
$$= o(1)$$

The expectation of samples likelihood function could be written in following form:

$$E\theta^i = \begin{cases} 1 & \text{if } i = 0 \\ \delta_{(x,y)} = 0 & \text{if } i = 1 \\ V_{(x,y)} + \delta_{(x,y)} = V_{(x,y)} & \text{if } i = 2 \end{cases} \qquad (A-4.3)$$

By (A-4.2)

$$\int_{|\theta|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x}, \tilde{y})d\theta = [\int_\Theta - \int_{|\theta|>\mu^{1-\frac{1}{2}\epsilon}}]\theta^i l(\theta|\tilde{x}, \tilde{y})d\theta$$
$$= E\theta^i + o(1)$$

Now for $i = 0, 2$

$$A_i^* = \int_{|\theta|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x}, \tilde{y})p_{\tau_*}(\theta)d\theta$$
$$= \frac{K_{l-1}}{\tau_*}\int_{|\theta|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^i l(\theta|\tilde{x}, \tilde{y})(1 + \frac{(\theta-\mu)^2}{(l-1)\tau_*})^{-l/2}d\theta \qquad (A-4.4)$$

Note $|\theta| \leq \mu^{1-\frac{1}{2}\epsilon}$, from (A-4.4),we can get following inequalities:

$$\frac{K_{l-1}}{\tau_*}(1+\frac{(\mu+\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2})^{-l/2}[E\theta^i+o(1)] \leq A_i^* \leq \frac{K_{l-1}}{\tau_*}(1+\frac{(\mu-\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2})^{-l/2}[E\theta^i+o(1)]$$

where

$$\tau_*^{l-1}\mu^{-l} = K_0^{l-1}\mu^{\frac{-l+m+n}{1-l}(l-1)-l} = K_0^{l-1}\mu^{-(m+n)}$$

$$K_0^{l-1}K_{l-1}(l-1)^{l/2} = k_0 K_{m-1}K_{n-1}$$

since

$$\begin{aligned}
\frac{K_{l-1}}{\tau_*}(1+\frac{(\mu\pm\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2})^{-l/2} &= K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}[\frac{(l-1)\tau_*^2}{\mu^2}+(1\pm\mu^{-\epsilon/2})^2]^{-l/2} \\
&= K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}O^*(1)
\end{aligned}$$

then

$$A_i^* = K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}(E\theta^i+o(1))O^*(1) \qquad (A-4.5)$$

For case $i=1$,

$$A_1^* = [\int_0^{\mu^{1-\epsilon/2}} \theta l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta - \int_0^{\mu^{1-\epsilon/2}} \theta l(-\theta|\tilde{x},\tilde{y})p_{\tau_*}(-\theta)d\theta]$$

then

$$
\begin{aligned}
0 \le |A_1^*| &\le |\tfrac{K_{l-1}}{\tau_*}[(1+\tfrac{(\mu-\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2})^{-l/2}\textstyle\int_0^{\mu^{1-\epsilon/2}}\theta l(\theta|\tilde{x},\tilde{y})d\theta \\
&- (1+\tfrac{(\mu+\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2})^{-l/2}\textstyle\int_0^{\mu^{1-\epsilon/2}}\theta l(-\theta|\tilde{x},\tilde{y})d\theta]| \\
&\le K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}|(1+o(1))\textstyle\int_0^{\mu^{1-\epsilon/2}}\theta l(\theta|\tilde{x},\tilde{y})d\theta \qquad (A-4.6)\\
&- (1-o(1))\textstyle\int_0^{\mu^{1-\epsilon/2}}\theta l(-\theta|\tilde{x},\tilde{y})d\theta| \\
&= K_{l-1}(l-1)^{l/2}\tau_*^{l-1}\mu^{-l}o(1)
\end{aligned}
$$

where, by Lemma 2.1, $|\int_0^{\mu^{1-\epsilon/2}}\theta l(\pm\theta|\tilde{x},\tilde{y})d\theta| < \infty$ implies

$$o(1)\int_0^{\mu^{1-\epsilon/2}}\theta l(\pm\theta|\tilde{x},\tilde{y})d\theta = o(1)$$

and by $E\theta = \delta_{(x,y)} = 0$,

$$\int_0^{\mu^{1-\epsilon/2}}\theta l(\theta|\tilde{x},\tilde{y})d\theta + \int_{-\mu^{1-\epsilon/2}}^{0}\theta l(\theta|\tilde{x},\tilde{y})d\theta = o(1)$$

From (A-4.3), (A-4.5) and (A-4.6), we get (A-4.1).

step 2) To prove

$$B_i^* = k_0 K_{m-1}K_{n-1}\mu^{i-m-n}(1+o(1)) \qquad (A-4.7)$$

For $i=0$

$$
\begin{aligned}
\int_{|\theta-\mu|>\mu^{1-\frac{1}{2}\epsilon}} p_{\tau_*}(\theta)d\theta &\overset{t=\frac{\theta-\mu}{\tau_*}}{=\!=\!=} \int_{|t|>\frac{\mu^{1-\epsilon/2}}{\tau_*}} K_{l-1}(1+\tfrac{t^2}{l-1})^{-l/2}dt \\
&= o(1)
\end{aligned}
$$

because of $\frac{\mu^{1-\epsilon/2}}{\tau_*} = \frac{\mu^{\epsilon/2}}{O^*(1)} \geq O^*(1)\mu^{\epsilon/2}$

then

$$\int_{|\theta-\mu|\leq\mu^{1-\frac{1}{2}\epsilon}} p_{\tau_*}(\theta)d\theta = 1 - o(1) \qquad (A-4.8)$$

For $i = 1$

Since $p_{\tau_*}(\theta)$ is the symmetric of function with respect to μ,

$$\int_{|\theta-\mu|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta p_{\tau_*}(\theta)d\theta = \mu$$

For $i = 2$

set $t = \frac{\theta-\mu}{\tau_*}$, expand $(\mu + t\tau_*)^2$, note

$$\int_{|t|\leq\frac{\mu^{1-\epsilon/2}}{\tau_*}} 2t\tau_*\mu K_{l-1}(1+\frac{t^2}{l-1})^{-l/2}dt = 0,$$

$$\begin{aligned}
\int_{|\theta-\mu|\leq\mu^{1-\frac{1}{2}\epsilon}} \theta^2 p_{\tau_*}(\theta)d\theta &= \mu^2(1-o(1)) + K_{l-1}\tau_*^2(l-1)\int_{|t|\leq\frac{\mu^{1-\epsilon/2}}{\tau_*}} \tfrac{t^2}{l-1}(1+\tfrac{t^2}{l-1})^{-l/2}dt \\
&= \mu^2(1-o(1)) - K_{l-1}\tau_*^2(l-1)\int_{|t|\leq\frac{\mu^{1-\epsilon/2}}{\tau_*}}(1+\tfrac{t^2}{l-1})^{-l/2}dt \\
&+ K_{l-1}\tau_*^2(l-1)\int_{|t|\leq\frac{\mu^{1-\epsilon/2}}{\tau_*}}(1+\tfrac{t^2}{l-1})^{-\frac{l}{2}+1}dt \\
&\leq [\mu^2 - K_{l-1}\tau_*^2(l-1)](1-o(1)) + K_{l-1}\tau_*^2(l-1)\int_{|t|\leq\frac{\mu^{1-\epsilon/2}}{\tau_*}} 1dt \\
&= \mu^2(1-o(1))^2 + 2K_{l-1}(l-1)\tau_*\mu^{1-\epsilon/2} \\
&\leq \mu^2(1+o(1))
\end{aligned}$$

$(A-4.9)$

Combining (A-4.8) and (A-4.9), we obtain

$$\int_{|\theta-\mu|\le\mu^{1-\epsilon/2}} \theta^i p_{T_*}(\theta)d\theta = \mu^i(1+o(1)) \qquad (A-4.10)$$

Now

$$\frac{c_{mn}}{[1+c_m(\mu+\mu^{1-\epsilon/2}-\bar{x})^2]^{m/2}[1+c_n(\mu+\mu^{1-\epsilon/2}-\bar{y})^2]^{n/2}}\int_{|\theta-\mu|\le\mu^{1-\frac{1}{2}\epsilon}} \theta^i p_{T_*}(\theta)d\theta$$

$$\le B_i^* \le$$

$$\frac{c_{mn}}{[1+c_m(\mu-\mu^{1-\epsilon/2}-\bar{x})^2]^{m/2}[1+c_n(\mu-\mu^{1-\epsilon/2}-\bar{y})^2]^{n/2}}\int_{|\theta-\mu|\le\mu^{1-\frac{1}{2}\epsilon}} \theta^i p_{T_*}(\theta)d\theta$$

and

$$\begin{aligned}\frac{c_{mn}}{[1+c_m(\mu\pm\mu^{1-\epsilon/2}-\bar{x})^2]^{m/2}[1+c_n(\mu\pm\mu^{1-\epsilon/2}-\bar{y})^2]^{n/2}} &= c_{mn}c_m^{-m/2}c_n^{-n/2}\mu^{-m-n}\frac{1}{1+o(1)} \\ &= K_{m-1}K_{n-1}k_0\mu^{-m-n}\frac{1}{1+o(1)}\end{aligned} \qquad (A-4.11)$$

(A-4.10), (A-4.11) implies (A-4.7)

step 3) To prove

$$\left[\int_{-\infty}^{-\mu^{1-\epsilon/2}} + \int_{\mu^{1-\epsilon/2}}^{\mu-\mu^{1-\epsilon/2}} + \int_{\mu+\mu^{1-\epsilon/2}}^{\infty}\right]\theta^i l(\theta|\tilde{x},\tilde{y})p_{T_*}(\theta)d\theta = (A_i^*+B_i^*)o(1) \qquad (A-4.12)$$

For i=0,1,2

a)

$$
\begin{aligned}
|\int_{-\infty}^{-\mu^{1-\epsilon/2}} \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta| &\leq \frac{K_{l-1}}{\tau_*}\left(1+\frac{(\mu+\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2}\right)^{-l/2} c_{mn}c_m^{-m/2}c_n^{-n/2}\int_{\mu^{1-\epsilon/2}}^{-\infty}\theta^{i-m-n}d\theta \\
&= \frac{K_{l-1}}{\tau_*}\left(1+\frac{(\mu+\mu^{1-\epsilon/2})^2}{(l-1)\tau_*^2}\right)^{-l/2} c_{mn}c_m^{-m/2}c_n^{-n/2}\frac{\mu^{(1-\epsilon/2)(i-m-n+1)}}{i-m-n+1} \\
&\leq c_1\frac{\mu^i}{\mu^{m+n}}\mu^{\alpha_1}
\end{aligned}
\qquad (A-4.13)
$$

where

$$c_1 \geq K_0^{l-1}\frac{K_{l-1}c_{mn}c_m^{-m/2}c_n^{-n/2}}{i-m-n+1}\left[\frac{\tau_*^2}{\mu^2}+\frac{(1+\mu^{-\epsilon/2})^2}{l-1}\right]^{-l/2}$$

and

$$
\begin{aligned}
\alpha_1 &= -\tfrac{1}{2}\epsilon i-(1-\epsilon/2)(m+n)+1-\epsilon/2 \\
&= -\tfrac{1}{2}\epsilon i-(1-\epsilon/2)(m+n-1) \\
&< 0
\end{aligned}
$$

b)

$$\int_{\mu^{1-\epsilon/2}}^{\mu-\mu^{1-\epsilon/2}} \theta^i l(\theta|\tilde{x},\tilde{y})p_{\tau_*}(\theta)d\theta$$

$$\leq c_{mn}c_m^{-m/2}c_n^{-n/2}h_{\tau_*}(\mu^{1-\epsilon/2}+\mu)\int_{\mu^{1-\epsilon/2}}^{\mu-\mu^{1-\epsilon/2}}(\theta-\bar{y})^{i-m-n}d\theta$$

$$= K_{l-1}\left[\frac{\tau_*^2}{(\mu^{1-\epsilon/2})^2}+(l-1)^{-1}\right]^{-l/2}c_{mn}c_m^{-m/2}c_n^{-n/2}\frac{1}{m+n-i-1}\frac{\tau_*^{l-1}}{(\mu^{1-\epsilon/2})^l}$$

$$\times\mu^{(1-\epsilon/2)(i-m-n+1)}\left[\left(1-\frac{\bar{y}}{\mu^{1-\epsilon/2}}\right)^{i-m-n+1}-\left(\mu^{\epsilon/2}-1-\frac{\bar{y}}{\mu^{1-\epsilon/2}}\right)^{i-m-n+1}\right]$$

$$\leq K_{l-1}[\tfrac{\tau_*^2}{(\mu^{1-\epsilon/2})^2} + (l-1)^{-1}]^{-l/2} c_{mn} c_m^{-m/2} c_n^{-n/2} K_0^{l-1}(1+o(1))$$

$$\times \tfrac{1}{m+n-i-1} \tfrac{\mu^i}{\mu^{m+n}} \mu^{-(m+n-1)(1-\epsilon/2)-\frac{\epsilon}{2}i+\frac{\epsilon}{2}l}$$

$$\leq c_2 \tfrac{\mu^i}{\mu^{m+n}} \mu^{\alpha_2} \qquad (A-4.14)$$

where

$$c_2 \geq K_{l-1}[\frac{\tau_*^2}{(\mu^{1-\epsilon/2})^2} + (l-1)^{-1}]^{-l/2} c_{mn} c_m^{-m/2} c_n^{-n/2} K_0^{l-1}(1+o(1))$$

and

$$\begin{aligned} \alpha_2 &= -(m+n-1)(1-\epsilon/2) - \tfrac{\epsilon}{2}i + \tfrac{\epsilon}{2}l \\ &< 0 \end{aligned}$$

c)

Similar as b), we can get

$$\int_{\mu+\mu^{1-\epsilon/2}}^{\infty} \theta^i l(\theta|\tilde{x},\tilde{y}) p_{\tau_*}(\theta) d\theta \leq c_3 \frac{\mu^i}{\mu^{m+n}} \mu^{\alpha_3} \qquad (A-4.15)$$

where

$$\begin{aligned} c_3 &\geq c_{mn} c_m^{-m/2} c_n^{-n/2} K_{l-1}[\tfrac{\tau_*^2}{(\mu^{1-\epsilon/2})^2} + \tfrac{1}{l-1}]^{-l/2} K_0^{l-1}(1+\mu^{-\epsilon/2} - \tfrac{\bar{y}}{\mu})^{i-m-n+1} \\ &\times (1+o(1)) \\ &> 0 \end{aligned}$$

$$
\begin{aligned}
\alpha_3 &= \tfrac{1}{2}\epsilon + 1 - m - n \\
&< \alpha_2 \\
&< 0
\end{aligned}
$$

Combine (A-4.13), (A-4.14) and (A-4.15),

$$
\begin{aligned}
\frac{[\int_{-\infty}^{-\mu^{1-\epsilon/2}} + \int_{\mu^{1-\epsilon/2}}^{\mu-\mu^{1-\epsilon/2}} + \int_{\mu+\mu^{1-\epsilon/2}}^{\infty}]\theta^i l(\theta|\tilde{x},\tilde{y})p_{T_*}(\theta)d\theta}{A_i+B_i} &\leq \frac{\frac{\mu^i}{\mu^{m+n}}\sum_{i=1}^{3} c_i\mu^{\alpha_i}}{B_i} \\
&= \frac{\sum_{i=1}^{3} c_i\mu^{\alpha_i}}{k_0 K_{m-1} K_{n-1}(1+o(1))} \\
&= o(1)
\end{aligned}
$$

Which implies (A-4.12)

step 4) Using (A-4.12),

$$
\int_{\Theta} \theta^i l(\theta|\tilde{x},\tilde{y})p_{T_*}(\theta)d\theta = (A_i^* + B_i^*)(1 + o(1))
$$

step 5) Now we define $m^*_{(x,y)}$ and $\delta^*_{(x,y)}$ and $V^*_{(x,y)}$ as following formula:

$$
\begin{aligned}
m^*_{(x,y)} &\stackrel{\text{def}}{=} (A_0^* + B_0^*)(1 + o(1)) \\
&= 2K_{m-1}K_{n-1}k_0\mu^{-(m+n)}(1 + o(1))
\end{aligned}
$$

$$
\begin{aligned}
\delta^*_{(x,y)} &\stackrel{\text{def}}{=} \frac{(A_1^*+B_1^*)(1+o(1))}{m^*_{(x,y)}} \\
&= \frac{K_{m-1}K_{n-1}k_0\mu^{-(m+n)}[o(1)+\mu(1+o(1))]}{2K_{m-1}K_{n-1}k_0\mu^{-(m+n)}(1+o(1))} \\
&= \tfrac{\mu}{2}(1+o(1))
\end{aligned}
$$

$$
\begin{aligned}
V^*_{(x,y)} &\stackrel{\text{def}}{=} \frac{(A_2^*+B_2^*)(1+o(1))}{m^*_{(x,y)}} - [\delta^*_{(x,y)}]^2 \\
&= \tfrac{\mu^2}{4}(1+o(1))[\tfrac{2V_{(x,y)}}{\mu^2} + 2(1+o(1)) - 1] \\
&= \tfrac{\mu^2}{4}(1+o(1))
\end{aligned}
$$

Note $\quad L = \mu \quad$ here,

$$
lim_{\mu\to\infty} Sup_\tau \frac{V^\pi_{(x,y)}}{\frac{L^2}{4}} \geq lim_{\mu\to\infty} \frac{V^*_{(x,y)}}{\frac{L}{4}} = 1
$$

Part ii) To prove

$$
lim_{L\to\infty} Sup_\tau \frac{V^\pi_{(x,y)}}{\frac{\mu^2}{4}} \leq 1
$$

Set $\quad I = (-\mu^q, \mu + \mu^q), \quad$ where $0 < q = \frac{l-1}{l} < 1$.

Define

$$
c = [\int_I \pi(\theta|\tilde{x}, \tilde{y})d\theta]^{-1}
$$

$$\pi^*(\theta|\tilde{x},\tilde{y}) = \begin{cases} c\pi(\theta|\tilde{x},\tilde{y}) & \text{if } \theta \in I \\ 0 & \text{if } \theta \in I^c \end{cases}$$

$$E^{\pi^*}\theta^i = \int_\Theta \pi^*(\theta|\tilde{x},\tilde{y})d\theta = c\int_I \theta^i \pi(\theta|\tilde{x},\tilde{y})d\theta$$

$$\delta^{\pi^*}_{(x,y)} = \int_\Theta \theta\pi^*(\theta|\tilde{x},\tilde{y})d\theta$$

$$V^{\pi^*}_{(x,y)} = \int_\Theta \theta^2\pi^*(\theta|\tilde{x},\tilde{y})d\theta - (\int_\Theta \theta\pi^*(\theta|\tilde{x},\tilde{y})d\theta)^2$$

step 1) Note that

$$\pi(\theta|\tilde{x},\tilde{y}) = \frac{(s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta-\mu)^2}{l-1})^{-l/2}}{\int_\Theta(s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta-\mu)^2}{l-1})^{-l/2}d\theta} \qquad (A-4.16)$$

and

$$\int_\Theta(s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta-\mu)^2}{l-1})^{-l/2}d\theta$$

$$\geq (\tau^2 + \frac{\mu^2}{l-1})^{-l/2}\int_0^1(s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}d\theta$$

$$\geq k(\tau^2 + \frac{\mu^2}{l-1})^{-l/2}$$

$$(A-4.17)$$

where k is a positive value just less than $\int_0^1 (s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2} d\theta$

step 2) From Theorem 4.3.2,

$$\hat{\delta}^{\pi}_{(x,y)} = \mu(1 - w_L), \quad 0 \le w_L \le 1, \text{and } |\delta^{\pi}_{(x,y)} - \hat{\delta}^{\pi}_{(x,y)}| = o(1),$$

so when L is large,

$$0 \le \delta^{\pi}_{(x,y)} \le \mu$$

step 3) To prove

$$|\int_{J^c} \theta^i (s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta-\mu)^2}{l-1})^{-l/2} d\theta$$

$$\le (\tau^2 + \frac{\mu^{2q}}{l-1})^{-l/2} \mu^{r(i-m-n+1)} v_i, \qquad \text{for i=1,2}$$

$$(A-4.18)$$

when i=1

$$|\int_{J^c} \theta (s_1^2 + \frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta-\mu)^2}{l-1})^{-l/2} d\theta|$$

$$\le \int_{\mu^q}^{\infty} \theta (s_1^2 + \frac{m(\theta+\bar{x})^2}{m-1})^{-m/2}(s_1^2 + \frac{n(\theta+\bar{y})^2}{n-1})^{-n/2}(\tau^2 + \frac{(\theta+\mu)^2}{l-1})^{-l/2} d\theta$$

$$+\int_{\mu+\mu^q}^{\infty}\theta(s_1^2+\frac{m(\theta-\bar{x})^2}{m-1})^{-m/2}(s_1^2+\frac{n(\theta-\bar{y})^2}{n-1})^{-n/2}(\tau^2+\frac{(\theta-\mu)^2}{l-1})^{-l/2}d\theta$$

$$\leq(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}$$

$$\times\ [(\tau^2+\frac{(\mu^q+\mu)^2}{l-1})^{-l/2}\int_{\mu^q}^{\infty}\theta(\theta+\bar{x})^{-m}(1+\frac{s_1^2(m-1)}{m(\theta+\bar{x})^2})^{-m/2}(\theta+\bar{y})^{-n}(1+\frac{(n-1)s_1^2}{n(\theta+\bar{y})^2})^{-n/2}d\theta$$

$$+(\tau^2+\frac{\mu^{2q}}{l-1})^{-l/2}\int_{\mu+\mu^q}^{\infty}\theta(\theta-\bar{x})^{-m}(1+\frac{s_1^2(m-1)}{m(\theta-\bar{x})^2})^{-m/2}(\theta-\bar{y})^{-n}(1+\frac{(n-1)s_1^2}{n(\theta-\bar{y})^2})^{-n/2}d\theta]$$

$$\leq(\tau^2+\frac{(\mu^q+\mu)^2}{l-1})^{-l/2}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}\int_{\mu^q}^{\infty}\theta^{1-m-n}d\theta$$

$$+(\tau^2+\frac{\mu^{2q}}{l-1})^{-l/2}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}\int_{\mu+\mu^q}^{\infty}(\theta-\bar{y})^{1-m-n}d\theta$$

$$\leq(\tau^2+\frac{\mu^{2q}}{l-1})^{-l/2}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}\frac{\mu^{q(2-m-n)}}{m+n-2}$$

$$+(\tau^2+\frac{\mu^{2q}}{l-1})^{-l/2}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}\frac{(\mu+\mu^q-\bar{y})^{2-m-n}}{m+n-2}$$

$$=(\tau^2+\frac{\mu^{2q}}{l-1})^{-l/2}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}(m+n-2)^{-1}$$

$$\times\ 4[\mu^{q(2-m-n)}+(\mu+\mu^q-\bar{y})^{2-m-n}]$$

$$= (\tau^2 + \tfrac{\mu^{2q}}{l-1})^{-l/2} \mu^{q(2-m-n)} v_1$$

where

$$\begin{aligned} v_1 &= (\tfrac{m-1}{m})^{m/2} (\tfrac{n-1}{n})^{n/2} (m+n-2)^{-1} [1 + (1 + \mu^{1-q} - \bar{y}\mu^{-q})^{2-m-n}] \\ &= (\tfrac{m-1}{m})^{m/2} (\tfrac{n-1}{n})^{n/2} (m+n-2)^{-1} (1 + o(1)) \end{aligned}$$

when i=2, similarly,

$$\int_{I^c} \theta^2 (s_1^2 + \frac{m(\theta - \bar{x})^2}{m-1})^{-m/2} (s_1^2 + \frac{n(\theta - \bar{y})^2}{n-1})^{-n/2} (\tau^2 + \frac{(\theta - \mu)^2}{l-1})^{-l/2} d\theta$$

$$\leq (\tau^2 + \frac{\mu^{2q}}{l-1})^{-l/2} \mu^{q(3-m-n)} v_2$$

where

$$v_2 = (\frac{m-1}{m})^{m/2} (\frac{n-1}{n})^{n/2} (m+n-3)^{-1} (1 + o(1))$$

step 4) From (A-4.16), (A-4.17), (A-4.18), for $i = 1, 2$,

$$\begin{aligned} |\textstyle\int_{I^c} \theta^i \pi(\theta|\tilde{x}, \tilde{y}) d\theta| &\leq \tfrac{v_i \mu^{q(1-m-n+i)}}{k(\tau^2 + \frac{\mu^2}{l-1})^{-l/2}} (\tau^2 + \tfrac{\mu^{2q}}{l-1})^{-l/2} \\ &\leq k_i \mu^{(1-q)l} \mu^{q(1-m-n+i)} && (A-4.19) \\ &= k_i \mu^{1+q(1-m-n+i)} \end{aligned}$$

where $k_i = \frac{v_i}{k}$.

step 5) By (A-4.19), when i=2,

$$\int_{I^c} \theta^2 \pi(\theta|\tilde{x}, \tilde{y}) d\theta \leq p_2 \mu^{l_2}$$

where

$$\begin{aligned} l_2 &= 1 + q(3 - m - n) \\ &\leq -q + 1 \\ &= \tfrac{1}{l} \\ &< 1 \end{aligned}$$

$$p_2 = k_2$$

when i=1, using $0 \leq \delta^{\pi}_{(x,y)} \leq \mu$, and (A-4.19)

$$\begin{aligned} |[\textstyle\int_I \theta\pi(\theta|\tilde{x}, \tilde{y}) d\theta]^2 - (\delta^{\pi}_{(x,y)})^2| &= |\textstyle\int_{I^c} \theta\pi(\theta|\tilde{x}, \tilde{y}) d\theta| \times |2\delta^{\pi}_{(x,y)} - \textstyle\int_{I^c} \theta\pi(\theta|\tilde{x}, \tilde{y}) d\theta| \\ &\leq k_1 \mu^{1+q(2-m-n)} (2\mu + k_1 \mu^{1+q(2-m-n)}) \\ &\leq (2k_1 + k_1^2) \mu^{2+q(2-m-n)} \\ &= p_1 \mu^{l_1} \end{aligned}$$

$$(A - 4.20)$$

where

$$\begin{aligned} l_1 &= 2 + q(2 - m - n) \\ &\leq 2 + \tfrac{l-1}{l}(-2) \\ &= 2/l \\ &\leq 1 \end{aligned}$$

$$p_1 = 2k_1 + k_1^2$$

step 6) Now, from (A-4.16), (A-4.17)

$$\frac{\int_{I^c} \pi(\theta|\bar{x},\bar{y})d\theta}{\int_I \pi(\theta|\bar{x},\bar{y})d\theta} \le \frac{\int_{I^c} \pi(\theta|\bar{x},\bar{y})d\theta}{\int_0^1 \pi(\theta|\bar{x},\bar{y})d\theta}$$

$$\le k^{-1}(\tau^2 + \tfrac{\mu^2}{l-1})^{l/2}(\tfrac{m-1}{m})^{m/2}(\tfrac{n-1}{n})^{n/2}$$

$$\times[(\tau^2 + \tfrac{(\mu^r+\mu)^2}{l-1})^{-l/2} \int_{\mu^q}^{\infty} \theta^{-m-n} d\theta + (\tau^2 + \tfrac{\mu^{2q}}{l-1})^{-l/2} \int_{\mu+\mu^q}^{\infty} (\theta - \bar{y})^{-m-n} d\theta]$$

$$= k^{-1}(\tau^2+\tfrac{\mu^2}{l-1})^{l/2}(\tfrac{m-1}{m})^{m/2}(\tfrac{n-1}{n})^{n/2}[(\tau^2+\tfrac{(\mu^q+\mu)^2}{l-1})^{-l/2}\tfrac{(\mu^q)^{1-m-n}}{m+n-1}+(\tau^2+\tfrac{\mu^{2q}}{l-1})^{-l/2}\tfrac{(\mu+\mu^q-\bar{y})^{1-m-n}}{m+n-1}]$$

$$\le k^{-1}(m+n-1)^{-1}(\tau^2 + \tfrac{\mu^2}{l-1})^{l/2}(\tfrac{m-1}{m})^{m/2}(\tfrac{n-1}{n})^{n/2}(\tau^2 + \tfrac{\mu^{2q}}{l-1})^{-l/2}\mu^{q(1-m-n)}$$

$$\times[1 + (\mu^{1-q} + 1 - \tfrac{\bar{y}}{\mu^q})^{1-m-n}]$$

$$\le p_0 \mu^{l_0}$$

where

$$p_0 = k^{-1}(m+n-1)^{-1}(\frac{m-1}{m})^{m/2}(\frac{n-1}{n})^{n/2}[1+(\mu^{1-q}+1-\frac{\bar{y}}{\mu^q})^{1-m-n}]$$

$$\begin{aligned} l_0 &= 1+q(1-m-n) \leq 1-3q \\ &= 1-3+\tfrac{3}{l} \\ &= \tfrac{3-2l}{l} \\ &< 0 \end{aligned}$$

So $\forall \epsilon > 0 \quad \exists M > 0$ such that when $|\mu| \geq M$,

$$\frac{\int_{I^c} \pi(\theta|\tilde{x},\tilde{y})d\theta}{\int_I \pi(\theta|\tilde{x},\tilde{y})d\theta} < \epsilon$$

This implies $1 \leq c \leq \frac{1}{1-\epsilon}$:

$$\begin{aligned} (1-\epsilon) &= (1-\epsilon)(\int_{I^c} \pi(\theta|\tilde{x},\tilde{y})d\theta + \int_I \pi(\theta|\tilde{x},\tilde{y})d\theta \\ &\leq (1-\epsilon^2)\int_I \pi(\theta|\tilde{x},\tilde{y})d\theta \\ &\leq \int_I \pi(\theta|\tilde{x},\tilde{y})d\theta \\ &= c^{-1} \\ &\leq 1 \end{aligned}$$

step 7) To prove

$$\frac{|V^{\pi}_{(x,y)} - V^{\pi^*}_{(x,y)}|}{\mu^2} \quad \text{converges to 0as } \mu \to \infty.$$

Now

$$\begin{aligned}\int_\Theta \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta - \int_\Theta \theta^2\pi^*(\theta|\tilde{x},\tilde{y})d\theta &\le \int_\Theta \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta - \int_I \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta \\ &= \int_{I^c} \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta \\ &\le p_2\mu^{l_2}\end{aligned} \tag{A-4.21}$$

and

$$\int_\Theta \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta - \int_\Theta \theta^2\pi^*(\theta|x,y)d\theta \ge \int_\Theta \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta - \frac{\int_I \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta}{1-\epsilon} \tag{A-4.22}$$

Since ϵ could be arbitrary small, from (A-4.21) and (A-4.22), we get the following limit form

$$\begin{aligned}|E^\pi(\theta^2) - E^{\pi^*}(\theta^2)| &= |\int_{I^c} \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta| \\ &\le |p_2\mu^{l_2}| + o(1)\end{aligned}$$

also, from (A-4.20)

$$|(\delta^\pi_{(x,y)})^2 - (\delta^{\pi^*}_{(x,y)})^2| \le p_1\mu^{l_1} + o(1)$$

So

$$\begin{aligned}\lim_{\mu\to\infty} \frac{|V^\pi_{(x,y)} - V^{\pi^*}_{(x,y)})|}{\mu^2} &\le \lim_{\mu\to\infty} \mu^{-2}[|\int_\Theta \theta^2\pi(\theta|\tilde{x},\tilde{y})d\theta - \int_\Theta \theta^2\pi^*(\theta|\tilde{x},\tilde{y})d\theta| \\ &+ |(\delta^\pi_{(x,y)})^2 - (\delta^{\pi^*}_{(x,y)})^2|] \\ &\le \lim_{\mu\to\infty} \mu^{-2}[p_2\mu^{l_2} + p_1\mu^{l_1}] \\ &= 0\end{aligned}$$

because of $l_i \le 1$, i=1,2.

step 8)

Since any distribution on an interval (0, ν) has variance less than or equal to $\frac{\nu^2}{4}$, then $\forall\tau$

$$\begin{aligned} V^{\pi^*}_{(\bar{x},\bar{y})} &= \int_{-\mu^q}^{0}(\theta - \delta^{\pi^*}(\tilde{x},\tilde{y}))^2\pi^*(\theta|\tilde{x},\tilde{y})d\theta + \int_0^{\mu+\mu^q}(\theta - \delta^{\pi^*}(\bar{x},\bar{y}))^2\pi^*(\theta|\tilde{x},\tilde{y})d\theta \\ &\leq \frac{(\mu^q)^2}{4} + \frac{(\mu+\mu^q)^2}{4} \leq \frac{(\mu+2\mu^q)^2}{4} \end{aligned}$$

step 9)

$$\begin{aligned} \lim_{\mu\to\infty} Sup_\tau \frac{V^{\pi}_{(\bar{x},\bar{y})}}{\mu^2/4} &= \lim_{\mu\to\infty}[Sup_\tau \frac{V^{\pi^*}_{(\bar{x},\bar{y})}}{\mu^2/4} + o(1)] \\ &\leq \lim_{\mu\to\infty}[Sup_\tau \frac{(\mu+2\mu^{2q})^2}{4\mu^2/4} + o(1)] \\ &= 1 \end{aligned}$$

From the result of part i) and part ii), we get Theorem 4.3.5.

References

[1] Chaubey, Yogendra.P. and Gabor, G.(1981). Another Look at Fisher's Solution to the Problem of the Weighted Mean. *Commun. Statist. -Thero. Meth.*, A10(12), 1225-1237.

[2] Fan,Tsai-Hung and Berger, James O. (1992). Behaviour of the Posterior Distribution and Inferences for a Normal Mean with t Prior Distributions. *Statistics and Decisions* **10**, 99-120.

[3] Fan, Tsai-Hung and Gerger, James O.(1990). Exact Convolution of t Distributions, with Applications to Bayesian Inference for a Normal Mean with t Prior Distributions. *J.Statist.Computation Simulation* **36**, 209-228.

[4] Box, George E.P. and Tiao, George C.(1973). Bayesian Inference in Statistical Analysis. *Addison Wesley, New York.*

[5] Manzoor Ahmad, Chaubey, Y.P. and Sinha, B.K.(1991). Estimation of A Common Mean of Several Univariate Inverse Gaussian Populations. *Ann. Inst. Statist. Math.* Vol. 43, No. 2, 357-367.