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He also wishes to thank Rita Levig for the typing of this report.

TABLE OF CONTENTS

Page

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS

AWGN	Additive White Gaussion Noise
BSC	Binary Symmetric Channel
CFSK	Coherent FSK
CPSK	Coherent PSK
DPSK	Differentially Coherent PSK
FEC	Foward Error Control
FSK	Frequency Shift Keying
NCG	Net Coding Gain
PSK	Phase Shift Keying
QC	Quasi-Cyclic (codes)
SNR	Signal to Noise Ratio

LIST OF SYMBOLS

A	Amplitude of a periodic waveform
d	Minimum distance (between codewords)
dB	Decibel
Eb	Energy per bit
erf	Error function
erfc	Complementary erf
k	Number of information bits in a block
K	Transmission rate of k bits per second
M(h,1,s)	Number of error patterns of weight 1, at distance s from a codeword of weight h
n	Length of block (bits)
N	Transmission rate of n bits per second
No	Noise power per unit bandwidth
P	Transition probability (bit error probability)
Pi	Pre-decoding bit error probability-protected channel
Po	Post-decoding bit error probability-protected channel
Pu	Bit error probability-unprotected channel
Pe	Probability of error (in general)
q	1-P
r	Code rate (k/n)
R	Redundancy factor (10 log n/k)
Si	Normalized energy per bit-predecoding, protected channel (dB)

So	Equivalent normalized energy per bit-protected channel (dB)
Su	Normalized energy per bit-unprotected channel (dB)
W(h)	Number of codewords of weight h in a given code.
Wc	Carrier angular frequency

CHAPTER 1

INTRODUCTION

1.1 Scope of Work

In the selection of an error-control coding technique for application in a digital communication system, the performances of alternative code designs are compared on the basis of various probabilistic measures. A typical basis for comparison is the post-decoding error rate. These measures, however, do not provide the communication engineer with the necessary trade-off factors of increased modulation rate versus improvement in probability of error.

The concept of Net Coding Gain (NCG) attempts to express the improvement attained by coding in terms of effective Signal to Noise Ratio (SNR), based on error-rate performance, while taking into account the increased modulation rate (or decreased information rate) due to redundant coding bits. NCG figures can be compared in cost-benefit terms with other expedients such as increased energy-per-bit etc.

The block codes discussed here are quasi-cyclic rate $p/(p+1)$ in conjunction with noncoherent Frequency Shift Keying (NCFSK) modulation, Coherent Frequency Shift Keying (CFSK) modulation and Phase Shift Keying (PSK) modulation techniques subject to Additive White Gaussian Noise (AWGN). The analysis is valid for any (n, k) block code in conjunction with a binary modulation in a memoryless channel and subject to AWGN, provided the relationship between Eb/No to the error rate is known. The NCG calculated initially is based on the approximated figure of block error rate derived from the so called "minimum distance bound". However, if the weight distribution for the code is known, the bound can be improved and closer approximation of NCG can be calculated.

The analysis consists of two parts:

i) We fix the rate of codes, vary the block length and evaluate the NCG.

ii) We fix the block length, vary the code rate and evaluate the NCG.

A tentative conclusion is that codes with longer length and rates around 0.6 tend to give a better NCG.

For a given coding scheme and modulation technique, this analysis demonstrates the trade-off between increased modulation rate (decreased energy per bit) and increased correction power of the code.

1.2 Summary of Past Work

Error detecting and correcting codes were first studied by Hamming (1) and Golay (13) in 1949-1950. The class of codes called "linear block codes" were investigated by Slepian (25) while the sub-class called "cylic codes" were first introduced by Prange in 1957, (2) and (14). Cyclic codes and their implementations are systematically presented in (28) and (29).

The concept of NCG as introduced by Blythe & Edgecombe in (3) follows measurements proposed by Ralphs (4) and by Neuman & Lumb (5). Coding/Modulation systems are also studied in (22), (23) and (30).

The set of codes analyzed in this paper were discovered by Chen, Peterson & Weldon (6). Their weight distribution was presented by Stein & Bhargava (7) The analysis of these codes follows the method introduced by Michelson (8) and Huntoon & Michelson (24).

CHAPTER 2

DIGITAL COMMUNICATION CHANNELS

2.1 Channel Coding

In digital communication systems, source information is generated as (or transformed into) a sequence of binary symbols. This data must be transmitted through a noisy channel and as a result errors are introduced. One of the characteristics of a digital communication system is the probability of error. Various units can be employed, such as message error rate, block error rate, word error rate, bit error rate, etc. - depending on the application of the system. Whenever the data is organized in words, word error rate may be used as the measure of performance. Similarly, all other error rate units may be appropriate measures of performance for applications where the data is organized accordingly.

Error Control techniques are usually divided into two types: 1. Forward Error Control (FEC) schemes - where no feedback channel is required and retransmission is not employed, 2. error detection with retransmission (referred to as Automatic Repeat Request - ARQ).

Additional system criteria to be considered are bandwidth and power constraints as well as tolerable delays in the processing of the data. Trade-offs between complexity and performance must be made in the implementation of the communication system.

Coding technique in FEC systems is based on introduction of redundancy in the message by the encoder.

The added redundant digits carry no new information and their function is to enable the decoder to detect and even correct errors introduced by the noisey coding channel.

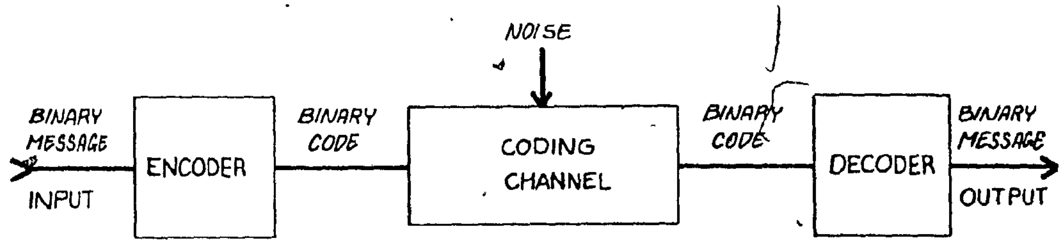


Figure 2.1-1 Block Diagram of a Digital Communication System

A physical coding channel includes in addition to the transmission link medium, a modulator which transforms each binary digit of the code into a physical waveform. At the receiving ~~end this~~ waveform is transformed by a demodulator into a binary digit. The sequence of binary digits, called the received sequence, is used by the decoder to reconstruct the binary message originated by the binary source (see Figure 2.1-2). The sequence received might not match the binary code generated by the encoder, due to noise disturbance in the transmission link medium (channel noise).

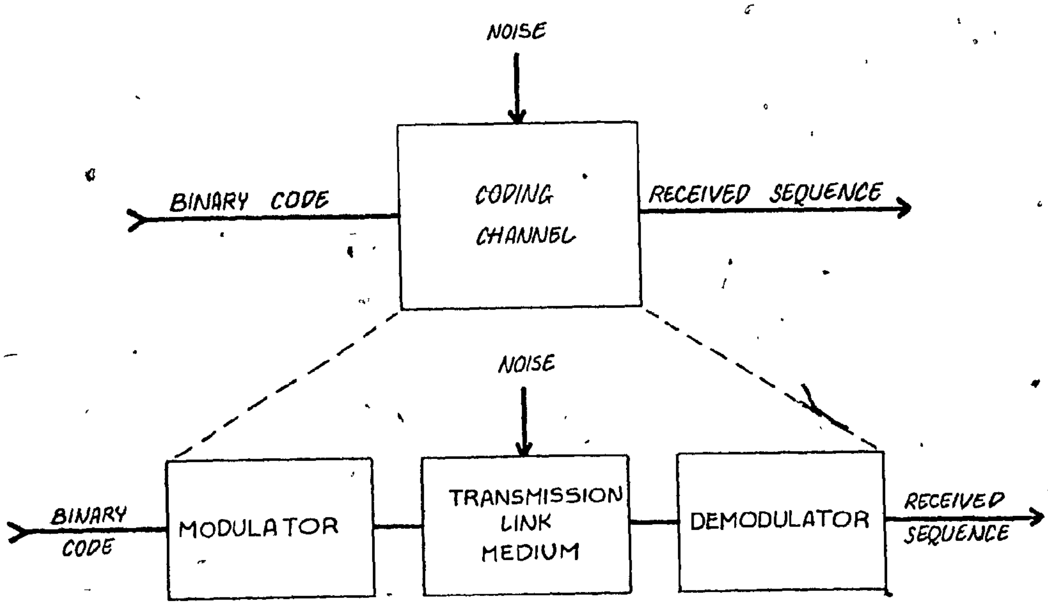


Figure 2.1-2 Physical Coding Channel Model

2.2 Block Coding

In the following chapters we explore one family of FEC codes, namely block codes. We will restrict our attention to a subclass of this group, called linear block codes.

Linear block coding consists of two basic steps:

1. The information stream (a sequence of binary digits) is segmented into message blocks of k successive information bits.
2. The encoder, according to the encoding rules, transforms a message block into a longer block of n bits ($n > k$), called a codeword. The encoding of each block is independent of preceding or succeeding blocks of information. The excess n-k bits (called redundant bits) allow error detection and even error correction by the decoder.

The produced code is an (n,k) block code.

Since each message block consists of k bits, there are 2^k possible distinct message blocks with corresponding 2^k possible codewords. A codeword is also called a code vector since it is an n-tuple from the vector space Vn of all n-tuples. A set of vectors of length n is called a (n,k) linear block code if and only if it is a k-dimensional subspace of the vector space of n-tuples.

For each (n,k) linear block code we define:

k	number of information bits
n	block length of the code
n-k	number of parity (redundant) bits
$r = k/n$	Code rate

We should note at this time that since the coded message includes a larger number of bits than the information segment, the encoded transmission rate of K bits/sec will be increased to a rate of N bits/sec ($N > K$) if an equal information rate is to be assumed.

The distance (or Hamming distance) between two codewords is the number of bits in which they differ. The smallest distance between all possible pairs of codewords is called the minimum distance of the code (d).

Weight (or Hamming weight) of a code word is the number of non-zero bits in the word. The distance between two codewords equals the weight of some other codeword. (The module-2 addition of two codewords produces another codeword whose weight is equal to the distance between the added codewords). Thus, the minimum distance d equals the minimum weight of the non-zero codewords in the code.

A linear block code with minimum distance d is capable of detecting all patterns of up to d-1 errors and correcting all patterns of up to t errors such that $d = 2t + 1$. A systematic introduction to linear block codes is presented in (9), chapter 3.

2.3 Quasi-Cyclic Codes

Quasi-cyclic (QC) codes are linear block codes with the additional property that the cyclic shift of a code vector by a fixed number of components yields another code vector. These were defined as such by Townsand and Weldon (15).

Mathematically, a rate $p/(p+1)$ quasi-cyclic code can be specified as the row space of the "generator matrix" (16).

$$G = \begin{bmatrix} & C_1 \\ & C_2 \\ I_{mp} & \vdots \\ & C_p \end{bmatrix} \qquad (2.3\text{-}1)$$

where I_{mp} is an identity matrix of order mp x mp and each Ci is an m x m binary circulant matrix of the form

$$C = \begin{bmatrix} c_0 & c_1 & \cdots & c_{m-1} \\ c_{m-1} & c_0 & \cdots & c_{m-2} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix} \qquad (2.3\text{-}2)$$

Due to their inherent algebriac structure these codes can be encoded (17) and decoded (18) by shift registers and simple logic circuitry.

In particular a (2m, m) rate ½ QC code is generated by the matrix,

$$G = \left[I_m, C \right] \qquad (2.3\text{-}3)$$

Thus, if $\underline{v} = \left[i_o, i_1, \ldots \, i_{m-1}; \, p_o, p_1 \ldots p_{m-1} \right]$

is a code word in the (2m, m) code, so is the vector,

$$\underline{v}' = \left[i_{m-1}, i_o, \ldots \, i_{m-2}; \, p_{m-1}, p_o, \ldots, p_{m-2} \right]$$

2.4 MODEMS

In section 2.1 we mentioned that a physical coding channel includes a modem (modulator-demodulator) and a transmission link. The modulation may be considered as an additional step in matching the source information to the communication channel (e.g. specified spectral window, physical nature of the transmission link, etc.). All modulation techniques are subject to transmission through noisy channels, and hence questions of the efficiency and capacity of any coding-modulation schemes arise.

Before discussing the selection of modems to be used with error-control coding, we will make some assumptions regarding the transmission link. We will assume that a binary symbol at the input of the link will be received correctly at the output with probability q (or will be erroneously interpreted as the opposite binary symbol with probability P = 1-q). We will also assume that the channel acts on transmitted symbols independently (a memoryless channel). Such a channel is called a Binary Symmetric Channel (BSC). See Figure 2.4-1.

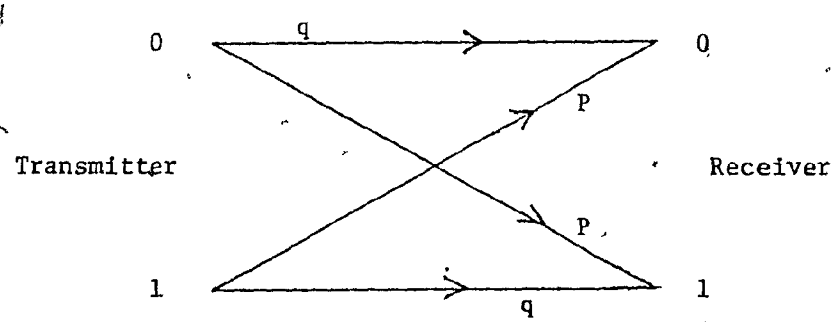


Figure 2.4-1 The Binary Symmetric Channel

The probability of error P in a BSC is a parameter dependent on such factors as the modulation technique employed in the channel, the power contents of each binary symbol (or Energy per bit Eb) and the Noise power per unit bandwidth (No). We will now attempt to describe these relationships for various modulation schemes.

Frequency Shift Keying (FSK)

In this technique the binary signal is transformed by the modulator into one of two frequencies depending on whether the bit is 1 or 0. The waveform is:

$$V_{FSK}(t) = A \cdot \cos(W_c \pm \Omega)t \quad (2.4\text{-}1)$$

The probability of error for a coherent system of FSK signal reception is calculated in (20) chap. 11.8 as

$$P \simeq (\tfrac{1}{2}) \operatorname{erfc}\left[0.6 \, (Eb/No)\right]^{\frac{1}{2}} \quad (2.4\text{-}2)$$

while the probability of error for a noncoherent system of FSK signal reception is ((20) chap. 11.9):

$$P = (\tfrac{1}{2}) \exp\left[-(\tfrac{1}{2}) \, (Eb/No)\right] \quad (2.4\text{-}3)$$

The main difference between the two reception schemes is that the coherent system is based on a correlator realization of the matched filter (20) while the noncoherent system uses a filter receiver with bandpass filters centered at the source-signal frequencies. The correlation receiver of the coherent scheme necessitates synchronized local waveforms (see Fig. 2.4-2) $S_1(t)$ and $S_2(t)$.

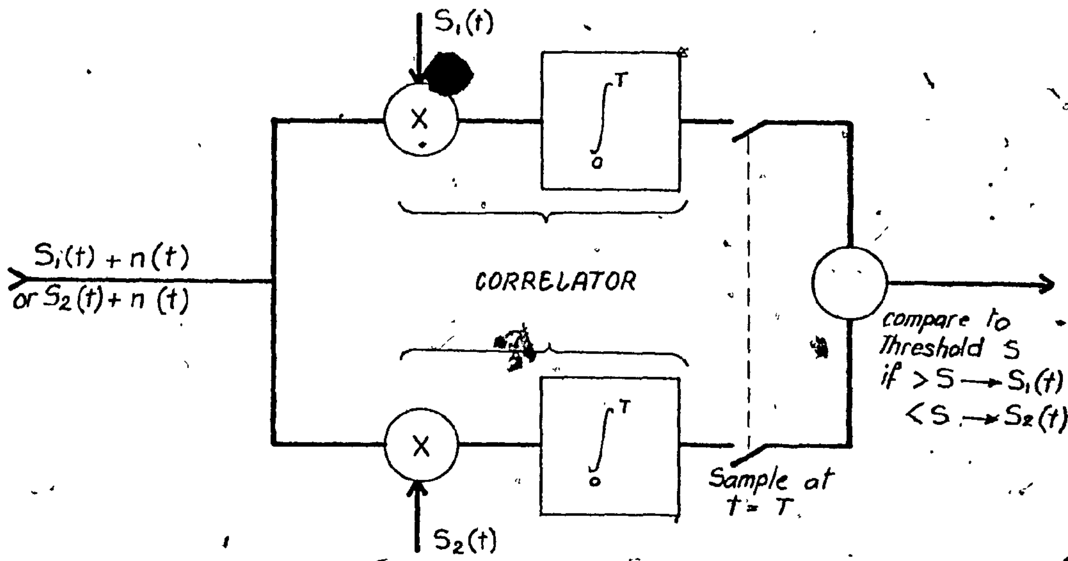


Figure 2.4-2 Optimum Coherent Detector (Correlation Receiver)

On the other hand the noncoherent filter-receiver makes no use of the phase of the incoming signal, hence the probability of detecting the signal is reduced. Approximately, a 2-dB increase in energy is needed to have the NCFSK yield the same probability of error as the coherent FSK reception scheme (26).

Phase Shift Keying (PSK)

The binary signal is transformed by the modulator into one of two waveforms:

$$V_{PSK}(t) = A \cdot COS\ (W_o t + \varphi\ (t)) = \pm\ A \cdot COS\ (W_o t) \qquad (2.4\text{-}4)$$

The probability of error of the PSK receiver is identical to that of the matched filter:

$$P = (\tfrac{1}{2}) \cdot erfc\ (Eb/No)^{\frac{1}{2}} \qquad (2.4\text{-}5)$$

as long as the local signal $2A \cdot COS(W_o t)$ is perfectly synchronized. In that case, the energy required to produce the same probability of error is approximately 2dB less than for the coherent FSK system. The reason for the superior performance of the PSK receiver is that the waveforms produced by the modulator follow the optimal choice yielding a maximum output signal for a given signal energy $(S_1(t) = A \cdot COS(W_o t) = -S_2(t))$

Differential Phase Shift Keying (DPSK)

This modulation technique is a modification of PSK which avoids the necessity of providing synchronous carrier required by the receiver for demodulating a PSK signal. In this technique the binary message is first transformed into another stream in which two identical consecutive digits represent 1 in the original message (The transformed sequence contains one bit more than the original message). The transformed sequence is then modulated as in the PSK case.

The DPSK demodulator can recover the message stream without the use of a local carrier. However, the simplicity of the demodulator circuitry should be weighted against the fact that errors no longer are independent of preceding or succeeding information as in the other modulation techniques presented here. Further discussion of the applicability of this modulation technique to the NCG analysis is provided in Section 3.2.

Numerical values of the ratio Eb/No for the three typical error rates were calculated using the modulation techniques mentioned above. Results are given in Table 2.4-1.

Probability of Error P	Eb/No (dB)		
	PSK	CFSK	NCFSK
10^{-3}	6.8	9.0	10.9
10^{-5}	9.6	11.8	13.4
10^{-7}	11.3	13.5	14.9

Table 2.4-1 Variation of Eb/No versus P for Several Modulation Schemes

Graphic representation of the error rate of data transmission systems is provided in Figure 2.4-3.

Finally, we note that in our search for a modem requiring minimal Eb/No ratio for a given probability of error, a limit was established by Shannon in 1948 (21). An arbitrarily low error rate can be achieved with appropriate coding using Eb/No of -1.6 dB. (Infinite bandwidth memoryless AWGN channel is assumed).

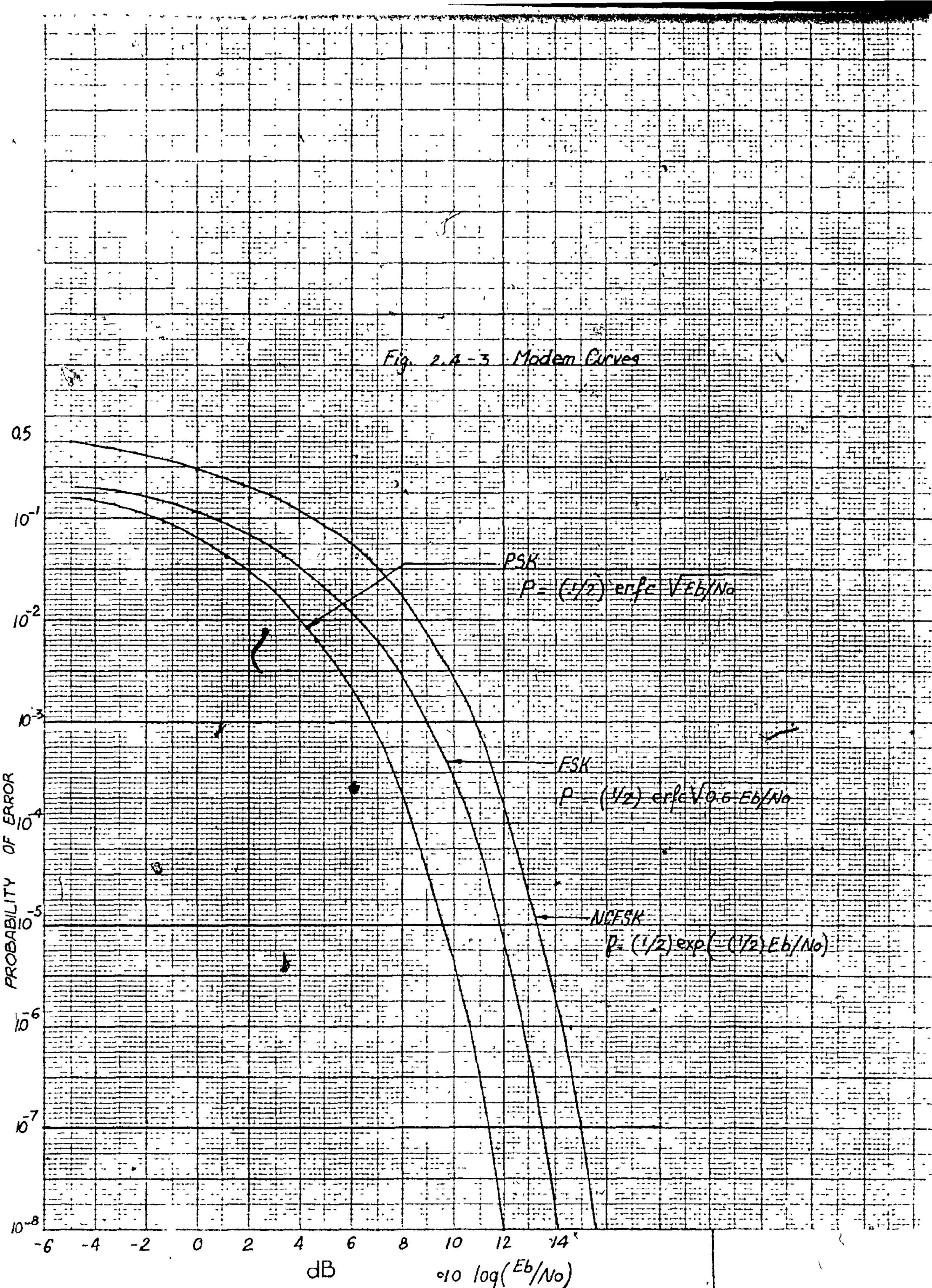


Fig. 2.4-3 Modem Curves
0.5
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
PROBABILITY OF ERROR
PSK
$P = (1/2)\ \mathrm{erfc}\ \sqrt{Eb/No}$
FSK
$P = (1/2)\ \mathrm{erfc}\sqrt{0.6\ Eb/No}$
NCFSK
$P = (1/2)\exp(-(1/2)Eb/No)$
-6 -4 -2 0 2 4 6 8 10 12 14
dB
$10\log(Eb/No)$

The communication engineer is now faced with two separate criteria in order to select a digital communication system:

1. Modulation technique (i.e. relationship between the ratio Eb/No to the error rate)
2. Coding scheme (i.e. correction power of the code in relation to the required increase in modulation rate).

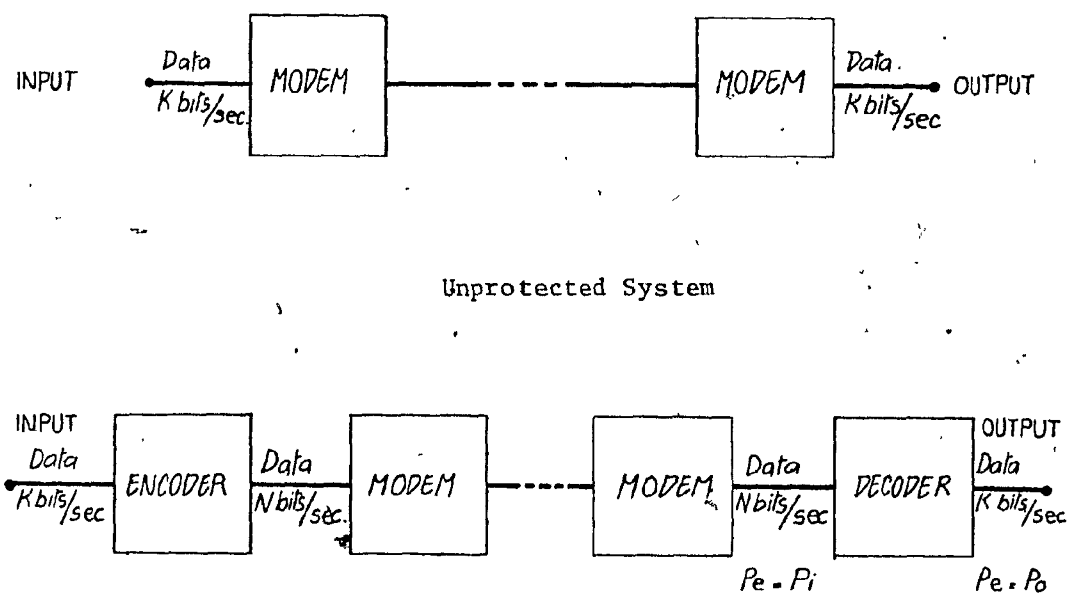
In the next chapter, we provide a unified treatment of the coding/mudulation trade-offs in digital communication systems.

CHAPTER 3

THE NET CODING GAIN

3.1 Definition of Net Coding Gain

In chapters 1 and 2 we dealt with probability of error as the only measure for comparison between various communications systems. Such measures neglect the relationship between the coding technique and the modulation (i.e. for increased correction power by the coder the transmission rate by the modulator is also increased, resulting in reduction of symbol energy and increase in probability of error - see Figure 3.1-1).



Unprotected System

Protected System

Figure 3.1-1 Redundancy in Coding Systems

In this chapter we will define another measure - the Net Coding Gain - which will express system characteristics in terms of effective SNR. The NCG, then, can be used as a criterion for a comparison of various communication systems.

In order to define the NCG let us recall the relationship between the probability of error and the normalized energy per bit Eb/No. Any modulation technique with such known relationship can be used for this analysis, assuming a memoryless transmission system.

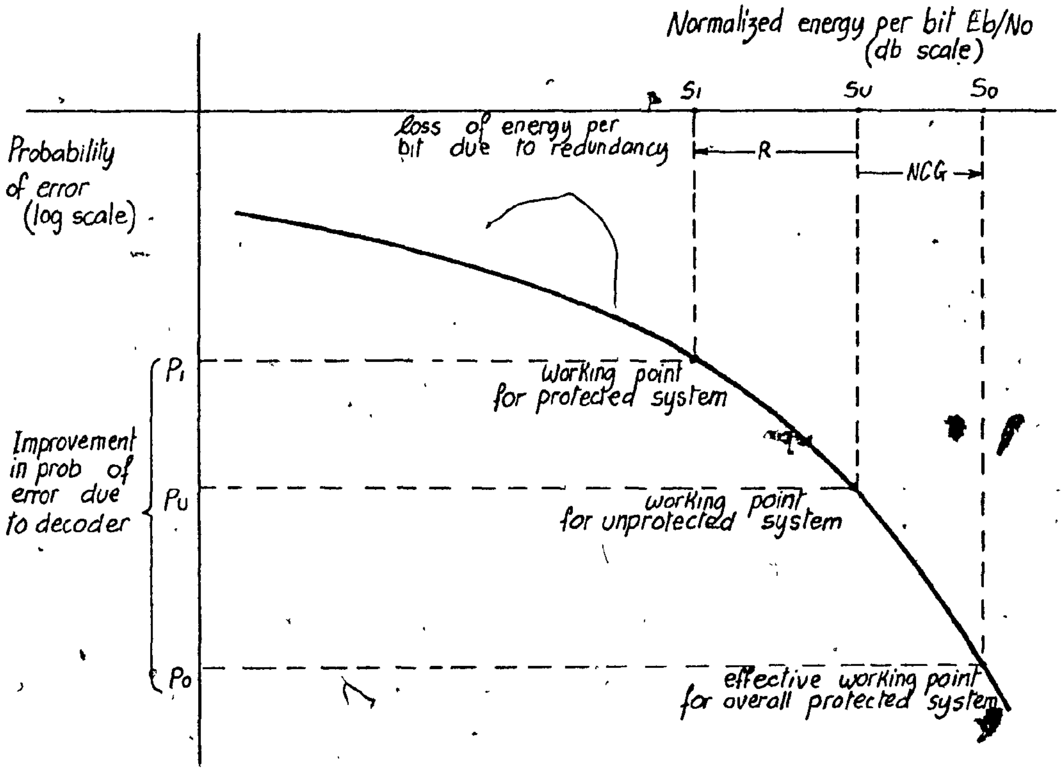


Figure 3.1-2 Definition of Net Coding Gain

In an uncoded system the operating point will be set along the modem curve according to power or error-rate constraints. We will assume a working point at (see Figure 3.1-2):

Su = energy per bit available with uncoded system, measured in dB

$$Su = 10 \log (Eb/No) \qquad (3.1\text{-}1)$$

Pu = equivalent bit error-rate on modem curve

Since the introduction of coding to the system requires redundancy in the binary sequence, the resultant energy per bit in the coded case is rEb/No (i.e. (k/n) Eb/No). On a logarithmic scale this value will become:

Si = energy per bit when redundant coding is used, measured in dB

$$Si = Su - R \qquad (3.1\text{-}2)$$

$$\text{where } R = 10 \log n/k \qquad (3.1\text{-}3)$$

Pi = resulting bit error rate

Given this probability of bit error we can calculate the post decoding bit error rate:

Po = bit error rate after decoding

So = equivalent energy per bit (in dB)

If the equivalent normalized energy per bit So is greater than the normalized energy per bit for the uncoded case Su, the result is a higher effective SNR which can be compared in cost-benefit terms with other expedients, such as increased energy per bit:

$$NCG\ (dB) = So - Su \qquad (3.1\text{-}4)$$

3.2 The Decoder

In order to calculate Po, the preformance of the decoder must be considered. For (n, k) block codes of rate r the value of Si can be derived according to Eq. 3.1-2. The calculation of Po is more involved, and as a first step we will approximate the value of post decoding error rate. A more accurate method is discussed in section 3.5.

In a BSC the probability of receiving ℓ bits without any bit-corruption due to noise is:

$$P_{\text{correct message}} = (1 - P_i)^{\ell} \quad (3.2\text{-}1)$$

where the transmission probability (or cross-over probability) Pi can be derived from Eq. 2.4-2, 2.4-3 and 2.4-5. Similarly the probability of one or more errors taking place in a transmission of ℓ bits through a BSC is:

$$P_{\text{incorrect transmission}} = 1 - P_{\text{correct message}} = 1 - (1 - P_i)^{\ell} \quad (3.2\text{-}2)$$

Since the errors in a block are assumed to be binomially distributed, the probability of exactly ℓ errors in a block of length n is:

$$P_i^{\ell}\,(1-P_i)^{n-\ell} \quad (3.2\text{-}3)$$

We point out that the binomial distribution of errors is not applicable to the case of DPSK modulation since differential encoding and detection of successive bits cause interdependence of errors (27). The question of conditional probability of an adjacent error in a binary DPSK system was investigated by Salz & Saltzberg (22) and by Oberst and Schilling (23). These works, however, do not provide a general analysis of block-error distributions.

For blocks with more than t errors (all blocks of t or less erroneous bits can be corrected by the decoder and hence produce correct post-decoding messages) we conclude that the probability of error in decoding of a block of n bits is upperbounded by:

$$P_{block\ error} = \sum_{\ell=t+1}^{n} \binom{n}{\ell} P_i^{\ell} (1 - P_i)^{n-\ell} \qquad (3.2\text{-}4)$$

This error rate is an overbound since, in general, not every pattern of t + 1 errors will produce a decoding error. Moreover, in this analysis we considered every block of t + 1 and more errors as an error producing block, ignoring the fact that some of these blocks can be detected as erroneous by the decoder and flagged as unreliable data for the user at the receiving end.

In order to complete the calculation of Po we have to determine the bit error rate, given the above block error rate: As an approximate value we can assume that in every n - bit codeword selected by the decoder there are d errors. (To a block containing t + 1 errors the decoder may add up to t errors in order to produce a valid codeword within distance d from the transmitted codeword). Thus, the k information bits will contain on the average kd/n errors and the bit error rate at the decoder output can be approximated as:

$$Po = (kd/n) \sum_{\ell=t+1}^{n} \binom{n}{\ell} P_i^{\ell} (1 - P_i)^{n-\ell} \qquad (3.2\text{-}5)$$

This value will be called here "minimum distance bound".

Using this value in conjunction with the modem curve will produce the equivalent energy per bit So. The NCG then, can be calculated from Eq. 3.1-4.

3.3 Coding Gain of Rate ½ Quasi-Cyclic Codes

In table 3.3-1 calculated values of the NCG for three typical error rates are presented. The codes used for this analysis are listed in the table as well:

CODE (n, k)	d	t	NCG at Pu 10^{-3}		NCG at Pu 10^{-5}		NCG at Pu 10^{-7}	
			NCFSK	CFSK	NCFSK	CFSK	NCFSK	CFSK
(6, 3)	3	1	-0.99	-0.35	-0.56	-0.09	-0.39	-0.01
(8, 4)	4	1	-1.54	-0.93	-0.86	-0.39	-0.58	-0.21
(10, 5)	4	1	-1.77	-1.17	-0.98	-0.52	-0.67	-0.30
(12, 6)	4	1	-1.95	-1.37	-1.08	-0.62	-0.73	-0.36
(14, 7)	4	1	-2.10	-1.54	-1.17	-0.71	-0.79	-0.42
(16, 8)	5	2	-0.33	0.61	0.62	1.30	0.99	1.52
(18, 9)	6	2	-0.65	0.29	0.46	1.15	0.89	1.42
(20, 10)	6	2	-0.81	0.13	0.38	1.07	0.83	1.37
(22, 11)	7	3	0.56	1.72	1.67	2.51	2.13	2.76
(24, 12)	8	3	0.32	1.49	1.58	2.40	2.05	2.69
(26, 13)	7	3	0.26	1.42	1.55	2.37	2.03	2.67
(28, 14)	8	3	0.03	1.20	1.44	2.27	1.97	2.60
(30, 15)	8	3	-0.10	1.07	1.38	2.21	1.93	2.56
(32, 16)	8	3	-0.22	0.95	1.32	2.15	1.89	2.53
(34, 17)	8	3	-0.34	0.83	1.26	2.09	1.85	2.49
(36, 18)	8	3	-0.45	0.72	1.21	2.04	1.82	2.46
(38, 19)	8	3	-0.56	0.61	1.16	1.99	1.79	2.43
(40, 20)	9	4	0.55	1.89	2.2	3.12	2.81	3.50
(42, 21)	10	4	0.38	1.73	2.13	3.05	2.76	3.40
(46, 23)	11	5	1.23	2.70	2.96	3.94	3.57	*
(48, 24)	12	5	1.08	2.56	2.89	3.88	3.53	*

Table 3.3-1

Net Coding Gain of Rate ½ Quasi-Cyclic Codes

The value of the NCG for the coded system in conjunction with CFSK and NCFSK modulation techniques are presented. However, other theoretical equations which differ only in some ratio of to the normalized energy per bit Eb/No will yield identical results. For example, CPSK and CFSK differ only in ratio 0.6 of the normalized energy per bit:

CFSK (see Eq. 2.4-2) $Pu = (\frac{1}{2}) \text{ erfc } [0.6 (Eb/No)]^{\frac{1}{2}}$

CPSK (see Eq. 2.4-5) $Pu = (\frac{1}{2}) \text{ erfc } [Eb/No]^{\frac{1}{2}}$

These modem curves are shifted along the abscissa (on a log-log scale of probability of error to normalized energy-per-bit) by the logarithm of the factor 0.6 as illustrated by Fig. 2.4-3. Hence, the procedure described in section 3.1 will be identical: (i) the starting operating point is shifted by the amount indicated above, (ii) redundancy loss is identical and (iii) error-rate improvement depends on the code only. The overall NCG figure is unchanged.

For selected codes we present the NCG in a form of a graph; see Figure 3.3-1 (NCFSK modem) and Figure 3.3-2 (CFSK modem). The codes considered are given in Table 3.3-2.

CODE NO.	CODE	d	t
1	(6, 3)	3	1
2	(16, 8)	5	2
3	(22, 11)	7	3
4	(40, 20)	9	4
5	(46, 23)	11	5

Table 3.3-2 Codes presented in Fig. 3.3-1 and Fig. 3.3-2

From Table 3.3-1 and Figures 3.3-1; 3.3-2, it can be seen that for all the analyzed codes, NCG figures for channels with small probability of error Pu are higher than the NCG figures for channels with higher probability of error. Most coding schemes cross into the positive NCG figure domain between $Pu=10^{-2}$ and $Pu=10^{-3}$.

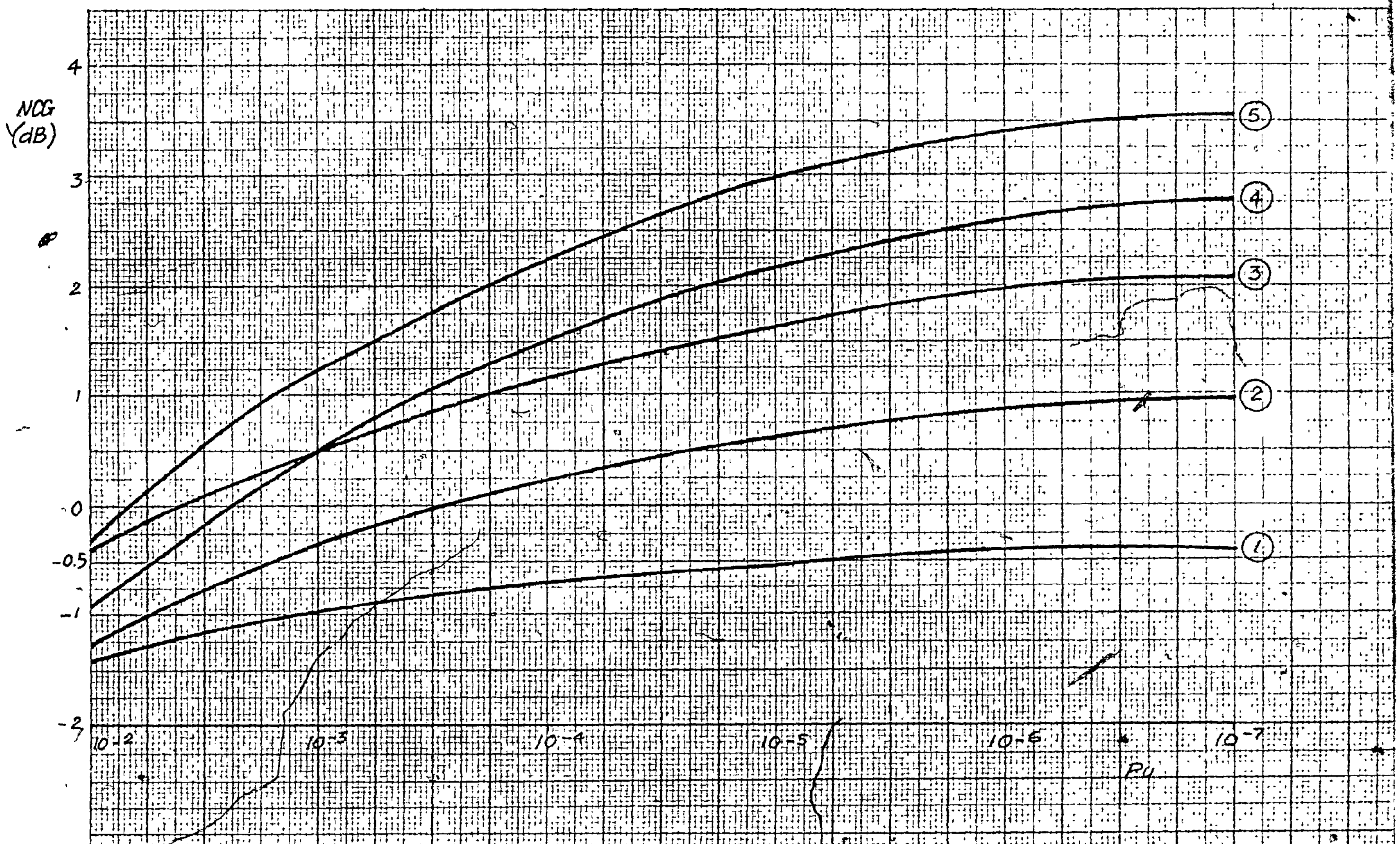


Fig 3.3-1 NCG vs error rate for selected rate 1/2 Quasi-Cyclic codes NCF 5K modem

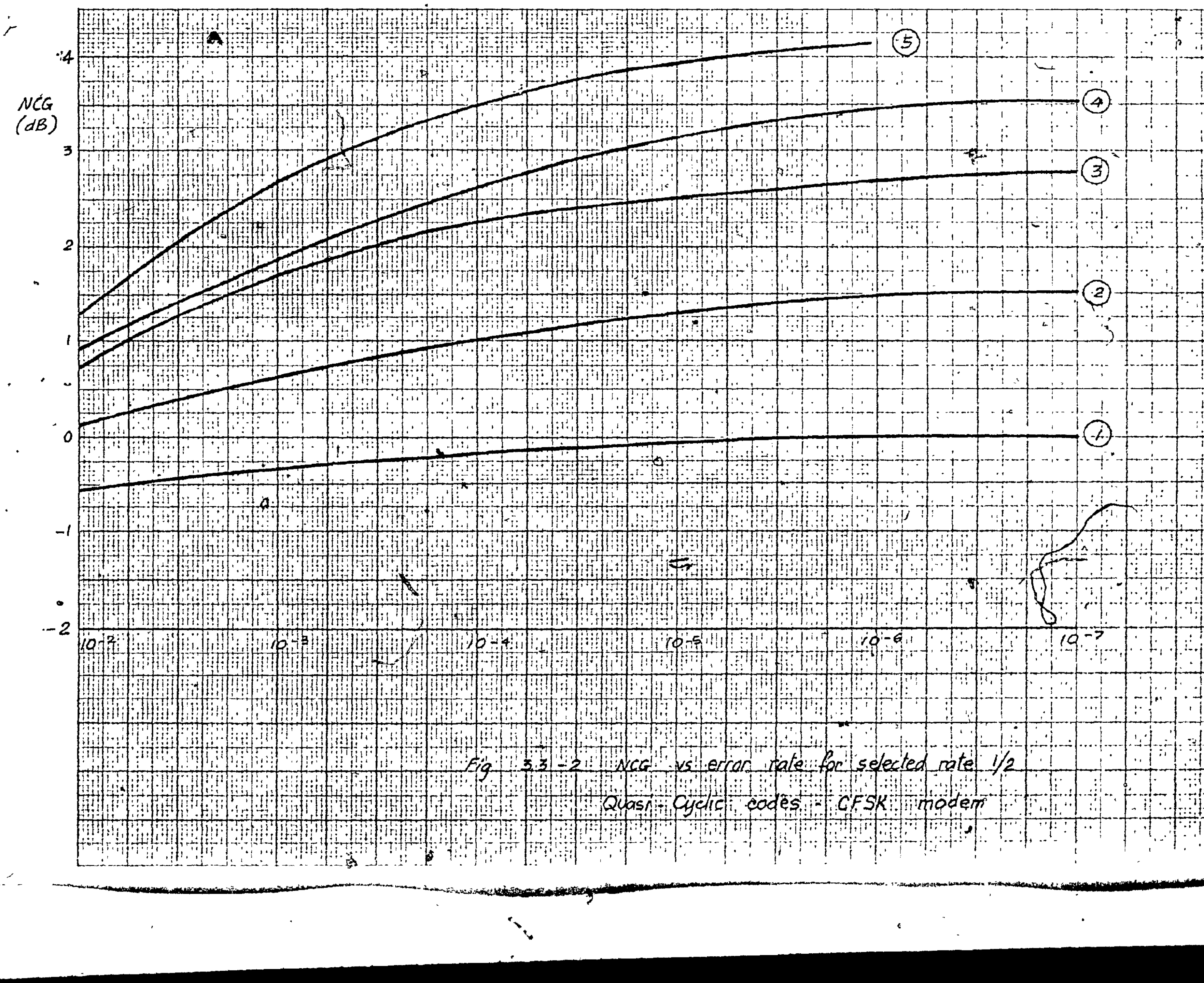


Fig 3.3-2 NCG vs error rate for selected rate 1/2 Quasi-Cyclic codes - CFSK modem

It is also evident that longer codes yield better NCG figures whenever accompanied by increased correction power of the code (t). For codes with equal correction power (t), longer codes yield smaller NCG figure due to decreased performance of the decoder. The difference between the "best" (or longest) and "worst" (or shortest) codes analyzed here in terms of NCG is in the order of 4 db.

The superiority of the CFSK/PSK modem discussed in section 2.4 is maintained in the results of the NCG analysis.

3.4 Coding Gain of Constant Length Block Codes

In section 3.3 we analyzed a set of rate ½ quasi-cyclic codes with variable block length (beginning with $n = 6$ up to $n = 48$). This analysis allowed us to observe the improvement, in terms of NCG, obtained by selecting longer - and more complex - codes. It is also of interest to consider, within a fixed block length, what degree of redundancy gives the best results.

Bandwidth expansion of ratio 2 to 1 was required for all codes analyzed in section 3.3 (since code rate of ½ was assumed). In this section we wish to examine various code rates from the point of view of bandwidth vs NCG.

The selected codes are listed in tables 3.4-1 to 3.4-6 together with their NCG performance for two modulation techniques at three typical values of bit error rate Pu.

Figure 3.4-1 is a graphic representation of the NCG vs the code rate for block codes of length 45. Figures 3.4-2 and 3.4-3 illustrate the NCG vs bit error rate for selected length 45 codes using NCFSK and CFSK modem respectively. The results indicate that codes with redundancy between 18dB to 3dB perform better at low error rates. For Pu higher than 10^{-3}, where the NCG is in any case small, a lower redundancy is advisable. Similarly, with very low probability of error Pu, code rates as low as 1/3 yield positive NCG figures.

CODE (n, k)	d	t	r	NCG at Pu = 10^{-3} NCFSK	CFSK	NCG at Pu = 10^{-5} NCFSK	CFSK	NCG at Pu = 10^{-7} NCFSK	CFSK
(15, 11)	3	1	0.73	0.40	0.83	0.97	1.31	1.19	1.46
(15, 10)	4	1	0.67	-0.36	0.09	0.37	0.74	0.65	0.95
(15, 9)	4	1	0.60	-1.01	-0.52	-0.19	0.20	0.12	0.44
(15, 8)	4	1	0.53	-1.76	-1.22	-0.84	-0.41	-0.48	-0.13
(15, 7)	5	2	0.47	-0.68	0.31	0.28	1.00	0.66	1.23
(15, 6)	6	2	0.40	-1.90	-0.77	-0.69	0.13	-0.21	0.43
(15, 5)	7	3	0.33	-1.43	0.11	-0.27	0.83	0.22	1.07
(15, 4)	8	3	0.27	-3.20	-1.37	-1.72	-0.41	-1.08	-0.06
(15, 2)	10	4	0.13	-6.96	-3.86	-5.02	-2.72	-4.10	-2.27

Table 3.4-1 NCG of length 15 block codes

CODE (n, k)	d	t	r	NCG at Pu = 10^{-3} NCFSK	CFSK	NCG at Pu = 10^{-5} NCFSK	CFSK	NCG at Pu = 10^{-7} NCFSK	CFSK
(21, 16)	3	1	0.76	0.40	0.80	1.05	1.38	1.30	1.56
(21, 15)	4	1	0.71	-0.18	0.22	0.61	0.95	0.91	1.19
(21, 14)	4	1	0.67	-0.62	-0.19	0.24	0.60	0.57	0.86
(21, 13)	4	1	0.62	-1.09	-0.69	-0.17	0.21	0.19	0.90
(21, 12)	5	2	0.57	0.18	1.01	1.18	1.79	1.56	2.04
(21, 9)	8	3	0.43	-0.55	0.77	0.78	1.71	1.30	2.01
(21, 5)	10	4	0.24	-3.88	-1.56	-1.90	-0.26	-1.04	0.22

Table 3.4-2 NCG of length 21 block codes

CODE (n, k)	d	t	r	NCG at Pu = 10^{-3} NCFSK	CFSK	NCG at Pu = 10^{-5} NCFSK	CFSK	NCG at Pu = 10^{-7} NCFSK	CFSK
(30, 25)	3	1	0.83	0.72	1.06	1.41	1.70	1.66	1.91
(30, 20)	5	2	0.67	0.79	1.51	1.84	2.37	2.22	2.65
(30, 15)	8	3	0.50	-0.09	1.01	1.38	2.21	1.93	2.56
(30, 10)	11	5	0.33	-1.02	0.99	0.83	2.19	1.56	2.59
(30, 5)	15	7	0.17	-5.14	-1.53	-2.53	0.00	-1.36	0.57

Table 3.4-3 NCG of length 30 block codes

CODE (n, k)	d	t	r	NCG at Pu = 10^{-3}		NCG at Pu = 10^{-5}		NCG at Pu = 10^{-7}	
				NCFSK	CFSK	NCFSK	CFSK	NCFSK	CFSK
(33, 21)	4	1	0.64	-1.27	-0.88	-0.21	0.15	0.20	0.50
(33, 20)	4	1	0.61	-1.60	-1.19	-0.49	-0.11	-0.05	0.26
(33, 13)	10	4	0.39	-1.08	0.57	0.82	1.95	1.54	2.40
(33, 12)	10	4	0.36	-1.80	-0.02	0.26	1.48	1.06	1.98
(33, 11)	11	5	0.33	-1.35	0.69	0.66	2.04	1.45	2.50
(33, 10)	12	5	0.30	-2.32	-0.08	-0.06	1.45	0.84	1.98
(33, 2)	22	10	0.06	-14.83	-6.24	-9.41	-4.08	-7.40	-3.20

Table 3.4-4 NCG of length 33 block codes

CODE (n, k)	d	t	r	NCG at Pu = 10^{-3}		NCG at Pu = 10^{-5}		NCG at Pu = 10^{-7}	
				NCFSK	CFSK	NCFSK	CFSK	NCFSK	CFSK
(35, 28)	4	1	0.80	0.19	0.515	1.05	1.35	1.37	1.62
(35, 27)	4	1	0.77	-0.04	0.29	0.85	1.16	1.19	1.44
(35, 25)	4	1	0.71	-0.55	-0.19	0.42	0.75	0.79	1.06
(35, 24)	4	1	0.69	-0.81	-0.45	0.20	0.53	0.58	0.86
(35, 22)	4	1	0.63	-1.40	-1.01	-0.30	0.06	0.12	0.42
(35, 20)	6	2	0.57	-0.64	0.16	0.78	1.39	1.30	1.78
(35, 19)	6	2	0.54	-1.04	-0.20	0.46	1.10	1.01	1.52
(35, 17)	6	2	0.49	-1.91	-1.00	-0.26	0.45	0.38	0.94
(35, 16)	7	3	0.46	-1.02	0.23	0.69	1.59	1.35	2.04
(35, 15)	8	3	0.43	-1.70	-0.36	0.21	1.17	0.94	1.67
(35, 10)	10	4	0.29	-4.27	-2.14	-1.67	-0.12	-0.57	0.59
(35, 7)	14	6	0.20	-5.69	-2.39	-2.61	-0.30	-1.27	0.45
(35, 4)	15	7	0.11	-10.34	-5.89	-6.86	-3.19	-4.95	-2.10
(35, 3)	20	9	0.09	-12.23	-5.61	-7.68	-3.17	-5.69	-2.20

Table 3.4-5 NCG of length 35 block codes

CODE (n, k)	d	t	r	NCG at $Pu = 10^{-3}$		NCG at $Pu = 10^{-5}$		NCG at $Pu = 10^{-7}$	
				NCFSK	CFSK	NCFSK	CFSK	NCFSK	CFSK
(45, 35)	4	1	0.78	-0.15	0.15	0.81	1.11	1.17	1.42
(45, 33)	4	1	0.73	-0.55	-0.22	0.48	0.79	0.87	1.13
(45, 30)	4	1	0.67	-1.19	-0.84	-0.07	0.27	0.77	0.65
(45, 29)	5	2	0.64	0.07	0.78	1.41	1.96	1.89	2.33
(45, 28)	6	2	0.62	-0.32	0.40	1.14	1.70	1.667	2.12
(45, 27)	6	2	0.60	-0.61	0.14	0.91	1.50	1.46	1.93
(45, 26)	6	2	0.58	-0.91	0.14	0.67	1.28	1.25	1.73
(45, 23)	7	3	0.51	-0.60	0.54	1.18	2.01	1.84	2.47
(45, 22)	8	3	0.49	-1.10	0.09	0.84	1.70	1.55	2.21
(45, 21)	8	3	0.47	-1.51	-0.27	0.52	1.42	1.28	1.96
(45, 20)	8	3	0.44	-1.95	-0.66	0.17	1.12	0.98	1.70
(45, 18)	8	3	0.40	-2.91	-1.53	-0.60	0.45	0.32	1.12
(45, 17)	8	3	0.38	-3.42	-2.01	-1.04	0.08	-0.05	0.79
(45, 16)	10	4	0.36	-2.99	-1.16	-0.41	0.90	0.61	1.58
(45, 15)	10	4	0.33	-3.62	-1.72	-0.92	0.47	0.18	1.21
(45, 14)	10	4	0.31	-4.30	-2.34	-1.50	-0.02	-0.30	0.81
(45, 12)	10	4	0.27	-5.75	-3.76	-2.85	-1.10	-1.43	-0.13
(45, 11)	10	4	0.24	-6.48	-4.58	-3.65	-1.76	-2.11	-0.68
(45, 10)	12	5	0.22	-6.75	-4.18	-3.52	-1.31	-1.89	-0.26
(45, 9)	12	5	0.20	-7.66	-5.23	-4.57	-2.14	-2.77	-0.[illegible]5
(45, 8)	12	5	0.18	-8.49	-6.42	-5.78	-3.11	-3.82	-1.76
(45, 7)	15	7	0.16	-8.66	-4.98	-5.11	-1.94	-3.15	-0.78
(45, 6)	18	8	0.13	-10.02	-5.39	-5.9	-2.22	-3.77	-1.02
(45, 5)	21	10	0.11	-10.24	-4.46	-5.89	-1.74	-3.82	-0.70
(45, 4)	24	11	0.09	-12.80	-5.44	-7.50	-2.57	-5.19	-1.47

Table 3.4-6 NCG of length 45 block codes

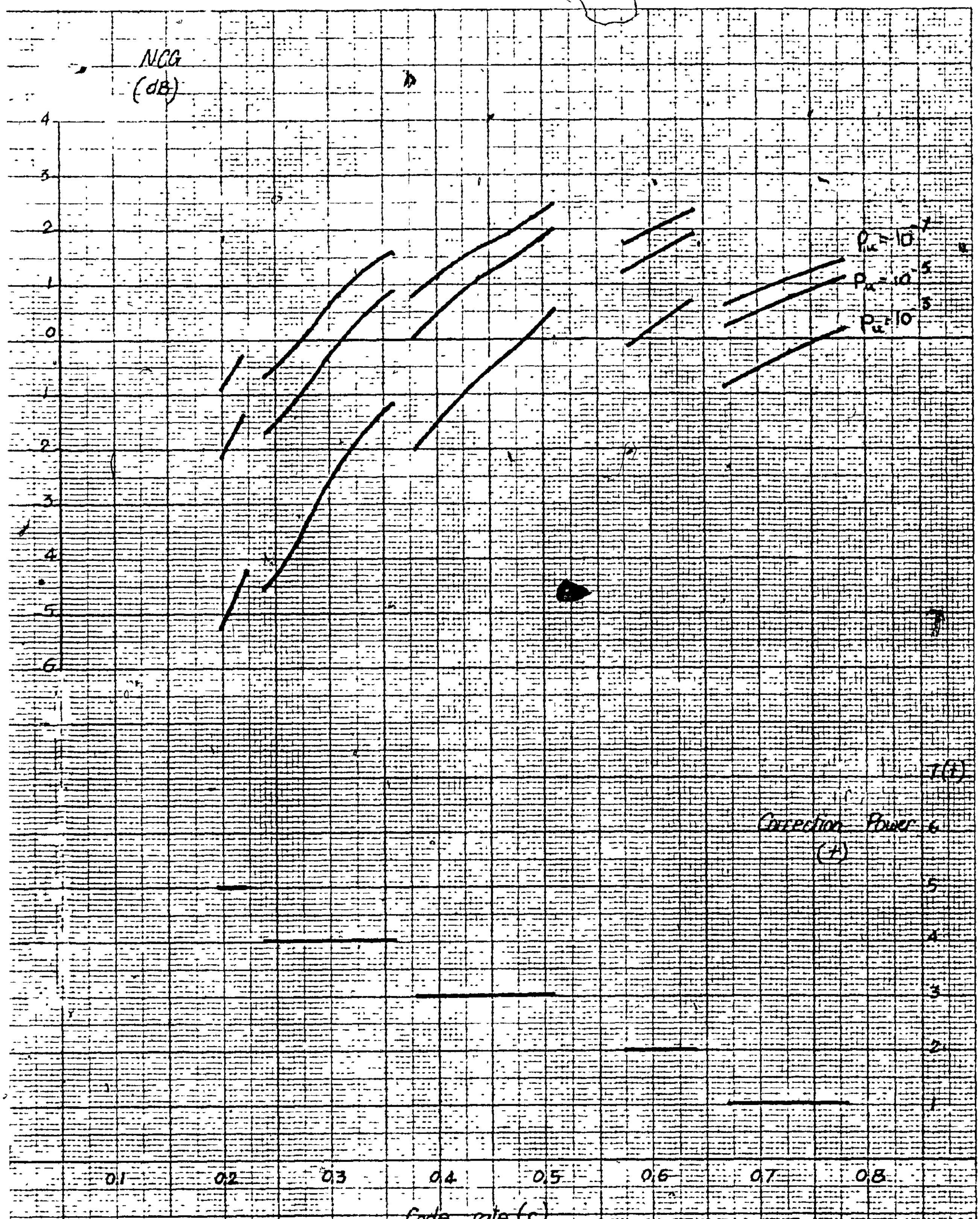


Fig 3.4-1 NCG vs Code rate
for block codes of lenght 45

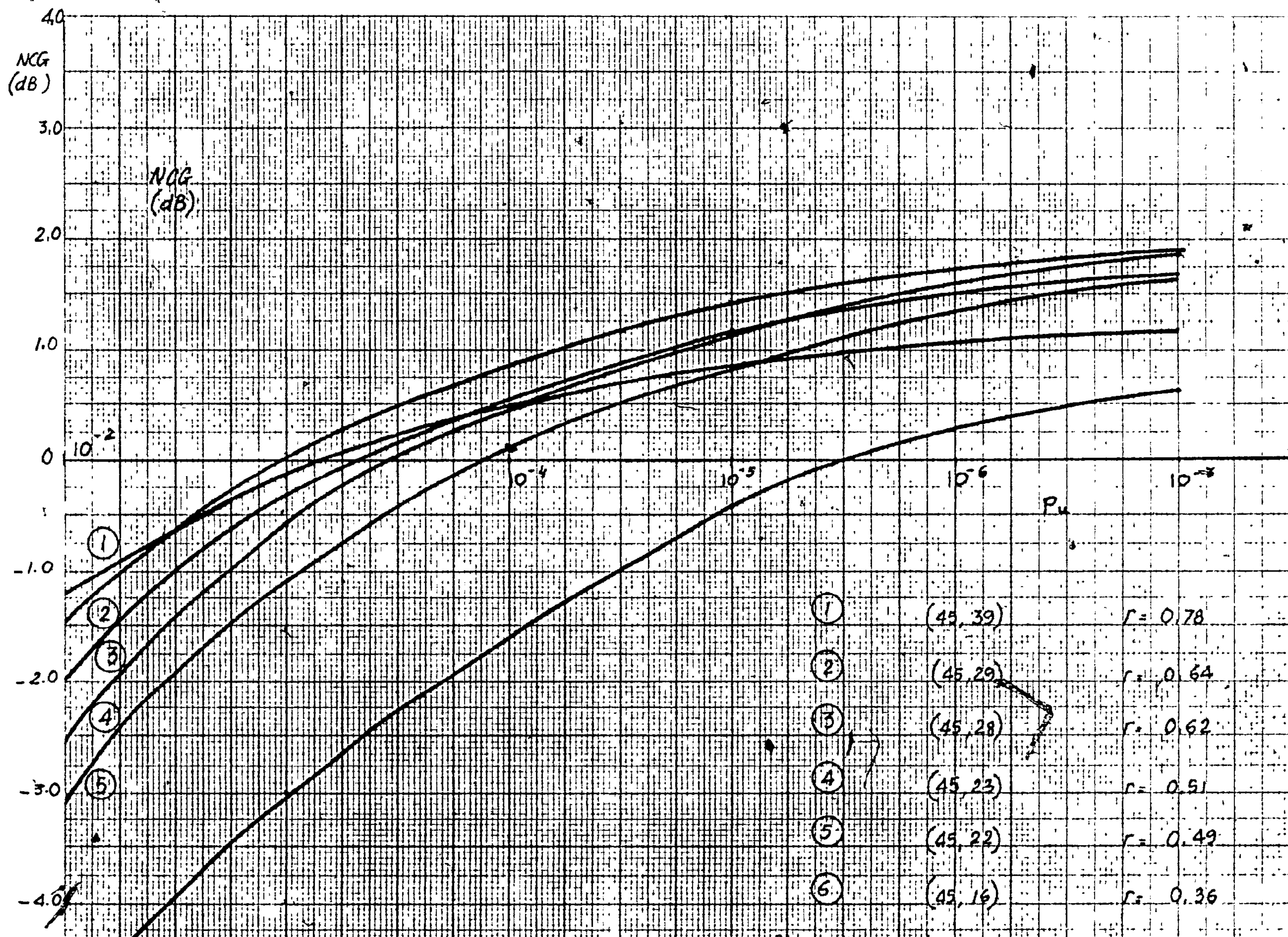


Fig. 3.4-2 NCG vs error rate for selected length 45 codes
NCFSK modem

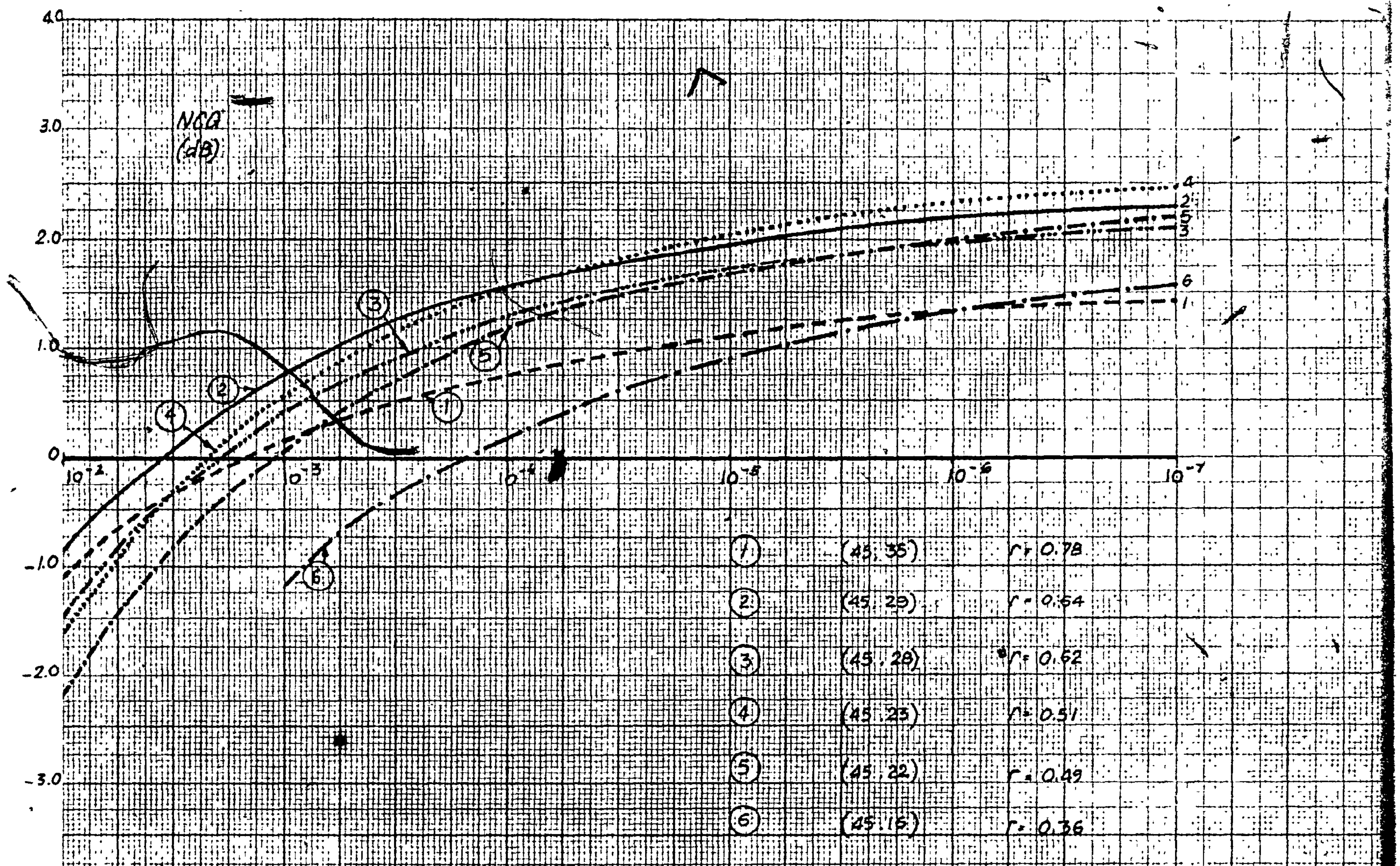


Fig. 3.4-3 NCG vs Error rate for selected length 45 codes CFSK Modem

3.5 Weight Distribution And The Net Coding Gain

In section 3.2 we assumed (Equation 3.2-3) binomial distribution of errors at the decoder's input. Hence, the probability of correct decoding is:

$$P_{CD} = \sum_{j=0}^{t} \binom{n}{j} P_i^{\,j} (1 - P_i)^{n-j} \qquad (3.5\text{-}1)$$

where P_i, as defined in section 3.1, is the transition probability in the coding channel of a protected communication system. Patterns (or blocks) of more than t errors can not be corrected by the decoder. If the received word contains more than t errors, the decoder may either detect the pattern as erroneous or may decode incorrectly into a wrong codeword.

In this section, we shall concentrate on the analysis of incorrect decoding so as to achieve closer approximation. Detected erroneous patterns are not included in the analysis since they may be flagged or suppressed by the decoder. Our analysis closely follows the work of Michelson (8) and Huntoon & Michelson (24).

If P_{ICD} (h) is the probability of incorrect decoding into a codeword of weight h, the probability of incorrect decoding is:

$$P_{ICD} = \sum_{h=d}^{n} P_{ICD}(h) \qquad (3.5\text{-}2)$$

For a group code (codes with group properties) and BSC we may assume that the all-zero word is transmitted by the encoder. An incorrect decoding will occur when the received word is interpreted by the decoder as a codeword other than the all-zero codeword. This occurs when the error pattern is located within a distance t of a non-zero codeword. From the group property of the code, it follows that the error probability analysis is independent of the transmitted or intended codeword, and the use of the all-zero word simplifies the bookkeeping.

In order to determine P_{ICD} (h) we defind M (h, ℓ; s) as the number of error patterns of weight ℓ which are at a distance s from a given codeword of weight h (i.e. $h - s \leq \ell \leq h + s$). Those patterns for which $s \leq t$ will be decoded into the codeword of weight h. If the error pattern is outside this range for all possible codewords of the code, error detection will be declared and no correction will be attempted.

The probability of occurrence of any one of the M (h, ℓ; s) error patterns is (see Equation 3.2-3):

$$P_{(\ell)} = P_i^{\ell} (1 - P_i)^{n - \ell}$$

Since the errors are independent M (h, ℓ, s) is the same for all codewords of weight h. Denoting W (h) the number of codewords of weight h in the code, we can write:

$$P_{ICD} (h) = W(h) \sum_{s = 0}^{t} \sum_{\ell = h - s}^{h + s} M (h, \ell; s) P_{(\ell)} \qquad (3.5\text{-}3)$$

In order to obtain the expression M (h, ℓ; s) for a particular codeword, let us denote a codeword C as

$$(C_1 \; C_2 \; C_3 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; C_n) \qquad (3.5\text{-}4)$$

with C_j the bit in the j^{th} position. An input error pattern E can be denoted as: $(e_1 \; e_2 \; . \; . \; . \; . \; . \; . \; . \; e_n)$ (3.5-5)
with e_j the bit in the j^{th} position.

If we designate μ as the measure of a set, we can write -

The weight of the error pattern E: $\ell = \mu \{ j : e_j \neq 0 \}$ (3.5-6)

The weight of the codeword C: $h = \mu \{ j : C_j \neq 0 \}$ (3.5-7)

Now we can define:

$$u = \mu \{ j : C_j = e_j \neq 0 \} \quad (3.5\text{-}8)$$

$$v = \mu \{ j : C_j \neq 0, \; e_j = 0 \} \quad (3.5\text{-}9)$$

$$w = \mu \{ j : C_j = 0, \; e_j \neq 0 \} \quad (3.5\text{-}10)$$

Since the sets are disjoint we can say: $\ell = u + w$ (3.5-11)

$$h = u + v \quad (3.5\text{-}12)$$

The distance between the error pattern and the codeword is the measure (cardinality) of the set of locations in which the two differ:

$$s = v + w \quad (3.5\text{-}13)$$

For a specific codeword of weight h, the number of possible error patterns with given values of u and w relative to the codeword is (24):

$$M(h, \ell; u, w) = \binom{h}{u} \cdot \binom{n - h}{w} \quad (3.5\text{-}14)$$

but, from Eq. 3.5-11, 3.5-12 and 3.5-13, we observe that

$$u = \frac{\ell + h - s}{2} \quad (3.5\text{-}15)$$

$$w = \frac{\ell - h + s}{2} \quad (3.5\text{-}16)$$

Hence, $M(h, \ell; s) = \binom{h}{\frac{h + \ell - s}{2}} \binom{n - h}{\frac{\ell - h + s}{2}}$ (3.5-17)

where $\binom{a}{b} = 0$ for b not an integer.

The probability of incorrect decoding to a codeword of weight h is:

$$P_{ICD}(h) = W(h) \sum_{s=0}^{t} \sum_{\ell = h - s}^{h + s} \binom{h}{\frac{h + \ell - s}{2}} \binom{n - h}{\frac{\ell - h + s}{2}} \cdot P(\ell) \quad (3.5\text{-}18)$$

The post decoding bit error rate can be obtained from:

$$Po = \frac{1}{n} \sum_{h = d}^{n} h \, P_{ICD}(h) \quad (3.5\text{-}19)$$

In order to illustrate the above procedure let us follow the block code (23, 12) as an example:

This code has minimum distance (d) of 7 and error correction capability (t) of 3. The weight distribution of this code (The Golay Code) is shown in table 3.5-1.

CODEWORD WEIGHT (h)	NUMBER OF WORDS OF WEIGHT (h)
0	1
7	253
8	506
11	1288
12	1288
15	506
16	253
23	1
all other h	0

Table 3.5-1 The Weight Distribution of the (23, 12) Golay Code.

The calculation of M (h, ℓ; s) can be done either by using the equations presented previously in this section, or by enumeration of all possible decoding error events as follows.

Since we assumed earlier that the all-zero codeword is transmitted and the decoder can successfully correct up to three errors, let us begin with the case of four errors.

Four Errors:

The decoder can add or delete up to three (t) errors per decode attempt and hence words of weight 1 through 7 can result. However, table 2.5-1 indicates codewords with weight 7 for the Golay Code. Thus, the only error event possible is decoding into a weight 7 codeword. A weight 7 codeword can be decoded from weight 4 error pattern only if the decoder attempts three bit changes. Each codeword of weight 7 (There are 253 such patterns, see table 3.5-1) contains $\binom{7}{4} = 35$ weight 4 patterns which will be decoded as such a codeword.

Five Errors:

Similarly, there are $\binom{7}{5}$ weight 5 patterns which will be decoded as a weight - 7 codeword through 2-bit change by the decoder. Also there are $\binom{8}{5}$ weight 5 patterns which will be decoded as a weight 8 codeword through 2-bit change by the decoder (There are 506 weight - 8 codewords, see table 3.5-1).

Six Errors:

These patterns will be decoded as weight 7 codewords through 1-bit change by the decoder. Each weight 7 codeword contains $\binom{7}{6} = 7$ possible weight 6 patterns. Another possibility is erroneous decoding of these patterns into weight 7 codewords through 3-bit changes by the decoder as follows:

2-bits are "0" to "1" changes while third bit is a "1" to "0" change. There are $\binom{7}{5}\cdot\binom{23-7}{1}$ = 336 such patterns. The last possibility involves decoding into weight 8 codewords through 2-bit changes by the decoder. Each weight 8 codeword contains $\binom{8}{6}$ = 28 possible weight 6 patterns.

Seven Errors:

This pattern can be decoded erroneously as a codeword of weight 7. It can also be decoded as a codeword of weight 8 through one bit change by the decoder (Each weight 8 codeword contains 8 possible weight 7 patterns). The last possible misinterpretation is into a weight 8 codeword through 3-bit changes by the decoder as follows: 2 bits are "0" to "1" changes while the third bit is a "1" to "0" change. There are $\binom{8}{6}\cdot\binom{23-8}{1}$ = 420 such patterns.

A table can be constructed to enumerate all possible error patterns (see table 3.5-2). The analysis of decoding events for h > 11 can be obtained from table 3.5-2 by symmetry arguments.

Finally the value P_{ICD} (h) can be obtained from equations 3.5-18 and 3.2-3. As a result Po dan be calculated and used for NCG evaluation. Since the Golay Code (23, 12) is a perfect code - no improvement in the previously calculated NCG is achieved.

Bit Error Rate P(e)	NCG (dB)
10^{-3}	0.79
10^{-5}	1.90
10^{-7}	2.33

Table 3.5-3 NCG for the (23, 12) Golay Code (NCFSK)

Codeword Weight h	Error Pattern Weight	Distance to Codeword	Number of Possible Decoding Errors
7	7	0	$\binom{7}{7} = 1$
	6	1	$\binom{7}{6} = 7$
	8	1	$\binom{23-7}{1} = 16$
	5	2	$\binom{7}{5} = 21$
	7	2	$\binom{7}{6}\binom{23-7}{1} = 112$
	9	2	$\binom{23-7}{2} = 120$
	4	3	$\binom{7}{4} = 35$
	6	3	$\binom{7}{5}\binom{23-7}{1} = 336$
	8	3	$\binom{7}{6}\binom{23-7}{2} = 840$
	10	3	$\binom{23-7}{3} = 560$
8	8	0	$\binom{8}{8} = 1$
	7	1	$\binom{8}{7} = 8$
	9	1	$\binom{23-8}{1} = 15$
	6	2	$\binom{8}{6} = 28$
	8	2	$\binom{8}{7}\binom{23-8}{1} = 120$
	10	2	$\binom{23-8}{2} = 105$
	5	3	$\binom{8}{5} = 56$
	7	3	$\binom{8}{6}\binom{23-8}{1} = 420$
	9	3	$\binom{8}{7}\binom{23-8}{2} = 840$
	11	3	$\binom{23-8}{3} = 455$
11	11	0	$\binom{11}{11} = 1$
	10	1	$\binom{11}{10} = 11$
	12	1	$\binom{23-11}{1} = 12$
	9	2	$\binom{11}{9} = 55$

Codeword Weight h	Error Pattern Weight	Distance to Codeword	Number of Possible Decoding Errors
	11	2	$\binom{11}{10}\binom{23-11}{1} = 132$
	13	2	$\binom{23-11}{2} = 66$
	8	3	$\binom{11}{8} = 165$
	10	3	$\binom{11}{9}\binom{23-11}{1} = 660$
	12	3	$\binom{11}{10}\binom{23-11}{2} = 726$
	14	3	$\binom{23-11}{3} = 220$

Table 3.5-2 Decoding Errors in the (23, 12) Golay Code

The procedure detailed in this section was then used for the analysis of a (48, 24) linear block code using NCFSK modulation. The weight distribution of this code is shown in Table 3.5-4.

h	W (h)
0	1
12	17296
16	535095
20	3995376
24	7681680
28	3995376
32	535095
36	17296

Table 3.5-4 The Weight Distribution of the (48, 24) Code

A table of enumeration of all possible error patterns is provided in Appendix B.

Following Eq. 3.5-18 and Eq. 3.2-3 the NCG figures were found to be

Bit Error Rate P(e)	NCG dB
10^{-3}	2.77
10^{-5}	3.90
10^{-7}	4.46

Table 3.5-5 NCG for the (48, 24) Code (NCFSK)

We can conclude that using the weight distribution of the code and the method outlined above provides a better approximation to the calculation of NCG.

CHAPTER 4

CONCLUSION

4.1 Conclusion

The concept of Net Coding Gain was introduced as a measure of performance of a digital communications system. It was shown that this measure incorporates both modulation and coding considerations, and permits the study of trade-offs between increased modulation rate and improved error-rate performance of a code.

The block codes studied here, quasi-cyclic rate $\frac{1}{2}$ and others in conjunction with various modulation techniques were shown to be most effective with longer block length and low probability of error. Superiority of CFSK(PSK) modems was demonstrated. The same measure, namely the NCG, was used in conjunction with the same modulation technique to deduce the optimal level of redundancy in block codes. It was observed that redundancy between 1.8 db to 3 db provides for highest NCG when used at low error rates.

Figures of NCG determined using the "minimum distance bound" were improved upon by using the weight distribution of the specific code in use. Analysis of the weight distribution of the code, as conducted here, provides better figures of NCG for all block codes which are not "perfect".

4.2 Suggestions for Further Work

This study dealt with binary modulation techniques in memoryless channels. The scope of this work may be enhanced by providing the NCG figures for M-ary modulation techniques on one hand and for channels with memory on the other hand. The modem curves for some M-ary systems are known (e.g. QPSK, 8 level PSK, etc.) The probability of occurrence of M errors in a sequence of n digits using DPSK system is also known.

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REFERENCES

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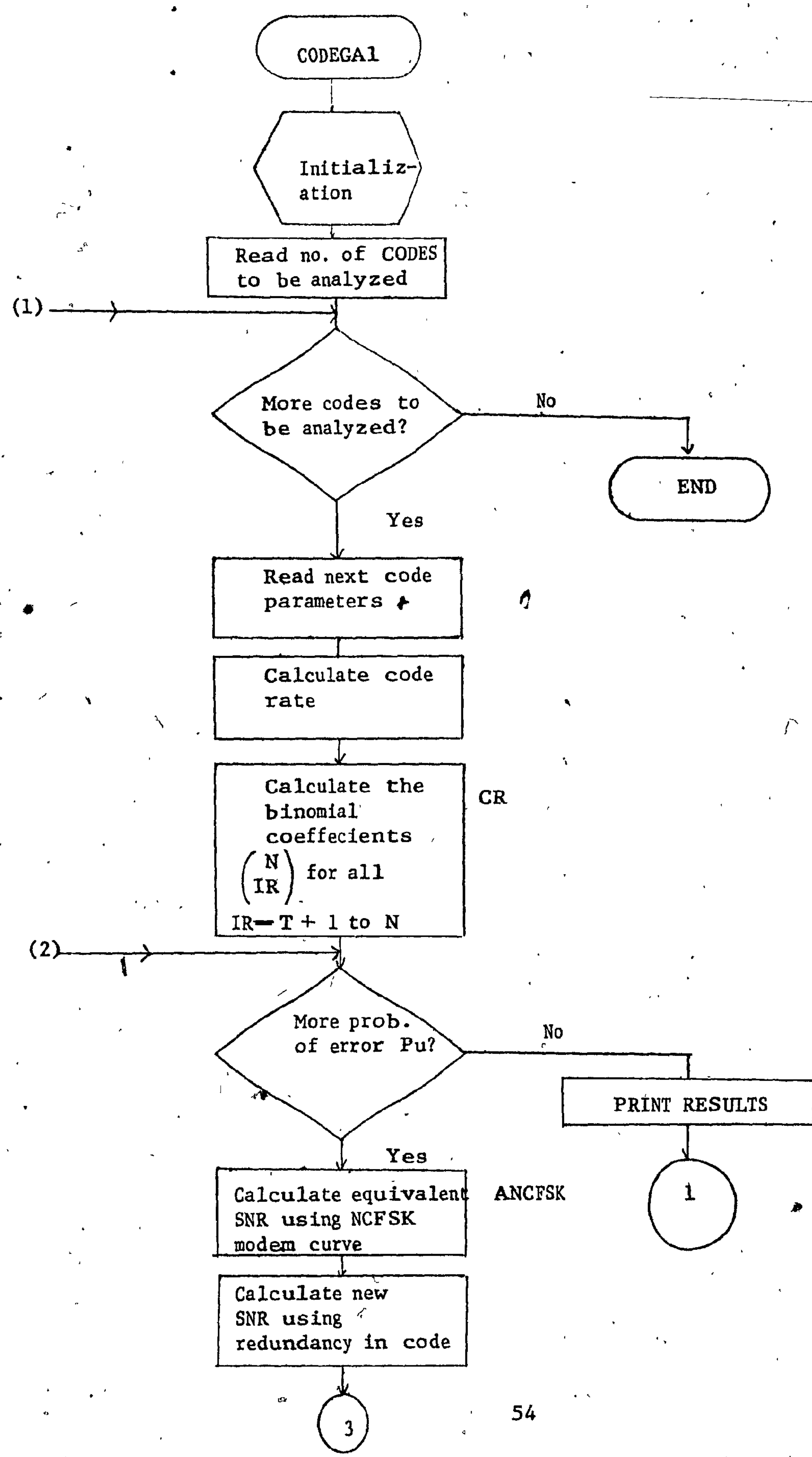
APPENDIX

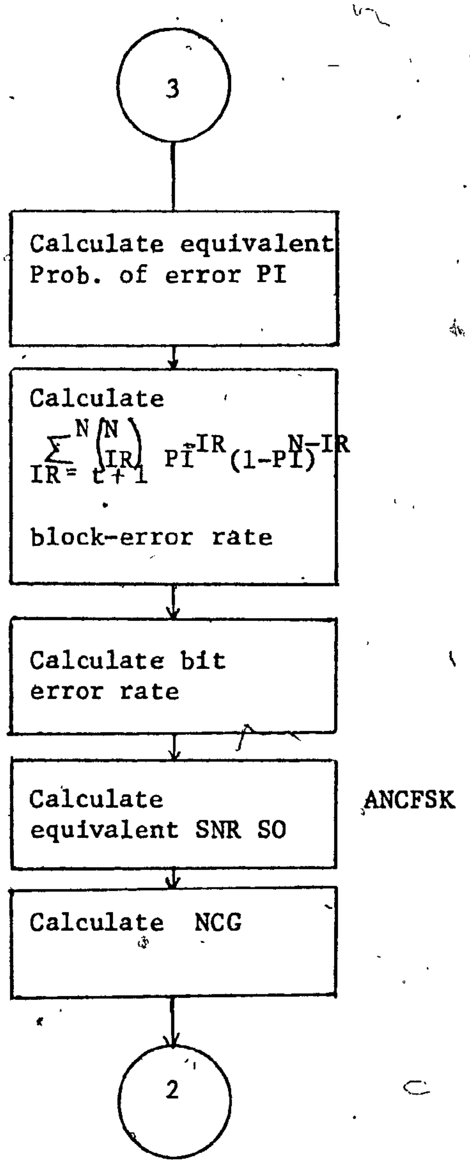
APPENDIX A

COMPUTER PROGRAMS

The analysis of the NCG made use of computer programs as described below:

CODEGA1	Calculation of NCG using NCFSK modem. Result is in a table format where the parameter is the probability of error for the uncoded case.
CODEGA2	Similar to CODEGA1 with the exception of using CFSK modulation technique. Replace routine ANCFSK with ACFSK and replace NCFSK with routine CFSK.
GRAPCG1	Similar to CODEGA1 with the exception of result is in a graph format.
GRAPCG 2	Similar to CODEGA2. Result is in a graph format.
NCGWD1	Similar to CODEGA1 with the exception of using weight distribution of the code instead of "minimum distance bound" approximation.


CODEGA1
Initialization
Read no. of CODES to be analyzed
(1)
More codes to be analyzed?
No
END
Yes
Read next code parameters
Calculate code rate
Calculate the binomial coeffecients
$\binom{N}{IR}$ for all
IR = T + 1 to N
CR
(2)
More prob. of error Pu?
No
PRINT RESULTS
1
Yes
Calculate equivalent SNR using NCFSK modem curve
ANCFSK
Calculate new SNR using redundancy in code
3


3
Calculate equivalent Prob. of error PI
Calculate
$\sum_{IR=t+1}^{N} \binom{N}{IR} PI^{IR}(1-PI)^{N-IR}$
block-error rate
Calculate bit error rate
Calculate equivalent SNR SO
ANCFSK
Calculate NCG
2

Subroutine **CR** (Used in all programs)

$$\binom{N}{IR} = \exp\left\{ \ln\left[\Gamma(n+1)\right] - \ln\left[\Gamma(IR+1)\right] - \ln\left[\Gamma(N - IR + 1)\right] \right\}$$

Return

Subrountine **ANCFSK** (Used in CODEGA1, GRAPCG1 and NCGWD1)

$$S = 10 \log_{10}(-2 \, a\ln (2P))$$

Return

Subroutine **NCFSK** (Used in CODEGA1, GRAPCG1 and NCGWD1)

$$P = (\tfrac{1}{2}) \exp\left\{ (\tfrac{1}{2})(-10^{S/10}) \right\}$$

Return

Subroutine **ACFSK** (Used in CODEGA2 and GRAPCG 2)

$$S = 10 \log_{10} \left\{ \left[\mathrm{erfc}^{-1}(2P) \right]^{2} / 0.6 \right\}$$

Return

Subroutine **CFSK** (Used in CODEGA2 and GRAPCG 2)

$$P = \left(\tfrac{1}{2}\right) \mathrm{erfc} \left[0.6 \; 10^{S/10} \right]^{\frac{1}{2}}$$

Return

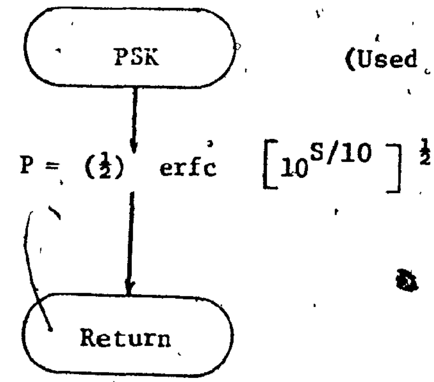
Subroutine **APSK** (Used in CODEGA3 and GRAPCG3)

$$S = 10 \log_{10} \left\{ \left[\mathrm{erfc}^{-1} (2P) \right]^{2} \right\}$$

Return

Subroutine

(Used in CODEGA3 and GRAPCG 3)



APPENDIX B

PROGRAM LISTINGS AND RESULTS

```
      PROGRAM NCGWD1(INPUT,OUTPUT)
C*    THIS PROGRAM CALCULATES THE NCG USING NCFSK MODEM.
C*            N IS THE BLOCK LENGTH
C*            K IS THE INFORMATION BITS IN EACH BLOCK N
C*            D IS THE MINIMUM DISTANCE
C*            T IS THE CORRECTION POWER OF THE CODE
C*            IGRAPH IS THE NUMBER OF CODES ANALYZED
C*   AUTHOR     M. AVNI
C
C*************************************************************************
C*************INITIALIZATION**********************************************
C*************************************************************************
      DIMENSION CE(50)
      DIMENSION X(3),Y(3,30)
      INTEGER D,T,H,W(50)
      COMMON N,K,D,T,W
      READ 300,IGRAPH
  300 FORMAT(4I3)
      PRINT 310,IGRAPH
  310 FORMAT("1NO OF CODES ANALYZED=",I2)
      DO 480 IG=1,IGRAPH
      READ 300,N,K,D,T
      DO 80 I=1,50
      W(I)=0
   80 CONTINUE
  130 READ 200,H,W(H)
  200 FORMAT(I3,I8)
      IF (W(50))250,120,120
  120 PRINT 220,H,W(H)
  220 FORMAT(1X,2I10)
      GO TO 130
  250 CONTINUE
      PRINT 240,(W(I),I=1,50)
  240 FORMAT(1X,10I10)
      FN=FLOAT(N)
      R=K/FN
      ITP1=T+1
      DO 360 IR=ITP1,N
      FR=FLOAT(IR)
      CALL CR(FN,FR,C)
      CE(IR)=C
  360 CONTINUE
      PINC=1.0
      DO 400 I=1,3
      PINC=PINC+2.0
      PU=10.0**(-PINC)
      CALL ANCFSK(PU,SU)
      SI=SU+10.0*ALOG10(R)
      CALL NCFSK(SI,PI)
      CALL EPRATE(I,PI,PO)
      CALL ANCFSK(PO,SO)
      G=SO-SU
      X(I)=PINC
      Y(I,IG)=G
  400 CONTINUE
        PRINT 320,IG,N,K,D,T,R,(Y(I,IG),I=1,3)
  320 FORMAT(" CODE",I2," N=",I2," K=",I2," ID=",I2," IT=",I2," R=",
```

```
      CF4.2,3F7.3)
  480 CONTINUE
      STOP
      END
```

73/172 OPT=1 FTN 4.6+446

```
      SUBROUTINE CR(FN,FR,C)
C*          THIS SUBROUTINE CALCULATES THE BINOMIAL COEFFICIENTS
C*          INPUT-FN,FR
C*          OUTPUT-C
      FN1=FN+1.0
      FR1=FR+1.0
      FJ1=FN-FR+1.0
      CALL MLGAMA(FN1,FAN,IER)
      CALL MLGAMA(FR1,FAR,IER)
      CALL MLGAMA(FJ1,FAJ,IER)
      FACT=FAN-FAR-FAJ
      C=EXP(FACT)
      RETURN
      END
```

73/172 OPT=1 FTN 4.6+446

```
      SUBROUTINE ANCFSK(P,S)
C***   THIS SUBROUTINE CALCULATES A NCFSK MODEM
C***   INPUT-POWER PER BIT TO NOISE   S
C***   OUTPUT- PROBABILITY OF ERROR P
      S=10.0*ALOG10(-2.0*ALOG(2.0*P))
      RETURN
      END
```

73/172 OPT=1 FTN 4.6+446

```
      SUBROUTINE NCFSK(S,P)
C***   THIS SUBROUTINE CALCULATES A NCFSK MODEM
C***   INPUT- POWER PER BIT TO NOISE  S
C***OUTPUT-PROBABILITY OF ERROR  P
      ARG=-10.0**(S/10.0)
      P=0.5*EXP(0.5*ARG)
      RETURN
      END
```

```
      SUBROUTINE ERRATE(I,PI,PO)
      INTEGER D,T,H,S,T1,W(50)
      COMMON N,K,D,T,W
      SUM3=0.0
      DO 100 H=D,N
      EMH=0.0
      IF(W(H))160,100
160   SUM2=0.0
      T1=T+1
      DO 200 IS=1,T1
      S=IS-1
      SUM1=0.0
      IS2=2*S+1
      DO 300 IL=1,IS2
      L=IL+H-S-1
      IF(L+S-N)110,110,300
110   PL=(PI**L)*(1.0-PI)**(N-L)
      JM=MOD((H+L-S),2)
      JN=MOD((L-H+S),2)
      IF(JM)300,120
120   IF(JN)300,140
140   FL1=(H+L-S)/2.0
      FL2=(L-H+S)/2.0
      FL3=FLOAT(H)
      FL4=FLOAT(N-H)
      CALL CR(FL3,FL1,C1)
      CALL CR(FL4,FL2,C2)
      EM=C1*C2
      IF(I.EQ.1)150,170
150   PRINT 330,EM,L,H,S
330   FORMAT(1X,E13.7,3I6)
170   EMP=EM*PL
      SUM1=SUM1+EMP
      EMH=EMH+EM
300   CONTINUE
      SUM2=SUM2+SUM1
200   CONTINUE
      PICD=SUM2*W(H)
      HPI=H*PICD
      SUM3=SUM3+HPI
      PRINT 330,EMH
100   CONTINUE
      PO=SUM3/N
      RETURN
      END
```

.1000000E+01	12	12	0
.1200000E+02	11	12	1
.3600000E+02	13	12	1
.6600000E+02	10	12	2
.4320000E+03	12	12	2
.6300000E+03	14	12	2
.2200000E+03	9	12	3
.2376000E+04	11	12	3
.7560000E+04	13	12	3
.7140000E+04	15	12	3
.4950000E+03	8	12	4
.7920000E+04	10	12	4
.4158000E+05	12	12	4
.8568000E+05	14	12	4
.5890500E+05	16	12	4
.7920000E+03	7	12	5
.1782000E+05	9	12	5
.1386000E+06	11	12	5
.4712400E+06	13	12	5
.7068600E+06	15	12	5
.3769920E+06	17	12	5
.1925357E+07			
.1000000E+01	16	16	0
.1600000E+02	15	16	1
.3200000E+02	17	16	1
.1200000E+03	14	16	2
.5120000E+03	16	16	2
.4960000E+03	18	16	2
.5600000E+03	13	16	3
.3840000E+04	15	16	3
.7936000E+04	17	16	3
.4960000E+04	19	16	3
.1820000E+04	12	16	4
.1792000E+05	14	16	4
.5952000E+05	16	16	4
.7936000E+05	18	16	4
.3596000E+05	20	16	4
.4368000E+04	11	16	5
.5824000E+05	13	16	5
.2777600E+06	15	16	5
.5952000E+06	17	16	5
.5753600E+06	19	16	5
.2013760E+06	21	16	5
.1925357E+07			
.1000000E+01	20	20	0
.2000000E+02	19	20	1
.2800000E+02	21	20	1
.1900000E+03	18	20	2
.5600000E+03	20	20	2
.3780000E+03	22	20	2
.1140000E+04	17	20	3

.5320000E+04	19	20	3
.7560000E+04	21	20	3
.3276000E+04	23	20	3
.4845000E+04	16	20	4
.3192000E+05	18	20	4
.7182000E+05	20	20	4
.6552000E+05	22	20	4
.2047500E+05	24	20	4
.1550400E+05	15	20	5
.1356600E+06	17	20	5
.4309200E+06	19	20	5
.6224400E+06	21	20	5
.4095000E+06	23	20	5
.9828000E+05	25	20	5
.1925357E+07			
.1000000E+01	24	24	0
.2400000E+02	23	24	1
.2400000E+02	25	24	1
.2760000E+03	22	24	2
.5760000E+03	24	24	2
.2760000E+03	26	24	2
.2024000E+04	21	24	3
.6624000E+04	23	24	3
.6624000E+04	25	24	3
.2024000E+04	27	24	3
.1062600E+05	20	24	4
.4857600E+05	22	24	4
.7617500E+05	24	24	4
.4857600E+05	26	24	4
.1062600E+05	28	24	4
.4250400E+05	19	24	5
.2550240E+06	21	24	5
.5586240E+06	23	24	5
.5586240E+06	25	24	5
.2550240E+06	27	24	5
.4250400E+05	29	24	5
.1925357E+07			
.1000000E+01	28	28	0
.2800000E+02	27	28	1
.2000000E+02	29	28	1
.3780000E+03	26	28	2
.5600000E+03	28	28	2
.1900000E+03	30	28	2
.3276000E+04	25	28	3
.7560000E+04	27	28	3
.5320000E+04	29	28	3
.1140000E+04	31	28	3
.2047500E+05	24	28	4
.6552000E+05	26	28	4
.7182000E+05	28	28	4
.3192000E+05	30	28	4
.4845000E+04	32	28	4
.9828000E+05	23	28	5
.4095000E+06	25	28	5
.6224400E+06	27	28	5
.4309200E+06	29	28	5
.1356600E+06	31	28	5
.1550400E+05	33	28	5