Connectedness of the Attractor
of an
Iterated Function System

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ABSTRACT

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We will study the action of a finite family, $\{F_i\}_{i=1}^m$, of contractive mappings on a compact subset of a complete metric space. The family $\{F_i\}_{i=1}^m$ is known as an iterated function system (IFS). Although many of the results presented here can be generalized to an arbitrary complete metric space, we will restrict ourselves to the metric space ($\mathbb{R}^n$, Euclidean distance). Using the notion of Minkowski sausages, we will define a distance function, h (the Hausdorff distance), on the space of all compact subsets of $\mathbb{R}^n$ denoted by $\mathcal{H}(\mathbb{R}^n)$. The pair $(\mathcal{H}(\mathbb{R}^n), h)$ forms a complete metric space in which we will establish the existence of a unique "point" $\mathcal{A}$ (compact subset of $\mathbb{R}^n$) satisfying the equation $\bigcup_{i=1}^m F_i(\mathcal{A}) = \mathcal{A}$. We will explore various characterizations of the compact set $\mathcal{A}$ which is referred to as the attractor of the IFS $\{F_i\}_{i=1}^m$.

The topological properties of an attractor depend on the contractive mappings constituting the corresponding IFS. The main purpose of this study is to investigate conditions under which an attractor will have certain connectivity properties. Among these properties will be considered total disconnectedness, connectedness and arcwise connectedness. We will see that in fact, the notions of connectedness and arcwise connectedness coincide in the case of an attractor. Some other topological properties of attractors including property S, local connectedness and semi-local connectedness will also be discussed.

To the Beautiful
Forests and Wildlife
of Canada

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1 Preliminaries in topology and analysis

In this chapter we will briefly review some well-known notions of topology and real analysis. Note that we have selected only the concepts that we will need in the following chapters. Eventhough we will consider the general concept of topological spaces in Section 1.1, we will be almost merely dealing with metric spaces throughout our discussion.

1.1 Topological spaces

Let $\mathcal{X}$ be a set. We say that a family $\mathcal{J}$ of subsets of $\mathcal{X}$ defines a *topology* on $\mathcal{X}$ provided that

- $\emptyset$ and $\mathcal{X}$ belong to $\mathcal{J}$,
- The union of any collection of sets of $\mathcal{J}$ is in $\mathcal{J}$,
- The intersection of any finite number of sets of $\mathcal{J}$ is in $\mathcal{J}$.

The pair $(\mathcal{X}, \mathcal{J})$ is called a *topological space*. When there is no ambiguity as in the topology chosen on $\mathcal{X}$, the set $\mathcal{X}$ itself is referred to as the topological space. The elements (sets) of $\mathcal{J}$ are called *open* sets. A set $\mathcal{F} \subseteq \mathcal{X}$ is *closed* if $\mathcal{X} \setminus \mathcal{F}$ is open.

Proposition 1

- *The intersection of any collection of closed sets is closed.*
- *The union of any finite collection of closed sets is closed.*

Although there is no ambiguity in the above definition of an open set, one needs to be careful when using the term "open subset". In fact, we have the following definition to distinguish between the two notions:

Let M be any subset of a topological space $\mathcal{X}$. A subset D of M is said to be *open in M* or *open relative to M* provided that there exists an open set $V \subset \mathcal{X}$ such that

$D = M \cap V$. So, by saying that D is open in M, we do not necessarily mean that D is a member of the topology defined on $\mathcal{X}$.

The following lemma gives another characterization of a set which is open relative to another set, but we will first have to introduce the concept of a neighborhood of a point:

Let $\mathcal{X}$ be a topological space. Any open set containing the point $x \in \mathcal{X}$ is called a *neighborhood* of x.

Lemma 1

Let M be a subset of a topological space $\mathcal{X}$. Let $K \subset M$ and suppose that for any point $x \in K$, there exists a neighborhood V_x of x such that $M \cap V_x \subset K$. Then, K is open in M.

Proof:

Since there exists a neighborhood V_x of x such that $M \cap V_x \subset K$ for every $x \in K$,

$$\bigcup_{x \in K} M \cap V_x \subset K$$

which implies that

$$M \cap \bigcup_{x \in K} V_x \subset K.$$

On the other hand, for any $x \in K$, we have

$$x \in M \quad \text{and} \quad x \in V_x,$$

so that $K \subset M \cap \bigcup_{x \in K} V_x$. Therefore, $M \cap \bigcup_{x \in K} V_x = K$. But $\bigcup_{x \in K} V_x$ is a union of open sets and is therefore open. So, K is open in M. ♠

We say that a collection C of sets *covers* a set E if $E \subset \bigcup_{\mathcal{O} \in C} \mathcal{O}$. In this case, the collection C is called a *covering* of E. If each $\mathcal{O} \in C$ is open, we call C an *open covering* of E. If C contains only a finite number of sets, we call C a *finite covering*.

A topological space $\mathcal{X}$ is said to be *compact* if every open covering of $\mathcal{X}$ contains a finite subcovering of $\mathcal{X}$.

1.2 Metric spaces

We will be interested in topological spaces with a special structure, namely, metric spaces.

A function $d : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is called a *metric* on a set $\mathcal{X}$ if for all x, y and z in $\mathcal{X}$,

- $d(x,y) \geq 0$, and $d(x,y) = 0$ iff $x = y$,
- $d(x,y) = d(y,x)$ (symmetry),
- $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality).

The real number $d(x,y)$ is called the *distance* between the points x and y, relative to the metric d. The pair $(\mathcal{X}, d)$ is called a *metric space*. When there is no confusion about d, we denote the metric space $(\mathcal{X}, d)$ simply by $\mathcal{X}$.

We will now list the following definitions concerning a metric space $(\mathcal{X}, d)$. Note that some concepts such as open and compact sets have already been defined in a topological space. However, for the purpose of our discussion, we will more often adopt their characterizations in a metric space, as stated below.

1. A *δ-neighborhood* of a point $x \in \mathcal{X}$ is a ball $B_\delta(x)$ with radius δ centered at x:

$$B_\delta(x) = \{y \in \mathcal{X} \mid d(x,y) < \delta\}.$$

2. A set $\mathcal{O} \subset \mathcal{X}$ is *open* if

$$\forall x \in \mathcal{O}, \ \exists \delta > 0 \ \text{ such that } \ B_\delta(x) \subset \mathcal{O}.$$

3. The *distance* between a point $x \in \mathcal{X}$ and a set $E \subset \mathcal{X}$ is defined as

$$d(x, E) = \inf_{y \in E} d(x,y).$$

4. The *diameter* of a set $E \subset \mathcal{X}$ is defined as

$$\operatorname{diam}(E) = \sup\{d(x,y) |\ x \in E,\ y \in E\}.$$

5. Let $E \subset \mathcal{X}$. Then $x \in \mathcal{X}$ is called a *limit point* (*point of accumulation* or *cluster point*) of E if

$$\forall \delta > 0, \quad (B_\delta(x) \setminus \{x\}) \cap E \neq \emptyset.$$

6. The *closure* of a set $E \subset \mathcal{X}$, denoted by $\overline{E}$ or $\mathrm{Cl}(E)$, is defined by

$$\begin{aligned} \overline{E} &= E \cup \{\text{limit points of E}\} \\ &= \{x \in \mathcal{X} |\ \forall \delta > 0,\ B_\delta(x) \cap E \neq \emptyset\}. \end{aligned}$$

   A point $x \in \overline{E}$ is called a *point of closure* of E. Therefore, every limit point of E is also a point of closure of E. Notice that E is closed if and only if $E = \overline{E}$.

7. The *interior* of a set $E \subset \mathcal{X}$, denoted by $\mathrm{Int}(E)$, is defined by $\mathcal{X} \setminus \overline{\mathcal{X} \setminus E}$. The set $\mathrm{Bd}(E) = \overline{E} \cap \overline{\mathcal{X} \setminus E}$ is called the *boundary* of E.

8. A sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is said to be *convergent* if

$$\exists y \in \mathcal{X} \text{ such that } \lim_{n\to\infty} d(x_n, y) = 0.$$

   We say that $\{x_n\}_{n=1}^{\infty}$ converges to y and write $\lim_{n\to\infty} x_n = y$, or $x_n \underset{n\to\infty}{\longrightarrow} y$. The point y is called the *limit* of $\{x_n\}_{n=1}^{\infty}$.

9. A sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is called a *Cauchy* sequence if

$$\forall \epsilon > 0, \quad \exists N \text{ such that } \forall n \geq N, \quad \forall m \geq N, \quad d(x_m, x_n) \leq \epsilon.$$

   It follows that every convergent sequence is a Cauchy sequence.

10. If every Cauchy sequence in $\mathcal{X}$ is convergent, then $\mathcal{X}$ is called a *complete* metric space.

11. Let $E \subset \mathcal{X}$. Then $x \in E$ is an *isolated point* of E if

$$\exists \delta > 0 \text{ such that } B_\delta(x) \cap E = \{x\}.$$

Equivalently, $x \in E$ is an isolated point of E if it is not a limit point of E.

12. A set $E \subset \mathcal{X}$ is *perfect* if E={limit points of E}.
In other words, E is perfect if and only if E is closed and contains no isolated point.

13. A set $K \subset \mathcal{X}$ is compact if and only if it is *sequentially compact*; i.e., if every sequence in K has a convergent subsequence in K.

1.3 Continuous functions

A function $f : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be *continuous at* $x \in \mathcal{X}$ if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that } d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon.$$

We say that f is *continuous* if it is continuous at each $x \in \mathcal{X}$.

Theorem 1

The following are equivalent:

- *$f : \mathcal{X} \longrightarrow \mathcal{X}$ is continuous.*
- *If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence,*
  then $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$.
- *If $\mathcal{O} \subset \mathcal{X}$ is open, then $f^{-1}(\mathcal{O})$ is also open.*

Theorem 2

Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous. If $K \subset \mathcal{X}$ is compact, then $f(K)$ is also compact.

Lemma 2

Let K be a compact subset of a metric space $\mathcal{X}$ and let $x \in \mathcal{X}$. Then there exists a point $y \in K$ such that $d(x, y) = d(x, K)$.

Proof:

Since $d(x, K) = \inf\{d(x, y) | \, y \in K\}$, we can choose a sequence of points $\{y_n\}_{n=1}^{\infty} \subset K$ such that $\lim_{n\to\infty} d(x, y_n) = d(x, K)$. Since K is compact, $\{y_n\}_{n=1}^{\infty}$ has a subsequence $\{y_{n_j}\}_{n_j=1}^{\infty}$ which converges to a point $y \in K$.

Finally, by continuity of the distance function d, we get

$$\begin{aligned} d(x, K) &= \lim_{n\to\infty} d(x, y_n) \\ &= \lim_{n_j\to\infty} d(x, y_{n_j}) \\ &= d(x, \lim_{n_j\to\infty} y_{n_j}) \\ &= d(x, y). \quad \spadesuit \end{aligned}$$

A function $f : \mathcal{X} \longrightarrow \mathcal{X}$ is called a *homeomorphism* on $\mathcal{X}$ if it is bijective (one-to-one and onto) and both f and f^{-1} are continuous.

Proposition 2

Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a homeomorphism on $\mathcal{X}$ and let $\mathcal{O}$ be an open subset of $\mathcal{X}$. Then $f(\mathcal{O})$ is also open.

Proof:

Since f^{-1} is a continuous function and $\mathcal{O}$ is open, by Theorem 1, $(f^{-1})^{-1}(\mathcal{O})$ is open. But, $(f^{-1})^{-1}(\mathcal{O}) = f(\mathcal{O})$. $\spadesuit$

2 Connectedness

One can study different topological properties of a given set. Among those, the concept of connectedness is one of the most intuitive ones. There are several types of connectedness. In this chapter we will study the main properties of a few of them (see [22], [23], [14]).

2.1 Connected sets

A *separation* of a set M is a pair $\mathcal{O}_1$, $\mathcal{O}_2$ of open sets such that

- $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$,
- $M \subset \mathcal{O}_1 \cup \mathcal{O}_2$,
- $M \cap \mathcal{O}_1 \neq \emptyset$ and $M \cap \mathcal{O}_2 \neq \emptyset$.

A set M is *connected* if there exists no separation of M. A compact connected set is called a *continuum*.

Example 2.1.1

Any closed interval in $\mathbb{R}$ is a continuum. ♡

A set U is called a *component* of a set M if U is a connected subset of M which is not contained in any other connected subset of M.

Example 2.1.2

On the real line $\mathbb{R}$, consider the set $M := (0,1) \cup (2,3) \cup \{4\}$. Then, the components of M are the intervals (0,1) and (2,3) as well as the singleton $\{4\}$. ♡

The following useful results on connectedness are well-known:

Theorem 3 (See [14])

Let M be a connected set. If $M \subset B \subset \bar{M}$, then B is also connected.

Theorem 4 (See [14])

The union of a collection of connected sets that have a point in common is connected.

Theorem 5

Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be continuous. If $M \subset \mathcal{X}$ is connected, then so is $f(M)$.

2.2 Locally connected sets

A set M is *locally connected at* $x \in M$ if for every neighborhood U of x, there is a neighborhood V of x such that $V \subset U$ and each point of $M \cap V$ lies together with x in a connected subset of $M \cap U$. If M is locally connected at each of its points, it is said to be *locally connected.*

Example 2.2.1

The topologist's sine curve (Fig. 1),

$$\{(x, sin\frac{1}{x})|\ x > 0\} \bigcup (\{0\} \times [-1, 1]),$$

is locally connected everywhere except at the points belonging to $\{0\} \times [-1, 1]$. ♡

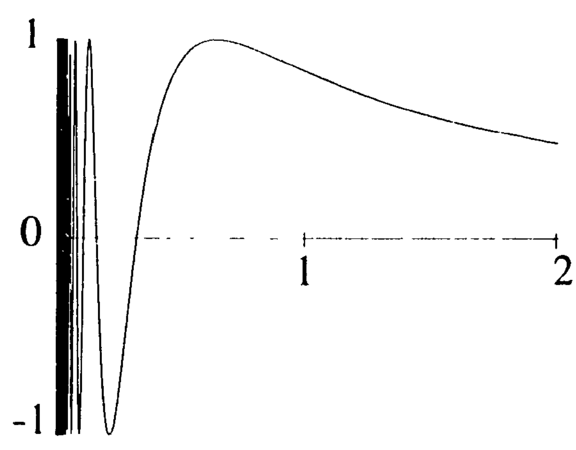


Figure 1: The topologist's sine curve.

Example 2.2.2

The "deleted comb" (Fig. 2),

$$([0,1]\times 0)\cup(\{\frac{n}{n+1}\}_{n=1}^{\infty}\times[0,1])\cup(\{1\}\times\{1\}),$$

is locally connected everywhere except at the point $\{1\}\times\{1\}$. ♡

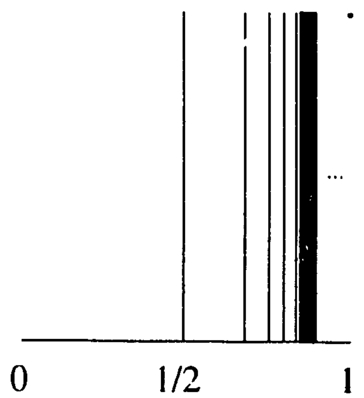


Figure 2: The deleted comb.

Example 2.2.3

Any open set $\mathcal{O}$, as well as its closure $\overline{\mathcal{O}}$ are locally connected. ♡

Let M be a set. A connected set $R\subset M$ is called a *region* in M if it is relatively open in M.

Theorem 6

A set M is locally connected if and only if for each point $x\in M$ and each neighborhood U of x, there exists a region R in M containing x and lying in U. Equivalently, in a metric space, each point of M lies in an arbitrary small region in M.

Proof: (See [23])

($\Longrightarrow$):

Suppose that M is locally connected. Let $x\in M$ and let U be a neighborhood of x.

Let K be the component of $M \cap U$ which contains x. We claim that K is a region in M. Indeed, $K \subset M \cap U \subset M$ and K is connected. We only have to show that K is open relative to M. Now, by the local connectedness of M, there is a neighborhood V_x of x such that $V_x \subset U$ and $M \cap V_x$ lies in a connected subset of $M \cap U$. But then, $M \cap V_x$ must be contained in K; otherwise K would not be maximal. We conclude that $M \cap V_x$ is an open set in M and satisfies $x \in M \cap V_x \subset K$. Since this is true for every $x \in M$ (and so, for every $x \in K \subset M$), by Lemma 1, K is open in M. In fact, K is the desired region since $K \subset M \cap U \subset U$.

($\Longleftarrow$):

Suppose that for each $x \in M$ and any neighborhood U of x, we can find a region R satisfying the above conditions. Since R is open in M, there exists an open set V such that $R = M \cap V$. Now, consider the open set $W = V \cap U$. We have

- $x \in R = M \cap V$ and $x \in U$; therefore, $x \in W$.
- $W \subset U$.
- $M \cap W = M \cap (V \cap U) = (M \cap V) \cap U = R \cap U \subset R$.
- By the hypothesis, R is a connected subset of $M \cap U$.

Therefore, M is locally connected at x which implies, by arbitrariness of $x \in M$, that M is locally connected. ♠

2.3 Property S

A set M is said to have *property S* provided that for each $\epsilon > 0$, M can be written as a finite union of connected sets each of diameter less than ϵ.

Example 2.3.1

The set $M = \{\{\frac{1}{n}\} \times [0, \frac{1}{n}]\}_{n=1}^{\infty} \cup ([0,1] \times \{0\})$, has property S (see Fig. 3). ♡

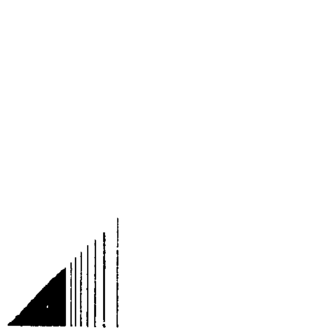


Figure 3: A set with property S.

Example 2.3.2

The topologist's sine curve (see Example 2.2.1) fails to have property S. ♡

Theorem 7

If a set has property S, then it is locally connected.

Proof: (See [22])

Let M be a set with property S. So, given $\epsilon > 0$, we can write $M = \bigcup_{i=1}^{n} M_i$, where M_i is a connected set with $\text{diam}(M_i) < \frac{\epsilon}{2}$ for $i = 1, ..., n$. Let $x \in M$ and let K be the union of all those M_i's which either contain x or have x as a limit point. So, by Theorems 3 and 4, K is connected. Also, $\text{diam}(K) < \epsilon$, since for $p, q \in K$, we have

$$\begin{aligned} d(p,q) &\leq d(p,x) + d(x,q) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

It follows, from the definition of K, that x cannot be a limit point of $M \setminus K$. Therefore, $d(x, M \setminus K) > 0$. This implies that for every $\epsilon > 0$, there exists a $\delta < \epsilon$ such that $B_\delta(x) \cap (M \setminus K) = \emptyset$. In other words, $B_\delta(x) \cap M = B_\delta(x) \cap K$.

So, we have shown that for every ϵ-neighborhood of x, there exists a δ-neighborhood of x such that

- $B_\delta(x) \cap M$ is connected, since by Theorem 4, $B_\delta(x) \cap K$ is connected,
- $B_\delta(x) \cap M \subset B_\delta(x) \subset B_\epsilon(x)$.

Since x was an arbitrary point of M, it follows that M is locally connected. ♠

The converse of this theorem need not be true. Table 1 provides such an example.

2.4 Semi-locally connected sets

A connected set M is said to be *semi-locally connected at x* if for any $\epsilon > 0$, there exists a neighborhood V of x of diameter less than ϵ such that $M \setminus V$ has only a finite number of components. The set M is semi-locally connected if it is semi-locally connected at each of its points.

Example 2.4.1

The topologist's sine curve (see Example 2.2.1) is semi-locally connected everywhere except at the points belonging to $\{0\} \times [-1, 1]$. ♡

Example 2.4.2

Let T be the triangle with vertices (0,0), (1,0) and (0,1). For each integer $i \geq 1$, let S_i be the segment with endpoints $(1/i, 0)$ and (0,1). The set $M = T \cup \bigcup_{i=1}^{\infty} S_i$ is semi-locally connected (see Fig. 4). ♡

There are many examples where a connected set which has property S is also semi-locally connected (for instance, see Example 2.3.1). However, we were unable to prove this true in general or disprove it with a counterexample. Therefore, we will make the following conjecture:

Conjecture 1

If a connected set has property S, then it is semi-locally connected.

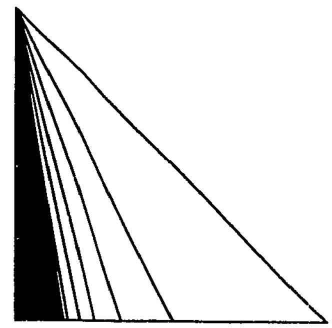


Figure 4: A semi-locally connected set.

2.5 Arcwise connected sets

Let x and y be elements of a set M. A continuous function $\Gamma : [0,1] \longrightarrow M$ is called an *arc* in M from x to y provided that $\Gamma(0) = x$ and $\Gamma(1) = y$. We sometimes denote this as $\Gamma = x \frown y$.

Lemma 3

If Γ is an arc in M from x to y, then there exists an arc in M from y to x.

Proof:

Indeed, the function $\Lambda : [0,1] \longrightarrow M$ defined by

$$\Lambda(x) = \Gamma(1 - x)$$

is an arc in M from y to x. ♠

Lemma 4

If Γ and Λ are arcs in M from x to y and from y to z, respectively, then there exists an arc in M from x to z.

Proof:

We are given that Γ and Λ are continuous functions form [0,1] to M and

$$\Gamma(0) = x \quad \Gamma(1) = \Lambda(0) = y \quad \Lambda(1) = z.$$

Take $\Delta : [0,1] \longrightarrow M$ to be

$$\Delta(x) = \begin{cases} \Gamma(2x) & \text{if} \quad 0 \le x \le 1/2 \\ \Lambda(2x-1) & \text{if} \quad 1/2 < x \le 1. \end{cases}$$

The function Δ is an arc in M from x to z. ♠

A set M is said to be *arcwise connected* if for every $x, y \in M$, there exists an arc from x to y. Note that every arc is arcwise connected.

Lemma 5

Let M be an arcwise connected set and let f be a continuous function on M. Then, $f(M)$ is also arcwise connected.

Proof:

Let y_1 and y_2 be any two points in $f(M)$. There exist points x_1 and x_2 in M such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since M is arcwise connected, there exists a continuous function $\Gamma : [0,1] \longrightarrow M$ such that $\Gamma(0) = x_1$ and $\Gamma(1) = x_2$. Therefore, $f(\Gamma) : [0,1] \longrightarrow f(M)$ is continuous,

$$f(\Gamma(0)) = f(x_1) = y_1 \qquad \text{and} \qquad f(\Gamma(1)) = f(x_2) = y_2.$$

Hence, $f(\Gamma)$ is an arc in $f(M)$ from y_1 to y_2. ♠

Example 2.5.1

A square in $\mathbb{R}^2$ is arcwise connected. ♡

The following relationship between the notions of connectedness and arcwise connectedness is a well-known result. Its proof can be found, for instance, in [14].

Theorem 8

If a set is arcwise connected, then it is connected.

The converse does not necessarily hold as shown in the next example.

Example 2.5.2

The *topologist's sine curve* (see Example 2.2.1) is connected but not arcwise connected. ♡

However, a partial converse of Theorem 8 can be obtained as states the next theorem. To see a proof of the latter we refer the reader to [23].

Theorem 9

Let $\mathcal{X}$ be a metric space. Every locally connected continuum in $\mathcal{X}$ is arcwise connected

2.6 Totally disconnected sets

A set is *totally disconnected* if its only nonempty connected subsets are one-point sets (singletons).

Example 2.6.1

The set of rationals $\mathcal{Q}$ is totally disconnected. ♡

Table 1 provides some more examples of sets with various connectivity properties. The properties considered are connectedness, property S, local connectedness, semi-local connectedness and arcwise connectedness.

3 Contractive mappings

In this section we will study a particular class of transformations on metric spaces, namely the contractive mappings (contractions). We will see that in a complete metric space, iterative use of a contractive mapping on any set results in "shrinking" that set to a unique point of the space. The metric space we deal with is the space of all compact sets of $\mathbb{R}^n$. Our main interest is to consider the simultaneous action of a finite number of contractions on a given point (a compact subset of $\mathbb{R}^n$) of this space.

3.1 Contractions

Roughly speaking, the term contraction means that points are moved closer together when one contraction is applied. The precise definition is given below:

Let $(\mathcal{X}, d)$ be a metric space. A mapping $f : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be *contractive* or a *contraction* provided that its *Lipschitz constant*,

$$\text{Lip}(f) = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x, y)},$$

satisfies $\text{Lip}(f) < 1$.

Example 3.1.1

i) $f(x) = \sqrt{x}$ is a contraction of the metric space $([1, \infty)$, Euclidean distance) since

$$\frac{d(f(x), f(y))}{d(x, y)} = \frac{\sqrt{x} - \sqrt{y}}{x - y} = \frac{1}{\sqrt{x} + \sqrt{y}} < \frac{1}{2}$$

for all $x, y \in [1, \infty)$. So, $\text{Lip}(f) < 1$.

ii) $g(x) = x$ is not a contraction of ($\mathbb{R}$, Euclidean distance). In fact, $\text{Lip}(g) = 1$ since for all $x, y \in \mathbb{R}$,

$$\frac{d(g(x), g(y))}{d(x, y)} = \frac{x - y}{x - y} = 1. \quad \heartsuit$$

Proposition 3

Every contraction of $(\mathcal{X}, d)$ is continuous.

We can find contractions in different classes of functions. Although we will discuss contractions in general, there is a particular type of mappings, namely similitudes, which will appear regularly in our examples. We define similitudes through affine transformations. A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is called an *affine transformation* provided that $f(x) = Ax + B$, where A and B are $n \times n$ and $n \times 1$ matrices, respectively. The matrix B is called the translation vector.

The nature of an affine transformation is determined by the matrix A. Below are some examples of A in $\mathbb{R}^2$ as well as the action of the corresponding affine transformation:

- Rotation about the origin, through an angle of θ:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

  We sometimes denote the above rotation matrix by R_θ.

- Reflection about the x-axis:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Reflection about the y-axis:

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Projection on the x-axis:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

- Projection on the y-axis:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Shear in the x direction with factor k:

$$A = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

- Shear in the y direction with factor k:

$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

If $A = \rho T$ for some orthogonal matrix T and $\rho \in \mathbb{R}$, $\rho \neq 0$, then f is called a *similitude* or a *similarity transformation.* If $0 < \rho < 1$, the similitude is contractive. A similitude transforms every subset of R^n to a geometrically similar set through a composition of a translation, a rotation and perhaps a reflection. Note that a similitude maintains angles unchanged.

3.2 Minkowski sausages

For $\epsilon > 0$, the *Minkowski ϵ-sausage* of a set E is defined as

$$\begin{aligned} E(\epsilon) &= \{x \mid d(x, E) \leq \epsilon\} \\ &= \bigcup_{x \in E} \overline{B_\epsilon(x)}. \end{aligned}$$

We will use the Minkowski sausages to define the Hausdorff distance between two sets in the next section. We will now list some of the properties of $E(\epsilon)$. Further details as well as the proof of the following properties can be found in [17].

1. If E is a compact subset of $\mathbb{R}^n$, then $E(\epsilon)$ is compact.

2. $E_1 \subset E_2 \Longrightarrow E_1(\epsilon) \subset E_2(\epsilon)$.

3. $E(\epsilon_1)(\epsilon_2) \subset E(\epsilon_1 + \epsilon_2)$.

4. $(E_1 \cup E_2)(\epsilon) = E_1(\epsilon) \cup E_2(\epsilon)$.

5. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a contraction with Lipschitz constant L, then $f(E(\epsilon)) \subset f(E)(L\epsilon)$.

6. If $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence of positive numbers converging to 0, then $E = \bigcap_{n=1}^{\infty} E(\epsilon_n)$.

Lemma 6

Let M be a closed subset of a metric space X. Then, M is not connected if and only if there exist closed sets $C_1 \subset X$ and $C_2 \subset X$ such that

- $C_1 \cap C_2 = \emptyset$,
- $M \subset C_1 \cup C_2$,
- $M \cap C_1 \neq \emptyset$ and $M \cap C_2 \neq \emptyset$.

Proof:

($\Longleftarrow$):

Let C_1 and C_2 be closed sets as described above.

Let $\epsilon = \min \{d(x,y) \mid x \in C_1,\ y \in C_2\}$. Then, the interior of the Minkowski $\epsilon/4$-sausages of C_1 and C_2 form a separation of M. Indeed, if

$$\mathcal{O}_1 = \mathrm{Int}(C_1(\frac{\epsilon}{4})) \quad \text{and} \quad \mathcal{O}_2 = \mathrm{Int}(C_2(\frac{\epsilon}{4})),$$

then $\mathcal{O}_1$ and $\mathcal{O}_2$ are two open sets such that:

- $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ (by definition of ϵ),
- $M \subset C_1 \cup C_2 \subset \mathcal{O}_1 \cup \mathcal{O}_2$,
- $M \cap \mathcal{O}_1 \neq \emptyset$ since $M \cap C_1 \neq \emptyset$ and $C_1 \subset \mathcal{O}_1$,
- $M \cap \mathcal{O}_2 \neq \emptyset$ since $M \cap C_2 \neq \emptyset$ and $C_2 \subset \mathcal{O}_2$.

($\Longrightarrow$):

Suppose that M is not connected and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be a separation of M. Let

$$M_1 = M \cap \mathcal{O}_1 \quad \text{and} \quad M_2 = M \cap \mathcal{O}_2.$$

Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are open, we have

$$\overline{\mathcal{O}_1} \cap \mathcal{O}_2 = \emptyset \quad \text{and} \quad \mathcal{O}_1 \cap \overline{\mathcal{O}_2} = \emptyset.$$

We will show below that one can set $C_1 = \overline{M_1}$ and $C_2 = \overline{M_2}$.

- We first show that $\overline{M_1} \cap \overline{M_2} = \emptyset$. Since M is closed, we have

  $$M_1 \subset M \Longrightarrow \overline{M_1} \subset \overline{M} = M,$$

  and

  $$M_2 \subset M \Longrightarrow \overline{M_2} \subset \overline{M} = M.$$

  Therefore,

  $$\overline{M_1} \cap \overline{M_2} \subset M = M_1 \cup M_2.$$

  Now,

  $$\begin{aligned}
  \overline{M_1} \cap \overline{M_2} &= (\overline{M_1} \cap \overline{M_2}) \cap (M_1 \cup M_2) \\
  &= (\overline{M_1} \cap \overline{M_2} \cap M_1) \cup (\overline{M_1} \cap \overline{M_2} \cap M_2) \\
  &= (M_1 \cap \overline{M_2}) \cup (\overline{M_1} \cap M_2) \\
  &\subset (\mathcal{O}_1 \cap \overline{\mathcal{O}_2}) \cup (\overline{\mathcal{O}_1} \cap \mathcal{O}_2) \\
  &= \emptyset \cup \emptyset \\
  &= \emptyset.
  \end{aligned}$$

- Next, notice that $M = M_1 \cup M_2 \subset \overline{M_1} \cup \overline{M_2}$.

- Finally, $M_1 \subset M \cap \overline{M_1}$ so that $M \cap \overline{M_1} \neq \emptyset$. Similarly, $M \cap \overline{M_2} \neq \emptyset$. ♠

3.3 The Hausdorff distance between compact sets

Let $\mathcal{X}$ be a complete metric space. The set of all nonempty compact subsets of $\mathcal{X}$ is denoted by $\mathcal{H}(\mathcal{X})$. We will restrict our attention to the case where $\mathcal{X} = \mathbb{R}^n$.

We will equip $\mathcal{H}(\mathbb{R}^n)$ with a distance function $h : \mathcal{H}(\mathbb{R}^n) \times \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathbb{R}$ defined as follows:
For any $K_1, K_2 \in \mathcal{H}(\mathbb{R}^n)$,

$$h(K_1, K_2) = \inf\{\epsilon \geq 0 |\ K_1 \subset K_2(\epsilon) \text{ and } K_2 \subset K_1(\epsilon)\}.$$

This definition is equivalent to the more popular one:

$$h(K_1, K_2) = \max[\sup\{d(x, K_1)|\ x \in K_2\},\ \sup\{d(x, K_2)|\ x \in K_1\}].$$

Since the latter definition is usually difficult to use in practice, we will adopt the former one in most of our discussion. It can be shown that the so defined h is indeed a distance function for the space $\mathcal{H}(\mathbb{R}^n)$. The pair $(\mathcal{H}(\mathbb{R}^n), h)$ is called the *Hausdorff metric space* and h is referred to as the *Hausdorff distance.*

In other words, two compact sets K_1 and K_2 are within Hausdorff distance r of each other if every point of K_1 is within distance r of some point of K_2, and every point of K_2 is within distance r of some point of K_1.

Example 3.3.1

i) Let K_1 be the square $ABCD$ where $A = (0,0)$, $B = (1,0)$, $C = (1,1)$ and $D = (0,1)$. Let K_2 be the diagonal BD (see Fig. 5). Then, $h(K_1, K_2) = \sqrt{2}/2$.

ii) Let D_1 and D_2 be two discs centered at (0,0) with radii 1 and 3, respectively (see Fig. 5). Then, $h(D_1, D_2) = 2$. ♡

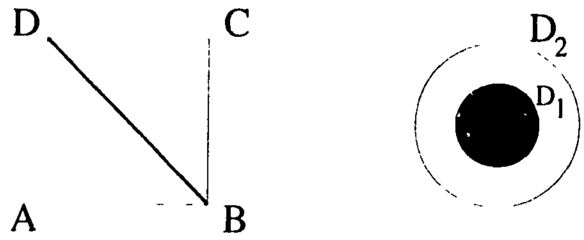


Figure 5: Illustration of the Hausdorff distance (see Example 3.3.1).

Lemma 7

Let A_i and B_i be compact sets for $i = 1, ..., m$. Then,

$$h(\bigcup_{i=1}^{m} A_i \ , \ \bigcup_{i=1}^{m} B_i) \leq \max_{1 \leq i \leq m} h(A_i, B_i).$$

Proof:

We will prove this result by induction:

Set $m = 2$. Let $\epsilon = \max\{h(A_1, B_1), \ h(A_2, B_2)\}$. Then,

- $\epsilon \geq h(A_1, B_1) \Longrightarrow \begin{cases} A_1 \subset B_1(\epsilon) & (1) \\ B_1 \subset A_1(\epsilon) & (2) \end{cases}$
- $\epsilon \geq h(A_2, B_2) \Longrightarrow \begin{cases} A_2 \subset B_2(\epsilon) & (3) \\ B_2 \subset A_2(\epsilon) & (4) \end{cases}$

From (1) and (3) it follows that

$$A_1 \cup A_2 \subset B_1(\epsilon) \cup B_2(\epsilon) = (B_1 \cup B_2)(\epsilon).$$

Also, from (2) and (4) we conclude that

$$B_1 \cup B_2 \subset A_1(\epsilon) \cup A_2(\epsilon) = (A_1 \cup A_2)(\epsilon).$$

Therefore, $\epsilon \geq h(A_1 \cup A_2, \ B_1 \cup B_2)$.

Next, supposet that

$$h(\bigcup_{i=1}^{k} A_i \ , \ \bigcup_{i=1}^{k} B_i) \leq \max_{1 \leq i \leq k} h(A_i, B_i).$$

Now,

$$
\begin{aligned}
h(\bigcup_{i=1}^{k+1} A_i \ , \ \bigcup_{i=1}^{k+1} B_i) &= h(\bigcup_{i=1}^{k} A_i \cup A_{k+1} \ , \ \bigcup_{i=1}^{k} B_i \cup B_{k+1}) \\
&\leq \max\{h \ (\bigcup_{i=1}^{k} A_i \ , \ \bigcup_{i=1}^{k} B_i) \ , \ h(A_{k+1}, B_{k+1})\} \\
&\leq \max_{1\leq i\leq k}\{h(A_i, B_i) \ , \ h(A_{k+1}, B_{k+1})\} \\
&= \max_{1\leq i\leq k+1} h(A_i, B_i). \quad \spadesuit
\end{aligned}
$$

Theorem 10 (See [3])

The metric space $(\mathcal{H}(\mathbb{R}^n), h)$ *is complete.*

Moreover, if $\{K_n\}_{n=1}^{\infty} \subset \mathcal{H}(\mathbb{R}^n)$ *is a Cauchy sequence, then*

$\lim_{n\to\infty} K_n = \{x \in \mathbb{R}^n|$ *There exists a Cauchy sequence* $\{x_n\}_{n=1}^{\infty}$ *with* $x_n \in K_n$, *such that* $\lim_{n\to\infty} x_n = x\}$.

3.4 The attractor of a contraction

Let $f : \mathcal{X} \longrightarrow \mathcal{X}$ be a map. A point $p \in \mathcal{X}$ for which $f(p) = p$ is called a *fixed point* of f. We will denote the set of all fixed points of f by $\text{Fix}(f)$. However, if Fix $(f) = \{p\}$, we can write Fix $(f) = p$ where no ambiguity may rise.

Theorem 11 (Banach Fixed Point Theorem)

Let $(\mathcal{X}, d)$ *be a complete metric space. If* $f : \mathcal{X} \longrightarrow \mathcal{X}$ *is a contraction, then* f *has a unique fixed point.*

Applying the Banach fixed point theorem to the complete space $\mathcal{H}(\mathbb{R}^n)$ we obtain:

Proposition 4

Every contraction $F : \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ *has a unique fixed point* $\mathcal{A}$.

Since, in general, $\mathcal{A} = F(\mathcal{A})$ is not a singleton of $\mathbb{R}^n$, it is not convenient to call it a fixed "point". So we will refer to $\mathcal{A}$ as the *attractor* of F. The following result states three important characterizations of the attractor of a contraction in $\mathcal{H}(\mathbb{R}^n)$.

Notation: For any natural number n, the n-th iterate of a given function F is denoted by $F^{\circ n}$.

Proposition 5 (See [17])

Let $F : \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ be a contraction whose attractor is $\mathcal{A}$. For any $K \in \mathcal{H}(\mathbb{R}^n)$,

1. $h(\mathcal{A}, F^{\circ n}(K)) \underset{n\to\infty}{\longrightarrow} 0$.
2. $\mathcal{A} = \{x \in \mathbb{R}^n \,|\, x$ is a limit point of some sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in F^{\circ n}(K)\}$.
3. $\mathcal{A} = \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)})$.

Proof:

Let $K \in \mathcal{H}(\mathbb{R}^n)$.

1. It follows at once from the Banach fixed point theorem.
2. $(\subset)$:

   Let $x \in \mathcal{A}$ and let $n \geq 1$. Since F is continuous, $F^{\circ n}(K)$ is compact. So, by Lemma 2, there exists a point $x_n \in F^{\circ n}(K)$ such that $d(x, x_n) = d(x, F^{\circ n}(K))$. So we can write

$$\begin{aligned} d(x, x_n) &= d(x, F^{\circ n}(K)) \\ &\leq \sup\{d(a, F^{\circ n}(K)) \,|\, a \in \mathcal{A}\} \\ &\leq \max[\sup\{d(a, F^{\circ n}(K)) \,|\, a \in \mathcal{A}\}, \ \sup\{d(y, \mathcal{A}) \,|\, y \in F^{\circ n}(K)\}] \\ &= h(\mathcal{A}, F^{\circ n}(K)). \end{aligned}$$

   But $h(\mathcal{A}, F^{\circ n}(K)) \underset{n\to\infty}{\longrightarrow} 0$. Therefore, $d(x, x_n) \underset{n\to\infty}{\longrightarrow} 0$ which means that $x = \lim_{n\to\infty} x_n$, where $x_n \in F^{\circ n}(K)$.

   $(\supset)$:

   Let x be a limit point of a sequence $\{x_n\}_{n=1}^{\infty}$ where $x_n \in F^{\circ n}(K)$. So,

$\{x_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{n_k=1}^{\infty}$ such that $x = \lim_{n_k \to \infty} x_{n_k}$. Now, for any point $\alpha \in \mathcal{A}$, we have

$$\begin{aligned} d(x, \mathcal{A}) &\leq d(x, \alpha) \\ &\leq d(x, y) + d(y, \alpha) \qquad \text{for any } y. \end{aligned}$$

Since $\alpha \in \mathcal{A}$ is arbitrary, we get, for any y:

$$\begin{aligned} d(x, \mathcal{A}) &\leq d(x, y) + \inf_{\alpha \in \mathcal{A}} d(y, \alpha) \\ &\leq d(x, y) + d(y, \mathcal{A}). \end{aligned}$$

In particular, for each $k = 1, 2, \ldots$, we can take $y = x_{n_k} \in F^{\circ n_k}(K)$ to get

$$\begin{aligned} d(x, \mathcal{A}) &\leq d(x, x_{n_k}) + d(x_{n_k}, \mathcal{A}) \\ &\leq d(x, x_{n_k}) + \sup_{x_{n_k} \in F^{\circ n_k}(K)} d(x_{n_k}, \mathcal{A}) \\ &\leq d(x, x_{n_k}) + h(F^{\circ n_k}(K), \mathcal{A}). \end{aligned}$$

Now, $d(x, x_{n_k}) \underset{n_k \to \infty}{\longrightarrow} 0$ since $x = \lim_{n_k \to \infty} x_{n_k}$. Also, from Part 1 we have $h(\mathcal{A}, F^{\circ n_k}(K)) \underset{n_k \to \infty}{\longrightarrow} 0$. Therefore, $d(x, \mathcal{A}) = 0$. But $\mathcal{A}$ is compact and therefore closed. Hence, $x \in \mathcal{A}$.

3. ($\subset$):

Let $x \in \mathcal{A}$. So, there exists a sequence $\{x_j\}_{j=1}^{\infty}$ converging to x where $x_j \in F^{\circ j}(K)$. Now, let $i \geq 1$ be fixed. For any $j \geq i$, we have

$$x_j \in F^{\circ j}(K) \subset \bigcup_{n=j}^{\infty} F^{\circ n}(K) \subset \bigcup_{n=i}^{\infty} F^{\circ n}(K).$$

In other words, $\{x_j\}_{j=i}^{\infty} \subset \bigcup_{n=i}^{\infty} F^{\circ n}(K)$; therefore, $x \in \overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)}$. This is true for any $i \geq 1$, so that $x \in \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)})$.
So, $\mathcal{A} \subset \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)})$.

($\supset$):

Let $x \in \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)})$. So, $x \in \overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)}$ for all $i \geq 1$. We claim that there exists a sequence $\{y_{n_i}\}_{i=1}^{\infty}$ such that for all $i \geq 1$,

- $n_i \leq n_{i+1}$,
- $y_{n_i} \in F^{\circ n_i}(K)$,
- $d(y_{n_i}, x) < \frac{1}{n_i}$.

We will use the principle of induction to justify this claim:

Since $x \in \overline{\bigcup_{n=1}^{\infty} F^{\circ n}(K)}$, there exists a sequence $\{x_i^1\}_{i=1}^{\infty} \subset \bigcup_{n=1}^{\infty} F^{\circ n}(K)$ such that $x_i^1 \xrightarrow[i \to \infty]{} x$. Then, for sufficiently large i, we have $d(x_i^1, x) < 1$ where $x_i^1 \in F^{\circ n_1}(K)$ for some $n_1 \geq 1$. Let $y_{n_1} = x_i^1$.

Next, suppose that a sequence $\{n_i\}_{i=1}^{L}$ is obtained such that $n_i \leq n_{i+1}$ for all $1 \leq i < L$, and

- $y_{n_i} \in F^{\circ n_i}(K)$,
- $d(y_{n_i}, x) < \frac{1}{n_i}$,

for all $1 \leq i \leq L$.

Since $x \in \overline{\bigcup_{n=n_L}^{\infty} F^{\circ n}(K)}$, there exists a sequence

$$\{x_i^{L+1}\}_{i=1}^{\infty} \subset \bigcup_{n=n_L}^{\infty} F^{\circ n}(K),$$

such that $x_i^{L+1} \xrightarrow[i \to \infty]{} x$. Then, for sufficiently large i, we have

$$d(x_i^{L+1}, x) < \frac{1}{L+1},$$

where $x_i^{L+1} \subset F^{\circ n_{L+1}}(K)$ for some $n_{L+1} \geq n_L$.

Let $y_{n_{L+1}} = x_i^{L+1}$. This completes the inductive argument.

Now, for $n \notin \{n_1, n_2, n_3, ...\}$, let y_n be any element of $F^{\circ n}(K)$. The sequence $\{y_n\}_{n=1}^{\infty} = \{y_1, y_2, ..., y_{n_1}, ..., y_{n_2}, ...\}$ has a subsequence, namely $\{y_{n_1}, y_{n_2}, y_{n_3}, ...\}$ which converges to x. Hence, x is a limit point of the sequence $\{y_n\}_{n=1}^{\infty}$ where $y_n \in F^{\circ n}(K)$ for each $n \geq 1$. So, $x \in \mathcal{A}$. Therefore, $\bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)}) \subset \mathcal{A}$. ♠

3.5 Iterated function systems (IFS's)

3.5.1 Some contractions of $\mathcal{H}(\mathbb{R}^n)$

1. Let f be a contraction of $\mathbb{R}^n$ with Lipschitz constant L and let K be a compact set. Define $F(K) = f(K)$. Then F is a contraction of $\mathcal{H}(\mathbb{R}^n)$ with Lipschitz constant L. The attractor of such a contraction consists of a single point in $\mathbb{R}^n$.

2. Any constant function on $\mathcal{H}(\mathbb{R}^n)$ is a contraction of $\mathcal{H}(\mathbb{R}^n)$ with Lipschitz constant 0. In this case, the attractor is obviously the constant image of the contraction.

3. More interesting contractions can be obtained from combining several contractions of $\mathcal{H}(\mathbb{R}^n)$ as follows:

   Let $F_1, F_2, ..., F_m$ be contractions of $\mathcal{H}(\mathbb{R}^n)$ with Lipschitz constants $L_1, L_2, ..., L_m$, respectively. The application $\mathbf{F} : \mathcal{H}(\mathbb{R}^n) \to \mathcal{H}(\mathbb{R}^n)$ defined by

$$\mathbf{F}(K) = \bigcup_{i=1}^{m} F_i(K)$$

   is a contraction of $\mathcal{H}(\mathbb{R}^n)$ with Lipschitz constant $L = \max\{L_i\}_{i=1}^{m}$. To see this, let K_1 and K_2 be two compact subsets of $\mathbb{R}^n$. Then,

$$\begin{aligned} h(\mathbf{F}(K_1),\ \mathbf{F}(K_2)) &= h(\bigcup_{i=1}^{m} F_i(K_1),\ \bigcup_{i=1}^{m} F_i(K_2)) \\ &\leq \max_{1 \leq i \leq m} h(F_i(K_1),\ F_i(K_2)) \quad \text{(by Lemma 7)} \\ &\leq \max_{1 \leq i \leq m} L_i h(K_1, K_2). \end{aligned}$$

3.5.2 Increasing contractions

A contractive mapping $F : \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ is said to be *increasing* provided that $K_1 \subset K_2$ implies $F(K_1) \subset F(K_2)$. Increasing contractions are also referred to as *isotone* contractions (see [9]). Note that whenever a contraction F satisfies

$$F(K_1 \cup K_2) = F(K_1) \cup F(K_2) \quad \forall K_1, K_2 \in \mathcal{H}(\mathbb{R}^n),$$

F is increasing. Indeed,

$$\begin{aligned} \forall K_1, K_2 \in \mathcal{H}(\mathbb{R}^n), \qquad F(K_1) &\subset F(K_1) \cup F(K_2) \\ &= F(K_1 \cup K_2). \end{aligned}$$

Now, if $K_1 \subset K_2$, then $K_1 \cup K_2 = K_2$.
Hence, $F(K_1) \subset F(K_2)$.

Remark: Contractions defined in section 3.5.1 are increasing.

Proposition 6

Suppose that F is an increasing contraction of $\mathcal{H}(\mathbb{R}^n)$ with attractor $\mathcal{A}$, and let K be a compact subset of $\mathbb{R}^n$. Then, for all $i \geq 1$,

1. *$K \subset F(K) \Longrightarrow \mathcal{A} = \overline{\bigcup_{n\geq i} F^{on}(K)}$. In particular, $K \subset \mathcal{A}$.*

2. *$F(K) \subset K \Longrightarrow \mathcal{A} = \bigcap_{n\geq i} F^{on}(K)$. In particular, $\mathcal{A} \subset K$.*

Proof:

1. Let $K \subset F(K)$. Since F is increasing, we have

$$K \subset F(K) \subset F^{o2}(K) \subset ... \subset F^{on}(K)$$

for all $n > 1$. This implies that for all $i \geq 1$,

$$\bigcup_{n=i}^{\infty} F^{on}(K) = \bigcup_{n=1}^{\infty} F^{on}(K).$$

Therefore, by part 3 of Proposition 5,

$$\mathcal{A} = \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)}) = \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=1}^{\infty} F^{\circ n}(K)})$$
$$= \overline{\bigcup_{n=1}^{\infty} F^{\circ n}(K)}$$
$$= \overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)} \quad \text{for all } i \geq 1.$$

2. Let $F(K) \subset K$. Since F is increasing,

$$K \supset F(K) \supset F^{\circ 2}(K) \supset ... \supset F^{\circ n}(K)$$

for all $n \geq 1$. We conclude that for all $i \geq 1$,

$$\bigcup_{n=i}^{\infty} F^{\circ n}(K) = F^{\circ i}(K) \quad \text{and} \quad \bigcap_{i=1}^{\infty} F^{\circ i}(K) = \bigcap_{n=i}^{\infty} F^{\circ n}(K).$$

Once again, by part 3 of Proposition 5,

$$\mathcal{A} = \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} F^{\circ n}(K)}) = \bigcap_{i=1}^{\infty} \overline{F^{\circ i}(K)}$$
$$= \bigcap_{i=1}^{\infty} F^{\circ i}(K)$$
$$= \bigcap_{n=i}^{\infty} F^{\circ n}(K) \quad \text{for all } i \geq 1. \quad \spadesuit$$

Proposition 6 is a useful tool to show that a given set is included in or contains the attractor of a contraction. We will see one of its applications in the following section (see Example 4.2.1).

3.5.3 Iterated function systems (IFS's)

Let $F_1, F_2, ..., F_m$ be contractions of a complete metric space $(\mathcal{X}, d)$. The family $\{F_1, F_2, ..., F_m\}$ is called an *iterated function system (IFS)* on $(\mathcal{X}, d)$ In our discussion, we will use the nomenclature "Iterated Function System" to refer to an IFS on the metric space ($\mathbb{R}^n$, Euclidean distance).

Given an IFS $\{F_i\}_{i=1}^m$, the mapping $\mathbf{F} : \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ defined by

$$\mathbf{F}(K) = \bigcup_{i=1}^{m} F_i(K)$$

is a contraction of $\mathcal{H}(\mathbb{R}^n)$ with Lipschitz constant $L = \max\{L_i\}_{i=1}^m$, where L_i is the Lipschitz constant of F_i for $1 \leq i \leq m$ (see Section 3.5.1). The attractor $\mathcal{A}$ of $\mathbf{F}$ is known as the attractor of the IFS.

Proposition 7

Let $F_1, F_2, ..., F_m$ be contractions of $\mathbb{R}^n$ and consider the IFS $\{F_1, F_2, ..., F_m\}$ with attractor $\mathcal{A}$.

For every point $\alpha \in \mathcal{A}$, there exists a sequence $\{j_k\}_{k=1}^\infty \subset \{1, ..., m\}$ such that $\alpha \in F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A})$ for all $k \geq 1$.

Conversely, for every sequence $\{j_k\}_{k=1}^\infty \subset \{1, ..., m\}$, there exists a point $\alpha \in \mathcal{A}$ such that $\alpha \in F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A})$ for all $k \geq 1$.

Proof:

Let $\alpha \in \mathcal{A}$. We will construct, using induction, a sequence $\{j_k\}_{k=1}^\infty \subset \{1, ..., m\}$ such that $\alpha \in F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A})$ for all $k \geq 1$.

Since $\mathcal{A} = \bigcup_{i=1}^m F_i(A)$, there exists j_1, with $1 \leq j_1 \leq m$, such that $\alpha \in F_{j_1}(\mathcal{A})$. Now,

$$\begin{aligned} F_{j_1}(\mathcal{A}) &= F_{j_1}(\bigcup_{i=1}^{m} F_i(A)) \\ &= \bigcup_{i=1}^{m} F_{j_1}(F_i(\mathcal{A})). \end{aligned}$$

Hence, $\alpha \in F_{j_1}(F_{j_2}(\mathcal{A}))$ for some $1 \leq j_2 \leq m$. Similarly, for any natural number k, if $\alpha \in F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A}))$, where $\{j_1, ..., j_k\} \subset \{1, ..., m\}$, we can write

$$\begin{aligned} F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A}) &= F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\bigcup_{i=1}^{m} F_i(A)) \\ &= \bigcup_{i=1}^{m} F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(F_i(\mathcal{A})). \end{aligned}$$

In other words, $\alpha \in F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k} \circ F_{j_{k+1}}(\mathcal{A})$ for some j_{k+1}, where $1 \leq j_{k+1} \leq m$. Continuing this way, we obtain the desired sequence.

Conversely, let $\{j_k\}_{k=1}^{\infty} \subset \{1, ..., m\}$. Then, since $F_{j_k}(\mathcal{A}) \subset \mathcal{A}$ for any $k \geq 1$, we get

$$\mathcal{A} \supset F_{j_1}(\mathcal{A}) \supset F_{j_1} \circ F_{j_2}(\mathcal{A}) \supset ... \supset F_{j_1} \circ F_{j_2} \circ ... \circ F_{j_k}(\mathcal{A}).$$

On the other hand, if $L = \max\{L_i\}_{i=1}^{m}$, then

$$\text{diam}(F_{j_1} \circ ... \circ F_{j_k}(\mathcal{A})) \leq L^k \text{diam}(\mathcal{A}).$$

Since $L < 1$, the right hand side of this inequality tends to zero, which implies that $\text{diam}(\bigcap_{k=1}^{\infty} F_{j_1} \circ ... \circ F_{j_k}(\mathcal{A})) = 0$. In other words, $\bigcap_{k=1}^{\infty} F_{j_1} \circ ... \circ F_{j_k}(\mathcal{A}) = \{\alpha\}$ for some $\alpha \in \mathbb{R}^n$. Since $F_{j_1} \circ ... \circ F_{j_k}(\mathcal{A}) \subset \mathcal{A}$ for all $k \geq 1$, we have $\alpha \in \mathcal{A}$. ♠

Remark: Given a point $\alpha \in \mathcal{A}$, a sequence $\{j_k\}_{k=1}^{\infty}$ as described in the above Proposition is called an *address* of the point α. It specifies a route one can follow to get from any given point of $\mathcal{A}$ to α via smaller and smaller copies of $\mathcal{A}$. This route is not necessarily unique as illustrated in the following example.

Example 3.5.1

Let $\{F_1, F_2, F_3\}$ be an IFS on $\mathcal{H}(\mathbb{R}^2)$ with attractor $\mathcal{A}$ where

$$\begin{aligned} F_1[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \\ F_2[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \\ F_3[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix}, \end{aligned}$$

where $x, y \in \mathbb{R}$. The attractor $\mathcal{A}$ is known as the *Sierpinski gasket* (see Fig. 6.).

The sequences $\{1, 2, 2, 2, ...\}$ and $\{2, 1, 1, 1, ...\}$ are two different addresses for the point $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \in \mathcal{A}$. Indeed,

$$\lim_{n \to \infty} F_1 \circ F_2^{\circ n}(\mathcal{A}) = F_1(\lim_{n \to \infty} F_2^{\circ n}(\mathcal{A})) = F_1(\text{Fix}(F_2)) = F_1[\begin{pmatrix} 1 \\ 0 \end{pmatrix}] = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$

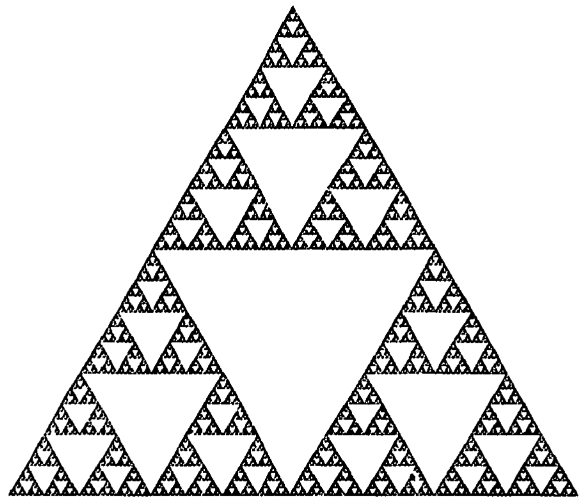


Figure 6: The Sierpinski gasket.

and

$$\lim_{n\to\infty} F_2 \circ F_1{}^{\circ n}(\mathcal{A}) = F_2(\lim_{n\to\infty} F_1{}^{\circ n}(\mathcal{A})) = F_2(\mathrm{Fix}(F_1)) = F_2[\begin{pmatrix}0\\0\end{pmatrix}] = \begin{pmatrix}1/2\\0\end{pmatrix}. \quad \heartsuit$$

Proposition 8

Suppose $\{F_i\}_{i=1}^m$ is an IFS with attractor $\mathcal{A}$. Let $\{F_{i_j}\}_{i_j=1}^{m'}$ $(m' \leq m)$ be a subset of $\{F_i\}_{i=1}^m$. Then, the attractor $\mathcal{A}'$ of $\{F_{i_j}\}_{i_j=1}^{m'}$ satisfies $\mathcal{A}' \subset \mathcal{A}$.

Proof:

Let $K \in \mathcal{H}(\mathbb{R}^n)$. Let

$$\mathbf{F}'(K) = \bigcup_{i_j=1}^{m'} F_{i_j}(K) \quad \text{and} \quad \mathbf{F}(K) = \bigcup_{i=1}^{m} F_i(K).$$

Then, $\mathbf{F}'(K) \subset \mathbf{F}(K)$, and hence,

$$\begin{aligned} \mathcal{A}' &= \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} \mathbf{F}'^{\circ n}(K)}) \\ &\subset \bigcap_{i=1}^{\infty}(\overline{\bigcup_{n=i}^{\infty} \mathbf{F}^{\circ n}(K)}) \\ &= \mathcal{A}. \quad \spadesuit \end{aligned}$$

Proposition 9

The attractor $\mathcal{A}$ of an IFS $\{F_1, F_2, ..., F_m\}$ can be characterized as

$$\mathcal{A} = \text{Cl}\ \{\text{Fix}(F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_r})|\ r \geq 1,\ 1 \leq i_j \leq m,\ 1 \leq j \leq r\}.$$

Proof:

Let $\Phi = \{\text{Fix}(F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_r}) \mid r \geq 1,\ 1 \leq i_j \leq m,\ 1 \leq j \leq r\}$.

We want to show that $\mathcal{A} = \overline{\Phi}$.

($\supset$):

Let $\phi \in \Phi$. Then, ϕ =Fix$(F_{i_1} \circ ... \circ F_{i_r})$ for some integer $r \geq 1$ where $1 \leq i_j \leq m$. Let $\{x_1\}$ =Fix(F_1). We have $\{x_1\} = F_1(\{x_1\}) \subset \bigcup_{i=1}^m F_i(\{x_1\}) = \mathbf{F}(\{x_1\})$; so, by Proposition 6, $\mathcal{A} = \overline{\bigcup_{p\geq 1} \mathbf{F}^{\circ p}(\{x_1\})}$.

Note that for any compact set K and any natural number p,

$$\mathbf{F}^{\circ p}(K) = \bigcup_{1\leq i_j \leq m} F_{i_1} \circ ... \circ F_{i_p}(K),$$

where $1 \leq j \leq p$. Therefore, $(F_{i_1} \circ ... \circ F_{i_r})^{\circ p}(\{x_1\}) \subset \mathbf{F}^{\circ pr}(\{x_1\})$. Consequently,

$$\begin{aligned}\bigcup_{p=1}^{\infty}(F_{i_1} \circ ... \circ F_{i_r})^{\circ p}(\{x_1\}) &\subset \bigcup_{p=1}^{\infty} \mathbf{F}^{\circ pr}(\{x_1\}) \\ &\subset \bigcup_{p=1}^{\infty} \mathbf{F}^{\circ p}(\{x_1\}) \\ &\subset \overline{\bigcup_{p=1}^{\infty} \mathbf{F}^{\circ p}(\{x_1\})} \\ &= \mathcal{A}.\end{aligned}$$

Hence, $\overline{\bigcup_{p=1}^{\infty}(F_{i_1} \circ ... \circ F_{i_r})^{\circ p}(\{x_1\})} \subset \overline{\mathcal{A}} \subset \mathcal{A}$ since $\mathcal{A}$ is closed. Now, $F_{i_1} \circ ... \circ F_{i_r}$ is a contraction of the complete metric space $\mathbb{R}^n$ and $\{x_1\}$ is compact. So, by the Banach fixed point theorem,

$$(F_{i_1} \circ ... \circ F_{i_r})^{\circ p}(\{x_1\}) \underset{p\to\infty}{\longrightarrow} \text{Fix}(F_{i_1} \circ ... \circ F_{i_r}) = \phi.$$

Therefore, $\phi \in \overline{\bigcup_{p=1}^{\infty}(F_{i_1} \circ ... \circ F_{i_r})^{\circ p}(\{x_1\})} \subset \mathcal{A}$. This shows that $\Phi \subset \mathcal{A}$ which implies $\overline{\Phi} \subset \overline{\mathcal{A}} = \mathcal{A}$.

($\subset$):

Let $\alpha \in \mathcal{A}$ and let $\{i_j\}_{j=1}^{\infty}$ be an address of α. Then,

$$\alpha \in F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_r}(\mathcal{A})$$

for any $r \geq 1$. On the other hand, we showed that $\overline{\Phi} \subset \mathcal{A}$; so, given $r \geq 1$, we have $\text{Fix}(F_{i_1} \circ ... \circ F_{i_r}) \in \mathcal{A}$, where $1 \leq i_j \leq m$. This implies that

$$\text{Fix}(F_{i_1} \circ ... \circ F_{i_r}) = F_{i_1} \circ ... \circ F_{i_r}(\text{Fix}(F_{i_1} \circ ... \circ F_{i_r})) \in F_{i_1} \circ ... \circ F_{i_r}(\mathcal{A}).$$

Let $L = \max\{L_i\}_{i=1}^{m}$. Since, $L < 1$, given $\epsilon > 0$, we can choose r so large that $L^r \text{diam}(\mathcal{A}) < \epsilon$. Hence,

$$\begin{aligned} d(\alpha, \text{Fix}(F_{i_1} \circ ... \circ F_{i_r})) &\leq \text{diam}(F_{i_1} \circ ... \circ F_{i_r}(\mathcal{A})) \\ &\leq L^r \text{diam}(\mathcal{A}) \\ &< \epsilon. \end{aligned}$$

Therefore, $\text{Fix}(F_{i_1} \circ ... \circ F_{i_r}) \underset{r \to \infty}{\longrightarrow} \alpha$; that is, $\alpha \in \overline{\Phi}$. Consequently, $\mathcal{A} \subset \overline{\Phi}$. ♠

Remark: The composite function $F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_r}$ $(1 \leq i_1, ..., i_r \leq m)$ is sometimes referred to as a "*word*" of length r.

4 Connectedness of the attractor of an IFS

We are now ready to take on the principal part of our discussion, the connectedness of the attractor of an IFS. We will once again focus on IFS's composed of contractions of $\mathbb{R}^n$. Our goal is to predict some of the topological properties (mostly connectedness) of the attractors of such IFS's by studying the contractive mappings forming them.

4.1 Totally disconnected attractors

To start this section, we would like to draw the readers' attention to the notion of a "totally disconnected IFS" considered by Barnsley in [3]. He defines an IFS to be totally disconnected if each point in its attractor possesses a unique address. We will show (see Example 4.1.4) that according to this definition, a totally disconnected attractor does not necessarily result from a totally disconnected IFS. For this reason, we will avoid using the concept of a totally disconnected IFS and instead refer to a totally disconnected attractor. By the latter we mean an attractor whose only nonempty connected subsets are singletons (usual definition in Topology).

Theorem 12 (See [19])

Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, ..., F_m\}$ and let L_i denote the Lipschitz constant of F_i for $i = 1, ..., m$. If $\sum_{i=1}^{m} L_i < 1$, then $\mathcal{A}$ is totally disconnected.

Proof:

Step I: There exists a closed and bounded set K (therefore, compact in $\mathbb{R}^n$), such that $\mathcal{A} \subset K$.

Proof (of Step I):

Let $d_i = d(\text{Fix}(F_1), \text{Fix}(F_i))$ and choose $\epsilon > 0$ such that $L_i(\epsilon + d_i) + d_i < \epsilon$ for $i = 1, ..., m$. Let $K = \text{Cl}\left[B_\epsilon\left(\text{Fix}\left(F_1\right)\right)\right]$. We will first show that $F_i(K) \subset K$ for each $i = 1, ..., m$.

Let y be any point in $F_i(K)$ and let $x \in K$ be such that $F_i(x) = y$. We have

$$\begin{aligned} d(y, \mathrm{Fix}(F_1)) &\leq d(y, \mathrm{Fix}(F_i)) + d(\mathrm{Fix}(F_i), \mathrm{Fix}(F_1)) \\ &= d(y, \mathrm{Fix}(F_i)) + d_i \\ &\leq L_i\, d(x, \mathrm{Fix}(F_i)) + d_i \\ &\leq L_i\, (c + d_i) + d_i \\ &< \epsilon. \end{aligned}$$

Therefore, $y \in K$ and so $F_i(K) \subset K$. It follows from Proposition 6 that $\mathcal{A} \subset K$.

Step II: Let W_r consist of all words (see Remark in Section 3.5.1) of length r. Then, for each $\epsilon > 0$, there exists an integer r such that

$$\sum_{w \in W_r} \mathrm{diam}(w(K)) < \epsilon.$$

Proof (of Step II):

Let $w = F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_r}$, where $i_1, \ldots, i_r \in \{1, \ldots, m\}$. We have

$$\mathrm{diam}(w(K)) \leq L_{i_1} L_{i_2} \ldots L_{i_r} \mathrm{diam}(K),$$

so that,

$$\begin{aligned} \sum_{w \in W_r} \mathrm{diam}(w(K)) &\leq \sum_{\text{all possible arrangements of } L_{i_j}\text{'s}} L_{i_1} \ldots L_{i_r} \mathrm{diam}(K) \\ &= (L_1 + L_2 + \ldots + L_m)^r \mathrm{diam}(K). \end{aligned}$$

Now, since $\sum_{i=1}^{m} L_i < 1$, we can choose r so large that $(\sum_{i=1}^{m} L_i)^r \mathrm{diam}(K) < \epsilon$.

Step III: For every integer r,

$$\mathcal{A} \subset K_r,$$

where $K_r = \bigcup_{w \in W_r} w(K)$.

Proof (of Step III):

We will first prove that the fixed point of any word of any length belongs to K_r for each r. Fix r and let w' be a word of length r'. If $r' \geq r$, then $w' = w \circ w''$ where w has length r. Hence,

$$\begin{aligned} w'(K) &= w(w''(K)) \\ &\subset w(K). \end{aligned}$$

The last inclusion holds since we showed, in Step I, that $F_i(K) \subset K$ for each i; consequently, $w''(K) \subset K$ for each word w''.

On the other hand,

$$\begin{aligned} \text{Fix}(w') &\subset \mathcal{A} \quad \text{(by Proposition 9)} \\ &\subset K \quad \text{(by Step I)}, \end{aligned}$$

so that, $\text{Fix}(w') = w'(\text{Fix}(w')) \in w'(K)$. Therefore,

$$\begin{aligned} \text{Fix}(w') &\in w'(K) \\ &\subset w(K) \\ &\subset K_r. \end{aligned}$$

Notice that $\text{Fix}(w') = \text{Fix}(w' \circ w' \circ \ldots \circ w')$. Now, if $r' < r$, the word $w' \circ w' \circ \ldots \circ w'$ (r times) is of length at least r. Therefore, by above (case $r' \geq r$), $\text{Fix}(w') \subset K_r$.

Recall that, by Proposition 9, the attractor $\mathcal{A}$ is the closure of the fixed points of all words (see Remark after Proposition 9). So, it follows from above that $\mathcal{A} \subset K_r$ for every r. But K_r is a finite union of closed sets and therefore closed. Hence, $\mathcal{A} \subset K_r$. This completes the proof of Step III.

Now, let $\epsilon > 0$ and choose an integer r such that $\sum_{w \in W_r} \text{diam}(w(K)) < \epsilon$ (see Step II). Let M be any nonempty connected subset of $\mathcal{A}$. So, $M \subset \mathcal{A} \subset \bigcup_{w \in W_r} w(K)$. Consider the set $T = \{w(K) \cap M \mid w \in W_r,\ w(K) \cap M \neq \emptyset\}$. Let us enumerate the elements of this set as $\Omega_1, \Omega_2, \ldots, \Omega_t$, where t is the cardinality of T. Note that $t \leq m^r$

since there are m^r words w of length r. So, we can write

$$M = \bigcup_{i=1}^{t} \Omega_i(K).$$

But $\overline{M}$ is a closed subset of the compact set $\mathcal{A}$ and so it is compact. Consequently, there exist $x, y \in \overline{M} = \bigcup_{i=1}^{t} \overline{\Omega_i(K)}$ such that $\text{diam}(\overline{M}) = d(x, y)$. On the other hand, by Theorem 3, $\overline{M}$ is also connected; so, we may relabel Ω_i's in such a way that

$$x \in \overline{\Omega_1(K)}, \quad y \in \overline{\Omega_t(K)} \quad \text{and} \quad \bigcup_{i=1}^{t'} \overline{\Omega_i(K)} \cap \bigcup_{i=t'+1}^{t} \overline{\Omega_i(K)} \neq \emptyset,$$

for every $t' \in \{1, ..., t\}$. This is possible since if for some t' we had

$$\bigcup_{i=1}^{t'} \overline{\Omega_i(K)} \cap \bigcup_{i=t'+1}^{t} \overline{\Omega_i(K)} = \emptyset,$$

then since $\bigcup_{i=1}^{t'} \overline{\Omega_i(K)}$ and $\bigcup_{i=t'+1}^{t} \overline{\Omega_i(K)}$ are closed (finite union of closed sets), by Lemma 6, $\overline{M}$ would not be connected. Hence, we can choose a "chain" of points $\{p_i\}_{i=1}^{t}$ such that

$$p_1 = x \in \overline{\Omega_1(K)}, \quad p_t = y \in \overline{\Omega_t(K)} \quad \text{and} \quad p_i \in \overline{\Omega_{i-1}(K)} \cap \overline{\Omega_i(K)},$$

for every $i \in \{2, ..., t-1\}$. So we have

$$\begin{aligned}
\text{diam}(M) = \text{diam}(\overline{M}) &= d(x, y) \\
&= d(p_1, p_t) \\
&\leq d(p_1, p_2) + d(p_2, p_3) + ... + d(p_{t-1}, p_t) \\
&\leq \text{diam}(\overline{\Omega_1(K)}) + \text{diam}(\overline{\Omega_2(K)}) + ... + \text{diam}(\overline{\Omega_{t-1}(K)}) \\
&\leq \sum_{w \in W_r} \text{diam}(w(K)) \\
&< \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, this implies that $\text{diam}(M) = 0$. Therefore, every nonempty connected subset of $\mathcal{A}$ is a singleton; i.e., $\mathcal{A}$ is totally disconnected. ♠

Example 4.1.1

Let $\{F_1, F_2\}$ be an IFS on $\mathcal{H}(\mathbb{R})$ with attractor $\mathcal{A}$ where

$$F_1(x) = \frac{1}{3}x \quad \text{and} \quad F_2(x) = \frac{1}{3}x + \frac{2}{3}$$

for $x \in \mathbb{R}$. The attractor $\mathcal{A}$ is called the (triadic) *Cantor set* (see Fig. 7).

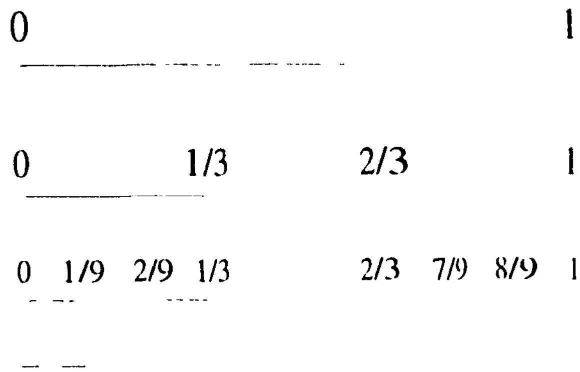


Figure 7: Initial steps in the construction of the Cantor set.

If L_1 and L_2 are Lipschitz constants of F_1 and F_2, respectively, we have

$$L_1 + L_2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

Therefore, by Theorem 12, the Cantor set is totally disconnected. Notice that the same conclusion would hold if we chose $F_2(x) = \frac{1}{3}x + a$ for any $a \neq 0, a \in \mathbb{R}$. $\heartsuit$

Example 4.1.2

Consider $\{F_1, F_2\}$, an IFS on $\mathcal{H}(\mathbb{R}^2)$ with attractor $\mathcal{A}$, where

$$\begin{aligned} F_1(z) &= \frac{1}{2}R_{30}(z), \\ F_2(z) &= \frac{9}{20}z + \begin{pmatrix} 11/20 \\ 11/20 \end{pmatrix}, \end{aligned}$$

for $z \in \mathbb{R}^2$ (see Fig. 8).

By Theorem 12, the attractor $\mathcal{A}$ is totally disconnected since

$$L_1 + L_2 = \frac{1}{2} + \frac{9}{20} = \frac{19}{20} < 1,$$

where L_1 and L_2 are Lipschitz constants of F_1 and F_2, respectively. $\heartsuit$

Figure 8: A totally disconnected attractor.

The following example illustrates a case where Theorem 12 is inconclusive. Although it consists of a small variation in the IFS of the preceeding example, we are unable to decide on the connectedness of the new attractor.

Example 4.1.3

Let $\{F_1, F_2\}$ be an IFS on $\mathcal{H}(\mathbb{R}^2)$ with attractor $\mathcal{A}$, where

$$\begin{aligned} F_1(z) &= \frac{1}{2}R_{30}(z), \\ F_2(z) &= \frac{1}{2}z + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \end{aligned}$$

for $z \in \mathbb{R}^2$ (see Fig. 9).

Figure 9: Is this attractor connected or totally disconnected?

Since the sum of the Lipschitz constants of F_1 and F_2 is equal to 1, Theorem 12 cannot be applied here. We will see later (Corollary 2) that this attractor must be either connected or totally disconnected. ♡

Theorem 13

Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, F_2, ..., F_m\}$ and let K be a compact set such that

- $\bigcup_{i=1}^{m} F_i(K) \subset K$,
- $F_i(K) \cap F_j(K) = \emptyset$ *for all* $i \neq j$.

Then $\mathcal{A}$ is totally disconnected.

Proof:

Let T be any subset of $\mathcal{A}$ containing at least two points and let ϵ =diam(T). Let $L = \max\{L_i\}_{i=1}^{m}$ where L_i is the Lipschitz constant of F_i for $i = 1, ..., m$. Define $\mathbf{F}(K) = \bigcup_{i=1}^{m} F_i(K)$. By the hypothesis, $\mathbf{F}(K) \subset K$. Hence, by Proposition 6, $T \subset \mathcal{A} \subset K$.

We claim that $\mathbf{F}^{\circ n}(K)$ is composed of m^n disjoint compact sets each of diameter less than or equal to L^ndiam(K). We will show this by induction:
The truth of this statement for $n = 1$ follows at once from the hypothesis. Suppose that it holds for $n = p$. In other words, suppose that $\mathbf{F}^{\circ p}(K) = \bigcup_{j=1}^{m^p} C_j$, where C_j's are disjoint compact subsets of K each of the form $F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_p}(K)$ for $\{i_1, i_2, ..., i_p\} \subset \{1, 2, ..., m\}$, and diam$(C_j) \leq L^p$diam$(K)$ for all $j = 1, ..., m^p$.

Now, for $\{i, l\} \in \{1, ..., m\}$ and for all $j = 1, ..., m^p$, the set $F_i(C_j) \subset K$ is compact and $F_i(C_j) \cap F_l(C_j) = \emptyset$ if $i \neq l$. So, $\mathbf{F}^{\circ p+1}(K) = \mathbf{F}(\mathbf{F}^{\circ p}(K))$ is composed of $m.m^p = m^{p+1}$ disjoint compact sets each of the form $F_{i_1} \circ F_{i_2} \circ ... \circ F_{i_{p+1}}(K)$ with diameter less than or equal to L^{p+1}diam(K), where $\{i_1, i_2, ..., i_{p+1}\} \subset \{1, ..., m\}$. This completes the proof of the claim.

Next, let N be a natural number such that $L^N \text{diam}(K) < \epsilon$. The set T intersects at least two of the disjoint compact sets which form $\mathbf{F}^{\circ N}(K)$. Let $\tilde{C}$ be one of them. Then, $\mathbf{F}^{\circ N}(K) \setminus \tilde{C}$ is the union of $m^N - 1$ compact sets and so it is compact. Now,

$$\begin{aligned} T &\subset \mathcal{A} \\ &= \mathbf{F}^{\circ N}(\mathcal{A}) \qquad \text{(since } \mathcal{A} \text{ is the fixed point of } \mathbf{F}\text{)} \\ &\subset \mathbf{F}^{\circ N}(K) \qquad \text{(since } \mathcal{A} \subset K\text{)}. \end{aligned}$$

Hence, $\tilde{C}$ and $\mathbf{F}^{\circ N}(K) \setminus \tilde{C}$ are two disjoint compact sets covering T. Therefore, by Lemma 6, T is not connected. Since T was an arbitrary subset of $\mathcal{A}$ with at least two points, we have shown that $\mathcal{A}$ is totally disconnected. ♠

Corollary 1

Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, ..., F_m\}$.
If $F_i(\mathcal{A}) \cap F_j(\mathcal{A}) = \emptyset$ for all $i \neq j$, then $\mathcal{A}$ is totally disconnected.

Proof:

Take $K = \mathcal{A}$ in Theorem 13. ♠

The converse of this result need not be true as shown in the next example.

Example 4.1.4

Let $\{F_1, F_2, F_3\}$ be an IFS on $\mathcal{H}(\mathbb{R}^2)$ with attractor $\mathcal{A}$, where

$$\begin{aligned} F_1(z) &= \frac{2}{5} R_{120}(z) + \begin{pmatrix} 9/10 \\ 0 \end{pmatrix}, \\ F_2(z) &= \frac{2}{5} R_{-120}(z) + \begin{pmatrix} 3/10 \\ \sqrt{3}/5 \end{pmatrix}, \\ F_3(z) &= \frac{3}{20} z + \begin{pmatrix} 17/40 \\ 17\sqrt{3}/40 \end{pmatrix}, \end{aligned}$$

for $z \in \mathbb{R}^2$ (see Fig. 10).

We have $L_1 + L_2 + L_3 = 2/5 + 2/5 + 3/20 = 19/20 < 1$; so, by Theorem 12, $\mathcal{A}$ is totally disconnected. On the other hand, $\text{Fix}(F_3) = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix} \in \mathcal{A}$. Therefore,

$$F_1(\text{Fix}(F_3)) \in F_1(\mathcal{A}) \quad \text{and} \quad F_2(\text{Fix}(F_3)) \in F_2(\mathcal{A}).$$

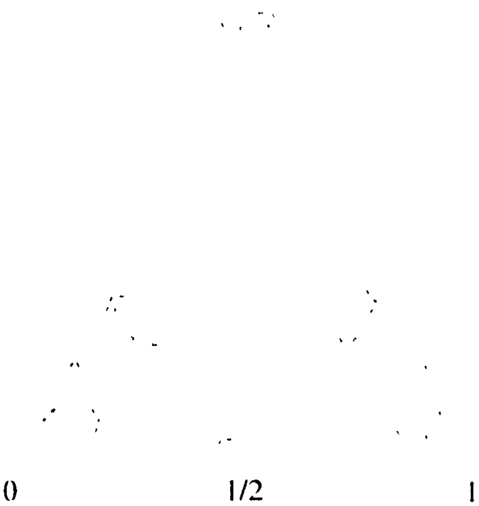


Figure 10: A totally disconnected attractor where not every point has a unique address.

But, $F_1(\text{Fix}(F_3)) = F_2(\text{Fix}(F_3)) = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$, so that $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) \neq \emptyset$.

This example also illustrates a case where the attractor of an IFS is totally disconnected but yet, not every point in the attractor has a unique address (see the opening paragraph of this section). Indeed, the sequences $\{1, 3, 3, 3, ...\}$ and $\{2, 3, 3, 3, ...\}$ are both addresses of the point $\begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$. ♡

4.2 Connected attractors

Theorem 14

Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, F_2\}$. If $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is connected.

Proof:

Suppose that $\mathcal{A}$ is not connected. We will show that $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) = \emptyset$. Let $\mathbf{F} : \mathcal{H}(\mathbb{R}^n) \longrightarrow \mathcal{H}(\mathbb{R}^n)$ be defined by

$$\mathbf{F}(K) = F_1(K) \cup F_2(K)$$

with Lipschitz constant $L = \max\{L_1, L_2\}$, where L_1 and L_2 are the Lipschitz constants of F_1 and F_2, respectively. Since $\mathcal{A}$ is not connected, by Lemma 6, there exist two disjoint closed sets C_1 and C_2, such that $\mathcal{A} \subset C_1 \cup C_2$, $\mathcal{A} \cap C_1 \neq \emptyset$ and $\mathcal{A} \cap C_2 \neq \emptyset$. Let

$$\delta = \inf\{d(x,y) |\ x \in C_1 \text{ and } y \in C_2\}.$$

Let k be the smallest integer such that any set of the form $F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A})$, with $\{i_1, \ldots, i_k\} \subset \{1, 2\}$, is entirely contained in either C_1 or C_2 (but not in both). Note that such a k exists since

$$\begin{aligned} \operatorname{diam}(F_{i_1} \circ \ldots \circ F_{i_\eta}(\mathcal{A})) &\leq L^\eta \operatorname{diam}(\mathcal{A}) \\ &\leq \delta \quad \text{(for sufficiently large } \eta). \end{aligned}$$

Now, we claim that there exists a sequence $\{j_1, \ldots, j_{k-1}\} \subset \{1, 2\}$ such that the sets $F_{j_1} \circ \ldots \circ F_{j_{k-1}} \circ F_1(\mathcal{A})$ and $F_{j_1} \circ \ldots \circ F_{j_{k-1}} \circ F_2(\mathcal{A})$ do not both lie in the same C_i for $i = 1, 2$. This is true because otherwise we could assume, without loss of generality, that for any sequence $\{i_1, \ldots, i_{k-1}\} \subset \{1, 2\}$,

$$F_{i_1} \circ \ldots \circ F_{i_{k-1}} \circ F_1(\mathcal{A}) \subset C_1 \quad \text{and} \quad F_{i_1} \circ \ldots \circ F_{i_{k-1}} \circ F_2(\mathcal{A}) \subset C_1.$$

But this would imply that

$$\begin{aligned} F_{i_1} \circ \ldots \circ F_{i_{k-1}}(A) &= F_{i_1} \circ \ldots \circ F_{i_{k-1}}[F_1(\mathcal{A}) \cup F_2(A)] \\ &= F_{i_1} \circ \ldots \circ F_{i_{k-1}} \circ F_1(\mathcal{A}) \bigcup F_{i_1} \circ \ldots \circ F_{i_{k-1}} \circ F_2(\mathcal{A}) \\ &\subset C_1, \end{aligned}$$

contradicting the fact that k was the smallest integer with this property. This completes the proof of the claim.

Let us assume, without loss of generality, that

$$F_{j_1} \circ \ldots \circ F_{j_{k-1}} \circ F_1(\mathcal{A}) \subset C_1 \quad \text{and} \quad F_{j_1} \circ \ldots \circ F_{j_{k-1}} \circ F_2(\mathcal{A}) \subset C_2.$$

We have

$$\begin{aligned} F_{j_1} \circ \ldots \circ F_{j_{k-1}}[F_1(\mathcal{A}) \cap F_2(\mathcal{A})] &\subset \bigcap_{i=1}^{2} F_{j_1} \circ \ldots \circ F_{j_{k-1}}(F_i(\mathcal{A})) \\ &\subset C_1 \cap C_2 \\ &= \emptyset. \end{aligned}$$

This is possible only if $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) = \emptyset$. ♠

The analysis of the connectedness of an IFS with only two contractions is considerably simpler than that of one with three or more contractions. Indeed, from Corollary 1 and Theorem 14 we obtain at once

Corollary 2

Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, F_2\}$. Then

- $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) = \emptyset \Longrightarrow \mathcal{A}$ *is totally disconnected.*
- $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) \neq \emptyset \Longrightarrow \mathcal{A}$ *is connected.*

The following example exhibits the more complicated dynamics resulted by adding a third contraction to the above IFS.

Example 4.2.1

Let

$$\begin{aligned} F_1[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \\ F_2[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \\ F_3[\begin{pmatrix} x \\ y \end{pmatrix}] &= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \end{aligned}$$

where $x, y \in \mathbb{R}$. Let $\mathcal{A}$ be the attractor of the IFS $\{F_1, F_2, F_3\}$ and define the function $\mathbf{F} : \mathcal{H}(\mathbb{R}^2) \longrightarrow \mathcal{H}(\mathbb{R}^2)$ by $\mathbf{F}(K) = \bigcup_{i=1}^{3} F_i(K)$. Let K be the rectangle $[-1, 1] \times [0, 1]$

(see Fig. 11). Observe that

$$\begin{aligned} F_1(K) &= [0,1] \times \{0\}, \\ F_2(K) &= [-1,0] \times \{0\}, \\ F_3(K) &= [-1/2, 1/2] \times [1/2, 1]. \end{aligned}$$

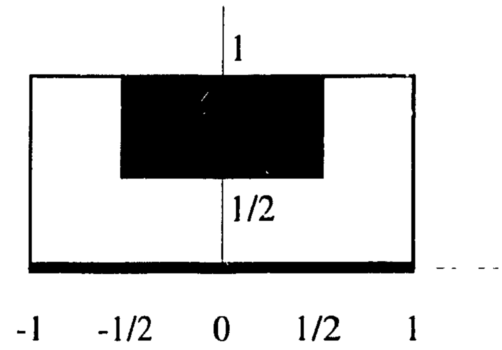


Figure 11: Determining the connectedness of an attractor (see Example 4.2.1).

We have $\mathbf{F}(K) \subset K$ and hence, by Proposition 6, $\mathcal{A} \subset K$. Also, by Proposition 8, the attractor $\mathcal{A}'$ of the IFS $\{F_1, F_2\}$ is contained in $\mathcal{A}$. Notice that the set $[-1,1] \times \{0\}$ is the attractor (fixed point) of the function $\mathbf{F}' : \mathcal{H}(\mathbb{R}^2) \longrightarrow \mathcal{H}(\mathbb{R}^2)$ defined by $\mathbf{F}'(K) = F_1(K) \cup F_2(K)$. Therefore, $\mathcal{A}' = [-1,1] \times \{0\} \subset \mathcal{A}$. We have

- $F_1(\mathcal{A}) \cap F_2(\mathcal{A}) \neq \emptyset$, since

$$\begin{aligned} \{0\} &= F_1(\mathcal{A}') \cap F_2(\mathcal{A}') \\ &\subset F_1(\mathcal{A}) \cap F_2(\mathcal{A}) \quad \text{(since } \mathcal{A}' \subset \mathcal{A}). \end{aligned}$$

- $F_1(\mathcal{A}) \cap F_3(\mathcal{A}) = \emptyset$, since

$$\begin{aligned} F_1(\mathcal{A}) \cap F_3(\mathcal{A}) &\subset F_1(K) \cap F_3(K) \quad \text{(since } \mathcal{A} \subset K) \\ &= \emptyset. \end{aligned}$$

Similarly,

- $F_2(\mathcal{A}) \cap F_3(\mathcal{A}) = \emptyset$.

We will show below that $\mathcal{A}$ is neither connected nor totally disconnected.

- $\mathcal{A}$ is not connected since, on one hand,

$$\begin{aligned} \mathcal{A} \subset K &\implies \mathbf{F}(\mathcal{A}) \subset \mathbf{F}(K) \\ &\implies \mathcal{A} \subset \mathbf{F}(K) \quad \text{and } \mathbf{F}(K) \text{ is not connected,} \end{aligned}$$

  and on the other hand, the points (1,0) and (0,1) which belong to $\mathcal{A}$ (since they are the fixed points of F_1 and F_3, respectively) are in different components of $\mathbf{F}(K)$.

- $\mathcal{A}$ is not totally disconnected since $\mathcal{A}' = [-1,1] \times \{0\} \subset \mathcal{A}$. ♡

4.3 Attractors with other types of connectedness

Theorem 15 (See [17])

Let $\{F_i\}_{i=1}^m$ be an IFS with attractor $\mathcal{A}$. Let $\mathbf{F}(K) = \bigcup_{i=1}^m F_i(K)$ for $K \in \mathcal{H}(\mathbb{R}^n)$. Suppose that there exists an arc $\Gamma \subset \mathbb{R}^n$ such that

- $\Gamma \subset \mathbf{F}(\Gamma)$,
- $\mathbf{F}(\Gamma)$ *is arcwise connected.*

Then, A is arcwise connected.

Proof:

Let y and z be any two points in $\mathcal{A}$. We have to show that there exists an arc in $\mathcal{A}$ from y to z. Notice that by Proposition 6,

$$\Gamma \subset \mathbf{F}(\Gamma) \implies \Gamma \subset \mathcal{A}.$$

Let $x_0 \in \Gamma$. We will first construct an arc in $\mathcal{A}$ from y to x_0. Let $\{i_j\}_{j=1}^\infty$ be an address of y. So, $y \in F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathcal{A})$ for all $j \geq 1$. Let $x_j = F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(x_0)$. Now,

$$x_0 \in \Gamma \subset \mathcal{A} \implies x_j \in F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathcal{A}).$$

Therefore, $d(x_j, y) \leq L^j \text{diam}(\mathcal{A})$ where L is the Lipschitz constant of $\mathbf{F}$. This means that $x_j \underset{j\to\infty}{\longrightarrow} y$. Now,

- $x_0 \in \Gamma \subset \mathbf{F}(\Gamma)$,
- $x_1 \in F_{i_1}(\Gamma) \subset \mathbf{F}(\Gamma)$,
- $\mathbf{F}(\Gamma)$ is arcwise connected.

Hence, there exists an arc $\Gamma_0 = x_0 \frown x_1 \subset \mathbf{F}(\Gamma) \subset \mathcal{A}$. Next,

- $x_1 \in F_{i_1}(\Gamma) \subset F_{i_1}(\mathbf{F}(\Gamma))$ (since $\Gamma \subset \mathbf{F}(\Gamma)$),
- $x_2 \in F_{i_1} \circ F_{i_2}(\Gamma) \subset F_{i_1}(\mathbf{F}(\Gamma))$,
- By Lemma 5, $F_{i_1}(\mathbf{F}(\Gamma))$ is arcwise connected.

So, there exists an arc $\Gamma_1 = x_1 \frown x_2 \subset F_{i_1}(\mathbf{F}(\Gamma)) \subset \mathcal{A}$. Similarly, for any $j \geq 1$,

- $x_j \in F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\Gamma) \subset F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathbf{F}(\Gamma))$,
- $x_{j+1} \in F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j} \circ F_{i_{j+1}}(\Gamma) \subset F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(F(\Gamma))$,
- By Lemma 5, $F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathbf{F}(\Gamma))$ is arcwise connected.

Therefore, there exists an arc $\Gamma_j = x_j \frown x_{j+1} \subset F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathbf{F}(\Gamma)) \subset \mathcal{A}$. Now, consider the function $\gamma : [0,1] \longrightarrow \mathcal{A}$ defined as follows:

$$\gamma(0) = y \quad \text{and for all } j \geq 0, \quad \begin{cases} \gamma(\frac{1}{j+1}) = x_j \\ \gamma([\frac{1}{j+2}, \frac{1}{j+1}]) = \Gamma_j. \end{cases}$$

We claim that γ is an arc in $\mathcal{A}$ form y to x_0. Indeed,

- $\gamma(0) = y$,
- $\gamma(1) = x_0$,
- $\gamma([0,1]) = \{y\} \cup \bigcup_{j=1}^{\infty} \Gamma_j \subset \mathcal{A}$,

- For any $j \geq 0$, the function γ is continuous on the interval $[\frac{1}{j+2}, \frac{1}{j+1}]$ and $\gamma(\frac{1}{j+2}) = x_{j+1} \in \Gamma_j \cap \Gamma_{j+1}$, so that γ is continuous on $(0,1]$.

We still have to show that γ is continuous at 0. In other words, we need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $d(\gamma(x), y) < \epsilon$ whenever $0 < x < \delta$. Notice that

$$\Gamma_j \subset F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathbf{F}(\Gamma)) \subset F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_j}(\mathcal{A});$$

therefore,

$$\operatorname{diam}(\Gamma_j) \leq L^j \operatorname{diam}(\mathcal{A}) \underset{j \to \infty}{\longrightarrow} 0.$$

On the other hand, we showed that $x_j \underset{j \to \infty}{\longrightarrow} y$ where $x_j \in \Gamma_j$. So, we can find an integer k such that for all $j \geq k$,

$$d(x_j, y) < \epsilon/2 \qquad \text{and} \qquad \operatorname{diam}(\Gamma_j) < \epsilon/2.$$

Let $\delta = \frac{1}{k+1}$. So, whenever $0 < x < \delta$, we have $\gamma(x) \in \Gamma_j$ for some $j > k$ and therefore,

$$\begin{aligned} d(\gamma(x), y) &\leq d(\gamma(x), x_j) + d(x_j, y) \\ &\leq \operatorname{diam}(\Gamma_j) + d(x_j, y) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence, γ is continuous on $[0,1]$.

Similarly, we can construct an arc in $\mathcal{A}$ from z to x_0 and therefore, by Lemma 3, there exists an arc in $\mathcal{A}$ from x_0 to z. Finally, by Lemma 4, there exists an arc in $\mathcal{A}$ from y to z. ♠

Example 4.3.1

Consider the triangle ABC where $A = (0,0)$, $B = (1,0)$ and $C = (1/2, \sqrt{3}/2)$. Let F, E and D denote the midpoints of the segments AB, BC and AC, respectively. Let Γ be the arc made by the union of the segments AB and AC (see Fig. 12). Now, consider the IFS we used in Example 3.5.1 to introduce the Sierpinski gasket. We have

- $F_1(\Gamma)$ is the union of the segments AD and AF,
- $F_2(\Gamma)$ is the union of the segments EF and FB,
- $F_3(\Gamma)$ is the union of the segments CD and DE.

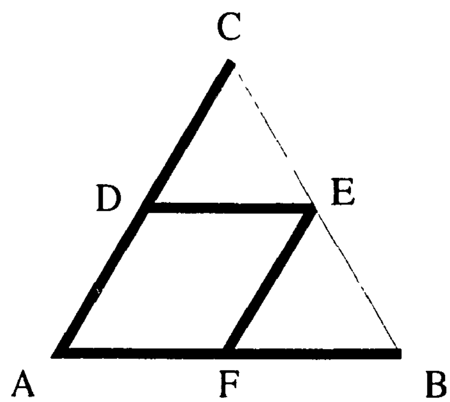


Figure 12: The arcwise connectedness of the Sierpinski gasket.

Therefore, $\mathbf{F}(\Gamma) = \bigcup_{i=1}^{3} F_i(\Gamma)$ is the union of the segments AC, AB, DE and EF. So, $\mathbf{F}(\Gamma)$ is arcwise connected and contains Γ. By Theorem 15, the Sierpinski gasket is arcwise connected. ♡

Example 4.3.2

Let $\{F_1, F_2, F_3\}$ be an IFS, where

$$F_1(z) = \frac{1}{2}R_{90}(z) + \begin{pmatrix} -1/2 \\ 0 \end{pmatrix},$$

$$F_2(z) = \frac{1}{2}R_{-90}(z) + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},$$
$$F_3(z) = \frac{1}{2}z + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},$$

for $z \in \mathbb{R}^2$. The attractor $\mathcal{A}$ of this IFS is shown in Fig. 13.

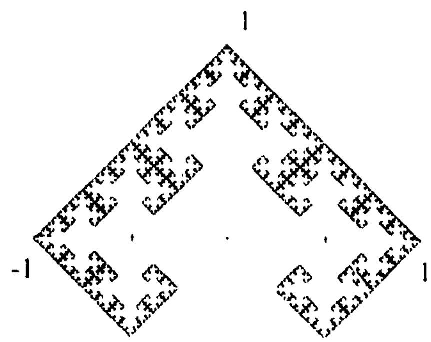


Figure 13: An arcwise connected attractor.

To see that $\mathcal{A}$ is arcwise connected, let Γ be the arc composed of the union of the segments BD and DF (see Fig. 14). Then,

- $F_1(\Gamma)$ is the union of the segments EF and FG,
- $F_2(\Gamma)$ is the union of the segments AB and BC,
- $F_3(\Gamma)$ is the union of the segments CD and DE.

Hence, $\mathbf{F}(\Gamma) = \bigcup_{i=1}^{3} F_i(\Gamma)$ is the union of the segments AB, BD, DF and FG and is therefore, arcwise connected. Also, $\Gamma \subset \mathbf{F}(\Gamma)$ so that by Theorem 15, the attractor $\mathcal{A}$ is arcwise connected. ♡

Theorem 16

Suppose that $\{F_i\}_{i=1}^{m}$ is an IFS with attractor $\mathcal{A}$. If $\mathcal{A}$ is connected, then it also has property S.

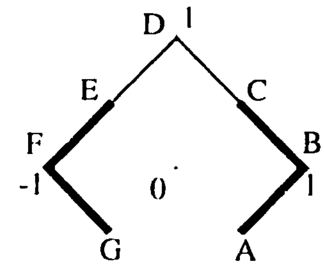


Figure 14: The arcwise connectedness of the attractor shown in Fig. 13.

Proof:

Let $\mathbf{F}(K) = \bigcup_{i=1}^{m} F_i(K)$ for $K \in \mathcal{H}(\mathbb{R}^n)$ and let L be the Lipschitz constant of $\mathbf{F}$. Since A is the fixed point of $\mathbf{F}$, for any $k \geq 1$ we have

$$\begin{aligned} \mathcal{A} &= \mathbf{F}^{ok}(\mathcal{A}) \\ &= \bigcup_{1 \leq i_j \leq m} F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A}). \end{aligned}$$

Now, $\operatorname{diam}(F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A})) \leq L^k \operatorname{diam}(\mathcal{A}) \underset{k \to \infty}{\longrightarrow} 0$. So, given $\epsilon > 0$, we can choose k so large that $\operatorname{diam}(F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A})) \leq \epsilon$. Furthermore, each $F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A})$ is connected (by Theorem 5). Finally, since there are m^k words of length k (see Remark after Proposition 9), we have shown that $\mathcal{A}$ can be expressed as a finite union of connected sets each of diameter less than ϵ. Therefore, $\mathcal{A}$ has property S. ♠

Combining Theorems 16 and 7, we immediately get

Corollary 3

Suppose that $\{F_i\}_{i=1}^{m}$ is an IFS whose attractor $\mathcal{A}$ is connected. Then, $\mathcal{A}$ is locally connected.

Theorem 17

Suppose that $\{F_i\}_{i=1}^m$ is an IFS whose attractor $\mathcal{A}$ is connected. Then, $\mathcal{A}$ is semi-locally connected.

Proof:

Let x be any point in $\mathcal{A}$. We would like to show that $\mathcal{A}$ is semi-locally connected at x. Let $\epsilon > 0$. By Corollary 3, $\mathcal{A}$ is locally connected. Therefore, by Theorem 6, we can write $\mathcal{A} = \bigcup_{\alpha \in \mathcal{A}} V(\alpha)$ where $V(\alpha)$ is a region in $\mathcal{A}$ containing α with $\text{diam}(V(\alpha)) < \epsilon$. So, the collection of sets $[V(\alpha)]_{\alpha \in \mathcal{A}}$ is an open covering of $\mathcal{A}$. Since $\mathcal{A}$ is compact, some finite subcollection $[W_i]_{i=1}^p$ of $[V(\alpha)]_{\alpha \in \mathcal{A}}$ covers $\mathcal{A}$ where $p \geq 1$; that is, $\mathcal{A} = \bigcup_{i=1}^p W_i$.

Now, $x \in W_j$ for some $1 \leq j \leq p$, and

$$\mathcal{A} \setminus W_j = \bigcup_{i=1}^{j-1} W_i \cup \bigcup_{i=j+1}^{p} W_i.$$

Hence, W_j is a neighborhood of x with diameter less than ϵ such that $\mathcal{A} \setminus W_j$ is a finite union of connected sets and therefore has a finite number of components (at most $p - 1$). Since $\epsilon > 0$ was arbitrary, $\mathcal{A}$ is semi-locally connected at x. ♠

Corollary 4

Suppose that $\{F_i\}_{i=1}^m$ is an IFS with attractor $\mathcal{A}$. Then $\mathcal{A}$ is connected if and only if $\mathcal{A}$ is arcwise connected.

Proof:

($\Longleftarrow$):

By Theorem 8, any arcwise connected set is connected.

($\Longrightarrow$):

Suppose that $\mathcal{A}$ is connected. Then, by Corollary 3, $\mathcal{A}$ is locally connected and since it is compact, we have that $\mathcal{A}$ is a locally connected continuum. Now it follows, from Theorem 9, that $\mathcal{A}$ is arcwise connected. ♠

Example 4.3.3

We saw in Example 4.3.1 that the Sierpinski gasket is arcwise connected. So, by Corollary 4, it is also connected. Then, it also follows from Theorem 16, Corollary 3 and Theorem 17, that the Sierpinski gasket has property S and is both locally connected and semi-locally connected. ♡

4.4 Other properties of attractors

Theorem 18 (See [9])

Let $\mathcal{A}$ be the attractor of an IFS $\{F_i\}_{i=1}^m$ where F_i is injective for $1 \leq i \leq m$. If $\mathrm{Fix}(F_p) \neq \mathrm{Fix}(F_q)$ for some $p \neq q$, then $\mathcal{A}$ is perfect.

Proof:

Suppose that $\mathrm{Fix}(F_p) \neq \mathrm{Fix}(F_q)$ for some $p \neq q$ and let $\mathbf{F}(K) = \bigcup_{i=1}^m F_i(K)$ for any $K \in \mathcal{H}(\mathbb{R}^n)$. Let $K = \{\mathrm{Fix}(F_p), \mathrm{Fix}(F_q)\}$. Then, since $K \subset \mathbf{F}(K)$, by Proposition 6,

$$\mathcal{A} = \overline{\bigcup_{n\geq 1} \mathbf{F}^{\circ n}(K)}.$$

We will first show that the set $\Lambda = \bigcup_{n\geq 1} \mathbf{F}^{\circ n}(K)$ has no isolated point. Let $\lambda \in \Lambda$. Then for some $r \geq 1$,

$$\begin{aligned} \lambda &\in \mathbf{F}^{\circ r}(K) \\ &= \bigcup_{1\leq i_1,\ldots,i_r\leq m} F_{i_1} \circ \ldots \circ F_{i_r}(K). \end{aligned}$$

So, $\lambda \in F_{i_1} \circ \ldots \circ F_{i_r}(K)$ for some sequence $\{i_j\}_{j=1}^r \subset \{1, \ldots, m\}$. Without loss of generality, we can assume

$$\lambda = F_{i_1} \circ \ldots \circ F_{i_r}(\mathrm{Fix}(F_p)).$$

Now, $\mathrm{Fix}(F_p) = \lim_{n\to\infty} F_p^{\circ n}(C)$ for any compact set C. In particular, by continuity of $F_{i_1} \circ \ldots \circ F_{i_r}$, we have

$$\begin{aligned} \lim_{n\to\infty} F_{i_1} \circ \ldots \circ F_{i_r} \circ F_p^{\circ n}(\mathrm{Fix}(F_q)) &= F_{i_1} \circ \ldots \circ F_{i_r}(\lim_{n\to\infty} F_p^{\circ n}(\mathrm{Fix}(F_q))) \\ &= F_{i_1} \circ \ldots \circ F_{i_r}(\mathrm{Fix}(F_p)) \\ &= \lambda. \end{aligned}$$

Note that $F_{i_1} \circ ... \circ F_{i_r} \circ F_p^{\circ n}(\text{Fix}(F_q)) \in \mathbf{F}^{\circ n+r}(K) \subset \Lambda$ for all $n \geq 1$. This means that λ is the limit of a sequence in Λ. Moreover, since $F_1, ..., F_m$ are injective, for all $n \geq 1$,

$$\begin{aligned}\text{Fix}(F_q) \neq \text{Fix}(F_p) &\Rightarrow F_p^{\circ n}(\text{Fix}(F_q)) \neq F_p^{\circ n}(\text{Fix}(F_p)) = \text{Fix}(F_p) \\ &\Rightarrow F_{i_1} \circ ... \circ F_{i_r} \circ F_p^{\circ n}(\text{Fix}(F_q)) \neq F_{i_1} \circ ... \circ F_{i_r}(\text{Fix}(F_p)) \\ &\Rightarrow F_{i_1} \circ ... \circ F_{i_r} \circ F_p^{\circ n}(\text{Fix}(F_q)) \neq \lambda.\end{aligned}$$

In other words, the sequence $\{F_{i_1} \circ ... \circ F_{i_r} \circ F_p^{\circ n}(\text{Fix}(F_q))\}_{n=1}^{\infty}$ is a sequence in Λ which converges to, but does not contain, λ. Hence, λ is a limit point of Λ. Since λ was arbitrary, we have shown that Λ and therefore $\overline{\Lambda}$ have no isolated points. Finally, since $\overline{\Lambda} = \mathcal{A}$ is closed, we conclude that $\mathcal{A}$ is perfect. ♠

Example 4.4.1

Consider once again the Cantor set, defined in Example 4.1.1. The two functions $F_1(x) = \frac{1}{3}x$ and $F_2(x) = \frac{1}{3}x + \frac{2}{3}$ are both injective. Also, $\text{Fix}(F_1) = 0$ whereas $\text{Fix}(F_2) = 2/3$. Hence, by Theorem 18, the Cantor set is perfect. ♡

Theorem 19

Let $\mathcal{A}$ be the attractor of an IFS $\{F_i\}_{i=1}^m$ where F_i is a homeomorphism of $\mathbb{R}^n$ for $i = 1, ..., m$. If $Int(\mathcal{A}) \neq \emptyset$, then $\mathcal{A} = \overline{Int(\mathcal{A})}$.

Proof:

$(\supset)$:

$$\begin{aligned}\text{Int}(\mathcal{A}) \subset \mathcal{A} \Longrightarrow \overline{\text{Int}(\mathcal{A})} &\subset \overline{\mathcal{A}} \\ &= \mathcal{A} \quad \text{(since } \mathcal{A} \text{ is closed)}.\end{aligned}$$

$(\subset)$:

Let $x \in \mathcal{A}$ and let $\{i_k\}_{k=1}^{\infty} \subset \{1, ..., m\}$ be an address of the point x. This means that

$x = \bigcap_{k=1}^{\infty} F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A})$. Next, let $\mathcal{O}$ be an open subset of $\mathcal{A}$ and let $x_0 \in \mathcal{O}$. For every integer $k \geq 1$, let $x_k = F_{i_1} \circ \ldots \circ F_{i_k}(x_0)$. Notice that for all $k \geq 1$,

- $$\begin{aligned} \mathcal{O} \subset \mathcal{A} \Longrightarrow F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{O}) &\subset F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{A}) \\ &\subset \mathcal{A}. \end{aligned}$$

- Since the composition of homeomorphisms is itself a homeomorphism, by Proposition 2, $F_{i_1} \circ \ldots \circ F_{i_k}(\mathcal{O})$ is open.

Hence, $x_k \in \text{Int}(\mathcal{A})$ for all $k \geq 1$. On the other hand,

$$d(x_k, x) \leq L^k \text{diam}(\mathcal{A}),$$

where L is the Lipschitz constant of the IFS $\{F_i\}_{i=1}^{m}$. In other words, $x_k \underset{k\to\infty}{\longrightarrow} x$ which implies that $x \in \overline{\text{Int}(\mathcal{A})}$. Therefore, $\mathcal{A} \subset \overline{\text{Int}(\mathcal{A})}$. ♠

Remark: By the above theorem, when the contractions forming a given IFS are homeomorphisms of $\mathbb{R}^n$ (for instance, affine transformations with nonzero determinants), the attractor $\mathcal{A}$ of that IFS

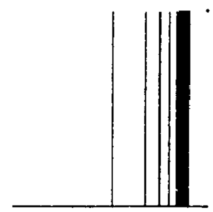
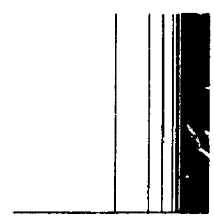
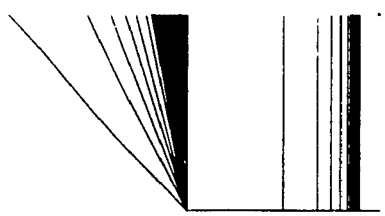
- either has an empty interior ($\text{Int}(\mathcal{A}) = \emptyset$),
- or is the closure of an open set ($\mathcal{A} = \overline{\text{Int}(\mathcal{A})}$).

Conclusion

In this exposition, characterizations and topological properties (primarily connectedness) of the attractor of an iterated function system were studied.

The attractor $\mathcal{A}$ of an IFS $\{F_i\}_{i=1}^{m}$, where F_i's are contractions of $\mathbb{R}^n$, can be connected (e.g., the Sierpinski gasket), totally disconnected (e.g., the Cantor set) or neither one (see Example 4.2.1). In the case of two contractions ($m = 2$), the attractor $\mathcal{A}$ is either connected or totally disconnected. If $\mathcal{A}$ is connected, then it has property S and is both locally connected and semi-locally connected. We also noted that the concepts of connectedness and arcwise connectedness are equivalent for $\mathcal{A}$. A sufficient condition for arcwise connectedness of $\mathcal{A}$ is provided in Theorem 15. However, this result has limited scope in the sense that it is only useful where $\mathcal{A}$ contains a segment of a straight line; for instance in the case of the Sierpinski gasket (see Example 4.3.1) or the attractor in Example 4.3.2.

To close our discussion, it is worth mentioning that the study of the connectedness of an attractor is still in its early stages. Indeed, as observed in Example 4.1.3, the available methods are insufficient to decide on the connectedness of the attractor of even some simple looking IFS's.

SET	GRAPH	PROPERTY*				
		1	2	3	4	5
Finite union of disjoint intervals: $(0,1) \cup (1,2)$			√	√		
Infinite union of disjoint intervals: $\bigcup_{n=1}^{\infty}(n, n+1)$				√		
The deleted comb (see Example 2.2.2)		√			√	
Union of the deleted comb (see Example 2.2.2) with the segment $\{1\} \times [0,1]$		√				√
Union of the deleted comb (see Example 2.2.2) with the set $\{(x, -nx) \mid -1 \leq x \leq 0, n \in \mathbb{N}\}$		√				

* 1= Connected, 2= Property S, 3= Locally connected, 4= Semi-locally connected, 5= Arcwise connected.

Table 1: Connectivity properties

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