Coupled Mode Theory and Network Applications

Raffi Antepyan

A Thesis

in

The Department

of

Electrical Engineering

Presented in Partial Fulfillment of the Requirements

for the degree of Master of Engineering at

Concordia University

Montreal, Québec, Canada

March 1984

ABSTRACT

Coupled Mode Theory and Network Applications

Raffi Antepyan

This thesis presents an analysis and summary of network representation techniques for various coupled systems, and the conditions and laws which define the system characteristics. Coupled mode analysis is used to describe distributed systems supporting weakly coupled modes. It is applicable to systems where a fraction of the powers is exchanged between propagating modes. The analysis is carried out for 2n-port networks and 4-port applications are given in detail. A method to evaluate the transfer matrix of lossy and nonuniform forward and backward couplers is given, where Jones and Mueller calculi are used in conjunction with coupled mode theory to obtain phase, amplitude, and power relations. Examples are cited from the areas of microwaves, integrated optics, and surface acoustic wave devices.

To my parents

and to Karen

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Dr. Otto Schwelb for his guidance and help throughout this work.

TABLE OF CONTENTS

LIST OF FIGURES

I. INTRODUCTION

Many branches of science and engineering involve the concept of wave motion and the related concepts of impedance, power flow, phase and group velocities, and coupling of modes. Coupled mode theory can be used to treat a number of problems in physics and engineering characterized by the above mentioned concepts.

The literature on coupled mode formalism is quite extensive. In references (10) to (16), one can find the basic principles of this concept. Some of the applications of coupled mode theory are given in references (1) to (9). Devices to which coupled mode formalism can be applied include directional couplers, distributed parametric circuits (2), electron beam devices (3,4), electro-optic modulators, acousto-optic beam deflectors, and various other devices (5-9).

Another important concept used in this thesis is the Jones and Mueller calculi, which are covered in the Appendix of reference (21). Jones calculus was first introduced by R. C. Jones in a series of papers in 1941 (30-32). Originally used to describe the polarization of light, represented by a two component state vector (Jones vector), here the concept is extended to cover systems having a dimensionality higher than two (22,23). Its advantage is that the four component polarization or Stokes vector whose elements are determined by the components of the complex electric field, and the generalization of Stokes vector to systems of dimensionality higher than two contain valuable information on the modal powers, phase differences between the wave amplitudes, and the local reflection coefficients of the counterpropagating waves (24).

The properties of a linear network can be expressed in closed form in terms of its terminal matrix representations. Analysis provided in this thesis shows that the properties of a linear distributed network can also be expressed in terms of its system matrix which describes the evolution of the field parameters inside the distributed system. There are various representations of a network; commonly used forms are impedance, admittance, scattering, and transfer matrices. Commonly known properties of a network are losslessness, reciprocity, and bilateral and transversal symmetry. Conditions for these and other network properties are given in this thesis for terminal matrix representations and for the system coupling matrix of a distributed network.

In Section 2.1, the fundamental concept of coupled mode theory is given in conjunction with the generalized Jones and Mueller calculi. β and γ coupling, the two basic forms of couplings encountered in a distributed system to produce effects of propagation, amplification, and decay, are also the subject of this section. Various matrix representations, and the conditions for various network properties are presented in tabular form in Section 2.2.

Section 2.3 covers 4-port network analysis, utilizing the techniques of the previous sections. Here, examples are cited from diverse fields for 4-port networks. Also, the concept of antireciprocal distributed systems is briefly discussed in this section.

Jones and Mueller calculi and their application to codirectionally and contradirectionally coupled systems are introduced in Chapter 3. The analysis given here includes the effect of loss in the normal mode propagation. Also, a nonuniform system analysis is presented for two specific types of couplers, namely tapered and chirped. A more general form of 2x2 nonuniform system analysis, involving a linear transformation of the dependent variables and a double diagonalization process can be found in reference (14) and is omitted here. The concepts introduced in Chapter 2 are successfully applied here for both codirectional and contradirectional couplers.

In Appendix I, the properties and construction of certain special matrices and vectors are given. Appendix II is on the methods of obtaining the power of certain types of matrices. Appendix III includes the listing and user manual of the computer program CONVRT4, written in FORTRAN5 which implements matrix conversion and testing for violation of conservation properties.

II. COUPLED MODE ANALYSIS AND CONSERVATION LAWS

2.1. Coupled Mode Formalism

Coupled mode analysis is used in cases when a system can be described by a first order vector differential equation of the form

$$\frac{d\bar{a}(z)}{dz} = -j\, R_a\, \bar{a}(z) \,, \qquad (2.1.1)$$

where $\bar{a}(z)^T = (a_1(z), a_2(z), \ldots, a_{2n}(z))$ is the state vector and R_a is the system matrix. For a weakly coupled system R_a must be diagonally dominant, such that a small fraction of power in the i^{th} mode is coupled to the j^{th} mode and vice versa. The solution for (2.1.1) is given as

$$\bar{a}(z) = M(z)\, \bar{a}(0) \,, \qquad (2.1.2)$$

where M(z) is the so called transfer matrix. Eq. (2.1.1) can be used to describe a system of n coupled transmission lines or a transmission system supporting a number of coupled propagation modes. In either case, assuming that the system supports bidirectional propagation, i.e. the waves can propagate in both positive or negative z direction, the transfer matrix M(z) must be nonsingular, since

$$\bar{a}(0) = M(z)^{-1}\, \bar{a}(z) \qquad (2.1.3)$$

must correspond to a physically realizable situation (33). It will be assumed that R_a can be diagonalized by the similarity transformation

$$\Lambda_r = U^{-1} R_a U = \text{diag}(\lambda_{r1}, \lambda_{r2}, \ldots, \lambda_{r2n}), \qquad (2.1.4)$$

where U is the modal matrix whose columns are the eigenvectors of R_a, and λ_{ri} is the i^{th} eigenvalue of R_a.

If an nxn matrix has a total of n linearly independent eigenvectors, regardless of degeneracy, then the matrix is said to be semisimple. Assuming that R_a is semisimple*, it can be expanded in terms of the set of projectors $\{K_i\}$ on the eigenspace of R_a as

$$R_a = \sum_{i=1}^{2n} \lambda_{ri} K_i , \qquad (2.1.5)$$

where K_i's are also the metrics of the space in which the vector $\bar{a}(z)$ is 'embedded'. A projector is a linear homogeneous operator having the following properties:

$$K_i^2 = K_i \text{ (idempotency)} , \qquad (2.1.6)$$

$$K_i K_j = 0 , \text{ for } i \neq j , \qquad (2.1.7)$$

$$\sum_{i=1}^{2n} K_i = E_{2n} , \qquad (2.1.8)$$

where E_{2n} is the 2n x 2n identity matrix. A treatment of projectors can be found in Chapter XI of reference (15). From (2.1.1) and (2.1.2) one obtains

$$\frac{d}{dz} M(z) = -j R_a M(z) , \qquad (2.1.9)$$

---

* For nonsemisimple matrices one cannot expand R_a in terms of the projectors on the eigenspaces. This case is treated in reference (15), pp. 270, 272.

with $M(0) = E_{2n}$. (2.1.10)

Assuming that $M(z)$ can also be expanded as

$$M(z) = \sum_{i=1}^{2n} \lambda_{mi}(z)\, K_i \, , \tag{2.1.11}$$

by (2.1.8), the boundary condition in (2.1.10) will be satisfied if

$$\lambda_{mi}(0) = 1 \quad \text{for all } i \, . \tag{2.1.12}$$

Substituting (2.1.11) in (2.1.9) and using the properties in (2.1.6) and (2.1.7), one obtains

$$\sum_{i=1}^{2n} \frac{d\lambda_{mi}}{dz}(z)\, K_i = -j \sum_{i=1}^{2n} \lambda_{ri}\, K_i \sum_{i=1}^{2n} \lambda_{mi}(z)\, K_i$$

$$= -j \sum_{i=1}^{2n} \lambda_{ri}\, \lambda_{mi}(z)\, K_i \, . \tag{2.1.13}$$

Multiplying both sides by K_j eliminates the summation. The solution satisfying (2.1.12) then is

$$\lambda_{mi}(z) = \exp\,(-j\, \lambda_{ri}\, z) \, , \tag{2.1.14}$$

and

$$M(z) = \sum_{i=1}^{2n} \exp\,(-j\, \lambda_{ri}\, z)\, K_i \, . \tag{2.1.15}$$

From (2.1.6) and (2.1.7), it can be seen that R_a and $M(z)$ commute, implying that they have a common set of eigenvectors such that

$$\Lambda_m(z) = U^{-1} M(z)\, U = \text{diag}(\, \exp(-j\, \lambda_{r1}\, z),\ \exp(-j\, \lambda_{r2}\, z), \ldots$$
$$\ldots,\ \exp(-j\, \lambda_{r2n}\, z)\ ) \ . \qquad (2.1.16)$$

Two forms of the system matrix R_a are of particular interest. Namely

(i) $R_a = K_0\, R_a^{\dagger}\, K_0$, (2.1.17)

and (ii) $R_a = K_1\, R_a^{\dagger}\, K_1$, (2.1.18)

where $K_0 = E_{2n}$, and † represents Hermitean conjugation.

$$K_1 = \begin{bmatrix} E_n & 0 \\ 0 & -E_n \end{bmatrix},$$

and R_a in partitioned form is given as

$$R_a = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}. \qquad (2.1.19)$$

Although no restrictions are imposed on the size of the submatrices of R_a, in this report interest is focused on cases when R_{12} and R_{21} are both nxn matrices. The more general case, when R_{12} and R_{21} are rectangular is treated in reference (13).

<u>case (i) : $R_a = R_a^{\dagger}$</u>

A matrix satisfying (2.1.17) is called Hermitean. The eigenvalues of a Hermitean matrix are real. Eq. (2.1.17) can be written in terms of its submatrices as

$$R_{11} = R_{11}^{\dagger} , \tag{2.1.20}$$

$$R_{22} = R_{22}^{\dagger} , \tag{2.1.21}$$

and $$R_{12} = R_{21}^{\dagger} . \tag{2.1.22}$$

The eigenvalues of R_a determine the type of propagation and coupling between the modes. To determine the eigenvalues of R_a, a suitable method is based on expressing the eigenvalues of R_a in terms of the eigenvalues of the submatrices R_{11} and R_{22} (13). Letting

$$R_{11} \bar{u}_1 = \lambda_1 \bar{u}_1 , \tag{2.1.23}$$

$$R_{22} \bar{u}_2 = \lambda_2 \bar{u}_2 , \tag{2.1.24}$$

and $$R_a \bar{u} = k\bar{u} , \tag{2.1.25}$$

where $$\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \\ \bar{u}_2 \end{bmatrix} , \tag{2.1.26}$$

eqs. (2.1.23) to (2.1.26) can be written as

$$\begin{bmatrix} (\lambda_1 - k) E_n & R_{12} \\ \\ R_{12}^{\dagger} & (\lambda_2 - k) E_n \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \\ \bar{u}_2 \end{bmatrix} = 0 . \tag{2.1.27}$$

Solving this for $\bar{u}_1$ and $\bar{u}_2$, one obtains

$$\bar{u}_1 = \frac{-1}{(\lambda_1 - k)} R_{12} \bar{u}_2 , \tag{2.1.28}$$

and $$\bar{u}_2 = \frac{-1}{(\lambda_2 - k)} R_{12}^{\dagger} \bar{u}_1 . \tag{2.1.29}$$

Substituting first from (2.1.29) into (2.1.28) and then vice versa,

$$R_{12}\, R_{12}^{\dagger}\, \bar{u}_1 = \kappa^2\, \bar{u}_1 \,, \tag{2.1.30}$$

and $$R_{12}^{\dagger}\, R_{12}\, \bar{u}_2 = \kappa^2\, \bar{u}_2 \,, \tag{2.1.31}$$

where $$\kappa^2 = (\lambda_1 - k)(\lambda_2 - k) \;. \tag{2.1.32}$$

It can be shown that the RHS of (2.1.32) is always positive or zero. Premultiplying (2.1.30) by $\bar{u}_1^{\dagger}$, (2.1.29) by $\bar{u}_2^{\dagger}$, and equating yields

$$\bar{u}_2^{\dagger}\, \bar{u}_2\, |(\lambda_1 - k)|^2 = \kappa^2\, \bar{u}_1^{\dagger}\, \bar{u}_1 \;. \tag{2.1.33}$$

Since $\bar{u}_2^{\dagger}\, \bar{u}_2$, $\bar{u}_1^{\dagger}\, \bar{u}_1$, and $|(\lambda_1 - k)|^2$ are all nonnegative, κ^2 must also be nonnegative. Solving (2.1.32) for k yields

$$k = \tfrac{1}{2}(\lambda_1 + \lambda_2) \pm 2\,\kappa\, \left\{ \left(\frac{\lambda_1 - \lambda_2}{2\,\kappa} \right)^2 + 1 \right\}^{\frac{1}{2}} , \tag{2.1.34}$$

which is real whenever λ_1 and λ_2 are real. It should be noted that when the dimension of R_a is 2n x 2n, there will be n (λ_1, λ_2) pairs and 2n eigenvalues of R_a (k's), 2 for each (λ_1, λ_2) pair.

<u>case (ii) : $R_a = K_1\, R_a^{\dagger}\, K_1$</u>

Following a similar approach as before, (2.1.18) results in

$$R_{11} = R_{11}^{\dagger} \,, \tag{2.1.35}$$

$$R_{22} = R_{22}^{\dagger} \,, \tag{2.1.36}$$

and $$R_{12} = -R_{21}^{\dagger} \;. \tag{2.1.37}$$

Eq. (2.1.27) now becomes

$$\begin{bmatrix} (\lambda_1 - k)\, E_n & R_{12} \\ -R_{12}^{\dagger} & (\lambda_2 - k)\, E_n \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = 0 \,, \qquad (2.1.38)$$

which results in

$$R_{12}\, R_{12}^{\dagger}\, \bar{u}_1 = -\kappa^2\, \bar{u}_1 \,, \qquad (2.1.39)$$

and $$R_{12}^{\dagger}\, R_{12}\, \bar{u}_2 = -\kappa^2\, \bar{u}_2 \,, \qquad (2.1.40)$$

where κ^2 is given by (2.1.32). Premultiplying (2.1.40) by $\bar{u}_2^{\dagger}$, the first row of (2.1.38) by $\bar{u}_1^{\dagger}$, and equating the expressions so obtained yields

$$|\lambda_1 - k|^2\, \bar{u}_2^{\dagger}\, \bar{u}_2 = -\kappa^2\, \bar{u}_1^{\dagger}\, \bar{u}_1 \,. \qquad (2.1.41)$$

Since $|\lambda_1 - k|^2$, $\bar{u}_2^{\dagger}\, \bar{u}_2$, and $\bar{u}_1^{\dagger}\, \bar{u}_1$ are all nonnegative, κ^2 must be nonpositive. Writing $\kappa = j\kappa'$, where κ' is real, instead of (2.1.34) one now obtains

$$k = \tfrac{1}{2}(\lambda_1 + \lambda_2) \pm 2\,\kappa' \left\{ \left(\frac{\lambda_1 - \lambda_2}{2\,\kappa'} \right)^2 - 1 \right\}^{\frac{1}{2}} \,, \qquad (2.1.42)$$

which is complex whenever

$$\left| \frac{\lambda_1 - \lambda_2}{2\,\kappa'} \right| < 1 \,, \qquad (2.1.43)$$

for real λ_1 and λ_2. A complex eigenvalue of R_a indicates exponentially growing and decaying waves. Despite the fact that the propagation constants of each individual mode is real, the system as a whole produces amplification and decay in the mode pairs which correspond to the complex conjugate eigenvalue pairs of R_a. This phenomenon is known as 'γ coupling' or 'active coupling' (11). When the condition in (2.1.43) is not satisfied, then there is 'β coupling' or 'passive coupling', where the modes vary sinusoidally with distance.

2.2. Network Representations and Conservation Laws

Closed form expressions are derived to express conservation of energy (losslessness condition), reciprocity, bilateral and transversal symmetry, semireciprocity, and antireciprocity in 2n-port networks. Although these expressions are in terms of the port quantities, their extension to distributed systems such as the one shown in Figure 2.1 is of particular interset and will be discussed here.

Defining the input and output voltage and current vectors as $\bar{v}_1^T = (V_1, V_2, .., V_n)$, $\bar{v}_2^T = (V_{n+1}, V_{n+2}, .., V_{2n})$, $\bar{i}_1^T = (I_1, I_2, .., I_n)$, $\bar{i}_2^T = (I_{n+1}, I_{n+2}, .., I_{2n})$, and denoting the corresponding incident and reflected wave vectors as $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$, and $\bar{b}_2$ respectively, the following eight matrices are defined in n x n block partitioned form. These matrices and their definitions are

Z (impedance)

$$\begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \bar{i}_1 \\ \bar{i}_2 \end{bmatrix} \qquad (2.2.1)$$

Y (admittance)

$$\begin{bmatrix} \bar{i}_1 \\ \bar{i}_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} \qquad (2.2.2)$$

Q or ABCD (impedance transfer)

$$\begin{bmatrix} \bar{v}_1 \\ \bar{i}_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \bar{v}_2 \\ -\bar{i}_2 \end{bmatrix} \qquad (2.2.3)$$

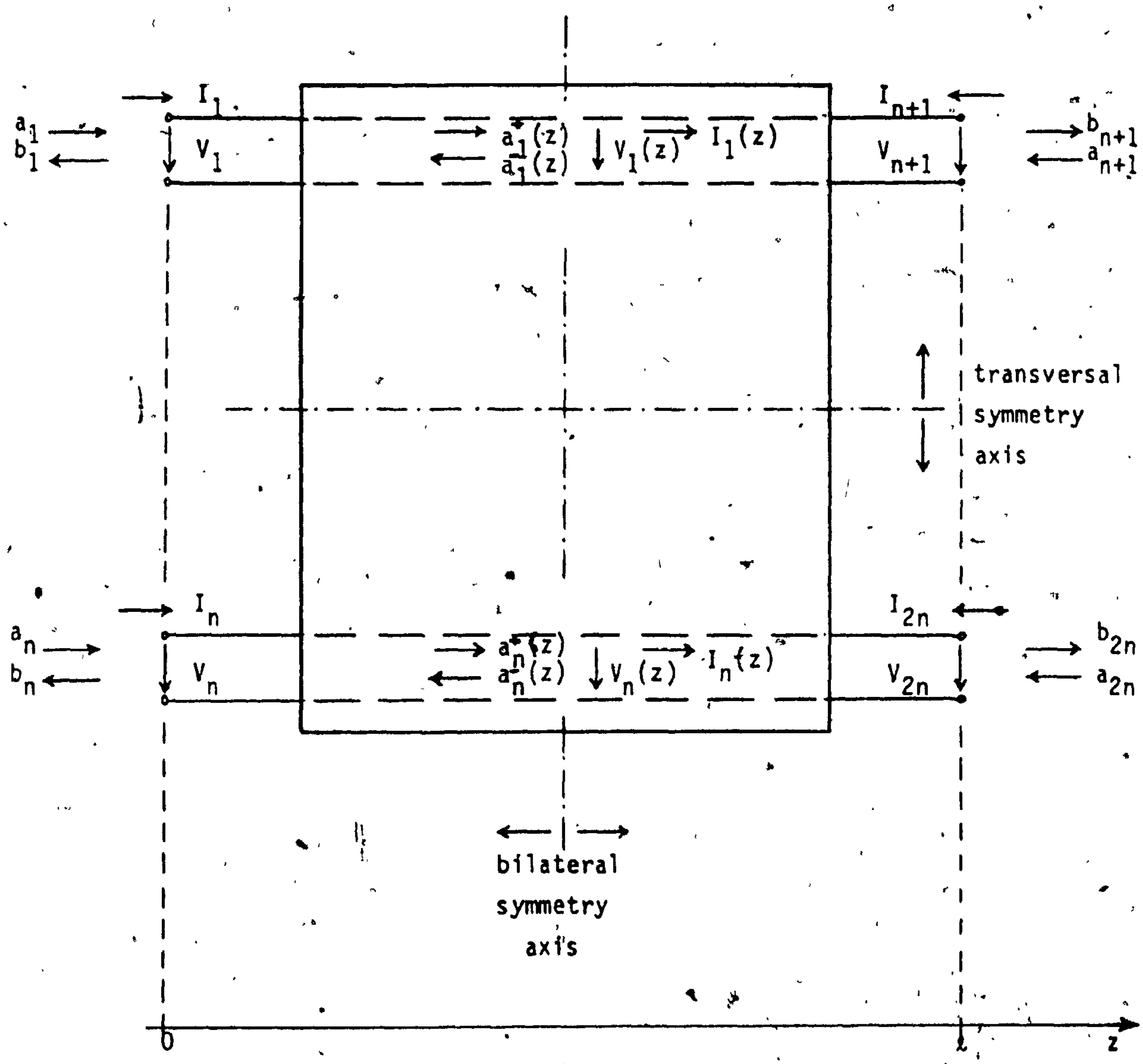

Figure 2.1. Representation of n coupled lines as a 2n-port network.

S (scattering)

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} \tag{2.2.4}$$

T (scattering transfer)

$$\begin{bmatrix} \bar{a}_1 \\ \bar{b}_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \bar{b}_2 \\ \bar{a}_2 \end{bmatrix} \tag{2.2.5}$$

In addition to these, three more representations are considered. These are

$$G = \tilde{Q}^{-1} , \tag{2.2.6}$$

$$A = \tilde{T} , \tag{2.2.7}$$

and

$$M = A^{-1} = \tilde{T}^{-1} , \tag{2.2.8}$$

where the symbol ~ signifies the tilde transform, defined as

$$\tilde{X} = \Pi_{2n} \, X \, \Pi_{2n}^{T} , \tag{2.2.9}$$

$$\Pi_{2n} = \left[\begin{array}{cccccccccccc}
\multicolumn{6}{c}{\longleftarrow n \longrightarrow} & \multicolumn{6}{c}{\longleftarrow n \longrightarrow} \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 & 0 & & 0 \\
0 & 1 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & & 0 \\
0 & 0 & 0 & 0 & & 0 & 0 & 1 & 0 & 0 & & 0 \\
0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 & & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & \vdots & & & & & & \vdots \\
0 & & & & & 0 & & & & & & 0 \\
0 & 0 & & & & 0 & & & & & 1 & 0 \\
0 & 0 & 0 & & & 1 & 0 & & & & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 1
\end{array} \right] \tag{2.2.10}$$

is a permutation matrix, where $\Pi_{2n}^{-1} = \Pi_{2n}^{T} \neq \Pi_{2n}$:

Since some representations use voltages (V) and currents (I) while others use waves (a,b), the relation between these two types of terminal parameters must be given. In this report, the so-called travelling wave representation, defined as

$$a_i(z)^{\pm} = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{Z_i}} V_i(z) \pm \sqrt{Z_i}\, I_i(z) \right) , \quad i = 1 \text{ to } n \qquad (2.2.11)$$

shall be used. Here, the normalizing impedance Z_i is the characteristic impedance of the i^{th} channel. Equation (2.2.11) can be written in matrix form as

$$\bar{g}(z) = \begin{bmatrix} V_1(z) \\ I_1(z) \\ V_2(z) \\ I_2(z) \\ \vdots \\ V_n(z) \\ I_n(z) \end{bmatrix} = \Omega \begin{bmatrix} a_1(z)^+ \\ a_1(z)^- \\ a_2(z)^+ \\ a_2(z)^- \\ \vdots \\ a_n(z)^+ \\ a_n(z)^- \end{bmatrix} = \bar{a}(z) , \qquad (2.2.12)$$

where the impedance transformation matrix Ω, and its inverse are given as

$$\Omega = \frac{1}{\sqrt{2}} \begin{bmatrix} \Omega_1 & 0 & 0 \ldots & 0 \\ 0 & \Omega_2 & & \\ 0 & & \ddots & \\ \vdots & & \Omega_i & \\ 0 & & & \ddots \, \Omega_n \end{bmatrix} , \qquad (2.2.13)$$

$$\Omega^{-1} = \sqrt{2} \begin{bmatrix} \Omega_1^{-1} & 0 & 0 & \ldots & 0 \\ 0 & \Omega_2^{-1} & & & \\ 0 & & \ddots & & \\ \vdots & & & \Omega_i^{-1} & \\ 0 & & & & \ddots \; \Omega_n^{-1} \end{bmatrix} , \qquad (2.2.14)^*$$

and

$$\Omega_i = \begin{bmatrix} \sqrt{Z_i} & \sqrt{Z_i} \\ \frac{1}{\sqrt{Z_i}} & -\frac{1}{\sqrt{Z_i}} \end{bmatrix} . \qquad (2.2.15)$$

It is advantageous to introduce the terminal parameter vectors $\bar{a}$, $\bar{b}$, $\bar{V}$, and $\bar{I}$, where for example $\bar{a} = (\bar{a}_1^T , \bar{a}_2^T)^T$, and the following relations

$$\bar{a} = F_1 (\bar{V} + F_2 \bar{I}) , \qquad (2.2.16)$$

and $$\bar{b} = F_1 (\bar{V} - F_2 \bar{I}) , \qquad (2.2.17)$$

where $F_1 = (2^{-\frac{1}{2}}) \text{ diag } (Z_1^{-\frac{1}{2}}, Z_2^{-\frac{1}{2}}, \ldots, Z_n^{-\frac{1}{2}}, Z_1^{-\frac{1}{2}}, Z_2^{-\frac{1}{2}}, \ldots, Z_n^{-\frac{1}{2}})$, (2.2.18)

and $$F_2 = \text{diag } (Z_1, Z_2, \ldots, Z_n, Z_1, Z_2, \ldots, Z_n) . \qquad (2.2.19)$$

By using the definition of the impedance, admittance, and scattering matrix; $\bar{V} = Z \bar{I}$, $\bar{I} = Y \bar{V}$, and $\bar{b} = S \bar{a}$, respectively in conjunction with the above transformation, one finds the conversion expression linking the Z, Y, and S representations. These are given as

$$S = F_1 (Z - F_2) (Z + F_2)^{-1} F_1^{-1} =$$

$$= F_1 (E_{2n} - F_2 Y) (E_{2n} + F_2 Y)^{-1} F_1^{-1} , \qquad (2.2.20)$$

* It should be noted that every element in (2.2.13) and (2.2.14) represents a 2 x 2 matrix.

$$Z = \tfrac{1}{2} F_1^{-1} (E_{2n} - S)^{-1} (E_{2n} + S) F_1^{-1} , \qquad (2.2.21)$$

and, $$Y = 2 F_1 (E_{2n} + S)^{-1} (E_{2n} - S) F_1 . \qquad (2.2.22)$$

It should be noted that the requisite inverses might not exist, necessitating the use of an alternate expression or indeed an alternate route. For example, if $E_{2n}-S$ is singular but $E_{2n}+S$ is not, Y can be computed to ascertain whether or not Z exists from the nonsingular or singular nature of Y. Alternatively, one can also attempt to obtain Z via the $S \rightarrow T \rightarrow Q \rightarrow Z$ route.

Generalized Pauli matrices given as

$$\sigma_1 = \begin{bmatrix} 0 & E_n \\ E_n & 0 \end{bmatrix} , \; \sigma_2 = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix} , \; \sigma_3 = \begin{bmatrix} E_n & 0 \\ 0 & -E_n \end{bmatrix} \qquad (2.2.23)$$

are used in the closed form expressions of conservation. The definitions and conversion from one type of matrix to another type are given in Table 2.1. The diagonal entries of the Table give the matrix definitions and the properties. Other entries give either the specific transformation relationship, or in the case of very complicated formulas, instructions for successive transformations.

The relationship between the impedance transfer matrix and the impedance matrix on the one hand, and the scattering transfer matrix and the scattering matrix on the other is described by

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} A\,C^{-1} & A\,C^{-1}\,D - B \\ C^{-1} & C^{-1} - D \end{bmatrix}, \quad (2.2.24)$$

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} T_{21}\,T_{11}^{-1} & T_{22} - T_{21}\,T_{11}^{-1}\,T_{12} \\ T_{11}^{-1} & -\,T_{11}\,T_{12}^{-1} \end{bmatrix}, \quad (2.2.25)$$

provided that the appropriate inverses exist.

The impedance and admittance matrix of a reciprocal network must be symmetric, or

$$Z = Z^T, \quad (2.2.26)$$

and $$Y = Y^T. \quad (2.2.27)$$

From (2.2.24), for reciprocal networks $A\,C^{-1}$, $C^{-1}\,D$, $D\,B^{-1}$, and $B^{-1}\,A$ must all be symmetric and in addition $(A\,C^{-1}\,D - B)^T = -C^{-1}$ and $(D\,B^{-1}\,A - C)^T = B^{-1}$ must hold. Thus $A^T\,C$, $C\,D^T$, $D^T\,B$, and $B\,A^T$ must likewise be symmetric and in addition $A\,D^T - B\,C^T = E_n$ and $D^T\,A - B^T\,C = E_n$ must hold. The latter set of conditions are expressed as

$$Q^{-1} = \sigma_2\,Q^T\,\sigma_2\,. \quad (2.2.28)$$

The scattering matrix of a reciprocal network is also symmetric, or

$$S = S^T. \quad (2.2.29)$$

From (2.2.25), for reciprocal networks $T_{21}\,T_{11}^{-1}$ and $T_{11}^{-1}\,T_{12}$ must be symmetric and $(T_{22} - T_{21}\,T_{11}^{-1}\,T_{12})^T = T_{11}^{-1}$ must hold. The conditions are satisfied if and only if $T_{11}\,T_{12}^T$, $T_{22}\,T_{21}^T$, $T_{22}^T\,T_{12}$ and $T_{11}^T\,T_{21}$ are symmetric and

	Z	Y	Q	G
Z	$\begin{bmatrix}\tilde{V}_1\\ \tilde{V}_2\end{bmatrix} = \begin{bmatrix}Z_{11} & Z_{12}\\ Z_{21} & Z_{22}\end{bmatrix}\begin{bmatrix}\tilde{I}_1\\ \tilde{I}_2\end{bmatrix}$ r : $Z=Z^T$ ar: $Z=-Z^T$ bs: $Z=\sigma_1 Z\sigma_1$ sr: $Z=\sigma_3 Z^T\sigma_3$ l : $Z=-Z^\dagger$ ts: $Z=\tilde{\sigma}_1 Z\tilde{\sigma}_1$	Y^{-1}	AC^{-1} $AC^{-1}D-B$ C^{-1} $C^{-1}D$	G → Q → Z
Y	Z^{-1}	$\begin{bmatrix}\tilde{I}_1\\ \tilde{I}_2\end{bmatrix} = \begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix}\begin{bmatrix}\tilde{V}_1\\ \tilde{V}_2\end{bmatrix}$ r : $Y=Y^T$ ar: $Y=-Y^T$ bs: $Y=\sigma_1 Y\sigma_1$ sr: $Y=\sigma_3 Y^T\sigma_3$ l : $Y=-Y^\dagger$ ts: $Y=\tilde{\sigma}_1 Y\tilde{\sigma}_1$	DB^{-1} $C-DB^{-1}A$ $-B^{-1}$ $B^{-1}A$	G → Q → Y
Q	$Z_{11}Z_{21}^{-1}$ $Z_{11}Z_{21}^{-1}Z_{22}-Z_{12}$ Z_{21}^{-1} $Z_{21}^{-1}Z_{22}$	$-Y_{21}^{-1}Y_{22}$ $-Y_{21}^{-1}$ $Y_{12}-Y_{11}Y_{21}^{-1}Y_{22}$ $-Y_{11}Y_{21}^{-1}$	$\begin{bmatrix}\tilde{V}_1\\ \tilde{I}_1\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\tilde{V}_2\\ -\tilde{I}_2\end{bmatrix}$ r : $Q^{-1}=\sigma_2 Q^T\sigma_2$ | ar : $Q^{-1}=\sigma_1 Q^T\sigma_1$ bs : $Q^{-1}=\sigma_3 Q\sigma_3$ | sr : $Q^{-1}=\sigma_{32}Q^T\sigma_{23}$ l : $Q^{-1}=\sigma_1 Q^\dagger\sigma_1$ | ts : $Q=\tilde{\sigma}_1 Q\tilde{\sigma}_1$	$\Pi_{2n}^T G^{-1}\Pi_{2n}$
G	Z → Q → G	Y → Q → G	$\tilde{Q}^{-1}$	$\begin{bmatrix}\vdots\\ V_j\\ -I_j\\ \vdots\end{bmatrix} = \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}\vdots\\ V_i\\ I_i\\ \vdots\end{bmatrix}$ i=1 to j=n+ to 2 r : $G^{-1}=\sigma_2 G^T\sigma_2$ | ar : $G^{-1}=\sigma_1 G^T\sigma$ bs : $G^{-1}=\sigma_3 G\sigma_3$ | sr : $G^{-1}=\sigma_4 G^T\sigma$ l : $G^{-1}=\sigma_1 G^\dagger\sigma_1$ | ts : $G=\tilde{\sigma}G\tilde{\sigma}$ | $\tilde{\sigma}=\sigma_2$
S	Z → Q → T → S or $F_1(Z-F_2)(Z+F_2)^{-1}F_1^{-1}$	Y → Q → T → S or $F_1(E_{2n}-F_2Y)(E_{2n}+F_2Y)^{-1}F_1^{-1}$	Q → T → S	G → T → S
T	Z → Q → T or Z → S → T	Y → Q → T or Y → S → T	$\Pi_{2n}^T\Omega^{-1}\tilde{Q}\Omega\Pi_{2n}$	$\Pi_{2n}^T\Omega^{-1}G^{-1}\Omega\Pi_{2n}$
A	Z → Q → A	Y → Q → A	$\Omega^{-1}\tilde{Q}\Omega$	$\Omega^{-1}G^{-1}\Omega$
M	Z → Q → M	Y → Q → M	$\Omega^{-1}\tilde{Q}^{-1}\Omega$	$\Omega^{-1}G\Omega$

Table 2.1. Universal Table of 2nx2n matrix representations r:reciprocity, l:los

	S	T	A	M
Z	$S \to T \to Q \to Z$ or $F_1^{-1}(E_{2n}-S)^{-1}(E_{2n}+S)F_2F_1$	$T \to Q \to Z$ or $T \to S \to Z$	$A \to Q \to Z$	$M \to Q \to Z$
Y	$S \to T \to Q \to Y$ or $F_1^{-1}F_2^{-1}(E_{2n}+S)^{-1}(E_{2n}-S)F_1$	$T \to Q \to Y$ or $T \to S \to Y$	$A \to Q \to Y$	$M \to Q \to Y$
	$S \to T \to Q$	$\Pi_{2n}^T \Omega \bar{T} \Omega^{-1} \Pi_{2n}$	$\Pi_{2n}^T \Omega A \Omega^{-1} \Pi_{2n}$	$\Pi_{2n}^T \Omega M^{-1} \Omega^{-1} \Pi_{2n}$
$\begin{bmatrix} \vdots \\ v_i \\ i_j \\ \vdots \end{bmatrix}$ i=1 to n, j=n+1 to 2n; $G^{-1}=\sigma_1 G^T \sigma_1$; $G^{-1}=\sigma_4 G^T \sigma_4$; $G=\bar{\sigma} G \bar{\sigma}$, $\bar{\sigma}=\sigma_{23}$	$S \to T \to G$	$\Omega \bar{T}^{-1} \Omega^{-1}$	$\Omega A^{-1} \Omega^{-1}$	$\Omega M \Omega^{-1}$
S	$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix}$ r: $S = S^T$ ar: $S^{-1} = S^T$ bs: $S = \sigma_1 S \sigma_1$ sr: $S = \sigma_3 S^T \sigma_3$ l: $S^{-1} = S^+$ ts: $S = \sigma_1 S \sigma_1$	$T_{21}T_{11}^{-1}$; $T_{22}-T_{21}T_{11}^{-1}T_{12}$; T_{11}^{-1} ; $-T_{11}^{-1}T_{12}$	$A \to T \to S$	$M \to T \to S$
$\ldots^{-1} \Omega \Pi_{2n}$	S_{21}^{-1} ; $-S_{21}^{-1}S_{22}$; $S_{11}S_{21}^{-1}$; $S_{12}-S_{11}S_{21}^{-1}S_{22}$	$\begin{bmatrix} \bar{a}_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ \bar{a}_2 \end{bmatrix}$ r: $T^{-1}=\sigma_2 T^T \sigma_2$ ar: $T^{-1}=\sigma_3 T^T \sigma_3$ bs: $T^{-1}=\sigma_1 T \sigma_1$ sr: $T^{-1}=\sigma_{32} T \sigma_{23}$ l: $T^{-1}=\sigma_3 T^+ \sigma_3$ ts: $T=\sigma_1 T \sigma_1$	$\Pi_{2n}^T A \Pi_{2n}$	$\Pi_{2n}^T M^{-1} \Pi_{2n}$
	$S \to T \to A$	$\bar{T}$	$\begin{bmatrix} \vdots \\ a_i \\ b_i \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \vdots \\ b_j \\ a_j \\ \vdots \end{bmatrix}$ i=1 to n, j=n+1 to 2n; r: $A^{-1}=\sigma_2 A^T \sigma_2$ ar: $A^{-1}=\sigma_3 A^T \sigma_3$ bs: $A^{-1}=\sigma_1 A \sigma_1$ sr: $A^{-1}=\sigma_4 A^T \sigma_4$ l: $A^{-1}=\sigma_3 A^+ \sigma_3$ ts: $A=\bar{\sigma}_1 A \bar{\sigma}_1$	M^{-1}
	$S \to T \to M$	$\bar{T}^{-1}$	A^{-1}	$\begin{bmatrix} \vdots \\ b_j \\ a_j \\ \vdots \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \vdots \\ a_i \\ b_i \\ \vdots \end{bmatrix}$ r: $M^{-1}=\sigma_2 M^T \sigma_2$ ar: $M^{-1}=\sigma_3 M^T \sigma_3$ bs: $M^{-1}=\sigma_1 M \sigma_1$ sr: $M^{-1}=\sigma_4 M^T \sigma_4$ l: $M^{-1}=\bar{\sigma}_3 M^+ \bar{\sigma}_3$ ts: $M=\bar{\sigma}_1 M \bar{\sigma}_1$

..., l:losslessness, bs,ts:bilateral and transversal symmetry, ar,sr:anti- and semireciprocity

2 of 2

$T_{11} T_{22}^T - T_{12} T_{21}^T = E_n$ and $T_{22}^T T_{11} - T_{12}^T T_{21} = E_n$ hold.
The second set of conditions is equivalent to saying

$$T^{-1} = \sigma_2 T^T \sigma_2 . \qquad (2.2.30)$$

Conservation of reciprocity is expressed in terms of other matrix representations as

$$G^{-1} = \tilde{\sigma}_2 G^T \tilde{\sigma}_2 , \qquad (2.2.31)$$

$$A^{-1} = \tilde{\sigma}_2 A^T \tilde{\sigma}_2 , \qquad (2.2.32)$$

$$M^{-1} = \tilde{\sigma}_2 M^T \tilde{\sigma}_2 . \qquad (2.2.33)$$

The impedance and admittance matrices of lossless networks are skew hermitean, or

$$Z = -Z^\dagger , \qquad (2.2.34)$$

and

$$Y = -Y^\dagger , \qquad (2.2.35)$$

whereas the scattering matrix is unitary, or

$$S^{-1} = S^\dagger . \qquad (2.2.36)$$

In a lossless network, the net power dissipation must be zero. In terms of port voltages and currents the power dissipated in the network is

$$P_d = \tfrac{1}{2} \sum_{i=1}^{2n} \mathrm{Re}\,(V_i^* I_i) = \tfrac{1}{4} (\bar{V}_2^\dagger , -\bar{I}_2^\dagger) \{ Q^\dagger \sigma_1 Q - \sigma_1 \} \begin{bmatrix} \bar{V}_2 \\ -\bar{I}_2 \end{bmatrix} \qquad (2.2.37)$$

where (2.2.3) and its hermitian conjugate have been used in obtaining the expression on the right. Setting P_d equal to zero yields

$$Q^{-1} = \sigma_1 \, Q^{\dagger} \, \sigma_1 \quad . \tag{2.2.38}$$

In terms of normalized incident and reflected waves the dissipated power is

$$P_d = \sum_{i=1}^{2n} \left(|a_i|^2 - |b_i|^2 \right) = \left(\bar{b}_2^{\,\dagger} \; ; \; \bar{a}_2^{\,\dagger} \right) \left\{ T^{\dagger} \sigma_3 T - \sigma_3 \right\} \begin{bmatrix} \bar{b}_2 \\ \bar{a}_2 \end{bmatrix} , \tag{2.2.39}$$

where the expression on the right has been obtained by using (2.2.5) and its hermitian conjugate. When P_d is forced to vanish for an arbitrary excitation, one obtains

$$T^{-1} = \sigma_3 \, T^{\dagger} \, \sigma_3 \quad . \tag{2.2.40}$$

In terms of other network representations, conservation of energy is expressed as

$$G^{-1} = \tilde{\sigma}_1 \, G^{\dagger} \, \tilde{\sigma}_1 \quad , \tag{2.2.41}$$

$$A^{-1} = \tilde{\sigma}_3 \, A^{\dagger} \, \tilde{\sigma}_3 \quad , \tag{2.2.42}$$

$$M^{-1} = \tilde{\sigma}_3 \, M^{\dagger} \, \tilde{\sigma}_3 \quad . \tag{2.2.43}$$

The condition of bilateral symmetry is derived by stipulating that a representation be unchanged when the corresponding 4-port is rotated around the bilateral symmetry axis shown in Figure 2.1. Denoting the transformation by a sub-tilde, one writes in the case of the scattering transfer representation

$$\begin{bmatrix} \bar{a}_2 \\ \bar{b}_2 \end{bmatrix} = \underset{\sim}{T} \begin{bmatrix} \bar{b}_1 \\ \bar{a}_1 \end{bmatrix} , \tag{2.2.44}$$

in the case of the impedance transfer representation

$$\begin{bmatrix} \bar{v}_2 \\ \bar{i}_2 \end{bmatrix} = \underset{\sim}{Q} \begin{bmatrix} \bar{v}_1 \\ -\bar{i}_1 \end{bmatrix} , \qquad (2.2.45)$$

and in the case of the scattering representation

$$\begin{bmatrix} \bar{b}_2 \\ \bar{b}_1 \end{bmatrix} = \underset{\sim}{S} \begin{bmatrix} \bar{a}_2 \\ \bar{a}_1 \end{bmatrix} , \qquad (2.2.46)$$

By forcing the complement to be equal to the original matrix, one obtains the condition for bilateral symmetry. In the various representations it is as follows

$$Z = \sigma_1 Z \sigma_1 , \qquad (2.2.47)$$

$$Y = \sigma_1 Y \sigma_1 , \qquad (2.2.48)$$

$$Q^{-1} = \sigma_3 Q \sigma_3 , \qquad (2.2.49)$$

$$G^{-1} = \tilde{\sigma}_3 G \tilde{\sigma}_3 , \qquad (2.2.50)$$

$$S = \sigma_1 S \sigma_1 , \qquad (2.2.51)$$

$$T^{-1} = \sigma_1 T \sigma_1 , \qquad (2.2.52)$$

$$A^{-1} = \tilde{\sigma}_1 A \tilde{\sigma}_1 , \qquad (2.2.53)$$

$$M^{-1} = \tilde{\sigma}_1 M \tilde{\sigma}_1 , \qquad (2.2.54)$$

The condition of transversal symmetry is derived by stipulating that a representation be unchanged when the corresponding 2n-port is rotated around the transversal symmetry axis shown in Figure 2.1. Letting a caret denote the transversal complement, for the scattering representation one obtains

$$\begin{bmatrix} b_n \\ b_{n-1} \\ \vdots \\ b_1 \\ b_{2n} \\ b_{2n-1} \\ \vdots \\ b_{n+1} \end{bmatrix} = \hat{S} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_{2n} \\ a_{2n-1} \\ \vdots \\ a_{n+1} \end{bmatrix} \qquad (2.2.55)$$

Equating the complement to the original matrix yields the condition for transversal symmetry. In various representations, this condition is given as

$$Z = \tilde{\sigma}_1 \, Z \, \tilde{\sigma}_1 \quad , \qquad (2.2.56)$$

$$Y = \tilde{\sigma}_1 \, Y \, \tilde{\sigma}_1 \quad , \qquad (2.2.57)$$

$$Q = \tilde{\sigma}_1 \, Q \, \tilde{\sigma}_1 \quad , \qquad (2.2.58)$$

$$G = \tilde{\sigma}_1 \, G \, \tilde{\sigma}_1 \quad , \qquad (2.2.59)$$

$$S = \tilde{\sigma}_1 \, S \, \tilde{\sigma}_1 \quad , \qquad (2.2.60)$$

$$T = \tilde{\sigma}_1 \, T \, \tilde{\sigma}_1 \quad , \qquad (2.2.61)$$

$$A = \tilde{\sigma}_1 \, A \, \tilde{\sigma}_1 \quad , \qquad (2.2.62)$$

$$M = \tilde{\sigma}_1 \, M \, \tilde{\sigma}_1 \quad . \qquad (2.2.63)$$

It should be noted that transversal symmetry implies that

$$Z_i = Z_{n+1-i} \quad ; \quad i < \tfrac{1}{2}n \ (\text{ n even }) \qquad$$

$$i < \tfrac{1}{2}(n-1) \ (\text{ n odd }) \qquad (2.2.64)$$

and consequently that

$$\tilde{\Omega} = \tilde{\sigma}_1 \; \tilde{\Omega} \; \tilde{\sigma}_1 \quad . \tag{2.2.65}$$

Out of the six conditions included in Table 2.1, perhaps the most interesting one in terms of a distributed network is the antireciprocity condition. The antireciprocity concept has been previously defined from a circuit theoretical point of view (17,18). An antireciprocal network is a network consisting of pure gyrators, such as a matched 4-port equal power divider, magic T (18), or a circulator. However, these examples of circuits are all lumped networks, rather than distributed. Antireciprocity condition in various representations is given as

$$Z = -\; Z^T \quad , \tag{2.2.66}$$

$$Y = -\; Y^T \quad , \tag{2.2.67}$$

$$Q^{-1} = \sigma_1 \; Q^T \; \sigma_1 \quad , \tag{2.2.68}$$

$$G^{-1} = \tilde{\sigma}_1^{\;-1} \; G^T \; \tilde{\sigma}_1^{\;-1} \quad , \tag{2.2.69}$$

$$S^{-1} = S^T \quad , \tag{2.2.70}$$

$$T^{-1} = \sigma_3 \; T^T \; \sigma_3 \quad , \tag{2.2.71}$$

$$A^{-1} = \tilde{\sigma}_3 \; A^T \; \tilde{\sigma}_3 \quad , \tag{2.2.72}$$

$$M^{-1} = \tilde{\sigma}_3 \; M^T \; \tilde{\sigma}_3 \quad . \tag{2.2.73}$$

A semireciprocal network is one consisting of a pure reciprocal network connected in series with a pure antireciprocal network. Semireciprocity condition in various representations is given as

$$Z = \tilde{\sigma}_3 \, Z^T \, \tilde{\sigma}_3 \quad , \tag{2.2.74}$$

$$Y = \tilde{\sigma}_3 \, Y^T \, \tilde{\sigma}_3 \quad , \tag{2.2.75}$$

$$Q^{-1} = \tilde{\sigma}_3 \, \sigma_2 \, Q^T \, \sigma_2 \, \tilde{\sigma}_3 \quad , \tag{2.2.76}$$

$$G^{-1} = \sigma_4 \, G^T \, \sigma_4 \qquad (\sigma_4 = \sigma_2 \tilde{\sigma}_3 = \sigma_4^{-1}) \quad , \tag{2.2.77}$$

$$S = \tilde{\sigma}_3 \, S^T \, \tilde{\sigma}_3 \quad , \tag{2.2.78}$$

$$T^{-1} = \tilde{\sigma}_3 \, \sigma_2 \, T^T \, \sigma_2 \, \tilde{\sigma}_3 \quad , \tag{2.2.79}$$

$$A^{-1} = \sigma_4 \, A^T \, \sigma_4 \quad , \tag{2.2.80}$$

$$M^{-1} = \sigma_4 \, M^T \, \sigma_4 \quad . \tag{2.2.81}$$

One can derive the conditions pertaining to the system matrix R_a by first differentiating a given condition for the transfer matrix M(z) in Table 2.1, and then substituting from (2.1.9) for $M(z)'$ (19). As an example the losslessness condition for the R_a matrix will be derived. From Table 2.1,

$$E_{2n} = M(z) \, \tilde{\sigma}_3 \, M(z)^{\dagger} \, \tilde{\sigma}_3 \tag{2.2.82}$$

is true for lossless networks. Assuming that the network is a linear, z dependent distributed system as in Figure 2.1, differentiating (2.2.82) yields

$$0 = M(z)' \, \tilde{\sigma}_3 \, M(z)^{\dagger} \, \tilde{\sigma}_3 + M(z) \, \tilde{\sigma}_3 \, M(z)^{\dagger\prime} \, \tilde{\sigma}_3 \quad , \tag{2.2.83}$$

which upon substitution from (2.1.9) becomes

$$R_a = \tilde{\sigma}_3 \, R_a^{\dagger} \, \tilde{\sigma}_3 \quad . \tag{2.2.84}$$

Equation (2.2.84) must hold for lossless systems. These and other conditions for R_a and $M(z)$, as well as R_g and $G(z)$ matrix pairs are summarized in Table 2.2.

A users manual and the listing of a computer program which implements the conversions and conservation laws of Table 2.1 for 4-port networks are given in Appendix III.

	$G(z)$	R_g
$G(z)$	$\bar{g}(z) = G(z)\,\bar{g}(0)$ $\bar{g}(z) = (V_1(z), I_1(z), \ldots, V_n(z), I_n(z))^T$ for conservation laws see Table 2.1	$U_{rg}\,\Lambda_t\,U_{rg}^{-1}$ $\Lambda_t = \mathrm{diag}(\exp(-j\lambda_{r1}z), \ldots, \exp(-j\lambda_{r2n}z))$
R_g	$j\,G(z)'\,G(z)^{-1} = j\,G(z)^{-1}\,G(z)'$	$\frac{d\bar{g}}{dz}(z) = -j\,R_g\,\bar{g}(z)\,,\quad U_{rg}\,\Lambda_r\,U_{rg}^{-1}\,,$ $U_{rg} = \Omega\,U_{ra}$ r : $R_g = -\tilde{\sigma}_2 R_g^T \tilde{\sigma}_2$ ar: $R_g = -\tilde{\sigma}_1 R_g^T \tilde{\sigma}_1$ l : $R_g = \tilde{\sigma}_1 R_g^{\dagger} \tilde{\sigma}_1$ sr: $R_g = -\tilde{\sigma}_{23} R_g^T \tilde{\sigma}_{32}$ bs: $R_g = -\tilde{\sigma}_3 R_g \tilde{\sigma}_3$ ts: $R_g = \sigma_1 R_g \sigma_1$
$M(z)$	$\Omega^{-1}\,G(z)\,\Omega$	$R_g \rightarrow U_{rg}, \Lambda_r \rightarrow U_{ra}, \Lambda_t \rightarrow M(z)$
R_a	$G(z) \rightarrow R_g \rightarrow R_a$	$\Omega^{-1}\,R_g\,\Omega$

Table 2.2. Conservation Laws and Conversion Routes for System and

	M(z)	R_a
))	$\Omega\, M(z)\, \Omega^{-1}$	$R_a \rightarrow U_{ra}, \Lambda_r \rightarrow U_{rg}, \Lambda_t \rightarrow G(z)$
2	$M(z) \rightarrow R_a \rightarrow R_g$	$\Omega\, R_a\, \Omega^{-1}$
	$\bar{a}(z) = M(z)\, \bar{a}(0)$ $a(z) = (a_1(z), a_2(z), \ldots, a_{2n}(z))^T$ for conservation laws see Table 2.1	$U_{ra}\, \Lambda_t\, U_{ra}^{-1}$
	$j\, M(z)'\, M(z)^{-1} = j\, M(z)^{-1}\, M(z)'$	$\frac{d\bar{a}}{dz}(z) = -j\, R_a\, \bar{a}(z)$, $U_{ra}\, \Lambda_r\, U_{ra}^{-1}$, $U_{ra} = \Omega^{-1}\, U_{rg}$ r : $R_a = -\sigma_2 R_a^T \sigma_2$ ar: $R_a = -\sigma_3 R_a^T \sigma_3$ l : $R_a = \sigma_3 R_a^{\dagger} \sigma_3$ sr: $R_a = -\sigma_2\sigma_3 R_a^T \sigma_3\sigma_2$ bs: $R_a = -\sigma_1 R_a \sigma_1$ ts: $R_a = \sigma_1 R_a \sigma_1$

n and Transfer Matrices

2 of 2

2.3. Four-port Distributed Networks

Four-ports comprise one of the basic building blocks in microwave and optical engineering. The classical example is the directional coupler. Others include the electro-optical coupler in which e.g. a TE and a TM mode interact, the acousto-optical coupler in which a surface acoustic wave deflects an optical ray, the travelling-wave tube and the backward wave oscillator where an electromagnetic transmission line is coupled to an electromechanical transmission line, etc. Since a number of integrated devices are four- ports, or can be viewed as such, as for example the anisotropic slab waveguide supporting a TE-TM hybrid mode, this section is devoted to the examples of distributed four-port networks.

The notation adopted in the previous section will be used here to describe four-ports, where n is now 2. The permutation matrix given in (2.2.10) in this case is

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad (2.3.1)$$

the impedance transformation matrix given in (2.2.13) and its inverse are

$$\Omega = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{Z_1} & \sqrt{Z_1} & 0 & 0 \\ \frac{1}{\sqrt{Z_1}} & \frac{-1}{\sqrt{Z_1}} & 0 & 0 \\ 0 & 0 & \sqrt{Z_2} & \sqrt{Z_2} \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & \frac{-1}{\sqrt{Z_2}} \end{bmatrix} , \qquad (2.3.2.a)$$

$$\Omega^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{Z_1}} & \sqrt{Z_1} & 0 & 0 \\ \frac{1}{\sqrt{Z_1}} & -\sqrt{Z_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & \sqrt{Z_2} \\ 0 & 0 & \frac{1}{\sqrt{Z_2}} & -\sqrt{Z_2} \end{bmatrix} . \qquad (2.3.2.b)$$

A classical example of a 4-port distributed network is the directional coupler, modeled as in Figure 2.2. Assuming exp ($j \omega t$) time dependence, straightforward nodal and mesh analysis yields

$$\frac{d V_1}{dz}(z) = -(R_{11} + j \omega L_{11}) I_1(z) - (R_{12} + j \omega L_{12}) I_2(z) , \qquad (2.3.3)$$

$$\frac{d V_2}{dz}(z) = -(R_{22} + j \omega L_{22}) I_2(z) - (R_{12} + j \omega L_{12}) I_1(z) , \qquad (2.3.4)$$

$$\frac{d I_1}{dz}(z) = -(G_{11} + j \omega C_{11}) V_1(z) + (G_{12} + j \omega C_{12}) V_2(z) , \qquad (2.3.5)$$

$$\frac{d I_2}{dz}(z) = -(G_{22} + j \omega C_{22}) V_2(z) + (G_{12} + j \omega C_{12}) V_1(z) . \qquad (2.3.6)$$

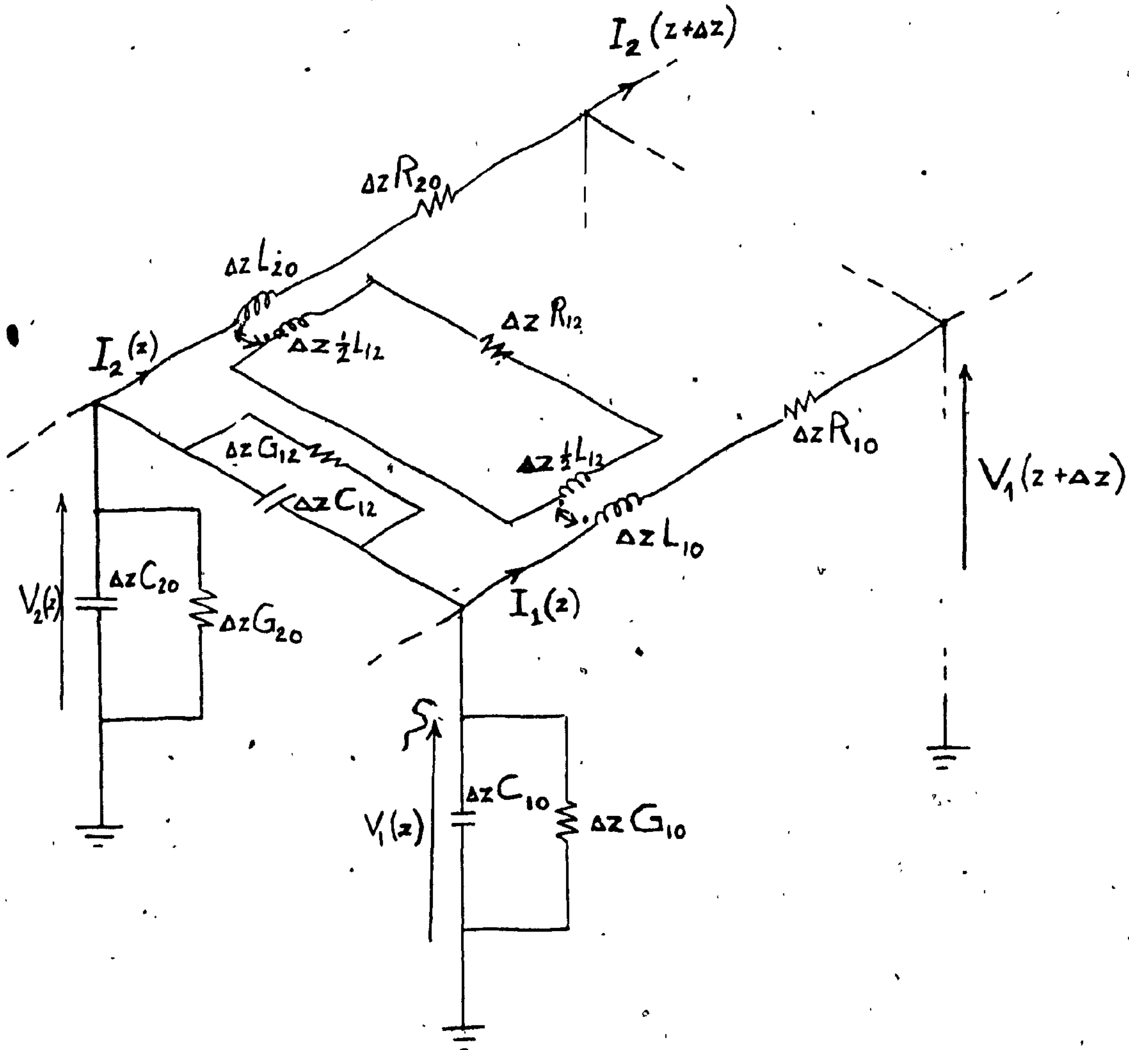

Figure 2.2 Circuit model of two coupled lossy transmission lines

where $L_{11} = L_{10} - L_{12}$, $L_{22} = L_{20} - L_{12}$, $R_{11} = R_{10} - R_{12}$, $R_{22} = R_{20} - R_{12}$, $C_{11} = C_{10} + C_{12}$, $C_{22} = C_{20} + C_{12}$, $G_{11} = G_{10} + G_{12}$, and $G_{22} = G_{20} + G_{12}$.

In matrix form, (2.3.3) - (2.3.6) can be written as

$$\frac{d\,\bar{g}(z)}{dz} = -j\,R_g\,\bar{g}(z) , \qquad (2.3.7)$$

where

$$R_g = \omega \begin{bmatrix} 0 & Z_{11} & 0 & Z_{12} \\ Y_{11} & 0 & Y_{12} & 0 \\ 0 & Z_{12} & 0 & Z_{22} \\ Y_{12} & 0 & Y_{22} & 0 \end{bmatrix} , \qquad (2.3.8)$$

and $Z_{11} = L_{11} - j\,R_{11}/\omega$, $Z_{12} = L_{12} - j\,R_{12}/\omega$, $Y_{11} = G_{11} - j\,C_{11}/\omega$, and $Y_{12} = G_{12} - j\,C_{12}/\omega$. Using superposition, i.e. setting I_2, $V_2 = 0$ and solving for V_1/I_1 and vice versa, the characteristic impedances of the unloaded lines 1 and 2 are found to be

$$Z_1 = \sqrt{Z_{11}/Y_{11}} , \qquad (2.3.9)$$

and

$$Z_2 = \sqrt{Z_{22}/Y_{22}} . \qquad (2.3.10)$$

These impedances are used in (2.3.2.a,b) to obtain $R_a = \Omega^{-1} R_g \Omega$. Thus

$$R_a = \begin{bmatrix} \gamma_1 & 0 & (a+b) & (a-b) \\ 0 & -\gamma_1 & -(a-b) & -(a+b) \\ (a+b) & (a-b) & \gamma_2 & 0 \\ -(a-b) & -(a+b) & 0 & -\gamma_2 \end{bmatrix}$$

(2.3.11)

where $\gamma_1 = \omega\sqrt{Z_{11}Y_{11}}$, $\gamma_2 = \omega\sqrt{Z_{22}Y_{22}}$, $a = \frac{1}{2}\omega Y_{12}\sqrt{Z_1 Z_2}$,

$b = \frac{1}{2}\omega Z_{12}/\sqrt{Z_1 Z_2}$.

The above results are valid for two uniform coupled transmission lines only. When the system is nonuniform, R_g, Ω, and R_a are no longer constant but functions of z. Differentiating (2.2.12), one obtains

$$\bar{g}(z)' = \Omega(z)'\,\bar{a}(z) + \Omega(z)\,\bar{a}(z)' , \qquad (2.3.12)$$

Since $\bar{g}(z)' = -j\,R_g(z)\,\bar{g}(z)$ and $\bar{a}(z)' = -j\,R_a(z)\,\bar{a}(z)$, (2.3.12) can be solved for R_a to yield

$$R_a(z) = \Omega^{-1}(z)\,R_g(z)\,\Omega(z) + j(\Omega^{-1}(z))'\,\Omega(z) =$$
$$= \Omega^{-1}(z)\,R_g(z)\,\Omega(z) - j\,\Omega^{-1}(z)\,\Omega(z)' . \qquad (2.3.13)$$

Since

$$(\Omega^{-1}(z))'\,\Omega(z) = \begin{bmatrix} 0 & -\{\ln\sqrt{Z_1}\}' & 0 & 0 \\ -\{\ln\sqrt{Z_1}\}' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\{\ln\sqrt{Z_2}\}' \\ 0 & 0 & -\{\ln\sqrt{Z_2}\}' & 0 \end{bmatrix} , \qquad (2.3.14)$$

the general form of R_a will no longer contain zero elements at entries 12, 21, 34, and 43 for nonuniform systems.

Another example of an uniform distributed 4-port is a surface acoustic wave (SAW) multistrip coupled filter consisting of a piezoelectric substrate supporting the SAW and a set of thin metallic strips deposited on the surface of the substrate perpendicular to the direction of SAW propagation. The S, T, and Q matrices for the unit cell of such a device are

$$S = \frac{1}{1\pm\alpha\pm\beta} \begin{bmatrix} ja\,\exp(-j\phi_A) & j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) & (1-jb)\exp(j\phi_B) & j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) \\ j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) & jb\,\exp(-j\phi_B) & j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) & (1-ja)\exp(j\phi_A) \\ (1-jb)\exp(j\phi_B) & j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) & ja\,\exp(-j\phi_A) & j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) \\ j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) & (1-ja)\exp(j\phi_A) & j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) & jb\,\exp(-j\phi_B) \end{bmatrix} \tag{2.3.15}$$

$$T = \begin{bmatrix} (1-ja)\exp(j\phi_A) & j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) & -ja & j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A-\phi_B)) \\ j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A+\phi_B)) & (1-jb)\exp(j\phi_B) & j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A-\phi_B)) & -jb \\ ja & -j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A-\phi_B)) & (1+ja)\exp(-j\phi_A) & -j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) \\ j\sqrt{ab}\,\exp(j\tfrac{1}{2}(\phi_A-\phi_B)) & jb & -j\sqrt{ab}\,\exp(-j\tfrac{1}{2}(\phi_A+\phi_B)) & (1+jb)\exp(-j\phi_B) \end{bmatrix} \tag{2.3.16}$$

$$Q = \begin{bmatrix} \cos\phi_A + a\sin\phi_A & -2\left[\frac{Z_1}{Z_2}\right]^{\frac{1}{2}} \sqrt{ab}\,\sin\tfrac{1}{2}\phi_A\cos\tfrac{1}{2}\phi_B & jZ_1\sin\phi_A\,(1+a\tan\tfrac{1}{2}\phi_A) & -j2\sqrt{Z_1Z_2}\,\sqrt{ab}\,\sin\tfrac{1}{2}\phi_A\sin\tfrac{1}{2}\phi_B \\ -2\left[\frac{Z_2}{Z_1}\right]^{\frac{1}{2}} \sqrt{ab}\,\cos\tfrac{1}{2}\phi_A\sin\tfrac{1}{2}\phi_B & \cos\phi_B + b\sin\phi_B & -j2\sqrt{Z_1Z_2}\,\sqrt{ab}\,\sin\tfrac{1}{2}\phi_A\sin\tfrac{1}{2}\phi_B & jZ_2\sin\phi_B\,(1+b\tan\tfrac{1}{2}\phi_B) \\ jY_1\sin\phi_A\,(1-a\cot\tfrac{1}{2}\phi_A) & j2\sqrt{Y_1Y_2}\,\sqrt{ab}\,\cos\tfrac{1}{2}\phi_A\cos\tfrac{1}{2}\phi_B & \cos\phi_A + a\sin\phi_A & -2\left[\frac{Z_2}{Z_1}\right]^{\frac{1}{2}} \sqrt{ab}\,\cos\tfrac{1}{2}\phi_A\cos\tfrac{1}{2}\phi_B \\ j2\sqrt{Y_1Y_2}\,\sqrt{ab}\,\cos\tfrac{1}{2}\phi_A\cos\tfrac{1}{2}\phi_B & jY_2\sin\phi_B\,(1-b\cot\tfrac{1}{2}\phi_B) & -2\left[\frac{Z_1}{Z_2}\right]^{\frac{1}{2}} \sqrt{ab}\,\sin\tfrac{1}{2}\phi_A\cos\tfrac{1}{2}\phi_B & \cos\phi_B + b\sin\phi_B \end{bmatrix} \qquad (2.3.17)$$

where a and b are normalized, purely reactive circuit parameters associated with track A and B of the SAW respectively, and ϕ_A and ϕ_B are phase shifts on the corresponding tracks between two metallic strips. It can be shown that the above matrices satisfy the reciprocity, the losslessness, and the bilateral symmetry conditions.

Further examples of 4-port networks include microwave and integrated optical filter elements using forward and reverse couplers with feedback (5). Figure 2.3 illustrates a forward coupler in which ports 2 and 4 are interconnected via a linear two-port, characterized by its transfer matrix M_2:

$$\begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = M_2 \begin{bmatrix} b_4 \\ a_4 \end{bmatrix} . \qquad (2.3.18)$$

Straightforward partitioning of the transfer matrix representing the 4-port: A , and subtitution of the relationship existing between ports 2 and 4 on account of M_2 yields the A_R matrix of the reduced 2-port:

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = A_R \begin{bmatrix} b_3 \\ a_3 \end{bmatrix} , \qquad (2.3.19)$$

where $A_R = A_{11} + A_{12} (M_2 - A_{22})^{-1} A_{21}$.

If port 2 is coupled to port 3, by M_2, instead of port 4, then the reduced 2-port is characterized by the expression

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \hat{A}_R \begin{bmatrix} b_4 \\ a_4 \end{bmatrix} , \qquad (2.3.20)$$

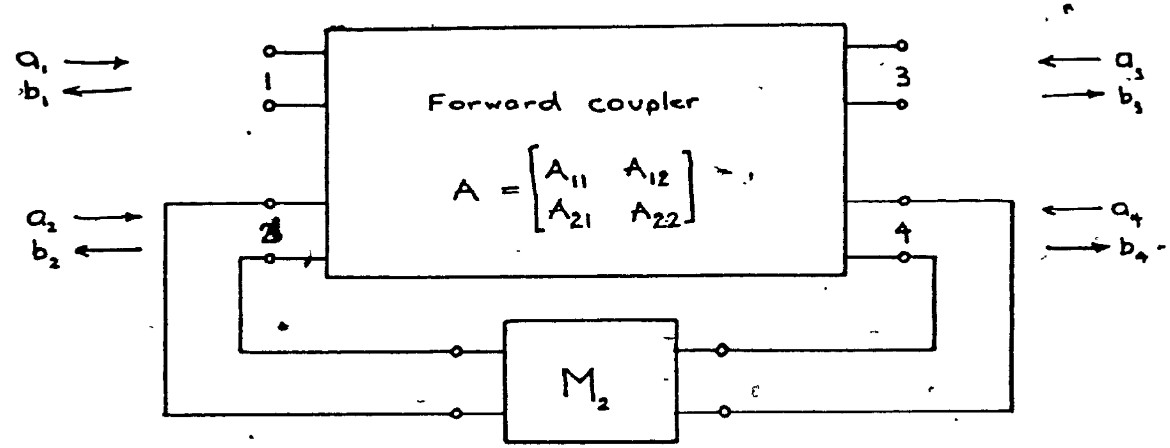

Fig. 2.3. Filter element consisting of a forward coupler and a feedback network.

where $A_R = A_{12} + A_{11} (M_2 - A_{21})^{-1} A_{22}$.

Utilizing now a reverse (or backward) coupler characterized by A one can construct a filter element by connecting port 4 via a linear 2-port with either port 3 or with port 2. In the first case, letting the linear 2-port be characterized by

$$\begin{bmatrix} b_3 \\ a_3 \end{bmatrix} = A_2 \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} , \qquad (2.3.21)$$

one obtains

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = A_R \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} , \qquad (2.3.22)$$

where $A_R = (A_{11} A_2 + A_{12} \sigma) (A_{21} A_2 + A_{22} \sigma)^{-1} \sigma$, and

$$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \qquad (2.3.23)$$

In the second case

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = A_2 \begin{bmatrix} a_4 \\ b_4 \end{bmatrix} , \qquad (2.3.24)$$

and

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = A_R \begin{bmatrix} b_3 \\ a_3 \end{bmatrix} , \qquad (2.3.25)$$

with $A_R = A_{11} + A_{12} (\sigma A_2 \sigma - A_{22})^{-1} A_{21}$.

The admittance matrix of an interdigitated directional coupler can be represented by denoting the electrical length of the interdigitated transmission line sections by θ and certain characteristic admittances defined in reference (28) by M and N as

$$Y = j \begin{bmatrix} -\cot\theta & \csc\theta \\ \csc\theta & -\cot\theta \end{bmatrix} \times \begin{bmatrix} M & N \\ N & M \end{bmatrix} , \qquad (2.3.26)$$

where the x operation indicates Kronecker, or direct product. From the Y matrix results the Q matrix:

$$Q = \cos\theta \, E_4 + j \sin\theta \, K , \qquad (2.3.27)$$

where

$$K = \begin{bmatrix} 0 & 0 & Z_0^2 M & -Z_0^2 N \\ 0 & 0 & -Z_0^2 N & Z_0^2 M \\ M & N & 0 & 0 \\ N & M & 0 & 0 \end{bmatrix} , \qquad (2.3.28)$$

and $Z_0^2 \triangleq 1/(M^2 - N^2)$. Since $K^2 = E_4$, $Q = \exp(j\theta K)$, and consequently $Q^n = \exp(j n \theta K)$ (see Appendix II). Finally, the transverse scattering matrix is found to be

$$T = \cos\theta \, E_4 + j \sin\theta \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} ; \qquad (2.3.29)$$

where

$$U = \tfrac{1}{2} \begin{bmatrix} M \xi_1^+ & N \eta^- \\ N \eta^- & M \zeta^+ \end{bmatrix} , \quad V = \tfrac{1}{2} \begin{bmatrix} M \zeta_1^- & N \eta^+ \\ N \eta^+ & M \zeta_2^- \end{bmatrix} , \qquad (2.3.30.a,b)$$

$\zeta_i^{\pm} = Z_i \pm Z_0^2/Z_i$, $i = 1,2$ and $\eta^{\pm} = \sqrt{Z_1 Z_2} \pm Z_0^2/\sqrt{Z_1 Z_2}$. Noting that $\begin{bmatrix} U & V \\ -V & -U \end{bmatrix} = E_4$,

the T matrix can also be expressed as a matrix exponential

$$T = \exp \{ j \theta \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} \} . \quad (2.3.31)$$

Consequently, the scattering transfer matrix of n identical interdigitated couplers in cascade is simply

$$T^n = \cos (n\theta) E_4 + j \sin (n\theta) \begin{bmatrix} U & V \\ -V & -U \end{bmatrix} =$$

$$= \exp (j n\theta \begin{bmatrix} U & V \\ -V & -U \end{bmatrix}) . \quad (2.3.32)$$

The condition of antireciprocity discussed in Section 2.2 carries over to the system matrices R_a and R_g as

$$R_a = - \tilde{\sigma}_3 R_a^T \tilde{\sigma}_3 \text{ and } R_g = - \tilde{\sigma}_1 R_g^T \tilde{\sigma}_1 . \quad (2.3.33.a,b)$$

Hence, R_a and R_g matrices of an antireciprocal system must have the following forms:

$$R_a = \begin{bmatrix} 0 & r_{a12} & r_{a13} & r_{a14} \\ r_{a12} & 0 & r_{a23} & r_{a24} \\ -r_{a13} & r_{a23} & 0 & r_{a34} \\ r_{a14} & -r_{a24} & r_{a34} & 0 \end{bmatrix} , \quad (2.3.34)$$

$$R_g = \begin{bmatrix} r_{g11} & 0 & r_{g13} & r_{g14} \\ 0 & -r_{g11} & r_{g23} & r_{g24} \\ -r_{g24} & -r_{g14} & r_{g33} & 0 \\ -r_{g23} & -r_{g13} & 0 & -r_{g33} \end{bmatrix} . \qquad (2.3.35)$$

To investigate the properties of a distributed antireciprocal network, one can separate Maxwell's equations for a 3 layered anisotropic slab waveguide shown in Figure 2.4, and write them as a set of coupled first order linear differential equations involving only transverse field components. Assuming harmonic time dependence ($e^{j\omega t}$), Maxwell's curl equations reduce to

$$\nabla \times \bar{H} = j\omega\varepsilon_0 \,\bar{\bar{\varepsilon}}\, \bar{E} \ , \qquad (2.3.36)$$

and

$$\nabla \times \bar{E} = -j\omega\mu_0 \,\bar{\bar{\mu}}\, \bar{H} \ , \qquad (2.3.37)$$

where

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \kappa_{xx} & \kappa_{xy} & \kappa_{xz} \\ \kappa_{yx} & \kappa_{yy} & \kappa_{yz} \\ \kappa_{zx} & \kappa_{zy} & \kappa_{zz} \end{bmatrix} , \qquad (2.3.38)$$

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} . \qquad (2.3.39)$$

Considering that $\partial/\partial y = 0$ and that the waveguide is uniform in the z direction, i.e. $\partial/\partial z = -j\beta_z$, one obtains from the Maxwell's equations a set of linear coupled differential equations which can be cast in the form:

$$\frac{d\bar{g}(x)}{dx} = -j\, R_g\, \bar{g}(x) \quad , \tag{2.3.40}$$

where $\bar{g}(x) = (\, E_y(x),\, H_z(x),\, E_z(x),\, -H_y(x)\,)$, (2.3.41)

$$R_g = \begin{bmatrix} -\frac{\beta_z \mu_{zx}}{\mu_{xx}} & \frac{\omega\mu_0(\mu_{xx}\mu_{zz}-\mu_{xz}\mu_{zx})}{\mu_{xx}} & 0 & \frac{\omega\mu_0(\mu_{xy}\mu_{zx}-\mu_{xx}\mu_{zy})}{\mu_{xx}} \\ \frac{-\beta_z^2\kappa_{xx}-k_0^2\mu_{xx}(\kappa_{xy}\kappa_{yx}-\kappa_{xx}\kappa_{yy})}{\omega\mu_0\mu_{xx}\kappa_{xx}} & -\frac{\beta_z\mu_{xz}}{\mu_{xx}} & \frac{\omega\mu_0(\kappa_{xx}\kappa_{yz}-\kappa_{xz}\kappa_{yx})}{\kappa_{xx}} & \frac{\beta_z(\kappa_{xx}\mu_{xy}-\kappa_{yx}\mu_{xx})}{\kappa_{xx}\mu_{xx}} \\ \frac{\beta_z(\kappa_{xx}\mu_{yx}-\kappa_{xy}\mu_{xx})}{\kappa_{xx}\mu_{xx}} & \frac{\omega\mu_0(\mu_{xz}\mu_{yx}-\mu_{xx}\mu_{yz})}{\mu_{xx}} & -\frac{\beta_z\kappa_{xz}}{\kappa_{xx}} & \frac{k_0^2\kappa_{xx}(\mu_{xx}\mu_{yy}-\mu_{xy}\mu_{yx})}{\omega\varepsilon_0\mu_{xx}\kappa_{xx}} \\ \frac{\omega\varepsilon_0(\kappa_{xx}\kappa_{zy}-\kappa_{xy}\kappa_{zx})}{\kappa_{xx}} & 0 & \frac{\omega\varepsilon_0(\kappa_{xx}\kappa_{zz}-\kappa_{xz}\kappa_{zx})}{\kappa_{xx}} & -\frac{\beta_z\kappa_{xz}}{\kappa_{xx}} \end{bmatrix} , \tag{2.3.42}$$

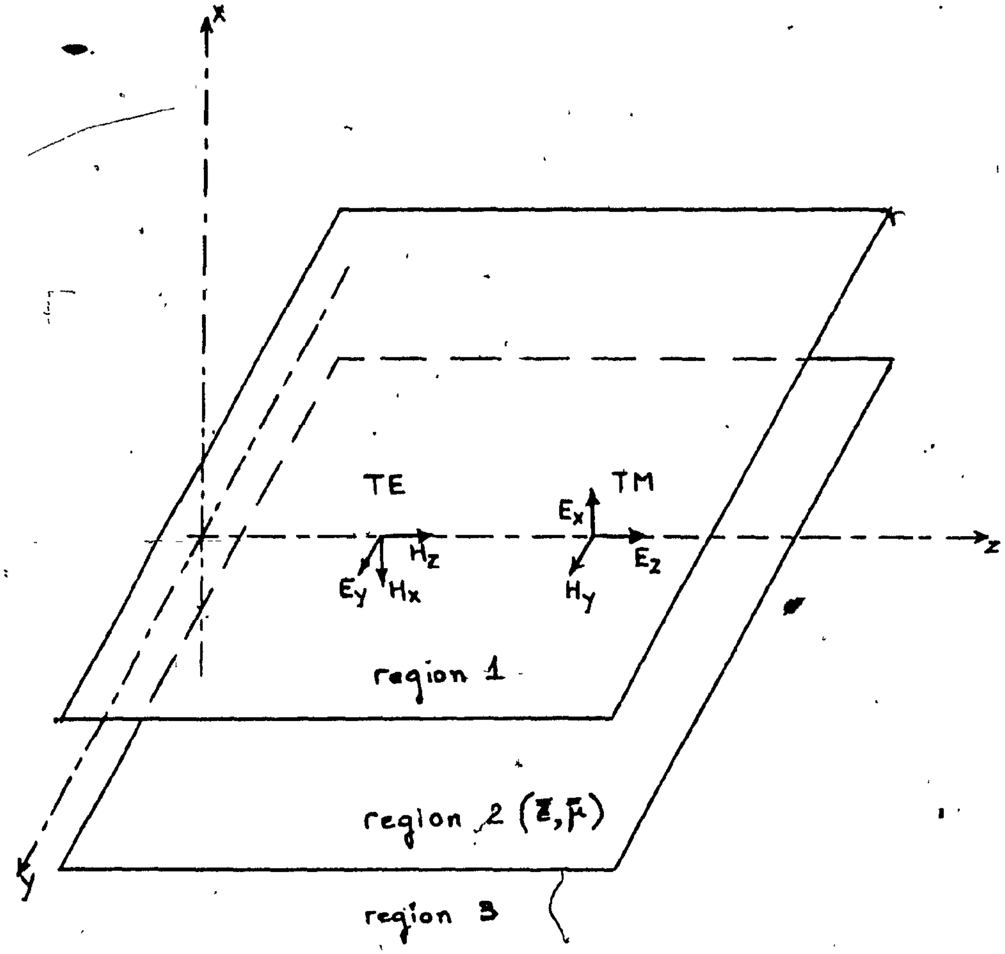

Figure 2.4 Three layered anisotropic slab waveguide.

and $k_0^2 = \omega^2\mu_0\epsilon_0$. The remaining x components of the electric and magnetic fields are related to the transverse components as

$$\begin{bmatrix} E_x(x) \\ H_x(x) \end{bmatrix} = \begin{bmatrix} \frac{-\kappa_{xy}}{\kappa_{xx}} & 0 & \frac{-\kappa_{xz}}{\kappa_{xx}} & \frac{-\beta_z}{\omega\epsilon_0\kappa_{xx}} \\ \frac{-\beta_z}{\omega\mu_0\mu_{xx}} & \frac{-\mu_{xz}}{\mu_{xx}} & 0 & \frac{\mu_{xy}}{\mu_{xx}} \end{bmatrix} \begin{bmatrix} E_y(x) \\ H_z(x) \\ E_z(x) \\ -H_y(x) \end{bmatrix}. \quad (2.3.43)$$

Comparing (2.3.42) and (2.3.35), one can see that for a lossless antireciprocal distributed device $\bar{\epsilon}$ and $\bar{\mu}$ must have the form:

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \kappa & 0 & j\kappa \\ 0 & \kappa_{yy} & j\kappa_{yz} \\ -j\kappa & -j\kappa_{yz} & \kappa \end{bmatrix}, \quad (2.3.44)$$

and

$$\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & j\mu \\ 0 & \mu\kappa_{yy}/\kappa & j\mu_{yz} \\ -j\mu & -j\mu_{yz} & \mu \end{bmatrix}, \quad (2.3.45)$$

where κ, μ, κ_{yz} and μ_{yz} are real and μ_{yz} and/or κ_{yz} can be taken as zero. For a system to be antireciprocal, (2.3.44) and (2.3.45) must hold simultaneously.

The foregoing analysis formulates the conditions for an antireciprocal distributed 4-port. To the best of the author's knowledge, no material can satisfy (2.3.44) and (2.3.45) simultaneously.

III. CODIRECTIONAL AND CONTRADIRECTIONAL COUPLERS

3.1 Stokes Vector and Mueller Matrix

In this section, Jones and Mueller calculi are introduced in the analysis of a system shown in Figure 3.1. Here, each channel is assumed to carry one wave; Some of the waves may be coupled codirectionally, others contradirectionally. The waves are orthogonal, therefore the total power flow in the device is the net sum of the powers propagating forward in the individual channels. This means that when power propagates in the negative z direction in a particular channel, that power flow must be prefixed by a negative sign. To keep the analysis consistent with Chapter 2, it is assumed that the number of channels is 2n. When this is the case, (2.1.1) and (2.1.2) can be used to describe the system shown in Figure 3.1.

Two additional vectors are introduced to characterize the device in Figure 3.1. These are the coherency vector $\bar{f}(z)$, and the generalized Stokes vector $\bar{s}(z)$. The elements of $\bar{f}(z)$ are defined by the expression

$$\mathrm{diag}\ (\ f_1,f_2,..,f_{2n},..,f_{(2n^2)}\ ) = \mathrm{diag}\ (\ a_1,a_2,..,a_{2n}\ ) \times \mathrm{diag}\ (\ a_1^*,a_2^*,..,a_{2n}^*\ )\ . \qquad (3.1.1)$$

Here, the x refers to the so-called direct product or Kronecker product of two matrices, and the star indicates complex conjugation. The vector $\bar{s}(z)$ is obtained from $\bar{f}(z)$ by a linear transformation (see Appendix I) given as

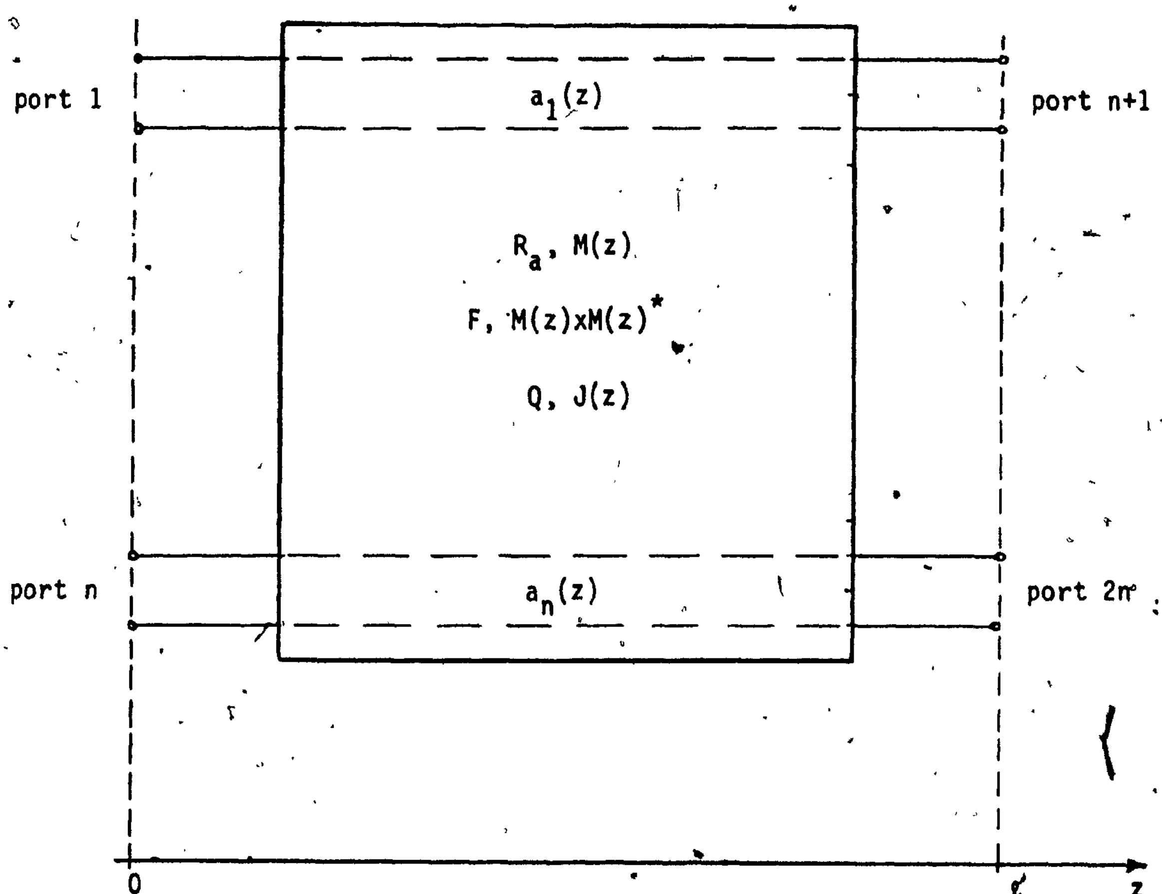

Figure 3.1. System of n coupled transmission lines. Each $a_i(z)$ propagates in either positive ($a_i(z)=a_i^+(z)$) or negative ($a_i(z)=a_i^-(z)$) z direction.

$$\bar{f}(z) = V\,\bar{s}(z) \quad , \tag{3.1.2}$$

where $\bar{s}(z)$ is real. Expanding (2.1.1) appropriately, one can show that $\bar{f}(z)$ and $\bar{s}(z)$ also obey a set of linear first order differential equations

$$\frac{d\bar{f}(z)}{dz} = -j\,F\,\bar{f}(z) \quad , \tag{3.1.3}$$

and $$\frac{d\bar{s}(z)}{dz} = -j\,Q\,\bar{s}(z) \quad , \tag{3.1.4}$$

whose solutions are

$$\bar{f}(z) = (\,M(z) \times M(z)^{*}\,)\,\bar{f}(0) \quad , \tag{3.1.5}$$

and $$\bar{s}(z) = J(z)\,\bar{s}(0) \quad , \tag{3.1.6}$$

respectively. The matrices F, Q, and J(z) are found to be*

$$F = R_a \times E_{2n} - E_{2n} \times R_a^{*} \quad , \tag{3.1.7}$$

$$Q = V^{-1}\,F\,V \quad , \tag{3.1.8}$$

and $$J(z) = V^{-1}\,(\,M(z) \times M(z)^{*}\,)\,V \quad , \tag{3.1.9}$$

where J(z) is sometimes called the Mueller matrix.

The pairwise properties of R_a and M(z), discussed in Section 2.1 are also satisfied by Q and J(z) on the one hand, and F and $(\,M(z) \times M(z)^{*}\,)$ on the other. The results are summarized in Table 3.1.

Matrix Pair	Diagonal Form	Modal Matrix	Properties
R_a	$\mathrm{diag}(\lambda_{r1},\lambda_{r2},\ldots,\lambda_{r2n})$	U	$R_a M(z) = M(z) R_a$
			$M(z)' = -j\, R_a M(z)$
$M(z)$	$\mathrm{diag}(\exp(-j\lambda_{r1}z),\exp(-j\lambda_{r2}z),\ldots,\exp(-j\lambda_{r2n}z))$		
F	$\mathrm{diag}(\lambda_{r1}-\lambda_{r1}^*,\ldots,\lambda_{r1}-\lambda_{r2n}^*,\lambda_{r2}-\lambda_{r1}^*,\ldots,\lambda_{r2}-\lambda_{r2n}^*,\ldots,\lambda_{r2n}-\lambda_{r2n}^*)$	$U \times U^*$	$F(M(z)\times M(z)^*) =$
			$= (M(z)\times M(z)^*)F$
$(M(z)\times M(z)^*)$	$\mathrm{diag}(\exp(-j(\lambda_{r1}-\lambda_{r1}^*)z),\ldots,\exp(-j(\lambda_{rj}-\lambda_{rj}^*)z),\ldots,\exp(-j(\lambda_{r2n}-\lambda_{r2n}^*)z))$		$(M(z)\times M(z)^*)' =$
			$= -j\, F(M(z)\times M(z)^*)$
Q	same as for F	$V^{-1}(U\times U^*)$	$QJ(z) = J(z)Q$
			$J(z)' = -j\, QJ(z)$
$J(z)$	same as for $(M(z)\times M(z)\ )$		

Table 3.1. Matrix pairs for uniform coupled systems

3.2. Uniform Couplers

In this section, two channel uniform couplers will be investigated. A reciprocal, bilaterally symmetric codirectional (or forward) coupler is characterized by a system matrix of the form

$$R_{a(f)} = \begin{bmatrix} k_1 & 0 & \kappa_f & 0 \\ 0 & -k_1 & 0 & -\kappa_f \\ \kappa_f & 0 & k_2 & 0 \\ 0 & -\kappa_f & 0 & -k_2 \end{bmatrix} , \qquad (3.2.1)$$

whereas a reciprocal, bilaterally symmetric contradirectional (or backward) coupler has a system matrix

$$R_{a(b)} = \begin{bmatrix} k_1 & 0 & 0 & \kappa_b \\ 0 & -k_1 & -\kappa_b & 0 \\ 0 & \kappa_b & k_2 & 0 \\ -\kappa_b & 0 & 0 & -k_2 \end{bmatrix} , \qquad (3.2.2)$$

where k_1, k_2, κ_f, and κ_b are complex. k_1 and k_2 are the propagation constants in lines 1 and 2, and κ_f (κ_b) are the forward (backward) coupling coefficients. The wave vector in (2.1.1) is now given as

$$\bar{a}(z) = (a_1(z)^+, a_1(z)^-, a_2(z)^+, a_2(z)^-)^T \qquad (3.2.3)$$

for both codirectional and contradirectional couplers. Figures 3.2.a and 3.2.b illustrate the possible routes of power flow in a codirectional and a contradirectional coupler, respectively.

For a forward coupler, the two waves propagating in channels

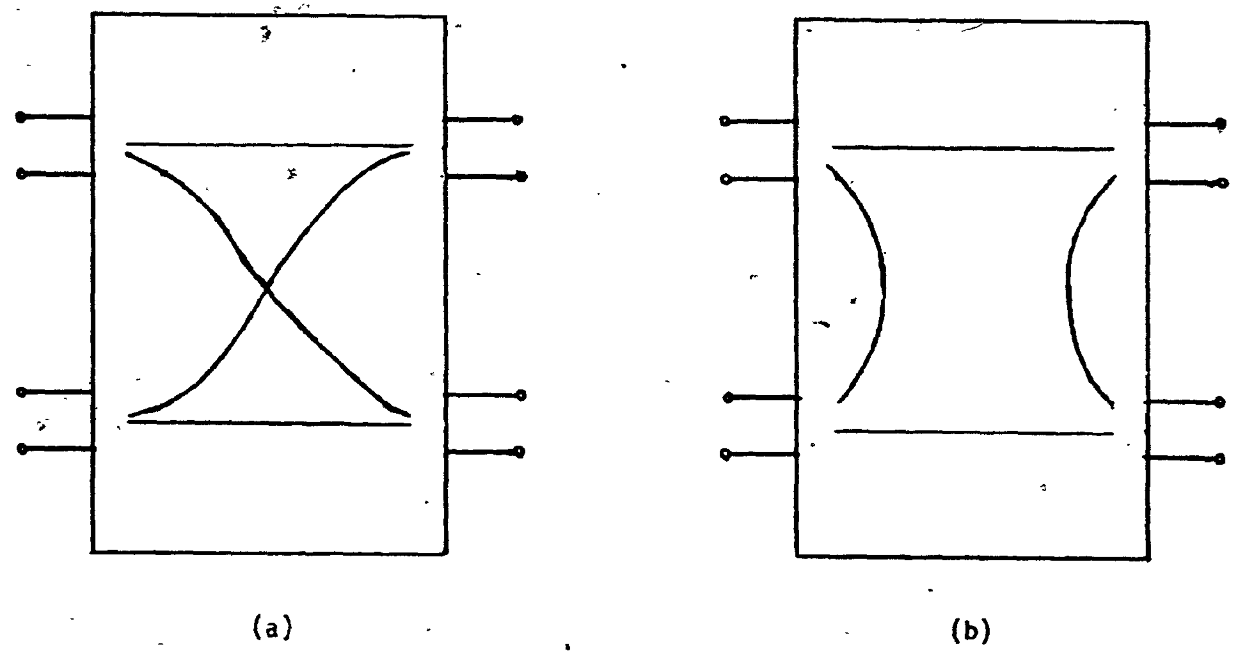

Figure 3.2. Possible routes of power flow in a (a) codirectional, (b) contradirectional coupler

1 and 2 are $a_1(z)^+$ and $a_2(z)^+$ both in the positive z direction or $a_1(z)^-$ and $a_2(z)^-$ both in the negative z direction. When this is the case, one can conveniently reduce the two 4x4 matrices given in (3.2.1) and (3.2.2) into a set of 2x2 matrices. This is possible since only four of the eight nonzero elements are needed to characterize the coupler. Thus in the forward case

$$\frac{d}{dz}\begin{bmatrix} a_1(z)^+ \\ a_2(z)^+ \end{bmatrix} = -j \begin{bmatrix} k_1 & \kappa_f \\ \kappa_f & k_2 \end{bmatrix}\begin{bmatrix} a_1(z)^+ \\ a_2(z)^+ \end{bmatrix} , \qquad (3.2.4)$$

whereas in the backward case

$$\frac{d}{dz}\begin{bmatrix} a_1(z)^+ \\ a_2(z)^- \end{bmatrix} = -j \begin{bmatrix} k_1 & \kappa_b \\ -\kappa_b & k_2 \end{bmatrix}\begin{bmatrix} a_1(z)^+ \\ a_2(z)^- \end{bmatrix} . \qquad (3.2.5)$$

Further simplification is possible by combining these and writing

$$R_a = \begin{bmatrix} k_1 & \kappa \\ \pm\kappa & k_2 \end{bmatrix} , \qquad (3.2.6)$$

where the upper (lower) sign corresponds to a codirectional (contradirectional) coupler, and it is understood that k_2 is the wavenumber in line 2 which is negative in the contradirectional case. The R_a matrix in (3.2.6) does not satisfy the losslessness condition. To satisfy conservation of energy,

$$\frac{d}{dz}\left(|a_1(z)|^2 \pm |a_2(z)|^2 \right) \qquad (3.2.7)$$

must vanish and as a result R_a must be

$$R_a = \begin{bmatrix} \beta_1 & \kappa \\ \pm\kappa^* & \beta_2 \end{bmatrix} , \qquad (3.2.8)$$

where β_1 and β_2 are real.

Losses can be of two basic types: Losses due to propagation in the lines, which are characterized by a complex loaded wavenumber, and losses in the coupling, where 12 and 21 elements of R_a are no longer complex conjugates of each other. Losses in the lines can be treated by letting

$$R_a = R_{a\beta} - j\,R_{a\alpha} \quad , \tag{3.2.9}$$

where

$$R_{a\alpha} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{bmatrix} \quad , \tag{3.2.10}$$

and $R_{a\beta}$ is as in (3.2.8).

The foregoing simplification can also be justified by evaluating the eigenvalues of the 4x4 system coupling matrix R_a, which are the propagation constants of the normal modes. Denoting the eigenvalues corresponding to the propagation constants in channel 1 as k_1^+ and k_1^-, and in channel 2 as k_2^+ and k_2^-, it can be shown that $k_1^+ = k_1^-$ and $k_2^+ = k_2^-$ for codirectional and contradirectional couplers. Hence the 2x2 system coupling matrix of (3.2.9) is sufficient to evaluate the performance of the coupler.

The appropriate wave vector in a forward coupler where propagation takes place in the positive z direction is

$$\bar{a}(z) = (\ a_1(z), a_2(z)\ )^T = (\ a_1(z)^+, a_2(z)^+\ )^T \quad . \tag{3.2.11}$$

The appropriate wave vector in a reverse coupler where the principal channel carries energy in the positive z direction is, on the other hand

$$\bar{a}(z) = (a_1(z), a_2(z))^T = (a_1(z)^+, a_2(z)^-)^T \quad . \qquad (3.2.12)$$

It can be seen that in the special cases discussed above a 2x2 matrix formalism is sufficient to analyze the properties of the coupler. In the general case where both forward and reverse coupling occurs, such a reduction cannot be implemented. Also, when the terminations at the output ports of the coupler are reflective, i.e. when $a_1(\ell)^-$ and/or $a_2(\ell)^-$ in the forward case, or $a_1(\ell)^-$ and $a_2(0)^+$ in the backward case are nonzero, it is necessary to retain the 4x4 matrix formalism.

For a uniform coupler, the system matrix R_a given in (3.2.9) is constant. In general, when the lines are lossy, k_1 and k_2 are the complex loaded wavenumbers given by

$$k_i = \beta_i - j\,\alpha_i \quad , i \quad 1,2 \ . \qquad (3.2.13)$$

Introducing the following notation

$$k_0 = \tfrac{1}{2}\,(k_1 + k_2) \ ,^{*}$$

$$\Delta k = \tfrac{1}{2}\,(k_1 - k_2) \ ,$$

$$\kappa = |\kappa|\, e^{j\phi} \ ,$$

$$Z = \frac{\Delta k}{|\kappa|} \ ,$$

$$W = (Z^2 \pm 1)^{\frac{1}{2}} \ ,$$

one can write the eigenvalues of R_a as $k^+ = k_0 + |\kappa|\, W$ and $k^- = k_0 - |\kappa|\, W$.

---

* not to be confused with free space wavenumber, defined as $\sqrt{\mu_0\varepsilon_0}\ \omega^2 = k_0^{\ 2}$

It can be seen that 'γ coupling', discussed in Section 2.1 occurs in contradirectional couplers whenever $\mathrm{Re}(\, Z^2\,) < 1$. The modal matrix of R_a can be chosen as

$$U = \begin{bmatrix} (\tfrac{1}{2}(1 + \tfrac{Z}{W}))^{\frac{1}{2}}\, e^{j\frac{1}{2}\phi} & \mp(\, \tfrac{1}{2}(1 - \tfrac{Z}{W}))^{\frac{1}{2}}\, e^{j\frac{1}{2}\phi} \\ (\pm\tfrac{1}{2}(1 - \tfrac{Z}{W}))^{\frac{1}{2}}\, e^{-j\frac{1}{2}\phi} & (\tfrac{1}{2}(1 + \tfrac{Z}{W}))^{\frac{1}{2}}\, e^{-j\frac{1}{2}\phi} \end{bmatrix} . \qquad (3.2.14)$$

Referring to Section 2.1, the proper metrics of a lossless system satisfying

$$R_a^{\dagger}\, K_i = K_i\, R_a \qquad (3.2.15)$$

are

$$K_1 = \frac{1}{2\,W} \begin{bmatrix} W + Z & e^{j\phi} \\ \pm e^{-j\phi} & W - Z \end{bmatrix} , \qquad (3.2.16)$$

and

$$K_2 = \frac{1}{2\,W} \begin{bmatrix} W - Z & -e^{j\phi} \\ \mp e^{-j\phi} & W + Z \end{bmatrix} . \qquad (3.2.17)$$

These metrics can be used to write R_a and $M(z)$ as

$$R_a = k^+\, K_1 + k^-\, K_2 \;, \qquad (3.2.18)$$

and $$M(z) = \exp\,(\, -jk^+ z\,)\; K_1 + \exp\,(\, -jk^- z\,)\; K_2 \;, \qquad (3.2.19)$$

The explicit form of the transfer matrix is

$$M(z) = e^{-jk_0 z} \begin{bmatrix} \cos|\kappa|Wz - j\frac{Z}{W}\sin|\kappa|Wz & -j\frac{e^{j\phi}}{W}\sin|\kappa|Wz \\ \mp j\frac{e^{-j\phi}}{W}\sin|\kappa|Wz & \cos|\kappa|Wz + j\frac{Z}{W}\sin|\kappa|Wz \end{bmatrix}$$

(3.2.20)

The foregoing so-called Jones calculus is inadequate to describe the state of polarization at location z. This information is contained in the Stokes vector defined in Section 3.1. As discussed in Section 3.1, the evolution of the Stokes vector is characterized either by the differential system matrix Q or by the terminal representation J(z).

For a lossy coupler

$$Q = Q_\beta - j\,Q_\alpha \quad , \tag{3.2.21}$$

where

$$Q_{\beta(\text{forward})} = j2|\kappa| \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\phi & \cos\phi \\ 0 & -\sin\phi & 0 & -\mathrm{Re}\{Z\} \\ 0 & -\cos\phi & \mathrm{Re}\{Z\} & 0 \end{bmatrix} , \tag{3.2.22}$$

$$Q_{\beta(\text{backward})} = j2|\kappa| \begin{bmatrix} 0 & 0 & \sin\phi & \cos\phi \\ 0 & 0 & 0 & 0 \\ \sin\phi & 0 & 0 & -\mathrm{Re}\{Z\} \\ \cos\phi & 0 & \mathrm{Re}\{Z\} & 0 \end{bmatrix} , \tag{3.2.23}$$

and

$$Q_\alpha = \begin{bmatrix} 2\mathrm{Im}\{k_0\} & 2\mathrm{Im}\{\Delta k\} & 0 & 0 \\ 2\mathrm{Im}\{\Delta k\} & 2\mathrm{Im}\{k_0\} & 0 & 0 \\ 0 & 0 & 2\mathrm{Im}\{k_0\} & 0 \\ 0 & 0 & 0 & 2\mathrm{Im}\{k_0\} \end{bmatrix} . \tag{3.2.24}$$

According to the lamellar representation suggested by Jones (34), a general anisotropic layer can be modelled as a cascade of eight layers, each representing a basic type of optical behaviour. The eight basic types are: isotropic refraction and absorbtion, linear birefringence and linear dichroism along the transverse coordinate axes, linear birefringence and linear dichroism along the bisectors of the transverse coordinate axes, circular birefringence and circular dichroism. These properties are summarized in Table 3.2, where Q and R_a are given for each type of optical behaviour.

The sum of the R_a matrices corresponding to the first 4 properties in Table 3.2 results in

$$\begin{bmatrix} \frac{\beta_1+\beta_2}{2} + \frac{\beta_1-\beta_2}{2} - j(\frac{\alpha_1+\alpha_2}{2} + \frac{\alpha_1-\alpha_2}{2}) & 0 \\ 0 & \frac{\beta_1+\beta_2}{2} - \frac{\beta_1-\beta_2}{2} - j(\frac{\alpha_1+\alpha_2}{2} - \frac{\alpha_1-\alpha_2}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} \beta_1 - j\alpha_1 & 0 \\ 0 & \beta_2 - j\alpha_2 \end{bmatrix} . \qquad (3.2.25)$$

The sum of the R_a matrices of the last 4 properties yields

$$\begin{bmatrix} 0 & (\kappa_{re} + \Delta\kappa) - j(\Delta\xi - \kappa_{im}) \\ (\kappa_{re} - \Delta\kappa) - j(\Delta\xi + \kappa_{im}) & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \kappa + \kappa_{loss} \\ \kappa^* - \kappa_{loss} & 0 \end{bmatrix} . \qquad (3.2.26)$$

where $\kappa_{loss} = \Delta\kappa - j\Delta\xi$. Letting κ_{loss} be zero, adding (3.2.25) and (3.2.26), and comparing the result to (3.2.9), one can see that R_a given as in (3.2.9) for codirectional couplers includes all the

	Q	R_a
Isotropic refraction	$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$	$\frac{1}{2}(\beta_1+\beta_2)\begin{bmatrix}1&0\\0&1\end{bmatrix}$
Isotropic absorption	$-j(\alpha_1+\alpha_2)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	$-j\frac{1}{2}(\alpha_1+\alpha_2)\begin{bmatrix}1&0\\0&1\end{bmatrix}$
Linear birefringence along the xy co-ordinate axes	$j(\beta_1-\beta_2)\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{bmatrix}$	$\frac{1}{2}(\beta_1-\beta_2)\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
Linear dichroism along the xy coordinate axes	$-j(\alpha_1-\alpha_2)\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$	$-j\frac{1}{2}(\alpha_1-\alpha_2)\begin{bmatrix}1&0\\0&-1\end{bmatrix}$
Linear birefringence along the bisectors of the xy coordinate axes	$j2\kappa_{re}\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix}$	$\kappa_{re}\begin{bmatrix}0&1\\1&0\end{bmatrix}$
Linear dichroism along the bisectors of the xy coordinate axes	$-j2\Delta\xi\begin{bmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{bmatrix}$	$-j\Delta\xi\begin{bmatrix}0&1\\1&0\end{bmatrix}$
Circular birefringence	$j2\kappa_{im}\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix}$	$j\kappa_{im}\begin{bmatrix}0&1\\-1&0\end{bmatrix}$
Circular dichroism	$j2\Delta\kappa\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}$	$\Delta\kappa\begin{bmatrix}0&1\\-1&0\end{bmatrix}$

Table 3.2. Q and R_a matrices for eight basic types of optical behaviour.(from Ref. (35))

properties of Table 3.2, except for linear dichroism along the bisectors of the xy coordinate axes, and circular dichroism. These two properties can be regarded as coupling losses, and in the presence of κ_{loss}, the losslessness condition $R_a = R_a^{\dagger}$ is not satisfied.

The general form of the R_a matrix, where all eight properties of Table 3.2 are included is

$$R_a = \begin{bmatrix} k_1 & \kappa + \kappa_{loss} \\ \kappa^* - \kappa_{loss} & k_2 \end{bmatrix} \qquad (3.2.27)$$

3.3. Nonuniform Couplers

For a nonuniform coupler, the system matrix R_a is not constant but a function of z. A closed form solution to

$$\frac{d\bar{a}(z)}{dz} = -j\, R_a(z)\, \bar{a}(z) \qquad (3.3.1)$$

exists only for some specific system matrices $R_a(z)$. Among the nonuniform system matrices for which a closed form solution to (3.3.1) is available is the so-called tapered and the so-called chirped distribution. These are treated in this section in detail. Other nonuniform couplers have been investigated by Chen and Ishimaru (14), Kogelnik (36), and Milton and Burns (7). Milton and Burns describe a transformation which is suitable to solve tapered nonuniform couplers. Their method is extended here to encompass a larger class of nonuniformities (22).

Starting with the linear transformation

$$\bar{a}(z) = U(z)\, \bar{d}(z) \quad , \qquad (3.3.2)$$

where U(z) is given in (3.2.14), and substituting this into (3.3.1), one obtains

$$\bar{d}(z)' = -j(\, \Lambda_r(z) - jU(z)^{-1}U(z)'\,)\bar{d}(z)$$

$$= -j\Lambda_r(z)\, \bar{d}(z) - U(z)^{-1}\, U(z)'\, \bar{d}(z) \quad , \qquad (3.3.3)$$

where $\Lambda_r(z) = \mathrm{diag}\,(\, k^+(z), k^-(z)\,)$, and the notation used in Section 3.2 is followed. For a uniform coupler, Λ_r and U are constant, hence $U' = 0$. The second term on the RHS of (3.3.3) vanishes, resulting in $\bar{d}(z)' = -j\Lambda_r\, \bar{d}(z)$. Hence for a uniform coupler (3.3.3) decouples the system equation.

In order to reduce the complexity of the coupling matrix, a second transformation is implemented:

$$\bar{d}(z) = H(z)\,\bar{h}(z) \quad , \tag{3.3.4}$$

where $H(z) = \text{diag}\,(\,h_{11}(z), h_{22}(z)\,)$. (3.3.5)

The system equation on the new basis is

$$\bar{h}(z)' = S(z)\,\bar{h}(z) \quad , \tag{3.3.6}$$

where $S(z) = -j\Lambda_r(z) - (\,U(z)H(z)\,)^{-1}\,(\,U(z)H(z))'$

$$= \begin{bmatrix} -jk^{+} - \dfrac{h_{11}(z)'}{h_{11}(z)} - j\tfrac{1}{2}\dfrac{Z}{W}\phi' & -\tfrac{1}{2}\dfrac{h_{22}(z)}{h_{11}(z)}\left(\dfrac{Z'}{W^2} - j\dfrac{\phi'}{W}\right) \\ \pm\tfrac{1}{2}\dfrac{h_{11}(z)}{h_{22}(z)}\left(\dfrac{Z'}{W^2} + j\dfrac{\phi'}{W}\right) & -jk^{-} - \dfrac{h_{22}(z)'}{h_{22}(z)} + j\tfrac{1}{2}\dfrac{Z}{W}\phi' \end{bmatrix}$$

(3.3.7)*

The problem is further simplified by choosing H(z) such that the diagonal elements of S(z) are eliminated. Thus

$$h_{11}(z) = \exp\left(-j\int_0^z \left(k^{+} + \tfrac{1}{2}\frac{Z}{W}\phi'\right) dz\right) \quad , \tag{3.3.8}$$

$$h_{22}(z) = \exp\left(-j\int_0^z \left(k^{-} - \tfrac{1}{2}\frac{Z}{W}\phi'\right) dz\right) \quad , \tag{3.3.9}$$

$$s_{12}(z) = -\frac{h_{22}(z)}{h_{11}(z)}\left(\tfrac{1}{2}\frac{Z'}{W^2} - j\tfrac{1}{2}\frac{\phi'}{W}\right) \quad , \tag{3.3.10}$$

$$s_{21}(z) = \pm\frac{h_{11}(z)}{h_{22}(z)}\left(\tfrac{1}{2}\frac{Z'}{W^2} + j\tfrac{1}{2}\frac{\phi'}{W}\right) \quad , \tag{3.3.11}$$

and $s_{11} = s_{22} = 0$. The S(z) matrix can then be written in terms of a new transformation variable u, where $u(z) = \int_0^z f(z')\,dz'$, for tapered nonuniformities characterized by

$$\frac{Z'}{W^2} = \text{constant, and } \phi' = 0 \ , \tag{3.3.12.a,b}$$

* not to be confused with the scattering matrix.

and for chirped nonuniformities, given by

$$\frac{\phi'}{W} = \text{constant} \ , \text{ and } Z' = 0 \ . \tag{3.3.13.a,b}$$

Whenever (3.3.12) or (3.3.13) is satisfied, S(z) may be written in terms of the new variable u, where f(z) is yet to be determined. One then seeks the solution of (3.3.6) in the form

$$\bar{h}(u) = D(u)\ N(u)\ \bar{h}(0) \ , \tag{3.3.14}$$

where D(u) is a diagonal matrix. Substituting (3.3.14) into (3.3.6),

$$\dot{D}(u)\ N(u) \pm D(u)\ \dot{N}(u) = S(u)\ D(u)\ N(u) \tag{3.3.15}$$

emerges as the condition a solution must satisfy, where the dot represents differentiation with respect to u. Assuming that a closed form solution to (3.3.1) can be found, the transfer matrix of the original basis can then be obtained by succesive back transformations from $\bar{h}(u)$ to $\bar{d}(z)$ to $\bar{a}(z)$, resulting in

$$M(z) = U(z)\ H(z)\ D(z)\ N(z)\ H(0)^{-1}\ U(0)^{-1} \ . \tag{3.3.16}$$

In the following two subsections, tapered and chirped couplers characterized by (3.3.12) and (3.3.13) respectively, are investigated.

3.3.1. Tapered Couplers

Tapered couplers are characterized by a linear variation of the normalized asynchronism parameter, i.e. by (3.3.12). H(z) and S(z) for tapers are

$$H(z) = \begin{bmatrix} e^{-j\int_0^z k^+ dz} & 0 \\ 0 & e^{-j\int_0^z k^- dz} \end{bmatrix} = e^{-j\int_0^z k_0 dz} \begin{bmatrix} e^{-j\frac{1}{2}u} & 0 \\ 0 & e^{j\frac{1}{2}u} \end{bmatrix} \tag{3.3.17}$$

and

$$S(u) = \frac{\gamma}{2} \begin{bmatrix} 0 & -e^{ju} \\ \pm e^{-ju} & 0 \end{bmatrix} , \tag{3.3.18}$$

respectively, where

$$u(z) = \int_0^z 2|\kappa| W \, dz , \tag{3.3.19}$$

$$\text{and } \gamma = \frac{Z'}{W^2} . \tag{3.3.20}$$

Condition (3.3.15) can be satisfied by the choice

$$D(u) = \begin{bmatrix} e^{j\frac{1}{2}u} & 0 \\ 0 & e^{-j\frac{1}{2}u} \end{bmatrix} , \tag{3.3.21}$$

and

$$N(u) = \begin{bmatrix} \cos\frac{1}{2}\Gamma u - j\frac{1}{\Gamma}\sin\frac{1}{2}\Gamma u & -\frac{\gamma}{\Gamma}\sin\frac{1}{2}\Gamma u \\ \pm\frac{\gamma}{\Gamma}\sin\frac{1}{2}\Gamma u & \cos\frac{1}{2}\Gamma u + j\frac{1}{\Gamma}\sin\frac{1}{2}\Gamma u \end{bmatrix} , \tag{3.3.22}$$

$$\text{where } \Gamma^2 = 1 \pm \gamma^2 . \tag{3.3.23}$$

Since $W^2 = Z^2 \pm 1$, (3.3.12 a) yields

$$Z' - \gamma Z^2 \mp \gamma = 0 . \tag{3.3.24}$$

Given the boundary values $Z(0)$ and $Z(\ell)$, the solution for the codirectional (upper sign) taper satisfying (3.3.24) is

$$Z(z) = \tan(\gamma z + \delta) \quad , \tag{3.3.25}$$

where $\delta = \tan^{-1} Z(0)$, (3.3.26)

and $\gamma = \frac{1}{\ell}(\tan^{-1} Z(\ell) - \delta)$. (3.3.27)

Keeping in mind that $Z(z)$ is complex for lossy couplers, i.e. $Z(z) = Z_{re}(z) + jZ_{im}(z)$, using the identity

$$\tan(a + jb) = \frac{\sin 2a + j \sinh 2b}{\cos 2a + \cosh 2b} \quad ; \tag{3.3.28}$$

where a and b are real, one obtains

$$\mathrm{Re}\{\delta\} = \tfrac{1}{2}\left(\sin^{-1}\left\{\frac{2\, Z_{re}(0)}{((1-|Z(0)|^2)^2 + 4Z_{re}(0)^2)^{\frac{1}{2}}}\right\} \pm 2n\Pi\right) \quad , \tag{3.3.29}$$

$$\mathrm{Im}\{\delta\} = \pm\tfrac{1}{2}\left(\cosh^{-1}\left\{1 + \frac{4\, Z_{im}(0)^2}{(1-|Z(0)|^2)^2 + 4Z_{re}(0)^2}\right\}^{\frac{1}{2}}\right) \quad , \tag{3.3.30}$$

where n is a nonnegative integer. The real and imaginary parts of γ can then be evaluated from

$$\mathrm{Re}\{\gamma\} = \frac{1}{\ell}\{\mathrm{Re}(\tan^{-1} Z(\ell)) - \mathrm{Re}(\delta)\} \quad , \tag{3.3.31}$$

and

$$\mathrm{Im}\{\gamma\} = \frac{1}{\ell}\{\mathrm{Im}(\tan^{-1} Z(\ell)) - \mathrm{Im}(\delta)\} \quad , \tag{3.3.32}$$

respectively. The term $\tan^{-1} Z(\ell)$ is obtained from (3.3.29) and (3.3.30) by replacing $Z(0)$ by $Z(\ell)$.

Similarly for contradirectional (lower sign) couplers, the solution for (3.3.24) is given as

$$Z(z) = \tanh(-\gamma z + \delta) \quad . \tag{3.3.33}$$

Following a similar procedure as the foregoing one, γ and δ can be obtained.

The results of this process are:

$$\mathrm{Re}\{\delta\} = \pm\tfrac{1}{2}\left(\cosh^{-1}\left\{1 + \frac{4\,Z_{re}(0)^2}{(1-|Z(0)|^2)^2 + 4Z_{im}(0)^2}\right\}^{\frac{1}{2}}\right) , \quad (3.3.34)$$

$$\mathrm{Im}\{\delta\} = \tfrac{1}{2}\left(\sin^{-1}\left\{\frac{2\,Z_{im}(0)}{((1-|Z(0)|^2)^2 + 4Z_{im}(0)^2)^{\frac{1}{2}}}\right\} \pm 2n\Pi\right) , \quad (3.3.35)$$

$$\mathrm{Re}\{\gamma\} = \frac{1}{\ell}\left(\mathrm{Re}\{\delta\} - \mathrm{Re}\{\tanh^{-1} Z(\ell)\}\right) , \quad (3.3.36)$$

$$\mathrm{Im}\{\gamma\} = \frac{1}{\ell}\left(\mathrm{Im}\{\delta\} - \mathrm{Im}\{\tanh^{-1} Z(\ell)\}\right) , \quad (3.3.37)$$

3.3.2. Chirped Couplers

Chirped couplers are characterized by a linear variation of the normalized phase of the coupling coefficient, expressed by conditions in (3.3.13.a,b). This condition simplifies (3.3.8) - (3.3.11) and as a result

$$H(z) = \begin{bmatrix} e^{-j\int_0^z (k^+ + \frac{Z\phi'}{2W})dz} & 0 \\ 0 & e^{-j\int_0^z (k^- - \frac{Z\phi'}{2W})dz} \end{bmatrix} =$$

$$= e^{-\int_0^z k_0 dz} \begin{bmatrix} e^{-j\frac{1}{2}u} & 0 \\ 0 & e^{j\frac{1}{2}u} \end{bmatrix}, \qquad (3.3.38)$$

and

$$S(u) = \tfrac{1}{2}\gamma \begin{bmatrix} 0 & e^{ju} \\ \pm e^{-ju} & 0 \end{bmatrix}, \qquad (3.3.39)$$

where now $u(z) = \int_0^z (2|\kappa|W + \frac{Z\phi'}{W})dz$ (3.3.40)

and $\gamma = j\,\frac{\phi'}{W}$. (3.3.41)

Choosing D(u) the same as in (3.3.21), one obtains

$$N(u) = \begin{bmatrix} \cos\tfrac{1}{2}\Gamma u - j\frac{1}{\Gamma}\sin\tfrac{1}{2}\Gamma u & \frac{\gamma}{\Gamma}\sin\tfrac{1}{2}\Gamma u \\ \pm\frac{\gamma}{\Gamma}\sin\tfrac{1}{2}\Gamma u & \cos\tfrac{1}{2}\Gamma u + j\frac{\gamma}{\Gamma}\sin\tfrac{1}{2}\Gamma u \end{bmatrix}, \qquad (3.3.42)$$

where $\Gamma^2 = 1 \mp \gamma^2$. (3.3.43)

From (3.3.13.b), Z is constant for chirped nonuniformities. Hence $W = (Z^2 \pm 1)^{\frac{1}{2}}$ is also a constant. From (3.3.13.a) ϕ' is constant, or $\phi(z) = Az + B$ where A, B are real.

For both tapered and chirped couplers, $H(z)\,D(z) = e^{-j\int_0^z k_0 dz}\,E_2$, and $H(0)^{-1} = E_2$. Hence (3.3.16) becomes

$$M(z) = e^{-j\int_0^z k_0 dz}\,U(z)\,N(z)\,U(0)^{-1} = e^{-j\int_0^z k_0 dz}\,M_0(z) \quad . \qquad (3.3.44)$$

For tapered couplers the ~~expanded~~ expression of $M_0(z)$ is

$$M_0(z) = \begin{bmatrix} [p_z p_0 (c - j\frac{1}{\Gamma} s) - p_0 q_z \frac{\gamma}{\Gamma} s + q_0 p_z \frac{\gamma}{\Gamma} s \pm q_0 q_z (c + j\frac{1}{\Gamma} s)] e^{-j\frac{1}{2}(\phi_0 - \phi_z)} & [p_0 q_z (c - j\frac{1}{\Gamma} s) \pm p_0 p_z \frac{\gamma}{\Gamma} s + q_0 q_z \frac{\gamma}{\Gamma} s - q_0 p_z (c + j\frac{1}{\Gamma} s)] e^{-j\frac{1}{2}(\phi_0 + \phi_z)} \\ [\pm p_z q_0 (c - j\frac{1}{\Gamma} s) \mp q_0 q_z \frac{\gamma}{\Gamma} s - p_0 p_z \frac{\gamma}{\Gamma} s \mp p_0 q_z (c + j\frac{1}{\Gamma} s)] e^{j\frac{1}{2}(\phi_0 + \phi_z)} & [\pm q_0 q_z (c - j\frac{1}{\Gamma} s) + q_0 p_z \frac{\gamma}{\Gamma} s - p_0 q_z \frac{\gamma}{\Gamma} s + p_0 p_z (c + j\frac{1}{\Gamma} s)] e^{j\frac{1}{2}(\phi_0 - \phi_z)} \end{bmatrix} \qquad (3.3.45)$$

where $p(z) = (\frac{1}{2}(1 + \frac{Z}{W}))^{\frac{1}{2}}$, $q(z) = (\pm\frac{1}{2}(1 - \frac{Z}{W}))^{\frac{1}{2}}$, $p_0 = p(0)$, $p_z = p(z)$, $q_0 = q(0)$, $q_z = q(z)$, $\phi_0 = \phi(0)$, $\phi_z = \phi(z)$, $c = \cos\frac{1}{2}\Gamma u$, $s = \sin\frac{1}{2}\Gamma u$, and as usual the upper (lower) sign refers to a forward (backward) coupler. The expression of $M_0(z)$ for chirped coupler is identical to (3.3.45) with the exception of a sign change in front of the third term in every one of the square brackets.

3.4. Numerical Results

Using the analysis of the previous section, a computer program has been written to carry out the evaluation of the transfer matrix $M_0(z)$, given in (3.3.45). The nonuniform coupler is then divided into N uniform sections in cascade as shown in Figure 3.3. The transfer matrix of each uniform section is evaluated from (3.2.20). The overall transfer matrix of the coupler can then be approximated by multiplying the transfer matrices of cascaded uniform sections. This is then divided by $e^{jk_0\ell}$, where k_0 is calculated from the approximated values of k_1 and k_2. The resulting matrix is an approximation of $M_0^{\sim}(\ell)$, given in (3.3.45). Some examples of the numerical results are given below.

Example #1. Codirectional, linear tapered coupler ($k_1(z) - k_2(z)$ is constant))

Data input to the program:

$$R_a(0) = \begin{bmatrix} 1.1 & .02 + j.03 \\ .02 - j.03 & 1.3 \end{bmatrix}, \quad R_a(\ell) = \begin{bmatrix} 1.2 & .024 + j.036 \\ .024 - j.036 & 1.4 \end{bmatrix}$$

length: $\ell = 22.5$ (normalized with respect to free space wavelength λ)

Results:

$$M_d(\ell) = \begin{bmatrix} (-.75076 + j.61368) & (.18829 - j.15584) \\ (-.18829 - j.15584) & (-.75076 - j.61368) \end{bmatrix}$$

Approximated by (N = 21) uniform sections, the resulting transfer matrix (without the phase term $\exp(-j\int_0^{\ell} k_0 dz)$ is

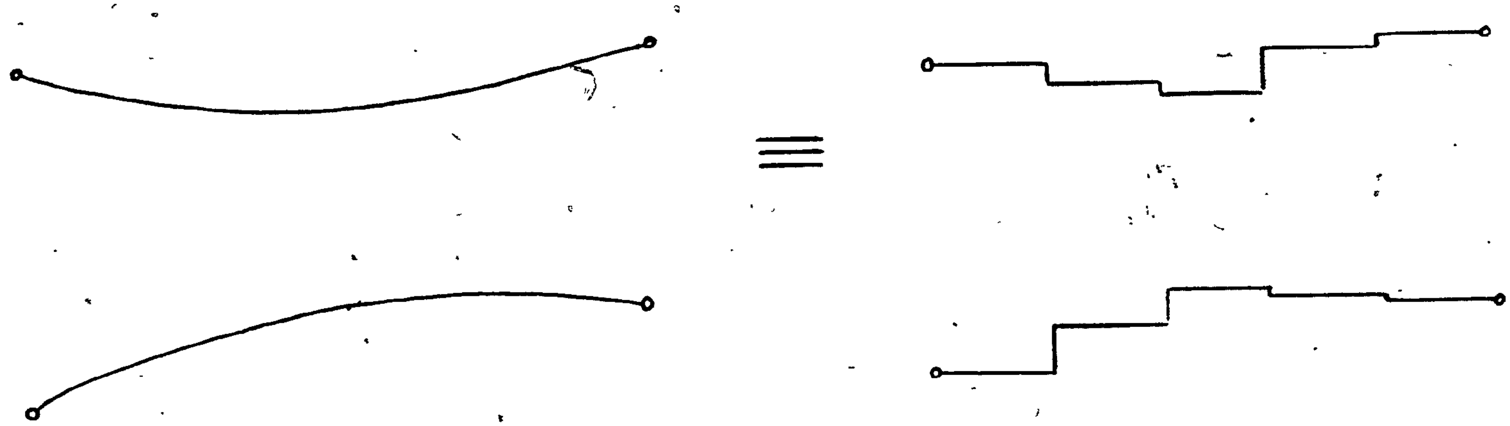

Figure 3.3. Nonuniform coupler approximated by cascaded uniform couplers

$$M_0(\ell) = \begin{bmatrix} (-.75084 + j.61315) & (.18488 - j.16153) \\ (-.18488 - j.16153) & (-.75084 - j.6.315) \end{bmatrix}.$$

Example #2. Codirectional, linear chirped coupler ($\Delta k(z)$, $|\kappa(z)|$ vary linearly.)

Data input to the program:

$$R_a(0) = \begin{bmatrix} 1.1 & .2 + j.2 \\ .2 - j.2 & 1.2 \end{bmatrix}, \quad R_a(\ell) = \begin{bmatrix} 1.0 & .04 + j.04 \\ .04 - j.04 & 1.02 \end{bmatrix}$$

Normalized length: $\ell = 123.5$ (*λ meters)

Results:

$$M_0(\ell) = \begin{bmatrix} (-.75996 + j.11314) & (.45258 - j.45258) \\ (-.45258 - j.45258) & (-.75996 - j.11314) \end{bmatrix}$$

Uniform section approximation with N = 21 results in

$$M_0(\ell) = \begin{bmatrix} (-.75996 + j.11314) & (.45258 - j.45258) \\ (-.45258 - j.45258) & (-.75996 - j.11314) \end{bmatrix}$$

Example #3: Contradirectional, linear tapered ($|\kappa|$ is constant.)

Data input to the program:

$$R_a(0) = \begin{bmatrix} 1.3 & .05 + j.04 \\ -.05 + j.04 & -1.1 \end{bmatrix}, \quad R_a(\ell) = \begin{bmatrix} 1.47 & .05 + j.04 \\ -.05 + j.04 & -1.25 \end{bmatrix}$$

Normalized length: $\ell = 15.5$ (*λ meters)

Results:

$$M_0(\ell) = \begin{bmatrix} (.61085 - j.79275) & (.02362 - j.03209) \\ (.02362 + j.03209) & (.61085 + j.79275) \end{bmatrix}$$

Uniform section approximation with N = 21 results in

$$M_0(\ell) = \begin{bmatrix} (.61887 - j.78649) & (.02623 - j.02978) \\ (.02623 + j.02978) & (.61887 + j.78649) \end{bmatrix}.$$

Example #4. Contradirectional, linear chirped ($\Delta k, |\kappa|$ are constant.)

Data input to the program:

$$R_a(0) = \begin{bmatrix} 1.1 & .02 + j.03 \\ -.02 + j.03 & -1.2 \end{bmatrix}, \; R_a(\ell) = \begin{bmatrix} 1.2 & .03 + j.02 \\ -.03 + j.02 & -1.1 \end{bmatrix}$$

Normalized length: $\ell = 10.5$ (*λ meters)

Results:

$$M_0(\ell) = \begin{bmatrix} (.87888 + j.47744) & (-.01367 + j.01367) \\ (-.01367 - j.01367) & (.87888 - j.47744) \end{bmatrix}$$

Uniform section approximation with N = 21 results in

$$M_0(\ell) = \begin{bmatrix} (.87887 + j.47746) & (-.01441 + j.01441) \\ (-.01441 - j.01441) & (.87887 - j.47746) \end{bmatrix}.$$

In all four examples, $M_0(\ell)$ satisfies the losslessness and the reciprocity condition.

IV. CONCLUSION

A general analysis has been introduced utilizing the principles of the coupled mode formalism and Jones and Mueller calculi. The analysis is applicable to various systems from diverse fields, some of which are cited in Section 2.3.

A tabular summary of various matrix representations of networks is given, including the conversion methods from one type of matrix to any other type. Conditions are given for reciprocity, losslessness, antireciprocity, semireciprocity, bilateral and transversal symmetry. These conditions are stated both in terminal matrix representation and system coupling matrix representation, applicable to distributed systems. Although the analysis is carried out for 2n-port networks using n by n block partitioned matrix representations, there is no obstacle in extending the formalism to the more general case where there are rectangular (nxm) blocks in the off diagonals.

In Chapter 3, codirectional and contradirectional couplers have been investigated, utilizing the methods of Chapter 2. Jones and Mueller calculi are used to describe the state of polarization in a codirectionalcoupler, citing the eight basic properties found in an anisotropic layer. Nonuniformities of specific types have been included and a simple analysis of these, valid for lossless as well as lossy, codirectional and contradirectional couplers is given. Numerical examples of 2 nonuniformly coupled lines are cited at the end of Chapter 3. The results of Chapter 3 are considered helpful in the design of directional couplers in the optical and microwave regime.

REFERENCES

1. Burman, R.,"Coupled Wave Equations for Propagation in Generally Inhomogeneous compressible Magnetoplasmas", Proc. IEEE, May 1967, 723-724.

2. Tien, P., "Parametric Amplification and Frequency Mixing in Propagating Circuits", Jour. Appl. Phys., 29, Sept 1958, 1347 -1357.

3. Pierce, J. R., "Traveling Wave Tubes", D. van Nostrand, Princeton N.J., 1950.

4. Johnson, H. R., "Backward-Wave Oscillators", Proc. Inst. Radio Engrs., 43, 1955, 684-697.

5. Schlaak, H. F.,"Periodic spectral filter with integrated optical directional coupler", Opt. and Quan. Elec., 13, 1981, 684-697.

6. Kogelnik, H., Schmidt, R. V., "Switched Directional Couplers with Alternating $\Delta\beta$",IEEE Jour. Q. E., vol. 12, No.7, 1976, 396 -401.

7. Milton, A. F., Burns, W. K., "Mode Coupling in Tapered Optical Waveguide Structures and Electro-optic Switches", IEEE Trans. Cir. Sys., vol.26, No.12, 1979, 1020-1028.

8. Pieper, R. J., Korpel, A., "Matrix formalism for the analysis of acoustooptic beam steering", Appl. Opt., vol.22, No.24, 1983, 4073-4081.

9. Burman, R., "Wave Propagation in a Stratified Compressible Magnetoplasma with a Static Pressure Gradient", Proc. IEEE, 1967, 1528-1529.

10. Pierce, J. R., "Coupling of Modes of Propagation", Jour. Appl. Phys., vol.25,1954, 179-183.

11. Louisell, W. H., "Coupled Mode and Parametric Electronics", John Wiley, N.Y., 1960.

12. Miller, S. E.,"Coupled Wave Theory and Waveguide Applications", Bell Syst. Tech. J., May 1954, 661-719.

13. Pease, M. C., "Generalized Coupled Mode Theory", Jour. Appl. Phys., vol.32, No.9, 1961, 1736-1743.

14. Chen, Y. S., Ishimaru, A., "On the General Solutions of Coupled -Mode Equations with varying Coefficients", Proc. IEEE, 54, 1966, 1071.

15. Pease, M. C., "Methods of Matrix Algebra", Academic Press, N.Y., 1965.

16. Yariv, A., "Coupled Mode Theory for Guided Wave Optics", IEEE Jour. Q. E., vol.9, No.9, 1973, 919-933.

17. Holt, A. G., Linggard, R., "The Multiterminal Gyrator", Proc. IEEE, Aug 1968, 1354-1355.

18. Carlin, H. J., Giordano, A. B., "Network Theory", Prentice-Hall, N.J., 1964.

19. Pease, M. C., "Conservation Laws of Uniform Linear Homogeneous Systems", Jour. Appl. Phys., vol.31, No.11, 1960, 1988-1996.

20. Arfken, G., "Mathematical Methods for Physicists", 2nd ed., Academic P., N.Y., 1970.

21. Simmons, J. W., Guttmann, M. J., "States, Waves and Photons: A Modern Introduction to Light", Addison-Wesley, Reading, Mass., 1970.

22. Schwelb, O., "Evolution of polarization in codirectional and contradirectional optical couplers", Jour. Opt. Soc. Am., vol.72, No.9, 1982, 1152-1158.

23. Schwelb, O., "Analysis of lossy multichannel uniform couplers", Internal Report, Concordia Un., Oct. 1982.

24. Born, M., Wolf, E., "Principles of Optics", 6th ed., Pergamon, Oxford, 1970, Secs. 1.4 and 10.8.

25. Schwelb, O., Antepyan R., "Properties and Representations of Four-ports", Internal Report, Concordia Un., Jan. 1984.

26. Schwelb, O., Antepyan, R., "Conservation Laws for Integrated Circuit Four-ports", to be published.

27. Schwelb, O., "Conditions for Transfer Matrices of Reciprocal and Lossless 2n-Port Networks", Internal Report, Concordia Un., Oct. 1982.

28. Ou, W.P., "Design Equations for an Interdigitated Directional Coupler", IEEE Trans. MTT, Feb. 1975, 253-255.

29. Schwelb, O., "Network Representation and Transverse Resonance for Layered Anisotropic Dielectric Waveguides", IEEE Trans. MTT, vol.30, No.6, 1982, 899-905.

30. Jones, R. C., "A New Calculus for the Treatment of Optical Systems, I. Description and Discussion of the Calculus", J. Opt. Soc. Am., vol.31, 1941, 488-493.

31. Jones, R. C., "A New Calculus for the Treatment of Optical Systems, II. Proof of Three General Equivalence Theorems", J. Opt. Soc. Am., vol.31, 1941, 493-499.

32. Jones, R. C., "A New Calculus for the Treatment of Optical Systems, III. The Schncke Theory of Optical Activity", J. Opt. Soc. Am., vol.31, 1941, 500-503.

33. Pipes, L. A., "Direct Computation of Transmission Matrices of Electrical Transmission Lines: PartI and II", J. Franklin Inst., vol.281, No.4, 1966, 275-292, 387-405.

34. Jones, R. C., "A New Calculus for the Treatment of Optical Systems, VII. Properties of the N-Matrices", J. Opt. Soc. Am., vol.38, 1948, 671-685.

35. Azzam, R. M. A., "Propagation of partially polarized light through anisotropic media with or without depolarization: A differential 4x4 matrix calculus", J. Opt. Soc. Am., vol.68, No.12, 1978, 1756-1767.

36. Kogelnik, H., "Filter Response of Nonuniform Almost-Periodic Structures", Bell Syst. Tech. J., vol.55, No.1, 1976, 109-126.

APPENDIX I

Properties of V

The matrix V, used in the transformation in (3.1.2) is chosen to be unitary ($V^{-1} = V^{\dagger}$). An additional property of V is obtained by taking the complex conjugate of (3.1.2), which yields

$$\bar{f}(z)^{*} = V^{*}\,\bar{s}(z)^{*} = V^{\dagger -1}\,\bar{s}(z)^{*} \quad , \tag{A.1.1}$$

and using the relation

$$\bar{f}(z)^{*} = P\,\bar{f}(z) \quad , \tag{A.1.2}$$

where P is a permutation matrix ($P = P^{-1}$). Since $\bar{s}(z)$ is real, from (A.1.1) and (A.1.2) one obtains

$$V\,V^{T} = P \ . \tag{A.1.3}$$

Summarizing these results, the properties of V can be written as

$$V\,V^{\dagger} = E_{2n} \quad (V^{-1} = V^{\dagger}) \quad , \tag{A.1.4}$$

$$V\,V^{T} = P = P^{-1} \quad , \tag{A.1.5}$$

$$V^{T}\,V = V^{-1}\,P\,V \neq V\,V^{T} \ . \tag{A.1.6}$$

Using (A.1.4) – (A.1.6), P, V, $\bar{f}(z)$, and $\bar{s}(z)$ will be constructed for a system of size 2 and 4 as specific examples.

(i) $2n = 2$, $\bar{a}(z) = (a_1(z), a_2(z))^{T}$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} , \quad V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & j \\ 1 & -1 & 0 & 0 \end{bmatrix} ,$$

$$\bar{f}(z) = \begin{bmatrix} |a_1|^2 \\ a_1 a_2^* \\ a_2 a_1^* \\ |a_2|^2 \end{bmatrix}, \quad \bar{s}(z) = \begin{bmatrix} |a_1|^2 + |a_2|^2 \\ |a_1|^2 - |a_2|^2 \\ 2\,(a_1 a_2^*)_{re} \\ 2\,(a_1 a_2^*)_{im} \end{bmatrix}.$$

(ii) $2n = 4$, $\bar{a}(z) = (a_1(z), a_2(z), a_3(z), a_4(z))^T$

$$P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$V = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & j & j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & j & j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & j & j \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -j & -j & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & j & 1 & -1 \\
0 & 0 & 0 & 0 & j & j & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -j & -j & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j & -j & 1 & -1 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & j & j & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -j & -j \\
0 & 0 & 0 & 0 & -j & -j & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j & -j & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$\bar{f}(z) = \begin{bmatrix} |a_1|^2 \\ a_1 a_2^* \\ a_1 a_3^* \\ a_1 a_4^* \\ a_2 a_1^* \\ |a_2|^2 \\ a_2 a_3^* \\ a_2 a_4^* \\ a_3 a_1^* \\ a_3 a_2^* \\ |a_3|^2 \\ a_3 a_4^* \\ a_4 a_1^* \\ a_4 a_2^* \\ a_4 a_3^* \\ |a_4|^2 \end{bmatrix} , \quad \bar{s}(z) = \frac{1}{2} \begin{bmatrix} |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2 \\ |a_1|^2 - |a_2|^2 + |a_3|^2 - |a_4|^2 \\ |a_1|^2 + |a_2|^2 - |a_3|^2 - |a_4|^2 \\ -|a_1|^2 + |a_2|^2 + |a_3|^2 - |a_4|^2 \\ 2\,(a_1 a_3^*)_{re} + 2\,(a_2 a_4^*)_{im} \\ -2\,(a_1 a_3^*)_{re} + 2\,(a_2 a_4^*)_{im} \\ 2\,(a_1 a_3^*)_{im} + 2\,(a_2 a_4^*)_{re} \\ 2\,(a_1 a_3^*)_{im} - 2\,(a_2 a_4^*)_{re} \\ 2\,(a_1 a_2^*)_{re} + 2\,(a_3 a_4^*)_{im} \\ -2\,(a_1 a_2^*)_{re} + 2\,(a_3 a_4^*)_{im} \\ 2\,(a_1 a_2^*)_{im} + 2\,(a_3 a_4^*)_{re} \\ 2\,(a_1 a_2^*)_{im} - 2\,(a_3 a_4^*)_{re} \\ 2\,(a_1 a_4^*)_{re} + 2\,(a_2 a_3^*)_{im} \\ -2\,(a_1 a_4^*)_{re} + 2\,(a_2 a_3^*)_{im} \\ 2\,(a_1 a_4^*)_{im} + 2\,(a_2 a_3^*)_{re} \\ 2\,(a_1 a_4^*)_{im} - 2\,(a_2 a_3^*)_{re} \end{bmatrix}$$

It should be noted that V chosen according to (A.1. 4) – (A.1.6) is not unique. The choice of V must be such that the resulting generalized Stokes vector contains useful parameters of the system, such as the sums and the differences of modal powers, phase differences between the amplitudes, reflection coefficients, etc.. For $2n = 2$ case the interpretation of the Stokes vector can be found in references (22) and (24).

APPENDIX II

Matrix Exponentials and the Power of Certain Transfer Matrices

Let K be an involutory square matrix, i.e. $K^2 = E$. Expanding the matrix exponential exp (αK), where α is a scalar, and making use of the involutory property of K results in

$$\exp(\alpha K) = \left(1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \ldots\right) E + \left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \ldots\right) K$$

$$= \cosh(\alpha) E + \sinh(\alpha) K . \qquad (A.2.1)$$

For $\alpha = j\theta$ one obtains $\exp(j\theta K) = \cos(\theta E) + j \sin(\theta K)$.

Transfer matrices are often expressible in this form. When a network or a device consists of N cascaded unit cells, the transfer matrix of the entire system is given by T^N, where T is the transfer matrix of the unit cell. Then, if T is of the form of (A.2.1),

$$T^N = \exp(N\alpha K) = \cosh(N\alpha) E + \sinh(N\alpha) K . \qquad (A.2.2)$$

Table A.1 lists several noteworthy examples.

If T is not of the form of (A.2.1), other methods can be used. One simple method consists of converting N to its binary equivalent and taking the power of T to 2^n, where n is the largest power of 2 that is still less than N. This method is best illustrated by an example. To find T^{200} one first expresses 200 as the sum of the powers of two:

$$200 = 2^7 + 2^6 + 2^3 = 128 + 64 + 8$$

and then generates the 128^{th} power of T by following the sequence

$$T\,T = T^2 ,\ T^2\,T^2 = T^4 ,\ \ldots$$

K	T	α
$(Z^2\pm1)^{-\frac{1}{2}}\begin{bmatrix} Z & e^{j\phi} \\ \pm e^{-j\phi} & -Z \end{bmatrix}$	$\cos\gamma\ell\ E \pm j\ \sin\gamma\ell\ K$	$\pm j\gamma\ell$
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \exp(\alpha) & 0 & 0 & 0 \\ 0 & \cosh\alpha & \sinh\alpha & 0 \\ 0 & \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 0 & \exp(\alpha) \end{bmatrix}$	α
$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & \cosh\alpha & \sinh\alpha \\ 0 & 0 & \sinh\alpha & \cosh\alpha \end{bmatrix}$	α
$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \cosh\alpha & 0 & \sinh\alpha & 0 \\ 0 & \cosh\alpha & 0 & \sinh\alpha \\ \sinh\alpha & 0 & \cosh\alpha & 0 \\ 0 & \sinh\alpha & 0 & \cosh\alpha \end{bmatrix}$	α
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} \exp(\alpha) & 0 & 0 & 0 \\ 0 & \exp(\alpha) & 0 & 0 \\ 0 & 0 & \exp(-\alpha) & 0 \\ 0 & 0 & 0 & \exp(-\alpha) \end{bmatrix}$	α
$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} \cos\psi + j\sin\psi & 0 \\ 0 & \cos\psi - j\sin\psi \end{bmatrix}$	$j\psi$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} \cos\psi & -j\sin\psi \\ -j\sin\psi & \cos\psi \end{bmatrix}$	$-j\psi$
$\begin{bmatrix} 0 & Z \\ Z^{-1} & 0 \end{bmatrix}$	$\begin{bmatrix} \cos\beta z & -j\sin\beta z \\ -jZ^{-1}\sin\beta z & \cos\beta z \end{bmatrix}$	$-j\beta z$
$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & \exp(\alpha) \end{bmatrix}$	α

Table A.1. Examples of matrices expressible in exponential form of (A.2.1).

seven consequtive times. During this process the program is ordered to store the result of the 6-fold and the 3-fold results, and to finally multiply the 7-fold, 6-fold, and 3-fold results. The number of matrix multiplications required to obtain T^{200} is thus only 7 + 1 + 1 = 9.

Subroutine MPOWER evaluates T^N using the latter method.

```
      SUBROUTINE EQT (A,B)
      COMPLEX A(4,4),B(4,4)
      DO 10 I=1,4
      DO 10 J=1,4
      B(I,J)=A(J,I)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE CMPLPR (N,A,B,PROD)
      COMPLEX A(N,N),B(N,N),PROD(N,N)
      DO 10 I=1,N
      DO 10 K=1,N
      PROD(I,K)=0.0
      DO 10 J=1,N
      PROD(I,K)=PROD(I,K)+A(I,J)*B(J,K)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MPOWER (E,N)
      INTEGER M(10)
      COMPLEX A(4,4),B(4,4),C(4,4),D(4,4,10),E(4,4)
      NN=N
      CALL EQT (E,A)
      DO 10 I=1,10
      DO 20 J=1,10
      K=NN-2**(J-1)
      IF (K.LT.0) THEN
      M(I)=J-2
      NN=NN-2**M(I)
      GO TO 10
      ELSE IF (K.EQ.0) THEN
      M(I)=J-1
      II=I
      GO TO 30
      ENDIF
   20 CONTINUE
   10 CONTINUE
   30 CALL EQT (A,B)
      PRINT *,' THE VALUES OF M ARE'
      PRINT *,(M(IJK),IJK=1,II)
      K=1
      IMI1=M(1)
      DO 50 I=1,IMI1
      CALL CMPLPR (4,A,B,C)
       DO 60 J=1,II
      IF (I.EQ.M(J)) THEN
      PRINT *,' K IN 60 =',K
      CALL EQT3D (C,D,K,1)
      K=K+1
      ENDIF
   60 CONTINUE
      CALL EQT (C,A)
      CALL EQT (C,B)
   50 CONTINUE
      IF (M(II).EQ.0) THEN
      PRINT *,' K OUT OF 50=',K
      CALL EQT3D (E,D,K,1)
      ENDIF
      CALL EQT3D (A,D,1,2)
      CALL EQT3D (B,D,2,2)
      DO 70 I=3,II+1
      CALL CMPLPR (4,A,B,C)
      CALL EQT3D (A,D,I,2)
      CALL EQT (C,B)
   70 CONTINUE
      PRINT *,' THE ',N,'TH POWER OF THE MATRIX IS'
      PRINT 404,((C(I,J),J=1,4),I=1,4)
  404 FORMAT(4(' (',G13.7,' , ',G13.7,')',1X))
      RETURN
      END

      SUBROUTINE EQT3D (A,D,K,II)
      COMPLEX A(4,4),D(4,4,10)
      DO 10 I=1,4
      DO 10 J=1,4
      IF (II.EQ.1) THEN
      D(I,J,K)=A(I,J)
      ELSE IF (II.EQ.2) THEN
      A(I,J)=D(I,J,K)
      ENDIF
   10 CONTINUE
      RETURN
      END
```

APPENDIX III

Program CONVRT4

Program CONVRT4 converts a 4x4 complex matrix of a given type into any other desired type within Table 2.1. It also has built-in tests for various conservation laws concerning four-port networks. The program is written in FORTRAN5.

The program is mostly self explanatory and can easily be used following a brief study. When it is run from a computer terminal interactively, the program describes to the user the required input data and the sequence in which it will accept it. Free formatted input makes the interactive use of the program simple and easy.

The input parameters for CONVRT4 are listed below.

NTIMES - An integer specifying the number of matrices to be converted.

FORMOUT - A character variable of size 1. When FORMOUT = 'Y', the matrix to be converted will be put into a format acceptable as input to the program.

ITYPE - An integer specifying the type of the given input matrix.

ICONVRT - An integer specifying the type of the desired output matrix. Parameter convention for matrix type is as follows.

Matrix type	ITYPE or ICONVRT
Impedance (Z)	1
Admittance (Y)	2
ABCD (Q)	3
Scattering (S)	4
Transfer (T)	5
Transfer (A)	6
Transfer (M)	7

ZYQSTAM - A 4x4 complex input matrix. Complex elements must be entered rowwise, real part preceding the imaginary part for each element. ZYQSTAM has therefore 32 real entries.

Z1 ,Z2 - The characteristic impedances for lines 1 and 2. As before, first the real then the imaginary part must be entered.

ISIGDIG - An integer specifying the number of significant digits used in subroutine CONSRV. If the tested matrix satisfies the test condition to ISIGDIG digits the test is passed, otherwise rejected.

Note: 'EOR' (end-of-record) must be entered at the end of each example.

Subroutine CONSRV tests the validity of five conditions. These are: reciprocity, bilateral symmetry, losslessness, semireciprocity, and antireciprocity. Both the input and output matrix is tested.

CONVRT4 uses the subroutine LEQ2C from the IMSL library package to evaluate the (high accuracy) inverse of a 4x4 matrix. However, two other 4x4 matrix inversion routines are included (INVERS4,INVERD4) which can be used whenever the IMSL library package is not available. Both of these routines evaluate the inverse using the partitioned form given as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

The only difference between these two routines is that INVERD4 uses double precision in the evaluation of the inverse of a 2x2 matrix. Both routines have given satisfactory results.

```fortran
      PROGRAM CONVRT4 (INPUT,OUTPUT)
      COMPLEX ZYGSTAM(4,4),MATSGYZ(4,4),OMEGA(4,4),OMEGA23(4,4),
     * MATRIX1(4,4),MATRIX2(4,4),Z1,Z2
      CHARACTER SIGN*3,FORMOUT*1
      COMMON IDET2
      COMMON /LINEIMP/Z1,Z2
      COMMON /TOLER/ISIGDIG
C-----------------------------------------------------------
C   INPUT FOUR BY FOUR COMPLEX MATRIX AND THE PARAMETERS
C   ICONVRT,ITYPE= 1 FOR Z
C                  2 FOR Y
C                  3 FOR G (OR ABCD)
C                  4 FOR S
C                  5 FOR T
C                  6 FOR A
C                  7 FOR M MATRICES.
C-----------------------------------------------------------
      PRINT *,' INPUT NUMBER OF EXAMPLES TO BE EVALUATED.'
      READ *,NTIMES
      PRINT *,' IF THE OUTPUT MATRIX IS DESIRED TO '
      PRINT *,' BE PUT IN AN INPUT FORMAT, TYPE "Y"'
      PRINT *,' ELSE TYPE "N".'
      READ 400,FORMOUT
      DO 9999 ITIMES=1,NTIMES
      DO 10 I=1,4
      DO 10 J=1,4
      OMEGA(I,J)=0.0
   10 CONTINUE
      PRINT *,' TYPE OF THE INPUT MATRIX'
      PRINT *,' INPUT 1 FOR Z,'
      PRINT *,'       2 FOR Y'
      PRINT *,'       3 FOR G (OR ABCD)'
      PRINT *,'       4 FOR S'
      PRINT *,'       5 FOR T'
      PRINT *,'       6 FOR A'
      PRINT *,'       7 FOR M MATRICES.'
      READ *,ITYPE
      PRINT *,' INPUT DESIRED TYPE TO BE CONVERTED.'
      PRINT *,' USE THE SAME CODE AS ABOVE'
      READ *,ICONVRT
      PRINT *,' INPUT THE ENTRIES OF THE COMPLEX 4X4'
      PRINT *,' MATRIX. ENTER ROWWISE FIRST THE REAL THAN THE'
      PRINT *,' IMAGINARY PART OF EACH ENTRY '
      DO 20 I=1,4
      DO 20 J=1,4
      READ *,ARE
      READ *,AIM
      ZYGSTAM(I,J)=CMPLX(ARE,AIM)
      MATRIX1(I,J)=ZYGSTAM(I,J)
   20 CONTINUE
      PRINT *,' INPUT THE DESIRED SIGNIFICANT '
      PRINT *,' DIGITS OF ACCURACY TO BE USED IN CONSRV'
      READ *,ISIGDIG
      PRINT *,ISIGDIG
      IF (ICONVRT.EQ.ITYPE) THEN
      PRINT *,' CONVERSION FROM ONE TO THE SAME TYPE'
      GO TO 9999
      ENDIF
      IF (((ITYPE.LE.3).AND.(ICONVRT.GE.4)).OR.
     * ((ITYPE.GE.4).AND.(ICONVRT.LE.3))) THEN
      PRINT *,' THE CHARACTERISTIC IMPEDANCES FOR LINES 1'
      PRINT *,' AND 2 FOR THE EVALUATION OF THE OMEGA '
      PRINT *,' MATRIX. ENTER FIRST THE REAL THEN THE '
      PRINT *,' IMAGINARY PARTS '
      READ *,ARE
      READ *,AIM
      Z1=CMPLX(ARE,AIM)
      PRINT *,' Z1=',Z1
      READ *,ARE
      READ *,AIM
      Z2=CMPLX(ARE,AIM)
      PRINT *,' Z2=',Z2
      OMEGA(1,1)=CSQRT(Z1/2.0)
      OMEGA(1,2)=OMEGA(1,1)
      OMEGA(2,1)=0.5/OMEGA(1,1)
      OMEGA(2,2)=-OMEGA(2,1)
      OMEGA(3,3)=CSQRT(Z2/2.0)
      OMEGA(3,4)=OMEGA(3,3)
      OMEGA(4,3)=0.5/OMEGA(3,3)
      OMEGA(4,4)=-OMEGA(4,3)
      CALL TILDE (OMEGA,OMEGA23)
C*    PRINT *,((OMEGA(I,J),J=1,4),I=1,4)
C*    PRINT *,((OMEGA23(I,J),J=1,4),I=1,4)
      ENDIF
      IF (ITYPE.EQ.1) THEN
      PRINT *,' THE INPUT IMPEDANCE MATRIX IS'
      PRINT 404,((ZYGSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.2) THEN
      CALL INVRS4 (ZYGSTAM,MATSGYZ)
C*    CALL INVERS4 (MATRIX1,MATRIX2)
C*    PRINT 404,((MATRIX2(I,J),J=1,4),I=1,4)
C*    CALL INVERD4 (MATRIX1,MATRIX2)
C*    PRINT 404,((MATRIX2(I,J),J=1,4),I=1,4)
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((MATSGYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.3) THEN
      CALL ZTOABCD (ZYGSTAM,MATSGYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING G (OR ABCD) MATRIX IS'
      PRINT 404,((MATSGYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL ZTOABCD (ZYGSTAM,MATSGYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSGYZ,ZYGSTAM,OMEGA23)
      CALL TTOS (ZYGSTAM,MATSGYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((MATSGYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL ZTOABCD (ZYGSTAM,MATSGYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSGYZ,ZYGSTAM,OMEGA23)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX IS'
      PRINT 404,((ZYGSTAM(I,J),J=1,4),I=1,4)
```

```
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.6) THEN
      CALL ZTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL TILDE (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL ZTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL TILDE (ZYQSTAM,MATSQYZ)
      CALL INVRS4 (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ENDIF
      ELSE IF (ITYPE.EQ.2) THEN
      PRINT *,' THE INPUT ADMITTANCE MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.3) THEN
      CALL YTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING G (OR ABCD) MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL YTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' CHECK 1 IN THE MAIN'
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL TTOS (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL YTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.6) THEN
      CALL YTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL TILDE (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL YTOABCD (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL ABCDTOT (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL TILDE (ZYQSTAM,MATSQYZ)
      CALL INVRS4 (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ENDIF
      ELSE IF (ITYPE.EQ.3) THEN
      PRINT *,' THE INPUT G (OR ABCD) MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL ABCDTOZ (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.2) THEN
      CALL ABCDTOY (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL ABCDTOT (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL TTOS (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL ABCDTOT (ZYQSTAM,MATSQYZ,OMEGA23)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX T IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.6) THEN
      CALL ABCDTOT (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL TILDE (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL ABCDTOT (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL TILDE (MATSQYZ,ZYQSTAM)
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ENDIF
      ELSE IF (ITYPE.EQ.4) THEN
      PRINT *,' THE INPUT SCATTERING MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL TTOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL ABCDTOZ (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.2) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
```

```
      CALL TTOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL ABCDTOY (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.3) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL TTOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      PRINT *,' THE CORRESPONDING Q (OR ABCD) MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.6) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL TILDE (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL STOT (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      CALL TILDE (MATSQYZ,ZYQSTAM)
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ENDIF
      ELSE IF (ITYPE.EQ.5) THEN
      PRINT *,' THE INPUT TRANSFER MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL TTOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL ABCDTOZ (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.2) THEN
      CALL TTOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL ABCDTOY (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.3) THEN
      CALL TTOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      PRINT *,' THE CORRESPONDING Q (OR ABCD) MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL TTOS (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.6) THEN
      CALL TILDE (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL TILDE (ZYQSTAM,MATSQYZ)
      CALL INVRS4 (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ENDIF
      ELSE IF (ITYPE.EQ.6) THEN
      PRINT *,' THE INPUT TRANSFER MATRIX A IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL ATOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL ABCDTOZ (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.2) THEN
      CALL ATOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      CALL ABCDTOY (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.3) THEN
      CALL ATOABCD (ZYQSTAM,MATSQYZ,OMEGA23)
      PRINT *,' THE CORRESPONDING Q (OR ABCD) MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL TILDE (ZYQSTAM,MATSQYZ)
      CALL TTOS (MATSQYZ,ZYQSTAM)
      IF (IDET2.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL TILDE (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.7) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX M IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ENDIF
      ELSE IF (ITYPE.EQ.7) THEN
      PRINT *,' THE INPUT TRANSFER MATRIX M IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      IF (ICONVRT.EQ.1) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      CALL ATOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL ABCDTOZ (ZYQSTAM,MATSQYZ)
      IF (IDET2.EQ.1) GO TO 9999
```

```
      PRINT *,' THE CORRESPONDING IMPEDANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.2) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      CALL ATOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      CALL ABCDTOY (ZYQSTAM,MATSQYZ)
      IF (IDETZ.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING ADMITTANCE MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ 3) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      CALL ATOABCD (MATSQYZ,ZYQSTAM,OMEGA23)
      PRINT *,' THE CORRESPONDING Q (OR ABCD) MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ.4) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      CALL TILDE (MATSQYZ,ZYQSTAM)
      CALL TTOS (ZYQSTAM,MATSQYZ)
      IF (IDETZ.EQ.1) GO TO 9999
      PRINT *,' THE CORRESPONDING SCATTERING MATRIX IS'
      PRINT 404,((MATSQYZ(I,J),J=1,4),I=1,4)
      ELSE IF (ICONVRT.EQ.5) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      CALL TILDE (MATSQYZ,ZYQSTAM)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX IS'
      PRINT 404,((ZYQSTAM(I,J),J=1,4),I=1,4)
      CALL EQT (ZYQSTAM,MATSQYZ)
      ELSE IF (ICONVRT.EQ 6) THEN
      CALL INVRS4 (ZYQSTAM,MATSQYZ)
      PRINT *,' THE CORRESPONDING TRANSFER MATRIX A IS'
      PRINT 404 ,((MATSQYZ(I,J),J=1,4),I=1,4)
      ENDIF
      ENDIF
      CALL EQT (MATSQYZ,MATRIX2)
      IF (FORMOUT.EQ 'Y') THEN
      CALL DATFORM (ICONVRT,ITYPE,MATRIX2)
      STOP
      ENDIF
      CALL CONSRV(MATRIX1,MATRIX2,ITYPE,ICONVRT)
  406 READ 405,SIGN
      IF (SIGN NE 'EOR') GO TO 406
 9999 CONTINUE
  403 FORMAT (A1)
  404 FORMAT(4(' (',G13.7,' , ',G13 7,')',1X))
  405 FORMAT (A3)
      STOP
      END

      SUBROUTINE TILDE (A,B)
      COMPLEX P(4,4),A(4,4),PROD(4,4),B(4,4),
     * AA(4,4),BB(4,4)
      DO 10 I=1,4
      DO 10 J=1,4
      P(I,J)=0.0
   10 CONTINUE
      P(1,1)=1.0
      P(2,3)=1.0
      P(3,2)=1.0
      P(4,4)=1.0
      CALL CMPLPR (4,P,A,PROD)
      CALL CMPLPR (4,PROD,P,B)
      RETURN
      END

      SUBROUTINE INVRS4 (A,B)
      COMPLEX A(4,4),B(4,4),WA(24),WKC(4)
      DO 10 I=1,4
      DO 10 J=1,4
      IF (J.EQ I) THEN
      B(I,J)=1 0
      ELSE
      B(I,J)=0 0
      ENDIF
   10 CONTINUE
      CALL LEQ2C (A,4,4,B,4,4,0,WA,WKC,IER)
      RETURN
      END

      SUBROUTINE ZTOABCD (Z,Q)
      COMPLEX Z(4,4),Q(4,4),PIN11(2,2),PIN12(2,2),
     * PIN21(2,2),PIN22(2,2),P21INV(2,2),
     * POUT11(2,2),POUT12(2,2),POUT21(2,2),POUT22(2,2)
      CALL PARTITN (Z,PIN11,PIN12,PIN21,PIN22,1)
      CALL INVRS2(PIN21,P21INV)
      CALL CMPLPR (2,PIN11,P21INV,POUT11)
      CALL CMPLPR (2,P21INV,PIN22,POUT22)
      CALL CMPLPR (2,POUT11,PIN22,POUT12)
      DO 10 I=1,2
      DO 10 J=1,2
      POUT21(I,J)=P21INV(I,J)
      POUT12(I,J)=POUT12(I,J)-PIN12(I,J)
   10 CONTINUE
      CALL PARTITN (Q,POUT11,POUT12,POUT21,POUT22,2)
      RETURN
      END

      SUBROUTINE PARTITN (A4BY4,A11,A12,A21,A22,IFLAG)
      COMPLEX A4BY4(4,4),A11(2,2),A12(2,2),A21(2,2),
     * A22(2,2)
      IF (IFLAG EQ.1) THEN
      DO 10 I=1,2
      DO 10 J=1,2
      A11(I,J)=A4BY4(I,J)
      A12(I,J)=A4BY4(I,J+2)
      A21(I,J)=A4BY4(I+2,J)
      A22(I,J)=A4BY4(I+2,J+2)
   10 CONTINUE
      ELSE IF (IFLAG EQ.2) THEN
      DO 20 I=1,2
      DO 20 J=1,2
      A4BY4(I,J)=A11(I,J)
      A4BY4(I,J+2)=A12(I,J)
      A4BY4(I+2,J)=A21(I,J)
      A4BY4(I+2,J+2)=A22(I,J)
   20 CONTINUE
      ENDIF
      RETURN
      END
```

```
      SUBROUTINE ABCDTOT (ABCD,T,OMEGA)
      COMPLEX ABCD(4,4),T(4,4),OMEGA(4,4),OMINV(4,4),
     $ PROD(4,4)
      CALL INVRS4 (OMEGA,OMINV)
      CALL CMPLPR (4,OMINV,ABCD,PROD)
      CALL CMPLPR (4,PROD,OMEGA,T)
      RETURN
      END

      SUBROUTINE TTOS (T,S)
      COMPLEX T(4,4),S(4,4),T11(2,2),T12(2,2),T21(2,2),
     $ T22(2,2),S11(2,2),S12(2,2),S21(2,2),S22(2,2)
      CALL PARTITN (T,T11,T12,T21,T22,1)
      CALL INVRS2 (T11,S21)
      CALL CMPLPR (2,T21,S21,S11)
      CALL CMPLPR (2,S21,T12,S22)
      CALL CMPLPR (2,S11,T12,S12)
      DO 10 I=1,2
      DO 10 J=1,2
      S22(I,J)=-S22(I,J)
      S12(I,J)=T22(I,J)-S12(I,J)
   10 CONTINUE
      CALL PARTITN (S,S11,S12,S21,S22,2)
      RETURN
      END

      SUBROUTINE YTOABCD (Y,Q)
      COMPLEX Y(4,4),Q(4,4),PIN11(2,2),PIN12(2,2),PIN21(2,2),
     $ PIN22(2,2),POUT11(2,2),POUT12(2,2),POUT21(2,2),
     $ POUT22(2,2)
      CALL PARTITN (Y,PIN11,PIN12,PIN21,PIN22,1)
      PRINT *,((PIN11(I,J),J=1,2),I=1,2)
      PRINT *,((PIN12(I,J),J=1,2),I=1,2)
      PRINT *,((PIN21(I,J),J=1,2),I=1,2)
      PRINT *,((PIN22(I,J),J=1,2),I=1,2)
      CALL INVRS2 (PIN21,POUT12)
      CALL CMPLPR (2,POUT12,PIN22,POUT11)
      CALL CMPLPR (2,PIN11,POUT12,POUT22)
      CALL CMPLPR (2,POUT22,PIN22,POUT21)
      DO 10 I=1,2
      DO 10 J=1,2
      POUT21(I,J)=PIN12(I,J)-POUT21(I,J)
      POUT11(I,J)=-POUT11(I,J)
      POUT12(I,J)=-POUT12(I,J)
      POUT22(I,J)=-POUT22(I,J)
   10 CONTINUE
      CALL PARTITN (Q,POUT11,POUT12,POUT21,POUT22,2)
      RETURN
      END

      SUBROUTINE ABCDTOZ (Q,Z)
      COMPLEX Q(4,4),Z(4,4)
      CALL ZTOABCD (Q,Z)
      RETURN
      END

      SUBROUTINE ABCDTOY (Q,Y)
      COMPLEX Q(4,4),Y(4,4),PIN11(2,2),PIN12(2,2),
     $ PIN21(2,2),PIN22(2,2),POUT11(2,2),POUT12(2,2),
     $ POUT21(2,2),POUT22(2,2)
      CALL PARTITN (Q,PIN11,PIN12,PIN21,PIN22,1)
      CALL INVRS2 (PIN12,POUT21)
      CALL CMPLPR (2,PIN22,POUT21,POUT11)
      CALL CMPLPR (2,POUT21,PIN11,POUT22)
      CALL CMPLPR (2,POUT11,PIN11,POUT12)
      DO 10 I=1,2
      DO 10 J=1,2
      POUT12(I,J)=PIN21(I,J)-POUT12(I,J)
      POUT21(I,J)=-POUT21(I,J)
   10 CONTINUE
      CALL PARTITN (Y,POUT11,POUT12,POUT21,POUT22,2)
      RETURN
      END

      SUBROUTINE STOT (S,T)
      COMPLEX S(4,4),T(4,4),S11(2,2),S12(2,2),S21(2,2),
     $ S22(2,2),T11(2,2),T12(2,2),T21(2,2),T22(2,2)
      CALL PARTITN (S,S11,S12,S21,S22,1)
      CALL INVRS2 (S21,T11)
      CALL CMPLPR (2,T11,S22,T12)
      CALL CMPLPR (2,S11,T11,T21)
      CALL CMPLPR (2,T21,S22,T22)
      DO 10 I=1,2
      DO 10 J=1,2
      T12(I,J)=-T12(I,J)
      T22(I,J)=S12(I,J)-T22(I,J)
   10 CONTINUE
      CALL PARTITN (T,T11,T12,T21,T22,2)
      RETURN
      END

      SUBROUTINE ATOABCD (A,Q,OMEGA)
      COMPLEX A(4,4),Q(4,4),OMEGA(4,4),OMINV(4,4),
     $ ATILDE(4,4),PROD(4,4)
      CALL TILDE (A,ATILDE)
      CALL INVRS4 (OMEGA,OMINV)
      CALL CMPLPR (4,OMEGA,ATILDE,PROD)
      CALL CMPLPR (4,PROD,OMINV,Q)
      RETURN
      END

      SUBROUTINE TTOABCD (T,ABCD,OMEGA)
      COMPLEX T(4,4),ABCD(4,4),OMEGA(4,4),OMINV(4,4),PROD(4,4)
      CALL INVRS4 (OMEGA,OMINV)
      CALL CMPLPR (4,T,OMINV,PROD)
      CALL CMPLPR (4,OMEGA,PROD,ABCD)
      RETURN
      END

      SUBROUTINE EOT (A,B)
      COMPLEX A(4,4),B(4,4)
      DO 10 I=1,4
      DO 10 J=1,4
      B(I,J)=A(I,J)
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE INVRS2(A,B)
      COMPLEX A(2,2),B(2,2),DET
      COMMON IDET2
      IDET2=0
      DET=A(1,1)*A(2,2)-A(1,2)*A(2,1)
      IF (CABS(DET).LT.1.0E-14) THEN
      PRINT *,' INVERSE OF 2 BY 2 MATRIX DOES NOT EXIST'
      IDET2=1
      GO TO 9999
      ENDIF
      B(1,1)=A(2,2)/DET
      B(2,2)=A(1,1)/DET
      B(1,2)=-A(1,2)/DET
      B(2,1)=-A(2,1)/DET
 9999 RETURN
      END

      SUBROUTINE CONSRV (MATRIX1,MATRIX2,ITYPE,ICONVRT)
      CHARACTER FIRST*7,SECOND*7
      COMPLEX ETA1(4,4),ETA2(4,4),ETA3(4,4),THETA1(4,4),
     * THETA2(4,4),THETA3(4,4),MATRIX1(4,4),MATRIX2(4,4),
     * I4(4,4),TRANS(4,4),PROD1(4,4),PROD2(4,4),TESTMAT(4,4),
     * MINUS1,PROD3(4,4),S3XS1(4,4)
      DATA FIRST/'FIRST '/
      DATA SECOND/'SECOND'/
      MINUS1=-1.0
      DO 10 I=1,4
      DO 10 J=1,4
      IF (J.EQ.I) THEN
      I4(I,J)=1.0
      ELSE
      I4(I,J)=0.0
      ENDIF
      ETA1(I,J)=ETA2(I,J)=ETA3(I,J)=0.0
      THETA1(I,J)=THETA2(I,J)=THETA3(I,J)=0.0
      S3XS1(I,J)=0.0
   10 CONTINUE
      ETA1(1,1)=ETA1(3,3)=THETA1(1,1)=THETA1(2,2)=1.0
      ETA1(2,2)=ETA1(4,4)=THETA1(3,3)=THETA1(4,4)=-1.0
      THETA2(1,3)=THETA2(2,4)=THETA2(3,1)=THETA2(4,2)=1.0
      ETA2(1,2)=ETA2(2,1)=ETA2(3,4)=ETA2(4,3)=1.0
      ETA3(1,2)=THETA3(1,3)=S3XS1(1,3)=S3XS1(4,2)=CMPLX(0.0,-1.0)
      ETA3(2,1)=THETA3(3,1)=S3XS1(2,4)=S3XS1(3,1)=CMPLX(0.0,1.0)
      ETA3(3,4)=ETA3(1,2)
      ETA3(4,3)=ETA3(2,1)
      THETA3(2,4)=THETA3(1,3)
      THETA3(4,2)=THETA3(3,1)
C---------------------------------------------------
C  TESTING FOR RECIPROCITY
C---------------------------------------------------
      II=ITYPE
      CALL EQT (MATRIX1,TESTMAT)
      GO TO 12
   11 II=ICONVRT
      CALL EQT (MATRIX2,TESTMAT)
   12 IF ((II.EQ.1).OR.(II.EQ.2).OR.(II.EQ.4)) THEN
      CALL TRNSPOS (TESTMAT,TRANS)
      CALL MATDIFF (TESTMAT,TRANS,IF1)
      IF (II.EQ.ITYPE) THEN
      PRINT 20,FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 20,SECOND
      ENDIF
      ELSE IF ((II.EQ.3).OR.(II.EQ.5)) THEN
      CALL CMPLPR (4,TESTMAT,THETA3,PROD1)
      CALL TRNSPOS (TESTMAT,TRANS)
      CALL CMPLPR (4,TRANS,THETA3,PROD2)
      CALL CMPLPR (4,PROD1,PROD2,TESTMAT)
      CALL MATDIFF (TESTMAT,I4,IF1)
      ELSE IF ((II.EQ.6).OR.(II.EQ.7)) THEN
      CALL CMPLPR (4,TESTMAT,ETA3,PROD1)
      CALL TRNSPOS (TESTMAT,TRANS)
      CALL CMPLPR (4,TRANS,ETA3,PROD2)
      CALL CMPLPR (4,PROD1,PROD2,TESTMAT)
      CALL MATDIFF (TESTMAT,I4,IF1)
      ENDIF
      IF (IF1.EQ.0) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 30,FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 30,SECOND
      ENDIF
      ELSE IF (IF1.EQ.1) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 31,FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 31,SECOND
      ENDIF
      ENDIF
      IF (II.EQ.ITYPE) GO TO 11
C---------------------------------------------------
C  TESTING FOR BILATERAL SYMMETRY
C---------------------------------------------------
      II=ITYPE
      CALL EQT (MATRIX1,TESTMAT)
      GO TO 22
   21 II=ICONVRT
      CALL EQT (MATRIX2,TESTMAT)
   22 IF ((II.EQ.1).OR.(II.EQ.2).OR.(II.EQ.4))THEN
      CALL CMPLPR (4,THETA2,TESTMAT,PROD1)
      CALL CMPLPR (4,TESTMAT,THETA2,PROD2)
      CALL MATDIFF (PROD1,PROD2,IF1)
      ELSE IF (II.EQ.3) THEN
      CALL CMPLPR (4,TESTMAT,THETA1,PROD1)
      CALL EQT (PROD1,TESTMAT)
      CALL CMPLPR (4,TESTMAT,PROD1,PROD2)
      CALL MATDIFF (I4,PROD2,IF1)
      ELSE IF (II.EQ.5) THEN
      CALL CMPLPR (4,TESTMAT,THETA2,PROD1)
      CALL EQT (PROD1,TESTMAT)
      CALL CMPLPR (4,TESTMAT,PROD1,PROD2)
      CALL MATDIFF (I4,PROD2,IF1)
      ELSE IF ((II.EQ.6).OR.(II.EQ.7)) THEN
      CALL CMPLPR (4,TESTMAT,ETA2,PROD1)
```

```
      CALL EQT (PROD1, TESTMAT)
      CALL CMPLPR (4, TESTMAT, PROD1, PROD2)
      CALL MATDIFF (I4, PROD2, IF1)
      ENDIF
      IF (IF1.EQ.0) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 32, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 32, SECOND
      ENDIF
      ELSE IF (IF1.EQ.1) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 33, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 33, SECOND
      ENDIF
      ENDIF
      IF (II.EQ.ITYPE) GO TO 21
C-------------------------------------------
C  TESTING FOR LOSSLESSNESS
C-------------------------------------------
      II=ITYPE
      CALL EQT (MATRIX1, TESTMAT)
      GO TO 42
41    II=ICONVRT
      CALL EQT (MATRIX2, TESTMAT)
42    IF ((II.EQ.1).OR.(II.EQ.2)) THEN
      CALL HERMCNJ (TESTMAT, PROD1)
      CALL MATCNPR (PROD1, MINUS1)
      CALL MATDIFF (TESTMAT, PROD1, IF1)
      ELSE IF (II.EQ.3) THEN
      CALL CMPLPR (4, TESTMAT, THETA2, PROD1)
      CALL HERMCNJ (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, THETA2, PROD2)
      CALL CMPLPR (4, PROD1, PROD2, TESTMAT)
      CALL MATDIFF (TESTMAT, I4, IF1)
      ELSE IF (II.EQ.4) THEN
      CALL HERMCNJ (TESTMAT, TRANS)
      CALL CMPLPR (4, TESTMAT, TRANS, PROD1)
      CALL MATDIFF (PROD1, I4, IF1)
      ELSE IF (II.EQ.5) THEN
      CALL CMPLPR (4, TESTMAT, THETA1, PROD1)
      CALL HERMCNJ (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, THETA1, PROD2)
      CALL CMPLPR (4, PROD1, PROD2, TESTMAT)
      CALL MATDIFF (TESTMAT, I4, IF1)
      ELSE IF ((II.EQ.6).OR.(II.EQ.7)) THEN
      CALL CMPLPR (4, TESTMAT, ETA1, PROD1)
      CALL HERMCNJ (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, ETA1, PROD2)
      CALL CMPLPR (4, PROD1, PROD2, TESTMAT)
      CALL MATDIFF (TESTMAT, I4, IF1)
      ENDIF
      IF (IF1.EQ.0) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 34, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 34, SECOND
      ENDIF
      ELSE IF (IF1.EQ.1) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 35, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 35, SECOND
      ENDIF
      ENDIF
      IF (II.EQ.ITYPE) GO TO 41
C-------------------------------------------
C  TESTING FOR SEMIRECIPROCITY
C-------------------------------------------
      II=ITYPE
      CALL EQT (MATRIX1, TESTMAT)
      GO TO 52
51    II=ICONVRT
      CALL EQT (MATRIX2, TESTMAT)
52    IF ((II.EQ.1).OR.(II.EQ.2).OR.(II.EQ.4)) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, ETA1, PROD1)
      CALL CMPLPR (4, ETA1, TESTMAT, PROD2)
      CALL MATDIFF (PROD1, PROD2, IF1)
      ELSE IF ((II.EQ.3).OR.(II.EQ.5)) THEN
      CALL CMPLPR (4, ETA1, THETA3, PROD1)
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, PROD1, PROD2)
      CALL CMPLPR (4, TESTMAT, PROD1, PROD3)
      CALL CMPLPR (4, PROD3, PROD2, PROD1)
      CALL MATDIFF (PROD1, I4, IF1)
      ELSE IF ((II.EQ.6).OR.(II.EQ.7)) THEN
      CALL CMPLPR (4, THETA1, ETA3, PROD1)
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, PROD1, PROD2)
      CALL CMPLPR (4, TESTMAT, PROD1, PROD3)
      CALL CMPLPR (4, PROD3, PROD2, PROD1)
      CALL MATDIFF (PROD1, I4, IF1)
      ENDIF
      IF (IF1.EQ.0) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 36, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 36, SECOND
      ENDIF
      ELSE IF (IF1.EQ.1) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 37, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 37, SECOND
      ENDIF
      ENDIF
      IF (II.EQ.ITYPE) GO TO 51
C-------------------------------------------
C  TESTING FOR ANTIRECIPROCITY
C-------------------------------------------
      II=ITYPE
      CALL EQT (MATRIX1, TESTMAT)
```

```
      00 TO 62
   61 II=ICONVRT
      CALL EGT  (MATRIX2, TESTMAT)
   62 IF ((II.EQ.1).OR.(II.EQ.2)) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL MATCNPR (TRANS, MINUS1)
      CALL MATDIFF (TESTMAT, TRANS, IF1)
      ELSE IF (II.EQ.3) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, THETA2, PROD1)
      CALL CMPLPR (4, TESTMAT, THETA2, PROD2)
      CALL CMPLPR (4, PROD2, PROD1, PROD3)
      CALL MATDIFF (PROD3, I4, IF1)
      ELSE IF (II.EQ.4) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TESTMAT, TRANS, PROD1)
      CALL MATDIFF (PROD1, I4, IF1)
      ELSE IF (II.EQ.5) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, THETA1, PROD1)
      CALL CMPLPR (4, TESTMAT, THETA1, PROD2)
      CALL CMPLPR (4, PROD2, PROD1, PROD3)
      CALL MATDIFF (PROD3, I4, IF1)
      ELSE IF ((II.EQ.6).OR.(II.EQ.7)) THEN
      CALL TRNSPOS (TESTMAT, TRANS)
      CALL CMPLPR (4, TRANS, ETA1, PROD1)
      CALL CMPLPR (4, TESTMAT, ETA1, PROD2)
      CALL CMPLPR (4, PROD2, PROD1, PROD3)
      CALL MATDIFF (PROD3, I4, IF1)
      ENDIF
      IF (IF1.EQ.0) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 38, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 38, SECOND
      ENDIF
      ELSE IF (IF1.EQ.1) THEN
      IF (II.EQ.ITYPE) THEN
      PRINT 39, FIRST
      ELSE IF (II.EQ.ICONVRT) THEN
      PRINT 39, SECOND
      ENDIF
      ENDIF
      IF (II.EQ.ITYPE) GO TO 61
   20 FORMAT(/' FOR RECIPROCAL NETWORKS, THE ',A7,
     $ ' MATRIX MUST BE SYMMETRIC. HENCE IT MUST EQUAL',
     $ ' ITS TRANSPOSE.'/)
   30 FORMAT(/' THE ',A7,' MATRIX SATISFIES THE ',
     $ 'RECIPROCITY CONDITION.'/)
   31 FORMAT (/' THE ',A7,' MATRIX DOES NOT SATISFY',
     $ ' THE RECIPROCITY CONDITION.'/)
   32 FORMAT(/' THE ',A7,' MATRIX SATISFIES THE ',
     $ ' SYMMETRY CONDITION.'/)
   33 FORMAT (/' THE ',A7,' MATRIX DOES NOT SATISFY',
     $ ' THE SYMMETRY CONDITION.'/)
   34 FORMAT(/' THE ',A7,' MATRIX SATISFIES THE ',
     $ ' LOSSLESSNESS CONDITION.'/)
   35 FORMAT(/' THE ',A7,' MATRIX DOES NOT SATISFY',
     $ ' THE LOSSLESSNESS CONDITION.'/)
   36 FORMAT(/' THE ',A7,' MATRIX SATISFIES THE ',
     $ ' SEMIRECIPROCITY CONDITION.'/)
   37 FORMAT(/' THE ',A7,' MATRIX DOES NOT SATISFY',
     $ ' THE SEMIRECIPROCITY CONDITION.'/)
   38 FORMAT(/' THE ',A7,' MATRIX SATISFIES THE ',
     $ ' ANTIRECIPROCITY CONDITION.'/)
   39 FORMAT(/' THE ',A7,' MATRIX DOES NOT SATISFY',
     $ ' THE ANTIRECIPROCITY CONDITION.'/)
      RETURN
      END

      SUBROUTINE TRNSPOS (A,B)
      COMPLEX A(4,4),B(4,4)
      DO 10 I=1,4
      DO 10 J=1,4
      B(I,J)=A(J,I)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MATDIFF (A,B,IF1)
      COMPLEX A(4,4),B(4,4)
      COMMON /TOLER/IBIGDIG
      IF1=0
      TOL=10.0**(-IBIGDIG)
      DO 10 I=1,4
      DO 10 J=1,4
      DIFF=CABS(A(I,J)-B(I,J))
      IF (DIFF.GT.TOL) THEN
      IF1=1
      GO TO 9999
      ENDIF
   10 CONTINUE
 9999 RETURN
      END

      SUBROUTINE HERMCNJ (A,B)
      COMPLEX A(4,4),B(4,4)
      DO 10 I=1,4
      DO 10 J=1,4
      B(I,J)=CONJG(A(J,I))
   10 CONTINUE
      RETURN
      END

      SUBROUTINE MATCNPR (A,CONSTNT)
      COMPLEX A(4,4),CONSTNT
      DO 10 I=1,4
      DO 10 J=1,4
      A(I,J)=CONSTNT*A(I,J)
   10 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE INVERS4 (A,B)
      COMPLEX B(4,4),A11(2,2),A12(2,2),A21(2,2),
     $ A22(2,2),B11(2,2),B12(2,2),B21(2,2),B22(2,2),
     $ A11INV(2,2),A12INV(2,2),A21INV(2,2),A22INV(2,2),
     $ B11INV(2,2),B12INV(2,2),B21INV(2,2),B22INV(2,2),
     $ A(4,4)
      CALL PARTITN (A,A11,A12,A21,A22,1)
      CALL INVRS2 (A11,A11INV)
      CALL INVRS2 (A12,A12INV)
      CALL INVRS2 (A21,A21INV)
      CALL INVRS2 (A22,A22INV)
      CALL CMPLPR (2,A12,A22INV,B11)
      CALL CMPLPR (2,B11,A21,B11INV)
      CALL CMPLPR (2,A22,A12INV,B12)
      CALL CMPLPR (2,B12,A11,B12INV)
      CALL CMPLPR (2,A11,A21INV,B21)
      CALL CMPLPR (2,B21,A22,B21INV)
      CALL CMPLPR (2,A21,A11INV,B22)
      CALL CMPLPR (2,B22,A12,B22INV)
      DO 10 I=1,2
      DO 10 J=1,2
      B11INV(I,J)=A11(I,J)-B11INV(I,J)
      B12INV(I,J)=A21(I,J)-B12INV(I,J)
      B21INV(I,J)=A12(I,J)-B21INV(I,J)
      B22INV(I,J)=A22(I,J)-B22INV(I,J)
   10 CONTINUE
      CALL INVRS2 (B11INV,B11)
      CALL INVRS2 (B12INV,B12)
      CALL INVRS2 (B21INV,B21)
      CALL INVRS2 (B22INV,B22)
      CALL PARTITN (B,B11,B12,B21,B22,2)
      RETURN
      END

      SUBROUTINE INVERT2 (A,B)
      COMPLEX A(2,2),B(2,2)
      DOUBLE PRECISION REA(2,2),IMA(2,2),RED1,IMD1,
     $ RED2,IMD2,DEN
      DO 10 I=1,2
      DO 10 J=1,2
      REA(I,J)=REAL(A(I,J))
      IMA(I,J)=AIMAG(A(I,J))
   10 CONTINUE
      CALL DBLPR (REA(1,1),IMA(1,1),REA(2,2),IMA(2,2),
     $ RED1,IMD1)
      CALL DBLPR (REA(1,2),IMA(1,2),REA(2,1),IMA(2,1),
     $ RED2,IMD2)
      RED1=RED1-RED2
      IMD1=IMD2-IMD1
      DEN=RED1*RED1+IMD1*IMD1
      IF (DABS(DEN).LT.1.0E-25) THEN
      PRINT *, ' INVERSE OF 2 BY 2 MATRIX CANNOT BE FOUND'
      GO TO 9999
      ENDIF
      CALL DBLPR (REA(2,2),IMA(2,2),RED1,IMD1,RED2,IMD2)
      RED2=RED2/DEN
      IMD2=IMD2/DEN
      B(1,1)=CMPLX(RED2,IMD2)
      CALL DBLPR (REA(1,1),IMA(1,1),RED1,IMD1,RED2,IMD2)
      RED2=RED2/DEN
      IMD2=IMD2/DEN
      B(2,2)=CMPLX(RED2,IMD2)
      CALL DBLPR (REA(1,2),IMA(1,2),RED1,IMD1,RED2,IMD2)
      RED2=RED2/DEN
      IMD2=IMD2/DEN
      B(1,2)=-CMPLX(RED2,IMD2)
      CALL DBLPR (REA(2,1),IMA(2,1),RED1,IMD1,RED2,IMD2)
      RED2=RED2/DEN
      IMD2=IMD2/DEN
      B(2,1)=-CMPLX(RED2,IMD2)
 9999 RETURN
      END

      SUBROUTINE DBLPR (AR,AI,BR,BI,C,D)
      DOUBLE PRECISION C,D,BR,BI,AR,AI
      C=AR*BR-AI*BI
      D=AI*BR+BI*AR
      RETURN
      END

      SUBROUTINE INVERD4 (A,B)
      COMPLEX B(4,4),A11(2,2),A12(2,2),A21(2,2),
     $ A22(2,2),B11(2,2),B12(2,2),B21(2,2),B22(2,2),
     $ A11INV(2,2),A12INV(2,2),A21INV(2,2),A22INV(2,2),
     $ B11INV(2,2),B12INV(2,2),B21INV(2,2),B22INV(2,2),
     $ A(4,4)
      CALL PARTITN (A,A11,A12,A21,A22,1)
      CALL INVERT2 (A11,A11INV)
      CALL INVERT2 (A12,A12INV)
      CALL INVERT2 (A21,A21INV)
      CALL INVERT2 (A22,A22INV)
      CALL CMPLPR (2,A12,A22INV,B11)
      CALL CMPLPR (2,B11,A21,B11INV)
      CALL CMPLPR (2,A22,A12INV,B12)
      CALL CMPLPR (2,B12,A11,B12INV)
      CALL CMPLPR (2,A11,A21INV,B21)
      CALL CMPLPR (2,B21,A22,B21INV)
      CALL CMPLPR (2,A21,A11INV,B22)
      CALL CMPLPR (2,B22,A12,B22INV)
      DO 10 I=1,2
      DO 10 J=1,2
      B11INV(I,J)=A11(I,J)-B11INV(I,J)
      B12INV(I,J)=A21(I,J)-B12INV(I,J)
      B21INV(I,J)=A12(I,J)-B21INV(I,J)
      B22INV(I,J)=A22(I,J)-B22INV(I,J)
   10 CONTINUE
      CALL INVERT2 (B11INV,B11)
      CALL INVERT2 (B12INV,B12)
      CALL INVERT2 (B21INV,B21)
      CALL INVERT2 (B22INV,B22)
      CALL PARTITN (B,B11,B12,B21,B22,2)
      RETURN
      END
```

```
      SUBROUTINE CMPLPR (N,A,B,PROD)
      COMPLEX A(N,N),B(N,N),PROD(N,N)
      DO 10 I=1,N
      DO 10 K=1,N
      PROD(I,K)=0.0
      DO 10 J=1,N
      PROD(I,K)=PROD(I,K)+A(I,J)*B(J,K)
   10 CONTINUE
      RETURN
      END

      SUBROUTINE DATFORM (II,JJ,MATR)
      COMPLEX MATR(4,4),Z1,Z2
      COMMON /LINEINF/Z1,Z2
      PRINT *,'001'
      PRINT *,'N'
      PRINT *,II
      PRINT *,JJ
      DO 10 I=1,4
      DO 10 J=1,4
      PRINT *,REAL(MATR(I,J))
      PRINT *,AIMAG(MATR(I,J))
   10 CONTINUE
      PRINT *,REAL(Z1)
      PRINT *,AIMAG(Z1)
      PRINT *,REAL(Z2)
      PRINT *,AIMAG(Z2)
      RETURN
      END
```