DIGITAL WAVE ANALYZER

by

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ABSTRACT

Title: Digital Wave Analyzer

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The objective of this report is to simulate a Digital Wave Analyzer on the CDC 6600 digital computer.

A brief summary of digital filters and related topics is first provided, for example, Z-transforms, time-domain analysis, Fourier analysis, bilinear transformation, the Sampling theorem and Approximation.

Next, it will be shown, how by using a second order analog filter (a) a corresponding digital filter can be obtained and (b) using similar techniques, higher order digital filters can be realized as software.

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CHAPTER 1

DIGITAL FILTERS AND RELATED TOPICS

1.1 Introduction

Discrete-time signals, or signals represented by a sequence of numbers are processed by digital filters, in the same way as continuous-time signals are by continuous-time systems.

Digital filters can be realized as hard-ware devices, having input and output ports and responding in a prescribed manner to a certain specified excitation. Alternatively, soft-ware realizations of digital filters are possible. These are computer programs which simulate the behaviour of the filter on a computing system.

1.2 Z-Transform

Z-transform theory helps in the analysis and synthesis of digital filters and other discrete-time systems. The Z-transform is analogous to the Laplace transform theory which is used in the study of continuous-time systems.

1.3 Fourier Analysis

A complex waveform is considered to be the sum of a number of sinusiodal waveforms of suitable amplitude, period and relative phase. A continuous sinusoidal function ($\sin \omega t$) is considered to be a single frequency component of frequency

ω radians/second and the frequency-domain description of a signal requires its breakdown into a number of basic functions. The above method of signal analysis just described was conceived by Jean Baptiste Joseph, Baron de Fourier in the late eighteenth century. His theory was originally applied to the analysis of heat flow and has since been used in many areas of applied science, but it now forms a principle tool in the analysis of signals.

1.4 Time-Domain Analysis

This is used to compute the time-domain response of digital filters to a given excitation. Convolution summation can be employed which is analogous to the convolution integral in continuous-time systems.

1.5 Synthesis

The bilinear transformation is given by the equation

$$s = \frac{2}{T} \frac{z-1}{z+1}$$

which can be used to derive a digital filter from a corresponding analog filter. The most significant properties of this transformation are:

1. It represents a one-to-one mapping of the open left half s-plane onto the open unit circle in the z-plane.

2. Maps one-to-one, the imaginary axis including the point at infinity onto the unit circle.

3. It is a one-to-one mapping of the open right half-plane onto the complement in the complex plane of the closed unit circle.

4. The inverse transformation

$$z = \frac{\frac{2}{T} + s}{\frac{2}{T} - s}$$

has analogous reciprocal properties. Thus rational functions transform to rational functions, poles in the open left-half transform to poles within the unit circle and zeros on the imaginary axis and at infinity transform to zeros on the unit circle.

1.6 Sampling Theorem

The sampling theorem states that a band-limited continuous-time signal with a Fourier spectrum extending upto a frequency ω_c, where $\omega_c < \omega_o/2$ can be recovered by using a series of samples separated by a period $T = \frac{2\pi}{\omega_o}$ seconds.

Thus there is no loss of information, if we sample a signal at a rate at least twice as fast as the signals highest frequency component. For signals which are not band-limited, we attempt to determine a value of ω_c so that most of the signals energy (about 99%) lies in the range $o \leq \omega \leq \omega_c$. The sampling period should be selected to be four to ten times smaller than $\frac{2\pi}{\omega_c}$. This procedure then

leads to satisfactory results whereby the process of sampling introduces a negligible source of error. Typically we put the signal to be sampled through a continuous-time low-pass guard filter which removes most of the signal's frequency content above $\frac{\pi}{T}$ so as to reduce aliasing errors due to sampling. This band-limited signal is then sampled at the rate of $\frac{1}{T}$ samples per second.

1.7 Approximation

From the properties of the bilinear transformation it follows that for every well-defined digital approximation problem there is a corresponding analogue approximation problem. The two problems are equivalent and the solution for one exists if the other has a solution. Thus many methods for designing analogue filters are applied to design digital filters.

Digital filters have very wide ranging applications in various branches of science and engineering, such as filtering of signals received from space, speech processing, telemetry and in bio-medical electronics.

This report is concerned with the design of recursive filters. Recursive filters introduce phase distortion and thus equalization is employed to obtain a linear phase. However, in this report most of the discussion will be confined to the study of the amplitude response of digital filters with respect to time and frequency.

CHAPTER 2

DIGITAL FILTER DESIGN TECHNIQUES

2.1 Introduction

Linear filters form a class of system which plays an important role in signal processing. Although the term "filter" broadly speaking may be a frequency-selective device, in practice it is a system which transmits a certain range of frequencies, and rejects others. Such frequency ranges are called passbands and stopbands respectively. These filters may be analog or digital. Knowledge of electrical network theory is needed for the design of analog filters; the same is fortunately not true of sampled-data filters or digital filters. These may be realized by suitable programming of a digital computer which is fed with a sampled version of the input signal. The increasing interest in digital filters is largely due to the availability of the digital computer as a research tool, and the work done on Z-transforms and linear systems forms an adequate background for the realization of digital filters. As stated earlier this report will be restricted to the design of recursive filters only. The response of such filters is a function of the present and past values of the excitation and the past values of the response, whereas in non-recursive digital filters the response is independent of the

past values of the response. As shown in Fig. 1 and Fig. 2 the network for the recursive digital filter contains feed forward paths and feedback loops whereas the non-recursive digital filter contains only feed forward paths.

2.2 Design Techniques

The Z-transform theory is a very helpful tool for designing digital filters. When it is required that the digital filters have very good performance, the problem is basically of using as few poles and zeros as possible to approximate a desired frequency response characteristic. Thus as a general rule when the transfer function in the Z-domain H(Z) has several Z-plane poles and zeros, the resulting formula becomes complicated and the numerical computation uneconomical.

The large amount of information that is available from analog filters can be used to design digital filters with roughly the same characteritics. A simple way to convert an analog design into an equivalent digital filter is to use the technique known as impulse invariance. This technique consists of using a sampled version of the impulse response of the analog filter to define the impulse response of a digital filter. The resulting relationships between the responses of the analog and digital versions of a typical low-pass filter are illustrated in Fig. 3. It is evident from the figure that sampling an impulse response, which is

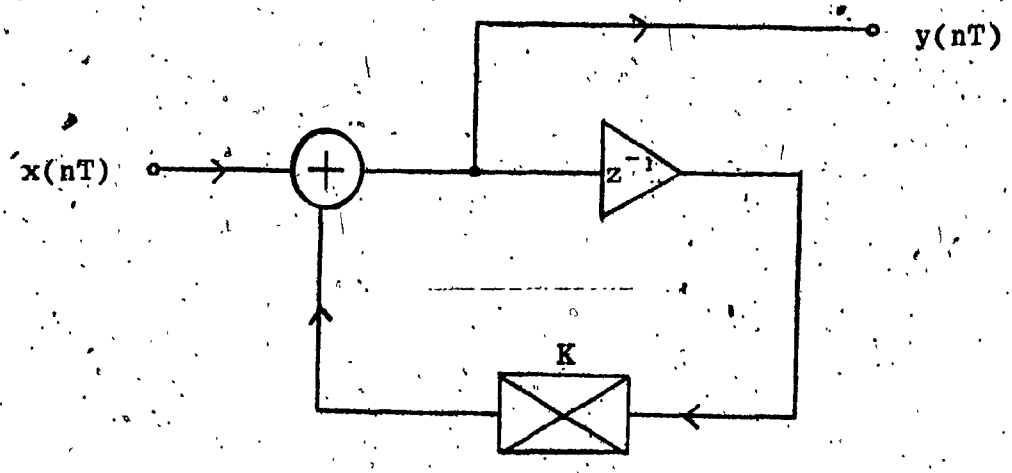


Fig. 1 Recursive Filter

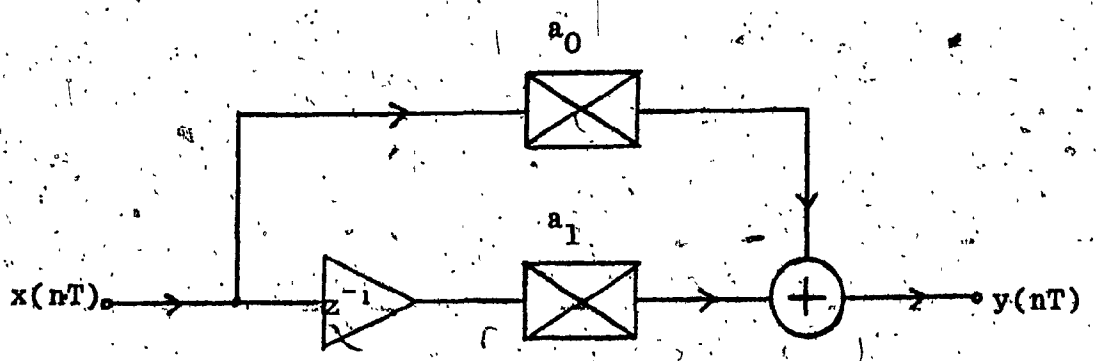


Fig. 2 Non-Recursive Filter

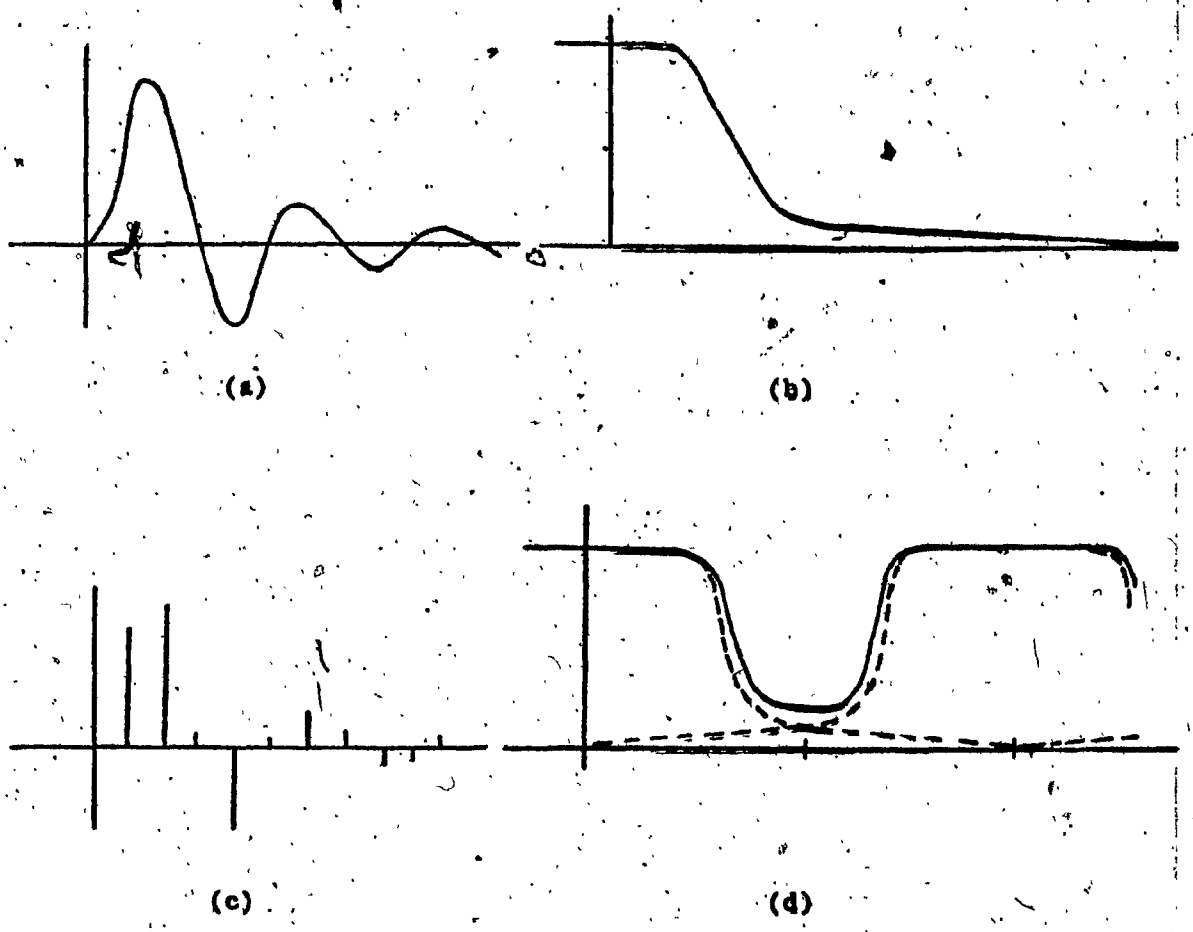


Fig. 3 (a), (b) Impulse and frequency responses of an analog low pass filter

and

(c), (d) their Impulse invariant digital equivalent.

a function of time, has the effect of repeating the corresponding frequency function indefinately at intervals in $\omega = \frac{2\pi}{T}$. From this, it follows that the digital filter will have a frequency response which is a repetitive version of the response of the analog filter. This effect reveals two major disadvantages of the impulse invariance technique:

(i) The frequency response of the digital filter in the range $o < \omega < \frac{\pi}{T}$ is a poor approximation of the response of the analog filter over the same range. This discrepancy could be minimized if the analog filter has an effective stopband throughout the frequency range $\frac{\pi}{T} < \omega < \infty$, which implies that the impulse invariance method cannot be used for high-pass filters.

(ii) The impulse response obtained must be used in a nonrecursive digital filtering operation. Since this response may contain many sample values of significant size, it can lead to a very uneconomic filter.

As discussed earlier in Chapter 1, the design technique employed here for realizing bandpass digital filters shall be by approximating corresponding analog filters using frequency and bilinear transformations.

When the same filtering requirement can be adequately met by several different digital filters, then the choice between them depends on the speed of execution of a computer program which

simulates the difference equation. An important factor in this speed is the number of multiplications. Some digital filters can essentially meet the same requirements as others with fewer multiplications per output sample and hence these filters are to be preferred. It is important to stress that speed of execution is one of the main limiting factors in the use of digital filtering methods.

2.3 Design of Second Order Bandpass Digital Filter

Given a transfer function representing a first order lowpass analog filter

$$T(s) = \frac{H}{s+1} \quad , \qquad H = 1 \tag{2.1}$$

the use of the lowpass to bandpass transformation

$$S = \frac{1}{\omega_B} \left(s + \frac{\omega_o^2}{s}\right) \tag{2.2}$$

will yield a transfer function representing a bandpass filter

$$T(s) = \frac{1}{\frac{1}{\omega_B}\left(s + \frac{\omega_o^2}{s}\right) + 1}$$

$$T(s) = \frac{\omega_B s}{s^2 + \omega_B s + \omega_o^2} \tag{2.3}$$

The following specifications shall be assigned to the above filter

$f_B = 0.2$ Hz

$f_o = 2$ Hz

$f_s = 20$ Hz

where f_B is the 3dB bandwidth

f_o is the centre frequency and

f_s is the sampling frequency

Thus

$\omega = 0.4\pi$ radian/sec

$\omega_o = 4\pi$ radian/sec

$T = \frac{1}{f_s} = \frac{1}{20} = 0.5$ sec

Putting the above values in equation (2.3) we obtain

$$T(s) = \frac{0.4\pi s}{s^2 + 0.4\pi s + 16\pi^2} \qquad (2.4)$$

The frequency response of the analog filter is obtained by putting $s = j\omega$

$$T(j\omega) = \frac{1.2566j\omega}{-\omega^2 + 1.2566j\omega + 157.9137}$$

$$= \frac{j\,(1.2566\omega)}{(157.9137 - \omega^2) + j\,(1.2566\omega)}$$

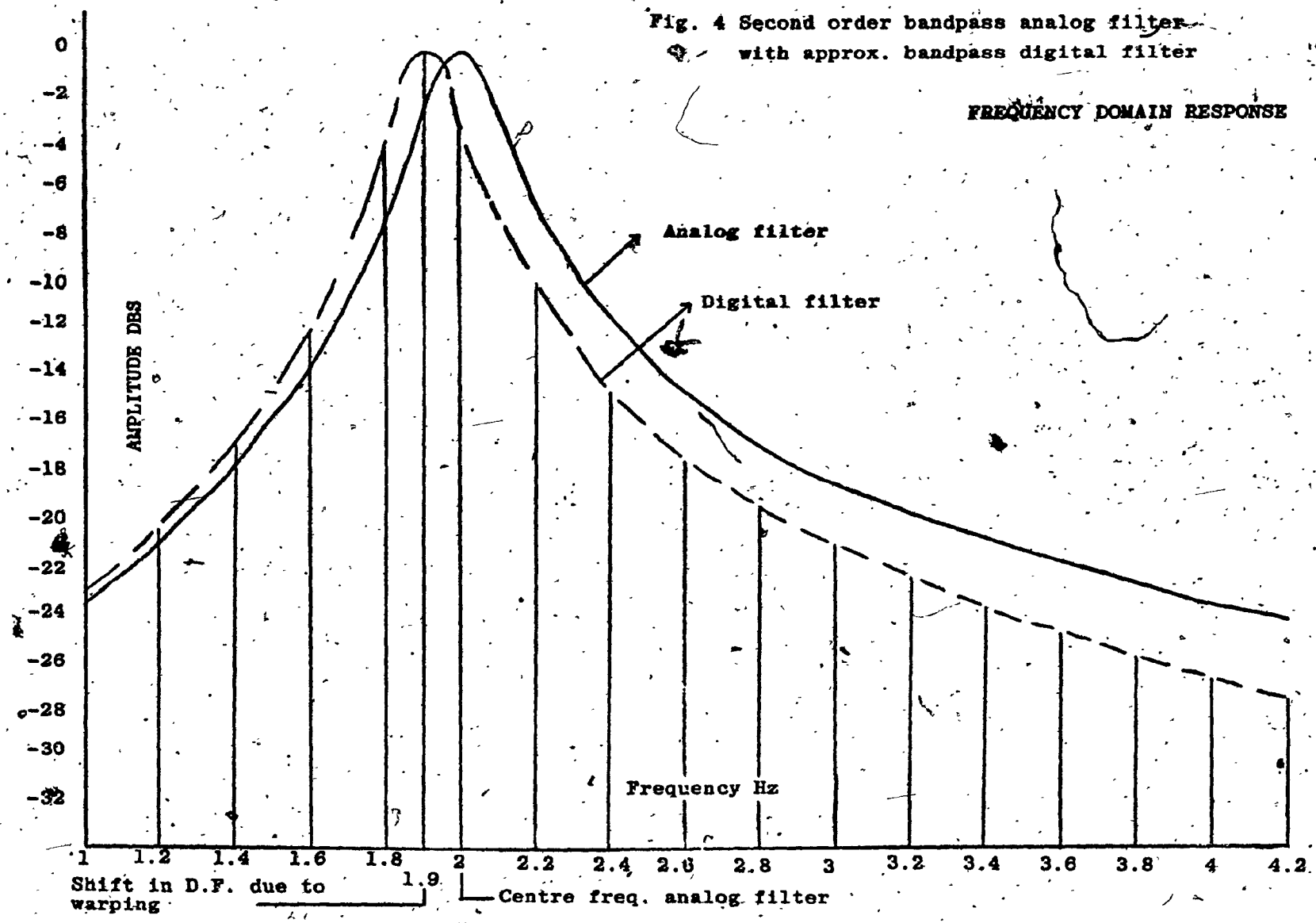
Computing the above by using a digital computer program we obtain the frequency response of the second order analog filter as shown in Fig. 4.

We shall now obtain a second order bandpass digital filter by using the bilinear transformation

$$s = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

Inserting the above in equation (2.4) we obtain the transfer function of the digital filter.

Fig. 4 Second order bandpass analog filter with approx. bandpass digital filter



$$H(z) = \frac{0.4\pi \; \frac{2}{T} \frac{(z-1)}{(z+1)}}{\frac{4}{T^2} \frac{(z-1)^2}{(z+1)^2} + 0.4\pi \; \frac{2}{T} \frac{(z-1)}{(z+1)} + 16\pi^2}$$

$$= \frac{0.4\pi \; T \times 2 \; (z-1) \; (z+1)}{4(z-1)^2 + 0.4\pi \; 2T(z-1) \cdot (z+1) + 16\pi^2 T^2 \; (z+1)^2}$$

$$= \frac{2.5133 \; T(z^2-1)}{(4 + 2.5133T + 156.9137T^2) \; z^2 + +(315.8273T^2 - 8)z + (157.9137T^2 - 2.5133t + 4)}$$

$$H(z) = \frac{\frac{2.5133T}{157.9137T^2 + 2.5133T + 4} \; (z^2-1)}{z^2 + \frac{315.8273T^2-8}{157.9137T^2+2.5133T+4} \; z + \frac{157.9137T^2-2.5133T+4}{157.9137+2.5133T+4}} \qquad (2.5)$$

or

$$H(z) = \frac{A1(z^2-1)}{z^2 + B1z + B2} \qquad (2.6)$$

where

$$A1 = \frac{2.5133T}{156.9136T^2 + 2.5133T + 4}$$

$$B1 = \frac{315.8273T^2 - 8}{157.9137T^2 + 2.5133T + 4}$$

$$B2 = \frac{157.9137T^2 - 2.5133T + 4}{157.913 \; T^2 + 2.5133T + 4}$$

2.31 Frequency Reponse

The amplitude and phase reponses of the digital filter are obtained by putting $z = e^{j\omega T}$ in H(z), equation (2.6)

$$H(e^{j\omega T}) = \frac{A1 \; (e^{j2\omega T} - 1)}{e^{j2\omega T} + B1 \; (e^{j\omega T}) + B2}$$

$$= \frac{A1 \; (\cos\omega 2 \; T + j \sin 2\omega T - 1)}{(\cos 2\omega T + j \sin 2\omega T) + B1 \; (\cos \omega T + j \sin \omega T) + B2}$$

$$H(e^{j\omega T}) = \frac{(A1 \cos 2\omega T - A1) + j\,(A1 \sin 2\omega T)}{(\cos 2\omega T + B1 \cos \omega T + B2) + j\,(\sin 2\omega T + B1 \sin \omega T)}$$

By using a computer program the amplitude response of the digital filter is as shown in Fig. 4.

Since our discussion involves the design of recursive digital filters which cause phase distortion, the study of phase distortion will not be included.

The amplitude response of the filter (Fig. 4) shows a shift at the resonant frequency as compared to its analog counterpart. The shift is due to frequency warping caused by bilinear transformation. This shift is called the Warping Effect, and will be discussed in detail later on.

The flow chart for the computer program used to determine the frequency response of the digital filters is shown in Fig. 5. Up until now the study was concerned with the frequency domain. The time domain response will now be examined.

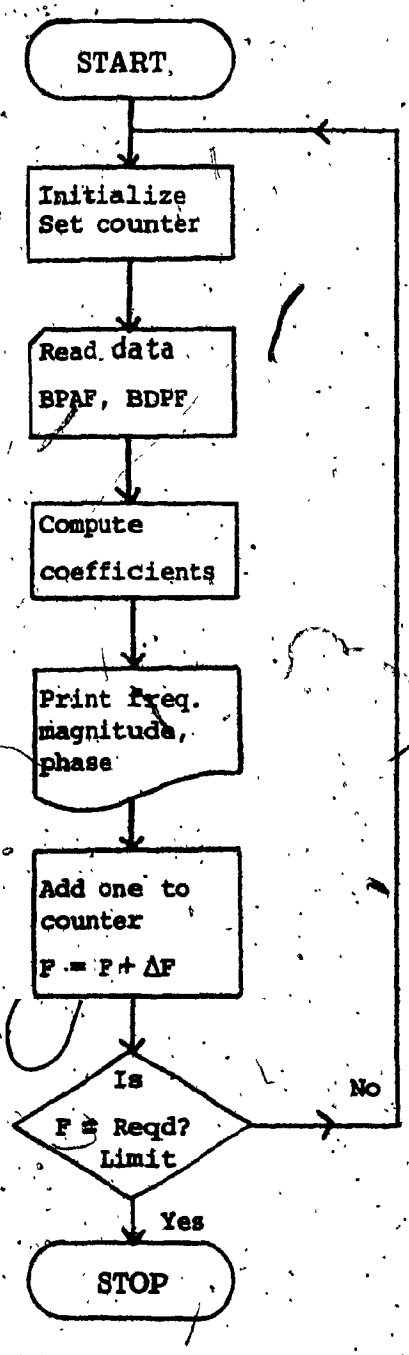
2.32 Time Domain Response

The transfer function of equation (2.6) can be used to obtain the difference equation required for the time-domain analysis of the filter. We can write

$$H(z) = \frac{A1\,(z^2-1)}{z^2 + B1 \cdot z + B2} = \frac{Y(z)}{X(z)}$$

$$\frac{Y(z)}{X(z)} = \frac{A1 z^2 - A1}{z^2 + B1 \cdot z + B2}$$

$$= \frac{A1 - A1 \cdot z^{-2}}{1 + B1 \cdot z^{-1} + B2 \cdot z^{-2}}$$



PROGRAM FLOWCHART FOR ANALOG AND DIGITAL BANDPASS FILTERS

Fig. 5.

$$Y(z) + B1 \cdot z^{-1}Y(z) + B2 \cdot z^{-2}Y(z) = A1 \cdot x(z) - A1 \cdot z^{-2}\, x(z)$$

$$Y(z) = A1 \cdot x(z) - A1 \cdot z^{-2}x(z) - B1 \cdot z^{-1}Y(z) - B2 \cdot z^{-2}Y(z)$$

or

$$Y(nT) = A1 \cdot x(nT) - A1 \cdot x(nT-2T) - B1 \cdot Y(nT-T) - B2 \cdot Y\,(nT-2T) \tag{2.7}$$

This is a second order difference equation representing a recursive bandpass digital filter where the coefficients A1, B1, B2 are real. Using a computer program described in the flow chart (Fig. 6), the difference equation can be simulated on the digital computer to give the time-domain responses of the filter.

Let a sinusoidal signal $x(nT) = \sin(\omega nT)$ be applied to the input of the bandpass filter

Input signal frequency f = 1.9 Hz

Sampling period T = 0.0645 sec

Sampling frequency f_s = 14.7059 Hz

Gain of the filter is unity

The response of the filter is as shown in Fig. 7. The output of the digital filter attains a steady state after 300 to 500 samples after which the response has the same amplitude as the input, indicating that the gain of the filter is unity, as expected.

A similar method can be employed to design higher order bandpass digital filters. The selectivity of such filters is significantly increased relative to that of low order filters. Bilinear transformation is applied to a high order analog

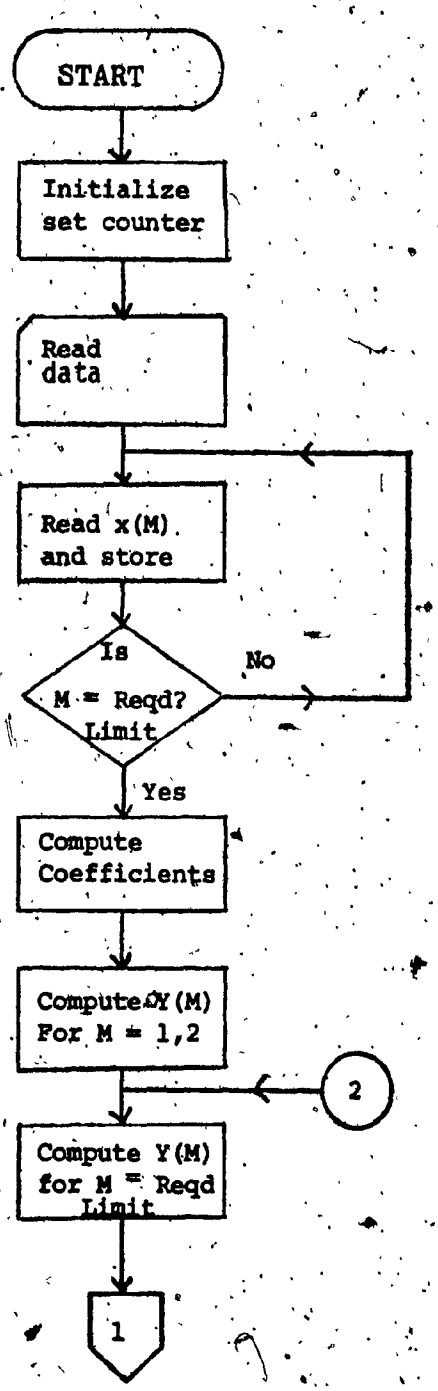
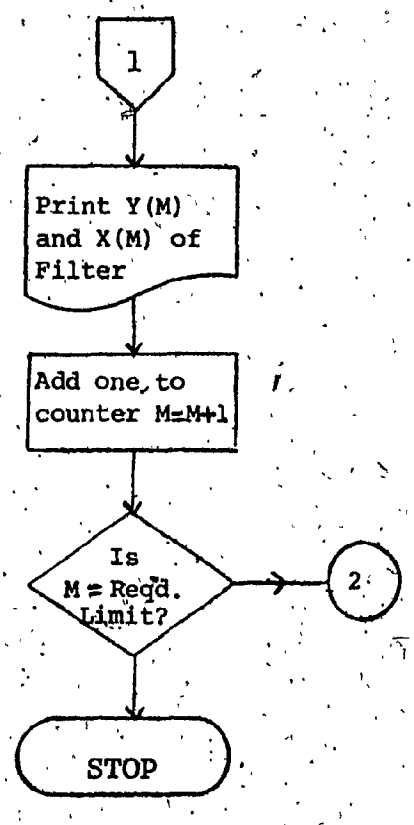

START
Initialize set counter
Read data
Read x(M) and store
Is M = Reqd? Limit
No
Yes
Compute Coefficients
Compute Y(M) For M = 1,2
2
Compute Y(M) for M = Reqd Limit
1

Fig. 6.



FLOWCHART OF TIMEDOMAIN RESPONSE OF BANDPASS DIGITAL FILTERS

Fig. 6

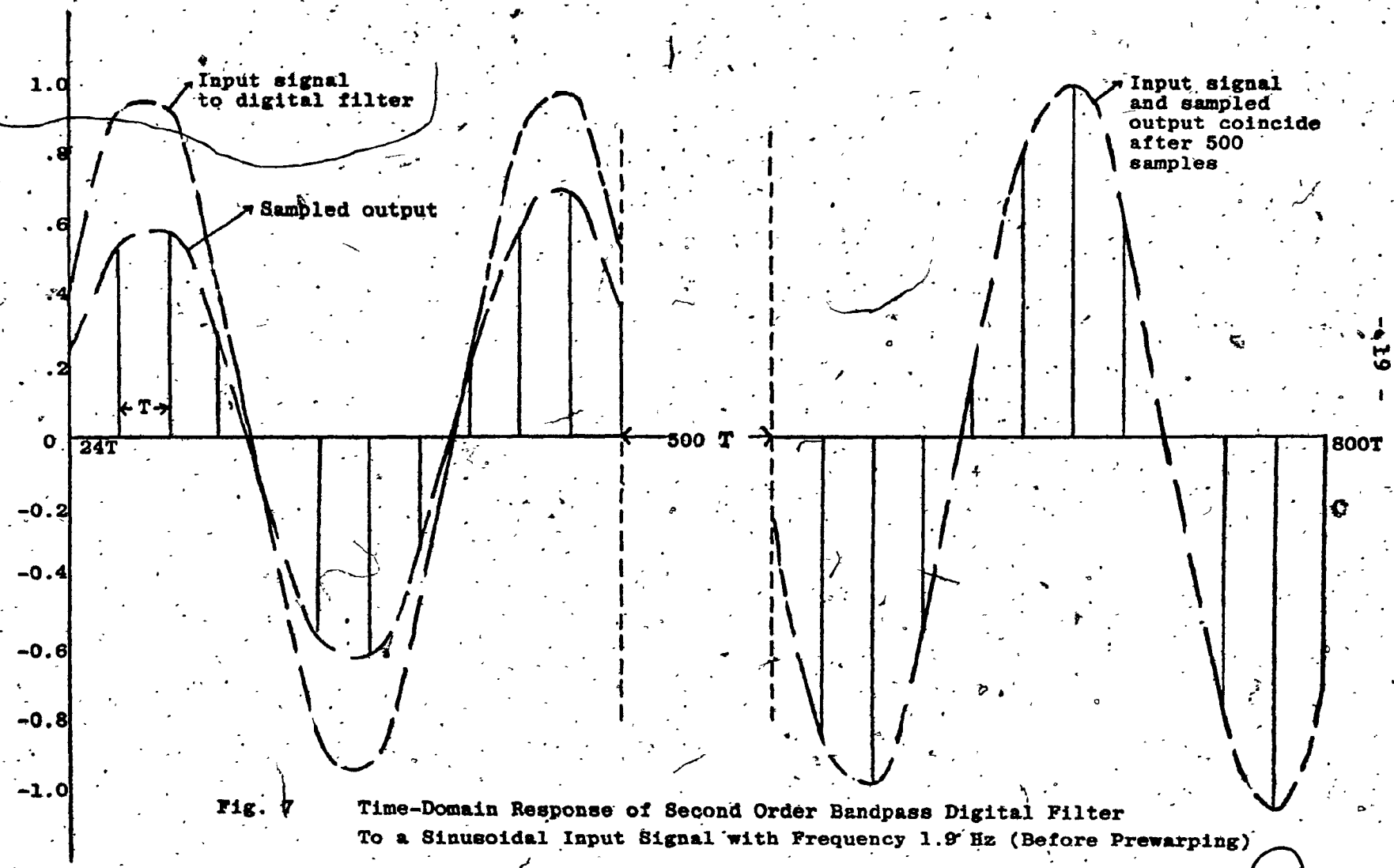


Fig. 7 Time-Domain Response of Second Order Bandpass Digital Filter To a Sinusoidal Input Signal with Frequency 1.9 Hz (Before Prewarping)

transfer function to obtain a corresponding digital transfer function. The design coefficients for such a filter can be obtained by (i) solving the polynomials in s for roots, using the Newton-Raphson algorithm and them factorising to obtain second order filters in cascade and (ii) deriving the coefficients for the digital filter directly from the numerator and denominator polynomials. Both the methods will be discussed in detail in the ensuing sections.

2.4 Sixth Order Bandpass Digital Filter

Third order analog lowpass filter is

$$T(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

frequency transformation for bandpass filter

$$s = \frac{1}{\omega_B}\left(s + \frac{\omega_o^2}{s}\right)$$

gives

$$T(s) = \frac{1}{\frac{1}{\omega_B^3}\left(s + \frac{\omega_o^2}{s}\right)^3 + 2\frac{1}{\omega_B^2}\left(s + \frac{\omega_o^2}{s}\right) + 2\frac{1}{\omega_B}\left(s + \frac{\omega_o^2}{s}\right) + 1}$$

which reduces to

$$T(s) = \frac{\omega_B^3 s^3}{s^6 + 2\omega_B s^5 + (3\omega_o^2 + 2\omega_B^2)s^4 + (4\omega_o^2\omega_B + \omega_B^3)s^3 + (3\omega_o^4 + 2\omega_B^2\omega_o^2)s^2 + 2\omega_B\omega_o^4 s + \omega_o^6} \qquad (2.8)$$

Equation (2.8) represents a 6th order bandpass analog filter

Let $f_1 = 1.9$ Hz

$f_2 = 2.1$ Hz

$f_o = 2.0$ Hz

$f_B = 0.2$ Hz

where f_1 and f_2 are the half power points

f_o is the centre frequency

f_B is the 3dB bandwidth

Thus $\omega_B = 0.4\pi$ rads/sec

$\omega_o = 4\pi$ rads/sec

$\omega_o = 2\pi f_o = 4\pi$ radians

Putting in equation (2.8)

$$T(s) = \frac{(0.4)^3 \pi^3 s^3}{s^6 + 0.8\pi s^5 + (48.32\pi^2)s^4 + (25.664\pi^3)s^3 + (3 \times 256\pi^4 + 5.12\pi^4)s^2 + 0.8 \times 256\pi^5 s + (4\pi)^6} \quad (2.9)$$

2.41 Frequency Response of the Analog Filter

Inserting $s = j\omega$

we have,

$$T(j\omega) = \frac{-j\,(0.4)^3\,\pi^3\,\omega^3}{[(4\pi)^6 - \omega^6 - (3 \times 256\pi^4 + 5.12\pi^4)\,\omega^2 + (48.32\pi^2)\omega^4] + j[0.8\pi\omega^5 - (25.664\pi^3)\,\omega^3 + 0.8 \times 256\pi^5\omega]} \quad (2.10)$$

The coefficients of the denominator polynomial $D(\omega)$ are

$$A1 = (4\pi)^6 - \omega^6 - (3 \times 356\pi^4 + 5.12\pi^4)\omega^2 + (48.32\pi^2)\omega^4$$

$$A2 = 0.8\pi\omega^5 - (25.664\pi^3)\omega^3 + 0.8 \times 256\pi^5\omega$$

The magnitudes of the numerator and denominator polynomials of (2.10) are

$$AMAGN = (0.4\pi\omega)^3$$

and $AMAGD = \sqrt{A1^2 + A2^2}$

We can now execute a computer program and plot the response of the 6th order bandpass analog filter against frequency as shown in Fig. 8. The 3dB points of the amplitude response 3dB occur approximately at the specified frequencies

$$f_1 = 1.9 \text{ Hz and } f_2 = 2.1 \text{ Hz}$$

From equation (2.10) to obtain the roots of the 6th order denominator polynomial D(s), we use a computer subroutine to solve D(s).

The real and imaginary roots are,

$$s = -0.3006 \pm j\ 12.0301$$

$$s = -0.6283 \pm j\ 12.5506$$

$$s = -0.3278 \pm j\ 13.1184$$

$$(s + 0.3006 - j\ 12.0301)\ (s + 0.3006 + j\ 12.0301)$$

$$= (s^2 + 0.3006s + j\ 12.0301s + 0.3006s + 0.0904$$

$$+ j\ 3.6162 - j\ 12.0301s - j\ 3.6162 + 144.7233)$$

$$= (s^2 + 0.6012s + 144.8137)$$

Similarly

$$(s + 0.3278 - j\ 13.1184)\ (s + 0.3278 + j\ 13.1184)$$

$$= (s^2 + 0.6556s + 172.1999)$$

and $(s + 0.6283 - j\ 12.5506)(s + 0.6283 + j\ 12.5506)$

$$= (s^2 + 1.2566s + 157.5175)$$

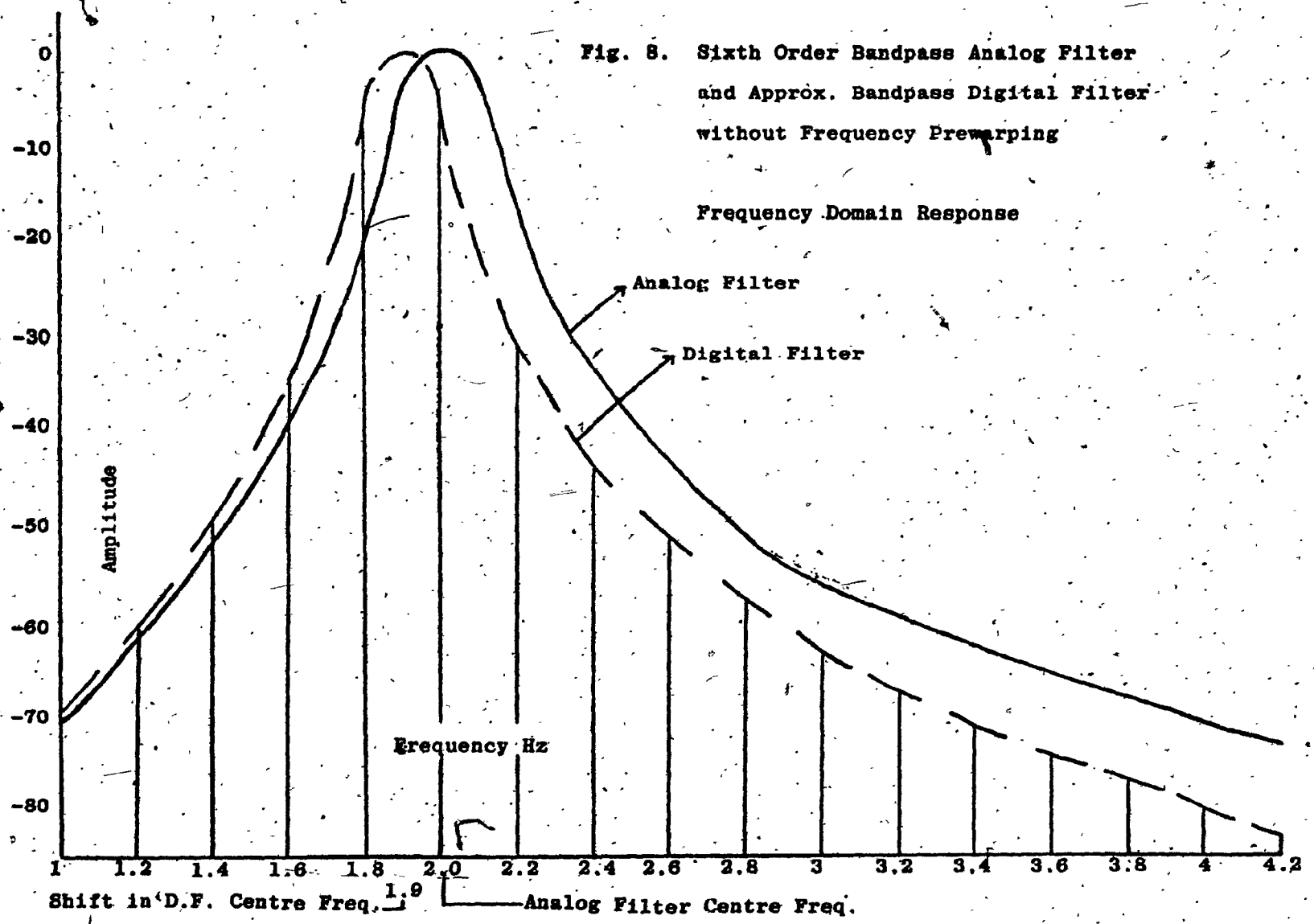


Fig. 8. Sixth Order Bandpass Analog Filter and Approx. Bandpass Digital Filter without Frequency Prewarping

Frequency Domain Response

Transfer function equation (2.10) after breaking into factors becomes,

$$T(s) = \frac{(0.4)^3 \pi^3 s^3}{(s^2 + 0.6012s + 144.8137)(s^2 + 0.7556s + 172.1999)(s^2 + 1.2566s + 157.9122)}$$

$$T(s) = \frac{(0.4)\pi s}{s^2 + 0.612s + 144.8137} \times \frac{(0.4)\pi s}{s^2 + 0.6556s + 172.1999} \times \frac{(0.4)\pi s}{s^2 + 1.2566s + 157.9122} \qquad (2.11)$$

Applying bilinear transformation

$s = \frac{2}{T}\frac{z-1}{z+1}$ to equation (2.11),

$$T_1(s) = \frac{0.4\pi s}{s^2 + 0.6012s + 144.8137}\Bigg|_{s = \frac{2}{T}\frac{z-1}{z+1}}$$

$$H_1(z) = \frac{0.4 \times \pi \times \frac{2}{T} \times \frac{z-1}{z+1}}{\frac{4}{T^2}\frac{(z-1)^2}{(z+1)^2} + 0.6012 \times \frac{2}{T}\frac{z-1}{z+1} + 144.8137}$$

After further manipulation,

$$H_1(z) = \frac{0.8 \times 3.16159T\, z^2 - 0.8 \times 3.14159 \times T}{(4 + 1.2024T + 144.8137T^2)\, z^2 - (8 - 144.8137 \times 2T^2)z + (4 - 1.2024T + 144.8137T^2)}$$

$$H_1(z) = \frac{2.5133 \times T\,(z^2 - 1)}{(4 + 1.2024T + 144.8136T^2)z^2 - (8 - 144.8137 \times 2T^2)z + (4 - 1.2024T + 144.8137T^2)}$$

$$H_1(z) = \frac{\frac{2.5133 \times T}{4 + 1.2024T + 144.8137T^2}\,(z^2 - 1)}{z^2 - \left(\frac{8 - 289.6274T^2}{4 + 1.2024T + 144.813T^2}\right)z + \frac{4 - 1.2024T + 144.8137T^2}{4 + 1.2024T + 144.8137T^2}}$$

Similarly

$$T_2(s) = \frac{0.4\,\pi\, s}{s^2 + 0.7556s + 172.1999} \Bigg|_{s = \frac{2}{T}\frac{z-1}{z+1}}$$

$$H_2(z) = \frac{\dfrac{2.5133 \times T}{4 + 1.3112T + 172.1999T^2}\,(z^2 - 1)}{z^2 - \left(\dfrac{8-344.3998T^2}{4+1.3112T+172.1999T^2}\right)z + \dfrac{4-1.3112T+172.1999T^2}{4+1.3112T\ 172,1999T^2}}$$

and

$$T_3(s) = \frac{0.4\,\pi\, s}{s^2 + 1.2566s + 157.9122} \Bigg|_{s = \frac{2}{T}\frac{z-1}{z+1}}$$

$$H_3(z) = \frac{\dfrac{2.5133 \times T}{4 + 2.5132T + 157.9122T^2}\,(z^2 - 1)}{z^2 - \left(\dfrac{8-315.8244T^2}{4+2.5132T+157.9122T^2}\right)z + \dfrac{4-2.5132T+157.9122T^2}{4+2.5132T+157.9122T^2}}$$

Thus transfer function equation (2.11)

$$T(s) = T_1(s) \times T_2(s) \times T_3(s)$$

transforms into a transfer function in the z-domain

$$H(z) = H_1(z) \times H_2(z) \times H_3(z)$$

2.42 Frequency Response of the Digital Filter

Frequency-domain analysis of the digital filter will be as follows:-

Putting the value $z = e^{j\omega T}$ in the three transfer functions $H_1(z)$, $H_2(z)$ and $H_3(z)$ just obtained,

$$H_1(z) = \frac{A1\ (z^2-1)}{z^2 - A2\ z + A3}$$

where A1, A2, A3 are the coefficients

$$H_1(e^{j\omega T}) = \frac{A1(e^{j2\omega T}-1)}{e^{j2\omega T} - A2\ (e^{j\omega T}) + A3}$$

which gives

$$H_1(e^{j\omega T}) = \frac{A1(\cos 2\omega T + j\sin 2\omega T - 1)}{(\cos 2\omega T + j\sin 2\omega T) - A2(\cos\omega T + j\sin\omega T) + A3}$$

$$= \frac{(A1\cos 2\omega T - A1) + j(A1\sin 2\omega T)}{(\cos 2\omega T - A2\cos\omega T + A3) + j(\sin 2\omega T - A2\sin\omega T)} \tag{2.12}$$

Similarly

$$H_2(z) = \frac{B1(z^2 - 1)}{z^2 - B2\,z + B3}$$

$$H_2(e^{j\omega T}) = \frac{(B1\cos 2\omega T - B1) + j(B1\sin 2\omega T)}{(\cos 2\omega T - B2\cos\omega T + B3) + j(\sin 2\omega T - B2\sin\omega T)} \tag{2.13}$$

and

$$H_3(z) = \frac{C1(z^2 - 1)}{z^2 - C2\,z + C3}$$

$$H_3(e^{j\omega T}) = \frac{(C1\cos 2\omega T - C1) + j(C1\sin 2\omega T)}{(\cos 2\omega T - C2\cos\omega T + C3) + j(\sin 2\omega T - C2\sin\omega T)} \tag{2.14}$$

where B1, B2, B3 and C1, C2, C3 are the coefficients of $H_2(z)$ and $H_3(z)$.

We can now solve equations (2.12), (2.13) and (2.14) by executing a computer program. From the computer printout, of amplitude in decibels and frequency in Hz obtained in tabular form, we plot the frequency response curve as shown in Fig. 8. It will be observed that the peak response at 2.0 Hz in the case of the 6th order analog filter has shifted to 1.9 Hz for the corresponding 6th order bandpass digital filter. This distortion of the frequency scale in the case of the digital filter in relation to that of the analog filter is due to the warping effect.

2.5 Time-Domain Analysis of the 6th Order Bandpass Digital Filter

From an earlier discussion,

$$H(z) = H_1(z) \times H_2(z) \times H_3(z) \tag{2.15}$$

or

$$\frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X_1(z)} \times \frac{Y_2(z)}{X_2(z)} \times \frac{Y_3(z)}{X_3(z)}$$

The three second order filters represented by $H_1(z)$, $H_2(z)$ and $H_3(z)$ are connected in cascade as shown in Fig. 9,

where

$$H_1(z) = \frac{Y_1(z)}{X_1(z)}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{Y_2(z)}{Y_1(z)}$$

and

$$H_3(z) = \frac{Y_3(z)}{X_3(z)} = \frac{Y_3(z)}{Y_2(z)}$$

now

$$\frac{Y_1(z)}{X_1(z)} = \frac{A1\ (z^2 - 1)}{z^2 - A2z + A3}$$

$$= \frac{A1z^2 - A1}{z^2 - A2z + A3}$$

$$\frac{Y_1(z)}{X_1(z)} = \frac{A1 - A1z^{-2}}{1 - A2z^{-1} + A3z^{-2}}$$

$$Y_1(z) = A1 \cdot X_1(z) - A1 \cdot z^{-2}X_1(z) + A2\ z^{-1}Y_1(z)$$

$$- A3 \cdot z^{-2}Y_1(z)$$

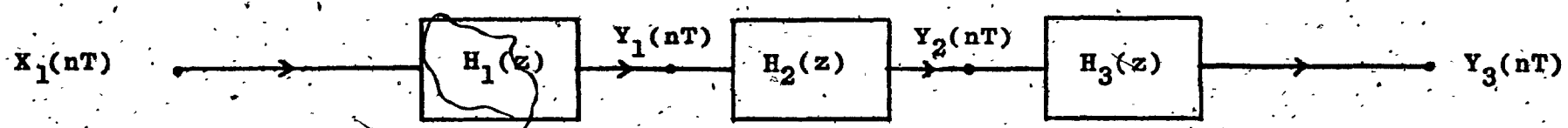


Fig. 9 Three Second Order Filters Connected in Cascade

Obtaining the inverse z-transform of above, we have

$$Y_1(nT) = A1 \cdot X_1(nT) - A1 \cdot X_1\,(nT-2T) + A2 \cdot Y_1\,(nT-T) - A3 \cdot Y_1\,(nT-2T) \qquad (2.16)$$

Similarly

$$Y_2(nT) = B1 \cdot Y_1(nT) - B1 \cdot Y_1\,(nT-2T) + B2 \cdot Y_2\,(nT-T) - B3 \cdot Y_2\,(nT-2T) \qquad (2.17)$$

and

$$Y_3(nT) = C1 \cdot Y_2(nT) - C1 \cdot Y_2\,(nT-2T) + C2 \cdot Y_3\,(nT-T) - C3 \cdot Y_3\,(nT-2T) \qquad (2.18)$$

The set of difference equations (2.16), (2.17) and (2.18) represent a 6th order bandpass digital filter having design coefficients A1, A2, A3; B1, B2, B3; and C1, C2, C3. These can be simulated by using a digital program which results in the time domain response of the digital filter shown in Fig. 10.

2.6 Frequency Pre-warping

The bilinear z-transformation

$$s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}} \qquad (2.19)$$

causes the entire left-hand side of the s-plane to be mapped into the interior of the unit circle in the z-plane as discussed in Section 1.5, Chapter 1. This transformation is useful for designing digital lowpass, bandpass, bandstop filters that have a piecewise constant magnitude function (passband and stopband).

Although the bilinear transformation is a straightforward technique that permits the design of broadband, high order

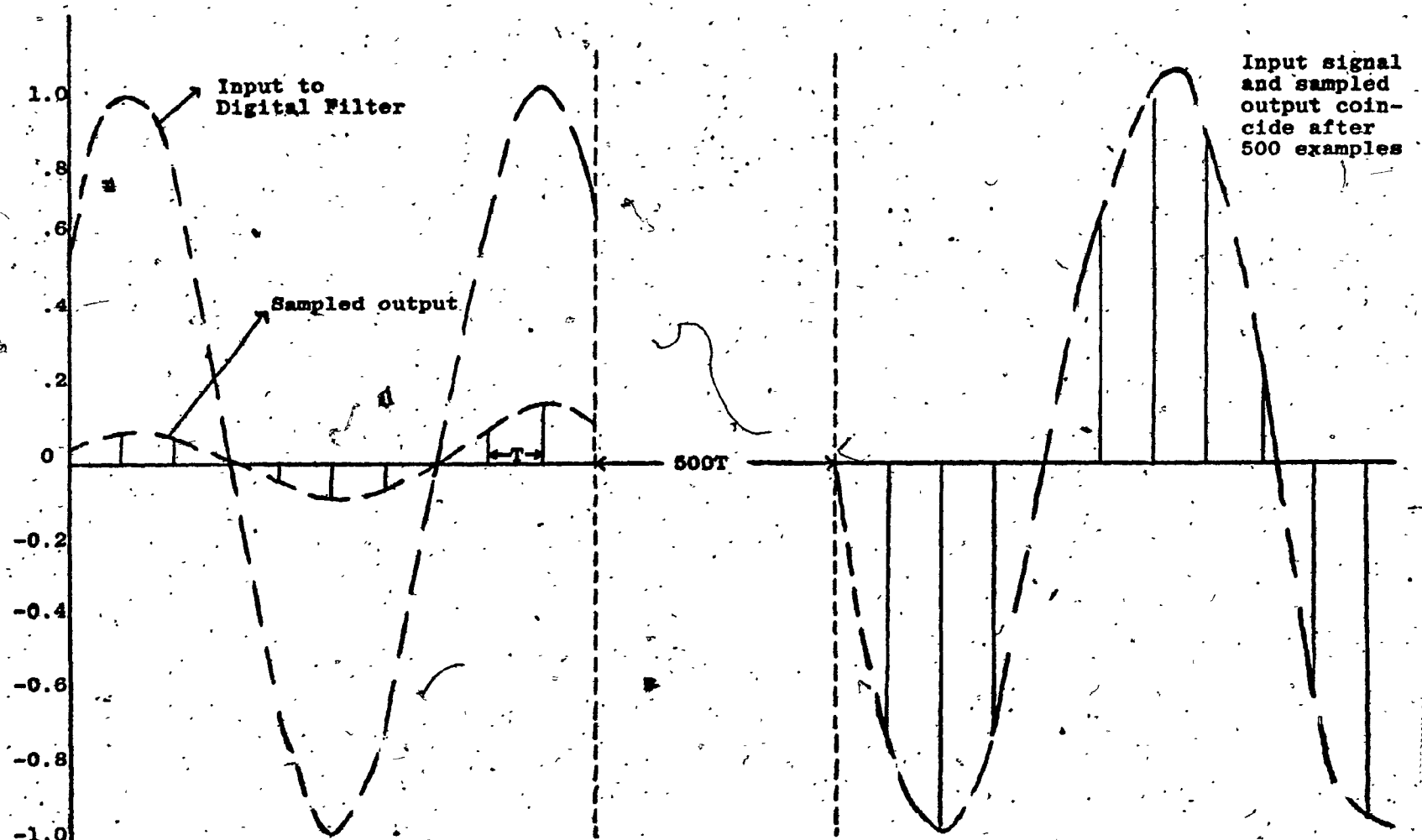


Fig. 10 Time-Domain Response of Sixth Order Bandpass Digital Filter to a Sinusoidal Input Signal with Frequency 2.0 Hz (After Prewarping)

filters, it matches only the magnitude response of the digital filter and the continuous-time prototype. This does not guarantee any preservation of the phase response and thus may not be suitable for applications where phase shift and phase delay are to be considered. Also due to the warping effect, the bilinear transformation cannot be used for the design of multi-pass band or linear-phase filters. For such filters, the continuous-time transfer function can be precorrected to cancel the effects of warping.

Consider a lowpass analog filter with a cutoff frequency ω_c. Applying the bilinear transformation gives rise to a digital filter with a cutoff frequency

$$\Omega_c = \frac{2}{T} \tan^{-1} \frac{\omega_c T}{2}$$

The cutoff frequency of the analog filter can be moved from ω_c to ω_c^1.

where $\omega_c^1 = \frac{2}{T} \tan \frac{\omega_c T}{2}$

Applying the frequency transformation, we have

$$s \rightarrow \frac{s}{\frac{2}{T\omega_c} \tan \frac{\omega_c T}{2}} \qquad (2.20)$$

For a bandpass analog transfer function, the above becomes

$$s \rightarrow \frac{s}{\frac{2}{T\omega_o} \tan \frac{\omega_o T}{2}} \qquad (2.21)$$

ω_o is the centre frequency in radians/sec

We use the relation (2.21) to prewarp the bandpass analog transfer function represented by the equation (2.11)

$\omega_o = 2\pi f_o = 4\pi$ rad., where $f_o = 2$ Hz

Hence (2.20) becomes

$$s \longrightarrow \frac{s}{\frac{2}{4\pi T} \tan \frac{4\pi T}{2}}$$

$$s = \frac{2\pi T s}{\text{Tan } 2\pi T} \qquad (2.22)$$

From equations (2.11) and (2.22), we have

$$T_1(s) = \frac{0.8\pi^2 \; T \tan (2\pi T)s}{4\pi^2 T^2 s^2 + 0.6012 \times 2\pi T \tan (2\pi T)s + 144.8137 \tan^2(2\pi T)}$$

Using bilinear transformation,

$$H_1(z) = \frac{\frac{1.6\pi^2 \tan (2\pi T)}{16\pi^2 + 0.6012 \times 4\pi \tan (2\pi T) + 144.8137 \tan^2(2\pi T)} (z^2 - 1)}{z^2 - \frac{32\pi^2 - 144.8137 \times 2 \tan^2 (2\pi T)}{16\pi^2 + 0.06012 \times 4\pi \tan (2\pi T) + 144.8137 \tan^2(2\pi T)} z + \frac{16\pi^2 - 0.6012 \times 4\pi \tan (2\pi T) + 144.8137 \tan^2(2\pi T)}{16\pi^2 + 0.6012 \times 4\pi \tan (2\pi T) + 144.8137 \tan^2(2\pi T)}}$$

Similarly

$$H_2(z) = \frac{\frac{1.6\pi^2 \tan (2\pi T)}{16\pi^2 + 0.6556 \times 4\pi \tan (2\pi T) + 172.1999 \tan^2(2\pi T)} (z - 1)}{z^2 - \frac{32\pi^2 - 172.1999 \times 2 \tan^2(2\pi T)}{16\pi^2 + 0.6556 \times 4\pi \tan (2\pi T) + 172.1999 \tan^2(2\pi T)} z + \frac{16\pi^2 - 0.6556 \times 4\pi \tan (2\pi T) + 172.999 \tan^2(2\pi T)}{16\pi^2 + 0.6556 \times 4\pi \tan (2\pi T) + 172.999 \tan^2(2\pi T)}}$$

and

$$H_3(z) = \frac{1.6\pi^2 \tan(2\pi T)}{16\pi^2 + 1.2566 \times 4\pi\tan(2\pi T) + 157.9122 \tan^2(2\pi T)}(z^2-1)$$

$$z^2 - \frac{32\pi^2 - 157.9122 \times 2\tan^2(2\pi T)}{16\pi^2 + 1.2566 \times 4\pi\tan(2\pi T) + 157.9122 \tan^2(2\pi T)} z$$

$$+ \frac{16\pi^2 - 1.2566 \times 4\pi\tan(2\pi T) + 157.9122 \tan^2(2\pi T)}{16\pi^2 + 1.2566 \times 4\pi\tan(2\pi T) + 157.9122 \tan^2(2\pi T)}$$

The prewarped, transformed transfer function in the z-domain obtained is,

$$H(z) = H_1(z) \times H_2(z) \times H_3(z).$$

Using the set of coefficients obtained from $H_1(z)$, $H_2(z)$, $H_3(z)$ and executing a computer program (see Appendix) we obtain the frequency domain response of the filter as shown in Fig. 11. It will be observed that as a result of compensating for frequency warping, the centre frequency of the digital filter is at 2.0 Hz as in the corresponding analog filter. Fig. 11 can be compared with the response curve shown in Fig. 8. The centre frequency of the digital filter was shifted to 1.9 Hz in the absense of prewarping.

Time domain analysis of the 6th order prewarped bandpass digital filter is performed in the same manner as discussed earlier except the coefficients used incorporate the prewarping feature. The frequency of the input signal is 2.0 Hz. The response curve obtained using a computer program is as shown in Fig. 10.

Up until now we had been studying the response of bandpass filters, with fixed centre frequencies to a sinusoidal signal having a fixed frequency. We shall now briefly discuss the

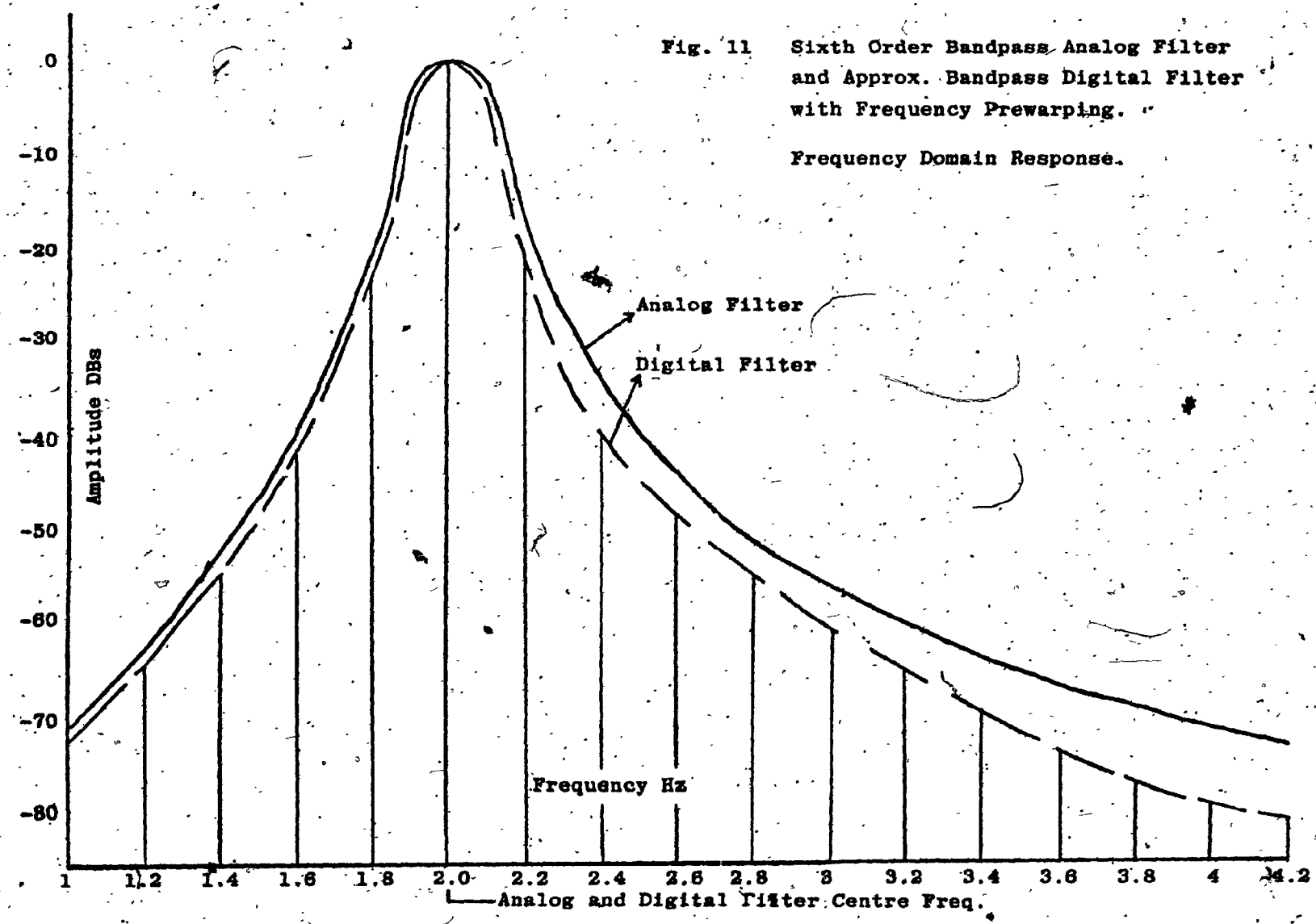


Fig. 11 Sixth Order Bandpass Analog Filter and Approx. Bandpass Digital Filter with Frequency Prewarping.

Frequency Domain Response.

response of filters having a fixed centre frequency but with the sinusoidal input signal having a varying frequency.

It was observed that when the frequency of the input signal was increased rapidly past the centre frequency of the filter, the amplitude response fell far below the required level of unity in the passband, Fig. 12. On slowly varying the input signal frequency, the response of the filter is unity in the passband, but the resonance is at a frequency less than the centre frequency of the filter, Fig. 13.

This peculiar characteristic of the computer simulated bandpass filter may be due to the Doppler effect. The following is a brief explanation of this phenomenon:

The number of points sampled between two highest adjacent peaks in the vicinity of the centre frequency = 8.

Let T be the time period between the peaks. Now

sampling period $T_s = 0.0645$ sec.

$$T = \text{number of samples} \times T_s$$

$$= 8 \times 0.0645$$

$$= 0.5160 \text{ sec}$$

signal frequency $f = \frac{1}{T} = \frac{1}{0.5160}$

$$= 2.0 \text{ Hz (approx.)}$$

Thus the peak of the response is actually at 2.0 Hz, the centre frequency of the filter and not at a much lower frequency as is apparent from the simulated response.

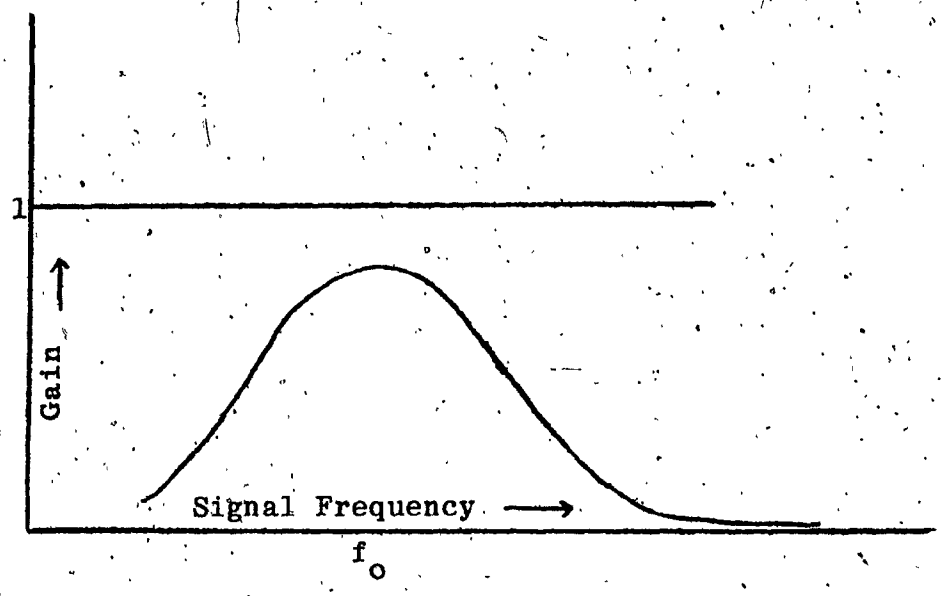


Fig.12 Response of filter to rapidly increasing signal frequency.

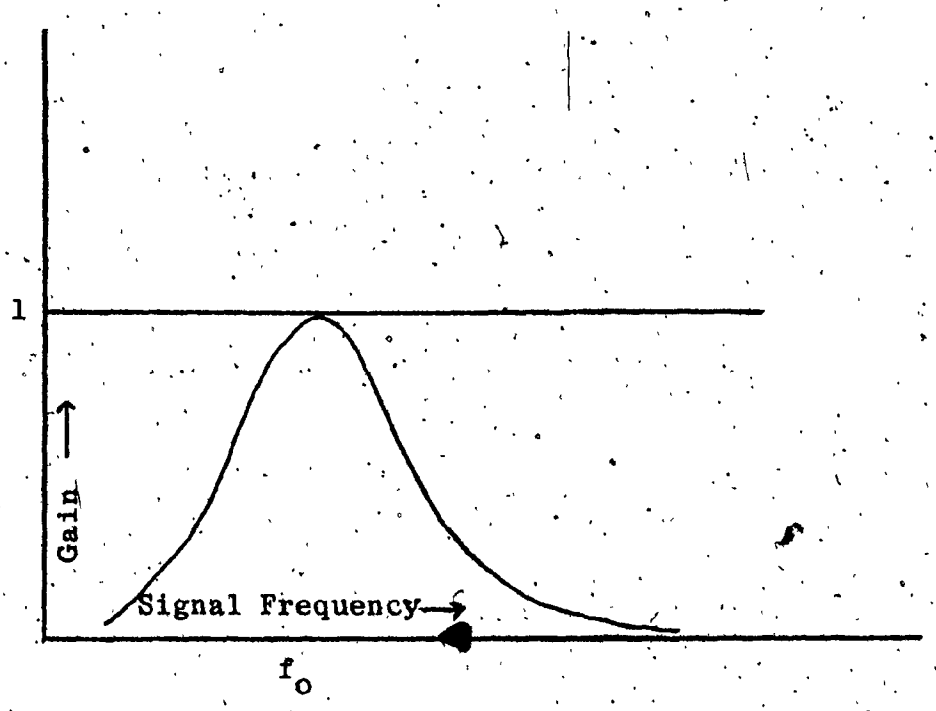


Fig. 13. Response of the filter to a slowly increasing signal frequency.

CHAPTER 3

DIGITAL WAVE ANALYZER

3.1 Introduction

In the field of electronic measurements, electrical signals may be analyzed in either the time or frequency domain.

In the time domain, the instantaneous value of a signal with respect to a time axis may be displayed on any suitable device. Information regarding signal amplitude, duration, periodicity, risetimes and fall times may be derived directly from the calibrated display. If the applied signal contains only one frequency component, information about that frequency can be obtained from a graph or from a CRT display. However, if the signal contains more than one frequency component, each of which might have a different amplitude, the various components will be combined into a single waveform and the individual characteristics of each component will then become difficult if not impossible to measure.

In Fig. 14a we have a combined frequency-time domain display of the components. The time-domain response is shown in Fig. 14b and the frequency-domain response is displayed in Fig. 14c.

A digital wave analyzer using a second order bandpass filter is simulated on the computer. A signal, consisting

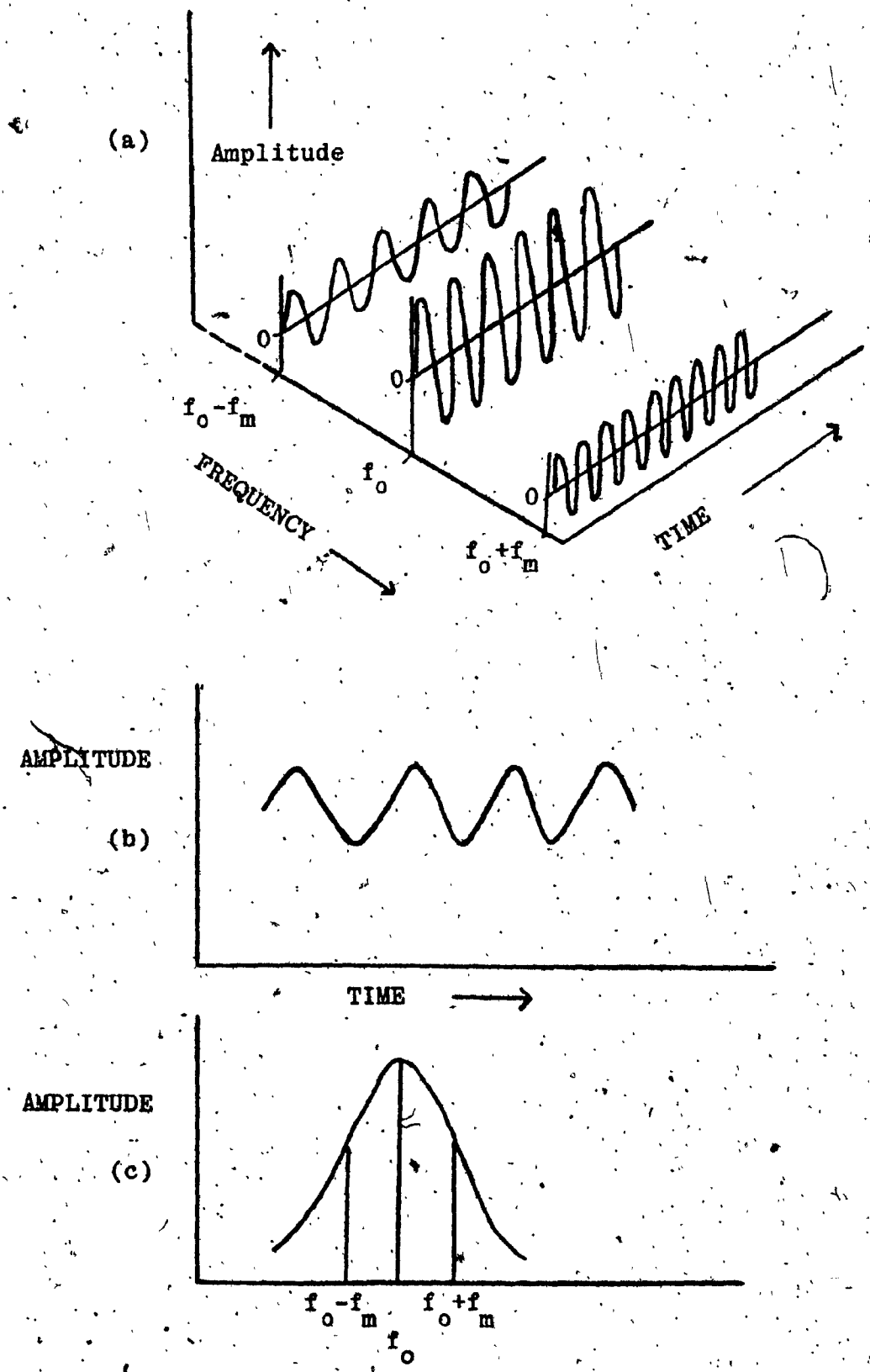


Fig. 14 Amplitude, Time and Frequency

of two different frequency components of 1.5 Hz and 1.9 Hz is applied as input to the wave analyzer. The response of the wave analyzer is observed, as the centre frequency is swept across the calibrated scale. It is found that the peaks on the response curve which should occur at the frequencies of the individual components are not sharp and accurate, resulting in the overlapping of the response curves. This problem can be overcome or considerably minimized by designing the wave analyzer with higher order filters or by using three or more second order bandpass filters connected in cascade, thereby greatly increasing the selectivity. We shall now study the responses of such a digital wave analyzer.

3.2 The Wave Analyzer

The design of a 6-th order bandpass digital filter with a slowly varying centre frequency will be discussed next.

A suitable 6-th order analog bandpass transfer function is,

$$T(s) = \frac{\omega_B^3 s^3}{s^6 + 2\omega_B s^5 + (3\omega_o^2 + 2\omega_B^2)s^4 + (4\omega_o^2\omega_B + \omega_B^3)s^3 + (3\omega_o^4 + 2\omega_B^2\omega_o^2)s^2 + 2\omega_B\omega_o^4 s + \omega_o^6} \quad (3.1)$$

where

$$\omega_o = 2\pi f_o \text{ rad/sec}$$

and

$$\omega_B = 2\pi \times f_B = 0.4\ \pi \text{ radians/sec}$$

are the centre frequency and the 3dB bandwidth, respectively.

substituting these values in equation (3.1), we obtain

$$T(s) = \frac{(0.4)^3\pi^3 s^3}{s^6 + 0.8\pi s^5 + (12\pi^2 f_o^{\,2} + 0.32\pi^2)s^4 + (6.4\pi^3 f_o^{\,2} + 0.064\pi^3)s^3 + (48\pi^4 f_o^{\,4} + 1.28\pi^4 f_o^{\,2})s^2 + 12.8\pi^5 f_o^{\,4} s + (2\pi f_o)^6} \qquad (3.2)$$

Inserting $s = \frac{\pi\ T\ Fos}{\tan\ (\pi TFo)}$ and $f_o = Fo$

in equation (3.2) to eliminate the warping effect we can write:

$$T(s) = \frac{(0.4)^3 \frac{\tan^3(\pi\ TFo)}{(TFo)^3} s^3}{s^6 + \frac{0.8\tan(\pi TFo)}{TFo} s^5 + (12\pi^2 Fo^2 + 0.32\pi^2)\frac{\tan^2(\pi TFo)}{(\pi TFo)^2} s^4 + (6.4\pi^3 Fo^2 + 0.06\pi^3)\frac{\tan^3(\pi TFo)}{(\pi TFo)^3} s^3 + (48\pi^4 Fo^4 + 1.28\pi^4 Fo^2)\frac{\tan^4(\pi TFo)}{(\pi TFo)^4} s^2 + 12.8\frac{\tan^5(\pi TFo)}{T^5 Fo} s + \left(\frac{2}{T}\right)^6 \tan^6(\pi TFo)}$$

Now by applying the bilinear transformation,

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

we have,

$$H(z) = \frac{B1\ \frac{(z-1)^3}{(z+1)^3}}{A1\ \frac{(z-1)^6}{(z+1)^6} + A2\ \frac{(z-1)^5}{(z+1)^5} + A3\ \frac{(z-1)^4}{(z+1)^4} + A4\ \frac{(z-1)^3}{(z+1)^3} + A5\ \frac{(z-1)^2}{(z+1)^2} + A6\ \frac{(z-1)^2}{(z+1)^2} + A7}$$

or

$$H(z) = \frac{B1\ (z-1)^3\ (z+1)^3}{A1\ (z-1)^6 + A2\ (z-1)^5\ (z+1) + A3\ (z-1)^4\ (z+1)^2 + A4\ (z-1)^3\ (z+1)^3 + A5\ (z-1)^2\ (z+1)^4 + A6\ (z-1)\ (z+1)^5 + A7\ (z+1)^6}$$

where

$$A1 = 64$$

$$A2 = \frac{25.6}{Fo} \tan(\pi TFo)$$

$$A3 = \frac{16\,(12\,Fo^2 + 0.32)}{Fo^2} \tan^2(\pi TFo)$$

$$A4 = \frac{8(6.4\,Fo^2 + 0.064)}{Fo^3} \tan^3(\pi TFo)$$

$$A5 = \frac{16\,(12\,Fo^2 + 0.32)}{Fo^2} \tan^4(\pi TFo)$$

$$A6 = \frac{25.6}{Fo} \tan^5(\pi TFo)$$

$$A7 = 64 \tan^6(\pi TFo)$$

$$B1 = \frac{0.512}{Fo^3} \tan^3(\pi TFo)$$

From equation (3.4) we get the numerator polynomial

$$N(z) = B1\,(z^6 - 3z^4 + 3z^2 - 1)$$

and the denominator polynomial

$$\begin{aligned} D(z) = {} & (A1 + A2 + A3 + A4 + A5 + A6 + A7)\,z^6 \\ & - (6A1 + 4A2 + 2A3 - 2A5 - 4A6 - 6A7)\,z^5 \\ & + (15A1 + 5A2 - A3 - 3A4 - A5 + 5A6 + 15A7)\,z^4 \\ & - (20A1 - 4A3 + 4A5 - 20A7)\,z^3 \\ & + (15A1 - 5A2 - A3 + 3A4 - A5 - 5A6 + 15A7)\,z^2 \\ & - (6A1 - 4A2 + 2A3 - 2A5 + 4A6 - 6A7)\,z \\ & + (A1 - A2 + A3 - A4 + A5 - A6 + A7) \end{aligned}$$

if

$$C1 = A1 + A2 + A3 + A4 + A5 + A6 + A7$$

$$C2 = 6A1 + 4A2 + 2A3 - 2A5 - 4A6 - 6A7$$

$$C3 = 15A1 + 5A2 - A3 - 3A4 - A5 + 5A6 + 15A7$$

$$C4 = 20A1 - 4A3 + 4A5 - 20A7$$

$$C5 = 15A1 - 5A2 - A3 + 3A4 - A5 - 5A6 + 15A7$$

$$C6 = 6A1 - 4A2 + 2A3 - 2A5 + 4A6 - 6A7$$
$$C7 = A1 - A2 + A3 - A4 + A5 - A6 + A7$$

the 6th order transfer function can be expressed as

$$H(z) = \frac{B1z^6 - 3B1z^4 + 3B1z^2 - B1}{C1z^6 - C2z^5 + C3z^4 - C4z^3 + C5z^2 - C6z + C7}$$

This gives

$$\frac{Y(z)}{X(z)} = \frac{B1 - 3B1z^{-2} + 3B1z^{-4} - B1 \cdot z^{-6}}{C1 - C2z^{-1} + C3z^{-2} - C4z^{-3} + C5z^{-4} - C6z^{-5} + C7z^{-6}}$$

consequently we can derive the difference equation,

$$Y(nT) = \frac{B1}{C1}\left[x(nT) - 3X(nT - 2T) + 3x(nT - 4T) - x(nT - 6T)\right] + \frac{C2}{C1}Y(nT - T) \frac{C3}{C1}Y(nT - 2T) + \frac{C4}{C1}Y(nT - 3T) - \frac{C5}{C1}Y(nT - 4T) + \frac{C6}{C1}Y(nT - 5T) - \frac{C7}{C1}Y(nT - 6T) \qquad (3.5)$$

This equation can be used to simulate a digital wave analyzer on the computer. A suitable flowchart is shown in Fig. 15.

3.3 Digital Wave Analyzer using Three Second Order Band-Pass Digital Filters Connected in Cascade.

In this case the second-order bandpass analog transfer function is

$$T(s) = \frac{\omega_B s}{s^2 + \omega_B s + \omega_o^2}$$

where

$$\omega_B = 0.4\pi \text{ rad/sec}$$
$$\omega_o = 2\pi F_o \text{ rad/sec}$$

are the 3dB bandwidth and centre frequency, respectively.

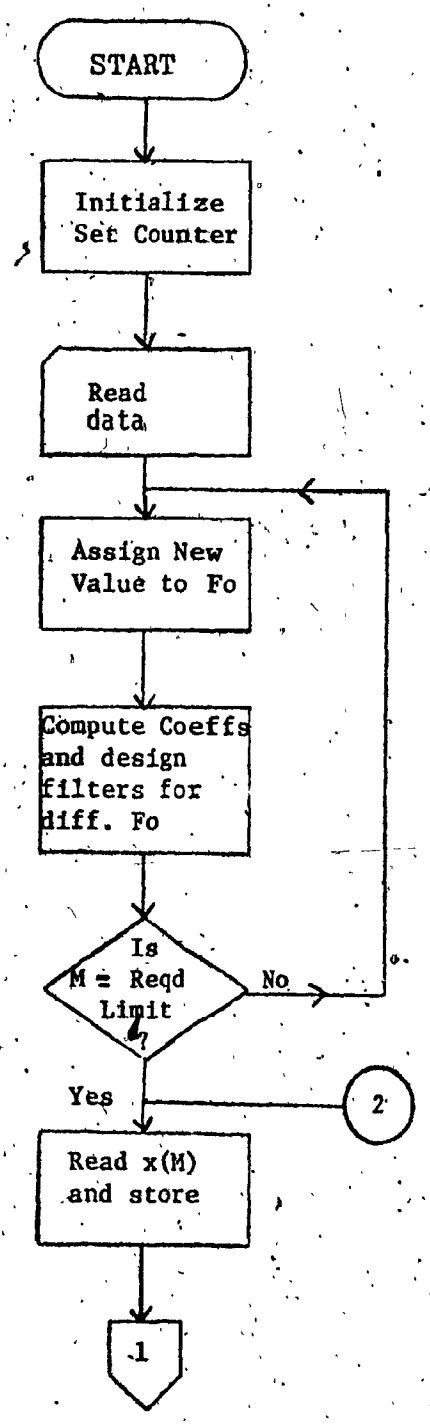


Fig. 15 Program Flowchart for Digital Wave Analyzer

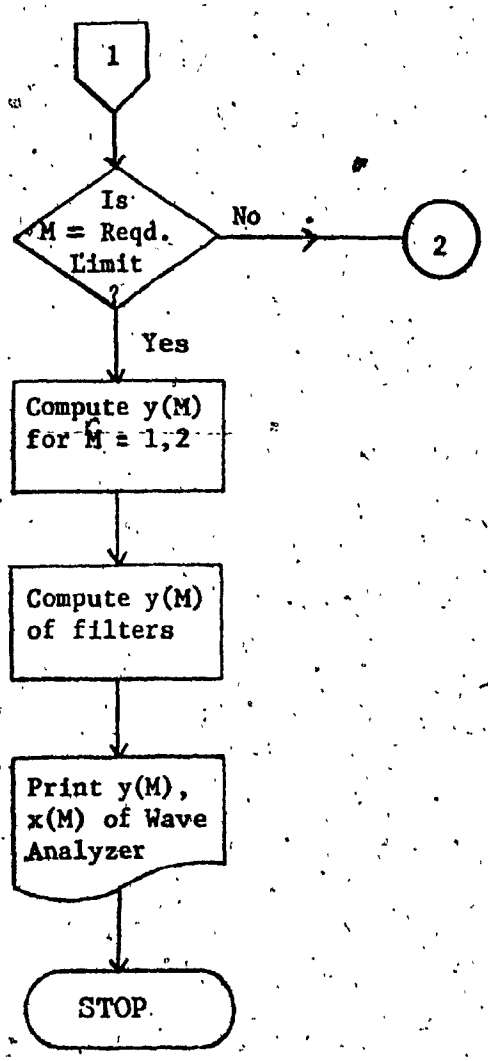


Fig. 15 Program Flowchart for Digital Wave Analyzer

Hence

$$T(s) = \frac{0.4\pi s}{s^2 + 0.4\pi s + 4\pi^2 Fo^2}$$

Compensating for frequency scale warping;

$$s \longrightarrow \frac{s}{\frac{2}{\omega_0 T} \tan \frac{\omega_0 T}{2}} = \frac{\pi T Fo}{\tan (\pi TFo)} s$$

and

$$T(s) = \frac{0.4\pi^2 TFo \tan (\pi TFo)\, s}{(\pi TFo)^2 s^2 + 0.4\pi^2 TFo \tan (\pi TFo) s + 4\pi^2 Fo^2 \tan^2(\pi TFo)} \qquad (3.6)$$

Transforming equation (3.6) into the z-domain using relation $s = \frac{2}{T}\frac{z-1}{z+1}$, we have

$$H(z) = \frac{\frac{0.8 \tan (\pi TFo)}{4Fo+0.8 \tan (\pi TFo) + 4Fo \tan^2(\pi TFo)} (z^2 - 1)}{z^2 - \frac{8Fo[1-\tan^2(\pi TFo)]}{4Fo+0.8 \tan (\pi TFo) + 4Fo \tan^2(\pi TFo)} z + \frac{4Fo-0.8 \tan (\pi TFo) + 4Fo \tan^2(\pi TFo)}{4Fo+0.8 \tan (\pi TFo) + 4Fo \tan^2(\pi TFo)}} \qquad (3.7)$$

where the coefficients of transfer function (3.7) are,

$$A1 = \frac{0.8 \tan (\pi TFo)}{4Fo + 0.8 \tan (\pi TFo) + 4Fo \tan^2 (\pi TFo)}$$

$$B1 = \frac{8Fo\,[1-\tan^2 (\pi TFo)]}{4Fo + 0.8 \tan (\pi TFo) + 4Fo \tan^2 (\pi TFo)}$$

$$B2 = \frac{4Fo - 0.8 \tan (\pi TFo) + 4Fo \tan^2 (\pi TFo)}{4Fo + 0.8 \tan (\pi TFo) + 4Fo \tan^2 (\pi TFo)}$$

Thus

$$H(z) = \frac{A1\,(z^2 - 1)}{z^2 - B1z + B2} \qquad (3.8)$$

This transfer function can be realized as shown in Fig. 16

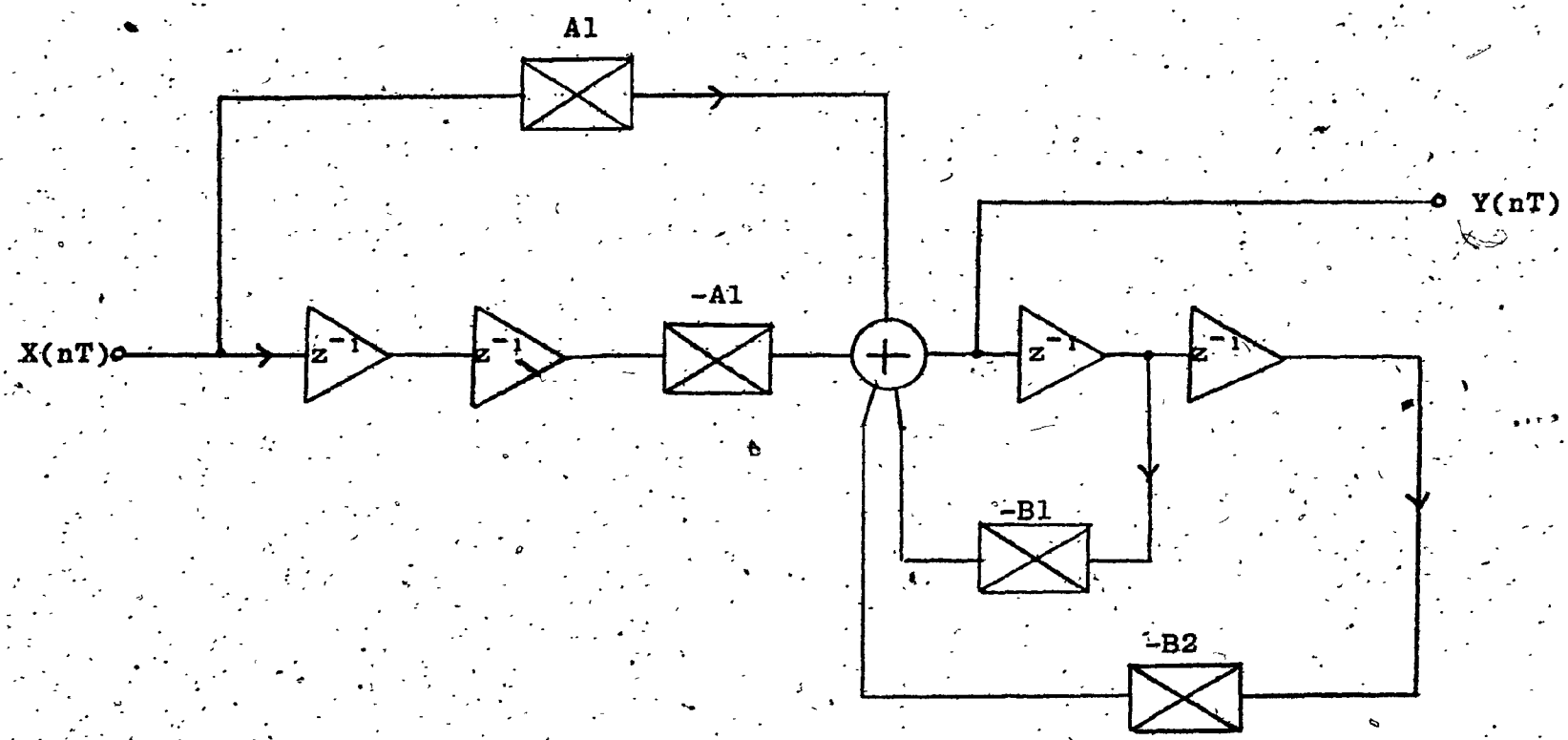


Fig. 16 Second Order Recursive Bandpass Digital Filter with Real Coefficients A1, -A1, -B1, -B2

From equation (3.8)

$$H(e^{j\omega T}) = \frac{A1\ (e^{2j\omega T}-1)}{e^{2j\omega T}-B1(e^{j\omega T}) + B2}$$

$$= \frac{A1\ (\cos 2\omega T + j \sin 2\omega T - 1)}{(\cos 2\omega T + j \sin 2\omega T) - B1\ (\cos\omega T + j \sin\omega T) + B2}$$

$$= \frac{(A1 \cos 2\omega T - A1) + j\ A1 \sin 2\omega T}{(\cos 2\omega T - B1 \cos\omega T + B2) + j\ (\sin 2\omega T - B1 \sin\omega T)}$$

$$= \frac{A2 + j\ A3}{A4 + j\ A5}$$

where

$$A2 = A1 \cos 2\omega T - A1$$

$$A3 = A1 \sin 2\omega T$$

$$A4 = \cos 2\omega T - B1 \cos\omega T + B2$$

$$A5 = \sin 2\omega T - B1 \sin\omega T$$

Now if three second order filters are connected in cascade, the overall transfer function will be

$$H(z) = H_1(z) \times H_2(z) \times H_3(z)$$

Thus

$$\frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X_1(z)} \times \frac{Y_2(z)}{X_2(z)} \times \frac{Y_3(z)}{X_3(z)}$$

Considering the output of the first filter as the input to the second and the output of the second filter as the input to the third filter, while the output of the third filter becomes the overall output of the cascaded network, we have

$$H1(z) = \frac{Y_1(z)}{X_1(z)}$$

$$H2(z) = \frac{Y_2(z)}{X_2(z)} = \frac{Y_2(z)}{Y_1(z)}$$

$$H3(z) = \frac{Y_3(z)}{X_3(z)} = \frac{Y_3(z)}{Y_2(z)}$$

Using the above information a computer program was constructed to obtain the frequency domain response of the cascaded filters. The response is as shown in Fig. 17. The set of three difference equations,

$$Y_1(nT) = A1X_1(nT) - A1 \cdot X_1(nT-2T) + B1Y_1(nT-T) - B2Y_1(nT-2T)$$

$$Y_2(nT) = A1 \cdot Y_1(nT) - A1 \cdot Y_1(nT-2T) + B1Y_2(nT-T) - B2Y_2(nT-2T)$$

$$Y_3(nT) = A1 \cdot Y_2(nT) - A1 \cdot Y_2(nT-2T) + B1Y_3(nT-T) - B2Y_3(nT-2T)$$

was simulated on the computer to obtain the time domain response of the filter.

It was observed that by cascading three bandpass filter, the amplitude response fell off at a much higher rate outside the passband given by $\omega_1 \leq \omega \leq \omega_2$.

The coefficients A1, B1, B2 of these equations depend on variable Fo, the centre frequency of the bandpass filters. In the frequency domain analysis of different waveforms the variable centre frequency Fo is sweeped across the spectrum of the given input signal. The response of the wave analyzer thus obtained has peaks at the different frequency components of the input signal. The positions of the centre frequency on the calibrated scale at which these peaks occur determine the frequency of the different components of the unknown input signal. Also the magnitude of these peaks are the amplitudes of the frequency components.

Our discussion will be concerned with the processing of periodic signals consisting of both simple and complex

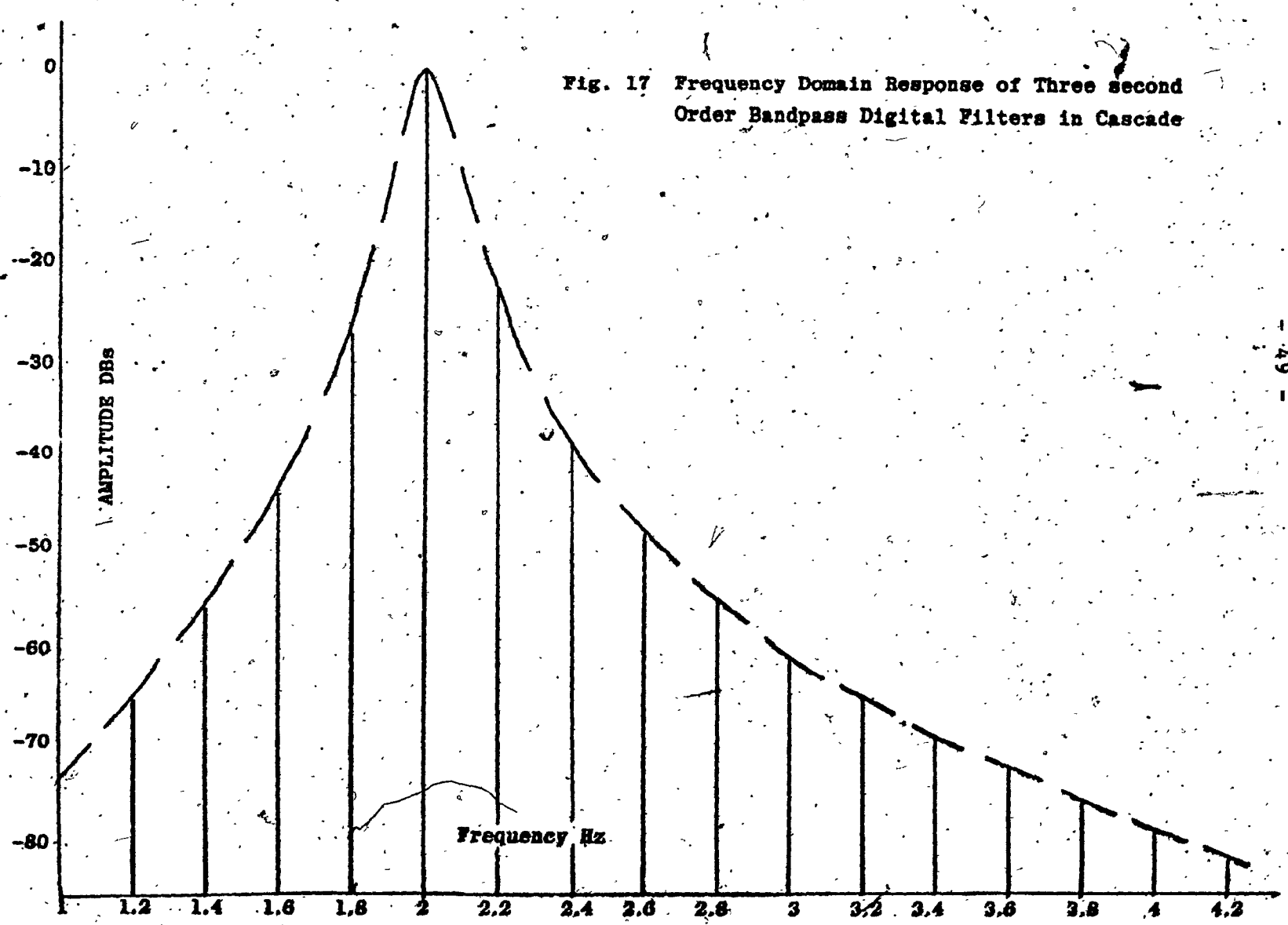


Fig. 17 Frequency Domain Response of Three second Order Bandpass Digital Filters in Cascade

waveforms. Firstly, we shall study the output of the digital wave analyzer when the input signal consists of one or more different sinusoidal components. The Input signal in this case will be,

$$x = \sin(2\pi fnT) \qquad (3.10)$$

where

f = frequency of input sinewave

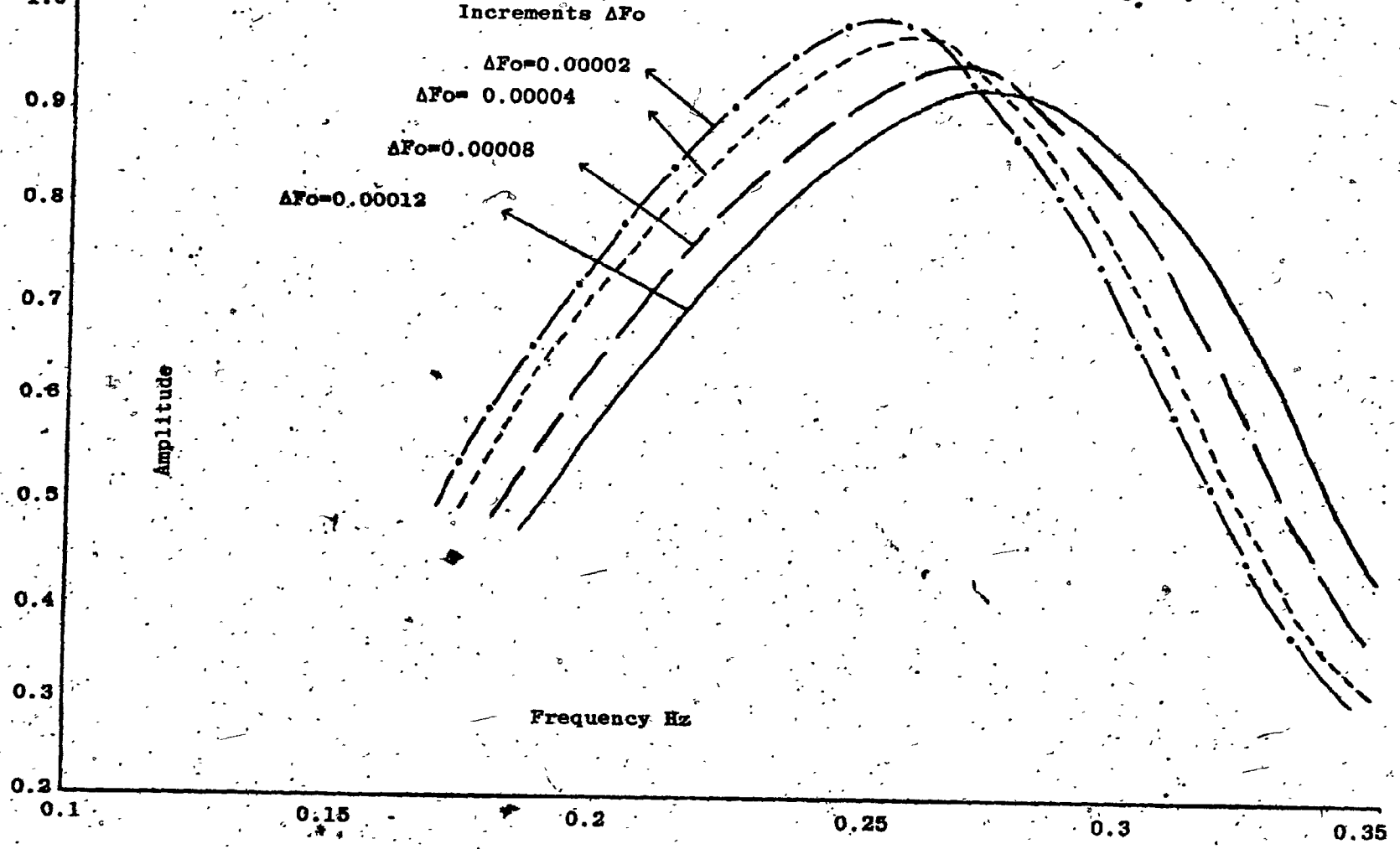
n = number of samples

T_s = sampling period

In our design of a digital wave analyzer we shall use the sampling frequency $f_s = 40.0$ Hz ($T_s = 0.025$ sec). As discussed earlier the sampling frequency was about 14.0 Hz ($T = 0.0645$) for bandpass digital filters with a fixed centre frequency f_o. In the case of a wave analyzer we have a moving centre frequency, and hence to compensate for this, the sampling frequency is kept at double the Nyquist frequency fN, where $fN = 2f_o$. This condition is required for the input signal to be sampled and that the maximum information be recovered from the sampled signal.

It is observed in Fig. 18 that when the centre frequency f_o is moved in large steps, the response of the wave analyzer for a particular frequency component occurs at a higher value on the calibrated frequency scale. Also the amplitude of the peak is less than the amplitude of the frequency components. The performance of the digital wave analyzer however improves

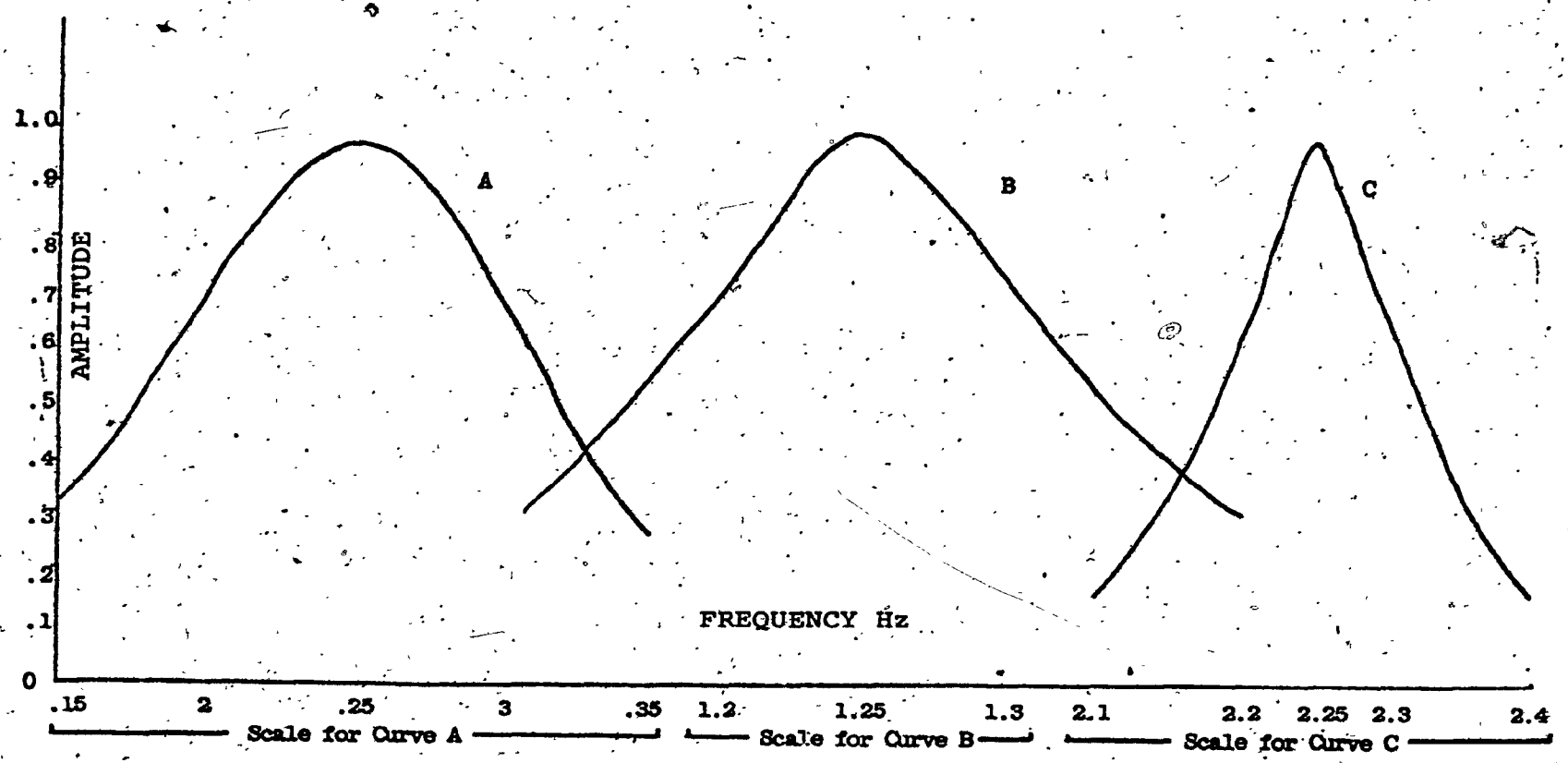
Fig. 18 Response of Digital Wave Analyzer to a Single Frequency Component for Four Different Values of Centre Frequency Increments ΔFo



by changing the centre frequency f_o by a very small increment and the increments Δf_o are decreased till the response of the wave analyzer is almost perfect, as shown for an input signal having a frequency 0.25 Hz. This particular study of the wave analyzer simulated on the computer agrees with practical observation, when using a Tetronix spectrum analyzer. When the centre frequency f_o is swept rapidly, the oscilloscope display shows that the centre frequency does not lock-on the signal freqency properly, resulting in a response low in amplitude and slightly higher in value on the calibrated frequency scale. Sweeping f_o slowly gives a sharp response and the position of the centre frequency on the calibrated frequency scale shows the proper value of the unknown signal frequency.

Next, the response of the wave analyzer was recorded for a signal consisting of three different sinusoidal components. Using $\Delta f_o = 0.00002$ as the increment in the centre frequency, the peaks are obtained in the response curve at points on the frequency scale where the centre frequency coincides with the frequency components of the signal being analyzed as shown in Fig. 19. There is no frequency folding or aliasing in this particular example, since the frequencies of the components are widely spaced and also the sampling frequency is above twice the Nyquist frequency. Aliasing problems can in practice

Fig. 19 Response of The Digital Wave Analyzer to an Input Signal Containing Three Frequency Components



be eliminated by using a lowpass filter at the input of the wave analyzer to bandlimit the continuous-time signal to be sampled.

Additional features can be incorporated in the wave analyzer. A rectifier and a peak detector can be employed as shown in Fig. 20. to obtain the envelopes of the sinusoidal variations in the response. This helps in eliminating the unwanted information of the response, thereby greatly reducing the computer printout time. Also it facilitates in simplifying the plotting of the response curves.

3.4 Analysis of Complex Waveforms

We shall now discuss the analysis of more complex but periodic waveforms. Such waveforms (for example rectangular, square, triangular, sawtooth waveforms) are made up of sinusoidal components of different frequencies and amplitudes. These waveforms can be easily represented by Fourier series and the larger the number of terms on the Fourier summation the nearer the resultant signal approaches the given wave-shape. Also, the sharper the discontinuity of the waveform the higher the frequency content of the signal.

There are some waveforms which display half wave symmetry. Fig. 21 shows examples of such waveforms. Fig. 21(b) exhibits symmetry while Fig. 21(a) does not.

The condition for a waveform to have symmetry is

$$F(t) = -f(t + T/2) \qquad (3.11)$$

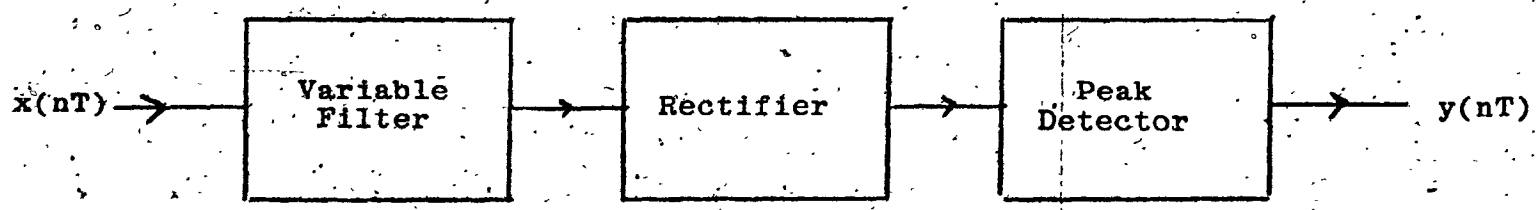


Fig. 20 Wave Analyzer Containing a Rectifier and a Peak Detector.

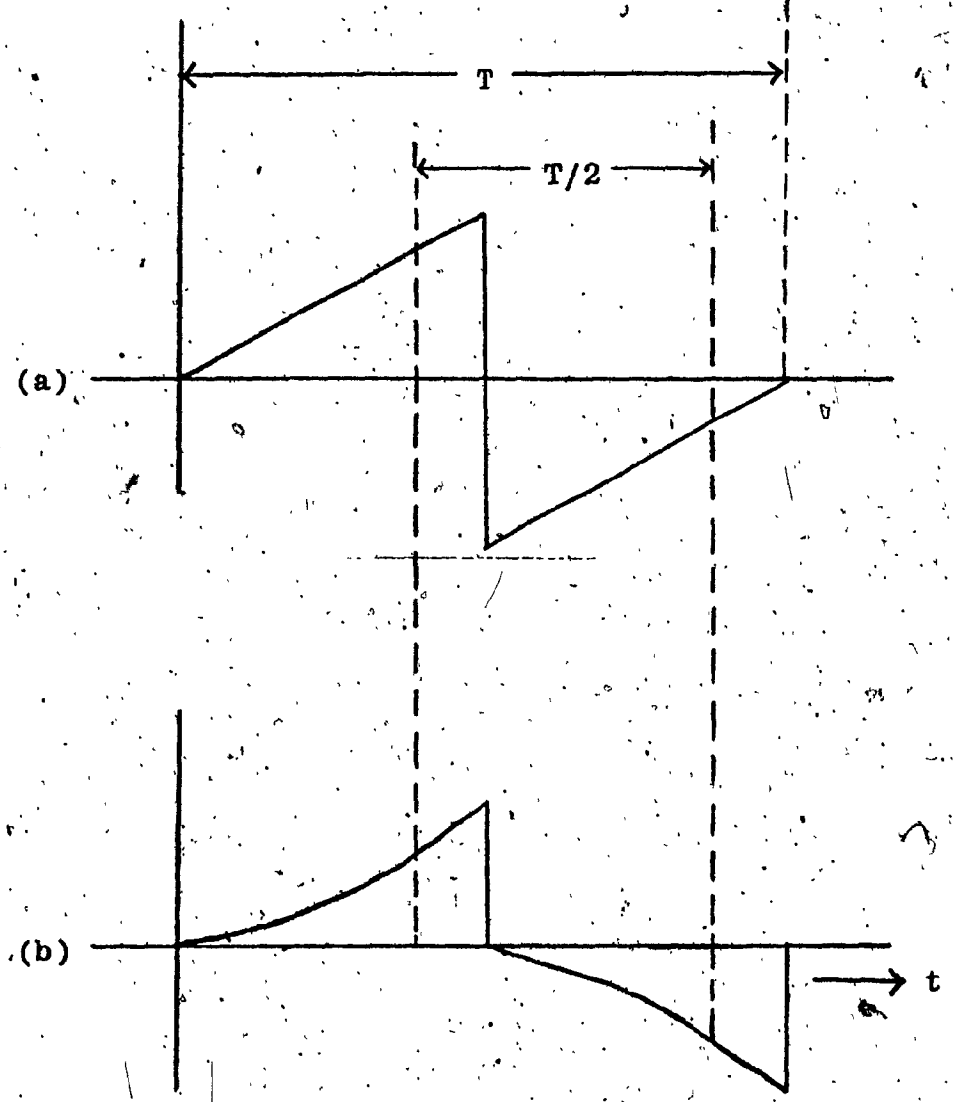


Fig. 21 (a) Non symmetric and
(b) symmetric waveforms

In other words, any two values of the waveform separated by T/2 will be equal in magnitude but opposite in sign. Generally, waveforms having only odd harmonics exhibit half-wave symmetry. Thus, a waveform of any complexity which shows such a symmetry cannot contain even harmonic components. Conversely, a waveform containing any second, fourth or other even harmonic components cannot display half-wave symmetry.

To illustrate the absence of even-order harmonics in a wave with half-wave symmetry, we obtain the Fourier analysis of a rectangular wave having half-wave symmetry using our digital wave analyzer comprizing a 6th order bandpass filter. The input to the wave analyzer is a rectangular wave with half-wave symmetry having a time period

$$T = N \times T_s$$
$$= 400 \times 0.025$$
$$= 10 \text{ sec}$$

where N is the number of samples contained per rectangular wave cycle, and T_s is the sampling period.

Using a computer program the amplitude response was obtained as shown in Fig. 22. The peaks at the first, third and fifth harmonics of 0.1 Hz, 0.3 Hz and 0.5 Hz respectively are not sharp and well-defined. This can be improved by using higher ordered band-pass filters which will increase the selectivity of the wave analyzer.

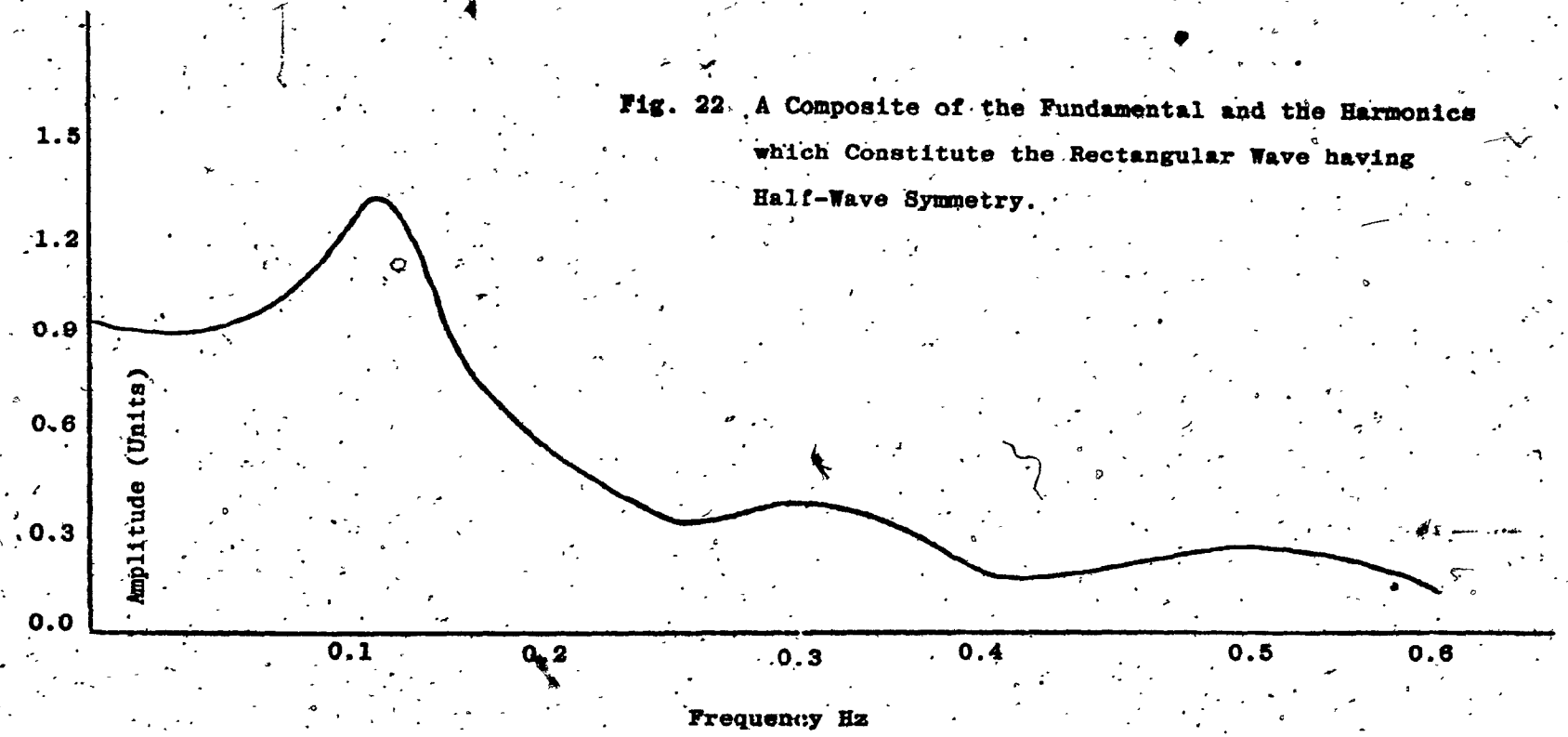


Fig. 22 A Composite of the Fundamental and the Harmonics which Constitute the Rectangular Wave having Half-Wave Symmetry.

Redesigning the wave analyzer by using five second order bandpass digital filters in cascade, the more selective amplitude response of Fig. 23 is obtained. The response of the wave analyzer to the rectangular wave of the previous example is obtained as shown in Fig. 24. As can be observed the response of the wave analyzer has improved greatly as compared to that of the 6th order wave analyzer.

The amplitudes at the fundamental, third and fifth harmonics agree, (see Table 1) with the Fourier series for a rectangular wave given by

$$f(t) = \frac{4}{\pi}\left(\cos\omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t + \ldots\ldots\right) \tag{3.12}$$

As expected, the wave contains no even harmonics. For our next example we decrease the period of the rectangular wave so as to increase the fundamental frequency from 0.1 Hz to 0.25 Hz, see Fig. 25(a), (b).

Centre frequency fo = 0.25 Hz

$$T = \frac{1}{f_o} = \frac{1}{0.25} = 4.0 \text{ sec}$$

Now $N \times \frac{1}{f_s} = N \times T_s = 4$

where N is the number of samples and T_s is the sampling period.

$$N \times 0.025 = 4$$

$$N = \frac{4}{.025} = \frac{4000}{25} = 160 \text{ samples}$$

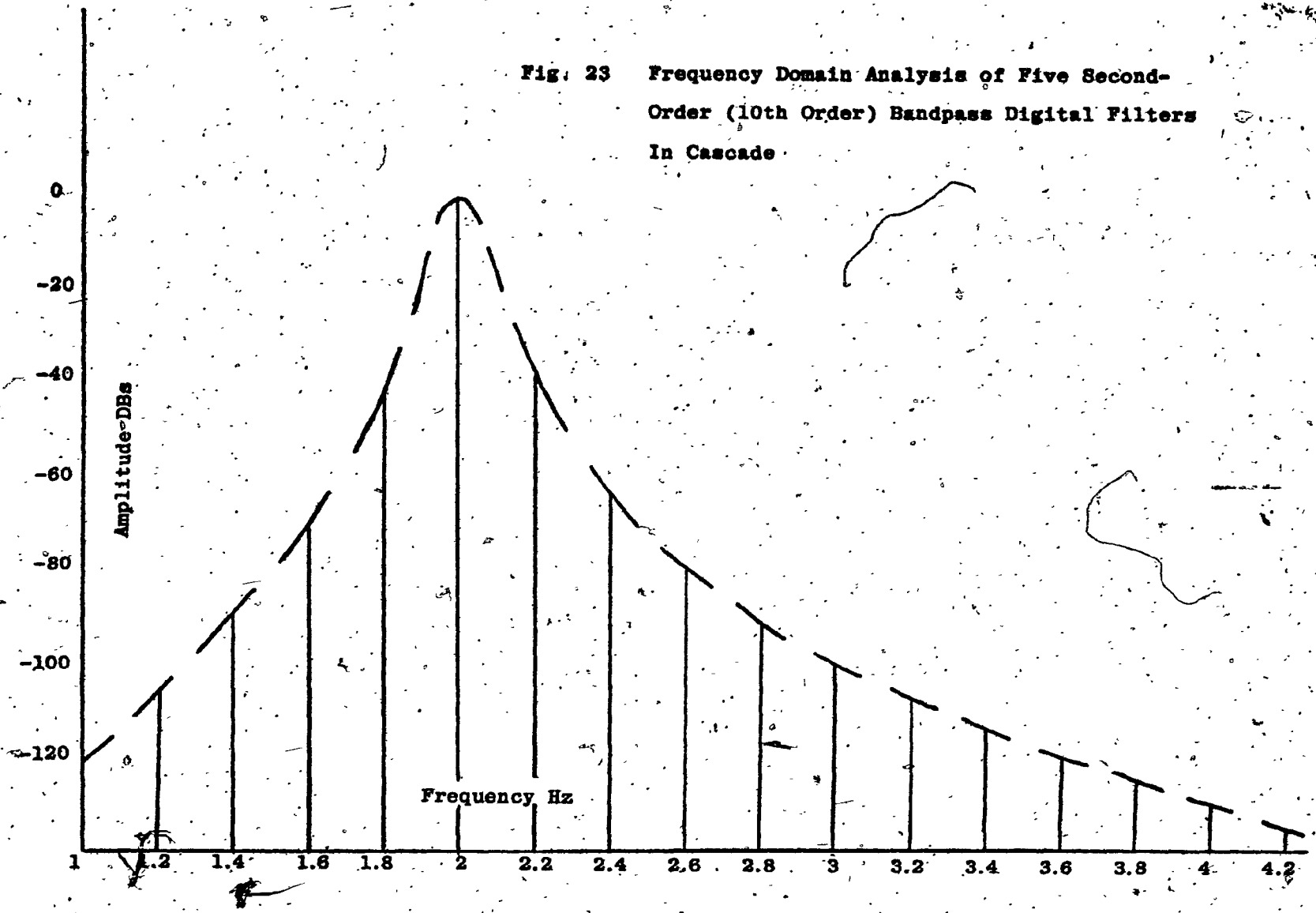


Fig. 23 Frequency Domain Analysis of Five Second-Order (10th Order) Bandpass Digital Filters In Cascade

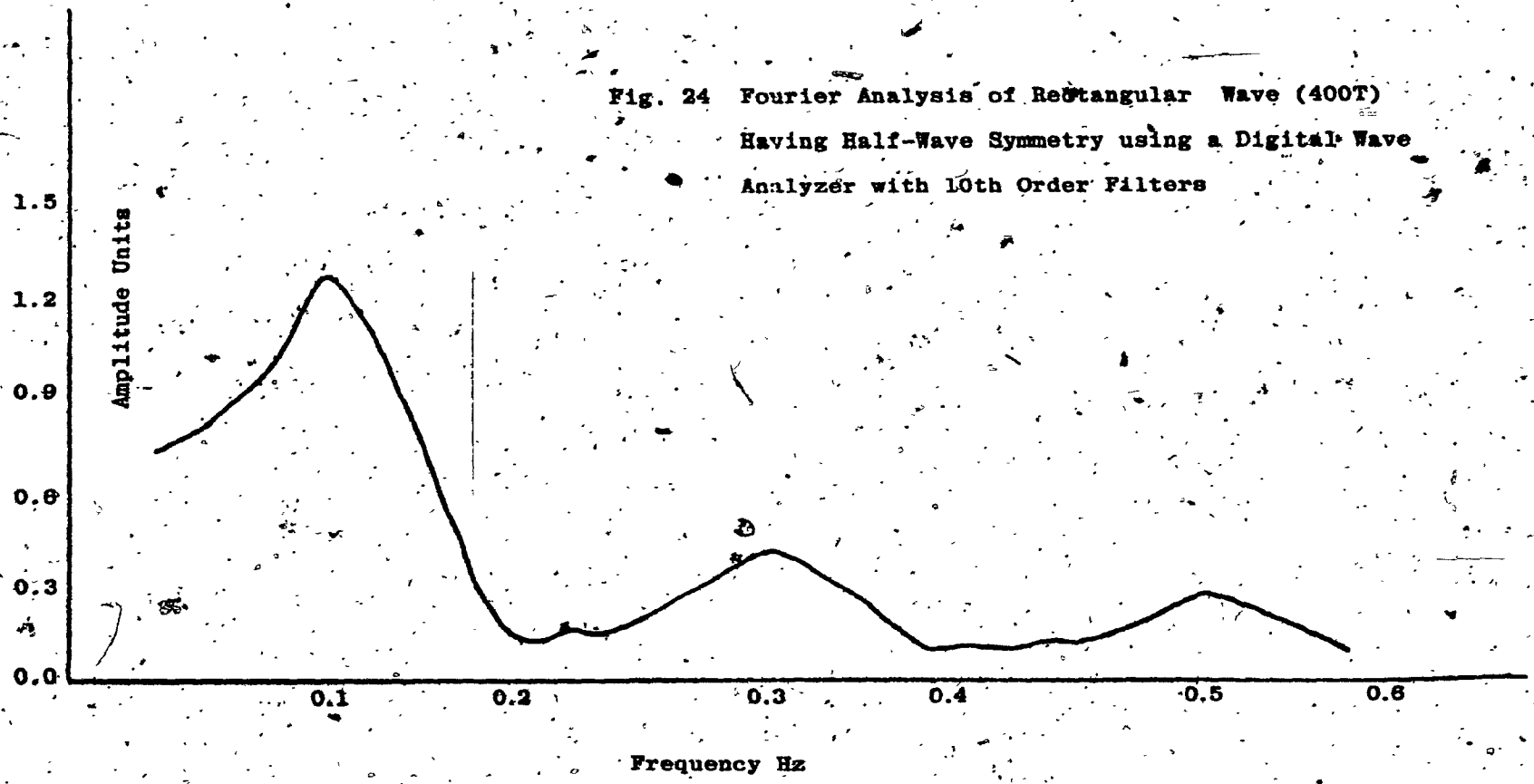


Fig. 24 Fourier Analysis of Rectangular Wave (400T) Having Half-Wave Symmetry using a Digital Wave Analyzer with 10th Order Filters

Table 1

Harmonic	Amplitude		Frequency	
	Measured	Theoretical	Measured	Theoretical
Fundamental	1.2689	1.2726	0.0999 Hz	0.1 Hz
Third	0.4013	0.4242	0.2998 Hz	0.3 Hz
Fifth	0.2594	0.2545	0.5007 Hz	0.5 Hz

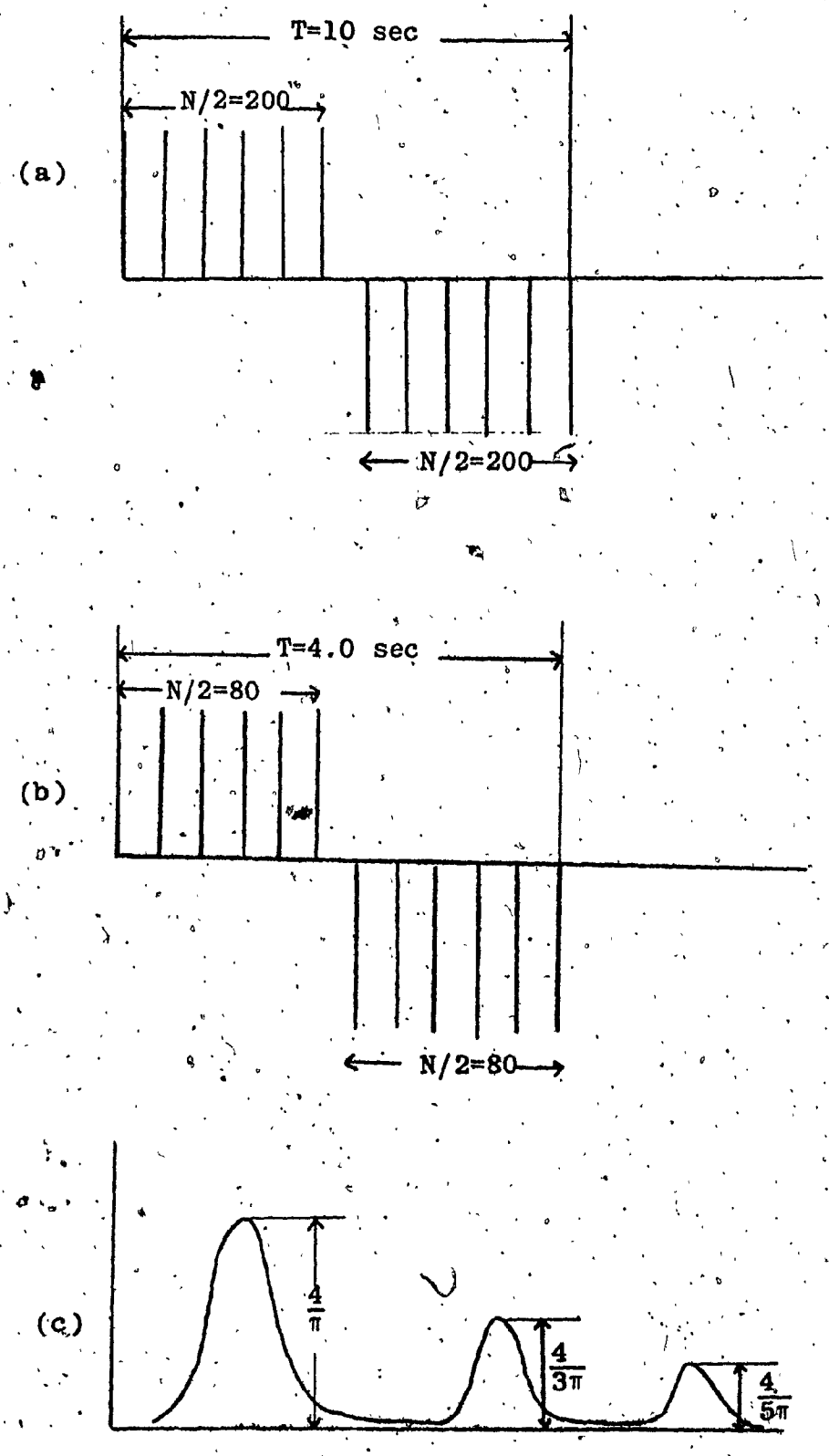


Fig 25 (a), (b) Rectangular Wave, and (c) The Wave Analyzer Response Representing its Fourier Series

Hence decreasing the period T of the rectangular wave and then performing the analysis as before, we obtain the Fourier response, Fig. 25(c) and Fig. 26 which agrees with the Fourier summation series given by equation (3.12). The fundamental is 0.25 Hz as expected and the third and fifth harmonies are 0.75 Hz and 1.25 Hz respectively with the amplitudes as required. Plots for the time-domain responses of the rectangular wave for the fundamental and third harmonic are shown in Fig. 27.

In a similar manner we can obtain the frequency analysis of other periodic complex waveforms.

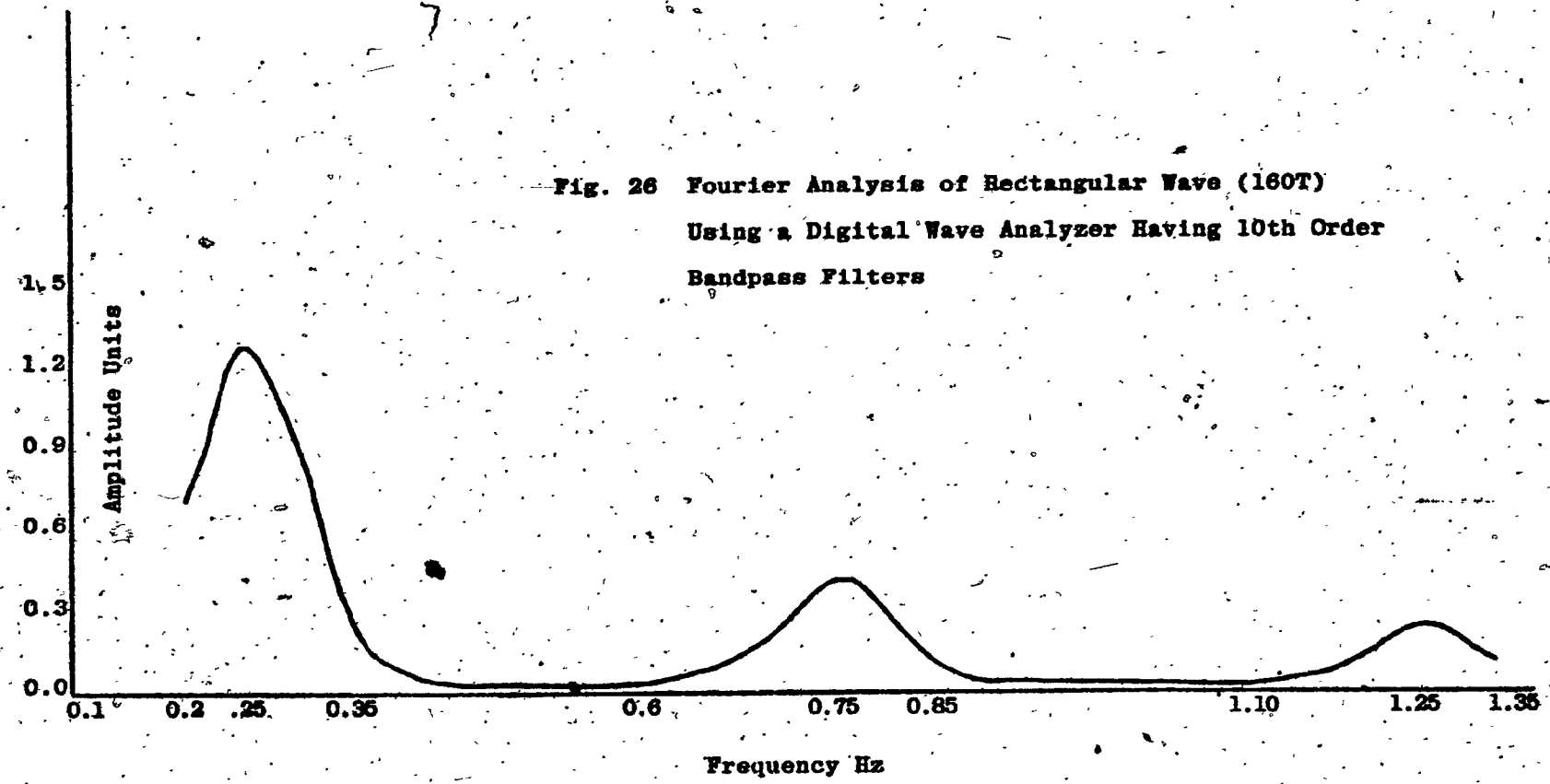
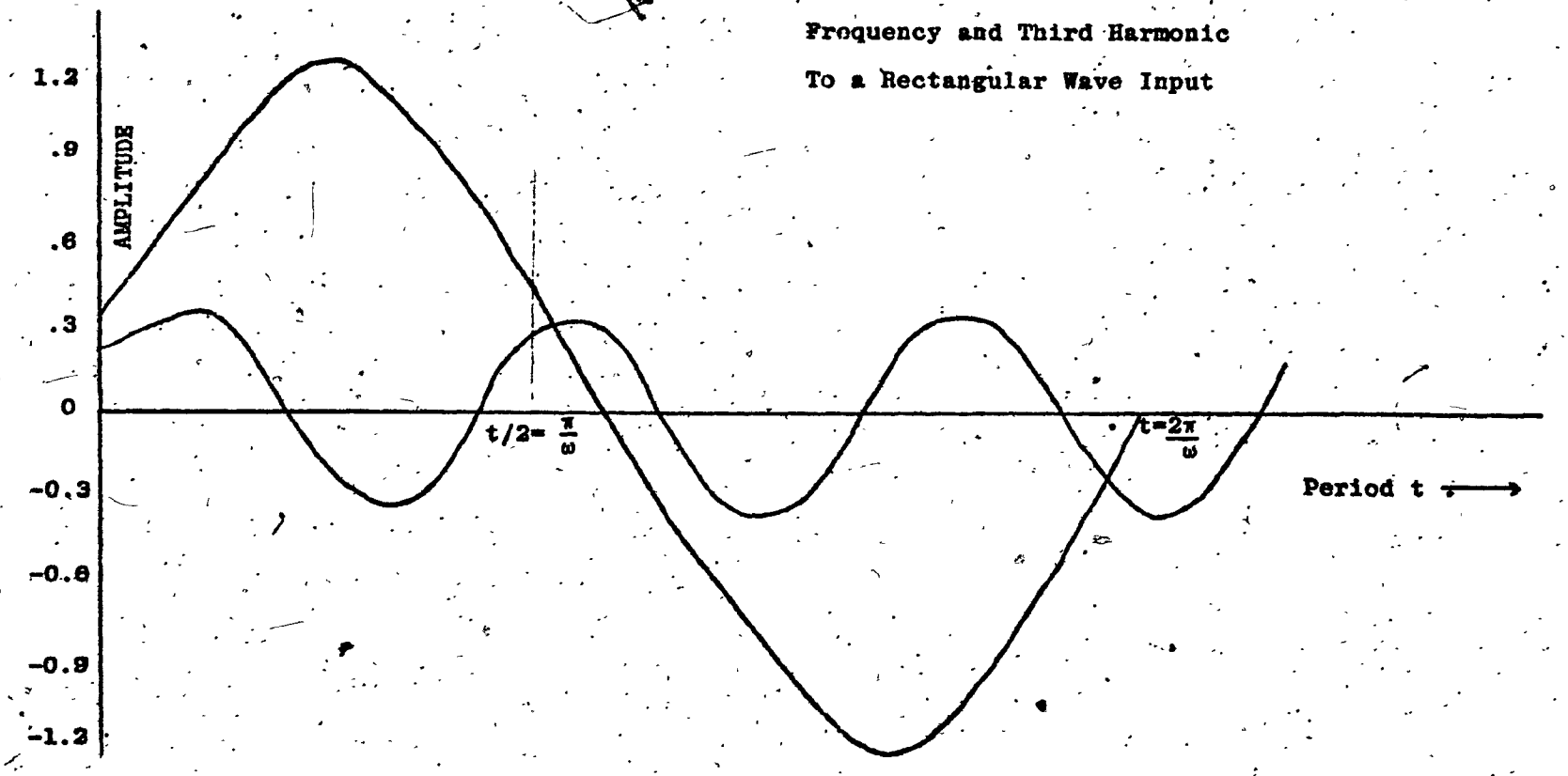


Fig. 26 Fourier Analysis of Rectangular Wave (160T) Using a Digital Wave Analyzer Having 10th Order Bandpass Filters

Fig. 27 Time Domain Response of The Digital Wave Analyzer Representing a Fundamental Frequency and Third Harmonic To a Rectangular Wave Input



CHAPTER 4

CONCLUSION

This report was primarily concerned with recursive bandpass digital filters and ultimately the simulation of a digital wave analyzer. The wave analyzer thus obtained was used successfully in the analysis of various signals consisting of simple sinusoidal as well as complex waveforms.

For simple periodic signals which can be described mathematically, the Fourier series remains the best method of analysis. For periodic signals, however, with no mathematical characterization, the wave analyzer method offers an alternative to the Fast Fourier Transform technique.

No attempt was made to discuss the purely theoretical aspects of digital filters. The aim was to obtain an understanding of the design of digital filter transfer functions and to illustrate the difference in the performance characteristics of low and high ordered digital filters. Hopefully, all these objectives are adequately met within the limited scope of this project.

APPENDIX: COMPUTER PROGRAMS

```
PROGRAM.BPDF        73/73    OPT=1                    FTN 4.2+P380        75/07/18. 1

                PROGRAM BPDF (INPUT,OUTPUT)
                REAL AMAGN,AMAGD
                T=0.0645
                FO=2.0
                A1=(0.8*TAN(3.14159*T*FO))/(4.0+FO+0.8*TAN(3.14159*T*FO)+4.0*FO*(T
               1AN(3.14159*T*FO))**2)
                B1=(8.0*FO-8.0*FO*(TAN(3.14159*T*FO))**2)/(4.0+FO+0.8*TAN(3.14159*
               1T*FO)+4.0*FO*(TAN(3.14159*T*FO))**2)
                B2=(4.0*FO-0.0*TAN(3.14159*T*FO)+4.0*FO*(TAN(3.14159*T*FO))**2)/(4
               1.0*FO+0.8*TAN(3.14159*T*FO)+4.0*FO*(TAN(3.14159*T*FO))**2)
                PRINT 150
                F=0.0
                DO 100 I=1,47
                X=I
                IF(X.LE.6.0) F=F+0.25
                IF(X.GT.6.0.AND.X.LE.26.0) F=F+0.05
                IF(X.GT.26.0.AND.X.LE.47.0) F=F+0.25
                W=2*3.14159*F
                A2=A1*COS(2*W*T)-A1
                A3=A1*SIN(2*W*T)
                A4=COS(2*W*T)-B1*COS(W*T)+B2
                A5=SIN(2*W*T)-B1*SIN(W*T)
                AMAGN=SQRT(A2**2+A3**2)
                AMAGD=SQRT(A4**2+A5**2)
                FMAG1=AMAGN/AMAGD
                FMAG=(FMAG1)**5
                H=20*ALOG10(FMAG)
                TETN1=ATAN(A3/A2)
                TETD1=ATAN(A5/A4)
                FTET1=TETN1-TETD1
                FTHET=(FTET1)**5
                PRINT 200,F,H,FTHET
            200 FORMAT(9X,F10.4,9X,F10.4,12X,F10.6)
            150 FORMAT(10X,*(FREQUENCY HZ)*,7X,*(MAGNITUDE DBS)*,7X,*PHASE*,//)
            100 CONTINUE
                STOP
                END
```


SIR GEORGE WILLIAMS U

```
PROGRAM     BPDF                                        CDC 6600 FTN V3.0-P296 OPT=1   75/027

            PROGRAM BPDF (INPUT,OUTPUT)
            DIMENSION X(300),Y(300)
            DIMENSION Y1(300),Y2(300),Y3(300)
            F=2.0
            T=0.0645
            DO 100 M=1,300
            X(M)=SIN(2*3.14159*F*M*T)
        100 CONTINUE
            A1=(1.6*3.14159**2*TAN(2*3.14159*T))/(16*3.14159**2+0.6012*4*3.141
           159*TAN(2*3.14159*T)+144.8137*(TAN(2*3.14159*T))**2)
            A2=(32*3.14159**2-144.8137*2*(TAN(2*3.14159*T)**2))/(16*3.14159**2
           1+0.6012*4*3.14159*TAN(2*3.14159*T)+144.8137*(TAN(2*3.14159*T))**2)
            A3=(16*3.14159**2-0.6012*4*3.14159*TAN(2*3.14159*T)+144.8137*(TAN(
           12*3.14159*T))**2)/(16*3.14159**2+0.6012*4*3.14159*TAN(2*3.14159*T)
           2+144.8137*(TAN(2*3.14159*T))**2)
            B1=(1.6*3.14159**2*TAN(2*3.14159*T))/(16*3.14159**2+0.6556*4*3.141
           159*TAN(2*3.14159*T)+172.1999*(TAN(2*3.14159*T))**2)
            B2=(32*3.14159**2-172.1999*2*(TAN(2*3.14159*T)**2))/(16*3.14159**2
           1+0.6556*4*3.14159*TAN(2*3.14159*T)+172.1999*(TAN(2*3.14159*T))**2)
            B3=(16*3.14159**2-0.6556*4*3.14159*TAN(2*3.14159*T)+172.1999*(TAN(
           12*3.14159*T))**2)/(16*3.14159**2+0.6556*4*3.14159*TAN(2*3.14159*T)
           2+172.1999*(TAN(2*3.14159*T))**2)
            C1=(1.6*3.14159**2*TAN(2*3.14159*T))/(16*3.14159**2+1.2566*4*3.141
           159*TAN(2*3.14159*T)+157.9122*(TAN(2*3.14159*T))**2)
            C2=(32*3.14159**2-157.9122*2*(TAN(2*3.14159*T)**2))/(16*3.14159**2
           1+1.2566*4*3.14159*TAN(2*3.14159*T)+157.9122*(TAN(2*3.14159*T))**2)
            C3=(16*3.14159**2-1.2566*4*3.14159*TAN(2*3.14159*T)+157.9122*(TAN(
           12*3.14159*T))**2)/(16*3.14159**2+1.2566*4*3.14159*TAN(2*3.14159*T)
           2+157.9122*(TAN(2*3.14159*T))**2)
            Y1(1)=A1*X(1)
            Y1(2)=A1*X(2)+A2*Y1(1)
            Y2(1)=B1*Y1(1)
            Y2(2)=B1*Y1(2)+B2*Y2(1)
            Y3(1)=C1*Y2(1)
            Y3(2)=C1*Y2(2)+C2*Y3(1)
            DO 200 M=3,300
            Y1(M)=A1*X(M)-A1*X(M-2)+A2*Y1(M-1)-A3*Y1(M-2)
            Y2(M)=B1*Y1(M)-B1*Y1(M-2)+B2*Y2(M-1)-B3*Y2(M-2)
            Y3(M)=C1*Y2(M)-C1*Y2(M-2)+C2*Y3(M-1)-C3*Y3(M-2)
            Y(M)=Y3(M)
        200 CONTINUE
            PRINT 20
         20 FORMAT(1H1,30X,*INPUTS ARE*,59X,*TIME DOMAIN RESPONSES ARE*,//)
            PRINT 30,(X(N),Y(N),N=1,300)
         30 FORMAT(//,300(30X,F10.6,64X,F10.6,/))
            STOP
            END
```

```
PROGRAM     DWANZ                                   CDC 6600 FTN V3.0-P296 OPT=1   75/0

            PROGRAM DWANZ (INPUT,OUTPUT)
            T=0.025
            F0=0.0009
            PRINT 150
            X1=0.0
            X2=0.0
            Y11=0.0
            Y12=0.0
            Y21=0.0
            Y22=0.0
            Y31=0.0
            Y32=0.0
            AL=0.0
            Y4=0.0
            I=0.0
            DO 25 J=I,20000
            ALL=AL
            AL=Y4
            A1=(0.8*TAN(3.14159*T*F0))/(4.0*F0+0.8*TAN(3.14159*T*F0)+4.0*F0*(T
           1AN(3.14159*T*F0))**2)
            B1=(8.0*F0-8.0*F0*(TAN(3.14159*T*F0))**2)/(4.0*F0+0.8*TAN(3.14159*
           1T*F0)+4.0*F0*(TAN(3.14159*T*F0))**2)
            B2=(4.0*F0-0.8*TAN(3.14159*T*F0)+4.0*F0*(TAN(3.14159*T*F0))**2)/(4
           1.0*F0+0.8*TAN(3.14159*T*F0)+4.0*F0*(TAN(3.14159*T*F0))**2)
            N=0.005*J
            L=J/200
            IF(L/2*2.EQ.L) X=1
            IF(L/2*2.NE.L) X=-1
            Y1=A1*X-A1*X2+B1*Y11-B2*Y12
            Y2=A1*Y1-A1*Y12+B1*Y21-B2*Y22
            Y3=A1*Y2-A1*Y22+B1*Y31-B2*Y32
            Y4=SQRT(Y3**2)
            IF(Y4.LT.0.10) GO TO 30
            IF(Y4.LT.AL.AND.ALL.LT.AL) PRINT 200,X,F0,Y4
         30 CONTINUE
            X2=X1
            X1=X
            Y12=Y11
            Y11=Y1
            Y22=Y21
            Y21=Y2
            Y32=Y31
            Y31=Y3
            F0=F0+0.00003
         25 CONTINUE
        150 FORMAT(1H1,30X,*INPUTS ARE*,22X,*CENTER FREQUENCY HZ*,22X,*FILTER
           1OUTPUTS ARE*,//)
        200 FORMAT(30X,F10.6,30X,F10.6,22X,F10.6)
            STOP
            END
```

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