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LA THÈSE A ÉTÉ MICROFILMÉE TELLE QUE NOUS L'AVONS RÉCEPTIONNÉE
Dynamic Analysis of Rotating Structures

Ashok Kaushal

A Thesis
in
The Department
of
Mechanical Engineering

Presented in Partial Fulfillment of the Requirements for the Degree of Master of Engineering at Concordia University Montréal, Québec, Canada

August 1985

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ABSTRACT

Dynamic Analysis of Rotating Structures

Ashok Kaushal

The objective of this thesis is to study the dynamic behavior of rotating structures. The structure, depending on the aspect ratio is modelled as either a rotating beam or a rotating plate.

Energy formulation and finite element modelling are carried out for a rotating beam type structure. Natural frequencies and mode shapes for such a structure are determined. The effect of the setting angle, hub radius and rotational speed on these natural frequencies and mode shapes are also presented and discussed.

Beam characteristic orthogonal polynomials are generated for analyzing rotating and non-rotating plates. Experimental results for a plate with all edges free are obtained and compared to validate the computed analytical results.
ACKNOWLEDGEMENT

The author is sincerely grateful to his supervisor Dr. R.B. Bhat for his enthusiastic guidance and continuous encouragement during the course of this work.

The help and assistance provided by friends and colleagues, especially by Mr. K.G. Balasubramaniam and Mr. Anil Dhir is greatly acknowledged. Thanks are also due to Ms. Lilian Jacob for her cooperation and understanding during the course of this work.

Further, the author is grateful to his parents for their abundant moral support and understanding throughout the course of this investigation.

Last but not least, the author is thankful to Mrs. Ilana Crawford for typing the thesis.
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NOMENCLATURE

$a, b$  plate side dimensions
$E$  Young's modulus of elasticity
$D$  flexural rigidity of the plate
$h$  thickness of the plate
$I_x$  moment of inertia of the beam cross-section
$L$  length of the beam
$p$  natural frequency
$R$  hub radius
$R_o$  dimensionless parameter
$\bar{R}$  dimensionless parameter
$R_1, R_2$ and $R_3$  radius of curvature
$T_f$  flexural kinetic energy
$T_R$  rotational kinetic energy
$U$  strain energy
$W$  deflection of the plate
$xyz$ and $XYZ$  coordinate axes
$\bar{z}$  dimensionless parameter
$\omega$  rotational speed
$\theta$  setting angle
$\nu$  Poisson's ratio
$\alpha$  dimensionless parameter
$\alpha_1$  plate side ratio
$\rho$  density of the material
$n$  dimensionless parameter
$E, \beta_1$  angles
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CHAPTER 1

INTRODUCTION, LITERATURE SURVEY AND OBJECTIVES
CHAPTER 1.

INTRODUCTION, LITERATURE SURVEY AND OBJECTIVES

1.1 General

The design of rotating structures such as helicopter rotors, long and flexible rotating space booms and wind turbines requires a good understanding of their vibrational behavior in operation. Centrifugal forces are set up in these structures due to their rotation, which deflect the structure and at the same time cause in-plane strains resulting in the stiffening of the structure.

Due to the change in structural characteristics with the speed of rotation, the natural frequencies and associated modes of free vibration will be much different from those under non-rotating conditions of such structures. The resonant frequencies of these rotating structures can only be established with absolute certainty by means of experiments in a spinning rig, in which the structure is excited by a piezo-electric element cemented to the rotating structure, and a detector crystal is used to measure the response.

1.2 Literature Survey

The determination of natural frequencies and mode shapes of rotating structures, such as turbine blades, is highly important in the design of turbomachines, because blade failures are normally attributed to fatigue which occurs when the blade vibrates under resonant conditions. It thus becomes imperative that one should be able to determine the natural frequencies and mode shapes of such rotating structures as accurately as possible.
Since the blades are idealized as beams for high aspect ratios and plates for low aspect ratios, the vibrations of rotating cantilever beams and plates have been studied in several investigations. The methods of solution of such rotating structures can broadly be classified as belonging to either the continuum model approach or the discrete model approach. In the continuum model approach, applications of the potential energy method, the complementary energy principle or the Galerkin process for the solution of such structures have been thoroughly investigated and developed. The resulting differential equations obtained using these methods are solved using a Runge-Kutta procedure. In the discrete model approach, the Holzer-Myklestad, Stodola, polynomial frequency equation transformation, station function, finite difference and finite element methods, are well developed. All these discrete model approaches suffer from the drawback of the discretization process of the distributed mass and elasticity, thus yielding lower bound solutions.

Various investigators in this field considered either one or a combination of a few of the blade parameters such as rotational speed, pretwist etc. to determine their effects on the natural frequencies. A general survey of such works is presented by Rao [1].

Energy expressions for a rotating beam undergoing transverse vibrations were derived by Carnegie [2] and he obtained the fundamental frequency of vibration using Rayleigh's method. Schilhansl [3] derived the equation of motion for the bending vibrations of a rotating cantilever beam with uniform cross-section and solved it by successive approximations to determine the effect of rotation on the fundamental frequency. Rubenstein and Stadler [4] and Pnueli [5] investigated the vibrations of
rotating cantilever beams and found that the rotation of the beam tended to increase the natural frequencies of flexural vibration compared to those for the non-rotating beam. Subrahmanyam, Kulkarni and Rao [6] used Reissner method to obtain the natural frequencies of rotating blades of a symmetric aerofoil cross-section with allowance for shear deflection and rotary inertia and showed that the method gives results which are superior to those obtained by using the potential energy expression in the Ritz method. Sutherland [7] has used a Myklestad type method by a suitable modification of the equations, relating the shears and the moments of consecutive stations on the beam, to take into account the effect of the centrifugal forces. Kumar [8] also used Myklestad method to obtain both out of plane and in-plane vibration frequencies of rotating beams with tip mass. Isabson and Eisley [9] used the extended Holzer-Myklestad procedure and Slypex [10] used the Stodala method to determine the natural frequencies of pretwisted cantilever blades. The method of Frobenius was used to solve for the natural frequencies and mode shapes of centrifugally stiffened beams by Wright, Smith, Thresher and Wang [11]. The effects of hub radius and tip mass on the bending natural frequencies of rotating beams were studied by Handleman, Boyce and Cohen [12], Lo, Goldberg and Bogdanoff [13] and Boyce and Handleman [14]. In these studies it was observed that the hub radius influenced the bending natural frequencies significantly in various modes. The effect of the setting angle on the natural frequencies was studied by Wang et al [15]. An improved strain energy formulation in Rayleigh-Ritz method was used by Kaushal and Bhat [16] to obtain the natural frequencies of a rotating cantilever beam. The shear distribution was integrated along the beam subject to appropriate boundary conditions to obtain the moment distribution and using this
information the strain energy was obtained. Bhat [17] studied the
transverse vibrations of a rotating cantilever beam with tip mass by using
beam-characteristic orthogonal polynomials in Rayleigh-Ritz method.

The idealization of rotating structures as rotating cantilever beams
is acceptable for high aspect ratios but at low aspect ratios these
structures are expected to behave like plates, rather than beams.

In 1909, Ritz postulated his famous variation method [18] and pre-
sented as a demonstration a solution for the force vibration of a com-
pletely free rectangular plate, the deflection function for which was
assumed to be a series of multiplications of free-free beam vibration mode
shapes [19]. His example has been followed by numerous investigators who
have used series of appropriate beam functions to numerically investigate
various types of plate problems.

The flexural vibration of rectangular plates by using beam vibration
mode shapes as admissible functions in both the Rayleigh and Rayleigh-Ritz
methods of analysis has been treated extensively by Young [20], Leissa
[21], and Dickinson [22].

The convergence and accuracy of Ritz's method has been discussed by
various authors including Trefftz [23], Courant [24], and Collatz [25].

While Ritz's method is well known, it has not been used as much as
might be expected for plate-vibration problems. This is probably due,
at least in part, to the great amount of computational labor which is
required both to set up and to solve the necessary equations. The amount
of computation involved depends to a large extent upon the set of functions
that are used to represent the plate deflection. For these functions,
some investigators have used combinations of the characteristic functions
which define the normal modes of a vibration of a uniform beam. Pickett [26] used polynomial functions as an application to a plate equilibrium problem. Bassily and Dickinson [27] used degenerated beam functions to study flexure problems concerning static deflection or free vibration of plates involving free edges. An alternative set of admissible functions, derived from the mode shapes of vibration of plates having two parallel edges simply supported and boundary conditions on the other two edges appropriate for the plate under consideration, was suggested by Dickinson and Li [28]. Even though these functions, which were called the "simply-supported plate functions", provided superior results for plates supported in some manner along all four edges, they did not yield satisfactory results when some of the plate edges were free. Laura [29] used "polynomial coordinate functions" to approximate the fundamental natural frequencies of systems, however, a basic limitation of this type of approximation is the difficulty in evaluating the higher frequencies. Goldfracht and Rosenhouse [30] used a bipolynomial series to approximate the deflection shapes of plates and obtained the natural frequencies and mode shapes of fixed plates with partial rotational flexibility of the edges employing Galerkin's method. They provided explicit algebraic expressions for the first nine modes. The deflection shapes defined by the bipolynomial series and are not orthogonal to each other and hence the resulting expressions become quite cumbersome. Bhat [31] proposed a set of beam characteristic orthogonal polynomials that can be used as plate deflection functions to obtain the natural frequencies and mode shapes of rectangular plates in Rayleigh-Ritz method. The orthogonal polynomials for the beam were constructed using Gram-Schmidt process [32], the first member of the set satisfying both the geometrical and natural boundary conditions of the
beam and all the rest satisfying the geometrical boundary conditions.

Finite element method was used by Putter and Manor [33] to solve for the natural frequencies in flexural vibrations for a rotating beam with tip mass. Hoa [34] also utilized the finite element technique to study the effect of setting angle and hub radii on a rotating beam with tip mass. Dokainish and Ratwani [35] using this technique determined the natural frequencies and the mode shapes of a cantilever plate mounted on the periphery of a rotating disc. From the results of computations carried out for various values of the aspect-ratio, hub radii and setting angles, they derived empirical formulae giving the effect of these parameters on the natural frequencies. Henry and Lalanne [36] derived the total potential energy of a thin rotating plate. They minimized this energy by using the finite element method and obtained the natural frequencies and mode shapes by a simultaneous iterative technique.

In summary, from a review of the existing literature, it is found that although the problem of rotating structures has been investigated, there is still a lot of scope for further research. In this investigation various new techniques are proposed to analyze such structures giving superior results as compared to the above mentioned. The details are discussed in the next section.

1.3 Scope of the Present Investigation

The objective of the present work is to analyze a class of rotating structures by using improved methods in calculating their natural frequencies and mode shapes. These structures depending on their aspect ratios, are idealized either as beams or plates.
In the present study, Chapter 2 deals with such a rotating type structure being modeled as a rotating beam. An improved strain-energy formulation in Rayleigh-Ritz method is carried out and the natural frequencies along with the mode shapes are obtained for different rotational speeds and various parametric combinations. The setting angle of the blade with respect to the plane of rotation and radius of the hub on which the structure is mounted are also taken into account in the formulation. The results are compared with the conventional Rayleigh-Ritz method and with those obtained using other methods.

In Chapter 3, a general purpose finite element package 'SPAR' is utilized to model this rotating beam type structure. The natural frequencies are obtained for different parametric combinations and compared.

The fourth Chapter deals with the rotating type structure being modeled as a plate. A class of beam characteristic orthogonal polynomials, constructed using Gram-Schmidt process, are employed as deflection functions for plates. Rayleigh-Ritz method is used to obtain their natural frequencies and mode shapes. Experimental results for a plate with all edges free, verifying the analytical formulation, are also presented in this Chapter.

Finally, conclusions and recommendations for future work are presented in the fifth Chapter.
CHAPTER 2

DYNAMIC ANALYSIS OF A ROTATING BEAM TYPE STRUCTURE USING
IMPROVED STRAIN ENERGY IN RAYLEIGH RITZ METHOD
CHAPTER 2

DYNAMIC ANALYSIS OF A ROTATING BEAM TYPE STRUCTURE USING IMPROVED STRAIN ENERGY IN RAYLEIGH RITZ METHOD

Dynamic behavior of rotating beam is of interest since it is possible to develop such a model for several engineering structures. Some examples are steam and gas turbines. Centrifugal forces are set up in these structures due to their rotation, which cause in-plane strains, resulting in the stiffening of the structure. In addition, Coriolis effects are also present which modify the structural characteristics.

Due to the change in structural characteristics with the speed of rotation, the natural frequencies of the structure will change and it is imperative that the speed of operation be away from the natural frequencies for satisfactory operation.

In this chapter, the dynamics of a rotating beam type structure is studied using conventional Rayleigh-Ritz method with an improved strain-energy formulation. The setting angle of the beam with respect to the plane of rotation and radius of the hub on which the structure is mounted are also taken into account in the formulation. The variation of natural frequencies and mode shapes with the speed of rotation are plotted for several parameter combinations such as setting angle, hub radius, etc.

2.1 Rayleigh-Ritz Method

The Rayleigh-Ritz method is widely used to obtain approximate values for the natural frequencies and mode shapes of structures. The method provides upper bound values for the system natural frequencies.

In this method, a deflection shape of the form,
\[ y = \sum_{n=1}^{N} C_n \phi_n(x) \]  

(2.1)

is chosen where \( \phi_n(x) \) are any admissible functions satisfying at least the geometrical boundary conditions and \( C_1, C_2, \ldots C_n \) are constant coefficients. The parameters \( C_n \)'s and \( \phi_n(x) \) form a generating set. In limiting the generating set to a finite number of functions, the analysis can be interpreted as approximating a continuous system as a \( n \)-degree of freedom discrete system. The coefficients are adjusted by minimizing the frequency with respect to each of the coefficients, which results in \( n \) algebraic equations in the \( n \) unknown coefficients, \( C_n \), involving \( p^2 \). The solution of these equations then gives the natural frequencies and mode shapes of the system.

For a conservative system, the maximum kinetic energy is equal to the maximum strain energy. By using this property, an expression for the natural frequency is obtained in terms of the undetermined coefficients \( C_n \) as,

\[ p^2 = \frac{U_{\text{max}}}{T_{\text{max}}^*} \]  

(2.2)

where \( T_{\text{max}}^* \) is the expression for kinetic energy without the term \( p^2 \), the square of the frequency of vibration. Minimizing \( p \) by differentiating it with respect to each of the coefficients, \( C_n \)

\[ \frac{dp^2}{dc_n} = 0 \]  

(2.3)

results in the equation

\[ \frac{dU_{\text{max}}}{dc_n} - p^2 \frac{dT_{\text{max}}^*}{dc_n} = 0 \]  

(2.4)
This is a standard eigen-value problem and the solution yields the natural frequencies and the corresponding coefficients $C_n$ can be used in equation (2.1) to obtain the approximate mode shapes.

2.2 Analysis

The cantilever beam considered is mounted on the periphery of a rotating disc as shown in Fig. 2.1. The xyz coordinate axis system is chosen such that x and y axes are in the plane of beam cross-section and are the principal centroidal axes of inertia in that plane. The z-axis is along the beam. XYZ is another set of orthogonal axis system where z-axis is along the beam and XZ-plane contains the plane of disc rotation. Origin of both xyz and XYZ coordinate systems are at the root of the beam where it is fixed to the disc. The angle $\theta$ between the Y and x axes is the setting angle.

Strain Energy

Let $M$ be the bending moment and $\beta$ be the slope of the elastic curve, the strain energy stored in an infinitesimal beam element is given by [37] as,

$$dU = \frac{1}{2} M dB$$  \hspace{1cm} (2.5)

Assuming small deflections from Fig. 2.2

$$\beta = \frac{dy}{dz}; \quad \frac{1}{R_1} = \frac{dB}{dz} = \frac{d^2y}{dz^2}$$  \hspace{1cm} (2.6)

From the theory of simple bending of beams, the flexure equation

$$\frac{1}{R_1} = \frac{M}{EI_x}$$  \hspace{1cm} (2.7)
Fig. 2.1: Rotating Cantilever Beam
Fig. 2.2: Geometric Parameters Considered for Deflection
is used where $R_1$ is the radius of curvature.

Substituting for $d\theta$ and $\frac{1}{R_1}$, the strain energy $U$ may be written as,

$$U_{\text{max}} = \frac{1}{2} \int \frac{M^2}{EI_x} \, dz = \frac{1}{2} \int EI_x \left( \frac{d^2y}{dz^2} \right)^2 \, dz$$  \hspace{1cm} (2.8)

where the integration is carried out along the length of the beam.

Let $\tilde{z} = \frac{z}{L}$ be the non-dimensional axial length. Substituting this in (2.8) and integrating along the length of the beam, the expression for strain energy is,

$$U_{\text{max}} = \frac{1}{2L^3} \int_0^L EI_x \left( \frac{d^2y}{dz^2} \right) \, d\tilde{z}$$  \hspace{1cm} (2.9)

**Kinetic Energy**

The total kinetic energy of the system is the sum of the kinetic energy due to the flexural motion of the beam $T_f$, and that due to rotation, $T_r$.

a) **Flexural Kinetic Energy**

The kinetic energy of the blade in flexural vibration can be considered to be made up of two parts. They are the translational kinetic energy, $T_f$, and that due to rotating inertia, $T_r$.

For an element $dz$, the instantaneous kinetic energy $dT_b$ of the mass concentrated at the centroid is given by

$$dT_b = \frac{1}{2} \, m \, y^2 \, dz$$  \hspace{1cm} (2.10)

where $m = \rho A$.

Hence, the total instantaneous kinetic energy $T_b$ is,
\[ T_b = \frac{1}{2}m \int_0^L \dot{y}^2 \, dz \quad (2.11) \]

The instantaneous kinetic energy \( dT_b \), due to rotation about the centroid, is,

\[ dT_b = \frac{\rho I_x \, dz (\dot{y}')^2}{2} \quad (2.12) \]

Therefore, the total instantaneous kinetic energy \( T_b \), thus becomes

\[ T_b = \frac{1}{2} \int_0^L \rho I_x (\dot{y}')^2 \, dz \quad (2.13) \]

The combined kinetic energy \( T_f \) is the sum of equations (2.11) and (2.13) and can be written as,

\[ T_f = \frac{1}{2} \int_0^L \left[ cA \dot{y}^2 + \rho I_x (\dot{y}')^2 \right] \, dz \quad (2.14) \]

**b) Kinetic Energy due to Rotation**

Figure 2.3 shows the blade mounted on the periphery of the rotating disc. For a short element \( dz \), each view shows both the rest position \( A \) and the deflected position \( B \), displacements occurring in both the \( n_z \) plane (Fig. 2.3(a)) and \( \xi_z \) plane (Fig. 2.3(b)).

When the blade deflects in the \( \xi_z \) plane, Fig. 2.3(b), the line of action of the centrifugal force, \( dF_{\xi} \), on the element \( dz \) remains parallel to the \( z \) axes. Hence the force component \( dF_{\xi} \) in the \( \xi \) direction is zero and the kinetic energy \( dT_{\xi} \) stored by the element is also zero. Thus,

\[ dT_{\xi} = 0 \quad (2.15) \]
Fig. 2.3: Beam-Hub Radii Assembly
With the element $dz$ in the deflected position $B$, Fig. 2.3(a), the force $dF$ can be resolved into two components $dF_\eta$ and $dF_z$, in the circumferential and radial directions $\eta$ and $z$ respectively.

The circumferential component $dF_\eta$ of the centrifugal force $dF$ is given by,

$$dF_\eta = dF \sin \beta_0$$  \hspace{1cm} (2.16)

where $dF = m\omega^2(R+z)\,dz$  \hspace{1cm} (2.17)

and $\sin \beta_0 = \frac{\eta_1}{R+z}$  \hspace{1cm} (2.18)

In the above equations, $R$ is the disc radius, $\omega$ is the angular velocity of rotation of disc and $m$ is mass of blade per unit length.

Substitution of equations (2.17) and (2.18) into (2.16) gives,

$$dF_\eta = m\omega^2 \eta_1 \,dz$$  \hspace{1cm} (2.19)

Since the $\eta_1$ component of the centrifugal force, $dF_\eta$, increases linearly from zero at the next position $A$, regarded as datum, to a value given by equation (2.19) at $B$, the average force during a displacement $\eta_1$ is thus, $dF_\eta/2$. With the force and motion in the same direction, the corresponding gain of the kinetic energy $d\mathcal{T}$ is given by,

$$d\mathcal{T}_\eta = \frac{dF_\eta \eta_1}{2} = \frac{m\omega^2 \eta_1^2 \,dz}{2}$$  \hspace{1cm} (2.20)

and for the entire blade $\mathcal{T}_\eta = \int_0^L \frac{\omega^2 \eta_1^2 \,dz}{2}$  \hspace{1cm} (2.21)

The centrifugal force $dF$ acting on the element in the undeflected position $A$ is given by (2.17). In the deflected position $B$ the force
component $dF_z$ in the $z$ direction is

$$dF_z = dF \cos \beta,$$  \hspace{1cm} (2.22)

For small displacements, $\cos \beta$ approaches unity; the force component in the $z$ direction can, thus, be regarded as constant and given by equation (2.17).

The $z$ component of kinetic energy $dT_z$ stored by the element can be written as,

$$dT_z = -dF_z \Delta = -dF \Delta$$  \hspace{1cm} (2.23)

where $\Delta$ is the total displacement of the element $dz$ in moving from rest position $A$ to the deflected position $B$.

To determine $\Delta$ from Fig. 2.4,

$$dz = d\delta^2 + (dz - d\delta)^2$$  \hspace{1cm} (2.24)

where $d\delta = [d\xi_1^2 + d\eta_1^2]^\frac{1}{2}$  \hspace{1cm} (2.25)

represents the increase of total transverse displacement $\delta$.

From (2.24),

$$d\Delta = \frac{1}{2} \left(\frac{d\delta}{dz}\right)^2$$  \hspace{1cm} (2.26)

Combining (2.25) and (2.26)

$$d\Delta = \frac{1}{2} \left(\left(\frac{d\xi_1}{dz}\right)^2 + \left(\frac{d\eta_1}{dz}\right)^2\right) dz$$  \hspace{1cm} (2.27)

and the total displacement $\Delta$, of the element at $z$ is, thus, given by

$$\Delta = \frac{1}{2} \int_0^z \left(\left(\frac{d\xi_1}{dz}\right)^2 + \left(\frac{d\eta_1}{dz}\right)^2\right) dz$$  \hspace{1cm} (2.28)

For small displacements, and making use of equations (2.17) and (2.28),
Fig. 2.4: Beam Element Before and After Deflection Showing the Relative Displacement $d\Delta$ of the Ends.
equation (2.23) can be written as

\[
dT_z = -m\omega^2(R+z)dz \int_0^Z \left\{ \left( \frac{\partial \xi_1}{\partial z} \right)^2 + \left( \frac{\partial \eta_1}{\partial z} \right)^2 \right\} dz
\]  

(2.29)

Thus, for the entire blade the kinetic energy \( T_z \), becomes

\[
T_z = -\frac{\omega^2}{2} \int_0^L \left[ m(R+z) \int_0^Z \left\{ \left( \frac{\partial \xi_1}{\partial z} \right)^2 + \left( \frac{\partial \eta_1}{\partial z} \right)^2 \right\} dz \right] dz
\]  

(2.30)

The total kinetic energy due to centrifugal effects,

\[
T_R = T_\xi + T_\eta + T_z
\]  

(2.31)

i.e. \( T_R = -\frac{\omega^2}{2} \int_0^L \left[ m(R+z) \int_0^Z \left\{ \left( \frac{\partial \xi_1}{\partial z} \right)^2 + \left( \frac{\partial \eta_1}{\partial z} \right)^2 \right\} dz + \mu^2 \right] dz
\]  

(2.32)

The above equation can be written as,

\[
T_R = -\frac{\omega^2}{2} \int_0^L \left[ \left( \frac{d \xi_1}{dz} \right)^2 + \left( \frac{d \eta_1}{dz} \right)^2 \right] \int_0^L \left[ m(R+z) + \mu^2 \right] dz
\]  

(2.33)

Now, using the relationships,

\[
\eta_1 = y \cos \theta - x \sin \theta
\]  

(2.34)

\[
\xi_1 = y \sin \theta + x \cos \theta
\]  

(2.35)

equation (2.33), can be written as

\[
T_R = \frac{\rho \omega^2}{2} \int_0^L \left\{ (x^2 \sin^2 \theta - 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta) - \left[ \left( \frac{\partial x}{\partial z} \right)^2 + \left( \frac{\partial y}{\partial z} \right)^2 \right] \right\} dz \int_0^L \left(R+z\right) \, dz
\]  

(2.36)
Neglecting coupling of \( x \) and \( y \) deflections and ignoring higher order terms,
\[
T_R = \frac{\rho A\omega^2}{2} \left[ \int_0^L y^2 \cos^2 \theta \, dz - \int_0^L \left( \frac{\partial y}{\partial z} \right)^2 \left( RL + \frac{L^2}{2} - R_z - \frac{z^2}{2} \right) \, dz \right] \tag{2.37}
\]

The total kinetic energy of the system in terms of the non-dimensional parameter \( \bar{z} \) is,
\[
T_{\text{TOTAL}} = T_f + T_r \tag{2.38}
\]

i.e.
\[
T_{\text{TOTAL}} = \frac{L}{2} \int_0^1 \rho \bar{A} (\bar{y})^2 \, d\bar{z} + \frac{1}{2L} \int_0^1 \rho \bar{I}_x (\bar{y}')^2 \, d\bar{z}
\]
\[
+ \frac{\rho A\omega^2}{2} \left[ L \int_0^1 y^2 \cos^2 \theta \, d\bar{z} - \frac{1}{L} \int_0^1 (\bar{y}')^2 \left( RL + \frac{L^2}{2} - R_z - \frac{L^2}{2} \right) \, d\bar{z} \right] \tag{2.39}
\]

where a dot represents differentiation with respect to time and \( (\cdot)' \) indicates differentiation with respect to \( \bar{z} \).

The equations of motion can be derived directly from the energy expressions by using the Lagrangian
\[
L = T - U \tag{2.40}
\]

Assuming harmonic motion,
\[
y = Y \exp (j\omega t) \tag{2.41}
\]

the time averaged value of the Lagrangian is obtained as
\[
\bar{L} = \int_0^{2\pi/\omega} L \, dt \tag{2.42}
\]

Incorporating the following dimensionless parameters
\[ \bar{R} = R/L \]
\[ \bar{r}^2 = I_x/AL^2 \]
\[ \alpha^2 = \lambda \omega^2 \]
\[ \lambda = \frac{pAL^4}{EI_x} \]
\[ n^2 = \lambda \rho^2 \]

(2.43)

the time averaged value of the Lagrangian can be written as

\[ \bar{L} = \frac{\Pi EI_x}{2pL^3} \int_0^1 \left[ \eta^2 \gamma^2 + n^2 \bar{r}^2 (\gamma')^2 + \alpha^2 \gamma^2 \cos^2 \Theta \right. \]
\[ \left. - \alpha^2 \gamma' \left\{ R(1 - \bar{z}) + \frac{1}{2} (1 - \bar{z}^2) - (\gamma'')^2 \right\} \right] d\bar{z} \]

(2.44)

To apply the Rayleigh-Ritz method, the shape function is assumed as a polynomial

\[ \gamma = \sum_{n=2}^{N} C_n \bar{z}^n \]

(2.45)

where \( C_n \) are arbitrary coefficients which are to be determined. Substituting from equation (2.45), and applying the Ritz process, the following homogenous simultaneous equations are obtained,

\[ \frac{\partial \bar{L}}{\partial C_{n_i}} = 0, \quad i = 2, 3, \ldots, N \]

(2.46)
$$\left[ \eta^2 \sum_{K=2}^{N} \frac{C_K}{(n+K+1)} + r^2 \sum_{K=2}^{N} \frac{\eta n C_K}{(n+K-1)} \right]$$

$$- \left[ -\alpha^2 \cos^2 \theta \sum_{K=2}^{N} \frac{C_K}{(n+K+1)} + \alpha^2 \int_0^1 \left\{ \bar{R} (1 - \bar{z}) + \frac{1}{2} (1 - \bar{z}^2) \right\} \sum_{K=2}^{N} \frac{\delta K C_K}{(n+K-1)} \right] + \sum_{K=2}^{N} \frac{nK(n-1)(K-1)C_K}{(n+K-3)} = 0$$ (2.47)

Equation (2.47) is in the form of a standard eigen-value problem.

$$\eta^2[A] - [B] = 0$$ (2.48)

and eigenvalues and eigenmodes are obtained by solving this equation. A computer algorithm developed to solve equation (2.48) is given in Appendix A.

2.3 Improved Strain Formulation in Rayleigh-Ritz Method

In the proposed method, a deflection shape of the form given in equation (2.45) is assumed

i.e. $$Y = \sum_{n=2}^{N} C_n \bar{z}^n$$ (2.49)

This shape function as before satisfies the geometrical boundary conditions at the fixed end of the beam i.e. the deflection and slope at $$\bar{z} = 0$$ are zero.

Instead of differentiating the deflection shape to be used in equations (2.8), (2.9), to obtain the strain energy, the deflection shape is used to obtain the distributed force over the beam, in the form,
\[ q(z) = -mp^2 \sum_{n=2}^{N} c_n z^n \]  

(2.50)

Integrating this distributed force to obtain the shear force at any section as [38].

\[ Q(z) = \int_{z}^{L} q(z) \, dz \]  

(2.51)

i.e. \( Q(z) = mp^2 \sum_{n=2}^{N} \frac{c_n}{(n+1)} \left[ \frac{z^{n+1}}{n+1} - 1 \right] \)  

(2.52)

Integrating the shear force once again gives the moment at any section as,

\[ M(z) = L^2 mp^2 \sum_{n=2}^{N} \frac{c_n}{(n+1)} \left[ \frac{1}{n+1} - \frac{z^{n+2}}{(n+1)(n+2)} - 1 + \frac{z}{n+1} \right] \]  

(2.53)

The strain energy of the beam is given by equations (2.8), (2.9) as,

\[ U = \frac{1}{2EI_x} \int_{0}^{l} M^2(z) \, dz \]  

(2.54)

To obtain the kinetic energy using equation (2.39) and assuming harmonic motion

\[ y = Y \cos \omega t \]  

(2.55)

\[ T_{TOTAL} = \frac{L}{2} \int_{0}^{1} \rho A Y^2 \dot{p}^2 \, dz + \frac{1}{2L} \int_{0}^{1} \rho I_x p^2 (Y')^2 \, dz \]

\[ + \frac{\rho A \omega^2}{2} \left[ L \int_{0}^{1} Y^2 \cos^2 \theta \, dz - \frac{1}{L} \int_{0}^{1} (Y')^2 \left( R_L + \frac{L^2}{2} - R_L - \frac{L^2 z^2}{2} \right) \, dz \right] \]  

(2.56)

where as before \(( \ )'\), indicates differentiation with respect to \( z \).
To obtain the slope ($y^*$)' and deflection ($y^*$) to be used in the above equation, the moment distribution, (2.53), is integrated twice, i.e.

$$y^* = \frac{L}{EI_x} \int_0^z M(z) \, dz$$

$$= -\frac{L^3}{EI_x} \frac{m_p}{2} \sum_{n=2}^N \frac{C_n}{(n+1)} \left[ \frac{z}{(n+2)} - \frac{z^{n+3}}{(n+2)(n+3)} - z + \frac{z^2}{2} \right] + \text{Constant} \quad (2.57)$$

at $z = 0$, ($y^*$)' = 0, therefore Constant = 0.

Integrating (2.57) again to obtain the deflection,

$$y^* = -\frac{L^4}{EI_x} \frac{m_p}{2} \sum_{n=2}^N \frac{C_n}{(n+1)} \left[ \frac{z^2}{2(n+2)} - \frac{z^{n+4}}{(n+2)(n+3)(n+4)} \right. $$

$$\left. - \frac{z^2}{2} + \frac{z^3}{6} \right] + \text{Constant} \quad (2.58)$$

at $z = 0$, $y^* = 0$, therefore Constant = 0.

Substituting equations (2.57), (2.58) in (2.56) gives,

$$T_{\text{TOTAL}} = \frac{L}{2} \int_0^1 \rho A y^2 p^2 d\bar{z} + \frac{L}{2L} \int_0^1 \rho I_x p^2 (y')^2 d\bar{z}$$

$$+ \frac{\rho A w^2 L}{2} \int_0^1 \left[ \frac{L^2}{EI_x} \sum_{n=2}^N \frac{C_n}{(n+1)} \left( \frac{\bar{z}^2}{2(n+2)} - \frac{\bar{z}^{n+4}}{(n+2)(n+3)(n+4)} \right. \right.$$}

$$\left. \left. - \frac{\bar{z}^2}{2} + \frac{\bar{z}^3}{6} \right) \cos^2 \theta d\bar{z} - \frac{\rho A w^2}{2L} \int_0^1 \left[ -L^4 m_p \sum_{n=2}^N \frac{C_n}{(n+1)} \left( \frac{\bar{z}}{(n+2)} - \frac{\bar{z}^2}{2} + \frac{\bar{z}^3}{6} \right) \right.$$}

$$\left. \left. - \frac{\bar{z}^{n+3}}{(n+2)(n+3)} - \bar{z} + \frac{\bar{z}^2}{z} \right] \right] \left( R \bar{L} \frac{\bar{L}^2}{2} - RL \bar{L} - \frac{L^2}{2} + \frac{\bar{z}^2}{z} \right) d\bar{z} \quad (2.59)$$
The Lagrangian is given by,

\[ L = T - U \]  

(2.60)

Applying the Ritz process, the following homogeneous equations are obtained,

\[ \frac{\delta L}{\delta C_n} = 0 \quad \text{for} \quad n = 2, 3, \ldots N \]  

(2.61)

Incorporating the dimensionless parameters given by equation (2.43), equation (2.61) can be written in the form of a standard eigenvalue problem as,

\[ \eta^2 \begin{bmatrix} A \end{bmatrix} - \begin{bmatrix} B \end{bmatrix} = 0 \]  

(2.62)

where the elements \( a_{nk} \) and \( b_{nk} \) of matrices \( [A] \) and \( [B] \) are given in Appendix B. The eigenvalues and eigenvectors are obtained by solving this equation.

A computer algorithm is developed to solve (2.62) for several parameter combinations of setting angle and hub radius and is given in Appendix C.

2.4 Discussion of the Results

Natural frequencies and mode shapes are obtained for a rotating cantilever beam at various rotational speeds for different values of setting angle and hub radius.

A convergence test was made by varying the number of terms considered in the polynomial shape functions. The results obtained using conventional Rayleigh-Ritz method with and without any improved strain energy formulation are shown in Tables 2.1-2.4. The variation of fundamental and second natural frequency for various values of Hub Radii and setting angle are presented in Tables 2.5-2.12.
Natural frequencies of the rotating beam for different rotational speeds and several parametrical combinations are shown in Figs. 2.5 through 2.16.

Variation of natural frequency, $n$, with rotational speed, $\alpha$, is shown in Figs. 2.5 and 2.6 for the first mode for various values of $\tilde{R}$ and setting angles. The increase in natural frequency is larger for higher setting angles and for any setting angle the increase is linear at higher rotational speeds.

The effect of hub radius $\tilde{R}$, on the natural frequencies of modes 1 through 5 are plotted against the rotational speed for setting angles of zero and 90° in Figs. 2.7 through 2.16. With higher value of $\tilde{R}$, the natural frequencies are higher and increase faster with the rotational speed. The results are compared with those from Kumar [8], Wang et al [15] and Hoa [34] for the first mode and with conventional Rayleigh-Ritz method for higher modes.

The variation of the fundamental mode shape with the rotational speed is shown in Fig. 2.17. The mode shapes are normalized to a value of unity at the tip. It is observed that as the rotational speed $\alpha$, increases, the beam tries to straighten itself more and more. The variation of the shape of second mode with rotational speed is shown in Fig. 2.18. The beam has a tendency to straighten itself as the rotational speed increases as in the case of fundamental mode.

In conclusion, natural frequencies and mode shapes of a rotating uniform cantilever beam are obtained using conventional Rayleigh-Ritz method with and without any improved strain energy formulation. A simple
polynomial with arbitrary coefficients is used as shape function. When the arbitrary coefficients are determined, along with the natural frequencies, using Ritz method, the normal mode shapes are obtained as simple polynomials. Natural frequencies and mode shapes are obtained for different rotational speeds and for different parametric combinations such as setting angle, hub radius, etc.

In the following chapter, the dynamic analysis of the rotating beam is carried out using the finite element approach. SPAR, a finite element package, is used to model the beam taking into account the hub-radius and the setting angle.
TABLE 2.1: Convergence of Natural Frequency with Number of Terms Using Conventional Rayleigh-Ritz Method

<table>
<thead>
<tr>
<th>Setting Angle</th>
<th>$\theta = 90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Velocity</td>
<td>$\alpha = 0$</td>
</tr>
<tr>
<td>Hub Radius</td>
<td>$\tilde{R} = 0$</td>
</tr>
<tr>
<td>Non-Dimensional Parameter</td>
<td>$\tilde{r}^2 = 2.5 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>MODE 1</th>
<th>MODE 2</th>
<th>MODE 3</th>
<th>MODE 4</th>
<th>MODE 5</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td>117.879</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>280.1555</td>
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</tr>
<tr>
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<td>22.0262</td>
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<td>128.2670</td>
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</table>
### TABLE 2.2: Convergence of Natural Frequency with Number of Terms Using Improved Strain Energy Formulation in Rayleigh-Ritz Method

<table>
<thead>
<tr>
<th>Setting Angle</th>
<th>( \theta = 90^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Velocity</td>
<td>( \alpha = 0 )</td>
</tr>
<tr>
<td>Hub Radius</td>
<td>( \bar{R} = 0 )</td>
</tr>
<tr>
<td>Non-Dimensional Parameter</td>
<td>( \bar{\eta}^2 = 2.5 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

<p>| Natural Frequency ( \eta = \rho \sqrt{\lambda} ) |</p>
<table>
<thead>
<tr>
<th>No. of Terms</th>
<th>MODE 1</th>
<th>MODE 2</th>
<th>MODE 3</th>
<th>MODE 4</th>
<th>MODE 5</th>
</tr>
</thead>
<tbody>
<tr>
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<td>22.0256</td>
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<td>22.0256</td>
<td>61.6389</td>
<td>120.4233</td>
<td>202.6457</td>
</tr>
</tbody>
</table>
TABLE 2.3: Convergence of Natural Frequency with Number of Terms Using Improved Strain Energy Formulation in Rayleigh-Ritz Method

<table>
<thead>
<tr>
<th>Setting</th>
<th>$\theta = 90^\circ$</th>
<th>Angular Velocity $\alpha = 5$</th>
<th>Hub Radius $R = 0$</th>
<th>Non-Dimensional Parameter $\tilde{r}^2 = 2.5 \times 10^{-5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Frequency $n = \rho \sqrt{\lambda}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No. of Terms</td>
<td>MODE 1</td>
<td>MODE 2</td>
<td>MODE 3</td>
<td>MODE 4</td>
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<td>65.1423</td>
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</tr>
</tbody>
</table>
**TABLE 2.4: Convergence of Natural Frequency with Number of Terms Using Improved Strain Energy Formulation in Rayleigh-Ritz Method**

<table>
<thead>
<tr>
<th>Setting</th>
<th>$\theta = 90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular Velocity</td>
<td>$\alpha = 10$</td>
</tr>
<tr>
<td>Hub Radius</td>
<td>$\bar{R} = 0$</td>
</tr>
<tr>
<td>Non-Dimensional Parameters</td>
<td>$\bar{r}^2 = 2.5 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Natural Frequency $n = p\sqrt{\lambda}$

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TABLE 2.6: Variation of Fundamental Natural Frequency for $R = 1$ and Various Values of Setting Angle using Improved Strain Energy Formulation in Rayleigh-Ritz Method

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</thead>
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TABLE 2.7: Variation of Fundamental Natural Frequency for $R = 5$ and Various Values of Setting Angle using Improved Strain Energy Formulation in Rayleigh Ritz Method

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<td>25.513</td>
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TABLE 2.8: Variation of Fundamental Natural Frequency for $\bar{R} = 10$ and Various Values of Setting Angle using Improved Strain Energy Formulation in Rayleigh Ritz Method

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TABLE 2.9: Variation of Second Natural Frequency for $\bar{R} = 0$ and Various Values of Setting Angle using Improved Strain Energy Formulation in Rayleigh Ritz Method

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TABLE 2.10: Variation of Second Natural Frequency for $\bar{R} = 1$ and Various Values of Setting Angle using Improved Strain Energy Formulation in Rayleigh-Ritz Method

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TABLE 2.11: Variation of Second Natural Frequency for $\bar{R} = 5$
and Various Values of Setting Angle using
Improved Strain Energy Formulation in Rayleigh-
Ritz Method

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Fig. 2.5: Variation of First Natural Frequency with Speed for Different Setting Angles. 
($\bar{R} = 0$)
Fig. 2.6: Variation of First Natural Frequency with Speed for Different Setting Angles.
($\phi = 1.0$)
Fig. 2.7: Variation of First Natural Frequency with Speed for Different Hub-Radii

\(-x-x-x-[8], \, \ldots-[15], \, \ldots-[34] \theta = 0^\circ\)
Fig. 2.8: Variation of First Natural Frequency with Speed for Different Hub-Radii

(-xx- [8], ----[15], ----[34], θ = 90°)
Fig. 2.9: Variation of Second Natural Frequency with Speed for Various Hub-Radii
($\theta = 0^o$)
Fig. 2.10: Variation of Second Natural Frequency with Speed for Various Hub-Radii
(θ = 90°)
Fig. 2.11: Variation of Third Natural Frequency for Various Hub-Radii

(XX - Rayleigh Ritz Method, θ = 0°)
Fig. 2.12: Variation of Third Natural Frequency for Various Hub-Radii

(XX = Rayleigh Ritz Method, \( \theta = 90^\circ \))
Fig. 2.13: Variation of Fourth Natural Frequency for Various Hub-Radii

(xx - Rayleigh-Ritz Method, $\theta = 0^\circ$)
Fig. 2.14: Variation of Fourth Natural Frequency for Various Hub-Radii

(xx - Rayleigh Ritz Method, \( \theta = 90^\circ \))
Fig. 2.15: Variation of Fifth Natural Frequency for Various Hub-Radii
(xx - Rayleigh Ritz Method, e = 0°)
Fig. 2.16: Variation of Fifth Natural Frequency for Various Hub-Radii

$xx$ - Rayleigh-Ritz Method, $\theta = 90^\circ$
Fig. 2.17: Variation of First Mode Shape with Speed

$(\bar{R} = 0, \theta = 90^\circ)$
Fig. 2.18: Variation of Second Mode Shape with Speed
\((\bar{R} = 0, \theta = 90^\circ)\).
CHAPTER 3

DYNAMIC ANALYSIS USING FINITE ELEMENT APPROACH
CHAPTER 3

DYNAMIC ANALYSIS USING FINITE ELEMENT APPROACH

In this chapter, the dynamics of a rotating beam type structure is studied using an analyzer package based on a finite element program called SPAR, developed by NASA [39]. This program is written in modular form, that is, its various modules can be used in any desired combination to perform the required computations.

The natural frequencies for the rotating beam are obtained for different Hub-Radii, setting angle, rotational speed and are compared to the results obtained in chapter 2.

Pertinent descriptions of the finite element analysis program are discussed in the following sections.

3.1 General Description of the Software

The finite element package SPAR is a versatile general purpose finite element program for structural dynamic analysis. SPAR is an array of separate absolute programs called processors or modules. These processors obtain input from two sources, firstly, input records from terminals, and secondly, from a data base. The data base consists of one or more direct access libraries, which contain the data sets' output from the different processors. These processors do not have to be executed in any particular order, provided all necessary source data sets reside in the data base. Each processor automatically extracts from the data base all of the data set it requires, and inserts into the data base the newly generated sets. Figure 3.1 shows the schematic of the general structure of the overall finite element program.
Fig. 3.1: SPAR Program Block Diagram
Each processor of SPAR performs a specific function e.g. processor TAB, contains an array of sub-processors which generate tables of material constants, section properties, constraint conditions and various other data sets comprising a substantial portion of the definition of the structure. Processor K assembles the unconstrained system stiffness matrices, processor M assembles the mass matrix of the structure etc. Further, the processor AUS performs matrix manipulations, such as multiplication, addition, transpose etc. This processor provides a great deal of flexibility to the user in adapting SPAR program to suit any computational needs. Other program and analysis details about the SPAR processors are contained in reference [39].

In SPAR, the execution on a processor is carried out by processor basis. Each processor is commanded by a separate explicit command. A string of such commands interleaved with the input numerical data is written by the user for a problem at hand, and is called a run stream. Various run streams can be executed in any order.

SPAR program contains many different types of elements, way to connect these elements and other features which are described in detail in the SPAR reference manual [39].

SPAR is an extremely flexible and efficient computer program; it is modular in nature, has data base capability and can be easily modified for any particular application. SPAR data base contains all the information relevant to the type of structure being analyzed. Data base is located on the drum storage, the location of a particular data in the data base is given by the library number and the data set name. A table of contents TDC is used to store and relate the names and addresses of all
data sets resident in the data base. A typical TOC listing is given in Appendix D.

3.2 Finite Element Model

The cantilever beam considered is mounted on the periphery of a rotating disc of radius \( R \). The beam has an area of cross-section \( A \), length \( L \), modulus of elasticity of material \( E \) and material density \( \rho \).

As shown in Fig. 3.2, \( xyz \) coordinate system is chosen such that the \( x \) and \( y \) axes are in the plane of the beam cross-section and are the principal centroidal axes of inertia in that plane. The \( z \) axis is along the beam. \( XYZ \) is another set of orthogonal axis system where \( Y \) is along the beam and the \( XY \) plane contains the plane of disc rotation. Origin of the \( XYZ \) coordinate axes is at the centre of the rotating disc while the \( xyz \) axes are located at the root of the beam where it is fixed to the disc. The angle \( \theta \), between the \( z \) and \( x \) axes is the setting angle.

The rotating beam is modelled having 16 node points. Node 1 is fixed to the rotating disc and hence all the motion components (displacement and rotation) are identically zero at this node. The stiffness matrix for the formulation is assembled by considering the effects of elastic stiffness, centrifugal force and initial stress.

A listing of the sample program and the run stream used to compute the natural frequencies for different Hub-radii, setting angles etc. is given in Appendix D.
Fig. 3.2: Rotating Cantilever Beam Modelled Using SPAR
3.3 Discussion of the Results

Natural frequencies are obtained for a rotating cantilever beam at various rotational speeds for different values of setting angle and hub radii by using a finite element package 'SPAR'.

Table 3.1 shows the variation of fundamental natural frequency for different rotational speeds, and several parameter combinations. The results are compared to the ones obtained using improved strain energy formulation in Rayleigh-Ritz in chapter 2.

The increase in natural frequency is larger for higher setting angles. With higher value of R, the natural frequency is higher and increase faster with the rotational speed. The results obtained using the finite element approach and the energy approach agree closely.

In this chapter, a finite element model of the rotating beam type structure is generated using SPAR. The natural frequencies are obtained for different rotational speed and several parameter combinations.

In the next chapter, the structure is modelled as a plate and the flexural vibrations of plates with different boundary conditions are studied using beam-characteristic orthogonal polynomials in Rayleigh-Ritz method.
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<th>Natural Frequency (Hz)</th>
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CHAPTER 4

DYNAMIC ANALYSIS OF ROTATING PLATES
CHAPTER 4

DYNAMIC ANALYSIS OF ROTATING PLATES

Rotating structures in many applications cannot be strictly modelled as beams and must be modelled as flat or curved plates. In this chapter, the vibrational behavior of stationary and rotating plate type structures is studied using Rayleigh-Ritz method.

A class of beam characteristic orthogonal polynomials, constructed using Gram-Schmidt process, are employed as deflection functions for plates in Rayleigh-Ritz method to obtain their natural frequencies and mode shapes. A plate with all edges free is analyzed to obtain confidence in the proposed method. Natural frequencies and mode shapes of plates with (i) all edges free and (ii) three edges free and one fixed (cantilevered plate) are presented and compared with those obtained by other methods. Experimental results obtained by using modal testing techniques are also presented for a plate with all edges free.

4.1 Beam Characteristic Orthogonal Polynomials

The first member of the orthogonal polynomial set $\phi_1(x)$ is chosen as the simplest polynomial of the least order that satisfies both the geometrical and the natural boundary conditions of the beam. The other members of the orthogonal set in the interval $a < x < b$ are generated using Gram-Schmidt process as follows:

$$\phi_2(x) = (x - B_2) \phi_1(x)$$

$$\phi_k(x) = (x - B_k) \phi_{k-1}(x) - C_k \phi_{k-2}(x)$$

where
\[
B_K = \left[ \int_a^b w(x) \phi_{K-1}^2(x) \, dx \right] \bigg/ \int_a^b w(x) \phi_{K-1}^2(x) \, dx \\
C_K = \left[ \int_a^b w(x) \phi_{K-1}(x) \phi_{K-2}(x) \, dx \right] \bigg/ \int_a^b w(x) \phi_{K-2}^2(x) \, dx
\]  

(4.3) \hspace{2cm} (4.4)

and \( w(x) \) is the weighting function. For uniform beams the weighting function, \( w(x) \), is unity. The polynomials \( \phi_k(x) \) satisfy the orthogonality condition,

\[
\int_a^b w(x) \phi_K(x) \phi_L(x) \, dx = 0 \quad \text{if} \quad K \neq L \\
= a_{KL} \quad \text{if} \quad K = L
\]  

(4.5)

Even though \( \phi_k(x) \) satisfies all the boundary conditions both geometric and natural, the other members of the orthogonal set satisfy only geometric boundary conditions and hence do not over-restrain the structure as in the case of classical beam functions.

4.2 Analysis

The deflection for a rectangular plate undergoing free flexural vibration can be expressed in terms of the beam characteristic orthogonal polynomials in \( x \) and \( y \) directions as

\[
W(x,y) = \sum_m \sum_n A_{mn} \phi_m(x) \phi_n(y)
\]  

(4.6)

where \( x = x/a \) and \( y = y/b \). \( x \) and \( y \) are the coordinates along the two sides of the plate and \( a \) and \( b \) are plate dimensions.

The strain energy of the plate can be obtained by assuming that the plate consists of a perfectly elastic, homogeneous, isotropic material. An element cut out of the plate by pairs of planes parallel to the \( xz \) and \( yz \) axes is shown in Fig. 4.1.
Fig. 4.1: Plate Element
The xy plane is taken as the middle plane of the plate and assuming that with small deflections, the lateral sides of an element cut out from the plate by planes parallel to zx and zy planes remain plane and rotate so as to be normal to the deflected middle surface of the plate. Then the strain in a thin layer of this element, indicated by the shaded area and distant z from the middle surface are equal to:

\[ e_{xx} = \frac{Z}{R_2} = -z \frac{\partial^2 W}{\partial x^2} \]  \hspace{1cm} (4.7)

\[ e_{yy} = \frac{Z}{R_3} = -z \frac{\partial^2 W}{\partial y^2} \]  \hspace{1cm} (4.8)

\[ e_{xy} = -2z \frac{\partial^2 W}{\partial x \partial y} \]  \hspace{1cm} (4.9)

where \( e_{xx}, e_{yy} \) are unit elongations in the x and y directions, \( e_{xy} \) is shear deformation in the xy plane, \( W \) is the deflection of the plate, and \( \frac{1}{R_2} \) and \( \frac{1}{R_3} \) are curvatures in the xz and yz planes respectively.

The corresponding stresses can be obtained by using Hooke's Law as:

\[ \sigma_x \left( e_{xx} + \nu e_{yy} \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right) \] \hspace{1cm} (4.10)

\[ \sigma_y = \frac{E}{1-\nu^2} \left( e_{yy} + \nu e_{xx} \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right) \] \hspace{1cm} (4.11)

\[ \tau = G e_{xy} \left( \frac{\partial^2 W}{\partial x \partial y} \right) \] \hspace{1cm} (4.12)

where \( \nu \) is the Poisson's ratio and \( E \) is the modulus of elasticity of the plate.
The potential energy accumulated in the shaded layer of the element during the deformation is,

\[
dU = \left( \frac{\epsilon_{xx} \sigma_x}{2} + \frac{\epsilon_{yy} \sigma_y}{2} + \frac{\epsilon_{xy} \tau}{2} \right) \, dx \, dy \, dz \quad (4.13)
\]

Using equations (4.7) through (4.12),

\[
dU = \frac{Ez^2}{2(1-\nu^2)} \left\{ \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right\} \, dx \, dy \, dz \quad (4.14)
\]

Integrating,

\[
U = \iiint dV = D \iiint \left\{ \left( \frac{\partial^2 W}{\partial x^2} \right)^2 + \left( \frac{\partial^2 W}{\partial y^2} \right)^2 + 2\nu \frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right\} \, dx \, dy \quad (4.15)
\]

In term of dimensionless parameters \( \hat{x} \) and \( \hat{y} \),

\[
U = \frac{1}{2} \frac{Dab}{a^b} \int_0^1 \int_0^1 \left[ W_{XX}^2 + \alpha_1 W_{YY}^2 + 2\nu \alpha_1^2 W_{XX} W_{YY} \right. \\
+ \left. 2(1-\nu) \alpha_1^2 W_{XY}^2 \right] \, d\hat{x} \, d\hat{y} \quad (4.16)
\]

The kinetic energy of the plate is

\[
T = \frac{1}{2} \rho h a b p^2 \int_0^1 \int_0^1 W^2(\hat{x}, \hat{y}) \, d\hat{x} \, d\hat{y} \quad (4.17)
\]
where $\rho$ - density of plate material

$h$ - thickness of the plate

$D$ - flexural rigidity of the plate

$\alpha$ - side ratio $a/b$

and the subscripts $x$ and $y$ refer to differentiation with respect to the subscript and the number of times the subscript appears denotes the order of differentiation.

The equations of motion are obtained by minimizing the Lagrangian with respect to the coefficient $A_{ij}$ as,

$$\frac{\partial T}{\partial A_{ij}} - \frac{\partial U}{\partial A_{ij}} = 0 \quad (4.18)$$

i.e.,

$$\rho a b p^2 \iint_0^1 \int_0^1 \frac{\partial W}{\partial A_{ij}} \, dx \, dy - \frac{abD}{\alpha^4} \iint_0^1 \int_0^1 \left[ \frac{\partial W}{\partial x} \frac{\partial W}{\partial A_{ij}} + x_i^2 \frac{\partial W}{\partial y} \frac{\partial W}{\partial A_{ij}} + \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} \frac{\partial W}{\partial A_{ij}} \right] \, dx \, dy = 0 \quad (4.19)$$

4.3 Plates with Three Edges Free and One Fixed (Cantilevered Plates)

Rotating structures with low aspect ratio can be idealized as a cantilevered plate. Figure 4.2 shows such a plate which consists of beam problems with clamped-free and free-free conditions. The first member of polynomials which satisfy the free-free conditions is taken to be unity. To determine the first member of polynomials satisfying the clamped-free condition consider the deflection function,

$$\phi_i(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \quad (4.19)$$
Fig. 4.2: Free-Free and Clamped-Free Beams used to Model the Cantilevered Plate
where \( x \) is the non-dimensional coordinate along the beam with the origin at one end of the beam. The boundary conditions for such a beam is that at \( x = 0 \), the deflection and slope are equal to zero and for \( x = 1 \), the moment and shear force are equal to zero, i.e., \( z(0) = z''(0) = z''(1) = z''(1) = 0 \) (4.25).

Substituting these boundary conditions in equation (4.19), the coefficients \( a_i \) are determined to yield,

\[ z(x) = a_i (6x^2 - 4x^3 - x^5) \] (4.21)

The coefficient \( a_i \) is appropriately chosen so as to normalize \( z_i(x) \) such that

\[ \int_0^1 z_i(x) \, dx = 1 \] (4.22)

The other polynomials are generated using Gram-Schmidt process as outlined in Section 4.2.

Substituting these deflection functions into the equation of motion outlined in Section 4.3, leads to the eigen-value problem.
\[
\iiint \left( \sum_{mn} \frac{m^2 - n^2}{m^2 + n^2} \right) \, dx \, dy
\]

\[
= \iiint \sum_{mn} A_{mn} \, dx \, dy - 2 \iiint \sum_{mn} \frac{m + n}{m^2 + n^2} \, dx \, dy
\]

\[
+ \iiint \sum_{mn} A_{mn} \, dx \, dy
\]

\[
- 2(1 - \ldots) \iiint \sum_{mn} A_{mn} \, dx \, dy = 0
\]

(4.23)

where \( \frac{\partial^2 u}{\partial x^2} \) and \( \frac{\partial^2 u}{\partial y^2} \) represent differentiation with \( x \) and \( y \) respectively.

This resulting eigen-value problem was solved by using the computer algorithm outlined in Appendix E.

4.4 Plates with all Edges Free

The plates with all edges free consist of beam problems with both ends free. The first member of the polynomials which satisfy all the boundary conditions, \( \phi_1(x) \), is chosen to be unity. This first member of the orthogonal polynomial set satisfies all the boundary conditions both geometric and natural. Using this first member, the other members of the orthogonal set are generated using the Gram-Schmidt process outlined in Section 4.2.
The resulting eigenvalue given by equation 4.23 is solved by using the computer algorithm given in Appendix F.

4.5 Experimental Verification Using Modal Testing Technique

Modal testing is the process of testing components to obtain their modal parameters. Essentially, modal testing involves identifying the natural frequencies of a structure and the associated deflected shapes when the structure is excited at one of the natural frequencies.

In this section, the natural frequencies and mode shapes for a square plate with free-free boundary conditions are obtained using an experimental modal analysis system. The experimental results obtained are compared with the computed analytical results.

4.6 General Description of the Software

The Modal Analysis program used consists of 13 transactions of which the 7 core ones are:

1) Geometry Entry.
2) Format data for display.
3) Identify Modal frequencies and damping.
4) Measure and extract Modal amplitudes.
5) Synthesize and plot transfer function.
6) Tabulate modal vectors or geometry.
7) Display mode shapes.

The core transactions shown above perform various portions of the experimental Modal Analysis procedure. The Geometry Transaction is used to store coordinates of the test points on the test structure. These test points form a "computer-sketch" of the test structure. The test points
are connected by straight-line fashion in the Format Transaction.

The Identify Transaction is used to determine the natural frequency and damping ratio for each mode to be analyzed. Measurements from anywhere on the structure can be used by the Identify Transaction.

When Geometry, Format, and Identify Transactions have been completed, the Measure Transaction is used to collect and store transfer function data for each test point. This transaction is also used to curve-fit this stored data to extract modal parameters.

After transfer function data at each test point has been curve-fit, the Format transaction is used to create the display files necessary to animate the calculated mode shapes. The Display Transaction can then access these files to generate animated mode shape displays on the system terminal.

The Synthesis Transaction can be used to generate and compare transfer functions. A synthesized transfer function, generated from extracted modal parameters, may be compared with a live measurement.

The Tabulate Transaction can be used to generate "report-ready" listings of test point coordinates and extracted modal parameters. Other program and analysis details about the structural analysis system are contained in reference [40].

4.7 Experimental Model

The flat-plate tested was suspended by two flexible threads at two corners such that the plate surface was in the vertical plane. The plate was divided into twenty five test points and each of these test points were excited by impulse excitation with response measurement being taken from
an accelerometer attached at the centre. The configuration is shown in Fig. 4.3.

Both the excitation signal and the input forcing signal were fed simultaneously into the FFT analyzer where the response time histories were transformed into frequency spectra. The ratio of these two functions which is the frequency response function of the structure was calculated and obtained as the output.

A complete experimental set up is given in Fig. 4.4. The frequency response function was further processed to obtain the modal parameters [39].

4.8 Natural Frequencies of a Rotating Plate

A simple configuration of a rotating plate is considered to use the orthogonal polynomial function approach. The setting angle is ninety degrees as shown in Fig. 4.5.

Consider an element of dimensions dx and dy. The radial displacement of this element towards the center due to deflection is

\[
\Delta r = \frac{1}{2} \int_0^y (\frac{dW}{dy})^2 \, dy
\]  

(4.24)

The work done on the element by the centrifugal forces acting on it during deflection is

\[
\Delta W = -m_w^2 (R+y) \, dx \, dy \, \frac{1}{2} \int_0^y (\frac{dW}{dy})^2 \, dy
\]

(4.25)

The energy corresponding to the work of the centrifugal forces will be obtained by summing that due to all such elements and is obtained as
Fig. 4.3: Measurement Points on Test Plate

X - accelerometer attachment point
Fig. 4.4: Schematic of the Experimental Procedure
Fig. 4.5 : Rotating Cantilever Plate
\[ U_r = - \int \int \Delta W \]
\[ = \frac{1}{2} m_0 \int \int (R+y) \, dx \, dy \int_0^l \left( \frac{dk}{dy} \right) \, dy \] 

The additional strain energy \( U_r \), due to centrifugal forces is added along with the bending strain energy \( U \), given in eq. (4.15) to solve for the natural frequencies of the rotating plate.

Substituting eq. (4.6) for \( W(x,y) \) in eq. (4.26), the strain energy due to rotation can be written as

\[ U_r = \int \int \int \left[ \int_0^l \left( \frac{R}{b} \right) \, dy \int_0^l \left( \frac{y}{n} \right) \, dy \right] \]

The natural frequencies of the rotating plate incorporating the strain energy due to rotation are numerically computed using Rayleigh's method. Results are tabulated in Table 4.3. Since only one term is used in the solution, flexure is considered only in the \( y \) direction and hence, the results are comparable to those for a rotating beam as can be verified by an inspection of Table 2.5. As to be expected, the natural frequency increases with speed of rotation showing the stiffening of the plate due to centrifugal forces.
4.9 Discussion of the Results

Natural frequencies, mode shapes and beam characteristic orthogonal polynomials are obtained for free-free and cantilevered plates.

A convergence test was carried out by taking equal number of terms for the series in both x and y directions. The results of this test are presented in Tables 4.1 and 4.2. The convergence is very rapid and the converged results are superior or equal to the comparison results. Experimental results obtained using modal analysis technique are also presented in Table 4.2.

The set of beam characteristic polynomials constructed using Gram-Schmidt process are presented and plotted in Tables 4.3-4.4 and Figs. 4.6-4.7 respectively.

The mode shape of the cantilevered plate is presented in Fig. 4.3. The animated mode shapes obtained using the experimental modal system are presented in Figs. 4.9 through 4.11.

The analytically computed first natural frequency could not be recorded experimentally due to the fact that the plate surface had a slight curvature when suspended by the flexible threads. Also, the first animated mode shape (Fig. 4.9) has a node point on one of its edges, probably due to an error in recording the transfer function. The other natural frequencies obtained experimentally agree closely with the analytical results and the results obtained by Leissa [21] and Dickinson [27] as shown in Table 4.2.

Natural frequencies of the rotating plate using Rayleigh's method are shown in Table 4.5. The natural frequencies increase with an increase
in rotational speed $\alpha$, and since flexure is considered only in the $y$

direction, the results are comparable to the one's obtained for a rotating beam as shown in Table 2.5.

In this chapter, a set of beam characteristic orthogonal polynomials are used to obtain the natural frequencies and mode shapes of rotating and non-rotating plates. Experimental modal analysis is also carried out on a plate with free-free edges and the results are compared to the analytical results.

In the final chapter, conclusions, recommendations and scope for future work are presented.
<table>
<thead>
<tr>
<th>Side Ratio $\alpha_1$</th>
<th>Mode No.</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>Ref. 27</th>
<th>Ref. 21</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>1</td>
<td>3.4881</td>
<td>3.4737</td>
<td>3.4712</td>
<td>3.4917</td>
<td>3.473</td>
<td>3.473</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>8.5472</td>
<td>8.5125</td>
<td>8.5089</td>
<td>8.5246</td>
<td>8.5089</td>
<td>8.5246</td>
</tr>
</tbody>
</table>

Table A.1: Frequency Parameters ($\omega_p^2a^2/\beta$) for Isotropic Cantilevered Plate ($v = 0.3$)
TABLE 4.2: Frequency Parameters \((\rho h^2 a^4/D)^{1/2}\) for Isotropic Free-Free Plate \((\nu = 0.3)\)

<table>
<thead>
<tr>
<th>Side Ratio (\alpha_1)</th>
<th>Mode No.</th>
<th>Values of (m) and (n) Taken in Deflection Function</th>
<th>Comparison Results</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>4</td>
<td>6</td>
<td>8</td>
</tr>
</tbody>
</table>
### TABLE 4.3: Orthogonal Polynomials for Clamped-Free Beam

\[ \phi_i(x) = \sum_{j=0}^{\infty} a_{ij} x^{j+2} \]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_{10} )</td>
<td>3.94676</td>
<td>20.0501</td>
<td>62.4439</td>
<td>152.106</td>
<td>317.665</td>
<td>596.162</td>
</tr>
<tr>
<td>( a_{11} )</td>
<td>-2.63117</td>
<td>-38.3606</td>
<td>-225.033</td>
<td>-868.760</td>
<td>-2613.85</td>
<td>-6647.44</td>
</tr>
<tr>
<td>( a_{12} )</td>
<td>6.57793</td>
<td>20.0043</td>
<td>259.361</td>
<td>1733.29</td>
<td>7900.08</td>
<td>28089.1</td>
</tr>
<tr>
<td>( a_{13} )</td>
<td>0.0</td>
<td>-4.16565</td>
<td>-115.023</td>
<td>-1512.98</td>
<td>-11321.0</td>
<td>-59107.0</td>
</tr>
<tr>
<td>( a_{14} )</td>
<td>0.0</td>
<td>0.0</td>
<td>21.1140</td>
<td>8191.069</td>
<td>8147.96</td>
<td>67318.5</td>
</tr>
<tr>
<td>( a_{15} )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-97.9188</td>
<td>-2861.11</td>
<td>-41733.8</td>
</tr>
<tr>
<td>( a_{16} )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>433.743</td>
<td>13350.5</td>
</tr>
<tr>
<td>( a_{17} )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>-1869.89</td>
</tr>
</tbody>
</table>
### Table 4.4: Orthogonal Polynomials for Free-Free Beam

\[
\phi_i(x) = \sum_{j=0}^{i} a_{ij} x^j
\]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{10})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{11})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{12})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{13})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{14})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{15})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{16})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(a_{17})</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 4.5: Variation of Frequency Parameters \( (\rho h^2 a^2/D)^{1/3} \)
for Isotropic Cantilevered Plate \( (\nu = 0.3; \theta = 90^\circ) \)

<table>
<thead>
<tr>
<th>Angular Velocity ((\alpha))</th>
<th>Frequency Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.531</td>
</tr>
<tr>
<td>1</td>
<td>3.692</td>
</tr>
<tr>
<td>2</td>
<td>4.142</td>
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<tr>
<td>3</td>
<td>4.797</td>
</tr>
<tr>
<td>4</td>
<td>5.588</td>
</tr>
<tr>
<td>5</td>
<td>6.464</td>
</tr>
<tr>
<td>6</td>
<td>7.395</td>
</tr>
<tr>
<td>7</td>
<td>8.149</td>
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<td>8</td>
<td>9.356</td>
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<tr>
<td>9</td>
<td>10.367</td>
</tr>
<tr>
<td>10</td>
<td>11.3913</td>
</tr>
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Fig. 4.6: Orthogonal Polynomial Deflection Functions for Clamped-Free Beam
Fig. 4.7: Orthogonal Polynomial Deflection Functions for Free-Free Beam
Fig. 4.8: Nodal Lines for Cantilevered Plates
Fig. 4.9: First Animated Mode Shape Using Experimental Modal System
Fig. 4.10: Second Animated Mode Shape Using Experimental Modal System
Fig. 4.11: Third Animated Mode Shape Using Experimental Modal System
CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS
CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

Dynamics of a class of rotating structures by using improved methods in calculating their natural frequencies and mode shapes has been proposed where the rotating structure depending on the aspect ratio is modelled either as a beam or a plate.

Rayleigh-Ritz method with improved strain energy formulation is used to obtain the natural frequencies and mode shapes of a rotating beam type structure. The setting angle of the beam with respect to the plane of rotation and radius of hub on which the structure is mounted are also taken into account in the formulation. The variation of natural frequencies and mode shapes presented in a non-dimensional form against the rotational speed.

The finite element program SPAR has also been used to model this rotating beam type structure. The natural frequencies for the structure are obtained for different Hub-Radii, setting angle and rotational speed.

At low-aspect ratios, the structure is modelled as a plate. A class of beam characteristic orthogonal polynomials, constructed using Gram-Schmidt process, are employed as deflection functions to obtain the natural frequencies and mode shapes for a stationary and rotating plate type structure. Natural frequencies and mode shapes of plates with all edges free and three edges free and one fixed are presented.

Modal analysis technique to obtain the modal parameters for a free-free plate is also presented.
5.1 Conclusions

The conclusions arrived on the basis of the results presented in the different chapters of the thesis are summarized and given below:

1) The natural frequencies of the rotating beam type structure increase for higher setting-angles and for any setting angle the increase is linear at higher rotational speeds.

2) With higher values of hub radius $R$, the natural frequencies are higher and increase faster with the rotational speed.

3) The beam tries to straighten itself more and more as the rotational speed $\alpha$ is increased.

4) A convergence test was made by varying the number of terms considered in the polynomial shape and the convergence was good.

5) The natural frequencies for the non-rotating plate type structure using the beam characteristic orthogonal polynomials are superior than using other beam functions.

6) Gram-Schmidt process is used to generate sets of orthogonal polynomial functions, the first member of each set satisfying the geometric and natural boundary conditions of an appropriate equivalent beam, and the remainder of the set satisfying only the geometric boundary conditions of the beam. The lack of satisfaction of the natural boundary conditions of the equivalent beam by the higher members of the set relaxes the over-restraint encountered in the use of true beam functions and permits the treatment of a plate with all edges free with a degree of accuracy.

7) The results obtained by modal analysis techniques agree closely to the analytical computed ones, hence, validating the analytical analysis.
5.2 Recommendations for Future Work

Some suggestions for possible future work are given below:

1) Pretwist, shear deformation, taper, etc., can be incorporated in the formulation for the rotating beam type structure.

2) The influence of a tip mass on the natural frequencies and mode shapes of a rotating beam type structure using improved strain energy formulation outlined can be investigated.

3) Experimental Modal analysis techniques can be utilized to determine the natural frequencies and mode shapes of a rotating beam to verify the analytical results.

4) Optimization of the beam can be carried out subject to the constraints of increase in natural frequency due to rotation, setting angle, hub radii, etc.

5) Rayleigh's method can be extended over to Rayleigh-Ritz method for rotating plates by incorporating the additional energy due to rotation.

6) Improved energy formulation can be carried out in Rayleigh-Ritz method for analyzing the plates.

7) Effect of setting angle, Hub-Radii on the natural frequencies of rotating plates can be studied.

8) Modal analysis techniques can be utilized to study rotating plates.
REFERENCES


APPENDIX A

CONVENTIONAL RAYLEIGH-RITZ PROGRAM

This Fortran program computes the natural frequencies and mode shapes for a rotating cantilever beam using conventional Rayleigh-Ritz method.

This program has been referenced in Section 2.2 of Chapter 2.
***************

C THIS PROGRAM IS USED TO COMPUTE EIGEN-VALUES AND EIGEN-VECTORS FOR A ROTATING STRUCTURE (CANTILEVER BEAM).
C THE SETTING ANGLE OF THE BLADE WITH RESPECT TO THE PLANE OF ROTATION AND THE HUB RADIUS (RBAR) ON WHICH THE BLADE IS MOUNTED ARE TAKEN INTO ACCOUNT IN THE FORMULATION.
C THE VARIATION OF NATURAL FREQUENCIES AND MODE SHAPES WITH THE SPEED OF ROTATION (ALPHA), SETTING ANGLE (THETA) AND C HUB RADIUS ARE STUDIED.

C*********************************************************************************************************************************************
C PROGRAM REF (INPUT, OUTPUT)
DIMENSION A(5,5), B(5,5), AREAL(5),
1 WK(800), BETA(5), BREAL(5), ZREAL(5,5), ZPLEX(5,5),
1 THETA(4), SUMY(10), EIGEN(10)
C COMPLEX; ALFA(5), Z(5,5)
C 2 READ*, LIM
LIM=5
LIMM=LIM-1
LIMN=LIM+1
C IF (LIM.EQ.0) STOP
PI=ATAN(1.0)*4.
THETA(1)=0.0
THETA(2)=PI/6.0
THETA(3)=PI/3.0
THETA(4)=PI/2.0
3 DO 5 MM=1,4
ANGLE=THETA(MM)*180.0/PI
PRINT 1000, ANGLE
1000 FORMAT (20X, *SETTING ANGLE=*, F5.2, 1X, *DEGREES*)
EMU=0.0
SIGMA=0.0
AREA=3.14E-4
LENGTH=0.1
EIXX=7.85E-9
SRBAR=SQRT(EIXX/AREA*LENGTH**2)
RBAR=10.0
ALPHA=0.0
5 DO 10 I=2, LIMN
DO 15 J=2, LIMN.
AT1=(COS(THETA(MM))**2/(I+J+1.))
AT2=I*J*(SRBAR+5.)/(I+J-1.)-RBAR/(I+J-.5/(I+J+1.))
AT3=((I*J)/(I+J-1.))**(SRBAR+1.)*EMU
AT4=EMU*(COS(THETA(MM)))**2
AT5=(I-1.)*(J-1.)*((I*J)/(I+J-3.))
BT1=(1./*(I+J+1.)*EMU
BT2=(SRBAR**2*I*J)/(I+J-1.)
BT3=SIGMA*I*J
AJ1=ALPHA**2*(-AT1+AT2+AT3-AT4)+AT5
BJ1=BT1+BT2+BT3
A((I-1.),(J-1.))=AJ1
B((I-1.),(J-1.))=BJ1
15 CONTINUE
10 CONTINUE
CALL EIGZF(A(5,5), B(5,5,2), ALFA, BETA, Z(5,5), WK, IER)

***************
DO 50 I=1,LIM
DO 55 J=1,LIM
ZREAL(I,J)=REAL(Z(I,J))
ZPLEX(I,J)=AIMAG(Z(I,J))
55 CONTINUE
50 CONTINUE
PRINT 100
100 FORMAT(2X,*NATURAL FREQUENCY WHEN*)
PRINT 150, ALPHA
150 FORMAT(2X,*ANGULAR VELOCITY=*,F5.2)
DO 30 I=1,LIM
AREAL(I)=REAL(ALFA(I))
BREAL(I)=BETA(I)
30 CONTINUE
DO 35 I=1,LIM
EIGEN(I)=SORT(AREAL(I)/BREAL(I))
35 CONTINUE
IF(LIM.EQ.1) GO TO 211
DO 9 I=1,LIMM
M=I+1
DO 11 J=M,LIM
XXX=(EIGEN(I)-EIGEN(J))
IF(XXX)12,13,13
TEMP=EIGEN(I)
EIGEN(I)=EIGEN(J)
EIGEN(J)=TEMP
12 CONTINUE
DO 14 N1=1,LIM
TEMP1=Z(N1,I)
*Z(N1,I)=Z(N1,J)
Z(N1,J)=TEMP1
14 CONTINUE
11 CONTINUE
9 CONTINUE
211 PRINT 400,(EIGEN(I),I=1,LIM)
400 FORMAT(2X,E16.8)
C5 CONTINUE
ALPHA=ALPHA+0.5
IF(ALPHA.GT.10.)GO TO 6
GO TO 4
6 CONTINUE
5 CONTINUE
C GO TO 2
END
APPENDIX B

ELEMENTS OF THE MATRICES

This appendix gives the elements $a_{nk}$ and $b_{nk}$ of matrices $[A]$ and $[B]$ as mentioned in Section 2.3 of Chapter 2.
The elements $a_{nk}$ and $b_{nk}$ of matrices [A] and [B] are given by:

$$a_{nk} = \frac{1}{(n + 1)(k + 1)} \left( A_1 + A_2 + \cdots + A_{10} \right) + \frac{72}{(n + 1)(k + 1)} (A_{11} + A_{12} + \cdots + A_{17})$$

$$b_{nk} = \frac{\alpha^2}{(n + 1)(k + 1)} \left( B_1 + B_2 + \cdots + B_{64} \right) + \frac{\alpha^2 \cos^2 \theta}{(n + 1)(k + 1)} \left( A_1 + A_2 + \cdots + A_{10} \right) + \frac{1}{(n + 1)(k + 1)} (B_{65} + B_{66} + \cdots + B_{72})$$

where

$$A_1 = \frac{1}{20(n + 2)(k + 2)}$$

$$A_2 = -\frac{1}{2(n + 1)(n + 3)(n + 4)(n + 2)(k + 2)}$$

$$A_3 = -\frac{1}{20(k + 2)}; \quad A_4 = \frac{1}{72(k + 2)}$$

$$A_5 = \frac{1}{2(n + 2)(k + 2)(k + 3)(k + 4)(k + 7)}$$

$$A_6 = \frac{1}{(n + k + 9)(k + 2)(k + 3)(k + 4)(n + 2)(n + 3)(n + 4)}$$

$$A_7 = \frac{1}{2(k + 2)(k + 3)(k + 4)(k + 7)}$$

$$A_8 = -\frac{1}{8(k + 2)(k + 3)(k + 4)(k + 8)}$$

$$A_9 = -\frac{1}{20(n + 2)}$$

$$A_{10} = \frac{1}{2(n + 2)(n + 3)(n + 4)(n + 7)}$$

$$A_{11} = \frac{1}{20}; \quad A_{12} = -\frac{1}{72}$$

$$A_{13} = \frac{1}{72(n + 2)}$$

$$A_{14} = -\frac{1}{6(n + 2)(n + 3)(n + 4)(n + 8)}$$

$$A_{15} = -\frac{1}{72}; \quad A_{16} = \frac{1}{252}; \quad A_{17} = \frac{1}{3(n + 2)(k + 2)}$$
\[ A_{18} = -\frac{1}{(n + 5)(n + 3)(n + 2)(k + 2)} \]
\[ A_{19} = -\frac{1}{3(k + 2)} \]
\[ A_{20} = \frac{1}{8(k + 2)} \]
\[ A_{21} = \frac{1}{(k + 5)(n + 2)(k + 3)(k + 2)} \]
\[ A_{22} = \frac{1}{(n + k + 7)(n + 3)(n + 2)(k + 3)(k + 2)} \]
\[ A_{23} = \frac{1}{(k + 5)(k + 3)(k + 2)} \]
\[ A_{24} = -\frac{1}{2(k + 6)(k + 3)(k + 2)} \]
\[ A_{25} = -\frac{1}{3(n + 2)} \]
\[ A_{26} = \frac{1}{(n + 5)(n + 3)(n + 2)} \]
\[ A_{27} = \frac{1}{3} \]
\[ A_{28} = \frac{1}{8} \]
\[ A_{29} = \frac{1}{8(n + 2)} \]
\[ A_{30} = -\frac{1}{2(n + 6)(n + 3)(n + 2)} \]
\[ A_{31} = -\frac{1}{8} \]
\[ A_{32} = \frac{1}{20} \]
\[ B_1 = \frac{\Box}{3(n + 2)(k + 2)} \]
\[ B_2 = -\frac{\Box}{(n + 5)(n + 3)(n + 2)(k + 2)} \]
\[ B_3 = -\frac{\Box}{3(k + 2)} \]
\[ B_4 = \frac{\Box}{8(k + 2)} \]
\[ B_5 = -\frac{\Box}{(k + 5)(n + 2)(k + 3)(k + 2)} \]
\[ B_6 = \frac{\Box}{(n + k + 7)(n + 3)(n + 2)(k + 3)(k + 2)} \]
\[ B_7 = \frac{\Box}{(k + 5)(k + 3)(k + 2)} \]
\[ B_8 = -\frac{\Box}{2(k + 6)(k + 3)(k + 2)} \]
\[ B_y = -\frac{n}{3(n + 2)} \]

\[ B_{40} = \frac{n}{(n + 5)(n + 3)(n + 2)} \]

\[ B_{11} = \frac{n}{3} \]

\[ B_{12} = -\frac{n}{8(n + 2)} \]

\[ B_{14} = -\frac{n}{2(n + 6)(n + 3)(n + 2)} \]

\[ B_{15} = -\frac{n}{8} \]

\[ B_{16} = \frac{n}{20} \]

\[ B_{17} = \frac{1}{6(n + 2)(n + 2)} \]

\[ B_{18} = \frac{1}{2(n + 5)(n + 3)(n + 2)(n + 2)} \]

\[ B_{19} = \frac{1}{6(n + 2)} \]

\[ B_{20} = \frac{1}{16(n + 3)(n + 2)(n + 3)(n + 2)} \]

\[ B_{21} = \frac{1}{2(n + 5)(n + 3)(n + 3)(n + 2)(n + 2)} \]

\[ B_{22} = \frac{1}{16(n + 3)(n + 2)(n + 3)(n + 2)} \]

\[ B_{23} = \frac{1}{2(n + 5)(n + 3)(n + 2)} \]

\[ B_{24} = \frac{1}{4(n + 5)(n + 3)(n + 2)} \]

\[ B_{25} = \frac{1}{6(n + 2)} \]

\[ B_{26} = \frac{1}{2(n + 5)(n + 3)(n + 2)} \]

\[ B_{27} = \frac{1}{6} \]

\[ B_{28} = \frac{1}{16} \]

\[ B_{29} = \frac{1}{16(n + 2)} \]

\[ B_{30} = \frac{1}{4(n + 6)(n + 3)(n + 2)} \]

\[ B_{31} = \frac{1}{16} \]

\[ B_{32} = \frac{1}{40} \]

\[ B_{33} = \frac{n}{4(n + 2)(n + 2)} \]

\[ B_{34} = \frac{n}{(n + 6)(n + 3)(n + 2)(n + 2)(n + 2)} \]
\[ B_{35} = \frac{1}{4(n + 2)} \quad ; \quad B_{36} = -\frac{1}{10(n + 2)} \]

\[ B_{37} = \frac{1}{(n + 6)(n + 2)(k + 3)(k + 2)} \]

\[ B_{38} = \frac{1}{(n + 6)(n + 3)(n + 2)(k + 3)(k + 2)} \]

\[ B_{39} = \frac{1}{(k + 6)(k + 3)(k + 2)} \]

\[ B_{40} = \frac{1}{2(k + 7)(k + 3)(k + 2)} \]

\[ B_{41} = \frac{1}{4(n + 2)} \quad ; \quad B_{42} = -\frac{1}{(n + 6)(n + 3)(n + 2)} \]

\[ B_{43} = \frac{1}{4} \quad ; \quad B_{44} = \frac{1}{10} \quad ; \quad B_{45} = -\frac{1}{10(n + 2)} \]

\[ B_{46} = \frac{1}{2(n + 7)(n + 3)(n + 2)} \]

\[ B_{47} = \frac{1}{10} \quad ; \quad B_{48} = -\frac{1}{24} \]

\[ B_{49} = \frac{1}{10(n + 2)(k + 2)} \]

\[ B_{50} = \frac{1}{2(n + 7)(n + 3)(n + 2)(k + 2)} \quad ; \quad B_{51} = \frac{1}{10(k + 2)} \]

\[ B_{52} = -\frac{1}{24(k + 2)} \quad ; \quad B_{53} = -\frac{1}{2(k + 6)(n + 2)(k + 3)(k + 2)} \]

\[ B_{54} = -\frac{1}{2(n + 6)(n + 3)(n + 2)(k + 3)(k + 2)} \]

\[ B_{55} = -\frac{1}{2(k + 7)(k + 3)(k + 2)} \quad ; \quad B_{56} = \frac{1}{4(k + 8)(k + 3)(k + 2)} \]

\[ B_{57} = \frac{1}{10(n + 2)} \quad ; \quad B_{58} = -\frac{1}{2(n + 7)(n + 3)(n + 2)} \]

\[ B_{59} = -\frac{1}{10} \quad ; \quad B_{60} = \frac{1}{24} \quad ; \quad B_{61} = -\frac{1}{24(n + 2)} \]

\[ B_{62} = \frac{1}{4(n + 8)(n + 3)(n + 2)} \quad ; \quad B_{63} = \frac{1}{24} \]

\[ B_{64} = -\frac{1}{56} \quad ; \quad B_{65} = \frac{1}{3} \quad ; \quad B_{66} = -\frac{1}{(k + 2)(n + 2)} \]
\[ B_{67} = -\frac{(n+k)}{2(k+2)(n+2)}; \quad B_{68} = \frac{(n+1)}{(k+2)(n+2)(k+3)} \]
\[ B_{69} = \frac{(k+1)}{(k+2)(n+2)(n+3)} \]
\[ B_{70} = -\frac{(n+2)}{(n+2)(k+2)(k+4)} \]
\[ B_{71} = -\frac{(k+2)}{(k+2)(n+2)(n+4)} \]
\[ B_{72} = \frac{1}{(n+2)(k+2)(n+k+5)} \]
APPENDIX C

PROGRAM FOR IMPROVED STRAIN ENERGY FORMULATION IN
RAYLEIGH-RITZ METHOD

This Fortran program is used to obtain the natural frequencies and
mode shapes for a rotating cantilever beam using improved strain energy
formulation in Rayleigh-Ritz method. This program was referenced in
Section 2.3 of Chapter 2.
C THIS PROGRAM IS USED TO COMPUTE EIGEN-VALUES AND EIGEN-
C VECTORS FOR A ROTATING STRUCTURE (CANTILEVER BEAM), AN
C IMPROVED STRAIN-ENERGY FORMULATION IN RAYLEIGH-RITZ
C METHOD IS CARRIED OUT BY SUCCESSIVE INTEGRATION OF THE
C ASSUMED DEFLECTION FUNCTION TO OBTAIN THE MOMENT
C DISTRIBUTION WHICH IS USED TO OBTAIN THE STRAIN ENERGY OF
C THE BLADE, THE SETTING ANGLE OF THE BLADE WITH RESPECT TO THE
C PLANE OF ROTATION AND THE HUB RADIUS (RBAR) ON WHICH THE BLADE
C IS MOUNTED ARE TAKEN INTO ACCOUNT IN THE FORMULATION.
C THE VARIATION OF NATURAL FREQUENCIES AND MODE SHAPES
C WITH THE SPEED OF ROTATION (ALPHA), SETTING ANGLE (THETA) AND
C HUB RADIUS ARE STUDIED.
C*******************************************************************************

C PROGRAM CANTI(INPUT, OUTPUT)
C COMPLEX ALFA(10), Z(10, 10)
C REAL LAB
C DIMENSION A(10, 10), B(10, 10), Beta(10), WK(300), EIG(10)
C 1, Theta(4), LAB(10), ZZ(5, 5), YY(10)
C
C READ*, LIM
C LIM=S
C LIMM=LIM-1
C LIMN=LIM+1
C IF (LIM.EQ.0) STOP
C PI=ATAN(1.0)*4
C Theta(1)=0.0
C Theta(2)=PI/6.0
C Theta(3)=PI/3.0
C Theta(4)=PI/2.0
C
C DO 5 MM=1, 4
C ANGLE=Theta(MM)*180.0/PI
C PRINT 1000, ANGLE
C 1000 FORMAT(20X, *SETTING ANGLE=*, F5.2, 1X, *DEGREES*)
C AREA=3.14E-4
C SLENGTH=1.0
C EIXX=7.85E-9
C SRBAR=(EIXX/(AREA*SLength**2))
C RBar=0.0
C DO 6 II=5, 20, 5
C Alpha=II-5
C 6
C DO 7 K=2, LIMN
C DO 7 N=2, LIMN
C KK=K-1
C NN=N-1
C P=K
C Q=N
C AT1=(COS(Theta(MM)))*2
C AT2=(1.)/(((Q+1.)*(P+1.))
C AT3=(1.)/((20.)*(Q+2.)*(P+2.))
C AT4=(1.)/((Q+7.)*(Q+4.)*(Q+3.)*(Q+2.)*(P+2.))
C AT5=(1.)/((20.)*(P+2.))
C AT6=(1.)/((72.)*(P+2.))
C AT7=(1.)/((2.)*(Q+2.)*(P+2.)*(P+3.)*(P+4.)*(P+7.))
C AT8=(1.)/((Q+P+9.)*(P+2.)*(P+3.)*(P+4.)*(Q+2.)*(Q+3.))
C AT9=(1.)/((Q+4.))
C
\[BT48 = (RBA) / 10.\]
\[BT49 = RBA / 24.\]
\[BT50 = (1.)/(10.(*(Q+2.))*(P+2.))\]
\[BT51 = (1.)/(2.(*(Q+7.))*(Q+3.))*(Q+2.))*(P+2.))\]
\[BT52 = (1.)/(10.(*(P+2.)))\]
\[BT53 = (1.)/(24.(*(P+2.)))\]
\[BT54 = (1.)/(2.(*(P+7.))*(Q+2.))*(P+3.))*(P+2.))\]
\[BT55 = (1.)/(2.(*(Q+9.))*(Q+3.))*(Q+2.))*(P+3.))*(P+2.))\]
\[BT56 = (1.)/(2.(*(P+7.))*(P+3.))*(P+2.))\]
\[BT57 = (1.)/(4.(*(P+8.))*(P+3.))*(P+2.))\]
\[BT58 = (1.)/(10.(*(Q+2.)))\]
\[BT59 = (1.)/(2.(*(Q+7.))*(Q+3.))*(Q+2.))\]
\[BT60 = (1.)/10.\]
\[BT61 = (1.)/24.\]
\[BT62 = (1.)/(24.*(*(Q+2.)))\]
\[BT63 = (1.)/(4.*(*(Q+8.))*(Q+3.))*(Q+2.))\]
\[BT64 = (1.)/24.\]
\[BT65 = (1.)/56.\]

\[CT1 = (1.)/((Q+1.))*(P+1.))\]
\[CT2 = (1.)/3.\]
\[CT3 = (1.)/((P+2.))*(Q+2.))\]
\[CT4 = 1.\]
\[CT5 = 0.5*(Q+F)\]
\[CT6 = ((Q+1.))/(P+3.))\]
\[CT7 = ((P+1.))/(Q+3.))\]
\[CT8 = ((Q+2.))/(P+4.))\]
\[CT9 = ((P+2.))/(Q+4.))\]
\[CT10 = (1.)/(Q+P+5,\)

\[DT1 = (1.)/((Q+1.))*(P+1.))\]
\[DT2 = (1.)/(3.(*(Q+2.))*(P+2.))\]
\[DT3 = (1.)/((Q+5.))*(Q+3.))*(Q+2.))*(P+2.))\]
\[DT4 = (1.)/(3.(*(P+2.))\]
\[DT5 = (1.)/(8.(*(P+2.)))\]
\[DT6 = (1.)/(P+5.))*(Q+2.))*(P+3.))*(P+2.))\]
\[DT7 = (1.)/((Q+P+7.))*(Q+3.))*(Q+2.))*(P+3.))*(P+2.))\]
\[DT8 = (1.)/(P+5.))*(P+3.))*(P+2.))\]
\[DT9 = (1.)/(2.(*(P+6.))*(P+3.))*(P+2.))\]
\[DT10 = (1.)/(3.(*(Q+2.)))\]
\[DT11 = (1.)/((Q+5.))*(Q+3.))*(Q+2.))\]
\[DT12 = (1.)/3.\]
\[DT13 = (1.)/8.\]
\[DT14 = (1.)/(8(*(Q+2.)))\]
\[DT15 = (1.)/(2.(*(Q+6.))*(Q+3.))*(Q+2.))\]
\[DT16 = (1.)/8.\]
\[DT17 = (1.)/20.\]
\[DT18 = AT2*(AT3-AT4-AT5+AT6-AT7+AT8+AT9-AT10-AT11\]
\[1+AT12+AT13-AT14+AT15-AT16-AT17+AT18\]
\[DT2 = BRB\&KDIJ*DT2-2DT3-2DT4-2DT5-2DT6+DT7+DT8+DT9-2DT10\]
\[1+DT11+DT12+DT13+DT14-2DT15-2DT16+2DT17\]
\[DT5 = DT11+DT2\]
\[AT5 = (ALPHA**2*(AT1*AT2))*(AT3-AT4-AT5+AT6-AT7+AT8+AT9-AT10-AT11\]
\[1+AT12+AT13-AT14+AT15-AT16-AT17+AT18\]

\[C\]
\[C\]
\[C\]
-110-

AT10 = (1.) / (6. * (P+2.) * (P+3.) * (P+4.) * (P+8.))
AT11 = (1.) / (20. * (Q+2.))
AT12 = (1.) / (2. * (Q+2.) * (Q+3.) * (Q+4.) * (Q+7.))
AT13 = (1.) / 20.
AT14 = (1.) / 72.
AT15 = (1.) / (72. * (Q+2.))
AT16 = (1.) / (6. * (Q+2.) * (Q+3.) * (Q+4.) * (Q+8.))
AT17 = (1.) / 72.
AT18 = (1.) / (252.)

**C**

**BT1** = (1.) / ((Q+1) * (P+1))
**BT2** = (RBAR) / (3. * (Q+2.) * (Q+2.))
**BT3** = (RBAR) / ((Q+5.) * (Q+3.) * (Q+2.) * (P+2.))
**BT4** = (RBAR) / (3. * (P+2.))
**BT5** = (RBAR) / (8. * (P+2.))
**BT6** = (RBAR) / ((P+5.) * (Q+2.) * (Q+3.) * (P+2.))
**BT7** = (RBAR) / ((Q+P+7.) * (Q+3.) * (Q+2.) * (P+3.) * (P+2.))
**BT8** = (RBAR) / ((P+5.) * (P+3.) * (P+2.))
**BT9** = (RBAR) / (2. * (P+6.) * (P+3.) * (P+2.))
**BT10** = (RBAR) / (3. * (Q+2.))
**BT11** = (RBAR) / ((Q+5.) * (Q+3.) * (Q+2.))
**BT12** = (RBAR) / 3.
**BT13** = (RBAR) / 8.
**BT14** = (RBAR) / (8. * (Q+2.))
**BT15** = (RBAR) / (2. * (Q+6.) * (Q+3.) * (Q+2.))
**BT16** = (RBAR) / 8.
**BT17** = (RBAR) / 20.
**BT18** = (1.) / (6. * (Q+2.) * (P+2.))
**BT19** = (1.) / (2. * (Q+5.) * (Q+3.) * (Q+2.) * (P+2.))
**BT20** = (1.) / (6. * (P+2.))
**BT21** = (1.) / (16. * (P+2.))
**BT22** = (1.) / (2. * (P+5.) * (Q+2.) * (Q+3.) * (P+2.))
**BT23** = (1.) / (2. * (Q+P+7.) * (Q+3.) * (Q+2.) * (P+3.) * (P+2.))
**BT24** = (1.) / (2. * (P+5.) * (P+3.) * (P+2.))
**BT25** = (1.) / (2. * (Q+P+6.) * (Q+P+3.) * (P+2.))
**BT26** = (1.) / (6. * (Q+2.))
**BT27** = (1.) / (2. * (Q+5.) * (Q+3.) * (Q+2.))
**BT28** = (1.) / 6.
**BT29** = (1.) / 16.
**BT30** = (1.) / (16. * (Q+2.))
**BT31** = (1.) / (4. * (Q+6.) * (Q+3.) * (Q+2.))
**BT32** = (1.) / 16.
**BT33** = (1.) / (40.)
**BT34** = (RBAR) / (4. * (Q+2.) * (P+2.))
**BT35** = (RBAR) / ((Q+6.) * (Q+3.) * (Q+2.) * (P+2.))
**BT36** = (RBAR) / (4. * (P+2.))
**BT37** = (RBAR) / (10. * (P+2.))
**BT38** = (RBAR) / ((P+6.) * (Q+2.) * (P+3.) * (P+2.))
**BT39** = (RBAR) / ((Q+P+8.) * (Q+3.) * (Q+2.) * (P+3.) * (P+2.))
**BT40** = (RBAR) / ((P+6.) * (P+3.) * (P+2.))
**BT41** = (RBAR) / (2. * (P+7.) * (P+3.) * (P+2.))
**BT42** = (RBAR) / (4. * (Q+2.))
**BT43** = (RBAR) / ((Q+6.) * (Q+3.) * (Q+2.))
**BT44** = (RBAR) / 4.
**BT45** = (RBAR) / 10.
**BT46** = (RBAR) / (10. * (Q+2.))
**BT47** = (RBAR) / (2. * (Q+7.) * (Q+3.) * (Q+2.))
BTT=(ALPHA**2)*(BT1)*(BT2+BT3+BT4+BT5+BT6+BT7+BT8+BT9+BT10
1+BT11+BT12+BT13+BT14+BT15+BT16+BT17+BT18+BT19+BT20
1+BT21+BT22+BT23+BT24+BT25+BT26+BT27+BT28+BT29+BT30
1-BT31-BT32-BT33-BT34-BT35-BT36-BT37-BT38-BT39-BT40
1-BT41-BT42-BT43-BT44-BT45-BT46-BT47-BT48-BT49-BT50
1+BT51+BT52+BT53+BT54+BT55+BT56+BT57+BT58+BT59+BT60+BT61
1-BT62+BT63+BT64+BT65))
CTT=((CT1)*(CT2+CT3*(-1)*CT4-CT5+CT6-CT7-CT8-CT9+CT10)))
A(KK,NN)=CTT+BTT-ATT
B(KK,NN)=BTT
7 CONTINUE
CALL EIGF(A,10,B,10,LIM,2,ALFA,BETA,Z,10,WK,IER)
PRINT 500
500 FORMAT(2X,*NATURAL FREQUENCY WHEN*)
PRINT 550,ALPHA
550 FORMAT(2X,*ANGULAR VELOCITY=*F5.2)
DO 8 I=1,LIM
EIG(I)=(ALFA(I))/BETA(I)
LAB(I)=SORT(EIG(I))
8 CONTINUE
IF(LIM.EQ.1)GO TO 211
DO 9 I=1,LIM
M=I+1
DO 10 J=M,LIM
XXX=(LAB(I)-LAB(J))
IF(XXX**2.LT.11.12)
10 TEMP=LAB(I)
LAB(I)=LAB(J)
LAB(J)=TEMP
CONTINUE
11 CONTINUE
DO 13 N1=1,LIM
TEMP1=Z(N1,I)
Z(N1,I)=Z(N1,J)
Z(N1,J)=TEMP1
13 CONTINUE
10 CONTINUE
9 CONTINUE
211 PRINT 600,(LAB(I),I=1,LIM)
600 FORMAT(E16.9)
PRINT 700
700 FORMAT(*/2X,*CORRESPONDING EIGEN VECTORS*/)
DO 14 I=1,LIM
DO 14 J=1,LIM
ZZ(I,J)=REAL(Z(I,J))
14 CONTINUE
DO 15 I=1,LIM
PRINT 800,(ZZ(I,J),J=1,LIM)
15 CONTINUE
C  ALPHA = ALPHA + 0.5
C  IF (ALPHA.GT.10.) GO TO 20
DO 16 J = 1, LIM
DO 17 LL = 1, 10
ZBAR = LL / 10.0
SUM = 0.0
DO 18 I = 1, 5
SUM = ZZ(I, J) * ZBAR**(I+1) + SUM
18 CONTINUE
YY(LL) = SUM
17 CONTINUE
DO 19 LL = 1, 10
YY(LL) = YY(LL) / ABS(YY(10))
19 CONTINUE
PRINT 900, (YY(LL), LL = 1, 10)
900 FORMAT (2X, *, Y = *, 1X, 10E12.4, /)
16 CONTINUE
20 CONTINUE
6 CONTINUE
5 CONTINUE
C  GO TO 2
END
APPENDIX D

SAMPLE LISTING OF TABLE OF CONTENTS AND A
RUN STREAM USED IN SPAR

This appendix gives a listing of table of contents stored in SPAR data base and a typical run stream used to calculate the natural frequencies for a rotating cantilever beam.

Each line of the table contains information denoting when the data was created, the name of the data set and the number of words in the data sets.

References were made to this appendix in Section 3.1 and Section 3.2 of Chapter 3 respectively.
### Example of TOC

#### TABLE OF CONTENTS: LIBRARY 2

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\texttt{SYSVEC\#WS}
\texttt{I=1 2; J=1,16; 3947.8418 3947.8418}
\texttt{KC=PROD(WS,DEM)}
\texttt{KEKC=SUM(K,-1.0 KC)}
\texttt{TABLE(NI=6, NJ=16)}
\texttt{P}
\texttt{TRAN(SRC\#JLOC, ILIM=3, JLIM=16, DSKIP=3)}
\texttt{APPLIED FORCE 1 0=PRODUCT(KC, P)}
\texttt{IXQT INV}
\texttt{RESET K=KEKC}
\texttt{IXQT SSOL}
\texttt{RESET K=KEKC}
\texttt{IXQT GSF}
\texttt{RESET EMBEP=1}
\texttt{IXQT KG}
\texttt{IXQT AUS}
\texttt{KECG=SUM(KEKC, KG)}
\texttt{IXQT DCU}
\texttt{COFY 2, 1 G SFAR 36 0}
\texttt{IXQT INV}
\texttt{RESET K=KECG}
\texttt{IXQT AUS}
\texttt{DEFINE V=VIBR MODE 1 1}
\texttt{DEFINE E=VIBR RVAL 1 1}
\texttt{RVEC CEIG 1 1=UNION(V)}
\texttt{KECG CEIG 1 1=UNION(E, V)}
\texttt{RVAL CEIG 1 1=UNION(0, E)}
\texttt{IVAL CEIG 1 1=SORT(E)}
\texttt{IXQT CEIG}
\texttt{RESET HIST=1}
\texttt{RESET NREQ=4}
\texttt{RESET CRAL=1}
\texttt{M=1 DEM}
\texttt{I=1 KECG}
\texttt{G=1 G}
\texttt{KINV=1 INV KECG}
\texttt{IXQT EXIT}
APPENDIX E

PROGRAM FOR COMPUTING NATURAL FREQUENCIES AND
MODE SHAPES FOR CANTILEVER PLATE USING ORTHOGONAL POLYNOMIALS

This Fortran program is used to calculate the natural frequencies and mode shapes for a cantilever plate using beam characteristic orthogonal polynomials as deflection functions in Rayleigh-Ritz method.

Reference was made to this program in Section 4.3 of Chapter 4.
PROGRAM PLTFF3(INPUT,OUTPUT)
C*********************************************************************/
C THIS PGM IS USED TO CALCULATE EIGEN VALUES AND */
C EIGEN VECTORS FOR A CANTILEVER PLATE */
C*********************************************************************/
EXTERNAL PHI,PHI11,PHI20,PHI22,PHIOX,PHIQY
COMPLEX ALFA(36),Z(36,36),TEMPG
DIMENSION ENO(12,12),AKS(36,36),AMS(36,36),
1WK(6000),BETA(36),DEFL(11),F(36)
COMMON/Para/A(12,12),C(12,12),L,M,N,LIM
READ*,ALPHA,ENF,MMM,NNN
PRINT 4,ALPHA,ENF,MMM,NNN
FORMAT(2X,'ALPHA=',F3.1,2X,'ENF=',F3.2,2X,'MMM=',I3,2X,'NNN=',I3)
LIM=1
LOOP=1
5 CONTINUE
DO 10 I=1,12
DO 10 J=1,12
A(I,J)=0.
10 CONTINUE
IF(LOOP.EQ.2) GO TO 7
A(1,3)=6.0
A(1,4)=-4.0
A(1,5)=1.0
IF(LOOP.EQ.1) GO TO 6
7 A(1,1)=1.0
6 CONTINUE
L=0
M=1
N=1
CALL QG10(0.1,PHI,E12)
DO 12 J=1,12
A(1,J)=A(1,J)/(SQRT(E12))
12 CONTINUE
PRINT 100,1
PRINT 110,(A(1,J),J=1,12)
CK=0.
K2=1
DO 20 K=2,12
M=K-1
N=K-1
L=1
CALL QG10(0.1,PHI,E11)
L=0
CALL QG10(0.1,PHI,E12)
BK=E11/E12
IF(K.EQ.2) GO TO 16
M=K-1
N=K-2
L=1
CALL QG10(0.1,PHI,E13)
M=K-2
N=K-2
L=0
20 CONTINUE
IF(K.EQ.3) GO TO 16
M=K-1
N=K-3
L=1
CALL QG10(0.1,PHI,E14)
M=K-2
N=K-3
L=0
16 CONTINUE
END
CALL QB10(0, 1, PHI, EI)
CK=EI3/E14
K2=K-2
16 K1=K-1
A(K,1)=-BK*A(K1,1)+CK*A(K2,1)
DO 25 J=2, 12
J1=J-1
A(K, J)=A(K1, J1)-BK*A(K1, J)-CK*A(K2, J)
25 CONTINUE
M=K
N=K
L=0
CALL QB10(0, 1, PHI, EN)
ENSQ=SORT(EN)
C*
FACTOR=A(K,1)/(ABS(A(K,1)))
DO 30 J=1, 12
A(K, J)=A(K, J)/ENSQ
30 CONTINUE
PRINT 100, K
100 FORMAT(2X, 'A(K,J)=K=', I3)
PRINT 110, (A(K, J), J=1, 12)
110 FORMAT(2X, 5E20.8)
20 CONTINUE
DO 50 I=1, 12
DO 50 J=1, 12
M=I
N=J
CALL QB10(0, 1, PHI, EN)
ENT(I, J)=EN
50 CONTINUE
PRINT 120
120 FORMAT(2X, 'ORTHOGONALITY PRINCIPLE')
PRINT 130, ((ENT(I, J), I=1, 12), J=1, 12)
130 FORMAT(2X, 5E20.8)
IF (LOOP, EQ, 2) GO TO 132
DO 131 I=1, 12
DO 131 J=1, 12
C(I, J)=A(I, J)
131 CONTINUE
LOOP=LOOP+1
IF (LOOP, EQ, 2) GO TO 5
132 CONTINUE
L=0
DO 70 II=1, MMMM
DO 70 JJ=1, NNNN
KK=JJ+NNNN*(II-1)
DO 75 MM=1, MMMM
DO 75 NN=1, NNNN
LL=NN+NNNN*(MM-1)
M=MM
N=II
LIM=1
CALL QB10(0, 1, PHI, TM1)
CALL QB10(0, 1, PHI22, TK1)
CALL QB10(0, 1, PHI20, TK2)
CALL QB10(0, 1, PHI11, TK3)
M=II
N=MM
CALL GG10(0.,1.,PHI20,TK4)
M=NN
N=JJ
LIM=2
CALL GG10(0.,1.,PHI,TK2),
CALL GG10(0.,1.,PHI22,TK5)
CALL GG10(0.,1.,PHI20,TK6)
CALL GG10(0.,1.,PHI11,TK7)
M=JJ
N=NN
CALL GG10(0.,1.,PHI20,TK8)
C PRINT 1001,TM1,TM2,TK1,TK2,TK3,TK4,TK5,TK6,TK7,TK8
C1001 FORMAT(2X,*TM,TK=*,E15.8)
AMS(KK,LL)=TM1*TM2
TEMP1=TK1*TM2
TEMP2=ALPHA**4*TK1*TM1
TEMP3=ENU*ALPHA**2*(TK2*TK8+TK4*TK6)
TEMP4=2.*(1.-ENU)*ALPHA**2*TK3*TK7
WKS(KK,LL)=TEMP1+TEMP2+TEMP3+TEMP4
75 CONTINUE
70 CONTINUE
MMNN=MMNN1=MNNN
C PRINT 150,(AMS(KK,LL),LL=1,MMNN),KK=1,MMNN)
CALL EIGZF(AMS,36,AMS,36,MMNN,2,ALFA,BETA,Z,36,WK,IER)
DO 90 I=1,MMNN
F(I)=REAL(ALFA(I))/BETA(I)
F(I)=SORT(F(I))
90 CONTINUE
PRINT 150,(F(I),I=1,MMNN)
MMNN1=MMNN-1
DO 105 I=1,MMNN1
L=I+1
DO 105 J=L,MMNN
IF(F(I)-F(J))105,105,115
115 TEMP=F(I)
F(I)=F(J)
F(J)=TEMP
DO 125 NI=1,MMNN
TEMPC=Z(NI,I)
Z(NI,I)=Z(NI,J)
Z(NI,J)=TEMPC
125 CONTINUE
105 CONTINUE
PRINT 140
140 FORMAT(2X,'EIGENVALUES AND EIGENVECTORS',/)
PRINT 150,(F(I),I=1,MMNN)
150 FORMAT(2X,BE12.6)
PRINT 2000
2000 FORMAT(2X,'ORTHOGONAL POLYNOMIAL FUNCTION VALUES')
DO 300 IM=1,MMM
M=IM
DO 300 I=1,21
X=(I-1)/20.
FVAL=POLY(X)
300 CONTINUE
PRINT 2001,X,FVAL
CONTINUE
DO 301 IN=1,NNNN
N=IN
DO 301 I=1,21
Y=(I-1)/20.
FVAL=POLLY(Y)
PRINT 2001,Y,FVAL
CONTINUE
2001 FORMAT(2X,FB.3,E15.5)
DO 400 K=1,6
DO 401 KX=1,11
X=(KX-1)/10.
DO 405 KY=1,11
Y=(KY-1)/10.
DEFL(KY)=0.
DO 402 I=1,MMMM.
M=I
TEMPX=POLLY(X)
DO 403 J=1,NNNN
IJ=J+NNNN*(I-1)
N=J
TEMPY=POLLY(Y)
DEFL(KY)=DEFL(KY)*IJ,K)*TEMPX*TEMPY
CONTINUE
403 CONTINUE
402 CONTINUE
405 CONTINUE
PRINT 3001,X,(DEFL(KY);KY=1,11)
401 CONTINUE
400 CONTINUE
3001 FORMAT(2X,F6.2,11F10.6,////////)
STOP
END
FUNCTION PHI(X)
COMMON/PARA/A(12,12),C(12,12),L*,M*,N*,LIM
PHIM=0.
PHIN=0.
DO 10 J=1,12
K=J-1
IF(LIM.EQ.2) GO TO 20
PHIM=A(M,J)***K+PHIM
PHIN=A(N,J)***K+PHIN
GO TO 30
20 PHIM=C(M,J)***K+PHIM
PHIN=C(N,J)***K+PHIN
30 CONTINUE
10 CONTINUE
IF(L.EQ.1) PHIM=PHIMX
PHI=PHIMX*PHIN
RETURN
END
FUNCTION POLY(X)  
COMMON/ PARA/ A(12, 12), C(12, 12), L, M, N, LIM
PHIM = A(M, 1)
DO 10 J = 2, 12
K = J - 1
PHIM = A(M, J) * X**K + PHIM
10 CONTINUE
POLY = PHIM
RETURN
END
FUNCTION POLLY(Y)  
COMMON/ PARA/ A(12, 12), C(12, 12), L, M, N, LIM
PHINT = C(N, 1)
DO 10 J = 2, 12
K = J - 1
PHINT = C(N, J) * Y**K + PHINT
10 CONTINUE
POLLY = PHINT
RETURN
END
FUNCTION PHI11(X)  
COMMON/ PARA/ A(12, 12), C(12, 12), L, M, N, LIM
PHIM1 = 0.
PHIN1 = 0.
DO 10 J = 2, 12
K = J - 1
KN = J - 2
IF(LIM .LE. 2) GO TO 20
PHIM1 = K * A(M, J) * X**KN + PHIM1
PHIN1 = K * A(N, J) * X**KN + PHIN1
20 GO TO 30
PHIM1 = K * C(M, J) * X**KN + PHIM1
PHIN1 = K * C(N, J) * X**KN + PHIN1
30 CONTINUE
10 CONTINUE
PHI11 = PHIM1 * PHIN1
RETURN
END
FUNCTION PHI20(X)  
COMMON/ PARA/ A(12, 12), C(12, 12), L, M, N, LIM
PHIM2 = 0.
PHIN = 0.
DO 10 J = 3, 12
IF(LIM .LE. 2) GO TO 30
K = J - 1
KN = J - 2
```
KM=J-3
PHIM2=K*KN*A(M,J)**KM+PHIM2
10 CONTINUE
   DO 20 J=1,12
      K=J-1
      PHIN=A(N,J)**K+PHIN
   20 CONTINUE
   GO TO 50
30 DO 60 J=3,12
   K=J-1
   KN=J-2
   KM=J-3
   PHIM2=K*KN*C(M,J)**KM+PHIM2
60 CONTINUE
   DO 70 J=1,12
      K=J-1
      PHIN=C(N,J)**K+PHIN
   70 CONTINUE
50 CONTINUE
   PH120=PHIM2*PHIN
   RETURN
END
FUNCTION PHI22(X)
COMMON/Para/A(12,12),C(12,12),L,M,N,LIM
PHIM2=0.,
   PHIN2=0.,
   DO 10 J=3,12
      K=J-1
      KN=J-2
      KM=J-3
      IF(LIM.EQ.2) GO TO 20
      PHIM2=K*KN*A(M,J)**KM+PHIM2
      PHIN2=K*KN*A(N,J)**KM+PHIN2
   20 CONTINUE
   GO TO 30
   30 CONTINUE
   10 CONTINUE
   PH122=PHIM2*PHIN2
   RETURN
END
```
APPENDIX F

PROGRAM FOR COMPUTING NATURAL FREQUENCIES AND
MODE SHAPES FOR FREE-FREE PLATE USING ORTHOGONAL POLYNOMIALS

This Fortran program is used to calculate the natural frequencies
and mode shapes for a free-free plate using beam characteristic
orthogonal polynomials as deflection functions in Rayleigh-Ritz method.

Reference was made to this program in Section 4.4 of Chapter 4.
PROGRAM PLFRFR(INPUT, OUTPUT)
EXTERNAL PHI, PHI11, PHI20, PHI22
COMPLEX ALFA(36), Z(36, 36), TEMPC
DIMENSION ENT(12, 12), AMS(36, 36), AKS(36, 36),
1WE(6000), BETA(36), DEFL(11), F(36)

C**********************************************************************
C THIS PROGRAM IS USED TO CALCULATE THE
C NATURAL FREQUENCIES AND MODE SHAPES OF A
C FREE-FREE PLATE.
C**********************************************************************
COMMON PARA/A(12, 12), L, M, N
READ *, ALFA, ENU, MMM, NNNN
PRINT 5, ALFA, ENU, MMM, NNNN
5 FORMAT(2X, 'ALPHA=', F3.1, 2X, 'ENU=', F3.2, 2X, 'MMM=', I1,
12X, 'NNNN=', I13)
DO 10 I = 1, 12
DO 10 J = 1, 12
A(I, J) = 0.
10 CONTINUE
A(1, 1) = 1.
L = 0
M = 1
N = 1
CALL DG10(0., 1.0, PHI, EI2)
DO 12 J = 1, 12
A(I, J) = A(I, J) / (SQR(EI2))
12 CONTINUE
PRINT 100, 1
PRINT 110, (A(I, J), J = 1, 12)
C2 = 0.
K2 = 1
DO 20 K = 2, 12
M = K - 1
N = K - 1
L = 1
CALL DG10(0., 1.0, PHI, EI1)
L = 0
CALL DG10(0., 1.0, PHI, EI2)
Bn = EI1 / EI2
IF (K = 2) GO TO 16
M = K - 1
N = K - 2
L = 1
CALL DG10(0., 1.0, PHI, EI3)
M = K - 2
N = K - 2
L = 0
CALL DG10(0., 1.0, PHI, EI4)
C2 = EI3 / EI4
K2 = K - 2
16 K1 = K - 1
A(K,1)=-BK*A(K1,1)-CK*A(K2,1)
DO 25 J=2,12
    J1=J-1
    A(K,J)=A(K1,J1)-BK*A(K1,J)-CK*A(K2,J)
25 CONTINUE
M=K
N=K
L=0
CALL QG10(0.,1.0,PHI,EN)
ENSQ=SORT(EN)
DO 30 J=1,12
    A(K,J)=A(K,J)/ENSQ
30 CONTINUE
PRINT 100,K
100 FORMAT(2X,'A(K,J), K=',I3)
PRINT 110,'(A(K,J), J=1,12)
110 FORMAT(2X,6E20.8)
20 CONTINUE:
DO 50 I=1,12
    DO 50 J=1,12
M=I
N=J
CALL QG10(0.,1.0,PHI,ENTEG)
ENT(I,J)=ENTEG
50 CONTINUE
PRINT 120
120 FORMAT(2X,'ORTHOGONALITY PRINCIPLE')
PRINT 130,'((ENT(I,J), I=1,12), J=1,12)
130 FORMAT(2X,6E20.8)
L=0
DO 70 II=1,MMMM
    DO 70 JJ=1,NNNN
        KK=JJ+NNNN*(II-1)
    DO 75 MM=I,MMMM
        DO 75 NN=I,NNNN
            LL=NN+NNNN*(MM-1)
M=MM
N=II
CALL QG10(0.,1.0,PHI,TM1)
CALL QG10(0.,1.0,PHI2,TM1)
CALL QG10(0.,1.0,PHI20,TM1)
CALL QG10(0.,1.0,PHI11,TM1)
M=II
N=MM
CALL QG10(0.,1.0,PHI20,TM2)
CALL QG10(0.,1.0,PHI22,TM2)
CALL QG10(0.,1.0,PHI20,TM2)
CALL QG10(0.,1.0,PHI11,TM2)
\begin{verbatim}
M=JI
N=NN
CALL QG10(0.1,0.1,PHI20,TK8)
PRINT 1001,TM1,TK2,TK3,TK4,TK5,TK6,TK7,TK8
1001 FORMAT(2X,'TH,TK=',5E15.8)
AM5(KK,LL)=TM1*TM2
TEMP1=TM1*TM2
TEMP2=ALPHA**4*TK5*TK1
TEMP3=ENU*ALPHA**2*(TK2*TK8+TK4*TK6)
TEMP4=2.*(1.-ENU)*ALPHA**2*TK3*TK7
AM5(KK,LL)=TEMP1+TEMP2+TEMP3+TEMP4
75 CONTINUE
70 CONTINUE
MMNN=MMMN*MMNN
PRINT 150,((AM5(KK,LL),LL=1,MMNN),KK=1,MMNN)
PRINT 150,((AM5(KK,LL),LL=1,MMNN),KK=1,MMNN)
CALL EIGEF(AM5,36,AM5,36,MMNN,2,ALFA,BETA,Z,36,WK*IER
PRINT *, 'ALFA AND BETA'
PRINT *, (ALFA(I), IXX=1,MMNN)
PRINT *, (BETA(I), IXX=1,MMNN)
DO 90 I=1,MMNN
F(I)=REAL(ALFA(I)/BETA(I))
F(I)=SORT(F(I))
90 CONTINUE
MMNN1=MMNN1-1
DO 105 I=1,MMNN1
L=I+1
DO 105 J=L,MMNN
XXX=(F(I)-F(J))
IF(XXX) 106,106,115
115 TEMP=F(I)
F(I)=F(J)
F(J)=TEMP
106 CONTINUE
DO 125 NI=1,MMNN
TEMPF=Z(NI,I)
Z(NI,I)=Z(NI,J)
Z(NI,J)=TEMPF
125 CONTINUE
105 CONTINUE
PRINT 140
140 FORMAT(2X,'EIGENVALUES AND EIGENVECTORS'/)
PRINT 150,((F(I),I=1,MMNN)
PRINT 150,((Z(I,J),I=1,MMNN),J=1,MMNN)
150 FORMAT(2X,8E12.6)
PRINT 2000
2000 FORMAT(2X,'ORTHOGONAL POLYNOMIAL FUNCTION VALUES')
DO 300 IM=1,6
M=IM
DO 300 I=1,21
X=(I-1)/20
300 CONTINUE
\end{verbatim}
FVAL=POLY(X)
PRINT 2001,X,FVAL
300 CONTINUE
2001 FORMAT(2X,F8.3,E15.5)
   DO 400 K=1,6
   DO 401 KX=1,11
   X=(KX-1)/10.
   DO 405 KY=1,11
   Y=(KY-1)/10.
   DEFL(KY)=0.
   DO 402 I=1,MMM
   M=I
   TEMPX=POLY(X)
   DO 403 J=1,NNNN
   IJ=J+NNNN*(I-1)
   M=J
   TEMPY=POLY(Y)
   DEFL(KY)=DEFL(KY)+Z(IJ,K)*TEMPX*TEMPY
   CONTINUE
   CONTINUE
   CONTINUE
   PRINT 3001,K,X,(DEFL(KY),KY=1,11)
401 CONTINUE
400 CONTINUE
3001 FORMAT(2X,I3,F6.2,11F10.3,//////)
STOP
END
FUNCTION PHI(X)
COMMON/ PARA/A(12,12),L,M,N
PHIM=0.
PHIN=0.
DO 10 J=1,12
   K=J-1
   PHIM=A(M,J)**KK+PHIM
   PHIN=A(N,J)**KK+PHIN
10 CONTINUE
IF(L.EQ.1)PHIM=PHIM*X
PHI=PHIM*PHIN
RETURN
END
FUNCTION POLY(X)
COMMON/ PARA/A(12,12),L,M,N
PHIM=A(M,1)
DO 10 J=2,12
   K=J-1
   PHIMT=A(M,J)**KK+PHIMT
10 CONTINUE
POLY=PHIMT
RETURN
END
FUNCTION PHI11(X)
COMMON/PARA/A(12,12),L,M,N
PHIM1=0.
PHIN1=0.
DO 10 J=2,12
  K=J-1
  KN=J-2
  PHIM1=K*A(M,J)**KN+PHIM1
  PHIN1=K*A(N,J)**KN+PHIN1
10 CONTINUE
PHI11=PHIM1*PHIN1
RETURN
END

FUNCTION PHI20(X)
COMMON/PARA/A(12,12),L,M,N
PHIM2=0.
PHIN=0.
DO 10 J=3,12
  K=J-1
  KN=J-2
  KM=J-3
  PHIM2=K**KN*A(M,J)**KM+PHIM2
10 CONTINUE
DO 20 J=1,12
  K=J-1
  PHIN=A(N,J)**PHIN
20 CONTINUE
PHI20=PHIM2*PHIN
RETURN
END

FUNCTION PHI22(X)
COMMON/PARA/A(12,12),L,M,N
PHIM2=0.
PHIN2=0.
DO 10 J=3,12
  K=J-1
  KN=J-2
  KM=J-3
  PHIM2=K**KN*A(M,J)**KM+PHIM2
  PHIN2=K**KN*A(N,J)**KM+PHIN2
10 CONTINUE
PHI22=PHIM2*PHIN2
RETURN
END