ELEMENT SPREAD MINIMIZATION OF A CASCADE OF SYMMETRIC STRUCTURES

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Abstract

Given a transfer function (open circuit voltage or short circuit current) of the form

$$T = \frac{(1 - U)^{n/2}}{Q(U)}$$

which is to be realized as a cascade of symmetric structures each having a chain matrix of the form

$$[a] = (1 - U)^{-1/2} \begin{bmatrix} 1 & a\,p \\ \frac{U}{ap} & 1 \end{bmatrix}$$

where p is the normalized short circuit impedance, and q is the normalized open circuit impedance at one part of the structure, and U = p/q, there are in general, many different realizations, giving rise to different sets of impedance scaling factors a_i. It is of practical interest to consider minimizing the ratio of the largest to the smallest of these factors. In this work we have carried out such a minimization for two, three and four section cascades, and have obtained easily applicable analytical results.

In the cases of the two and three line realizations the transfer functions are of the form

$$T = \frac{(1 - U)^{n/2}}{1 + KU}$$

where $n = 2$ or $n = 3$. In some applications the value of n is of no great importance, and we have therefore compared two and three line realizations, and determined for what ranges K_1 the two line realization has a smaller minimum ratio than the three line realization.

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Chapter I

Introduction

1.1 Statement of the Problem

The synthesis of a transfer function (open circuit voltage or short circuit current) as a cascade of symmetric structures is now well known [1]. It will be shown that the transfer function of a cascade of n symmetric structures is of the form

$$T = \frac{(1 - U)^{n/2}}{Q(U)}$$

where U is a dummy variable used in the synthesis procedure, and Q(U) is a polynomial having all its zeros on the negative real axis of the U-plane. The synthesis is carried out by choosing some other polynomial P(U) such that Q(U)/P(U) is an RC driving point function of U. Apart from this restriction, P(U) is completely arbitrary. Hence there are many possible realizations of the given transfer function, and one may introduce other criteria to select a best realization.

One important such criterion is element spread. If the symmetric structure is lumped, then the choice of P(U) will significantly affect the impedance scalings of the various sections of the cascade. If the symmetric structure is distributed, then this impedance scaling will be reflected in the widths of the various sections of the cascade. Obviously large element spreads will lead to practical difficulties. It is therefore of interest to attempt to minimize element variations.

In this work, we will examine two, three and four section cascades, and in each case, analytically determine configurations, i.e. impedance scalings for each section, which will minimize the ratio of the largest to the smallest scaling.

The method of approach will be to first analyze a cascade of sections. Next, the chain matrix parameter $A = 1/T_V$, which will now be known in terms of the scaling values a_i will be studied and optimum solutions obtained. By so proceeding the polynomial P(U) will automatically become fixed. The results obtained will also be immediately applicable to the synthesis of the short circuit transfer function T_I, as this has precisely the same form as T_V.

Finally, we note that since the major application of this work will be in cascade synthesis of transmission lines, we will in Chapters 3 - 5 not distinguish between "sections" and "lines".

1.2 Chain Matrix of Symmetric Structure

It has been shown (1) that a symmetric structure (SS) has a Z matrix of the form:

$$[Z] = \frac{a}{F_2}\begin{bmatrix} F_1 & 1 \\ 1 & F_1 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{1.1}$$

where Z, F_1 and F_2 are all functions of S, and a is the impedance scaling factor for the section. Hence, the transmission matrix [a] is given by:

$$[a] = \begin{bmatrix} F_1 & \dfrac{a(F_1^2 - 1)}{F_2} \\ \dfrac{F_2}{a} & F_1 \end{bmatrix} = F_1\begin{bmatrix} 1 & \dfrac{a(F_1^2 - 1)}{F_1F_2} \\ \dfrac{F_2}{aF_1} & 1 \end{bmatrix} \tag{1.2}$$

We define p, q as the normalized short and open circuit impedances of the SS. Hence,

$$p = 1/y_{11} = \frac{F_1^2 - 1}{F_1F_2} \tag{1.3}$$

$$q = Z_{11} = F_1/F_2 \tag{1.4}$$

It is convenient to define the variable U as

$$U = p/q = \frac{F_1^2 - 1}{F_1^2} \tag{1.5}$$

The chain matrix [a] now becomes:

$$[a] = \frac{1}{(1 - U)^{1/2}} \begin{bmatrix} 1 & a_p \\ \frac{U}{a_p} & 1 \end{bmatrix} \tag{1.6}$$

The variable U could have been defined in other ways [1]. For example letting:

$$U = \sqrt{\frac{p}{q}} = \sqrt{\frac{F_1^{\,2} - 1}{F_1}} \tag{1.7}$$

[a] becomes:

$$[a] = \frac{1}{(1 - U)^{1/2}} \begin{bmatrix} 1 & a\sqrt{pq} \;\; U \\ \frac{U}{a\sqrt{pq}} & 1 \end{bmatrix} \tag{1.8}$$

In this work we will use definition (1.5).

1.3 Examples of Symmetric Structures.

1.3.1 RC Symmetric Structure

Consider the lumped SS of Fig. 1.1

It can be easily seen that:

$$[Z] = \begin{bmatrix} a + \frac{a}{DS} & \frac{a}{DS} \\ \frac{a}{DS} & a + \frac{a}{DS} \end{bmatrix} = \frac{a}{DS} \begin{bmatrix} 1 + DS & 1 \\ 1 & 1 + DS \end{bmatrix} \tag{1.9}$$

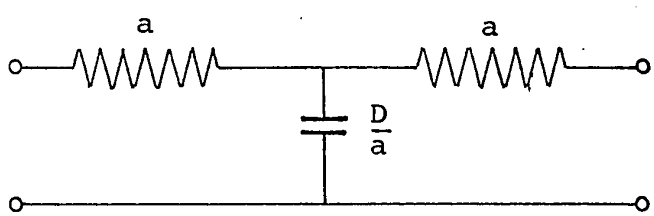


Figure 1.1 RC Lumped SS.

Comparing equation (1.9) to equation (1.1) we see that:

$$F_1 = 1 + DS \tag{1.10}$$

$$F_2 = DS \tag{1.11}$$

Using equations (1.10), (1.11) and (1.2) we obtain:

$$[a] = (1 + DS) \begin{bmatrix} 1 & \dfrac{a((1 + DS)^2 - 1)}{DS(1 + DS)} \\ \\ \dfrac{DS}{a(1 + DS)} & 1 \end{bmatrix} \tag{1.12}$$

Equations (1.3), (1.4) and (1.5) become:

$$p = \frac{F_1^{\,2} - 1}{F_1 F_2} = \frac{DS + 2}{DS + 1} \tag{1.13}$$

$$q = \frac{F_1}{F_2} = \frac{1 + DS}{DS} \tag{1.14}$$

$$U = \frac{p}{q} = \frac{DS(DS + 2)}{(DS + 1)^2} \tag{1.15}$$

1.3.2 Uniform RC - Transmission Lines

The RC-transmission line has become available as a network element, as a result of the appearance of microminiature circuits.

One method of constructing such lines consists of depositing a thin resistive film of tantalum on a glass substrate, converting a portion of this layer into tantalum oxide, and finally depositing a conducting layer of gold over the oxide. The uniform RC-line is one where the resistance and capacitance per unit length are constant over the length of the line. The network symbol for the uniform RC-line is given in Figure 1.2.

The [a] matrix of such a line has been calculated [2], and is

$$[a] = \begin{bmatrix} \cosh \theta & \frac{r_o}{\theta} \sinh \theta \\ \frac{\theta}{r_o} \sinh \theta & \cosh \theta \end{bmatrix} \qquad (1.16)$$

where $\theta = \sqrt{sRC}$

R is the total resistance of the line $= r_o \ell$

r_o is the resistance per unit length

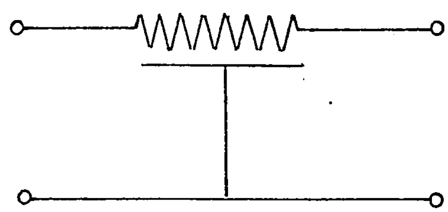
ℓ is the length of the line

C is the total capacitance of the line $= C_o \ell$

C_o is the capacitance per unit length

Equation (1.16) may be rewritten as

$$[a] = \cosh \theta \begin{bmatrix} 1 & \frac{r_o}{\theta_o} \tanh \theta \\ \frac{\theta}{r_o} \tanh \theta & 1 \end{bmatrix}$$



Uniform RC-line

Fig. 1.2.

The impedances p and q may be calculated, and are,

$$\left.\begin{aligned} q &= \frac{1}{\theta \tanh \theta} \\ p &= \frac{\tanh \theta}{\theta} \\ \text{Hence, } U &= \tanh^2\theta \end{aligned}\right\} \qquad (1.17)$$

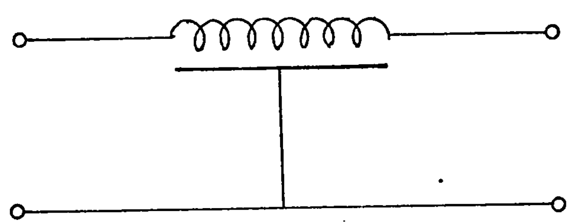
1.3.3 Uniform LC - Transmission Line

At high frequencies (the Kilomegacycle range) one example of an LC-line is the microstrip [3], which consists of a metallic strip conductor bonded to a dielectric sheet, to the other side of which, is bonded a metallic ground plate. Electrically, the microstrip is analogous to a two 2-wire line, in which, one of the wires is represented by the image in the ground plane, of the wire that is physically present. The symbol of the LC-line is given in Fig. 1.3.

The chain matrix of the uniform LC-line is identical to that of the RC line (1.16), except that:

$$\theta = s\sqrt{LC}$$

where L and C are the total inductance and capacitance of the line.



The Uniform LC-line

Figure 1.3

Chapter II

Analysis and Properties of a Cascade of Symmetric Structures

2.1 Analysis

As was indicated in the previous chapter, the chain matrix of a single symmetric structure can always be put into the following form:

$$[a] = (1 - U)^{-1/2} \begin{bmatrix} 1 & ap \\ \frac{U}{ap} & 1 \end{bmatrix} \qquad (2.1)$$

where a is the impedance scaling factor for the section. If we cascade n sections, each of which has an impedance scaling factor a_i, then the overall chain matrix is given by:

$$[a] = (1 - U)^{\frac{-n}{2}} \prod_{i=1}^{n} \begin{bmatrix} 1 & a_i p \\ \frac{U}{a_i p} & 1 \end{bmatrix}$$

The above may be rewritten as:

$$[a] = (1 - U)^{\frac{-n}{2}} \prod_{i=1}^{n} \begin{bmatrix} I + \alpha_i \end{bmatrix}$$

where

$$\alpha_i = \begin{bmatrix} 0 & a_i p \\ \frac{U}{a_i p} & 0 \end{bmatrix}$$

Hence,

$$[a] = (1 - U)^{-\frac{n}{2}} \left[I + \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_i \alpha_j \right.$$

$$\left. + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \alpha_i \alpha_j \alpha_k + --- \right] \tag{2.2}$$

The second term of (2.2) takes on the form:

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \begin{bmatrix} 0 & pa_i \\ \frac{U}{p} \frac{1}{a_i} & 0 \end{bmatrix} . \tag{2.3}$$

The third term takes on the form:

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_i \alpha_j = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} U \begin{bmatrix} \frac{a_i}{a_j} & 0 \\ 0 & \frac{a_j}{a_i} \end{bmatrix} \tag{2.4}$$

The fourth term takes on the form:

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \alpha_i \alpha_j \alpha_k$$

$$= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} U \begin{bmatrix} 0 & p \frac{a_i a_k}{a_j} \\ \frac{U}{p} \frac{a_j}{a_i a_k} & 0 \end{bmatrix} \tag{2.5}$$

Other terms may be obtained in a similar fashion.

2.2 Properties of a Cascade of Symmetric Structures.

It has been shown (1) that the chain matrix of a cascade of symmetric structures is of the form

$$[a]_n = (1 - U)^{-\frac{n}{2}} \begin{bmatrix} K_\alpha \prod_1^k (U + \alpha_i) & K_\beta \, p \prod_1^\ell (U + \beta_i) \\ K_\gamma \frac{1}{p} U \prod_1^\ell (U + \gamma_i) & K_\delta \prod_1^k (U + \delta_i) \end{bmatrix} \tag{2.6}$$

where $U = \frac{p}{q}$, and where the following necessary conditions hold:

(a) if n is even, $k = \frac{n}{2}$ and $\ell = \frac{n}{2} - 1$, while if n is odd,

$k = \ell = \frac{n-1}{2}$

(b) α_i and γ_i , δ_i and γ_i , δ_i and β_i , α_i and β_i all interlace on the negative real axis of the U plane. Further, $0 < \alpha_1 < \gamma_1$, $0 < \delta_1 < \gamma_1$, $0 < \delta_1 < \beta_1$, and $0 < \delta_1 < \beta_1$.

(c) If the symmetric structure is an RC-transmission line, then

$$K_\alpha = \frac{1}{\prod_1^k \alpha_i} \qquad K_\beta = \frac{\sum_1^n R_i}{\prod_1^\ell \beta_i}$$

$$K_{\gamma} = \frac{\sum_{1}^{n} \left(\frac{1}{r_i}\right)}{\prod_{1}^{\ell} \gamma_i} \qquad\qquad K_{\delta} = \frac{1}{\prod_{1}^{k} \delta_i}$$

As a consequence of the above results, we note that the open circuit voltage transfer function of the cascade is of the form:

$$T_V = \frac{(1 - U)^{\frac{n}{2}}}{K_{\alpha} \prod_{1}^{k} [U + \alpha_i]} \tag{2.7}$$

A similar form exists for the short circuit current transfer function.

We further note that the driving point immittance functions are of the form $pZ(U)$ or $\frac{1}{p}Y(U)$ where $Z(U)$ and $Y(U)$ are RC driving point functions in U.

It has been shown (1) that such functions can always be realized as a cascade of symmetric structures.

CHAPTER III

TWO AND THREE SECTION CASCADES

3.1 Introduction

As seen in Chapter II, the open circuit voltage transfer function of a cascade of two sections is of the form:

$$T_V = \frac{1}{A_2} = \frac{1-U}{1+KU} \tag{3.1}$$

while that of a three section cascade is of the form:

$$T_V = \frac{1}{A_3} = \frac{(1-U)^{3/2}}{1 + KU} \tag{3.2}$$

As there are many possible cascade realizations of equation (3.1), we have the liberty to choose a "best" realization.

We shall choose to minimize the ratio of the largest a_i to the smallest a_i, and this will be called the optimum realization.

In the case where the sections are transmission lines, this will correspond to minimizing the ratio of the largest to the smallest widths. Further, if we ignore the numerators of (3.1) and (3.2) we can compare the optima obtained in (3.2) with those of (3.1).

3.2 Two Section Cascade

As seen in Chapter II, we have the following relationships for two sections:

$$A_2 = (1-U)^{-1} \; (1 + \frac{a_1}{a_2} U) \tag{3.3}$$

and also:

$$D_2 = (1-U)^{-1} \; (1 + \frac{a_2}{a_1} U) \tag{3.4}$$

It is only necessary to analyze (3.3), because if we reverse a_1 and a_2, then from (3.2) we obtain (3.4). Therefore we need only consider the case where $K = \frac{a_1}{a_2}$. It is obvious that given K, there is only one possible ratio $\frac{a_1}{a_2}$ which will realize (3.3), and the only freedom we have is in a proportional scaling of a_1 and a_2.

3.3 Three Section Cascade

3.3.1 Introduction

As seen in Chapter II, we have the following relationships for three sections:

$$A_3 = (1-U)^{-3/2} \; [1 + U \, (\frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_1}{a_3}) \;] \tag{3.5}$$

and also

$$D_3 = (1-U)^{-3/2} \; [1 + U (\frac{a_2}{a_1} + \frac{a_3}{a_2} + \frac{a_3}{a_1}) \;] \tag{3.6}$$

It is only necessary to analyze (3.5), because if we reverse a_1 and a_3, then from (3.5), we obtain (3.6). Hence for a given value of K, the optimum realization of $T_V = \frac{1}{A_3}$ is also that of $T_i = \frac{1}{D_3}$.

3.3.2 Preliminary Results

The following inequalities will be needed:

a) If $\frac{1+x}{K-x} \gtrless 1$ Then $x \gtrless \frac{K-1}{2}$ (3.7)

b) If $(\frac{K-1}{2})^2 - (\frac{K-1}{4})\sqrt{(K+1)(K-3)} \leq 1$ Then $K \geq 3$ (3.8)

Proof:

The radical is real only if $K \geq 3$. Rearranging, we obtain:

$$(\frac{K-1}{2})^2 - 1 \leq \frac{K-1}{4}\sqrt{(K+1)(K-3)}.$$

Squaring both members and multiplying both sides of the inequalities by 4, we obtain: $(K-3)^2(K+1)^2 \leq (K-1)^2(K+1)(K-3)$. If $K \neq -1$ and also $K \neq 3$, we can divide both sides of the inequality by $(K+1(K-3)$ and obtain: $(K-3)(K+1) \leq (K-1)^2$ or $K^2 - 2K - 3 \leq K^2 - 2K + 1$, which is always true. This completes the proof.

c) For all K such that $K \geq -1$, we obtain that:

$$\frac{K-1}{2} < (-1 + \sqrt{1+K})^2 \qquad (3.9)$$

Equation (3.9) implies that:

$$\frac{K-1}{2} < (-1+\sqrt{1+K})^2 = K+2-2\sqrt{K+1}$$

After some manipulations we obtain:

$$0 < K+5 - 4\sqrt{K+1} \quad \text{or} \quad 4\sqrt{K+1} < K+5$$

Squaring both sides, we get: $16(K+1) < (K+5)^2 = K^2+10K+25$

Therefore $0 < K^2-6K+9 = (K-3)^2$, which is always true.

d) If $K > 3$, then the two roots of

$$x^2 + x(1-K) + 1 \qquad (3.10)$$

are real, positive and reciprocals of each other.

The roots are given by

$$x_{1,2} = \frac{K-1}{2} \pm \tfrac{1}{2}\sqrt{(K+1)(K-3)}$$

3.4 Method Used to Establish the Optimum

We will now proceed to establish the optimum. To achieve our goal, we take all the a's in pairs, assuming one to be the largest and the other to be the smallest. Each ratio will then be minimized. These minima will then be compared with each other in order to determine, for a given range of K, the true minimum relation.

Starting with the relation

$$K = \frac{a_1}{a_2} + \frac{a_2}{a_3} + \frac{a_1}{a_3} \qquad (3.11)$$

we let $a_1 = x$

$a_2 = 1$

and obtain $a_3 = \frac{1+x}{K-x}$

$$\left.\begin{aligned} a_1 &= x \\ a_2 &= 1 \\ a_3 &= \frac{1+x}{K-x} \end{aligned}\right\} \qquad (3.12)$$

Note that if all a_i are multiplied by the same arbitrary constant, the value of K is unchanged.

The different possible relationships among the magnitudes of the elements are given below.

(M = maximum and m = minimum)

Case I	$a_1 = M \geq a_3 \geq a_2 = m$
Case II	$a_2 = M \geq a_3 \geq a_1 = m$
Case III	$a_1 = M \geq a_2 \geq a_3 = m$
Case IV	$a_3 = M \geq a_2 \geq a_1 = m$
Case V	$a_2 = M \geq a_1 \geq a_3 = m$
Case VI	$a_3 = M \geq a_1 \geq a_2 = m$

We now proceed to the individual cases

3.4.1 Case I $a_1 = M \geq a_3 \geq a_2 = m$

We make use of Equations (3.12). We want to minimize $\frac{a_1}{a_2} = x$, with $x > 1$

$$a_1 \geq a_3 \text{ Gives us: } \quad x \geq \frac{1+x}{K-x} \qquad (3.13)$$

$$a_3 \geq a_2 \text{ Gives us: } 1 \leq \frac{1+x}{K-x} \tag{3.14}$$

Since the function x has no maximal point (i.e., point where the derivative is zero), the only way an optimum can occur is if one of the inequalities (3.13) or (3.14) becomes an equality. The other then becomes a strict inequality. We now proceed to analyze each case separately

Case I(a)

Inequality (3.13) becomes an equality, that is:

$$x = \frac{1+x}{K-x} \qquad (\text{thus } a_1 = a_3)$$

After some algebraic manipulations we obtain that x is a root of $x^2 + x(1-K) + 1 = 0$. Therefore:

$$x_{1,2} = \frac{(K-1) \pm \sqrt{(K+1)(K-3)}}{2} \quad , \qquad K \geq 3.$$

From inequality (3.14) we know that: $1 < \frac{1+x}{K-x}$. Hence $x > \frac{K-1}{2}$ (see 3.3.2 a)). Therefore the only acceptable root of (3.10) is the one whose radical is preceeded by a positive sign, and thus we obtain as an acceptable solution:

$$\left.\begin{aligned} a_1 = a_3 = x &= \frac{K-1}{2} + \frac{1}{2}\sqrt{(K+1(K-3)} \\ a_2 &= 1 \qquad \text{with } K \geq 3 \end{aligned}\right\} \tag{3.15}$$

$$\text{Optimum} = \frac{a_1}{a_2} = \frac{k-1}{2} + \tfrac{1}{2}\sqrt{(K+1)(K-3)} \qquad (3.16)$$

<u>Case I(b)</u>

Inequality (3.14) becomes an equality, that is:

$$1 = \frac{1+x}{K-x} \qquad (\text{thus } a_2 = a_3)$$

From the last equality we obtain: $x = \frac{K-1}{2}$. Combining $x = \frac{K-1}{2}$ with inequality (3.13) we obtain:

$$\frac{K-1}{2} > \frac{1+x}{K-x} = \frac{1 + \frac{K-1}{2}}{K - \frac{K-1}{2}} = 1$$

If $\frac{K-1}{2} > 1$ then necessarily, $K > 3$. The solution for this case will be:

$$\left.\begin{array}{c} a_1 = \frac{K-1}{2} \\ a_2 = a_3 = 1 \\ \text{with } K > 3 \end{array}\right\} \qquad (3.17)$$

and the optimum is:

$$\left.\begin{array}{c} \frac{a_1}{a_2} = \frac{K-1}{2} \\ \text{with } K > 3 \end{array}\right\} \qquad (3.18)$$

We now have to find out which of the two cases (3.15) and (3.17) is the optimum solution, both being valid for $K > 3$.

Supposing that (3.18) is a better solution than (3.16), then we should have: $\frac{K-1}{2} < \frac{K-1}{2} + \tfrac{1}{2}\sqrt{(K+1)(K-3)}$, which is true if K > 3.

Therefore Case I(b) is the optimum solution.

Hence for Case I:

$$\left.\begin{aligned} a_1 &= \frac{K-1}{2} = \text{optimum} \\ a_2 &= a_3 = 1 \\ &\text{with } K > 3 \end{aligned}\right\} \tag{3.19}$$

3.4.2 Case II $a_2 = M \geq a_3 \geq a_1 = m$

We make use of equations (3.11). We want to minimize $\frac{a_2}{a_1} = \frac{1}{x}$ where necessarily $x < 1$.

We should also have:

$$a_2 \geq a_3 \text{ , or } 1 \geq \frac{1+x}{K-x} \tag{3.20}$$

$$a_3 \geq a_1 \text{ , or } \frac{1+x}{K-x} \geq x \tag{3.21}$$

Since the function $\frac{1}{x}$ has no maximal point, the only way an optimum can be achieved, is if one of the inequalities (3.20) or (3.21) becomes an equality. The other then becomes a strict inequality. We now proceed to analyze each case separately.

Case II(a)

Inequality (3.20) becomes an equality and hence:

$$1 = \frac{1+x}{K-x} \qquad \text{(thus } a_2 = a_3\text{)}$$

or

$$x = \frac{K-1}{2} \tag{3.22}$$

From inequality (3.21) we should have that, $\frac{1+x}{K-x} > x$. Replacing x by $\frac{K-1}{2}$, we obtain the requirement that $K < 3$. Element values are:

$$\left.\begin{array}{l} a_1 = \frac{K-1}{2} \\ a_2 = a_3 = 1 \\ \text{with } 1 < K < 3 \end{array}\right\} \tag{3.23}$$

The optimum is then:

$$\left.\begin{array}{l} \frac{a_2}{a_1} = \frac{2}{K-1} \\ \\ \text{with } 1 < K < 3 \end{array}\right\} \tag{3.24}$$

<u>Case II(b)</u>

Inequality (3.21) becomes an equality, that is

$$x = \frac{1+x}{K-x} \qquad (\text{thus } a_1 = a_3)$$

After some algebraic manipulations we obtain that x is a root of the following equation:

$$x^2 + x(1-K) + 1 = 0 \tag{3.25}$$

Therefore:

$$x_{1,2} = \frac{(K-1) \pm \sqrt{(K+1)(K-3)}}{2} \quad , \quad K \geq 3$$

From inequality (3.20) we know that $1 > \frac{1+x}{K-x}$. Hence $x < \frac{K-1}{2}$. Therefore, the only acceptable root of equation (3.25) is the one whose radical is preceded by a negative sign, and thus we obtain as the only acceptable solution:

$$\left.\begin{aligned} a_1 &= a_3 = \frac{K-1}{2} - \tfrac{1}{2}\sqrt{(K+1)(K-3)} \\ a_2 &= 1, \quad K \geq 3 \end{aligned}\right\} \tag{3.25}$$

and the optimum ratio is

$$\frac{a_2}{a_1} = \frac{2}{K - 1 - \sqrt{(K+1)(K-3)}}, \quad K \geq 3 \tag{3.26}$$

Having already found a possible optimum for $K \geq 3$, we have to compare both cases, so as to determine the real optimum.

If we suppose that $\frac{K-1}{2}$ (Case I(b), (3.18)), is a better optimum than $\frac{2}{K - 1 - \sqrt{(K+1)(K-3)}}$ (Case II(b), (3.26)), then we should have: $\frac{K-1}{2} < \frac{2}{K - 1 - \sqrt{(K+1)(K-3)}}$, which means that: $\left(\frac{K-1}{2}\right)^2 - \frac{K-1}{4}\sqrt{(K+1)(K-3)} < 1$ which is true if $K > 3$ (for demonstration see equation (3.8)) and therefore Solution I(b) is still the best if $K \geq 3$.

3.4.3 Case III $a_1 = M \geq a_2 \geq a_3 = m$

Making use of equation (3.12), we want to minimize:

$$\frac{a_1}{a_3} = \frac{x(K-x)}{1+x} \tag{3.27}$$

This function has no minimum value, it has only a maximum,

(because $\left.\frac{d\,\frac{a_1}{a_3}}{dx}\right|_{x = \sqrt{K_2}} < 0$). Therefore we proceed as before.

One of the inequalities:

$$a_1 \geq a_3 \quad \text{or} \quad x \geq 1 \tag{3.28}$$

$$a_2 \geq a_3 \quad \text{or} \quad 1 \geq \frac{1+x}{K-x} \tag{3.29}$$

becomes an equality, and the other a strict inequality.

Case III(a)

Letting $x = 1$, we obtain from (3.29) that $K > 3$.

A possible case will be:

$$\left.\begin{aligned} a_1 &= a_2 = 1 \\ a_3 &= \frac{2}{K-1} \\ K &> 3 \end{aligned}\right\} \tag{3.30}$$

And the optimum will be:

$$\frac{a_1}{a_3} = \frac{K-1}{2} \text{ with } K > 3 \tag{3.31}$$

This case has the same optimum as Case I(b), and may be used as an alternate to it.

Case III(b)

Letting $a_2 = a_3 = 1$, we obtain as a possible solution:

$$\left.\begin{aligned} a_1 &= \frac{K-1}{2} \quad \text{with } K > 3 \\ a_2 &= a_3 = 1 \end{aligned}\right\} \tag{3.32}$$

And the optimum will be:

$$\frac{a_2}{a_3} = \frac{2}{K-1} \quad \text{with } K > 3 \tag{3.33}$$

Case III(b) is identical to case I(b), and can therefore be put aside.

3.4.4 Case IV $a_3 = M \geq a_2 \geq a_1 = m$

Making use of equations (3.12), we want to minimize $\frac{a_3}{a_1}$:

$$\frac{a_3}{a_1} = \frac{1+x}{x(K-x)} \tag{3.34}$$

This function possesses a minimum. To obtain it we take the derivative of (3.34) and set it equal to zero.

Doing so, we obtain:

$$\frac{d}{dx} \frac{1+x}{x(K-x)} = \frac{x^2 + 2x - K}{x^2(K-x)^2} = 0 \tag{3.35}$$

Hence the minimum occurs at one of the roots of

$$x^2 + 2x - K = 0 \tag{3.36}$$

Equation (3.36) has two roots, namely: $x_{1,2} = -1 \pm \sqrt{1+K}$.

Since $x > 0$, the only acceptable solution is:

$$x_1 = -1 + \sqrt{1+K} \tag{3.37}$$

Combining equations (3.37) with a_3 of equation (3.12), we obtain:

$$a_3 = \frac{1+x}{K-x} = \frac{1 - 1 + \sqrt{1+K}}{K + 1 - 1\sqrt{1+K}} = \frac{1}{-1 + \sqrt{1+K}}$$

The a's then will be:

$$\left.\begin{aligned} a_1 &= -1 + \sqrt{1+K} \\ a_2 &= 1 \\ a_3 &= \frac{1}{-1 + \sqrt{1+K}} \end{aligned}\right\} \tag{3.38}$$

The requirement $a_2 > a_1$ implies that $K < 3$. From $a_1 > 0$ we get $K > 0$. The minimum ratio obtained is:

$$\left.\begin{aligned} &\frac{a_3}{a_1} = \left(\frac{1}{-1 + \sqrt{1+K}}\right)^2 \\ &\text{with } 0 < K < 3 \end{aligned}\right\} \tag{3.39}$$

Because $K < 3$, we will have to compare Case IV with Case II(a), to find out which is the optimum solution.

If we suppose Case IV to be the real optimum, we should have:

$$\left(\frac{1}{-1 + \sqrt{1+K}}\right)^2 < \frac{2}{K-1} \quad \text{or} \quad (-1 + \sqrt{1+K})^2 > \frac{K-1}{2} \tag{3.40}$$

which is true (see equation (3.9)).

Therefore the new optimum for $1 < K < 3$ is now Case IV.

3.4.5 Case V $a_2 = M \geq a_1 \geq a_3 = m$

Making use of equations (3.12), we want to optimize:

$$\frac{a_2}{a_3} = \frac{K-x}{1+x} \tag{3.41}$$

We should also have:

$$a_2 \geq a_1 \qquad 1 \geq x \tag{3.42}$$

$$a_1 \geq a_3 \qquad x \geq \frac{1+x}{K-x} \tag{3.43}$$

Since the function $\frac{K-x}{1+x}$ has no maximal point, the only way an optimum can be achieved, is if one of the inequalities (3.42) or (3.43) becomes an equality. We now proceed to analyze each case separately.

Case V(a)

Letting inequality (3.42) become an equality, we obtain

$$1 = x \tag{3.44}$$

Equation (3.43) is now an inequality, therefore if in $x > \frac{1+x}{K-x}$ we let $x = 1$, we obtain $K > 3$.

We now obtain as a possible solution:

$$\left.\begin{array}{l} a_1 = a_2 = 1 \\ a_3 = \frac{2}{K-1} \\ \text{with } K > 3 \end{array}\right\} \tag{3.45}$$

The optimum is now:

$$\frac{a_2}{a_3} = \frac{K-1}{2} \tag{3.46}$$

We see that Case V(a) is an alternative to Case I(b), and both are acceptable solutions.

<u>Case V(b)</u>

Letting inequality (3.43) become an equality, we obtain:

$$x = \frac{1+x}{K-x} \qquad (\text{or } a_1 = a_3) \tag{3.47}$$

After some algebraic manipulations, we obtain that x is a root of the following equation:

$$x^2 + x(1-k) + 1 = 0 \tag{3.48}$$

Therefore:

$$x_{1,2} = \frac{(K-1) \pm \sqrt{(K+1)(K-3)}}{2}, \qquad K > 3 \tag{3.49}$$

From inequality (3.42) we know that $1 > x$ which means that the only acceptable root of equation (3.48) is the one whose radical is preceeded by a minus sign. Thus:

$$x_2 = \frac{K-1}{2} - \tfrac{1}{2}\sqrt{(K+1)(K-3)} \quad \text{with } K \geq 3 \tag{3.50}$$

Thus a possible solution is:

$$\left.\begin{array}{l} a_1 = a_3 = \frac{K-1}{2} - \tfrac{1}{2}\sqrt{(K+1)(K-3)} \\ a_2 = 1 \quad \text{and} \quad K \geq 3 \end{array}\right\} \tag{3.51}$$

The optimum is now:

$$\left.\begin{array}{l} \dfrac{a_2}{a_3} = \dfrac{1}{\frac{K-1}{2} - \frac{1}{2}\sqrt{(K+1)(K-3)}} = \dfrac{K-1}{2} + \tfrac{1}{2}\sqrt{(K+1)(K-3)} \\ \text{For } K \geq 3 \end{array}\right\} \tag{3.52}$$

We have to find out which of the two cases V(b) or I(b) has the optimum solution for $K > 3$. Supposing that Case I(b) (3.18) is better than Case V(b) (3.52), then we should have:

$\frac{K-1}{2} < \frac{K-1}{2} + \frac{1}{2}\sqrt{(K+1)(K-3)}$ which is necessarily true if $K > 3$. Therefore solution for Case I(b) is better than solution for Case V(b).

3.4.6 Case VI $a_3 = M \geq a_1 \geq a_2 = m$

Making use of equations (3.12), we want to minimize

$$\frac{a_3}{a_2} = \frac{1+x}{K-x} \tag{3.54}$$

We should also have:

$$a_3 \geq a_1 \quad \text{which means} \quad \frac{1+x}{K-x} \geq x \tag{3.55}$$

$$a_1 \geq a_2 \quad \text{which means} \quad x \geq 1 \tag{3.56}$$

Since the function has no maximal point, the only way an optimum can be achieved is if one of the inequalities (3.55) or (3.56) becomes an equality. The other then becomes a strict inequality. We now proceed to analyze each case separately.

Case VI(a)

Letting inequality (3.55) become an equality, we obtain:

$$x = \frac{1+x}{K-x} \quad (\text{thus } a_3 = a_1) \tag{3.57}$$

After some algebraic manipulations we obtain that x is a root of the following equation:

$$x^2 + x(1-K) + 1 = 0 \tag{3.58}$$

Therefore:

$$x_{1,2} = \frac{(K-1) \pm \sqrt{(K+1)(K-3)}}{2} \quad , \quad K > 3 \tag{3.59}$$

From inequality (3.56) we know that $x > 1$; therefore the only acceptable root of equation (3.58) is the one whose radical is preceded by a positive sign, and therefore we obtain as an acceptable value for x:

$$x = \frac{K - 1 + \sqrt{(K+1)(K-3)}}{2} \tag{3.60}$$

A possible solution will then be:

$$\left.\begin{aligned} &a_1 = a_3 = \frac{K-1}{2} + \tfrac{1}{2}\sqrt{(K+1)(K-3)} \\ &a_2 = 1 \quad \text{with } K \geq 3 \end{aligned}\right\} \tag{3.61}$$

The optimum then is:

$$\left.\begin{aligned} &\frac{a_3}{a_1} = \frac{K-1}{2} + \tfrac{1}{2}\sqrt{(K+1)(K-3)} \\ &\text{with } K \geq 3 \end{aligned}\right\} \tag{3.62}$$

We have the same optimum as for Case V(b) and we know already that Case I(b) is better.

Case VI(b)

Letting inequality (3.56) become an equality we obtain:

$$x = 1 \quad (\text{thus } a_1 = a_2) \tag{3.63}$$

Combining the inequality (3.55) with x = 1 we see that $K < 3$, and we obtain as a possible solution:

$$\left.\begin{aligned} &a_1 = a_2 = 1 \\ &a_3 = \frac{2}{K-1} \\ &\text{with } 1 < K < 3 \end{aligned}\right\} \tag{3.64}$$

The optimum is given by:

$$\frac{a_3}{a_2} = \frac{2}{K-1} \qquad \text{with } 1 < K < 3 \tag{3.65}$$

We have the same optimum $\frac{2}{K-1}$ (for $1 < K < 3$) as in Case II(a), but we already have shown that Case IV is better than Case II(a), therefore we can eliminate Case VI(b).

3.5 Summary of Results for Three Section Cascades

Depending on the value of K, we obtained three cases:

1. $1 < K < 3$

$$\left.\begin{aligned} a_1 &= -1 + \sqrt{1+K} \\ a_2 &= 1 \\ a_3 &= \frac{1}{-1 + \sqrt{1+K}} = \frac{1 + \sqrt{1+K}}{K} \\ \text{optimum ratio} &= \frac{a_3}{a_1} = \frac{1}{(-1 + \sqrt{1+K})^2} = \frac{2 + K + 2\sqrt{1+K}}{K^2} \end{aligned}\right\} \tag{3.66}$$

2. $K > 3$

$$\left.\begin{aligned} a_1 &= \frac{K-1}{2} \\ a_2 &= a_3 = 1 \\ \text{optimum ratio} &= \frac{K-1}{2} \end{aligned}\right\} \tag{3.67}$$

3. $$\left.\begin{aligned} K > 3 \qquad a_1 &= a_2 = 1 \\ a_3 &= \frac{2}{K-1} \\ \text{optimum ratio} &= \frac{K-1}{2} \end{aligned}\right\} \tag{3.68}$$

Remarks

1. When $K = 3$, we have an unique optimum: then $a_1 = a_2 = a_3 = 1$. Formulas (3.66), (3.67) and (3.68) are all acceptable.
2. As mentioned earlier, we should expect solutions which are symmetrical with respect to a_1 and a_3.

If we look at the sets of equations (3.66) we see that we can replace a_1 by $\frac{1}{a_3}$. And a_3 by $\frac{1}{a_1}$ without changing solution.

Looking at the sets of equations (3.67) and (3.68) we see that by switching a_1 with $\frac{1}{a_3}$ and a_3 by $\frac{1}{a_1}$ we get the other set of equation and vice-versa.

3.6 Comparison of Two and Three Section Cascades

There are two first degree equations in U (see equations (3.3) and (3.5)).

We wish to know for what range of K it is more advantageous to use a three section cascade instead of a two section cascade.

We know that if $1 < K < 3$, the optimum is given by

$\frac{1}{(-1 + \sqrt{1+k})^2}$ if we use three section cascade.

If we use only two sections cascade, the optimum is given by K.

We try first to determine the limiting case, where we could use indifferently two or three sections cascade. Therefore we have to solve:

$$K = \frac{1}{(-1 + \sqrt{1+K})^2} \tag{3.69}$$

or

$$K^4 - 2K^2 - 4K + 1 = 0 \tag{3.70}$$

This polynomial has only one root in the range [1,3] and this was found to be:

$$K_0 = 1.946965328 \tag{3.71}$$

Hence, if $K < K_0$, we should use two sections, and otherwise three. The results of this chapter are summarized in Table 3.1.

Table 3.1

First Degree in U

	Two Cascade Sections		Three Cascade Sections		
	$K < 1.95$		$1.94 < K < 3$	$3 < K$	
a_1	$a_1 = K$	$a_1 = 1$	$a_1 = -1 + \sqrt{1+K}$	$a_1 = \frac{K-1}{2}$	$a_1 = 1$
a_2	$a_2 = 1$	$a_2 = \frac{1}{K}$	$a_2 = 1$	$a_2 = 1$	$a_2 = 1$
a_3	——	——	$a_3 = \frac{1 + \sqrt{1+K}}{K}$	$a_3 = 1$	$a_3 = \frac{2}{K-1}$
optimum	K	K	$\frac{2 + K + 2\sqrt{1+K}}{K^2}$	$\frac{K-1}{2}$	$\frac{K-1}{2}$

CHAPTER IV

FOUR SECTION CASCADE

4.1 Introduction

As seen in Chapter II, cascades of four or five sections will give rise to transfer functions whose denominators are of the second degree in U.

It is the aim of this chapter to optimize the four section case with respect to the criteria set forth in the previous chapter.

4.2 Definition of the Problem

4.2.1 Definition of Ratios

The relationships for four sections are:

$$A_4 = (1-U)^{-2}\left[1 + U\left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_4}\right) + U^2\left(\frac{a_1 a_3}{a_2 a_4}\right)\right] \tag{4.1}$$

$$A_4 = (1-U)^{-2}\,[1 + UK_1 + U^2K_2]$$

$$D_4 = (1-U)^{2}\left[1 + U\left(\frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_2} + \frac{a_4}{a_3}\right) + U^2\left(\frac{a_2 a_4}{a_1 a_3}\right)\right] \tag{4.2}$$

It is only necessary to analyze Equation (4.1), because if, in Equation (4.1), we switch a_1 to a_4 and a_2 to a_3, then it becomes equation (4.2).

Letting

$$K_1 = R_{12} + R_{13} + R_{14} + R_{23} + R_{24} + R_{34}$$

$$K_2 = R_{12}R_{34}$$

where:

$$R_{ij} = \frac{a_i}{a_j}$$

and

$$R_{ik} \times R_{kj} = R_{ij}$$

then we obtain:

$$K_1 = R_{12} + R_{12}R_{23} + R_{12}R_{23}R_{34} + R_{23} + R_{23}R_{34} + R_{34}$$

$$K_2 = R_{12}R_{34}$$

We can express R_{23} and R_{34} in terms of R_{12}, K_1, and K_2. Doing so we find that:

$$R_{23} = \frac{-[R_{12}^2 - K_1R_{12} + K_2]}{R_{12}^2 + R_{12}(1+K_2) + K_2}$$

$$R_{34} = \frac{K_2}{R_{12}}$$

We note in passing, that the requirement that $R_{23} > 0$, bounds the range of values which R_{12} may assume: we must have that

$$R_{12} \varepsilon I$$

where

$$I \triangleq \left(\frac{K_1}{2} - \sqrt{(\frac{K_1}{2})^2 - K_2} \quad , \quad \frac{K_1}{2} + \sqrt{(\frac{K_1}{2})^2 - K_2} \right)$$

$$K_1 \geq 2\sqrt{K_2}$$

The impedance scaling values (a_i) of each secticn can now be expressed in terms of the ratios R_{12} and R_{23}.

Letting $a_1 = y$, $R_{12} = x$ and $R_{23} = A(x)$, simple manipulations lead to:

$$\left.\begin{aligned} a_1 &= y \\ a_2 &= \frac{y}{x} \\ a_3 &= \frac{y}{xA} \\ a_4 &= \frac{y}{K_2 A} \end{aligned}\right\} \tag{4.3a}$$

where $x \varepsilon I$,

$$I = \left(\frac{K_1}{2} - \sqrt{(\frac{K_1}{2})^2 - K_2} \quad , \quad \frac{K_1}{2} + \sqrt{(\frac{K_1}{2})^2 - K_2} \right) \tag{4.3b}$$

$$A = - \left(\frac{x^2 - K_1 x + K_2}{x^2 + x(1+K_2) + K_2} \right) \tag{4.3c}$$

$$K_1 \geq 2\sqrt{K_2} \tag{4.3d}$$

We note for future reference that it can easily be shown that $x = \sqrt{K_2} \in I$

The possible ratios to be considered are listed below

$$R_{12} = x \qquad R_{21} = \frac{1}{x}$$

$$R_{13} = xA \qquad R_{31} = \frac{1}{xA}$$

$$R_{14} = K_2A \qquad R_{41} = \frac{1}{K_2A}$$

$$R_{23} = A \qquad R_{32} = \frac{1}{A}$$

$$R_{24} = \frac{K_2A}{x} \qquad R_{42} = \frac{x}{K_2A}$$

$$R_{34} = \frac{K_2}{x} \qquad R_{43} = \frac{x}{K_2}$$

4.2.2 Minima of Ratios

As we shall be interested in minimizing ratios, we first consider which of the above ratios could possibly have a minimum, for some $x \in I$.

Beginning with A, we find its first derivative W.R.T. x,

$$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{-(x^2 - K_1x + K_2)}{x^2 + x(1+K_2) + K_2}\right) = \frac{(x^2-K_2)(1+K_1+K_2)}{(x^2 + x(1+K_2) + K_2)^2}$$

The first derivative is zero for $x = \pm\sqrt{K_2}$, of which, $x = \sqrt{K_2}$ is of interest.

One may determine whether we have a maximum of a minimum either by taking the sign of the second derivative,

or by examining a sketch of the function A (Fig. 4.1).
Obviously, $x = \sqrt{K_2}$ is a maximum.

Hence R_{23} possess no minimum within the required range. However, $R_{32} = \frac{1}{A}$ has a minimum for $x = \sqrt{K_2}$.

The minimum value for R_{32} is:

$$R_{32MIN} = \frac{1}{A(\sqrt{K_2})} = \left.\frac{x^2 + (1+K_2)x + K_2}{-x^2 + K_1 x - K_2}\right|_{x=\sqrt{K_2}} = \frac{(1 + \sqrt{K_2})^2}{(K_1 - 2\sqrt{K_2})}$$

It follows at once that R_{14} has no minimum in I, but that R_{41} does for $x = \sqrt{K_2}$,

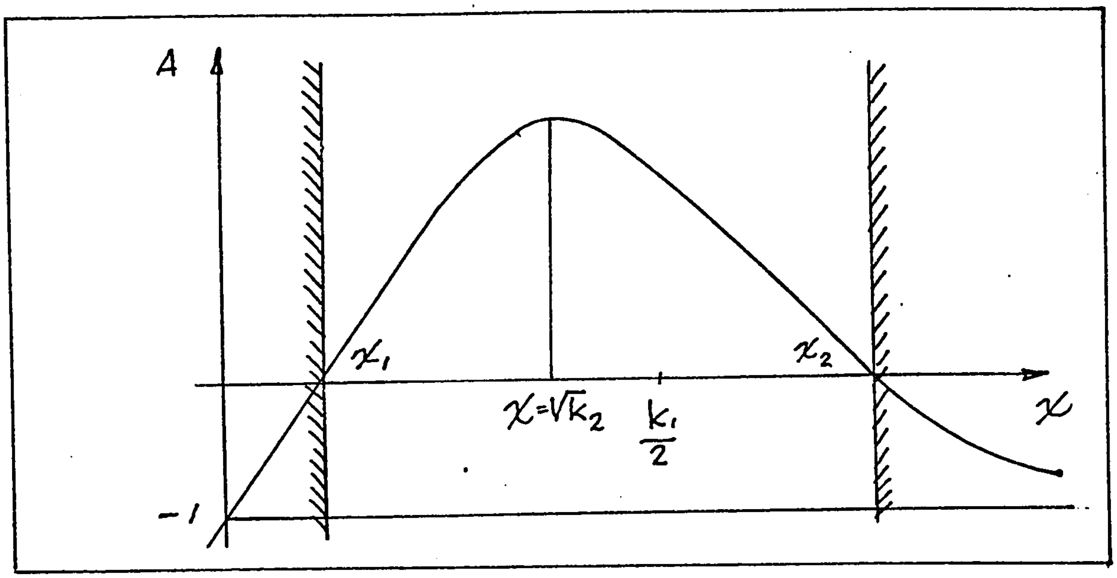
$$R_{41MIN} = \left.\frac{1}{K_2 A}\right|_{x=\sqrt{K_2}} = \left.\frac{x^2 + (1+K_2)x + K_2}{K_2(-x^2 + K_1 x - K_2)}\right|_{x=\sqrt{K_2}}$$

$$= \frac{(1 + \sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})}$$

We next consider $R_{13} = xA$.
A sketch of this function is given in Fig. 4.2

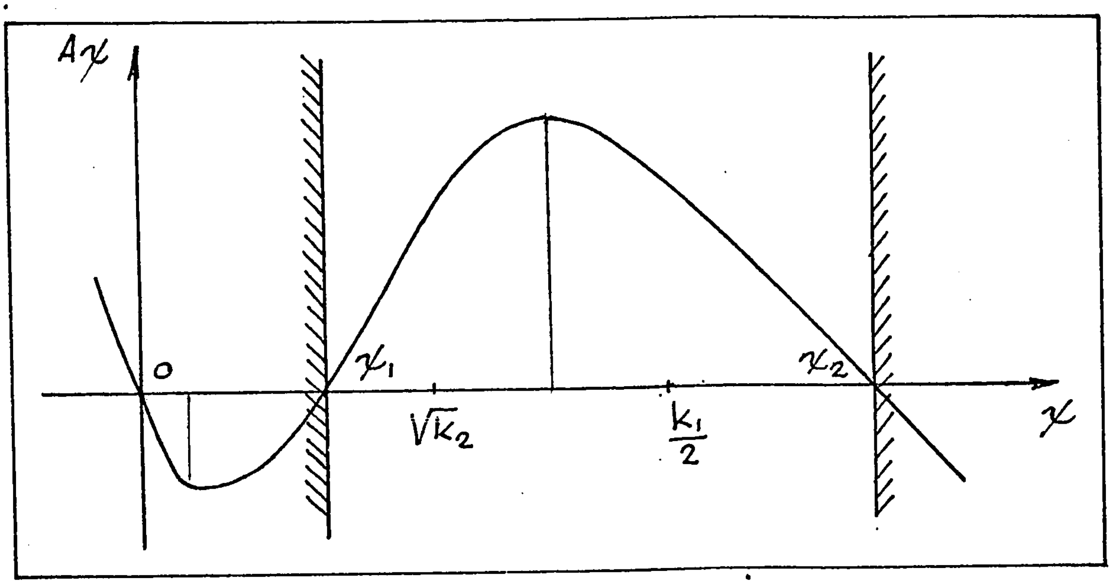
Obviously the function possesses a maximum in 1. This maximum occurs for some $x > \sqrt{K_2}$ since

$$\left.\frac{d}{dx}(Ax)\right|_{x=\sqrt{K_2}} = \left.(A'x + A)\right|_{x=\sqrt{K_2}} = A(\sqrt{K_2}) > 0$$



$$x_1 = \frac{K_1}{2} - \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2} \qquad x_2 = \frac{K_1}{2} + \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}$$

FIGURE 4.1 Sketch of $y = A$



Where $x_1 = \frac{K_1}{2} - \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}$ and $x_2 = \frac{K_1}{2} + \sqrt{\left(\frac{K_1}{2}\right)^2 - K_2}$

FIGURE 4.2 Sketch of y = AX

It thus follows that $R_{31} = \frac{1}{xA}$ has a minimum for some value of $x > \sqrt{K_2}$ such that $x \varepsilon I$.

Finally we consider $R_{24} = \frac{K_2 A}{x}$. A sketch is given in Fig. 4.3.

Obviously this function possesses a maximum in the range $x \varepsilon I$.

This maximum occurs for some $x < \sqrt{K_2}$, since

$$\frac{d}{dx}\left(\frac{K_2 A}{x}\right)\bigg|_{x=\sqrt{K_2}} = \frac{K_2[A'x - A]}{x^2}\bigg|_{x=\sqrt{K_2}} = -A(\sqrt{K_2}) < 0$$

It thus follows that $R_{42} = \frac{x}{K_2 A}$ has a minimum at some value of $x < K_2$ such that $x \varepsilon I$.

All the other functions of Table 4.1, namely R_{12}, R_{34}, R_{21} and R_{43} have no maxima or minima in the range I.

4.2.3 Some Required Relationships

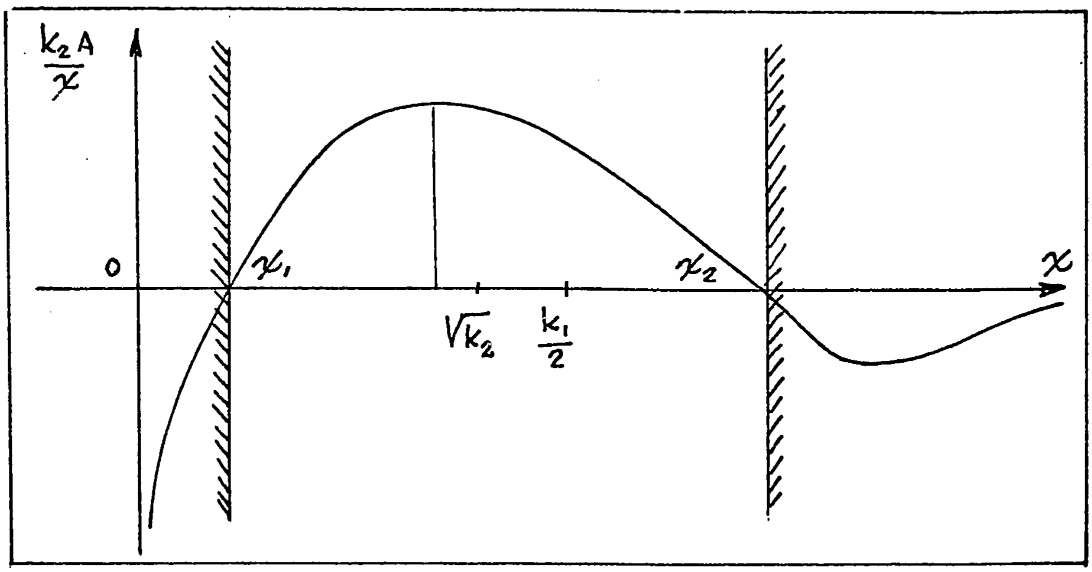
In order to simplify the discussion later in the Chapter, we now develop some useful results.

1) Define the Polynomial P_2 as:

$$P_2 = x^2 + \frac{(1 + K_2 - K_1)}{2} x + K_2 = (x - x_1)(x - x_2) \quad (4.4)$$

which arises from letting A = 1.

We shall be interested only in the case where P_2 has



$$x_1 = \frac{K_1}{2} - \sqrt{(\frac{K_1}{2})^2 - K_2} \qquad x_2 = \frac{K_1}{2} + \sqrt{(\frac{K_1}{2})^2 - K_2}$$

FIGURE 4.3 Sketch of $y = \frac{K_2 A}{x}$

real roots satisfying:

$$x_1 > K_2 \quad , \quad x_2 < 1$$
$$x_1 < 1 \quad , \quad x_2 > K_2 \tag{4.5}$$

Note that since $x_1 x_2 = K_2$ then $x_1 > K_2 \rightarrow x_2 < 1$ and vice-versa.

We now consider the following possibilities:

a) $\underline{x_2 > K_2 > 1 > x_1}$ - Hence $P_2(K_2) < 0$ and it may easily be shown that this implies that:

$$K_1 > 3(1 + K_2) \tag{4.6}$$

The same result would follow by considering $P_2(1) < 0$. The inequality (4.6) will be used for subcases: OR-4B, OR-7A, OR-11A and OR-19B.

b) $\underline{x_2 > 1 > K_2 > x_1}$ - The restriction on K_1 is again given by (4.6).

2) <u>Define the Polynomial P_2' as:</u>

$$P_2' = x^2 + \frac{(1 + K_2 - K_1K_2)}{1 + K_2} x + K_2 = (x - x_1)(x - x_2) \tag{4.7}$$

Which will arise from letting $AK_2 = 1$. We note that if $x_1 > K_2$, then $x_2 < 1$ and vice-versa.

As before we shall be interested only in real roots satisfying this requirement. Using the same argument as before, i.e., that x_1 and x_2 must lie outside the interval $[1, K_2]$ we obtain the requirement that:

$$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2} \tag{4.8}$$

The inequality (4.8) will be used for subcases: OR - 4A, OR - 6A, OR - 7B and OR - 16B*.

We note the following necessary results: If $K_1 > 3(1 + K_2)$ and also $K_1 > \dfrac{(1 + K_2)(2 + K_2)}{K_2}$, then for both P_2 and P_2' we have that:

(a) If $K_2 < 1$:

$$x_1 < \frac{2(1 + K_2)}{K_1 - K_2 - 1} < K_2 < 1 < \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)} < x_2 \tag{4.9}$$

(b) If $1 < K_2$:

$$x_1 < \frac{2(1 + K_2)}{K_1 - K_2 - 1} < 1 < K_2 < \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)} < x_2 \tag{4.10}$$

The proofs are tedious but straightforward.

---

* The different subcases referred to are defined in section 4.3.

3) Define the Polynomial P_3 as:

$$P_3 = x^3 + x^2(1 - K_1) + x(1 + 2K_2) + K_2$$

$$= (x - x_1)(x - x_2)(x - x_3) \qquad (4.11)$$

Which arises from letting $Ax = 1$.

We now show that P_3 always has a negative real root.

First, we note that $P_3(0) = K_2 > 0$. Now if $K_2 < 1$, then P_3 has a root, say x_1, in the interval $[-K_2, 0]$, since $P_3(-K_2) = -K_2^2(1 + K_1 + K_2) < 0$. Hence

$$|x_1x_2x_3| = K_2 \quad \text{and} \quad |x_1| < K_2 \rightarrow |x_2x_3| > 1 \qquad (4.12)$$

Further, if $K_2 > 1$, then P_3 has a root in the interval $(-1, 0)$, since $P_3(-1) = -(1 + K_1 + K_2) < 0$. Hence

$$|x_1x_2x_3| = K_2 \quad \text{and} \quad |x_1| < 1 \rightarrow |x_2x_3| > K_2 \qquad (4.13)$$

We shall be interested only in real positive roots, and we will consider in particular the following cases:

a) $K_2 < 1,\ x_2 \ \varepsilon\ (K_2, \sqrt{K_2}) \rightarrow x_3 > \dfrac{1}{\sqrt{K_2}}$

It may easily be verified that: $P_3(K_2) > 0$, and hence that $K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2}$, $P_3(\sqrt{K_2}) < 0$, and hence that $K_1 > \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$.

Therefore:

$$\left.\begin{array}{l} \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2} \\ \text{If } x_2 \ \varepsilon\ (K_2, \sqrt{K_2}) \quad \text{where} \quad K_2 < 1 \end{array}\right\} \tag{4.14}$$

Inequalities (4.14) apply to case OR-8B.

b) $K_2 < 1,\ x_2 \ \varepsilon\ (\sqrt{K_2}, 1) \rightarrow x_3 > 1$

It may easily be verified that $P_3(\sqrt{K_2}) > 0$, and hence that $K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$, $P_3(1) < 0$, and hence that $K_1 > 3(1 + K_2)$.

Therefore:

$$\left.\begin{array}{l} 3(1 + K_2) < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \\ \text{For } x_2 \ \varepsilon\ (\sqrt{K_2}, 1) \quad \text{where } K_2 < 1 \end{array}\right\} \tag{4.15}$$

Inequalities (4.15) apply to case OR-20B. It can easily be shown that when $K_2 < 1$, there always exist values of K_1 satisfying inequalities (4.14) and (4.15).

c) $\underline{K_2 > 1,\ x_2 \ \varepsilon\ (1, \sqrt{K_2}) \rightarrow x_3 > \sqrt{K_2}}$

It may easily be verified that:

$P_3(1) > 0$, and hence that $K_1 < 3(1 + K_2)$

$P_3(\sqrt{K_2}) < 0$, and hence that $K_1 > \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$

Therefore:

$$\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < 3(1 + K_2) \qquad (4.16)$$

If $x_2 \ \varepsilon\ (1, \sqrt{K_2})$ where $K_2 > 1$

Inequalities (4.16) apply to case OR-2A. It can be shown that there always exists values of K_1 satisfying inequality (4.16).

d) $\underline{K_2 > 1,\ x_2 \ \varepsilon\ (\sqrt{K_2}, K_2) \rightarrow x_3 > 1}$

It can be seen that there is no guarantee that x_3 is larger than x_2. We must therefore consider two subcases:

i) $\underline{x_3 > x_2}$

It may easily be verified that:

$P_3(\sqrt{K_2}) > 0$, and hence that $K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$

$P_3(K_2) < 0$, and hence that $K_1 > \dfrac{(1 + K_2)(2 + K_2)}{K_2}$

$$\left.\begin{aligned} &\frac{(1 + K_2)(2 + K_2)}{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \\ &x_2 \; \varepsilon \; (\sqrt{K_2}, K_2) \quad \text{where } x_3 > x_2 \end{aligned}\right\} \quad (4.17)$$

Inequalities (4.17) apply to case OR-5A.

ii) $\underline{x_2 > x_3}$

In this case the inequality becomes:

$$\left.\begin{aligned} &\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2} \\ &\text{If } x_2 \; \varepsilon \; (\sqrt{K_2}, K_2) \quad \text{where } x_2 > x_3 \end{aligned}\right\} \quad (4.18)$$

Inequalities (4.18) also apply to case OR-5A. It remains to determine for what values of K_2 inequality (4.17) or (4.18) is applicable. The two bounds intersect for a value of $K_2 = 7.0711$. It may be shown that for $1 < K_2 < 7.0711$, inequality (4.17) applies, and that for $K_2 > 7.0711$, inequality (4.18) is true. (See Fig. 4.4).

4) Define the Polynomial P_3' as:

$$P_3' = x^3 + x^2(1 + 2K_2) + xK_2(1 - K_1) + K_2^2$$

$$= (x - x_1)(x - x_2)(x - x_3). \tag{4.19}$$

Which arises from letting $AK_2 = x$. The polynomials P_3 and P_3' may be related as follows:

$$P_3'(x) = \frac{x^3}{K_2} P_3(\frac{K_2}{x}) \tag{4.20}$$

Hence if W is a root of P_3, then $\frac{K_2}{W}$ is a root of P_3'. Our consideration of P_3' is thus simplified.

a) $K_2 < 1,\ x_2 \varepsilon\ (K_2, \sqrt{K_2})$

This corresponds through the $\frac{K_2}{x}$ transformation to the interval for P_3 where $W_2\ \varepsilon(\sqrt{K_2}, 1)$, which is case b) of P_3.

Therefore:

$$\left.\begin{array}{l} 3(1 + K_2) < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \\ \\ x_2\ \varepsilon\ (K_2, \sqrt{K_2}), \quad \text{where } K_2 < 1 \end{array}\right\} \tag{4.21}$$

Inequalities (4.21) apply to case OR-8A.

b) $K_2 < 1,\ x_2\ \varepsilon(\sqrt{K_2}, 1)$

This corresponds through the $\frac{K_2}{x}$ transformation to the interval for P_3 where $W_2 \varepsilon(K_2, \sqrt{K_2})$, which is case a) of P_3.

Therefore:

$$\left.\begin{array}{l} \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2} \\ \\ x_2\varepsilon(\sqrt{K_2},\ 1) \quad \text{where } K_2 < 1 \end{array}\right\} \tag{4.22}$$

Inequalities (4.22) apply to case OR-9A.

c) $K_2 > 1,\ x_2\varepsilon(1, \sqrt{K_2})$

This corresponde through the $\frac{K_2}{x}$ transformation to the interval for P_3 where $W_2\varepsilon(\sqrt{K_2},\ K_2)$, which is case d) of P_3.

Therefore:

$$\text{i)}\quad \left.\begin{array}{l} \dfrac{(1 + K_2)(2 + K_2)}{K_2} < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \\ \\ x_2\varepsilon(1,\ \sqrt{K_2}) \text{ and also } 1 < K_2 < 7.0711 \end{array}\right\} \tag{4.23}$$

$$\text{ii)}\quad \left.\begin{array}{l} \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2} \\ \\ x_2\varepsilon(1,\ \sqrt{K_2}) \quad \text{and also } 7.0711 < K_2 \end{array}\right\} \tag{4.24}$$

Inequalities (4.23) and (4.24) apply to case OR-15B.

d) $K_2 > 1,\ \ x_2\varepsilon(\sqrt{K_2},\ K_2)$

This corresponds through the $\frac{K_2}{x}$ transformation to the interval for P_3 where $W_2 \varepsilon (1, \sqrt{K_2})$, which is case c) of P_3.

Therefore:

$$\left.\begin{aligned} &\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < 3(1 + K_2) \\ &x_2 \varepsilon (\sqrt{K_2}, K_2) \quad \text{where } K_2 > 1 \end{aligned}\right\} \tag{4.25}$$

Inequalities (4.25) apply to case OR-2B.

4.2.4 Minimization Procedure

The procedure to be followed in obtaining minimum ratios is similar to that used in the last Chapter.

We will assume the a_i's to be ordered with respect to decreasing magnitude, and consider all possible such orderings.

For simplicity the value of y will be adjusted to make the smallest $a_i = 1$.

For each ordering there will be a specific ratio a_i. If the a_i has a minimum in I, we will determine whether the minimum is acceptable or not. If the minimum is an acceptable one, then from the ordering relationships we find the restrictions on K_1 and K_2. If a_i has no acceptable minimum in I, or else if a_i has no minimum at all, the possible solution to the minimization problem will occur, when two a_i's become equal, or more specifically either the two largest or the two smallest a's will become equal.

As before we obtain from ordering relationships restrictions on K_1 and K_2.

In some instances we will find that two sets of element values will give rise to the same minimum ratio, and will be valid for the same ranges of K_1 and K_2. Such sets will be said to be equivalent.

Finally, we will compare the minima obtained for all cases where there are overlapping ranges for K_1 and K_2 in order to determine the true minimum for each range of K_1 and K_2.

4.3 Examination of all Possible Orderings

The various orderings will be referred to in the sequel as OR-1, OR-2, etc...

We now proceed to the definition and consideration of each of these.

OR-1 : $(a_1, a_2, a_3, a_4 = 1)$; Ratio $= R_{14} = K_2 A$

As $a_4 = 1 \rightarrow y = K_2 A$

Element values become:

$$a_1 = K_2 A$$

$$a_2 = \frac{K_2 A}{x}$$

$$a_3 = \frac{K_2}{x} \qquad (4.26)$$

$$a_4 = 1.0$$

Since R_{14} has no minimum in I, we must consider two subcases:

<u>OR-1A : $(a_1 = a_2, a_3, a_4 = 1)$</u>

From (4.26) we see that $x = 1$. Hence,

$$a_1 = K_2A = a_2$$

$$a_3 = K_2$$

$$a_4 = 1$$

where $A = \dfrac{K_1 - 1 - K_2}{2(1 + K_2)}$.

From the ordering relationships we have that:

$$a_1 = a_2 > a_3 \rightarrow K_1 > 3(1 + K_2)$$

$$a_3 > a_4 = 1 \rightarrow K_2 > 1$$

Since $x = 1$, $\quad R_{14} = K_2A = \dfrac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)} > K_2$

<u>OR-1B:: $(a_1, a_2, a_3 = a_4 = 1)$</u>

From (4.26) we see that: $x = K_2$. Hence,

$$a_1 = K_2A$$

$$a_2 = A$$

$$A_3 = a_4 = 1$$

It can easily be shown that $A(K_2) = A(1)$, the value of A being given by:

$$A = \frac{K_1 - 1 - K_2}{2(1 + K_2)}$$

From the ordering relationships, we have that:

$$a_1 > a_2 \rightarrow K_2 > 1$$

$$a_2 > a_3 = 1 \rightarrow A > 1 \rightarrow K_1 > 3(1 + K_2)$$

Hence OR-1A and OR-1B are <u>equivalent</u>, i.e., have same minimum ratio, and cover the same range of K_1 with respect to K_2 .

A summary is given in Table 4.1.

<u>OR - 2 : $(a_1, a_3, a_2, a_4 = 1)$</u>; Ratio = $R_{14} = K_2 A$

As $a_4 = 1 \rightarrow y = K_2 A$, element values become:

$$\left.\begin{aligned} a_1 &= K_2 A \\ a_2 &= \frac{K_2 A}{x} \\ a_3 &= \frac{K_2}{x} \\ a_4 &= 1.0 \end{aligned}\right\} \qquad (4.27)$$

Since R_{14} has no minimum in I, we must consider two subcases:

<u>OR - 2A : $(a_1 = a_3, a_2, a_4 = 1)$</u>

From $a_1 = a_3$ we see that: $A = \frac{1}{x}$. Hence,

Table 4.1

Summary of Case OR-1

	OR-1A	OR-1B	Remarks
a_1	K_2A	K_2A	Ratio = $R_{14} = K_2A > K_2$
a_2	K_2A	A	Range : $K_2 > 1$ $K_1 > 3(1 + K_2)$
a_3	K_2	1	with $A = \frac{K_1 - 1 - K_2}{2(1 + K_2)} > 1$
a_4	1	1	

$$a_1 = \frac{K_2}{x} = a_3$$

$$a_2 = \frac{K_2}{x^2}$$

$$a_4 = 1.0$$

From the ordering relationships we have that:

$$a_1 = a_3 > a_2 \rightarrow \quad x > 1$$

$$a_2 > a_4 = 1 \rightarrow \quad \sqrt{K_2} > x$$

Hence, $K_2 > \sqrt{K_2} > x > 1$ (4.28)

and $\sqrt{K_2} \leq R_{14} = K_2 A = \frac{K_2}{x} \leq K_2$

Since $A = \frac{1}{x}$, equation (4.3c) becomes:

$$\frac{-(x^2 - K_1\, x + K_2)}{x^2 + (1+K_2)\, x + K_2} = \frac{1}{x},$$

which can be put into the form:

$$x^3 + x^2\,(1-K_1) + x(1+2K_2) + K_2 = 0 \tag{4.29}$$

Taking into account equations (4.28) and (4.20) we know from previous study (equation 4.16) that:

$$\frac{1 + 2\sqrt{K_2} + 3\,K_2}{\sqrt{K_2}} < K_1 < 3(1 + K_2)$$

OR - 2B (a_1, a_3, $a_2 = a_4 = 1$)

From $a_2 = a_4$ we see that:

$$x = K_2 A \rightarrow \quad A = \frac{x}{K_2}$$

Hence:

$$a_1 = x$$

$$a_2 = 1 = a_4$$

$$a_3 = \frac{K_2}{x}$$

From the ordering relationships we have that:

$$a_1 > a_3 \rightarrow \quad x > \frac{K_2}{x} \rightarrow x > \sqrt{K_2}$$

$$a_3 > 1 \quad \rightarrow \quad \frac{K_2}{x} > 1 \rightarrow K_2 > x$$

The above inequalities give us:

$$K_2 > x > \sqrt{K_2} > 1 \tag{4.30}$$

and hence: $\sqrt{K_2} \leq R_{14} \leq K_2$

As $x = K_2 A$, we obtain:

$$x = K_2 \quad \frac{-(x^2 - K_1 x + K_2)}{x^2 + (1+K_2)\, x + K_2}$$

$$x^3 + x^2(1 + 2\, K_2) + x(K_2)(1 - K_1) + K_2{}^2 = 0 \tag{4.31}$$

From the combined conditions (4.30), (4.31) we know that (see equations 4.25):

$$3(1 + K_2) > K_1 > \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

Both OR - 2A and OR-2B cover the same range of K_1 and K_2. Hence OR - 2A and OR - 2B are equivalent.

A summary is given in Table 4.2.

<u>OR 3 (a_1, a_2, a_4, a_3 = 1)</u> ; Ratio = R_{13} = xA

Since $a_3 = 1$, we have that $y = xA$

Therefore, element values are:

$$\left.\begin{aligned} a_1 &= xA \\ a_2 &= A \\ a_3 &= 1 \\ a_4 &= \frac{x}{K_2} \end{aligned}\right\} \qquad (4.32)$$

Since R_{13} has no minimum in I, we must consider two subcases:

<u>OR 3 A ($a_1 = a_2$, a_4, a_3 = 1)</u>

From $a_1 = a_2$ we see that:

$x = 1$

If $x = 1$ then $A = \dfrac{K_1 - K_2 - 1}{2(1 + K_2)}$

Equation (4.32) now becomes:

Table 4.2

Summary of Case OR - 2

	OR-2A	OR-2B	Remarks
a_1	K_2/x	x'	$\sqrt{K_2} < R_{14} = \frac{K_2}{x}$
a_2	K_2/x^2	1.0	$K_2 > 1$; $\frac{1+2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < 3(1 + K_2)$
a_3	K_2/x	K_2/x'	x is a root of P_3 lying in interval $[1,\sqrt{K_2}]$
a_4	1.0	1.0	$x' = \frac{K_2}{x}$

$$a_1 = a_2 = \frac{K_1 - K_2 - 1}{2(1 + K_2)}$$

$$a_4 = \frac{1}{K_2}$$

$$a_3 = 1$$

From the ordering relationships we see that:

$$a_1 = a_2 > a_4 \quad \rightarrow \quad K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

$$a_4 > 1 \quad \rightarrow \quad 1 > K_2$$

It follows that $R_{13} = A > \frac{1}{K_2}$

<u>OR 3 B $(a_1, a_2, a_4 = a_3 = 1)$</u>

This case is identical to OR - 1B.

A summary is given in Table 4.3.

<u>OR - 4 $(a_1, a_4, a_2, a_3 = 1)$</u> ; Ratio = R_{13} = xA

Since $a_3 = 1$, we have that $y = xA$, elements value are:

$$\left.\begin{aligned} a_1 &= xA \\ a_2 &= A \\ a_3 &= 1 \\ a_4 &= \frac{x}{K_2} \end{aligned}\right\} \qquad (4.33)$$

Table 4.3

Summary of Case OR - 3A		
	OR-3A	Remarks
a_1	$\frac{K_1 - K_2 - 1}{2(1 + K_2)}$	$R_{13} = A = \frac{K_1 - K_2 - 1}{2(1 + K_2)} > \frac{1}{K_2}$
a_2	$\frac{K_1 - K_2 - 1}{2(1 + K_2)}$	Range $K_2 < 1$
a_3	1.0	$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$
a_4	$1/K_2$	

Since R_{13} has no minimum in I, we must consider two subcases -

OR - 4A ($a_1 = a_4$, a_2, $a_3 = 1$)

From (4.33) we see that: $A = \frac{1}{K_2}$ Hence,

$$a_1 = a_4 = \frac{x}{K_2}$$

$$a_2 = \frac{1}{K_2}$$

$$a_3 = 1$$

From the ordering relationships, we see that:

$$a_1 = a_4 > a_2 \quad \rightarrow \quad x > 1$$

$$a_2 > 1 \quad \rightarrow \quad 1 > K_2$$

From the above inequalities we see that: $x > 1 > K_2$ (4.34)

and $A = \frac{1}{K_2} = \frac{-(x^2 - K_1 x + K_2)}{x^2 + (1 + K_2) x + K_2}$

Therefore $x > K_2$ is a root of:

$$x^2 (1 + K_2) + x (1 + K_2 - K_1K_2) + K_2(1 + K_2) = 0$$

or $$x^2 + x \frac{(1 + K_2 - K_1K_2}{1 + K_2} + K_2 = 0 \qquad (4.35)$$

As $x > K_2$ is the largest root of P_2, we can only have

$$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

Further, $R_{13} = \frac{x}{K_2} > 1$

<u>OR - 4B $(a_1, a_4, a_2 = a_3 = 1)$</u>

As $a_2 = a_3$ we see that $A = 1$. Element values now become:

$$a_1 = x$$

$$a_2 = a_3 = 1$$

$$a_4 = \frac{x}{K_2}$$

From the ordering relationships, we see that:

$$a_1 > a_4 \quad \rightarrow \quad K_2 > 1$$

$$a_4 > 1 \quad \rightarrow \quad x > K_2$$

Hence, $x > K_2 > 1$ (4.36)

As $A = 1$, we know that x is a root of P_2. As $x > K_2 > 1$ we should have that (see inequality 4.6):

$$K_1 > 3(1 + K_2)$$

Because $A = 1$, we also have

$$R_{13} = xA = x > K_2$$

A summary is given in Table 4.4.

Table 4.4

Summary of Case OR - 4

	OR-4A	Remarks
a_1	X/K_2	$R_{13} = x/K_2 > 1$
a_2	$1/K_2$	$x > 1 > K_2$ & $K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$
a_3	1	x is the largest root of P_2'-
a_4	x/K_2	

	OR-4B	Remarks
a_1	x	$R_{13} = x > K_2$
a_2	1	$x > K_2 > 1$ & $K_1 > 3(1 + K_2)$
a_3	1	
a_4	x/K_2	x is the largest root of P_2

<u>OR - 5: (a_1, a_3, a_4, a_2 = 1)</u> ; Ratio $R_{12} = x$

As $a_2 = 1 \rightarrow y = x$, element values become:

$$a_1 = x$$

$$a_2 = 1$$

$$a_3 = \frac{1}{A}$$

$$a_4 = \frac{x}{K_2 A}$$

Since R_{12} has no minimum in I, we must consider two subcases:

<u>OR - 5 A: ($a_1 = a_3$, a_4, $a_2 = 1$)</u>

From $a_1 = a_3$, we see that $x = \frac{1}{A}$. Hence,

$$a_1 = x = a_3$$

$$a_2 = 1$$

$$a_4 = \frac{x^2}{K_2}$$

From the ordering relationships we have that:

$$a_1 = a_3 > a_4 \rightarrow K_2 > x$$

$$a_4 > 1 \rightarrow x > \sqrt{K_2}$$

Hence, $K_2 > x > \sqrt{K_2} > 1$ (4.37)

From the equation $Ax = 1$ we see that x is a root of equation P_3.

From equations (4.37) and equation P_3 we obtain: (see equations (4.17) and (4.18)):

a) if $1 < K_2 < 7.0711$

$$\text{then } \frac{(1 + K_2)(2 + K_2)}{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

b) if $7.0711 < K_2$

$$\text{then } \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

<u>OR - 5 B (a_1, a_3, $a_4 = a_2 = 1$)</u>

This case is the same as Case OR - 2B and has already been covered.

Summary of Case OR - 5A is given in Table 4.5.

<u>OR - 6: (a_1, a_4, a_3, $a_2 = 1$)</u> ; Ratio $R_{12} = x$

As $a_2 = 1 \longrightarrow y = x$, element values become:

$$\left.\begin{aligned} a_1 &= x \\ a_2 &= 1 \\ a_3 &= \frac{1}{A} \\ a_4 &= \frac{1}{K_2 A} \end{aligned}\right\} \qquad (4.38)$$

As R_{12} has no minimum in I, we must consider two subcaues:

Table 4.5

Summary of Case OR - 5A

	OR-5A	Remarks
a_1	x	$K_2 > R_{12} = x > \sqrt{K_2} > 1$
a_2	1	x is a root of P_3 lying in interval
a_3	x	$[\sqrt{K_2}, K_2]$.
a_4	x^2/K_2	If a) $1 < K_2 < 7.0711$ then $\dfrac{(1 + K_2)(2 + K_2)}{K_2} < K_1 <$ $\dfrac{1 + 2\sqrt{K_2} + 3\,K_2}{K_2}$ b) $7.0711 < K_2$ then $\dfrac{1 + 2\sqrt{K_2} + 3\,K_2}{\sqrt{K_2}} < K_1 <$ $\dfrac{(1 + K_2)(2 + K_2)}{K_2}$

OR 6A ($a_1 = a_4, a_3, a_2 = 1$)

As now $K_2A = 1$, equations (4.38) become:

$a_1 = x$

$a_2 = 1$

$a_3 = K_2$

$a_4 = x$

From the ordering relationships we have that:

$a_1 = a_4 > a_3 \quad \rightarrow \quad x > K_2$

$a_3 > 1 \quad \rightarrow \quad K_2 > 1$.

Hence $x > K_2 > 1$ (4.39)

Since $K_2A = 1$, and $x > K_2$, x must be the largest root of P_2', and hence (4.8):

$$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

OR - 6B ($a_1, a_4, a_3 = a_2 = 1$)

This case is the same as Case OR - 4B and has already been covered.

Summary of Case OR - 6A is given in Table 4.6.

OR - 7: ($a_2, a_3, a_4, a_1 = 1$) ; Ratio $R_{21} = \frac{1}{x}$

As $a_1 = 1 \rightarrow y = 1$, element values become:

Table 4.6

Summary of Case OR - 6A

	OR-6A	Remarks
a_1	x	Ratio $R_{12} = x > K_2$ Range $K_2 > 1$ $K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$ with $x > K_2 > 1$ being the largest root of P_2'.
a_2	1	
a_3	K_2	
a_4	x	

$$\left.\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{x} \\ a_3 &= \frac{1}{xA} \\ a_4 &= \frac{1}{K_2 A} \end{aligned}\right\} \qquad (4.40)$$

Since R_{21} has no minimum in I, we must consider two subcases:

OR - 7A: ($a_2 = a_3$, a_4, $a_1 = 1$)

As now $A = 1$, equations (4.40) become:

$$a_1 = 1$$

$$a_2 = \frac{1}{x} = a_3$$

$$a_4 = \frac{1}{K_2}$$

From the ordering relationships we have that:

$$a_2 = a_3 > a_4 \rightarrow K_2 > x$$

$$a_4 > 1 \rightarrow 1 > K_2$$

Hence, $1 > K_2 > x$ (4.41)

As $A = 1$, and $x < K_2$, we must have that x is the smallest root of P_2, and that (4.6):

$$K_1 > 3\ (1 + K_2)$$

The ratio R_{21} now becomes: $R_{21} = \frac{1}{x} > \frac{1}{K_2}$

OR - 7B ($a_2, a_3, a_4 = a_1 = 1$)

As now $a_4 = a_1$ we obtain: $1 = AK_2$, equations (4.49) become:

$$a_1 = a_4 = 1$$

$$a_2 = \frac{1}{x}$$

$$a_3 = K_2/x$$

From the ordering relationships we have that:

$$a_2 > a_3 \rightarrow 1 > K_2$$

$$a_3 > 1 \rightarrow K_2 > x$$

Hence $1 > K_2 > x$ (4.42)

As $AK_2 = 1$ and $K_2 > x$, x must be the smallest root of P_2', and hence (4.8):

$$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

The ratio R_{21} now becomes: $R_{21} = \frac{1}{x} > \frac{1}{K_2}$

Summary of Case OR - 7 is given in Table 4.7.

OR - 8: ($a_2, a_4; a_3, a_1 = 1$) ; Ratio = $R_{21} = 1/x$

As $a_1 = 1 \rightarrow y = 1$, element values become:

$$a_1 = 1$$

$$a_2 = 1/x$$

Table 4.7

Summary of Cases OR - 7

Case OR - 7A

	OR-7A	Remarks
a_1	1.0	Ratio $R_{21} = 1/x > 1/K_2$ $1 > K_2 > x$ with $K_2 < 1$ & $K_1 > 3(1 + K_2)$ where x is the smaller root of P_2.
a_2	$1/x$	
a_3	$1/x$	
a_4	$1/K_2$	

Case OR - 7B

	OR-7B	Remarks
a_1	1.0	Ratio $R_{21} = 1/x > 1/K_2$ $K_2 < 1$ & $K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$ $1 > K_2 > x$ where x is the smaller root of P_2'.
a_2	$1/x$	
a_3	K_2/x	
a_4	1.0	

$a_3 = 1/xA$

$a_4 = 1/K_2A$

Since $R_{21} = 1/x$ has no minimum in I, we must consider two subcases:

<u>OR - 8A ($a_2 = a_4$, a_3, $a_1 = 1$)</u>

From $a_2 = a_4$, we see that $x = K_2A$. Hence,

$a_1 = 1$

$a_2 = a_4 = 1/x$

$a_3 = K_2/x^2$

From the ordering relationships, we have that:

$a_2 = a_4 > a_3 \rightarrow x > K_2$

$a_3 > 1 \rightarrow x < \sqrt{K_2}$

Hence, $1 > \sqrt{K_2} > x > K_2$ (4.43)

As $x = K_2A$, we know that x is a root of P_3'.

Consideration of inequality (4.43) and the fact that x is a root of P_3', levels to the case of equation (4.21) to conclude that:

$$3(1 + K_2) < K_1 < (1 + 2\sqrt{K_2} + 3K_2)/\sqrt{K_2}$$

Finally, from inequalities (4.43), we see that R_{12} is bounded as follows:

$$\frac{1}{\sqrt{K_2}} < R_{21} = \frac{1}{x} < \frac{1}{K_2}$$

OR - 8B: $(a_2, a_4, a_3 = a_1 = 1)$

From $a_3 = a_1$, we see that $Ax = 1$. Hence,

$a_1 = a_3 = 1$

$a_2 = 1/x$

$a_4 = x/K_2$

From the ordering relationships, we have that:

$a_2 > a_4 \rightarrow x > \sqrt{K_2}$

$a_4 > 1 \rightarrow x > K_2$

Hence,

$$K_2 < x < \sqrt{K_2} < 1 \tag{4.44}$$

As $Ax = 1$, we know that x is a root of P_3. Using, therefore, inequality (4.14) we have that:

$$\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

Finally, from inequalities (4.44), we see that R_{21} is bounded as follows:

$$\frac{1}{K_2} > R_{21} = \frac{1}{x} > \frac{1}{\sqrt{K_2}}$$

Summary of Cases OR - 8 is given in Table 4.8.

Table 4.8

Case OR - 8A		
	OR-8A	Remarks
a_1	1.0	$\frac{1}{\sqrt{K_2}} < R_{21} = \frac{1}{x} < \frac{1}{K_2}$ $K_2 < x < \sqrt{K_2} < 1$ where x is a root of P_3' with $3(1 + K_2) < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$
a_2	$1/x$	
a_3	K_2/x^2	
a_4	$1/x$	

Case OR - 8B		
	OR-8B	Remarks
a_1	1.0	$\frac{1}{\sqrt{K_2}} < R_{21} = \frac{1}{x} < \frac{1}{K_2}$ $K_2 < x < \sqrt{K_2} < 1$ where x is a root of P_3 with $\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$
a_2	$1/x$	
a_3	1.0	
a_4	x/K_2	

OR - 9: $(a_2, a_4, a_1, a_3 = 1)$; Ratio $= R_{23} = A$

As $a_3 = 1 \rightarrow y = xA$, element values become:

$a_1 = xA$

$a_2 = A$

$a_3 = 1$

$a_4 = \frac{x}{K_2}$

Since R_{21} has no minimum in I, we must consider two subcases:

OR - 9A $(a_2 = a_4, a_1, a_3 = 1)$

From $a_2 = a_4$ we obtain: $x = AK_2$. Hence,

$a_1 = \frac{x^2}{K_2}$

$a_2 = \frac{x}{K_2}$

$a_3 = 1$

$a_4 = \frac{x}{K_2}$

From the ordering relationships we have that:

$a_2 = a_4 > a_1 \rightarrow 1 > x$

$a_1 > 1 \rightarrow x > \sqrt{K_2}$

Hence,

$$1 > x > \sqrt{K_2} > K_2 \tag{4.45}$$

Since $x = AK_2$ we see that x is a root of equation P_3'.

From equations (4.45) and (4.19) we obtain (4.22):

$$\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

<u>OR - 9B: $(a_2, a_4, a_1 = a_3 = 1)$</u>

This case is the same as Case OR - 8B and has already been covered.

Summary of Case OR - 9A is given in Table 4.9.

<u>OR - 10: $(a_2, a_1, a_4, a_3 = 1)$</u> ; Ratio $R_{23} = A$

As $a_3 = 1 \rightarrow y = xA$. Element values become:

$a_1 = xA$

$a_2 = A$

$a_3 = 1$

$a_4 = x/K_2$

Since R_{23} has no minimum in I, we must consider two subcases:

<u>OR - 10A: $(a_2 = a_1, a_4, a_3 = 1)$</u>

This case is the same as Case OR-3A and has already been covered.

Table 4.9

<table>
<tr><th colspan="3">Case OR - 9A</th></tr>
<tr><th></th><th>OR-9A</th><th>Remarks</th></tr>
<tr><td>a_1</td><td>x^2/K_2</td><td rowspan="4">$\frac{1}{\sqrt{K_2}} < R_{23} = \frac{x}{K_2} < \frac{1}{K_2}$

$K_2 < \sqrt{K_2} < x < 1$ where x is a root of P_3', and

$\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$</td></tr>
<tr><td>a_2</td><td>x/K_2</td></tr>
<tr><td>a_3</td><td>1</td></tr>
<tr><td>a_4</td><td>x/K_2</td></tr>
</table>

OR - 10B: $(a_2, a_1, a_4 = a_3 = 1$

From $a_4 = a_3$ we see tha $x = K_2$. Hence,

$a_1 = K_2A$

$a_2 = A$

$a_3 = a_4 = 1$

From the ordering relationships we have that:

$a_2 > a_1 \rightarrow \quad 1 > K_2$

$a_1 > 1 \rightarrow \quad K_2 > \frac{1}{A}$

Hence, $\frac{1}{A} < K_2 < 1$

From $x = K_2$ we obtain:

$$A = \left.\frac{-(x^2 - K_1 x + K_2)}{x^2 + (1 + K_2)\cdot x + K_2}\right|_{x=K_2} = \frac{K_1 - 1 - K_2}{2(1 + K_2)}$$

We should also have that $K_2 > \frac{1}{A}$. Therefore

$$\frac{K_1 - 1 - K_2}{2(1 + K_2)} > \frac{1}{K_2} \rightarrow K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

Table 4.10 summarizes the results of Case OR - 10B

OR - 11: $(_2, a_3, a_1, a_4 = 1)$; Ratio: $R_{24} = \frac{K_2A}{x}$

As $a_4 = 1$ $\quad y = K_2A$, element values become:

$a_1 = K_2A$

$a_2 = K_2A/x$

Table 4.10

Case OR - 10B		
	OR-10B	Remarks
a_1	K_2A	Ratio $R_{23} = A = \frac{K_1 - 1 - K_2}{2(1 + K_2)} > \frac{1}{K_2}$ $K_2 < 1$ and $K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$
a_2	A	
a_3	1.0	
a_4	1.0	

$a_3 = K_2/x$

$a_4 = 1$

Since R_{24} has no minimum in I, we must consider two subcases.

<u>OR - 11 A: ($a_2 = a_3$, a_1, $a_4 = 1$)</u>

From $a_2 = a_3$, we see that A = 1. Hence,

$a_1 = K_2$

$a_2 = K_2/x = a_3$

$a_4 = 1$

From the ordering relationships we have that:

$a_2 = a_3 > a_1 \rightarrow 1 > x$

$a_1 > 1 \rightarrow K_2 > 1$

Hence, $K_2 > \sqrt{K_2} > 1 > x$

Since A = 1, and $K_2 > x$, x must be the smallest root of P_2 and we must have (4.6) that:

$K_1 > 3(1 + K_2)$.

From A = 1 and x < 1 we obtain:

$$R_{24} = \frac{K_2A}{x} = \frac{K_2}{x} > K_2$$

<u>OR - 11B (a_2, a_3, $a_1 = a_4 = 1$)</u>

This case is the same as Case OR - 7B and has already been covered.

Table 4.11 summarizes Case OR - 11A.

Table 4.11

Case OR - 11A		
	OR-11A	Remarks
a_1	K_2	Ratio $R_{24} = K_2A/x > K_2$
a_2	K_2/x	$K_2 > \sqrt{K_2} > 1 > x$, $K_1 > 3(1 + K_2)$
a_3	K_2/x	where x is the smallest root of P_2
a_4	1.0	

<u>OR - 12: (a_2, a_1, a_3, $a_4 = 1$)</u> ; Ratio $R_{24} = K_2A/x$

As $a_4 = 1 \rightarrow y = K_2A$, element values become:

$a_1 = K_2A$

$a_2 = K_2A/x$

$a_3 = K_2/x$

$a_4 = 1$

Since R_{24} has no minimum in I, we must consider two subcases:

<u>OR - 12A: ($a_2 = a_1$, a_3, $a_4 = 1$)</u>

This case is the same as Case OR - 1A and has already been covered.

<u>OR - 12B: (a_2, a_1, $a_3 = a_4 = 1$)</u>

This case is the same as Case OR - 10B, and has already been covered.

<u>OR - 13: (a_3, a_4, a_1, $a_2 = 1$)</u> ; Ratio $R_{32} = 1/A$

As $a_2 = 1 \rightarrow y = x$, element values become:

$$\left.\begin{aligned} a_1 &= x \\ a_2 &= 1 \\ a_3 &= 1/A \\ a_4 &= x/K_2A \end{aligned}\right\} \qquad (4.46)$$

From the ordering relationships we see that:

$K_2 > x > 1$

We have found previously that R_{32} has a minimum in I for $x = \sqrt{K_2}$ and that:

$$R_{32} \text{ minimum} \equiv \frac{1}{A(\sqrt{K_2})} = \frac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$$

Using the values $x = \sqrt{K_2}$ and $A = \dfrac{K_1 - 2\sqrt{K_2}}{(1 + \sqrt{K_2})^2}$ in conjunction with equations (4.46) we obtain:

$$a_1 = \sqrt{K_2}$$

$$a_2 = 1$$

$$a_3 = \frac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$$

$$a_4 = \frac{(1 + \sqrt{K_2})^2}{\sqrt{K_2}(K_1 - 2\sqrt{K_2})}$$

From the ordering relationships we have that:

$$a_3 > a_4 \quad \rightarrow \quad K_2 > 1$$

$$a_4 > a_1 \quad \rightarrow \quad K_1 < \frac{(1 + K_2)(1 + 2\sqrt{K_2})}{K_2}$$

$$a_1 > 1 \quad \rightarrow \quad K_2 > 1$$

We know from inequality (4.3d) that $2\sqrt{K_2} < K_1$. Therefore we obtain as limits of K_1:

$$2\sqrt{K_2} < K_1 < \frac{(1 + K_2)(1 + 2\sqrt{K_2})}{K_2}$$

Table 4.12 summarizes Case OR - 13

Table 4.12

<table>
<tr><th colspan="3">Case OR - 13</th></tr>
<tr><th></th><th>OR-13</th><th>Remarks</th></tr>
<tr><td>a_1</td><td>$\sqrt{K_2}$</td><td rowspan="4">Ratio $(R_{32})_{min} = \dfrac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$
$K_2 > 1$
$2\sqrt{K_2} < K_1 < \dfrac{(1 + K_2)(1 + 2\sqrt{K_2})}{K_2}$</td></tr>
<tr><td>a_2</td><td>1.0</td></tr>
<tr><td>a_3</td><td>$(1+\sqrt{K_2})^2/(K_1-2\sqrt{K_2})$</td></tr>
<tr><td>a_4</td><td>$(1+\sqrt{K_2})^2/\sqrt{K_2}(K_1-2\sqrt{K_2})$</td></tr>
</table>

<u>OR - 14 (a_3, a_1, a_4, a_2 = 1)</u> ; Ratio $R_{32} = 1/A$

As $a_2 = 1 \rightarrow y = x$, element values become:

$$\left.\begin{aligned} a_1 &= x \\ a_2 &= 1 \\ a_3 &= 1/A \\ a_4 &= x/K_2A \end{aligned}\right\} \qquad (4.47)$$

We have found previously that R_{32} has a minimum in I for $x = \sqrt{K_2}$ and that:

$$R_{32\text{minimum}} = \frac{1}{A(\sqrt{K_2})} = \frac{(1+\sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$$

Using the values $x = \sqrt{K_2}$ and $A = \dfrac{K_1 - 2\sqrt{K_2}}{(1+\sqrt{K_2})^2}$ in conjunction with equations (4.47) we obtain:

$$a_1 = \sqrt{K_2}$$

$$a_2 = 1$$

$$a_3 = \frac{(1+\sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$$

$$a_4 = \frac{(1+\sqrt{K_2})^2}{\sqrt{K_2}(K_1 - 2\sqrt{K_2})}$$

From the ordering relationships we have that:

$$a_3 > a_1 \rightarrow K_1 < (1 + 2\sqrt{K_2} + 3K_2)/\sqrt{K_2}$$

$$a_1 > a_4 \rightarrow K_1 > (1 + K_2)(1 + 2\sqrt{K_2})/K_2$$

$$a_4 > 1 \rightarrow K_1 < (1 + 2\sqrt{K_2} + 3K_2)/\sqrt{K_2}$$

Therefore: $$\frac{(1 + K_2)(1 + 2\sqrt{K_2})}{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

Finally, $a_1 > 1 \rightarrow K_2 > 1$

Case OR - 14 is summarized in Table 4.13.

OR - 15: $(a_3, a_1, a_2; a_4 = 1)$; Ratio $R_{34} = K_2/x$

As $a_4 = 1 \rightarrow y = K_2A$, element values become:

$a_1 = K_2A$

$a_2 = K_2A/x$

$a_3 = K_2/x$

$a_4 = 1.0$

Since R_{34} has no minimum in I, we must consider two subcases:

OR - 15A: $(a_3 = a_1, a_2, a_4 = 1)$

This case is the same as Case OR - 2A, and has already been covered.

OR - 15B $(a_3, a_1, a_2 = a_4 = 1)$

Now $a_2 = a_4 \rightarrow x = K_2A$. Hence,

Table 4.13

Case OR - 14		
	OR-14	Remarks
a_1	$\sqrt{K_2}$	Ratio $(R_{32})_{min} = \dfrac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$ $K_2 > 1$ $\dfrac{(1 + K_2)(1 + 2\sqrt{K_2})}{K_2} < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$
a_2	1	
a_3	$(1+\sqrt{K_2})^2/(K_1-2\sqrt{K_2})$	
a_4	$(1+\sqrt{K_2})^2/\sqrt{K_2}(K_1-2\sqrt{K_2})$	

$$a_1 = x$$

$$a_2 = 1.0$$

$$a_3 = 1/A$$

$$a_4 = 1.0$$

From the ordering relationships we have that:

$$a_3 > a_1 \rightarrow 1/A > x$$

$$a_1 > 1 \rightarrow x > 1$$

Now $\frac{1}{A} = \frac{K_2}{x} \rightarrow \frac{K_2}{x} > x \rightarrow x = \sqrt{K_2}$.

Hence, $1 < x < \sqrt{K_2} < K_2$ (4.48)

As $x = K_2A$, we know that x is a root of P_3'. Because $1 < x < \sqrt{K_2}$, we should also have (4.23 and 4.24):

$$\frac{(1 + K_2)(2 + K_2)}{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

$x \; \varepsilon (1, \sqrt{K_2})$ $\qquad 1 < K_2 < 7.0711$

$$\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

$x \; \varepsilon (1, \sqrt{K_2})$ $\qquad K_2 > 7.0711$

From inequalities (4.48), we see that:

$$\sqrt{K_2} < R_{34} = \frac{K_2}{x} < K_2$$

A summary of Case OR - 15B is given in Table 4.14.

Table 4.14

Case OR - 15B		
	OR-15B	Remarks
a_1	x	$\sqrt{K_2} < R_{34} = K_2/x < K_2$
a_2	1	x is a root of P_3' lying in the interval: $[1, \sqrt{K_2}]$. If
a_3	K_2/x	a) $1 < K_2 < 7.0711$ then
a_4	1	$\frac{(1 + K_2)(2 + K_2)}{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$ b) $7.0711 < K_2$ then $\frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$

OR - 16: $(a_3, a_2, a_1, a_4 = 1)$; Ratio $R_{34} = K_2/x$

As $a_4 = 1 \rightarrow y = K_2x$, element values become:

$a_1 = K_2A$

$a_2 = K_2A/x$

$a_3 = K_2/x$

$a_4 = 1.0$

Since R_{34} has no minimum in I, we must consider two subcases:

OR - 16 A: $(a_3 = a_2, a_1, a_4 = 1)$

This case is the same as Case OR - 11A, which has already been covered.

OR - 16 B: $(a_3, a_2, a_1 = a_4 = 1)$

From $a_1 = a_4$, we obtain: $1 = K_2A$. Hence,

$a_1 = 1.0$

$a_2 = 1/x$

$a_3 = K_2/x$

$a_4 = 1.0$

From the ordering relationships we see that:

$a_3 > a_2 \rightarrow K_2 > 1$

$a_2 > a_1 = 1 \rightarrow x < 1$

Hence,

$$K_2 > \sqrt{K_2} > 1 > x$$

Since $AK_2 = 1$, and $x < K_2$, x must be the smallest root of P_2' and therefore we must have (4.8):

$$K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$$

From $x < 1$ we obtain:

Ratio $R_{34} = K_2/x > K_2$

Summary of Case OR - 16B is given in Table 4.15.

<u>OR - 17: (a_3, a_4, a_2, a_1 = 1)</u> ; Ratio $R_{31} = \frac{1}{xA}$

As $a_1 = 1 \rightarrow y = 1$, element values become:

$a_1 = 1.0$

$a_2 = 1/x$

$a_3 = 1/xA$

$a_4 = 1/K_2A$

We know from previous study (see section 4.2.2) that R_{31} possesses a minimum for some value of $x > \sqrt{K_2}$.

From the ordering relationships we have that:

$$\frac{1}{xA} > \frac{1}{K_2A} > \frac{1}{x} > 1.0 \qquad (4.49)$$

Table 4.15

Case OR - 16B		
	OR-16B	Remarks
a_1	1.0	$R_{34} = K_2/x > K_2$
a_2	$1/x$	$K_2 > 1$ & $K_1 > \frac{(1 + K_2)(2 + K_2)}{K_2}$
a_3	K_2/x	and $K_2 > \sqrt{K_2} > 1 > x$, where x is the
a_4	1.0	smaller root of P_2'

Hence, $x < 1$ and also $x < K_2$. Therefore $x < 1 < \sqrt{K_2} < K_2$ or $x < K_2 < \sqrt{K_2} < 1$. In either case the minimum of R_{31} which occurs for some $x > \sqrt{K_2}$ is unacceptable, and we therefore consider two subcases:

<u>OR - 17 A: ($a_3 = a_4$, a_2, $a_1 = 1$)</u>

From $a_3 = a_4$ we obtain: $x = K_2$. Hence,

$a_1 = 1.0$

$a_2 = 1/K_2$

$a_3 = 1/K_2A$

$a_4 = 1/K_2A$

where $A(K_2) = \dfrac{K_1 - 1 - K_2}{2(1 + K_2)}$

From the ordering relationships we have that:

$$a_3 = a_4 > a_2 \rightarrow 1 > A = \frac{K_1 - K_2 - 1}{2(1 + K_2)} \rightarrow K_1 < 3(1 + K_2)$$

$$a_2 > 1 \quad \rightarrow \quad 1 > K_2$$

And ratio $R_{31} = 1/xA = \dfrac{2(1 + K_2)}{K_2(K_1 - K_2 - 1)} > \dfrac{1}{K_2}$

<u>OR - 17 B (a_3, a_4, $a_2 = a_1 = 1$)</u>

Since $a_1 = a_2$, we have that $x = 1$. Hence,

$a_1 = 1.0$

$a_2 = 1.0$

$a_3 = 1/A$

$a_4 = 1/K_2A$

with $A(1) = \dfrac{K_1 - 1 - K_2}{2(1 + K_2)}$

From the ordering relationships we have that:

$$a_3 > a_4 \quad \rightarrow \quad K_2 > 1$$

$$a_4 > 1 \quad \rightarrow \quad K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

We have also: $R_{31} = \dfrac{1}{xA} = \dfrac{2(1 + K_2)}{K_1 - 1 - K_2} > K_2$

A summary of Cases OR - 17 is given in Table 4.16.

<u>OR - 18 : (a_3, a_2, a_4, a_1 = 1)</u> ; Ratio $R_{31} = \dfrac{1}{Ax}$

As $a_1 = 1 \rightarrow y = 1$, element values become:

$a_1 = 1$

$a_2 = 1/x$

$a_3 = 1/xA$

$a_4 = 1/K_2A$

Table 4.16

Case OR - 17A		
	OR-17A	Remarks
a_1	1.0	$R_{31} = 1/xA = \dfrac{2(1+K_2)}{(K_1 - 1 - K_2)\ K_2} > \dfrac{1}{K_2}$
a_2	$1/K_2$	
a_3	$1/K_2A$	$K_2 < 1$ and $2\sqrt{K_2} < K_1 < 3(1+K_2)$
a_4	$1/K_2A$	$A(K_2) = \dfrac{K_1 - 1 - K_2}{2(1+K_2)}$

Case OR - 17B		
	OR-17B	Remarks
a_1	1	$R_{31} = 1/xA = \dfrac{2(1+K_2)}{K_1 - 1 - K_2} > K_2$
a_2	1	
a_3	$1/A$	$K_2 > 1$ & $2\sqrt{K_2} < K_1 < \dfrac{(1+K_2)(2+K_2)}{K_2}$
a_4	$1/K_2A$	$A(1) = \dfrac{K_1 - 1 - K_2}{2(1+K_2)}$

We know from previous study that R_{31} has a minimum value for $x > \sqrt{K_2}$ where $x \in I$. (See section 4.2.2)

From the ordering relationships we have that:

$$a_3 > a_4 \quad \rightarrow \quad K_2 > x$$

$$a_2 > 1 \quad \rightarrow \quad 1 > x$$

Hence we obtain two possibilities:

(1) $K_2 > \sqrt{K_2} > 1 > x$

(2) $1 > \sqrt{K_2} > K_2 > x$

In both cases we see that $x < \sqrt{K_2}$, therefore the minimum of R_{31} when $x > \sqrt{K_2}$ is an unacceptable value.

Since R_{31} has no acceptable minimum, we must consider two subcases:

<u>OR - 18 A ($a_3 = a_2$, a_4, $a_1 = 1$)</u>

This case is the same as Case OR - 7A, which has already been analyzed.

<u>OR - 18 B (a_3, a_2, $a_4 = a_1 = 1$)</u>

This case is the same as Case OR - 16B, which has already been analyzed. Like both subcases of Case OR - 18 can be omitted, we can omit Case OR - 18 entirely.

OR - 19 : (a_4, a_1, a_2, a_3 = 1) ; Ratio $R_{43} = \frac{x}{K_2}$

As $a_3 = 1 \rightarrow y = xA$, element values become:

$a_1 = xA$

$a_2 = A$

$a_3 = 1.0$

$a_4 = x/K_2$

Since R_{43} has no minimum in I, we must consider two subcases:

OR - 19 A ($a_4 = a_1$, a_2, a_3 = 1)

This case is the same as Case OR - 4A, and has already been covered.

OR - 19 B (a_4, a_1, $a_2 = a_3 = 1$)

Now $a_2 = a_3 \rightarrow A = 1$. Hence,

$a_1 = x$

$a_2 = 1.0$

$a_3 = 1.0$

$a_4 = x/K_2$

From the ordering relationships we see that:

$$a_4 > a_1 \quad \rightarrow \quad 1 > K_2$$

$$a_1 > 1 \quad \rightarrow \quad x > 1$$

Hence,

$$x > 1 > \sqrt{K_2} > K_2$$

Since $A = 1$, and $x > K_2$, x is the largest root of P_2 and therefore we must have (see inequality 4.6):

$$K_1 > 3(1 + K_2)\ .$$

From $x > 1$ we obtain:

$$\text{Ratio } R_{43} = x/K_2 > \frac{1}{K_2}$$

Summary of Case OR - 19B is given in Table 4.17.

<u>OR - 20 : (a_4, a_2, a_1, a_3 = 1)</u> ; Ratio $R_{43} = \frac{x}{K_2}$

As $a_3 = 1 \quad \rightarrow \quad y = x_2A$, element values become:

$a_1 = xA$

$a_2 = A$

$a_3 = 1.0$

$a_4 = x/K_2$

Since R_{43} has no minimum in I, we must consider two subcases:

<u>OR - 20 A ($a_4 = a_2$, a_1, $a_3 = 1$)</u>

This case is the same as Case OR - 9A, which has already been covered.

Table 4.17

Case OR - 19B		
	OR-19B	Remarks
a_1	x	$R_{43} = x/K_2 > 1/K_2$
a_2	1.0	$x > 1 > \sqrt{K_2} > K_2$ and $K_1 > 3(1 + K_2)$
a_3	1.0	x is the larger root of P_2
a_4	x/K_2	

<u>OR - 20 B (a_4, a_2, $a_1 = a_3 = 1$)</u>

From $a_1 = a_3$, we obtain: $Ax = 1$. Hence,

$a_1 = 1$

$a_2 = \frac{1}{x}$

$a_3 = 1.0$

$a_4 = x/K_2$

From the ordering relationships we have that:

$a_4 > a_2 \rightarrow x > \sqrt{K_2}$

$a_2 > 1.0 \rightarrow 1 > x$

Hence,

$1 > x > \sqrt{K_2} > K_2$

Further, since $Ax = 1$, we conclude that x must be the smallest root of P_3, and hence (4.15) that

$$3(1 + K_2) < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

From $x < 1$ we obtain: Ratio $R_{43} = x/K_2 < 1/K_2$

A summary of Case OR - 20B is given in Table 4.18.

<u>OR - 21 (a_4, a_3, a_1, $a_2 = 1$)</u> ; Ratio $R_{42} = \frac{x}{AK_2}$

As $a_2 = 1 \rightarrow y = x$, element values become:

$a_1 = x$

$a_2 = 1.0$

Table 4.18

Case OR - 20B		
	OR-20B	Remarks
a_1	1	Ratio $R_{43} = x/K_2 < 1/K_2$
a_2	$1/x$	x is a root of P_3 in the interval
a_3	1	$[\sqrt{K_2}, 1]$.
a_4	x/K_2	with $K_2 < 1$ and
		$3(1 + K_2) < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$

$a_3 = 1/A$

$a_4 = x/K_2A$

We have seen previously that R_{42} has a minimum for $x < \sqrt{K_2}$ such that $x \in I$. (See section 4.2.2)

From the ordering relationships we have

$a_1 > 1 \rightarrow x > 1$

$a_4 > a_3 \rightarrow x > K_2$

There are two possible cases:

(1) $x > K_2 > \sqrt{K_2} > 1$

(2) $x > 1 > \sqrt{K_2} > K_2$

In either case $x > \sqrt{K_2}$, which means that the minimum for R_{42} cannot be utilized since it exists only for the range $x < \sqrt{K_2}$. We must therefore conclude that R_{42} has no minimum in the acceptable range of x and proceed to consider the following two subcases.

OR - 21 A ($a_4 = a_3$, a_1, $a_2 = 1$)

From $a_4 = a_3$ we obtain $x = K_2$. Hence,

$a_1 = K_2$

$a_2 = 1.0$

$a_3 = 1/A$

$a_4 = 1/A$

From the ordering relationships we should have:

$$a_4 = a_3 > a_1 \rightarrow A < \frac{1}{K_2}$$

$$a_1 > 1 \rightarrow K_2 > 1$$

Because $x = K_2$ we obtain:

$$A(K_2) = \frac{K_1 - 1 - K_2}{2(1 + K_2)}$$

From inequality $A < \frac{1}{K_2}$, we also have that:

$$A = \frac{K_1 - 1 - K_2}{2(1 + K_2)} < \frac{1}{K_2} \rightarrow K_1 < \frac{(1 + K_2)(2 + K_2)}{K_2}$$

From $x = K_2$ we obtain:

$$R_{42} = \frac{2(1 + K_2)}{K_1 - 1 - K_2} > K_2$$

<u>OR - 21 B (a_4, a_3, $a_1 = a_2 = 1$)</u>

From $a_1 = a_2$ we obtain $x = 1$. Hence,

$$a_1 = 1$$

$$a_2 = 1.0$$

$$a_3 = 1/A$$

$$a_4 = \frac{1}{K_2 A}$$

Note that $x = 1 \rightarrow A(1) = \frac{K_1 - 1 - K_2}{2(1 + K_2)}$

From the ordering relationships we have:

$a_4 > a_3 \rightarrow 1 > K_2$

$a_3 > 1 \rightarrow 1 > A$

From the combination of the conditions $x = 1$ and $1 > A$ we obtain:

$K_1 < 3(1 + K_2)$

Since $x = 1$ and $K_1 < 3(1 + K_2)$ we obtain $R_{42} = \frac{x}{AK_2} > \frac{1}{K_2}$

A summary of Cases OR - 21A and OR - 21B are given in Table 4.19.

<u>OR - 22 :(a_4, a_1, a_3, a_2 = 1)</u> ; Ratio $R_{41} = \frac{1}{K_2 A}$

As $a_2 = 1 \rightarrow y = x$, element values become:

$a_1 = x$

$a_2 = 1.0$

$a_3 = 1/A$

$a_4 = x/K_2A$

We have seen previously that R_{41} has a minimum for $x = \sqrt{K_2}$ - (see section 4.2.2). From the ordering relationships we obtain:

$$\left.\begin{aligned} a_1 > 1 &\rightarrow x > 1 \\ a_4 > a_3 &\rightarrow x > K_2 \end{aligned}\right\} \qquad (4.50)$$

Table 4.19

Case OR - 21A		
	OR-21A	Remarks
a_1	K_2	$R_{42} = x/K_2A = \dfrac{2(1 + K_2)}{K_1 - 1 - K_2} > K_2$ $K_2 > 1$ and $2\sqrt{K_2} < K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2}$
a_2	1.0	
a_3	$\dfrac{2(1 + K_2)}{K_1 - 1 - K_2}$	
a_4	$\dfrac{2(1 + K_2)}{K_1 - 1 - K_2}$	

Case OR - 21B		
	OR-21B	Remarks
a_1	1.0	$R_{42} = x/K_2A = \dfrac{2(1 + K_2)}{K_2(K_1 - 1 - K_2)} > \dfrac{1}{K_2}$ $K_2 < 1$ and $2\sqrt{K_2} < K_1 < 3(1 + K_2)$
a_2	1.0	
a_3	$\dfrac{2(1 + K_2)}{K_1 - 1 - K_2}$	
a_4	$\dfrac{2(1 + K_2)}{K_2(K_1 - 1 - K_2)}$	

From the above inequalities we obtain two possibilities:

(1) $x > K_2 > \sqrt{K_2} > 1$ (4.51)

(2) $x > 1 > \sqrt{K_2} > K_2$ (4.52)

Inequalities (4.51) and (4.52) both satisfy inequalities (4.50), but neither of the inequalities (4.51) and (4.52) can be satisfied by $x = \sqrt{K_2}$ unless $x = K_2 = \sqrt{K_2} = 1$ which is a particular case. Since R_{41} has no acceptable minimum, we must consider two subcases:

OR - 22 A: ($a_4 = a_1$, a_3, $a_2 = 1$)

This case is the same as Case OR - 6A, which has already been analyzed.

OR - 22 B: (a_4, a_1, $a_3 = a_2 = 1$)

This case is the same as Case OR - 19 B, which has already been analyzed.

OR - 23 : (a_4, a_2, a_3, $a_1 = 1$) ; Ratio $R_{41} = \frac{1}{K_2 A}$

As $a_1 = 1 \rightarrow y = 1$, element values become:

$a_1 = 1$

$a_2 = \frac{1}{x}$

$a_3 = \frac{1}{xA}$

$$a_4 = \frac{1}{K_2 A}$$

$R_{41} = 1/K_2$ has a minimum for $x = \sqrt{K_2}$ such that

$$R_{41} = \frac{(1 + \sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})}$$

We know from previous study (see section 4.2.2) that $x = \sqrt{K_2}$ implies that $x \varepsilon I$.

From the ordering relationships we see:

$$\frac{1}{K_2 A} > \frac{1}{x} > \frac{1}{xA} > 1.0$$

From the above inequalities we obtain:

$1 > x$ and $x > K_2$. Therefore:

$1 > x > K_2$

We now note that $x = \sqrt{K_2}$ lies in this range.

We now establish bounds for K_1. We have that:

$$\frac{1}{K_2 A} > \frac{1}{x} \rightarrow \frac{(1 + \sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})} > \frac{1}{\sqrt{K_2}} \rightarrow K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$

and

$$\frac{1}{x} > \frac{1}{xA} \rightarrow 1 > \frac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}} \rightarrow K_1 > (1 + \sqrt{K_2})^2 + 2\sqrt{K_2}$$

Bounds on the minimum value of R_{41} may also be found and are:

$$\frac{1}{\sqrt{K_2}} < R_{41} = \frac{(1 + \sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})} < \frac{1}{K_2}$$

Case OR - 23 is summarized in Table 4.20.

OR - 24 : $(a_4, a_3, a_2, a_1 = 1)$; Ratio $R_{41} = 1/K_2A$

As $a_1 = 1 \rightarrow y = 1$, element values become:

$a_1 = 1.0$

$a_2 = 1/x$

$a_3 = 1/xA$

$a_4 = 1/K_2A$

We have seen previously that R_{41} has a minimum for $x = \sqrt{K_2}$. From the ordering relationships we obtain:

$$\left.\begin{array}{lll} a_4 > a_3 & \rightarrow & x > K_2 \\ a_3 > a_2 & \rightarrow & 1 > A \\ a_2 > 1 & \rightarrow & 1 > x \end{array}\right\} \qquad (4.53)$$

From inequalities (4.53) we obtain $1 > x > K_2$. Hence $x = \sqrt{K_2}$ falls into this interval, where:

Table 4.20

<table>
<tr><th colspan="3">Case OR - 23</th></tr>
<tr><th></th><th>OR-23</th><th>Remarks</th></tr>
<tr><td>a_1</td><td>1.0</td><td rowspan="4">$$\frac{1}{\sqrt{K_2}} < R_{41} = \frac{(1+\sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})} < \frac{1}{K_2}$$
$K_2 < 1$ and
$$(1+\sqrt{K_2})^2 + 2\sqrt{K_2} < K_1 < \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}}$$</td></tr>
<tr><td>a_2</td><td>$1/\sqrt{K_2}$</td></tr>
<tr><td>a_3</td><td>$\frac{(1+\sqrt{K_2})^2}{\sqrt{K_2}(K_1-2\sqrt{K_2})}$</td></tr>
<tr><td>a_4</td><td>$\frac{(1+\sqrt{K_2})^2}{K_2(K_1-2\sqrt{K_2})}$</td></tr>
</table>

$$A(\sqrt{K_2}) = \frac{K_1 - 2\sqrt{K_2}}{(1 + \sqrt{K_2})^2}$$

As $a_3 > a_2 \rightarrow 1 > A$, we have that:

$$1 > \frac{K_1 - 2\sqrt{K_2}}{(1 + \sqrt{K_2})^2} \rightarrow K_1 < (1 + \sqrt{K_2})^2 + 2\sqrt{K_2}$$

The lower limit of K_1 is necessarily $2\sqrt{K_2}$.

Case OR - 24 is summarized in Table 4.21.

4.4 Derivation of Optimum Solution

In the previous section, we have examined all possible orderings with respect to magnitude of the section scaling co-efficients a_1, a_2, a_3 and a_4. In each case we have determined the bounds of K_1 and K_2 and the variable x, which would permit the existence of such an ordering. The ratio of the maximum a_i to the minimum a_j was then examined. If the ratio possessed a minimum for a value of x satisfying the ordering relationships, this value was adopted as a possible solution. Otherwise, the monotonic behaviour of the ratio with respect to x, within the confines imposed by the ordering relationships led to the conclusion that the ratio must be smallest

Table 4.21

Case OR - 24		
	OR-24	Remarks
a_1	1.0	$R_{41} = 1/K_2A = \frac{(1+\sqrt{K_2})^2}{K_2(K_1 - 2\sqrt{K_2})} > \frac{1}{K_2}$ $K_2 < 1$ and $2\sqrt{K_2} < K_1 < (1+\sqrt{K_2})^2 + 2\sqrt{K_2}$
a_2	$1/\sqrt{K_2}$	
a_3	$\frac{(1+\sqrt{K_2})^2}{\sqrt{K_2}(K_1-2\sqrt{K_2})}$	
a_4	$\frac{(1+\sqrt{K_2})^2}{K_2(K_1-2\sqrt{K_2})}$	

at the limit points of the ordering relationships, viz, either the two largest, or the two smallest a_i's must be equal.

In carrying out this process, we have established several possible solutions to the minimization problem for various ranges of K_1 and K_2. In this section we will compare these possibilities, and obtain the true solution to the minimization problem.

4.4.1 Optimum Solutions for $K_2 < 1$

An examination of the previous section reveals that there are five distinct ranges, R_1 to R_5, for K_1, and these are listed in Table 4.22. The relevant results from the pertinent orderings of the a_i's are also listed there.

The x marks indicate the ranges of K_1 for which each ordering is applicable, and the circled x marks show the optimum solution for each range, as derived below.

We now begin to compare for each range of K_1, the minimized maximum ratios obtained.

Table 4.22

Optimum Solutions for $K_2 < 1$

$0 < K_2 < 1$

a) $R_1 = \left\{ K_1 : 2\sqrt{K_2} < K_1 < (1 + \sqrt{K_2})^2 + 2\sqrt{K_2} \right\}$

b) $R_2 = \left\{ K_1 : (1 + \sqrt{K_2})^2 + 2\sqrt{K_2} < K_1 < 3(1 + K_2) \right\}$

c) $R_3 = \left\{ K_1 : 3(1 + K_2) < K_1 < \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \right\}$

d) $R_4 = \left\{ K_1 : \dfrac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} < K_1 < \dfrac{(1 + K_2)(2 + K_2)}{K_2} \right\}$

e) $R_5 = \left\{ K_1 : K_1 > \dfrac{(1 + K_2)(2 + K_2)}{K_2} \right\}$

Case	R_1	R_2	R_3	R_4	R_5	$R_{minimum}$	
OR-3A					x	$R_{13} = \dfrac{K_1 - K_2 - 1}{2(1+K_2)} > \dfrac{1}{K_2}$	
OR-4A					x	$R_{13} = \dfrac{x}{K_2} > \dfrac{1}{K_2}$	x is largest root of P_2'
OR-7A			x	x	x	$R_{21} = \dfrac{1}{x} > \dfrac{1}{K_2}$	x is smallest root of P_2
OR-7B					x	$R_{21} = \dfrac{1}{x} > \dfrac{1}{K_2}$	x is smallest root of P_2'
OR-8A			x			$R_{21} = \dfrac{1}{x} < \dfrac{1}{K_2}$	x is a root of P_3' such that $K_2 < x < \sqrt{K_2} < 1$
OR-8B				x		$R_{21} = \dfrac{1}{x} < \dfrac{1}{K_2}$	x is a root of P_3 such that $K_2 < x < \sqrt{K_2} < 1$

Table 4.22 (continued)

Case	R_1	R_2	R_3	R_4	R_5	$R_{minimum}$
OR-9A				x		$\frac{1}{\sqrt{K_2}} < R_{23} = \frac{x}{K_2} < \frac{1}{K_2}$ x is a root of P3' such that $K_2 < \sqrt{K_2} < x < 1$
OR-10B					x	$R_{23} = \frac{x}{K_2} = \frac{K_1-1-K_2}{2(1+K_2)} > \frac{1}{K_2}$
OR-17A	x	x				$R_{31} = \frac{1}{xA} = \frac{2(1+K_2)}{K_1-1-K_2} > \frac{1}{K_2}$
OR-19B			x	x	x	$R_{43} = \frac{x}{K_2} > \frac{1}{K_2}$ x is the largest root of P_2 such that $x > 1 > K_2$
OR-20B			x			$R_{43} = \frac{x}{K_2} < \frac{1}{K_2}$ x is a root of P_3 such that $K_2 < \sqrt{K_2} < x < 1$
OR 21B	x	x				$R_{42} = \frac{x}{K_2A} = \frac{2(1+K_2)}{K_2(K_1-1-K_2)} > \frac{1}{K_2}$
OR-23		x	x			$R_{41} = \frac{(1+\sqrt{K_2})^2}{K_2(K_1-2\sqrt{K_2})} < \frac{1}{K_2}$
OR-24	x					$R_{41} = \frac{(1+\sqrt{K_2})^2}{K_2(K_1-2\sqrt{K_2})} > \frac{1}{K_2}$

a) $K_2 < 1$ with $K_1 \varepsilon R_1$

It is obvious that cases OR-17A and OR-21B are equivalent. We also know from previous study that:

$$A(1) = A(K_2)$$

From Case OR-24, we see that:

$$R_{41} = \frac{1}{K_2 A(x_b)} \quad \text{where } x_b = \sqrt{K_2} \text{ minimizes } \frac{1}{K_2 A(x)}$$

It follows that:

$$\underset{\text{OR-24}}{(R_{41})} = \frac{1}{K_2 A(\sqrt{K_2})} < \underset{\text{OR-21B}}{(R_{42})} = \frac{1}{K_2 A(1)} = \frac{1}{K_2 A(K_2)} = \underset{\text{OR-17A}}{(R_{31})}$$

Hence OR-24 is the optimum solution for this range of K_1.

b) $K_2 < 1$ with $K_1 \varepsilon R_2$

Proceeding as above, we obtain:

$$\underset{\text{OR-23}}{(R_{41})} = \frac{1}{K_2 A(\sqrt{K_2})} < \underset{\text{OR-21B}}{(R_{42})} = \frac{1}{K_2 A(1)} = \frac{1}{K_2 A(K_2)} = \underset{\text{OR-17A}}{(R_{31})}$$

Hence OR-23 is the optimum solution for this range of K_1.

c) $K_2 < 1$ with $K_1 \varepsilon R_3$.

It is obvious that cases OR-8A and OR-20B are equivalent.

as are cases OR - 7A and OR - 19B. From Table 4.24 we have that:

$$\underset{\text{OR-8A}}{(R_{21})} = \frac{1}{x_a} < \frac{1}{K_2} < \underset{\text{OR-7A}}{(R_{21})} = \frac{1}{x_b}$$

Therefore Cases OR - 8A and OR - 20B are superior to Cases OR - 7A and OR - 19B.

From OR - 8A, we see that $R_{21} = \frac{1}{x_a}$ is a solution of P_3', where P_3' arises from the condition:

$$\frac{1}{K_2A(x_a)} = \frac{1}{x_a}$$

From OR - 23, we see that $R_{41} = \frac{1}{K_2A(x_b)}$, where $x_b = \sqrt{K_2}$ minimizes $\frac{1}{K_2A(x)}$

It therefore follows that:

$$R_{41} = \frac{1}{K_2A(\sqrt{K_2})} < \frac{1}{K_2A(x_a \neq \sqrt{K_2})} = \frac{1}{x_a} = R_{21}$$

Hence, OR - 23 is the optimum solution for this range of K_1.

d) $K_2 < 1$ with $K_1 \varepsilon R_4$

It is obvious that Cases OR - 7A and OR - 19B are equivalent, as are Cases OR - 8B and OR - 9A. We also know that:

$$\underset{\text{OR-8B}}{(R_{21})} = \underset{\text{OR-8B}}{\frac{1}{x}} < \frac{1}{K_2} < \underset{\text{OR-7A}}{\frac{1}{x}} = \underset{\text{OR-7A}}{(R_{21})}$$

Hence, OR - 8B or OR - 9A is the optimum solution for this range of K_1.

e) $K_2 < 1$ with $K_1 \varepsilon R_5$

It is obvious that Cases OR - 3A and OR - 10B, Cases OR - 7A and OR - 19B, and Cases OR - 7B and OR - 4A are equivalent pairs. From inequality (4.9) we know that:

$$(x_2)_{P_2'} > \frac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)}, \text{ where } x_2 > x_1.$$

Therefore

$$(R_{13})_{OR-4A} = \frac{x_{OR-4A}}{K_2} > \frac{K_1 - K_2 - 1}{2(1 + K_2)} = (R_{13})_{OR-3A}$$

From inequality (4.9) we see that:

$$(x_1)_{P_2} < \frac{2(1 + K_2)}{K_1 - K_2 - 1}$$

Therefore

$$(R_{13})_{OR-7A} = \frac{1}{(x_1)_{P_2}} > \frac{K_1 - K_2 - 1}{2(1 + K_2)}$$

Hence, OR - 3A and OR - 10B are optimum solutions for the above range of K_1.

4.4.2 Optimum solutions for $1 < K_2 < 7.0711$

An examination of the previous section reveals that there are five distinct ranges, R_6 to R_{10}, for K_1 and these are listed in Table 4.23.

The relevant results from the pertinent orderings of the a_i's are also listed there.

The x marks indicate the ranges of K_1 for which each ordering is applicable, and the circled x marks indicate optimum solutions as determined below.

We now begin to compare for each range of K_1, the minimized maximum ratios obtained.

Table 4.23

Optimum Solutions for $1 < K_2 < 7.0711$

$$1 < K_2 < 7.0711; \text{f) } R_6 = \left\{ K_1 : 2\sqrt{K_2} < K_1 < \frac{(1+K_2)(1+2\sqrt{K_2})}{K_2} \right\}$$

$$\text{g) } R_7 = \left\{ K_1 : \frac{(1+K_2)(1+2\sqrt{K_2})}{K_2} < K_1 < \frac{(1+K_2)(2+K_2)}{K_2} \right\}$$

$$\text{h) } R_8 = \left\{ K_1 : \frac{(1+K_2)(2+K_2)}{K_2} < K_1 < \frac{1+2\sqrt{K_2}+3K_2}{\sqrt{K_2}} \right\}$$

$$\text{i) } R_9 = \left\{ K_1 : \frac{1+2\sqrt{K_2}+3K_2}{\sqrt{K_2}} < K_1 < 3(1+K_2) \right\}$$

$$\text{j) } R_{10} = \left\{ K_1 : 3(1+K_2) < K_1 \right\}$$

Case	R_6	R_7	R_8	R_9	R_{10}	$R_{minimum}$
OR-1A					x	$R_{14} = K_2A = K_2 \frac{(K_1-1-K_2)}{2(1+K_2)}$
OR-1B					x	$R_{14} = K_2A = K_2 \frac{(K_1-1-K_2)}{2(1+K_2)}$
OR-2A				x		$\sqrt{K_2} < R_{14} = \frac{K_2}{x} < K_2$ x is smallest positive root of P_3 where $1<x<\sqrt{K_2}$
OR-2B				x		$\sqrt{K_2} < R_{14} = \frac{K_2}{x} < K_2$ x is smallest positive root of P_3 where $1<x<\sqrt{K_2}$
OR-4B					x	$R_{13} = x > K_2$ x is largest root of P_2 $x > K_2 > 1$
OR-5A			x			$K_2 > R_{12} = x > \sqrt{K_2} > 1$ x is largest positive root of P_3 where $1 < \sqrt{K_2} < x < K_2$

Table 4.23 (continued)

Case	R_6	R_7	R_8	R_9	R_{10}	$R_{minimum}$
OR-6A			x	x	x	$R_{12} = x > K_2$ x is largest root of P_2' where $x > K_2 > 1$
OR-11A					x	$R_{24} = \frac{K_2}{x} > K_2$ x is smallest root of P_2 where $K_2 > 1 > x$
OR-13	x					$R_{32} = \frac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$
OR-14		x	x			$R_{32} = \frac{(1 + \sqrt{K_2})^2}{K_1 - 2\sqrt{K_2}}$
OR-15B			x			$\sqrt{K_2} < R_{34} = \frac{K_2}{x} < K_2$ x is largest positive root of P_3 where $1 < x < \sqrt{K_2}$
OR-16B			x	x	x	$R_{34} = \frac{K_2}{x} < K_2$ x is smallest root of P_2' where $K_2 > 1 > x$
OR-17B	x	x				$R_{31} = \frac{2(1 + K_2)}{K_1 - 1 - K_2} > K_2$
OR-21A	x	x				$R_{42} = \frac{2(1 + K_2)}{K_1 - 1 - K_2} > K_2$

f) $1 < K_2 < 7.0711$ with $K_1 \varepsilon R_6$

It is obvious that Cases OR - 17B and OR - 21A are equivalent. We also know from previous study that:

$$A(1) = A(K_2)$$

From Case OR - 13, we see that:

$$R_{32} = \frac{1}{A(x_b)} \quad \text{where } x_b = \sqrt{K_2} \text{ minimizes } \frac{1}{A(x)}$$

It follows that:

$$R_{32} = \frac{1}{A(\sqrt{K_2})} < R_{31} = \frac{1}{A(1)} = \frac{K_2}{K_2 A(K_2)} = R_{42}$$

Hence OR - 13 is the optimum solution for this range of K_1.

g) $1 < K_2 < 7.0711$ with $K_1 \varepsilon R_7$

Proceeding as in f), we obtain:

$$R_{32} = \frac{1}{A(\sqrt{K_2})} < R_{31} = \frac{1}{x(1)\ A(1)} = \frac{x(K_2)}{K_2 A(K_2)} = R_{42}$$

Hence OR - 14 is the optimum solution for this range of K_1.

h) $1 < K_2 < 7.0711$ with $K_1 \varepsilon R_8$

It is obvious that Cases OR - 5A and OR - 15B are equivalent, as are Cases OR - 17B and OR - 21A. We already know that:

$$\underset{\text{OR-5A}}{R_{12}} = \underset{\text{OR-15B}}{(R_{34})} < K_2 < \underset{\text{OR-17B}}{(R_{31})} = \underset{\text{OR-21A}}{(R_{42})}$$

Therefore Cases OR - 5A and OR - 15B are better than Cases OR-17B and OR - 21A. From OR - 5A, we see that $R_{12} = x$ is a solution of P_3, and arose from the condition $\frac{1}{A(x_a)} = x_a$. From OR - 14, we see that $R_{32} = \frac{1}{A(x_b)}$, where $x_b = \sqrt{K_2}$ minimizes $\frac{1}{A(x)}$. It therefore follows that:

$$R_{32} = \frac{1}{A(\sqrt{K_2})} < \frac{1}{A(x_a \neq \sqrt{K_2})} = \frac{1}{x_a} = R_{12}$$

Hence, OR - 14 is the optimum solution for this range of K_1.

<u>i) $1 < K_2 < 7.0711$ with $K_1 \varepsilon R_9$</u>

It is obvious that Cases OR - 2A and OR - 2B are equivalent, as are Cases OR - 6A and OR - 16B.

We already know that:

$$\underset{\text{OR-2A}}{(R_{14})} = \underset{\text{OR-2B}}{(R_{14})} < K_2 < \underset{\text{OR-6A}}{(R_{12})} = \underset{\text{OR-16B}}{(R_{34})}$$

Hence OR - 2A and OR - 2B are the optimum solutions for this range of K_1.

<u>j) $1 < K_2 < 7.0711$ with $K_1 \varepsilon R_{10}$</u>

It is obvious that Cases OR - 1A and OR - 1B are equivalent, as are Cases OR - 4B and OR - 11A, and Cases OR - 6A and OR - 16B. From inequality (4.10) we see that:

$$\underset{P_2}{(x_2)} > \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)} \quad \text{and also: } \underset{P_2}{(x_2)} > \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)}$$

Therefore:

$$\underset{\text{OR-4B}}{(R_{13})} = \underset{P_2}{(x_2)} > \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)} = \underset{\text{OR-1A}}{(R_{14})}$$

From the same inequality we obtain:

$$\underset{\text{OR-6A}}{(R_{12})} = \underset{P_2'}{(x_2)} > \frac{K_2(K_1 - K_2 - 1)}{2(1 + K_2)} = \underset{\text{OR-1A}}{(R_{14})}.$$

Therefore: Case OR - 1A (≃ OR - 1B) is the best solution for the above range of K_1.

4.4.3 Optimum solutions where $7.0711 < K_2$

An examination of the previous section reveals that there are five distinct ranges, R_{11} to R_{15}, for K_1, and these are listed in Table 4.24.

The relevant results from the pertinent orderings of the a_i's are also listed there.

The x marks indicate the ranges of K_1, for which each ordering is applicable, and the circled x marks indicate optimum solutions as found below.

We now begin to compare for each range of K_1, the minimized maximum ratios obtained.

Table 4.24

Optimum Solutions For $7.0711 < K_2$

$K_2 > 7.0711$ k) $R_{11} = \left\{K_1 : 2\sqrt{K_2} < K_1 < \frac{(1+K_2)(1+2\sqrt{K_2})}{K_2}\right\}$

l) $R_{12} = \left\{K_1 : \frac{(1+K_2)(1+2\sqrt{K_2})}{K_2} < K_1 < \frac{1+2\sqrt{K_2}+3K_2}{\sqrt{K_2}}\right\}$

m) $R_{13} = \left\{K_1 : \frac{1+2\sqrt{K_2}+3K_2}{\sqrt{K_2}} < K_1 < \frac{(1+K_2)(2+K_2)}{K_2}\right\}$

n) $R_{14} = \left\{K_1 : \frac{(1+K_2)(2+K_2)}{K_2} < K_1 < 3(1+K_2)\right\}$

o) $R_{15} = \left\{K_1 : 3(1+K_2) < K_1\right\}$

Case	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	$R_{minimum}$
OR-1A					x	$R_{14} = K_2A = K_2 \frac{(K_1-1-K_2)}{2(1+K_2)}$
OR-1B					x	$R_{14} = K_2A = K_2 \frac{(K_1-1-K_2)}{2(1+K_2)}$
OR-2A			x	x		$\sqrt{K_2} < R_{14} = \frac{K_2}{x} < K_2$ x is smallest positive root of P_3 where $1<x<\sqrt{K_2}$
OR-2B			x	x		$\sqrt{K_2} < R_{14} = \frac{K_2}{x} < K_2$ x is smallest positive root of P_3 where $1<x<\sqrt{K_2}$
OR-4B					x	$R_{13} = x > K_2$ x is largest root of P_2 with $x > K_2 > 1$
OR-5A			x			$K_2 > R_{12} = x > \sqrt{K_2} > 1$ x is largest positive root of P_3 where $1<\sqrt{K_2}<x<K_2$

Table 4.24 (continued)

Case	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	$R_{minimum}$
OR-6A				x	x	$R_{12} = x > K_2$ x is largest root of P_2' where $x > K_2 > 1$
OR-11A					x	$R_{24} = \frac{K_2}{x} > K_2$ x is smallest root of P_2 where $K_2 > 1 > x$
OR-13	x					$R_{32} = \frac{(1+\sqrt{K_2})^2}{K_1-2\sqrt{K_2}}$
OR-14		x				$R_{32} = \frac{(1+\sqrt{K_2})^2}{K_1-2\sqrt{K_2}}$
OR-15B			x			$\sqrt{K_2} < R_{32} = x < K_2$ x is largest positive root of P_3 where $1 < \sqrt{K_2} < x < K_2$
OR-16B				x	x	$R_{34} = \frac{K_2}{x} > K_2$ x is smallest root of P_2' where $K_2 > 1 > x$
OR-17B	x					$R_{31} = \frac{2(1+K_2)}{K_1-1-K_2} > K_2$
OR-21A	x					$R_{42} = \frac{2(1+K_2)}{K_1-1-K_2} > K_2$

k) $7.0711 < K_2$ with $K_1 \in R_{11}$

It is obvious that cases Or - 17B and OR - 21A are equivalent. We also know from previous study that:

$$A(1) = A(K_2)$$

From Case OR - 13, we see that:

$$R_{32} = \frac{1}{A(x_b)} \text{ where } x_b = \sqrt{K_2} \text{ minimizes } \frac{1}{A(x)}$$

It follows that:

$$R_{32} = \frac{1}{A(\sqrt{K_2})} < R_{31} = \frac{1}{A(1)} = \frac{1}{A(K_2)} = R_{42}$$

Hence, OR - 13 is the optimum solution for this range of K_1.

l) $7.0711 < K_2$ with $K_1 \in R_{12}$

As OR - 14 is the only solution for this range of K_1, it is the optimum solution.

m) $7.0711 < K_2$ with $K_1 \in R_{13}$

It is obvious that Cases OR - 2A and OR - 2B are equivalent, as are Cases OR - 5A and OR - 15B. We must establish which of $\frac{K_2}{x_{OR-2A}}$ and x_{OR-5A} is smallest.

We know that x_{OR-2A} is the smallest positive root of P_3 and that x_{OR-5A} is the largest root of P_3.

From inequality (2.7) we have that:

$x_{OR-2A}\ x_{OR-5A} > K_2$ and therefore:

$$R_{14} = \frac{K_2}{x_{OR-2A}} < x_{OR-5A} = R_{12}$$

Hence, OR - 2A or OR - 2B is a better solution than OR - 5A or OR - 15B, for this range of K_1.

n) 7.0711 < K_2 with $K_1 \in R_{14}$

It is obvious that Cases OR -2A and OR - 2B are equivalent, as are Cases OR - 6A and OR - 16B. We also know that:

$$R_{14} = \frac{K_2}{x_{OR-2A}} < K_2 < x_{OR-6A} = R_{12}$$

Hence, OR - 2A or OR - 2B is the optimum solution for this range of K_1.

o) 7.0711 < K_2 with $K_1 \in R_{15}$

It is obvious that Cases OR - 1A and OR - 1B are equivalent, as are Cases OR - 4B and OR - 11A, and Cases OR - 6A and and OR - 16B. From inequality (4.9) we obtain:

$$(x_2)_{P_2} > \frac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)}$$ and also:

$$(x_2)_{P_2'} > \frac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)}$$

Therefore:

$$\underset{\text{OR-4B}}{(R_{13})} = \underset{P_2}{(x_2)} > \frac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)} = \underset{\text{OR-1A}}{(R_{14})} = \underset{\text{OR-1B}}{(R_{14})}$$

and also:

$$\underset{\text{OR-6A}}{(R_{12})} = \underset{P_2'}{(x_2)} \quad \frac{K_2(K_1 - 1 - K_2)}{2(1 + K_2)} = \underset{\text{OR-1A}}{(R_{14})} = \underset{\text{OR-1B}}{(R_{14})}$$

Therefore Cases OR - 1A and OR - 1B are the optimum solutions for the above range of K_1.

<u>4.4.4 Complete Solutions for Four Sections</u>

All previous findings are summarized in Tables 4.25 and 4.26.

Table 4.25

Summary for Four Section Cascade when $K_2 < 1$

* denotes largest a_i ; M_1, M_2 and M_3 are intervals on the x-axis, defined as follows:

$$M_1 = \left\{ 2\sqrt{K_2} \;,\; \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \right\}$$

$$M_2 = \left\{ \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \;,\; \frac{(1 + K_2)(2 + K_2)}{K_2} \right\}$$

$$M_3 = \left\{ \frac{(1 + K_2)(2 + K_2)}{K_2} \;,\; \infty \right\}$$

x is a root of $P_3 \equiv x^2 + \dfrac{(1 + K_2 - K_1)}{2}\, x + K_2$

Range of K_1	Case	a_1	a_2	a_3	a_4	Range of x
$K_1 \varepsilon M_1$	1	1.0	$\frac{1}{\sqrt{K_2}}$	$\frac{(1+\sqrt{K_2})}{K_2(K_1-2\sqrt{K_2})}$	$\frac{(1+\sqrt{K_2})^{2*}}{K_2(K_1-2\sqrt{K_2})}$	
$K_1 \varepsilon M_2$	2	$\frac{K_2}{x^2}$	$\frac{1}{x}$ *	1.0	$\frac{1}{x}$	$K_2 < x < \sqrt{K_2}$
	3	1.0	$\frac{1}{x}$ *	1.0	$\frac{x}{K_2}$	$K_2 < x < \sqrt{K_2}$
$K_1 \varepsilon M_3$	4	$\frac{K_1-K_2-1}{2(1+K_2)}$ *	$\frac{K_1-K_2-1}{2(1+K_2)}$ *	1.0	$\frac{1}{K_2}$	
	5	$\frac{K_2(K_1-K_2-1)}{2(1+K_2)}$	$\frac{K_1-K_2-1}{2(1+K_2)}$ *	1.0	1.0	

Table 4.26

Summary for Four Section Cascade when $1 < K_2$

* denotes largest a_i ; M_4, M_5 and M_6 are intervals on the x axis defined as follows:

$$M_4 = \left\{ 2\sqrt{K_2} \ , \ \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \right\}$$

$$M_5 = \left\{ \frac{1 + 2\sqrt{K_2} + 3K_2}{\sqrt{K_2}} \ , \ 3(1 + K_2) \right\}$$

$$M_6 = \left\{ 3(1 + K_2) \ , \ \infty \right\}$$

x is the root of $P_3 = x^2 + \frac{1 + K_2 - K_1}{2} x + K_2$ lying in the appropriate range.

Table 4.26 (continued)

Range of K_1	Case	a_1	a_2	a_3	a_4	Range of x
$K_1 \epsilon M_4$	6	$\sqrt{K_2}$	1.0	$\frac{(1+\sqrt{K_2})^2}{K_1-2\sqrt{K_2}}$ *	$\frac{(1+\sqrt{K_2})^2}{K_2(K_1-2\sqrt{K_2})}$	
$K_1 \epsilon M_5$	7	$\frac{K_2}{x}$ *	$\frac{K_2}{x^2}$	$\frac{K_2}{x}$ *	1.0	$1<x<\sqrt{K_2}$
	8	x	1.0	$\frac{K_2}{x}$ *	1.0	$1<x<\sqrt{K_2}$
$K_1 \epsilon M_6$	9	$\frac{K_2(K_1-1-K_2)}{2(1+K_2)}$ *	$\frac{K_2(K_1-1-K_2)}{2(1+K_2)}$ *	K_2	1.0	
	10	$\frac{K_2(K_1-1-K_2)}{2(1+K_2)}$ *	$\frac{K_1-1-K_2}{2(1+K_2)}$	1.0	1.0	

The results are displayed graphically in Fig. 4.4.

Chapter V

Conclusions

This work has been directed towards realizing transfer functions of the form

$$T = \frac{(1 - U)^{n/2}}{Q(U)} \tag{5.1}$$

as two, three, and four section cascades of symmetric structures, in such a way that the ratio of the impedance scaling factors a_i was in each case minimized. In the case of the two line cascade equation (5.1) is of the form

$$T = \frac{(1 - U)}{1 + KU} \tag{5.2}$$

it was found here that the ratio was unique, hence no optimization was possible.

In the case of the three line cascade, equation (5.1) becomes

$$T = \frac{(1 - U)^{3/2}}{1 + KU} \tag{5.3}$$

Here it was possible to obtain optimum solutions for various ranges of K, and these are given in Table 3.1. Further, since in some approximation techniques [4] it does not particularly matter whether the numerator of T is of the form (5.2) or (5.3), we have compared two line and three line realizations to see for what ranges of K, the two line realization would be superior to the three line.

In the case of the four line cascade the transfer function (5.1) takes on the form

$$T = \frac{(1 - U)^2}{1 + K_1 U + K_2 U^2} \tag{5.4}$$

This case was much more difficult to analyze. However, complete results were obtained and are summarized in Tables 4.26 and 4.27 and Fig. 4.4.

We would have wished to consider the five line cascade. However, the analytical difficulties were such that it was beyond the scope of this work. However, it is intended to carry out this study at a later date so that a comparison analogous to the two and three cascades could be made for the four and five line cascades.

It would be enormously difficult to use the approach adopted in this thesis to examine longer cascades, and a different approach would be necessary. Such an approach would of necessity involve using computerized optimization algorithms [4]. As such, simple analytical results could not be obtained.

However, in a large number of cases, for example, in active transfer function synthesis, it is rarely necessary to go beyond the four or five section cascade. Hence, this work, which gives simple analytical results, should be of significant practical value.

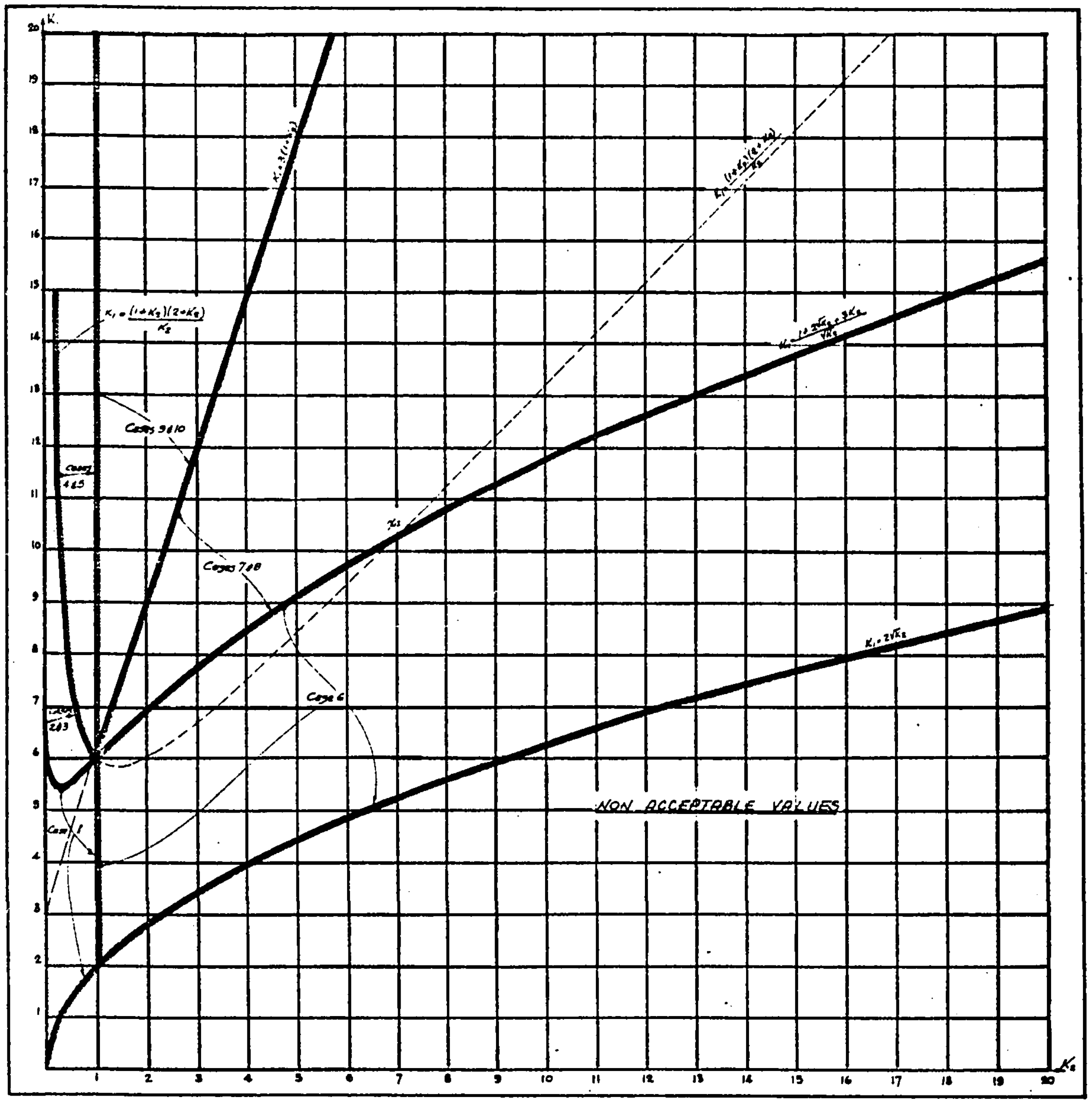


Figure 4.4

Solutions Ranges for Four Section Cascades

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