FREIGHT TRAIN OPTIMAL TRAJECTORY CALCULATION
BY LINEAR PROGRAMMING

Kenneth Porter

A Thesis
in
The Faculty
of
Engineering

Presented in Partial Fulfilment of the Requirements
for the Degree of Master of Engineering
at Sir George Williams University,
Montreal, Canada.


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<tbody>
<tr>
<td>A</td>
<td>((3n+1+1+\tau_n^{-1}+\omega_n^{-1}) \times (3n+1+1+\tau_n^{-1}+\omega_n^{-1}))</td>
<td>(\text{constant system matrix of the augmented discrete system:})</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>(x(k+1) = Ax(k) + Bu(k))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>a_{i}(t)</td>
<td>air flow rate in the brake cylinder pipe of the (i^{th}) train member</td>
<td>(\text{in}^3/\text{sec.})</td>
<td>26</td>
</tr>
<tr>
<td>a_{n+1}(t)</td>
<td>air flow rate in the pipe between the train brake pipe and auxiliary reservoir of train member J</td>
<td>(\text{in}^3/\text{sec.})</td>
<td>29</td>
</tr>
<tr>
<td>B</td>
<td>((3n+1+1+\tau_n^{-1}+\omega_n^{-1}) \times (1+2))</td>
<td>(\text{constant control matrix of the augmented discrete system:})</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>(x(k+1) = Ax(k) + Bu(k))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>b(k)</td>
<td>state variable constraint parameter</td>
<td></td>
<td>77</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
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<td></td>
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<td>--------</td>
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<td></td>
</tr>
<tr>
<td>$b_1(t)$</td>
<td>air braking effort developed at the $1^{st}$ train member; $b_1(t) = e_1p_1(t)$</td>
<td>lbs.</td>
<td>16</td>
</tr>
<tr>
<td>$c_{i}$</td>
<td>fluid capacitance of brake cylinder of the $i^{th}$ train member</td>
<td>in$^2$/lb.</td>
<td>26</td>
</tr>
<tr>
<td>$c_{n+1}$</td>
<td>fluid capacitance of auxiliary reservoir of train member $j$</td>
<td>in$^2$/lb.</td>
<td>29</td>
</tr>
<tr>
<td>$c$</td>
<td>$(1+2)^{k+1}$ - dimensional cost vector: $c = (0, 0, 0, \ldots, 0, 1)^T$</td>
<td>-</td>
<td>74</td>
</tr>
<tr>
<td>$D(k)$</td>
<td>$(3n+1+1+n^1+n^2) \times (1+2)^k$ matrix defined by: $D(k) = [A^{k-1}B, A^{k-2}B, \ldots, AB, B, 0, \ldots, 0]$ where $0$ is a $(3n+1+n^1+n^2)$ - dimensional null column vector</td>
<td>-</td>
<td>76</td>
</tr>
<tr>
<td>$d$</td>
<td>direction vector normal to the separating hyperplane defined as follows in the euclidean space $E^n$: $&lt;d, x&gt; = &lt;d,d&gt;$, where $&lt;d, x&gt;$ denotes the inner product of the $n$-dimensional vectors $d$ and $x$</td>
<td>-</td>
<td>72</td>
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<td>SYMBOL</td>
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<tr>
<td>$d(s_i(t))$</td>
<td>the effective grade at the position of the $i^{th}$ member</td>
<td>%</td>
<td>39</td>
</tr>
<tr>
<td>$d_2$</td>
<td>direction vector used to define velocity constraints</td>
<td>-</td>
<td>88</td>
</tr>
<tr>
<td>$d_{3</td>
<td>1+1}$</td>
<td>direction vector used to define objective constraints</td>
<td>-</td>
</tr>
<tr>
<td>$d_L$</td>
<td>direction vector used to define tractive effort constraints</td>
<td>-</td>
<td>88</td>
</tr>
<tr>
<td>$d_h$</td>
<td>direction vector used to define dynamic braking constraints</td>
<td>-</td>
<td>88</td>
</tr>
<tr>
<td>$d^*(t)$</td>
<td>disturbance input to the system (i.e. the time dependent function giving the effective grade at the head end of the train = actual grade + equivalent grade due to curvature)</td>
<td>%</td>
<td>40</td>
</tr>
<tr>
<td>$d^*(k)$</td>
<td>discrete time equivalent of $d^*(t)$</td>
<td>%</td>
<td>59</td>
</tr>
<tr>
<td>$E^n$</td>
<td>Euclidian space of dimension $n$</td>
<td>-</td>
<td>72</td>
</tr>
<tr>
<td>$E^+$</td>
<td>the half-space region in $E^n$ defined by: $E^+ = {x: &lt;d,x&gt; &gt; &lt;d,d&gt;}$</td>
<td>-</td>
<td>72</td>
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</tr>
<tr>
<td>$E^-$</td>
<td>the half-space region in $E^n$ defined by: $E^- = {x: \langle d, x \rangle &lt; \langle d, d \rangle }$</td>
<td></td>
<td>72</td>
</tr>
<tr>
<td>$e_i$</td>
<td>constant of proportionality between brake cylinder pressure and braking effort at the $i$-th train member</td>
<td>$\text{in}^2$</td>
<td>27</td>
</tr>
<tr>
<td>$F$</td>
<td>$m \times (3n+1+1+1+\omega_n^T, \omega_n^T)$ terminal constraint matrix of the augmented discrete system: $x(k+1) = Ax(k) + Bu(k); Fx(k) = f$</td>
<td></td>
<td>71</td>
</tr>
<tr>
<td>$f$</td>
<td>$m$ - dimensional terminal constraint vector of the augmented discrete system: $x(k+1) = Ax(k) + Bu(k); Fx(k) = f$</td>
<td></td>
<td>71</td>
</tr>
<tr>
<td>$f_i^*(t)$</td>
<td>tractive/dynamic braking effort developed by the $i$-th train member (identically zero for all $t$ if train member $i$ is not a powered locomotive)</td>
<td>lbs.</td>
<td>17</td>
</tr>
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\[ (x_1) \]
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<tbody>
<tr>
<td>$F_1(t)$</td>
<td>force of combined draft gears between train member $i$ and $(i+1)$</td>
<td>lbs.</td>
<td>16</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity</td>
<td>ft/sec²</td>
<td>39</td>
</tr>
<tr>
<td>$H(k)$</td>
<td>control variable constraint parameter</td>
<td></td>
<td>78</td>
</tr>
<tr>
<td>$h_i$</td>
<td>length of the $i$-th train member</td>
<td>ft.</td>
<td>19</td>
</tr>
<tr>
<td>$l$</td>
<td>the member number of the locomotive nearest train member $i$ which is capable of effecting a reduction in brake pipe pressure</td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>$l'$</td>
<td>index indicating member number of train; $l'$ indicates the leading member in the direction of train motion</td>
<td></td>
<td>16</td>
</tr>
<tr>
<td>$J$</td>
<td>the member number of the car furthest from any locomotive capable of effecting an increase in brake pipe pressure</td>
<td></td>
<td>29</td>
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(k11)
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<tr>
<td>j</td>
<td>index indicating locomotive number (note that it is not necessarily true that ( l = j ))</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>K</td>
<td>total number of discrete time intervals, each of duration ( T ), in the interval ( {t_0, t_f} )</td>
<td></td>
<td>57</td>
</tr>
<tr>
<td>KIPS</td>
<td>kilopounds</td>
<td>lbs</td>
<td>91</td>
</tr>
<tr>
<td>k</td>
<td>discrete time interval number</td>
<td></td>
<td>57</td>
</tr>
<tr>
<td>L''</td>
<td>tractive effort linear constraint parameter</td>
<td>lb-secs/ft</td>
<td>66</td>
</tr>
<tr>
<td>L'''</td>
<td>tractive effort linear constraint parameter</td>
<td>lbs.</td>
<td>66</td>
</tr>
<tr>
<td>l</td>
<td>total number of locomotives in the train (( n ) = ( n-1 ))</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>M'</td>
<td>dynamic braking linear constraint parameter</td>
<td>lbs-secs/ft</td>
<td>66</td>
</tr>
<tr>
<td>( \bar{m} )</td>
<td>dimension of terminal constraint vector ( f )</td>
<td></td>
<td>71</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
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<tr>
<td>$M''$</td>
<td>dynamic braking linear constraint parameter</td>
<td>lbs.</td>
<td>66</td>
</tr>
<tr>
<td>$m_i$</td>
<td>the effective mass of the $i^{th}$ train member (i.e., the actual mass of</td>
<td>slugs</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>the member plus some allowance for the rotational inertia of the wheelsets)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>total number of members in the train, including locomotives</td>
<td></td>
<td>29</td>
</tr>
<tr>
<td>$0$</td>
<td>$(3n+1)$ - dimensional null column vector</td>
<td></td>
<td>64</td>
</tr>
<tr>
<td>$0$</td>
<td>$(3n+1+1+\tau + \omega_n)$ - dimensional null column vector</td>
<td></td>
<td>76</td>
</tr>
<tr>
<td>$0_{31}$</td>
<td>3-dimensional null column vector</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>$0_{13}$</td>
<td>3-dimensional null row vector</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>$0_{33}$</td>
<td>3X3 null matrix</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>$P_1$</td>
<td>maximum horsepower rating of locomotive member 1</td>
<td>hp</td>
<td>33</td>
</tr>
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<td>DEFINITION</td>
<td>UNITS</td>
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<tr>
<td>$p_1(t)$</td>
<td>brake cylinder pressure at the $i^{th}$ train member</td>
<td>lbs/in$^2$</td>
<td>26</td>
</tr>
<tr>
<td>$p_{n+1}(t)$</td>
<td>reduction in auxiliary reservoir pressure of train member $j$ (when $p_{n+1}(t) = 0$, all auxiliary reservoirs are completely recharged)</td>
<td>lbs/in$^2$</td>
<td>29</td>
</tr>
<tr>
<td>$q(t)$</td>
<td>$(3n+1)$-dimensional state vector of the continuous system: $\dot{q} = \Delta q + \Omega r$</td>
<td>-</td>
<td>46</td>
</tr>
<tr>
<td>$q(k)$</td>
<td>$(3n+1)$-dimensional state vector of the discrete system: $q(k+1) = \Delta q(k) + \Omega r(k)$</td>
<td>-</td>
<td>58</td>
</tr>
<tr>
<td>$q'(k)$</td>
<td>$(1+\tau/n+\omega_n)$-dimensional vector appended to $q(k)$ to represent the time delays in the system</td>
<td>-</td>
<td>62</td>
</tr>
<tr>
<td>$r(t)$</td>
<td>$(2n+1)$-dimensional control vector of the continuous system: $\dot{r} = \Delta q + \Omega r$</td>
<td>-</td>
<td>47</td>
</tr>
<tr>
<td>$r(k)$</td>
<td>$(2n+1)$-dimensional control vector of the discrete system: $q(k+1) = \Delta q(k) + \Omega r(k)$</td>
<td>-</td>
<td>58</td>
</tr>
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<td>---------------------------------------------------------------------------</td>
<td>------------------------</td>
<td>------------------------</td>
</tr>
<tr>
<td>$r^*(t)$</td>
<td>reduction in equalizing reservoir pressure effected by the engineman, resulting from varying automatic brake valve handle positions</td>
<td>$lb/ln^2$</td>
<td>22</td>
</tr>
<tr>
<td>$r^*_i(t)$</td>
<td>reduction in brake pipe pressure at the $i^{th}$ train member</td>
<td>$lb/ln^2$</td>
<td>24</td>
</tr>
<tr>
<td>$R_C$</td>
<td>curve resistance</td>
<td>$lbs.$</td>
<td>41</td>
</tr>
<tr>
<td>$R_{F+A}$</td>
<td>friction and air resistance</td>
<td>$lbs.$</td>
<td>43</td>
</tr>
<tr>
<td>$R_G$</td>
<td>grade resistance</td>
<td>$lbs.$</td>
<td>39</td>
</tr>
<tr>
<td>$R_I$</td>
<td>fluid resistance of pipe connecting auxiliary reservoir and brake cylinder of the $i^{th}$ train member</td>
<td>$lbs-secs/ln^5$</td>
<td>26</td>
</tr>
<tr>
<td>$R_{n+1}$</td>
<td>fluid resistance of pipe connecting auxiliary reservoir and train brake pipe at train member $J$</td>
<td>$lbs-secs/ln^5$</td>
<td>29</td>
</tr>
<tr>
<td>$S_0$</td>
<td>reference position of train</td>
<td>$ft.$</td>
<td>16</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>UNITS</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>------------</td>
<td>-------</td>
<td></td>
</tr>
<tr>
<td>( s_i(t) )</td>
<td>position of the ( i )th train member relative to ( S_0 )</td>
<td>ft. 16</td>
<td></td>
</tr>
<tr>
<td>( T )</td>
<td>discrete time interval duration</td>
<td>secs. 57</td>
<td></td>
</tr>
<tr>
<td>( t )</td>
<td>the independent variable time ( (t_0, t_f ) denote initial and final times, respectively)</td>
<td>secs. 16</td>
<td></td>
</tr>
<tr>
<td>( t' )</td>
<td>dummy variable</td>
<td>57</td>
<td></td>
</tr>
<tr>
<td>( U )</td>
<td>((1+2)K + 1) - dimensional constant offset vector</td>
<td>86</td>
<td></td>
</tr>
<tr>
<td>( u )</td>
<td>((1+2)K + 1) - dimensional LRAW vector defined by: ( u = [u(0), u(1), \ldots, u(K-1)]^T )</td>
<td>74</td>
<td></td>
</tr>
<tr>
<td>( u' )</td>
<td>( u' = u + \mu )</td>
<td>86</td>
<td></td>
</tr>
<tr>
<td>( u(k) )</td>
<td>((1+2)) - dimensional control vector of the augmented discrete system: ( x(k+1) = Ax(k) + Bu(k) )</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>( \mu )</td>
<td>number of scalar control constraints</td>
<td>80</td>
<td></td>
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<tr>
<td>SYMBOL</td>
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<td>------------------------</td>
</tr>
<tr>
<td>( V' )</td>
<td>minimum permissible train speed</td>
<td>ft/sec.</td>
<td>67</td>
</tr>
<tr>
<td>( V'' )</td>
<td>maximum permissible train speed</td>
<td>ft/sec.</td>
<td>67</td>
</tr>
<tr>
<td>( v_1(t) )</td>
<td>velocity of the ( i^{th} ) train member</td>
<td>ft/sec.</td>
<td>16</td>
</tr>
<tr>
<td>( w_i )</td>
<td>total weight of the ( i^{th} ) member</td>
<td>lbs.</td>
<td>33</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>zero-force length of the draft gears and couplers between train member ( i ) and ( (i+1) )</td>
<td>ft.</td>
<td>19</td>
</tr>
<tr>
<td>( x(k) )</td>
<td>((3n+1+i+l+n+\omega_n)) - dimensional state vector of the augmented discrete system: ( x(k+1) = Ax(k) + Bu(k) )</td>
<td></td>
<td>60</td>
</tr>
<tr>
<td>( x_0 )</td>
<td>initial condition of ( x ); ( x_0 = x(0) )</td>
<td></td>
<td>71</td>
</tr>
<tr>
<td>( \bar{x} )</td>
<td>number of state trajectory constraints or minimax objectives</td>
<td></td>
<td>80</td>
</tr>
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<td>SYMBOL</td>
<td>DEFINITION</td>
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</tr>
<tr>
<td>$y_1(t)$</td>
<td>total resistance to motion of the $i$-th train member due to grades, curves, wind and friction</td>
<td>lbs.</td>
<td>17</td>
</tr>
<tr>
<td>$Z$</td>
<td>upper limit to dynamic braking level which applies to locomotives with extended range over the speed range 6-25 mph</td>
<td>lbs.</td>
<td>35</td>
</tr>
<tr>
<td>$z_1(t)$</td>
<td>extension (+) or compression (-) of draft gears between train members $i$ and $(i+1)$</td>
<td>ft.</td>
<td>19</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>spring constant of the combined draft gears between train members $i$ and $(i+1)$</td>
<td>lbs/ft</td>
<td>19</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>damping coefficient of the combined draft gears between train members $i$ and $(i+1)$</td>
<td>lbs·secs/ft.</td>
<td>19</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>response time of the automatic brake valve and equalizing reservoir</td>
<td>secs.</td>
<td>23</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
<td>UNITS</td>
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</table>
| $\Delta$ | $(3n+1) \times (3n+1)$ constant system matrix of the continuous system: 
$q = \Delta q + \Omega r$ | | 48 |
| $\Delta^*$ | $(3n+1) \times (3n+1)$ constant system matrix of the discrete system: 
$q(k+1) = \Delta^* q(k) + \Omega^* r(k)$ | | 58 |
| $\delta$ | transmission speed of the brake pipe reduction signal in the train line | ft/sec. | 23 |
| $\xi$ | low speed dynamic braking coefficient (i.e., constant of proportionality between velocity of locomotive and dynamic braking limit at low speeds) | lbs-secs/ft | 35 |
| $\eta$ | constant of proportionality between the time delayed equalizing reservoir reduction appearing at the $i^{th}$ train member and the supply pressure of that member's auxiliary reservoir. | | 24 |
vertical angle of the track
gradient  \( \theta \)  deg.  39

velocity coefficient in
linearized version of the
Davis formula for resistance,
due to air and friction of
member 1  \( \lambda_1 \)  secs/ft.  43

zero-velocity constant in
linearized version of the Davis
formula for resistance due to
air and friction of member 1
\( \lambda_0 = \lambda_0 \) is assumed constant for
all members of the train.  \( \lambda_0 \)  43

coefficient of friction between
wheel and rail (assumed constant
for each train member)  \( \mu \)  33

dummy scalar variable which is
to be minimized in the LP problem
corresponding to the original
minimax problem  \( \pi \)  74

(XX1)
<table>
<thead>
<tr>
<th>SYMBOL</th>
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<tbody>
<tr>
<td>$\rho_C(s_1(t))$</td>
<td>track curvature at position $s_1(t)$.</td>
<td>deg.</td>
<td>39</td>
</tr>
<tr>
<td>$\rho_g(s_1(t))$</td>
<td>track gradient at position $s_1(t)$</td>
<td>$%$</td>
<td>38</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>time constant of brake cylinder and connecting pipe system of the $i^{th}$ train member ($\sigma_1 = R_1C_1$)</td>
<td>secs.</td>
<td>27</td>
</tr>
<tr>
<td>$\sigma_{n+1}$</td>
<td>time constant of auxiliary reservoir and connecting pipe system of train member $J$ ($\sigma_{n+1} = R_{n+1}C_{n+1}$)</td>
<td>secs.</td>
<td>30</td>
</tr>
<tr>
<td>$\tau_1$</td>
<td>overall time delay which occurs between the reduction made by the engineman and the initiation of braking action at member $i$. ($\tau_1 = \gamma + \frac{1}{6} \left</td>
<td>s_i - s_1 \right</td>
<td>$)</td>
</tr>
<tr>
<td>$\bar{\tau}_1$</td>
<td>number of time intervals, each of duration $T$, represented by the time delay $\tau_1$ (i.e. $\bar{\tau}_1 = \tau_1/T$)</td>
<td></td>
<td>59</td>
</tr>
<tr>
<td>SYMBOL</td>
<td>DEFINITION</td>
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<tr>
<td>( \phi(t) )</td>
<td>state transition matrix of the system (for system matrix ( A )): ( \phi(t) = e^{At} )</td>
<td>57</td>
<td></td>
</tr>
<tr>
<td>( \Omega )</td>
<td>((3n+1) \times (2n+1)) constant control matrix of the continuous system: ( \dot{q} = \Delta q + \Omega r )</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>( \Omega^* )</td>
<td>((3n+1) \times (2n+1)) constant control matrix of the discrete system: ( q(k+1) = \Delta^* q(k) + \Omega^* r(k) )</td>
<td>58</td>
<td></td>
</tr>
<tr>
<td>( \omega_i )</td>
<td>the time delay between the appearance of the effective grade ( d^*(t) ) at ( s_0(t) ) and ( s_1(t) ) secs.</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>( \bar{\omega}_1 )</td>
<td>the number of time intervals, each of duration ( T ), represented by the time delay ( \omega_i ) (i.e., ( \bar{\omega}_1 = \omega_i / T ))</td>
<td>59</td>
<td></td>
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(xxii)
ABSTRACT

Freight train separations and derailments attributable to adverse longitudinal dynamics of the train moving over undulating track have become more frequent with the advent of longer and heavier trains in recent years. This thesis addresses the problem of attempting to compute the optimal control trajectory for a specific train over a specific track section, wherein the optimality criterion used is the minimization of the maximum coupler force appearing in the train over the duration of its operation.

A mathematical model of the train (including a representation of its braking systems) is first formulated, based on a third-order model for each train member. The discrete equivalent of this continuous model, incorporating the time delays inherent in the system, is then derived. The optimization problem is then reformulated as a linear programming problem and solved on a digital computer.

The complementary relationship of this work to the research in this area currently underway at the Canadian Institute of Guided Ground Transport at Queen's University, Kingston, Ontario, is also delineated.

The procedure developed in this thesis appears to meet the stated objective of designing a method of pre-computing the "best" way to operate a specific train over a particular territory.
ACKNOWLEDGEMENTS

The author gratefully acknowledges the encouragement and assistance given by Dr. V. Ramachandran and Dr. M.N.S. Swamy in the preparation of this thesis.

This work was carried out while in the employ of CP Rail and the author wishes to thank the many individuals in that organization who generously contributed advice, information and facilities to this effort. In particular, the encouragements offered by Mr. G.T. Fisher and Mr. R.B. Gillis are acknowledged and appreciated.

The author also wishes to thank Dr. P.J. McLane of the C.I.G.G.T. at Queen's University for his many helpful discussions and suggestions.

Finally, but perhaps most importantly, a special note of thanks is in order to my wife Helen for her artful blending of the two attributes: persistence and patience.
CHAPTER 1
INTRODUCTION

1.1 Background:

Modern freight trains, particularly those in bulk commodity service, are characterized by very high tonnage, great length and increasing speed of operation. This trend has been due to the necessity of increasing both the capacity of single-track main line operations and the utilization of the motive power fleet. Failure to realize these operational objectives would result in the requirement for capital expenditures of such magnitude that rail transport costs for these commodities would soon become prohibitive.

However, the move towards operation of heavier and longer freight trains has unfortunately been marked by a significant increase in the frequency of train separations and derailments. The experience of the Southern Railway \(^{(43)}\) is typical evidence of the correlation between train uncoupling rate and train length. One can only conclude that there must be even more frequent occurrences of excessive inter-car forces which, although not resulting in such catastrophic events as train derailments or separations, certainly contribute to the costly incidence of car and lading damage. Estimates of this cost run as high as a quarter of a billion dollars. \(^{(60)}\)

Although railbed, track and equipment failures are a major
cause of train derailments, a very significant portion of such incidents can be attributed mainly to the adverse dynamic effects of a train in motion, variously described as "train action", "slack action", "run-ins" and "run-outs." The suitability of such descriptive terms becomes immediately apparent when one considers the characteristics of a moving n-car train. Rather than being rigidly connected together, each car in the train is coupled with its neighbour in such a manner as to allow relative displacements between successive cars of approximately 0.5 feet. Hence, as a result of various unbalanced external forces, the individual cars comprising the train are likely to undergo relative velocities and accelerations. Indeed, a 200 car train can vary in length by anywhere up to 100 feet during motion.

Inter-car forces well in excess of 500,000 pounds resulting from such slack action in trains have been reported. (69) A serious problem currently confronting railways is to devise a method of operating trains such that these excessive slack action forces are minimized.

1.2 Prior Work:

A specific train moving over a particular section of track may be considered as a system with a specified set of parameter values being subjected to a specified set of input or control functions. In
the context of this system viewpoint, the problem defined above may be considered to be an optimization problem, wherein one is concerned with the selection of parameter values and/or control functions which will lead to optimum performance, as defined by an appropriate criterion.

Wilson (50) identifies the most important elements of this system, which may be divided into two groups, as follows:

**System Parameters**

1) track condition (i.e., effective coefficient of friction between wheel and rail)
2) freight car characteristics
3) locomotive characteristics
4) braking system characteristics
5) inter-car coupling and draft gear characteristics
6) train marshalling (i.e., sequential ordering of locomotives and cars)

**System Control Functions**

1) track profile (i.e., gradient and curvature as a function of time)
2) locomotive throttle action, as a function of time
3) train braking action, as a function of time

Practically speaking, the parameters (1) to (4) and control function (1) are fixed; that is, the system designer is usually
not free to select these parameter values and control function. Thus, the only system elements available for design in the optimization process are the system parameters (v) and (vi), and the system control functions (ii) and (iii).

Therefore, all prior work on the above-defined problem, whether empirical or analytical in nature, may be classified as effectively either parameter optimization or optimal control approaches.

1.2.1. Parameter Optimization:

It is intuitively evident that the optimum selection of control functions from some class of functions is a more difficult analytical or empirical problem than the optimum selection of parameter values from some admissible set. Perhaps for this reason, most prior work on the slack action problem has in fact been in the area of parameter optimization.

Parker (29) for example, has given an empirical method for estimating the optimum marshalling of locomotives in the train.

Train simulations have been used extensively in attempts to arrive at optimal draft gear characteristics and train marshalling policies. The work of Wilson (50) and Roggeveen (34 - 38) is representative of the efforts that have been made in this direction.
The effects of draft gear characteristics on both freight car lading and longitudinal train motions has also been the subject of much analytical work. For example, Pipes (32) has treated the analysis of longitudinal motions of trains by the electrical analog. Frudenstein (14) has completed a dynamic analysis which relates the principle design parameters of the draft gears to the forces experienced by a resilient lading.

No attempt will be made here to compile an exhaustive history of work in the area of parameter optimization. The American Association of Railroads (AAR) has undertaken a comprehensive ten year program of research on track/train dynamics. The first phase of this program will include the preparation of an extensive bibliography of work in this area and this reference is expected to be available very shortly.

1.2.2 Optimal Control:

Despite much work in the area of parameter optimization, it is generally concluded that train separations caused by excessive coupler forces resulting from slack actions are still very much a problem.

For example, the Southern Railway has decided to abandon using all motive power on the front of long unit trains; instead, they are distributing locomotives through the train by adopting
remote-control techniques. Even so, the use of mid-train locomotive units has not in itself eliminated slack action problems. (67) Parker (29) gives an excellent case study of train operation with remote-controlled locomotives and he too concludes that skillful handling of the train control functions is the most important factor in avoiding slack action problems.

In a 1968 panel discussion (67), S.H. Fillion, Chief Engineer for Waugh Equipment, a major producer of draft gears, expressed doubts that upgrading of the knuckle, coupler, yoke and draft gear can be the whole solution. Slack, he said, must be controlled because "if a run-in is going to buckle an essentially unstable train, then stronger inter-car hardware is not going to solve, and may actually complicate, the problem by causing structural failure or derailment."

Again, Wilson (50) recognized the need to establish "optimal operating programs for a given train on a given route," and that this implies development of optimal brake and locomotive throttle functions such that they be "coordinated to yield a combined energy input and dissipation system which provides the best attainable performance."

Many train handling rules have been empirically evolved as a result of extensive operating experience. Reference (53) contains
an excellent example of the practical train braking and throttle control procedures that have resulted from experimental field tests by CP Rail. Goldstone (15) reports that the Southern Railway is even instrumenting one coupler in some of its trains with a strain gauge in an attempt to provide the train engineer with some input regarding the internal dynamics of his moving train.

There have been relatively few analytical approaches to the problem of designing optimal braking and throttle control functions for a specific train over a specific route.

Ichikawa (17) has successfully applied optimization methods for determining the optimal control law which minimizes energy consumption for a train operating from one station to the next. Unfortunately, he has had to assume an extremely simple, lumped-mass mathematical model for the train as a whole in order to be able to solve the resulting bounded state variable problem using the calculus of variations. This approach is simply impractical when a more realistic (i.e., more complex) train model is assumed.

To the author's knowledge, by far the most promising research on the problem of designing optimal control functions for freight trains is the work of McLane and Peppard (25), currently in progress at the Canadian Institute of Guided Ground Transport (CIGGT) at Queen's University, Kingston, Ontario. The objective of this work is, through the application of linear regulator theory, to develop
a feedback controller which minimizes coupler forces and velocity deviations from some scheduled velocity. The model for the longitudinal dynamics of a train is somewhat deficient in that the air braking system of the train is not represented. Nevertheless, Dr. McLane has indicated to the author that the model will be altered so as to include this important aspect of train operation. When this is accomplished, a practical regulator capable of compensating for modeling inaccuracies and such system noise as wind and variable track conditions should result.

Although there is a good possibility that the work currently underway at the CIGGT will result in an implementable train regulator, there still remains unresolved the problem of designing the optimal "scheduled velocity" about which this regulator should function. It is precisely this problem which is the subject of this thesis — to develop a procedure for designing for a specific train an ideal velocity schedule (or more accurately, an optimal system trajectory) which may then serve as the reference input for an on-board train regulator such as that being developed by McLane and Peppard.

One very desirable feature of such a design procedure is that the track profile be taken into account directly in the process of determining the optimal trajectory of the system. This track profile, consisting of the gradient and curvature as functions of
distance, is one of the most important inputs to the system. Whereas McLane and Peppard are attempting to design a regulator which is capable of operating over non-level terrain by treating gradient and curvature effects as just additional noise to the system, it would seem a better approach to take advantage of the fact that this input is known in the process of deriving the optimal system trajectory. Intuitively, if grades and curves are accounted for in the specification of the "scheduled velocity" as a function of time, then the regulator would have a much better chance of succeeding at compensating for the many other true system noises, such as wind, slippery rail conditions, etc.

1.3 Scope of Thesis and Brief Outline of Approach:

Athans (1) gives an excellent outline of a systematic design process for controlling a non-linear uncertain system about a desired trajectory through the use of the stochastic linear-quadratic-gaussian (LQG) problem. In his paper, Athans identifies thirteen distinct steps in the design process, comprising the following six separate functions:

Part A: Deterministic Modelling
Part B: Stochastic Modelling
Part C: Linearization Modelling
Part D: Control Problem Calculations
Part E: Filtering Calculations
Part F: Construction of Linearized Dynamic Compensator.

* I am indebted to Dr. McLane for bringing this paper to my attention.
Athans also gives the structure of an overall control system which combines pre-computed signals with on-line measurements generated by the linear dynamic compensator. This structure is repeated here as Figure 1.1 for reference.

Whereas McLane and Peppard are concerned with Parts C, D and F, resulting in the construction of a proportional - integral - derivative dynamic compensator, this thesis addresses the problem of pre-computing the optimal trajectory of the system as defined by the set of vector functions \( \{ u_0(t), x_0(t), y_0(t) \} \). With reference to Athans' paper, the objective of this thesis is to provide the pre-computed ideal open-loop control input and system output, by accomplishing the following three steps of Part A:

**Part A: Deterministic Modelling**

**Step 1:**
Obtain deterministic model of actuators and plant. In state variable form, this yields the vector differential equation:

\[
\dot{x}(t) = f \{ x(t), u(t) \} \quad \ldots \ldots \ldots (1-1)
\]

**Step 2:**
Obtain deterministic model of plant and sensors:

\[
y(t) = g \{ x(t) \} \quad \ldots \ldots \ldots \ldots \ldots (1-2)
\]
Figure 11A.

Structure of overall control system, as given by Athans (1)

\[
\begin{align*}
\text{physcial process} & \quad \text{on-line command signal} \quad u(t) \quad \text{(on-line)} \\
\text{physical plant} & \quad \text{on-line control correction signal} \quad \dot{u}(t) \quad \text{(on-line)} \\
\text{linear dynamic compensator (part f)} & \quad \text{on-line input to dynamic compensator} \quad \delta z(t) \\
\text{ideal sensor output for desired state response} \quad y_0(t) \quad \text{(precomputed)} \\
\text{actuators} & \quad \text{on-line control input} \quad u_0(t) \quad \text{(precomputed)} \\
\text{ideal open-loop input signal} & \quad \text{precomputed output} \quad \tilde{y}_0(t) = \tilde{y}_0(t) \quad \text{(part A)} \\
\text{sensor data (on-line)} & \quad \text{sensor data (on-line)} \quad z(t) \\
\end{align*}
\]
Step 3:
Determine ideal input - state - output response:

\[ u_0(t) : \text{ideal (open-loop) input} \]

\[ x_0(t) : \text{ideal state response} \]

\[ y_0(t) : \text{ideal output response} \]

As pointed out by Athans, the modelling process (steps 1 and 2 combined) is extremely critical to the ultimate success of the control design, and much of this thesis will be concerned with this modelling process.

The usual procedure for accomplishing step 3 is to formulate an optimal control problem of the following form:

"Given the system described by equation (1-1), find an admissible control vector \( u_0(t) \) which transfers the system from some initial state \( x_0(t_0) \) to some final state \( x_0(t_f) \), such that \( x_0(t_f) \) is a subset of \( S \), where \( S \) is a target set of final states, and such that some cost functional defined by the designer is minimized."
It will be shown that the traditional means of obtaining a solution to the problem just formulated will not work in this instance, primarily because of the extremely high dimensionality of the system. A procedure first outlined by Enns (12) will then be employed to obtain a solution. The approach of Enns is to reformulate the given problem as a standard linear programming (LP) problem by an appropriate discretization of the system equation (1-1); the resulting LP problem may then be solved by any of the standard methods commonly available. An important attribute of this solution method is that the dimension of the LP problem is relatively independent of the dimensionality of the system (1-1).

The thesis then concludes with a worked example for a small sample train of three cars (or three sections), and a discussion of the results.

In essence, the problem under consideration in this thesis is to devise a method for controlling excessive inter-car forces caused by relative accelerations between successive cars that may result from the slack-action of a train in motion.
CHAPTER 2

PHYSICAL SYSTEM AND MODEL FORMULATION

2.1 Assumptions Regarding Overall System Model:

In this chapter, a model of the overall physical system will be formulated. The model so derived will be a one-dimensional representation in which only longitudinal motion and forces will be considered directly.

To be sure, the actual train undergoes vertical displacements as a result of track gradients; but it is only the horizontal components of the forces due to gravity that we are interested in and the effect of these forces is represented in the model as resistance to longitudinal motion. Also, the track itself is assumed to be continuous in that the impact of wheels on rail joints is neglected.

Similarly, lateral forces are certainly encountered between the flanges of the car wheels and the rail guideways, particularly through curved track sections, but the effect of these forces is again represented in the model simply as resistance to longitudinal motion.

Additional assumptions will be noted as each of the elements of this physical system is modelled in the following sections.
2.2. Model for Individual Train Members:

Roggeveen (34) has shown in freight car impact tests that a two-lump model of an individual train member must be assumed in order to depict the coupler force versus time characteristic reasonably accurately for all coupler force levels. However, he has also shown that a first-order lumped model gives a sufficiently accurate representation of the impact of successive train members up to a certain critical coupler force level, corresponding to the point where the spring of the inter-car draft gear is completely compressed. At force levels exceeding this critical value, the two train members are effectively in solid metal-to-metal contact, and it is in this region that the first-order model fails and a second-order representation becomes necessary.

However, the end objective of this whole exercise is to minimize the peak coupler forces in the train. Making the assumption that the optimal control derived in this dissertation will succeed in keeping coupler forces at least below this critical level (approximately 200,000 pounds for average draft-gears) permits us to assume the simpler first-order model. By way of justifying this assumption, Wilson (50) reports excellent agreement between test data from instrumented trains and computer predictions from a model which assumes a first-order representation for each train member.
Let us define the following nomenclature:

\( t \): the independent variable time (in seconds)

\( i \): the subscript indicating the member number in the train; \( i = 1 \) indicates the leading member in the direction of train motion.

\( m_i \): the effective mass (in slugs) of the \( i^{th} \) train member; this is the actual mass of the member plus some allowance for the rotational inertia of the wheelsets (see Taylor (16), for instance).

\( s_i(t) \): the position (in feet) of the \( i^{th} \) train member, relative to some reference position \( s_0 \).

\( v_i(t) \): the velocity (in ft/sec) of the \( i^{th} \) train member.

\( b_i(t) \): the air braking effort (in pounds) developed at the \( i^{th} \) train member.

\( f_i(t) \): the force (in pounds) on the \( i^{th} \) and \((i + 1)^{th}\) train members from the combined draft gears between them.
$f_i^*(t)$: the tractive or dynamic braking effort (in pounds) developed by the $i^{th}$ train member (identically zero for all $t$ if train member $i$ is not a powered locomotive).

$y_i(t)$: the total resistance to motion (in pounds) of the $i^{th}$ train member due to grades, curves, wind and friction.

Considering the model depicted in Figure 2.1, then from Newton's Second Law the equations of motion for the $i^{th}$ train member are:

$$\dot{s}_i(t) = v_i(t) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2-1a)$$

$$\dot{v}_i(t) = \frac{1}{m_i} \left( \mathcal{F}_{i-1}(t) - \mathcal{F}_i(t) - y_i(t) - b_i(t) \right) + f_i^*(t) \quad \cdots (2-1b)$$

where $(\cdot) \equiv \frac{d}{dt} (\cdot)$

2.3 Model For Draft Gears and Couplers between Train Members:

Smith (45) lists over twenty different types of draft gears, but for our purposes we shall assume that the important characteristics of the combined draft gears between any two successive train members
Figure 2.1: Model of the $i^{th}$ member of the train.
can be modelled by a translational pure spring and viscous damper arranged in parallel, as in Figure 2.2 wherein the following additional symbols are defined:

\( a_i \): the spring constant (in pounds/foot) of the combined draft gears between train members \( i \) and \((i + 1)\).

\( \beta_i \): the damping coefficient (in pounds-seconds/foot) of the combined draft gears between train members \( i \) and \((i + 1)\).

\( h_i \): the length (in feet) of train member \( i \).

\( w_i \): the zero-force length (in feet) of the draft gears between train members \( i \) and \((i + 1)\).

\( z_i(t) \): the extension or compression (in feet) of the draft gears between train members \( i \) and \((i + 1)\). Note that \( z_i(t) + w_i \) represents the total separation between the trailing wall of member \( i \) and the leading wall of member \((i + 1)\).

It should be noted that the non-linear effect (namely dead-zone) of slack in the draft gears, as described by Roggeveen (37),
Figure 2.2: Model of the combined draft gears between train members $i$ and $(i+1)$
is specifically not included in this model. Opinion is currently divided on the question of whether or not this non-linearity is an important factor which should be represented in the model; the current efforts of McLane and Peppard \(25\) in this regard perhaps will settle this issue.

From Figure 2.2, we have:

\[
\begin{align*}
\dot{z}_i(t) &= s_i(t) - s_{i+1}(t) - h_i - w_i \quad \ldots \quad (2-2) \\
\dot{z}_i(t) &= s_i(t) - s_{i+1}(t) - v_i(t) - v_{i+1}(t) \quad \ldots \quad (2-3)
\end{align*}
\]

since \(h_i\) and \(w_i\) are constants.

Thus, the mathematical model of the inter-car connection is:

\[
\begin{align*}
\mathcal{F}_i(t) &= a_i z_i(t) + b_i \dot{z}_i(t) \\
&= a_i z_i(t) + b_i \{ v_i(t) - v_{i+1}(t) \} \quad \ldots \quad (2-4)
\end{align*}
\]

2.4 Model for Train Air Brake System:

Reference \(65\) contains an excellent equipment glossary and overall description of the train air brake system. Other relevant material dealing with this important aspect of train operation is given in the bibliography.
To quote from reference (65):

"The primary source for slowing and stopping trains is the compressed air stored at each car in the train. This air is admitted to the brake cylinder, translated to a mechanical force acting on the brake rigging, and in turn forces the brake shoes against the wheels of the train to retard their rotation. The amount of braking is controlled by the reduction of brake pipe pressure, as determined by the automatic brake valve handle position in the locomotive. This brake pipe reduction throughout the train causes the control valve at each car to respond. The control valve then meters the stored air in the brake cylinders in proportion to the amount of drop in brake pipe pressure."

Let $r^*(t)$ represent the reductions in brake pipe pressure ($\text{lbs/in}^2$) called for by the engineman as a result of varying automatic brake valve handle positions. There are definite restrictions on the manner in which this brake valve handle position can be varied, but these constraints will be dealt with later.

Now consider the following sequence of events which occurs after the engineman has initiated the braking action by moving the brake valve handle from the release position:
the locomotive equalizing reservoir pressure reduces;

ii) this in turn causes the train brake pipe pressure at the locomotive to reduce;

iii) this pressure drop is then transmitted through the brake pipe of the train until it reaches the control valve on the \(i^{th}\) train member;

iv) the control valve then meters the supply of air from the auxiliary reservoir to the brake cylinder of the \(i^{th}\) train member.

Rather than model in detail the above sequence of events, let \(\tau_i\) represent the overall time delay which occurs between events (i) and (iv). Then, \(\tau_i\) is known to be of the form:

\[
\tau_i = \gamma + \frac{1}{\delta} \left| s_1(t) - s_i(t) \right| \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2-5)
\]

where symbols are defined as follows:

\(\gamma\) : response time (seconds) of the automatic brake valve and equalizing reservoir.

\(\delta\) : transmission speed (feet/second) of the pressure drop signal in the brake pipe of the train.
the member number of the locomotive nearest train member \( i \) which is capable of effecting a reduction in brake pipe pressure.

(Typical values for \( \gamma \) and \( \delta \) are 2 seconds and 500 feet/second, respectively).

The reduction in brake pipe pressure at the \( i^{th} \) train member, \( r_i^*(t) \), is therefore given by: \( r_i^*(t) = r^*(t - \tau_i) \). Thus, the auxiliary reservoir supply pressure to the brake cylinder pipe can be represented as \( \eta \cdot r^*(t - \tau_i) \), where \( \tau_i \) is defined by eq. (2-5) and \( \eta \) is a constant of proportionality between the time delayed equalizing reservoir pressure drop appearing at the control valve of the \( i^{th} \) train member and the supply pressure of that member's auxiliary reservoir.

The braking force generated at the wheels of member \( i \) is proportional to the pressure in its brake cylinder. In essence, the brake cylinder behaves approximately like a pure fluid capacitance in which energy is stored because of the compressibility of the fluid. The brake cylinder is supplied with air from the auxiliary reservoir through a long, relatively narrow, brake cylinder pipe. Hence, we shall assume the simple model for brake action as illustrated in Figure 2.3.
Figure 2.3: Model of brake action at train member 1.
The new nomenclature of Figure 2.3 is defined as follows:

\[ C_1 : \text{fluid capacitance (in } \frac{5}{\text{lb}} \text{) of the brake cylinder at train member 1.} \]

\[ R_1 : \text{fluid resistance (lb-secs/in } 5 \text{) of the brake cylinder pipe at train member 1.} \]

\[ a_1 (t) : \text{air flow rate (in } \frac{3}{\text{sec}} \text{) in the brake cylinder pipe of the } i^{th} \text{ train member.} \]

\[ p_1 (t) : \text{brake cylinder pressure (lbs/in}^2 \text{) at the } i^{th} \text{ train member.} \]

Note that the auxiliary reservoir supply pressure is modelled as an ideal pressure source. In actual fact, the auxiliary reservoir is a fixed volume container, but since its volume is significantly greater than that of the brake cylinder, this representation is a reasonable approximation.

From Figure 2.3, we obtain the mathematical model for brake action at train member 1 as:

\[ a_1 (t) = C_1 \dot{p}_1 (t) = \frac{1}{R_1} \left[ n \cdot r^* (t - \tau_1) - p_1 (t) \right] \ldots \ldots \ldots \ldots (2-6) \]

and

\[ \dot{p}_1 (t) = \frac{1}{a_1} \left[ n \cdot r^* (t - \tau_1) - p_1 (t) \right] \ldots \ldots \ldots \ldots (2-7) \]
where \( \sigma_1 = R_1 C_1 \) is the time constant (seconds) of the braking system.

Now define \( e_i \) to be the constant of proportionality (in \(^2\)) between brake cylinder pressure and the braking force developed at the \( i \)th train member. Therefore, \( e_i \) takes into account such factors as the brake rigging action, the coefficient of friction between the brake shoes and wheels of member \( i \), and the coefficient of friction between wheel and rail. Thus, the braking effort \( b_i(t) \) of Figure 2.1 is given by:

\[
b_i(t) = e_i p_i(t) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2-8)
\]

As mentioned previously, in order for the braking action model defined by eqs. (2-7) and (2-8) to be valid, \( r^*(t) \) must be suitably constrained so as to represent realistic train air brake applications and releases.

For instance, when an application of the train air brakes is called for, the minimum and maximum automatic brake valve handle positions are known (65) to correspond to equalizing reservoir reductions of 7 and 23 psi respectively. Thus we have the first constraint on \( r^*(t) \):

\[
\begin{align*}
either & \quad r^*(t) = 0 \\
& \quad or \quad 7 \leq r^*(t) \leq 23 
\end{align*}
\]

\[
\ldots \ldots \ldots \ldots \ldots \ldots \ldots (2-9)
\]
In addition, although brake application increases are permitted at any time, partial brake releases are not valid. That is, if $\Delta t$ is some small time increment, then:

\[
\begin{align*}
\text{either } r^k(t + \Delta t) &= 0 \\
\text{or } r^k(t) &\leq r^k(t + \Delta t) \leq 23
\end{align*}
\]  \hspace{1cm} (2-10)

One final constraint on $r^k(t)$ is required, in order to avoid undesired brake releases in the actual train. Peterson (30) gives an excellent account of the causes of such undesired releases, and illustrates that these can be attributed to inappropriate brake applications made under "false gradient" conditions.

Some explanation of terminology is in order before proceeding further. When the automatic brake valve handle is moved to the release position, this causes the brake valve to restore air pressure to the brake pipe. The increasing brake pipe pressure then causes the equipment at each member to move to release, thereby exhausting the brake cylinder pressure and recharging the auxiliary reservoir. Now, "true gradient" may be defined as the difference between the head end and rear car brake pipe pressures when the train is fully charged, and is primarily due to leakage in the piping of the system. Whenever the train is not fully charged (i.e., the recharging function alluded to previously has not been completed), a "false gradient" is said to exist and is defined as the difference between the actual gradient at that point in time and
the true gradient for that particular train.

In order to represent this final constraint on $r^n(t)$, it will therefore be necessary to represent in the model this recharging of the auxiliary reservoir farthest from the locomotive capable of supplying air to the brake pipe. To this end, let us define the following nomenclature:

- $n$: the total number of members in the train, including locomotives.
- $J$: the member number of the car furthest from any locomotive capable of effecting an increase in brake pipe pressure.
- $C_{n+1}$: fluid capacitance ($\text{in}^5/\text{lb}$) of auxiliary reservoir of train member $J$.
- $R_{n+1}$: fluid resistance ($\text{lb-secs/in}^5$) of pipe connecting auxiliary reservoir and brake pipe at train member $J$.
- $a_{n+1}(t)$: air flow rate ($\text{in}^3/\text{sec}$) in the pipe connecting the auxiliary reservoir and brake pipe at train member $J$.
- $p_{n+1}(t)$: reduction in auxiliary reservoir pressure ($\text{lbs/in}^2$) at train member $J$; as such, $p_{n+1}(t)$ is also the
"false" gradient existing at any time t. When

\[ p_{n+1}(t) = 0, \] all auxiliary reservoirs in the
train are completely recharged.

The model of this recharging function of the \( j \)th member's
auxiliary reservoir is therefore assumed as in Figure 2.4.

From Figure 2.4 we have:

\[ \frac{dp_{n+1}(t)}{dt} = \frac{1}{\sigma_{n+1}} \left\{ r^*(t) - p_{n+1}(t) \right\} \] ........................................(2-11)

where \( \sigma_{n+1} = R_{n+1} C_{n+1} \) is the time constant of the recharging system
(secs).

Peterson (30) shows that in order to avoid undesired train
brake releases, brake applications required must be increased by an
amount equal to any false gradient existing at the time of application.
Thus, the final constraints on \( r^*(t) \) may be expressed as:

\[ \begin{align*}
\text{if } r^*(t) &= 0, \text{ then}
\end{align*} \]

\[ \text{either: } r^*(t + \Delta t) &= 0 \]

\[ \text{or: } 7 + p_{n+1}(t) &\leq r^*(t + \Delta t) \leq 23 \]

........................................(2-12)
Figure 2.4: Model of train recharging action. When \( r^k(t) \) is brought to zero then brake release is initiated and when \( p_{n+1}(t) = 0 \), then train is fully recharged. Thus, \( p_{n+1}(t) \) is a measure of the "false" gradient in the train at any time \( t \).
For convenience, the constraints (7-9), (7-10) and (7-17)
are summarized here as follows:

I) \[ r^* (t) = 0, \text{ then}\]
\[ \text{either: } r^* (t + \Delta t) = 0 \]
\[ \text{or: } p_{n+1} (t) + 7 \leq r^* (t + \Delta t) \leq 23 \]

II) \[ r^* (t) \neq 0, \text{ then}\]
\[ \text{either: } r^* (t + \Delta t) = 0 \]
\[ \text{or: } r^* (t) \leq r^* (t + \Delta t) \leq 23 \]

(2-13)

2.5 Representation of Motive Power Constraints:

Each locomotive in the train represents a potential source of
both tractive and braking effort. As in Figure 2.1, let \( f_i^* (t) \)
represent the force output (lbs) of the locomotive member \( i \) at any
time \( t \). Then for \( f_i^* (t) > 0 \) the unit is "motoring", that is,
applying tractive force; and for \( f_i^* (t) < 0 \) the unit is in
"dynamic braking", that is, applying braking force. Obviously,
if \( f_i^* (t) = 0 \), then the unit is idling. In order to represent real
locomotives, \( f_i^* (t) \) must be suitably constrained, as follows.

2.5.1 Motoring Constraints (when \( f_i^* (t) > 0 \)):

The theoretical maximum force that a locomotive can transmit
to the rail is limited by two factors. These are (1) the adhesion
limit (coefficient of friction) between the wheel and the rail, beyond which wheel slip will occur, and (ii) the actual power of the locomotive:

1) Adhesion Limit:

If \( W \) is the effective weight of the locomotive (lbs) and \( \mu \) is the tractive adhesion coefficient (dimensionless), then:

\[
f^*_1(t) \leq \mu \cdot W, \quad \text{(2-14)}
\]

11) Motive Power Limit:

If \( P \) is the maximum horsepower rating of the locomotive, and \( v(t) \) is the velocity of the locomotive (ft/sec), then from reference (61):

\[
f^*_1(t) \leq \frac{451 \cdot P}{v(t)}, \quad \text{(2-15)}
\]

where the constant 451 is derived from 550 ft-lbs/sec per horsepower x 0.82 transmission efficiency of the electro-mechanical drive system.
2.5.2. Dynamic Braking Constraints \( (\text{when } f_{i}^{*}(t) < 0 ) \):

There are basically four constraints on the braking force, which can be generated by locomotive member \( i \):

1) Adhesion Limit:

As in motoring, the maximum braking force is limited by the adhesion between wheel and rail:

\[
f_{i}^{*}(t) \geq -\mu w_{i} \quad \quad \quad \quad \quad \quad (2-16)
\]

2) Motor Heating Limit:

In dynamic braking, the grid current (and hence the braking horsepower developed through \( I_{R}^{2}R \) loss (59, 66) in the braking resistor grids) is limited. For instance, from performance curves (65) for the SD-40 locomotive, this limit is known to be as follows:

\[
f_{i}(t) \geq \frac{638 p_{i}}{v_{i}(t)} \quad \quad \quad \quad \quad \quad (2-17)
\]
III. Motor Field Current Limit:

At low speeds, the motor armature current is less, as the wheels are turning slowly. Hence, the braking horsepower is less, since $I^2R$ is smaller due to the smaller grid current $I$. This limit on the available braking effort is of the form (65):

$$f_1^*(t) \geq -\xi v_1(t) \quad \ldots \ldots \ldots \ldots \ldots (2-18)$$

where the value of $\xi$ depends on whether dynamic braking is standard or extended range (see reference (65) for an explanation of these terms). Typical values for a 4 axle, 2250 horsepower unit are $\xi_{std} = 1110 \text{ lb-secs/ft}$, and $\xi_{ext} = 3960 \text{ lb-secs/ft}$.

iv) Extended Range Dynamic Braking Limit:

If extended range dynamic braking is available, then over the speed range of approximately 9 to 37 ft/sec, the following constraint applies:

$$f_1^*(t) \geq -z \quad \ldots \ldots \ldots \ldots \ldots (2-19)$$
Typically, \( Z = 40,000 \) lbs. for a GP-40 unit (4 axles, 2250 horsepower), and \( Z = 60,000 \) lbs. for an SD-40 unit (6 axles 3000 horsepower).

2.5.3. Summary of Motive Power Constraints:

For convenience, the constraints on \( f_1^*(t) \) are summarized here as follows:

\[
\begin{align*}
|f_1^*(t)| & \leq \mu \cdot W_1 \\
\frac{638P_1}{v_1(t)} & \leq f_1^*(t) \leq \frac{451P_1}{v_1(t)} \\
f_1^*(t) & \geq -\epsilon v_1(t) \\
f_1^*(t) & \geq -Z
\end{align*}
\]

It should be noted that some or all of the constraints of (2-30) apply for any particular class of power, depending on the specific values of \( \mu, P_1, \epsilon \) and \( Z \). However, in the most general problem, all these constraints may apply, in which case the allowable region for \( f_1^*(t) \) has the form represented by the hatched area in Figure 2.5.

Of course, if dynamic braking is not permitted for any particular train, then \( f_1^*(t) \geq 0 \) must be added as the final constraint.
Figure 2.5: Motive power constraints; hatched area is allowable region.
In addition, \( f^*_i(t) \) within the allowable region of Figure 2.5 must be quantized, since the engineman's throttle has only eight distinct positions and each corresponds to a specific power output. For instance, McLane and Peppard (25) treat the quantization of locomotive tractive effort, and Reference (53) contains an example of this in the case of a locomotive in dynamic braking. However, the approach taken here will be to assume that the allowable region is continuous and then effect the required quantization after the optimal control has been obtained.

2.6 Representation of the Resistance to Motion Function, \( y_1(t) \), for the \( i^{th} \) Member:

The function \( y_1(t) \) is taken to represent the total resistance to motion of the \( i^{th} \) train member due to grades, curves, friction and air resistance. Let us consider each component of \( y_1(t) \) in the following sections.

2.6.1. Grade Resistance:

If the member is ascending a grade (slope), its mass is being raised vertically. Let \( \rho_g(s_1(t)) \) be the grade, expressed in per cent, at position \( s_1(t) \). Since the grade \( \rho_g(s_1(t)) \) gives the vertical rise over 100 ft. of track, then

\[
\tan \theta = \frac{\rho_g(s_1(t))}{100}
\]

\[..................(2-21)\]
where $\theta$ is the angle of the grade in degrees. Also, for a member of weight $W_i$ ascending a grade of $\theta$ degrees, we have for the grade resistance, $R_G$, in lbs:

$$R_G = W_i \sin \theta \quad \text{(2-22)}$$

However, since $\theta$ is quite small, we have $\sin \theta = \tan \theta = 0.01 \rho_G \{s_i(t)\}$ by eq. (2-21). Hence:

$$R_G = 0.01 W_i \cdot \rho_G \{s_i(t)\} = 0.01 \rho_G \{s_i(t)\} \cdot \text{gm} \quad \text{(2-23)}$$

where $g$ is the acceleration due to gravity (ft/sec$^2$).

### 2.6.2. Curve Resistance:

The effect of varying degrees of curvature on resistance to motion has been determined empirically (46), and most simply stated, one degree of curvature offers approximately the same resistance to motion as a 0.04 per cent grade, that is 0.8 lbs/ton/degree of curvature. Hence, the effect of curve resistance is represented in the model by constructing an "effective" grade $d \{s_i(t)\}$ which is the sum of the actual grade $\rho_G \{s_i(t)\}$ and the equivalent grade due to curvature. Let $\rho_C \{s_i(t)\}$ represent the actual curvature (degrees) as a function of position on the track. Then we have:
\[ d(s_i(t)) = \rho_G(s_{i-1}(t)) + 0.04 \rho_C^{(1)}(a) \quad \text{(2-24)} \]

As eq. (2-24) now stands, the effective grade is expressed as an explicit function of position. Unfortunately, this is an inconvenient representation, since the functions \( \rho_G \) and \( \rho_C \) cannot be expressed in closed form and, as we shall see, \( s_i(t) \) is one of the state variables of the system. To circumvent this difficulty, we shall arbitrarily assume that the effective grade can be defined as an explicit function of time. Let:

\[ d^k(t - \omega_i) = d(s_i(t)) \quad \text{(2-25)} \]

where \( \omega_i \) is the time delay (seconds) which occurs before the grade input to the system appears at the \( i^{th} \) member. It will be found convenient to define \( d^k(t) \) with respect to some position \( s_0(t) \) one member length ahead of the first train member. If \( z_0, h_0 \) and \( w_0 \) denote the parameter values of this imaginary "zeroth" member, then we have for \( i = 1, 2, \ldots n \):

\[ s_i(t) = s_0(t) - \sum_{j=0}^{i-1} (z_j h_j + w_j) \quad \text{(2-26)} \]

and thus for a train travelling at constant speed, we have for \( \omega_i \):
\[ \omega_1 = \frac{\sum_{j=0}^{1} (z_j + h_j + w_j)}{v_1(t)} \]  \hspace{1cm} (2-27)

Since the objective of this whole exercise is to reduce the maximum absolute value of the \( z_1 \)'s, a reasonable approximation for \( \omega_1 \) is:

\[ \omega_1 \approx \frac{\sum_{j=0}^{1} (h_j + w_j)}{v_1(t)} \]  \hspace{1cm} (2-28)

Thus, we see that the specification of \( d^*(t) \) and \( \omega_1 \) for a particular train over a specific piece of track will probably require an iterative approach, since some beforehand assumptions regarding the trajectory of \( v_1(t) \) must be made. It is for this reason that the approximation sign is used in (2-25).

To summarize this and the preceding section, if \( R_C \) is the resistance due to curvature, then the sum of \( R_G \) and \( R_C \) for any member \( i \) is taken to be given by:

\[ R_G + R_C = 0.01 \, \text{gm}_i \, d^*(t-\omega_1) \]  \hspace{1cm} (2-29)

2.6.3: Friction and Air Resistance:

To quote from reference (61):

"A railway vehicle moving upon level, tangent track, in still
and at constant speed encounters certain resistances that must be overcome by the tractive effort of the locomotive.

These resistances include:

1) Rolling friction between wheel and rail:
This varies with the surface condition of the rail under load, the horizontal contour of the railhead, and contour and condition of the wheel tread. This can be considered a constant for a given quality of track.

2) Journal bearing friction:
This varies with the weight of each axle and, at low speed, the type, design and lubrication of the bearing.

3) Train dynamic losses:
These include flange effects which are associated with lateral motion and the resulting friction and impact of the wheel flanges against the gage side of the rail. They vary with speed, rail alignment, and the tracking effect of the trucks. Also, there are miscellaneous losses due to sway, concussion, buffing and slack action.
iv) Air resistance:

This varies directly with the cross-sectional area of the vehicle, its length and shape, and the square of its speed. It is also influenced by zones of turbulence related to shape.

Davis (10) gives an empirical formula for computing the total of the four classes of resistance noted above. This formula is a second degree polynomial in $v_i(t)$, the speed of the $i^{th}$ member. However, it is reasonable to use a linearized form of the Davis formula as follows (25):

$$R_{F+A} = g n_i \{ \lambda_{01} + \lambda_1 v_i(t) \} \quad (2-30)$$

where we have:

$R_{F+A}$: friction and air resistance (lbs) of the $i^{th}$ train member.

$\lambda_{01}$: zero-velocity constant (dimensionless) in linearized version of Davis formula for resistance due to friction and air resistance.

$\lambda_1$: velocity coefficient (secs/ft) in linearized version of Davis formula.
Hence, the total resistance to motion of the $i$-th train member, $y_i(t)$, is given by:

$$y_i(t) = R_G + R_F + R_{F+A} = 0.01 \cdot g \cdot d^*_i(t - \omega_i) + g \cdot m_i \cdot \left( \lambda_{0i} + \lambda_1 y_i(t) \right). \quad \ldots \ldots (2-31)$$

2.7 Overall Train Model:

By combining equations (2-1b), (2-3), (2-4), (2-7), (2-8) and (2-31), one derives the following system equations for the $i$-th member of the train:

$$\begin{align*}
\dot{v}_i(t) &= \frac{a_i - \gamma_i}{m_i} \cdot z_{i-1}(t) + \frac{\beta_{i-1}}{m_i} \cdot v_{i-1}(t) \\
&\quad - \frac{a_i}{m_i} \cdot z_i(t) - \frac{\beta_{i-1} + \beta_i}{m_i} \cdot v_i(t) \\
&\quad - \frac{e_i}{m_i} \cdot \rho_i(t) + \frac{\beta_i}{m_i} \cdot v_{i+1}(t) - g \cdot m_i \cdot \dot{v}_i(t) \\
&\quad - 0.01 \cdot g \cdot d^*(t - \omega_i) + \frac{1}{m_i} \cdot \rho_i(t) - f_i^*(t),
\end{align*} \quad \ldots \ldots (2-32)$$

$$\begin{align*}
\beta_i(t) &= -\frac{1}{\sigma_i} \cdot \rho_i(t) + \frac{n}{\sigma_i} \cdot r^*(t - \tau_i) \\
z_i(t) &= v_i(t) - v_{i+1}(t)
\end{align*}$$

Thus, equations (2-32) constitute the mathematical model for the train member $i$; however, a number of special situations exist which require modification of this system.
I) \( i=1 \):

In this situation there is obviously no coupler force on the leading draft gear of the first member, and therefore \( \alpha_{i=1} \) and \( \beta_{i=1} \) of eqns. (2-32) should be set to zero for \( i=1 \).

II) Member \( i \) is not a powered locomotive.

In this case, \( f_i(t) = 0 \) for all \( t \).

III) \( i=n \):

In this instance, there is obviously no coupler force on the trailing draft gear of the last member, and therefore \( \alpha_i \) and \( \beta_i \) should be set to zero for \( i=n \). Also, the last equation of system (2-32) should be deleted since \( z_i(t) \) is not defined for \( i=n \).

Now, equation (2-1a) with \( i=1 \), equation (2-11), and equations (2-32) for each train member, together constitute the overall train model.
2.8. **State Variable Representation of Train:**

Consideration of the system equations for the train, as derived in the preceding sections, immediately suggests a convenient set of physical variables (see Schultz and Melsa (39), for instance) which may be used to define the state vector of the system. Therefore, let:

\[
\begin{bmatrix}
  s_1(t) \\
  v_1(t) \\
  p_1(t) \\
  z_1(t) \\
  \vdots \\
  v_{n-1}(t) \\
  p_{n-1}(t) \\
  p_n(t) \\
  p_{n+1}(t)
\end{bmatrix} = \begin{cases}
  q_1 \\
  q_2 \\
  q_3 \\
  q_4 \\
  \vdots \\
  q_{3(n-1)} \\
  q_{3n-1} \\
  q_{3n} \\
  q_{3n+1}
\end{cases} = q(t) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots , (2-33)
\]

where \( q \) is the \((3n+1)\)-dimensional state vector of the \( n \) member system.
Let us further define:

\[ f^*_j(t) \]
\[ f^*_j(t_1) \]
\[ f^*_j(t_2) \]
\[ \vdots \]

\[ r^*(t-t_1) \]
\[ r^*(t-t_2) \]
\[ \vdots \]
\[ r^*(t-t_n) \]

\[ 0.01 d^*(t-\omega_1) + \lambda_{01} r_{1+n} \]
\[ 0.01 d^*(t-\omega_2) + \lambda_{02} r_{2+n} \]
\[ \vdots \]
\[ 0.01 d^*(t-\omega_1) + \lambda_{01} r_{1+n} \]
\[ 0.01 d^*(t-\omega_n) + \lambda_{0n} r_{2+n} \]

\[ \begin{bmatrix} r_1 \\ \vdots \\ r_{1+n} \end{bmatrix} = \begin{bmatrix} r_{1+n} \\ \vdots \\ r_{2+n} \end{bmatrix} = r(t) \quad (2-34) \]
where \( r \) is the \((2n+1)\) - dimensional control vector of the system of \( n \) members, \( l \) of which are powered locomotives. Note that the last \( n \) elements of \( r \) are treated as control variables, although in fact they are actually pre-defined inputs.

With the state and control vectors defined as in (2-33) and (2-34), the state variable representation of the train is given by:

\[
\dot{q} = \Delta q + \Omega r. \quad (2-35)
\]

where \( \Delta \) is the \((3n+1) \times (3n+1)\) constant system matrix:

\[
\Delta = \begin{bmatrix}
\Delta_{11} & \Delta_{12} & 0_{33} & \cdots & 0_{33} & 0_{31} \\
0_{33} & \ddots & \ddots & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
0_{33} & \cdots & \cdots & \cdots & \cdots & \Delta_{n+1,n+1}
\end{bmatrix}
\]

\[
......... (2-36)
\]
\[ \Delta_{i,1-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{\beta_{i-1}}{m_i} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad i = 2, 3, \ldots, n \quad \ldots \quad (2-37a) \]

\[ \Delta_{i,1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \left( \frac{\beta_i}{m_i} + g\lambda_i \right) - \frac{e_i}{m_i} \\ 0 & 0 & -\frac{1}{\sigma_i} \end{bmatrix} \quad i = 1 \]

\[ \Delta_{i,1} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{\alpha_{i-1}}{m_i} - \left( \frac{\beta_{i-1} + \beta_i}{m_i} + g\lambda_i \right) - \frac{e_i}{m_i} \\ 0 & 0 & -\frac{1}{\sigma_i} \end{bmatrix} \quad i = 2, 3, \ldots, n-1 \quad \ldots \quad (2-37b) \]

\[ \Delta_{i,1} = \begin{bmatrix} 0 & -1 & 0 \\ \frac{\alpha_{n-1}}{m_n} - \left( \frac{\beta_{n-1} + g\lambda_i}{m_n} \right) - \frac{e_n}{m_n} \\ 0 & 0 & -\frac{1}{\sigma_n} \end{bmatrix} \quad i = n \]

\[ \Delta_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{\alpha_1}{m_1} & \frac{\beta_1}{m_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad l = 1, 2, \ldots, n-1 \quad \ldots \quad (2-37c) \]

\[ \Delta_{n+1,n+1} = \frac{1}{\sigma_{n+1}} \quad \ldots \quad (2-37d) \]
and $O_{33}$, $O_{31}$ and $O_{13}$ are null matrices of order 3x3, 3x1 and 1x3 respectively.

Also, $\Omega$ in (2-35) is the $(3n+1) \times (2n+1)$ constant control matrix:

$$
\Omega = \{\Omega_1, \ldots, \Omega_{n+1}, \ldots, \Omega_{n+1}, \ldots, \Omega_{2n+1}\} \ldots (2-38)
$$

where $\Omega_1$ through $\Omega_{2n+1}$ are defined as follows:

1) $\Omega_1$ through $\Omega_i$:

Each of these $(3n+1)$-dimensional column vectors is associated with a powered locomotive in the train. Note that the subscript of $\Omega_i$ indicates only that there are $i$ such units in the train, and not necessarily that the $i^{th}$ unit is also the $i^{th}$ member of the train; that is, the $i$ powered locomotives could be dispersed throughout the train rather than grouped together at the head end.

Let the $j^{th}$ locomotive be the $i^{th}$ member of the train. Then the only non-zero element of $\Omega_j$ is the $(3i-1)\text{th}$, which has a value of $\frac{i}{m_i}$.
11) \( \mathbf{a}_{n+1} \) through \( \mathbf{a}_{n+1} \):

Each of these \( n (3n+1) \) - dimensional column vectors is associated with the braking system for each train member. For the \( i^{th} \) member, if \( i \neq j \), then the only non-zero element of \( \mathbf{a}_{i+1} \) is the \( (3i)^{th} \), which has a value of \( \frac{g}{\sigma_i} \). If \( i = j \), then \( \mathbf{a}_{i+1} \) has one other non-zero element, the \( (3n+1)^{th} \), which has a value of \( \frac{1}{\sigma_{n+1}} \).

11i) \( \mathbf{a}_{1+n+1} \) through \( \mathbf{a}_{2n+1} \):

Each of these \( n (3n+1) \) - dimensional column vectors is associated with the total resistance to motion at each train member due to grades, curves, friction and air resistance. For the \( i^{th} \) member, then, the only non-zero element of \( \mathbf{a}_{1+n+1} \) is the \( (3i-1)^{th} \), which has a value of \( g \).

In summary, with the definitions (2-33), (2-34), (2-36), (2-37) and (2-38), then the state variable representation of the train is given by system (2-35) along with the constraints indicated by (2-13) and (2-20).
3.1. The Optimal Control Problem:

Referring to the problem description given in section 1.1, we recall that our objective is to devise a control procedure which minimizes the maximum coupler force experienced anywhere in the train during its operation over a given section of track. This objective will be realized if we can minimize the maximum draft gear extension or compression, since coupler forces are largely proportional to this variable.

\[
\text{If } \max_{t_0 \leq t \leq t_f} \left[ \left| z_1(t) \right|, \left| z_2(t) \right|, \ldots, \left| z_1(t) \right|, \ldots, \left| z_{n-1}(t) \right| \right] 
\]

represents the maximum draft gear extension or compression experienced anywhere in the train between the start and end of operation at times \(t_0\) and \(t_f\) respectively, then the optimal control problem may be stated precisely as follows:
"Given the linear system (2-35), it is desired to find an admissible control vector \( r(t) \) which transfers the system from its initial state \( q(t_0) \) to some desired final state \( q(t_f) \) such that the performance measure

\[
\max_{t_0 \leq t \leq t_f} \left[ |z_1(t)|, |z_2(t)|, \ldots, |z_{n-1}(t)|, |z_n(t)| \right]
\]

is minimized."  

\[ (3-1) \]

### 3.2. Possible Approaches to the Problem:

The specific problem (3-1) has the following characteristics:

1) The state vector of system (2-35) is of very high dimension; for example, the representation of a 100 member train would require 30\( l \) state variables.

2) There are time delays in the control vector of system (2-35), as is evident in the definition of \( r \) given by (2-34). As a matter of fact, the usual description of such a system would be:

\[
q(t) = \Delta q(t) + \sum_{j=0}^{\tau} \{ \Omega' r'(t-\tau_j) + \Omega'' r''(t-\omega_j) \}
\]

where \( \Omega', \Omega'', r', \text{ and } r'' \) are control matrices and vectors of appropriate dimensions. Olbrot (28) illustrates a procedure for determining...
the absolute controllability of linear systems with time delays in control, and Sebakhy and Bayoumi (41) give a simplified criterion for the controllability of such systems. However, for the present, it will be convenient to retain the system description given by (2-35) and in particular the control vector definition (2-34).

(iii) The performance measure selected is not of the standard integral form.

(iv) The control vector \( r \) is rather severely constrained. In particular, the first \( l \) control variables are constrained as in (2-20), the next \( n \) as in (2-13), and the final \( n \) control variables are actually pre-determined inputs or forcing functions whose value over time is fixed a priori and not subject to manipulation by the optimization process.

(v) We do not seek the optimal control law \( r(q(t)) \); rather, we seek the optimal control function \( r(t) \). That is, it would be sufficient for our purposes to determine the optimal open-loop control, rather than the optimal feedback control.
This is consistent with the discussion in section 1.3 regarding the relationship of this work to that of McLane and Peppard (25).

Consideration of these problem characteristics in relation to the available optimization techniques (Pierre (31) provides an excellent catalogue, for example) led the author to the opinion that the only hope for a practical solution method would be to apply some suitable mathematical programming technique to a discrete representation of problem (3-1). For instance, the two point boundary value problem which results from the application of variational techniques is intractable for such large dimensional systems. Similarly, the application of dynamic programming, particularly the state increment form of Larson (22), would have been a very elegant approach, had not Bellman's "curse of dimensionality" made it impractical.

Canon et al (7), Dantzig (9), and Polak (33) provide rigorous treatments of the applicability of mathematical programming (MP) to optimal control problems, and Torn (47), Zadeh and Whalen (52) give specific examples of linear programming (LP) applications in particular.

Wheeler (49) demonstrates a fairly straightforward procedure for transforming any linear optimal control problem into either a linear, quadratic, or convex programming problem. Unfortunately, the method retains the state variables in the MP formulation, so
that for other than very low-order systems the matrices involved become extremely large. As an example, application of Wheeler's transformation of the discrete problem analogous to (3-1) for a 100 member train over 100 discrete time intervals would result in an LP matrix containing in the order of two billion elements.

In a procedure outlined by Enns (12,13) and Lack (21), a transformation from state space to control space permits an LP formulation with the crucial property that the dimension of the LP tableau is not directly dependent on the number of state variables. This solution method, to be described in section 3.4, will be applied to a discretized form of problem (3-1).

### 3.3. Discrete Representation of the Problem:

In order to apply an LP technique, the set of control trajectory values to be optimized must obviously be finite. A discrete form of problem (3-1) must therefore be derived.

#### 3.3.1. System Equations

Disregarding for the present the time delays present in some elements of the control vector defined in (2-34), let us consider the system (2-35). This representation implies that the control
vector is a continuously varying function of time, whereas, in the practical operation of the actual train, this will not be the case. Rather, it is much more realistic to assume that any particular control vector value will be held constant over some minimum time period. Hence, the control variables are actually piecewise constant signals, and this fact is emphasized by writing:

\[ r(t) = r(kT), \quad \left\{ \begin{array}{l} kT < t < (k+1)T \\ k = 0, 1, \ldots, K-1 \end{array} \right. \] \quad \ldots \ldots (3-2)

where \( T = t_f / K \) is the minimum time interval over which the control signals are assumed to remain constant, and \( K \) is an appropriately specified integer parameter.

It should be stressed that \( r(kT) \) is not merely an approximation of \( r(t) \); rather, the representation (3-2) is an exact equivalence so long as \( K \) is appropriately selected.

With \( r(t) \) defined as in (3-2) we no longer require the continuous time response, and in fact need only obtain the values of the state vector at the end of each time interval. To this end, let us first consider the continuous solution of the state equations (2-35), as given in any standard reference (see, for instance, Schwarz and Friedland (40)):

\[ q(t) = \Phi(t-t_0)q(t_0) + \int_{t_0}^{t} \Phi(t-t')qr(t') \, dt' \] \quad \ldots \ldots (3-3)

where \( t' \) is a dummy variable and \( \Phi(t) = e^{At} \) is the state transition matrix of the system (2-35).
Now, letting $t_0 = kT$ and $t = (k+1)T$ in (3-3), we have:

\[ q((k+1)T) = \Phi(T)q(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T-t') \Omega r(t') dt' \quad \cdots (3-4) \]

With $r$ as defined in (3-2):

\[ q_{t}((k+1)T) = \Phi(T)q(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T-t') \Omega dt' \cdot r(kT) \quad \cdots (3-5) \]

The integral term in (3-5) has the same value for all $k$, and therefore:

\[ \Delta q_{t}((k+1)T) = \Phi(T)q(kT) + \int_{0}^{T} \Phi(T-t') \Omega dt' \cdot r(kT) \quad \cdots (3-6) \]

For notational convenience, let us omit the argument $T$ and define:

\[ \Phi* = \Phi(T) \quad \cdots (3-7) \]

\[ \Omega* = \int_{0}^{T} \Phi(T-t') \cdot \Omega dt' \quad \cdots (3-8) \]

Then we have the set of $(3n+1)$ difference equations:

\[ q(k+1) = \Delta* q(k) + \Omega* r(k), \quad k = 0, 1, \ldots, k-1 \quad \cdots (3-9) \]

Enns (12) gives the following equivalent definitions for $\Delta*$ and $\Omega*$:
\[
\Delta^* = \sum_{n=0}^{\infty} \frac{1}{n!} (\Delta \tau)^n \quad \ldots \quad (3-10)
\]

\[
\Omega^* = I \left[ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (\Delta \tau)^n \right] \Omega \quad \ldots \quad (3-11)
\]

Equations (3-10) and (3-11) are useful for calculating \( \Delta^* \) and \( \Omega^* \) for small values of \( T \), and Johnson and Phillips (18) have derived a practical method of evaluating \( \Omega^* \) for large \( T \).

The algebraic recursion relation (3-9) is thus an exact discretized representation of system (2-35) for a suitably chosen value of \( k \).

3.3.2. Time Delays in Controls:

We may now turn our attention to the time delays in the control vector \( r \). Recalling definitions (2-34) and (3-7), we may write:

\[
r_{i+1}(t) = \Delta^*(t-T_1) = \Delta^*(k-T_1) \quad \ldots \quad (3-12)
\]

\[
r_{i+n+1}(t) = 0.01d^*(t-\omega_i) + \lambda_{0i}
\]

\[
= 0.01d^*(k-\omega_i) + \lambda_0 \quad \ldots \quad (3-13)
\]

for \( i = 1, 2, \ldots, n \), and where \( k \) has been specified so that \( T_1 = r_1/T \) and \( \omega_i = \omega_1/T \) are integers. Also, \( \lambda_0 = \lambda_{01} \) has been
assumed constant for all members of the train.

Therefore $r_{i+1}$ is merely the control variable $r^a(k)$ delayed in time by $\tau_i$ intervals. Similarly, $r_{i+n+1}$ is the control variable $\{0.01d^a(k)+\lambda_0\}$ delayed in time by $\bar{\tau}_i$ intervals. Since (3-9) is a set of difference equations, we may take advantage of relationships (3-12) and (3-13) to reduce the dimension of the control vector by incorporating the system time delays in the definitions of the system matrix and the state vector (see Noton (26) for example).

To this end, let us define the \((1+2)\)-dimensional control vector $\bar{u}(k)$ as follows:

$$
\begin{pmatrix}
\bar{r}_1(k) \\
\bar{r}_2(k) \\
\bar{r}_3(k) \\
\bar{r}_4(k) \\
0.01d^a(k)+\lambda_0
\end{pmatrix}
= u(k)
$$

(3-14)

Let us further define the \((3n+2+1+\bar{\tau}_i+\bar{u}_i)\)-dimensional state vector $\bar{x}(k)$ by augmenting the vector $q$ of definition (2-33) as follows:
where $q^j(k)$ is the vector appended to $q(k)$ and is defined by the following relationships:

\[
\begin{align*}
q^j(k+1) & = f^j_1(k) = u^j_1(k) \\
& \vdots \\
q^j(k+1) & = f^j_j(k) = u^j_j(k) \\
& \vdots \\
q^j(k+1) & = f^j_1(k) = u^j_1(k) \\
q^j(k+1) & = q^j(k) \\
& \vdots \\
q^j(k+1) & = q^j(k) \\
& \vdots \\
\lambda \left( k+1 \right) & = q^j(k) \\
& \vdots \\
q^j(k+1) & = q^j(k) \\
& \vdots \\
q^j(k+1) & = q^j(k) \\
& \vdots \\
q^j(k+1) & = 0.01 \lambda^j(k) + \lambda_0 = u^j_{1+2}(k)
\end{align*}
\]
Note that a time delay of \( \tau \) has been introduced in the control variable \( u_j(k) = f_j^*(k) \), \( j = 1, 2, \ldots, l \). This will facilitate the expression of constraints (2.20), since \( v_i(k) \) and \( f_j^*(k-1) \) are now both in state space and, as we shall see, linear state variable constraints can readily be accommodated in the LP formulation. This delay may either be regarded as representative of the reaction time of the engineman and the diesel locomotive, or may be discounted entirely by treating \( f_j^*(k-1) \) as the actual system input when the optimal control trajectory has been found.

With \( x(k) \) and \( u(k) \) defined as in (3-15) and (3-14), the relationships (3-16) may be included in the state variable description of the system (3-9) by writing:

\[
x(k+1) = Ax(k) + Bu(k), \quad k = 0, 1, \ldots, K-1 \quad \ldots \ldots (3-17)
\]

where the \((3n+1+l+\tau+\omega) \times (3n+1+l+\tau+\omega)\) constant system matrix \( A \) is given by:

\[
\hat{A} = \begin{bmatrix}
A_{11} & A_{12} \\
-\tau & -\omega \\
A_{21} & A_{22}
\end{bmatrix} \quad \ldots \ldots (3-18)
\]

and the various sub-matrices of \( A \) are defined as follows:
i) \( A_{11} \) is the \((3n+1) \times (3n+1)\) matrix:

\[
A_{11} = A^* \tag{3-19}
\]

\[
\begin{bmatrix}
\Omega_1^* & \cdots & \Omega_j^* & \cdots & \Omega_{n+1}^* & 0 & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \Omega_1^* & \cdots & 0 & \cdots & \Omega_{n+1}^* \\
0 & \cdots & 0 & \cdots & 0 & \cdots & \Omega_1^* & \cdots & 0 \\
\end{bmatrix}
\tag{3-20}
\]

In which \( \Omega_1^* \) through \( \Omega_{2n+1}^* \) are the column vectors of the matrix \( \Omega^* \).

Note particularly the inverted order of \( \Omega_{1+1}^* \) through \( \Omega_{n+1}^* \), and of \( \Omega_{1+n+1}^* \) through \( \Omega_{2n+1}^* \). Note also that if \( \tau_n > n \) there are:

\( (\tau_n - n) (3n+1) \) dimensional null vectors \( 0 \) dispersed through the sequence \( \Omega_{1+1}^* \) through \( \Omega_{n+1}^* \). Similarly, if \( \omega_n > n \) there are \( \omega_n - n \) \( 0 \) vectors dispersed through the sequence \( \Omega_{1+n+1}^* \) through \( \Omega_{2n+1}^* \). There are no \( 0 \) vectors for \( \tau_n + \omega_n \leq 2n \).

ii) \( A_{12} \) is the \((3n+1) \times (1+\tau_n + \omega_n)\) matrix:

\[
A_{12} = A^* \tag{3-20}
\]

iv) \( A_{22} \) is the \((1+\tau_n + \omega_n) \times (1+\tau_n + \omega_n)\) matrix:
\[ a_{ij} = \begin{cases} 
1, & i=(1+1), \ldots, (1+\tau_n-1); j=(1+2), \ldots, (1+\tau_n) \\
1, & i=(1+\tau_n+1), \ldots, (1+\tau_n+\omega_n-1); j=(1+\tau_n+2), \ldots, (1+\tau_n+\omega_n) \\
0, & \text{otherwise} 
\end{cases} \]

\[ \begin{array}{c}
\vspace{5mm}
\text{\begin{equation}(3-21)\end{equation}}
\end{array} \]

and where the \((3n+1+1+\tau_n+\omega_n) \times (1+2)\) constant control matrix

\[ B = [b_{ij}] \]

is defined by:

\[ b_{ij} = \begin{cases} 
1, & i=(3n+2), \ldots, (3n+1+1); j=1, \ldots, 1 \\
1, & i=(3n+1+1+\tau_n); j=1+1 \\
1, & i=(3n+1+1+\tau_n+\omega_n); j=1+2 \\
0, & \text{otherwise} 
\end{cases} \]

\[ \text{\begin{equation}(3-22)\end{equation}} \]

Thus, \((3-47)\) constitutes a discretization of system \((2-35)\) in which the time delays in control are represented by additional state variables.

3.3.3. System Constraints:

1) The constraints \((2-20)\) on \( u_j(k) = f_j(k), j=1, \ldots, 1; \)
cannot be expressed in this form since only linear constraints can be accommodated by an LP technique. Instead, the following approximations will be considered as the constraints on the tractive or braking effort:

\[
\begin{align*}
|x_{3n+1+j}(k)| & \leq u_j \\
-x_{3n+1+j}(k) + L^i x_2(k) & \leq L^ii \\
x_{3n+1+j}(k) + M^i x_2(k) & \leq M^i \quad j = 1, \ldots, 4 \\
x_{3n+1+j}(k) + \xi x_2(k) & \geq 0 \\
x_{3n+1+j}(k) & \geq -z
\end{align*}
\]

where \(L^i, L^ii, M^i\) and \(M^i\) are positive constants chosen so as to best approximate the constraints $(2-20)$ over the speed range of interest, and where \(x_{3n+1+j}(k) = u_j(k-1) = f_j^e(k-1)\). Since usually only a specific speed range is desired, not all of the constraints...
(3-23) may need to be applied. For example, if the speed of the train (as defined by \( x_2(k) = v_1(k) \)), is restricted to between \( V' \) and \( V'' \) in Figure 3.1, then only the indicated subset of (3.23) is required.

If a single train speed is specified rather than a range as in Figure 3.1, then obviously only absolute bounds on \( u_j(k), \ j=1, \ldots, 1, \) need be applied.

II) The constraints (2-13) on \( u_{1+1}(k) = r*(k) \) are difficult to express simultaneously. Instead, the problem will first be run with the constraint:

\[
0 \leq u_{1+1}(k) \leq 23, \ \text{k}=0,1,\ldots,K-1 \quad \ldots \ldots (3-24)
\]

Then by examining the optimal trajectory of \( u(k) \) and \( x(k) \), one or the other of the following constraints will be applied for values of \( k \) specified on the basis of the trajectory of \( x_{3n+1}(k) \):

\[
\left\{ \begin{array}{l}
\text{either } u_{1+1}(k) = 0 \\
\text{or } u_{1+1}(k-1) \leq u_{1+1}(k) \leq 23
\end{array} \right\} (k \text{ as specified}) \quad (3-25)
\]
Figure 3.1: Approximation of constraints (2-20) by linear constraints (3-23). Hatched area is allowable region for the indicated $V'$ and $V''$. Solid lines are constraints (2-20) from Figure 2.5.
With constraints (3-25) specified for appropriate values of \( k \), a new/optimal trajectory should then be calculated.

\( \text{III} \) Train speed will usually be restricted by one of the following constraints:

\[
\begin{align*}
\text{either } & \quad x_2(k) = V' \\
\text{or } & \quad V' \leq x_2(k) \leq V''
\end{align*}
\]

(3-26)

\( \text{III} \) As indicated in section 3.2(iv), the control variable \( u_{1+2}(k) \) is actually fixed a priori by the specification of the effective grade \( d^*(k) \) as a function of \( k \).

\[
u_{1+2}(k) = 0.01d^*(k) + \lambda_0
\]

(3-27)

If the first of the constraints (3-26) is specified, then \( x_1(k) = s_4(k) \) is known and hence \( d^*(k) \) as well as \( u_{1+2}(k) \) may be defined exactly. However, if the second of the constraints (3-26) is applied to the system then the following iterative approach is required:

1) guess at the probable trajectory of \( x_1(k) \).

2) based on (1), specify \( d^*(k) \) and hence \( u_{1+2}(k) \).
(III) calculate the optimal trajectories,

(iv) examine \( d^*(k) \) and \( x_1(k) \) from step (III) and re-specify \( d^*(k) \) accordingly.

(v) repeat steps (III) and (iv) until a satisfactory correlation of \( x_1(k) \) and \( d^*(k) \) is obtained.

3.3.4. Restatement of the Problem

"Given the linear discrete-time system \((3-17)\), it is desired to find a control vector \( u(k) \) which transfers the system from its initial state \( x(0) \) to some desired final state \( x(k) \) such that the specified constraints \((3-23)\) through \((3-27)\) are met, and such that the performance measure

\[
\max_{0 \leq k \leq K} \left\{ x_4(k), x_7(k), \ldots, x_{3n-2}(k) \right\}
\]

is minimized. \((3-28)\)

3.4. A Method of Solution

Lack (21) and Enns (12, 13) give a straightforward procedure for calculating the optimal control trajectory for problems such as \((3-28)\) by linear programming. The reader is referred to reference (3.5) in particular for an excellent detailed description.
of the method, but a brief summary of the procedure in terms of problem (3-28) is presented here for convenience.

3.4.1. Basic Solution Method:

Consider the system:

\[ x(k+1) = Ax(k) + Bu(k), \quad k = 0, 1, \ldots, K-1 \]  \hfill (3-27)

where \( x(k) \) is the \((3n+1+1)^{\text{st}}\) dimensional state vector, \( u(k) \) is the \((1+2)^{\text{nd}}\) dimensional control vector, and \( A \) and \( B \) are the constant matrices of the system as defined by (3-18) and (3-22) respectively.

The initial condition:

\[ x(0) = x_0 \]  \hfill (3-29)

and the terminal condition:

\[ f^T x(K) = f \]  \hfill (3-30)

must be satisfied, where \( f \) is an \( m \)-dimensional vector and \( F \) is an \( m \times (3n+1+1+1+1) \) matrix.
The optimality criterion is specified by defining a region \( E^+ \) in state space and requiring the state trajectory to stay as far from this region as possible. That is, we wish to find the control which maximizes the distance of closest approach of the trajectory \( x(k) \) to the region \( E^- \) while meeting conditions (3-29) and (3-30). The region \( E^+ \) is the half-space in \( \mathbb{E}^n \) defined by:

\[
E^+ = \{ x : \langle d, x \rangle > \langle d, d \rangle \} \quad \text{(3-31)}
\]

where \( d \) is a vector normal to the separating hyperplane, as illustrated for the two dimensional case in Figure 3.2. (The half-space \( E^- \) is defined analogously). The notation \( \langle d, x \rangle \) indicates the inner (scalar) product of the vectors \( d \) and \( x \); geometrically, \( \langle d, x \rangle \) may be considered to be the "projection" of the vector \( x \) on the direction vector \( d \).

For the specific problem (3-28), we ideally require 2\((n-1)\) such \( d \) vectors to be defined, each specifying the "danger region" for the state variable \( x_{3i+1} \), \( i = 1, 2, \ldots, n-1 \) in turn. This point will be discussed further in chapter 5.

Enns shows that the maximization of the minimum distance from the trajectory \( x(k) \) to the region \( E^+ \) is equivalent to the minimax problem:

\[
\text{maximize} \quad \text{minimize} \langle d, x(k) - e \rangle
\]
Figure 3.2: Illustration of optimality criterion for state space of dimension 2.
Find \( u(k) \) so as to:

\[
\begin{aligned}
\max_{u(k)} \min_k & \left[ \frac{\langle d, d-x(k) \rangle}{\langle d, d \rangle} \right] \\
\end{aligned}
\]

\( \ldots \ldots \ldots (3-32) \)

The key to transforming problem (3-32) to an LP problem is to introduce a dummy variable \( \pi \) (a scalar), and the set of constraints:

\[
\langle d, x(k) \rangle \leq \pi, \ k = 0, 1, \ldots, K \quad \ldots \ldots \ldots (3-33)
\]

Then (3-32) is equivalent to the constrained minimization problem:

\[
\begin{aligned}
\min \pi, \ \text{subject to} \ (3-33) \quad & \ldots \ldots \ldots (3-34) \\
u(k) & \\
\end{aligned}
\]

Accomplishing (3-34) amounts to "squashing" the state trajectory with a hyperplane normal to the \( d \) vector and parallel to the boundary of the region \( \mathbb{E}^+ \).

To put the problem (3-34) into an LP form, define the \((1+2K+1)\) dimensional vector \( u \) and the constant \((1+2K+1)\) dimensional vector \( c \) by:
\[ u = \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(K-1) \\ \pi \end{bmatrix} \]  \hspace{1cm} (3-35)

\[ c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \]  \hspace{1cm} (3-36)

where it is understood that \( u(k) \) is defined by (3-14).

Now, the crucial operation in this procedure is the transformation from state space to control space accomplished by writing the state vector \( x \) at any time \( k \) as a linear function of \( u \) by repeated use of (3-17) with (3-29):

\[
x(1) = Ax(0) + Bu(0) = Ax_0 + Bu(0)
\]

\[
x(2) = Ax(1) + Bu(1) = A^2x_0 + ABu(0) + Bu(1)
\]

\[
x(k) = Ax(k-1) + Bu(k-1) = A^kx_0 + A^{k-1}Bu(0) + A^{k-2}Bu(1) + \ldots + ABu(k-2) + Bu(k-1)
\]

or:

\[
x(k) = A^kx_0 + \sum_{j=0}^{k-1} A^{k-1-j}Bu(j) \]  \hspace{1cm} (3-37)
By defining the \((3n+1+1+n+m) \times (1+2K)\) matrix \(D(k)\) as:

\[
D(k) = \begin{bmatrix}
          A^{k-1}B & A^{k-2}B & \ldots & AB & B & 0 & \ldots & 0
        \end{bmatrix}
\]

then (3-37) may be written:

\[
x(k) = A^k x_0 + \begin{bmatrix} D(k) & 0 \end{bmatrix} u
\]

and hence the condition (3-30) as:

\[
\begin{bmatrix} F & D(K) & 0 \end{bmatrix} u = f - FA^k x_0
\]

and the constraints (3-33) as:

\[
\left[ \begin{bmatrix} d, D(k) \end{bmatrix} \mid -1 \right] u \leq -\left[ \begin{bmatrix} d, A^k x_0 \end{bmatrix} \right] , \quad k_d = 0, 1, \ldots, K
\]

where \(\left[ d, D(k) \right]\) indicates the inner product of \(d\) with each of the columns of \(D(k)\), the result being a row vector.

Thus the control problem (3-32) is shown by Enns to be equivalent to the problem:

\[
\begin{cases}
  \text{Find } u \text{ to } \min \quad \langle c, u \rangle \\
  \text{subject to } (3-40) \text{ and } (3-41)
\end{cases}
\]

\[
(3-42)
\]
This is the standard form of linear programming and the optimal control may be found directly by the simplex algorithm (see Künzi (20), for instance).

A distinct advantage of this method is that the problem (3.42) may be readily extended to include the linear constraints associated with problem (3.28) directly as additional LP constraints.

3.4.2 Extension of the Problem to Include State Variable Constraints:

Egnn shows that any linear equality or inequality relation for the state variables may be included as additional LP constraints by means of (3.37) and (3.38). In general, the constraint:

\[ <d, x(k) > \left[ \begin{array}{c} \vdots \\ \vdots \end{array} \right] b, \ k = 0, 1, \ldots, K \]  

may be formulated as the set of (K+1) scalar LP constraints:

\[ \left\{ \begin{array}{l}
0 \quad \text{b} - <d, x_0 > \\
[<d; D(1)> | 0] u \quad \text{b} - <d; Ax_0 > \\
[<d; D(k) > | 0] u \quad \text{b} - <d; A^k x_0 > \\
[<d; D(k) > | 0] u \quad \text{b} - <d; A^k x_0 > 
\end{array} \right\} \]
Note that these constraints may be time-varying, that is, \( h(k) \) in general.

### 3.4.3. Extension of the Problem to Include Control Variable Constraints:

Again, Enns shows that any linear equality or inequality constraint on the control may be included as additional LP constraints. Although constraints on the control variables of the type (3-43) may be similarly expressed, the more usual situation is that the constraints are simple bounds on the absolute value of a particular control variable. This type of constraint:

\[
\left| u_1(k) \right| \leq H(k) \tag{3-45}
\]

is equivalent to the two linear constraints:

\[
\begin{align*}
- H(k) & \geq u_1(k) \\
+ H(k) & \leq u_1(k)
\end{align*}
\tag{3-46}
\]

and hence may be readily included in problem (3-42) as additional LP constraints. The even simpler constraint:

\[
u_1(k) = H(k) \tag{3-47}\]
may obviously be incorporated in the LP formulation as well. Constraint (3-47) is the means by which a particular control variable is treated as a fixed input. The notation $H(k)$ has been used in (3-45) to (3-47) to emphasize the fact that the inequality or equality constraint on the control variable may be time varying.

3.5. Linear Programming Problem Equivalent to the Control Problem:

Providing that a suitable set of direction vectors $d$ can be defined, we have thus transformed the original control problem (3-1) to the linear programming problem:

\[
\text{Find } u \text{ to } \min <c, u> \text{ subject to constraints } (3-40) \\
\quad \text{and } (3-41), \text{ in } \quad \left\{\begin{array}{l}
(3-44), \\
(3-46) \text{ and } (3-47) \text{ as required.}\end{array}\right.\]

A remarkable feature of this method is that the dimension of the state vector of problem (3-28), the discrete equivalent of (3-1), does not enter into the dimension of the equivalent LP problem (3-48). In particular, if the:

1) number of control variables $= (1+2)$
ii) number of scalar control constraints = \( u \)

iii) number of state trajectory constraints
or minimax objectives = \( x \)

iv) number of state variable terminal constraints = \( m \)

v) number of discrete time intervals = \( K \)

then, as illustrated by Enns (12), the dimensions of the corresponding LP matrix are as follows:

\[
\begin{align*}
\text{Number of rows} & = \{(x+u)K + x+m\} \\
\text{Number of columns} & = \{(1+2)K + 1\}
\end{align*}
\] .............. (3-49)

Thus the limiting factor is now the total number of constraints and discrete time intervals rather than the dimension of state space. This characteristic will be elaborated in Chapter 5, but let us first illustrate the technique with a small example.
CHAPTER 4

EXAMPLE: A THREE-MEMBER TRAIN

To illustrate the solution technique, consider the simple case of a train consisting of just three "members." However, in order that reasonably large coupler forces will be realized, we shall assume that the leading train member actually comprises two 2,000-hp locomotives and ten freight cars, and that the other two members each comprise twelve freight cars. The gross weight of each freight car and locomotive is taken to be 80 tons and 190 tons respectively, and the overall length of each vehicle is assumed to be 44 feet. The velocity at some given initial position is taken to be 45 mph, and we require the train to travel an additional mile in 80 seconds. It should be noted that although we are requiring the average speed to be equal to the initial speed, we are not specifying what the velocity/trajectory should be over the run-time interval, except that a speed limit of 60 mph is assumed to apply over the given territory.

The parameter values of this 4000 hp, 3100 ton train are given in Table 4.1.
\[ m_1 = (2 \times 190) + (10 \times 80) = 1180 \text{ tons} = 73,292 \text{ slugs} \]
\[ m_2 = m_3 = (12 \times 80) = 960 \text{ tons} = 39,627 \text{ slugs} \]
\[ a_1 = a_2 = 500,000 \text{ lbs/ft.} \]
\[ \beta_1 = \beta_2 = 15,000 \text{ lbs-sec/ft.} \]
\[ e_1 = e_2 = e_3 = 1440 \text{ in.}^2 \]
\[ \lambda_0 = 0 \]
\[ \lambda_1 = 0.000054 \text{ secs/ft.} \]
\[ \lambda_2 = \lambda_3 = 0.000045 \text{ secs/ft.} \]
\[ \sigma_1 = \sigma_2 = \sigma_3 = 15 \text{ secs.} \]
\[ \sigma_4 = 60 \text{ secs.} \]
\[ \tau_1 = 4 \text{ secs.} \]
\[ \tau_2 = 8 \text{ secs.} \]
\[ \tau_3 = 12 \text{ secs.} \]
\[ \omega_1 = 8 \text{ secs} \]
\[ \omega_2 = 16 \text{ secs.} \]
\[ \omega_3 = 24 \text{ secs.} \]
\[ n = 2.5 \]

*Table 4.1: Train Parameter Values*
The $w_i$'s of Table 4.1 are calculated from (2.28), where

$$w_i = 12 \times 44 = 528 \text{ ft. for } i = 1, 2, 3.$$ 

Let the effective grade under consideration be such that its maximum frequency of change in value is once per 350 feet. At the given speed limit of 88 ft/sec., this implies that the effective grade input to each member will be constant for at least four seconds. Since it is reasonable to require the other inputs to the system (i.e., air brakes and throttle/dynamic brake applications) to be held constant for at least this maximum time period as well, then it is concluded that a discrete time interval of four seconds, would be a satisfactory assumption in this case. We therefore require the number of time intervals to be $K^* t_f / T = 80/4 = 20.$

Let the effective grade as a function of discrete time interval $k$ be specified as in Figure 4.1.

Note from Table 4.1 that $\lambda_0$ has been assumed to be zero. Thus the control variable $u_3(k)$, as defined by (3.27), is simply given as $0.01 d_x(k)$. This assumption for $\lambda_0$ was made merely to simplify the example; any other value would just serve to bias this function.
Figure 4.1: Effective grade as a function of discrete time interval $k$. $d^*(k)$ is specified relative to $s_0(k)$ (see discussion in section 2.6.2).
Let us now cast this problem in terms of the previous developments in Chapter 3.

For this case, we have \( n = 3, l = 1, \tau_n = \tau_3/T = 3 \), and \( \bar{\omega}_n = \omega_3/T = 6 \). Thus the dimension of the state vector \( x(k) \) is 20, and that of the control vector \( u(k) \) is 3.

As the initial condition \( x_0 \) of the train, we shall assume that the starting position is considered to be zero, the velocities of the three members are equal, the draft gears are neither extended nor compressed and the brake cylinders have all been released for some previous time sufficient to ensure that no false gradient presently exists. Then for \( x_0 \) we have:

\[
x_0 = \begin{bmatrix} 0 & 66 & 0 & 0 & 66 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}^T \quad \ldots \ldots \quad (4-1)
\]

Now consider for a moment the 61-dimensional vector \( u \), as defined by (3-35):

\[
u = \begin{bmatrix} u_1(0) & u_2(0) & u_3(0) & u_1(1) & u_2(1) & u_3(1) & \ldots & u_1(19) & u_2(19) & u_3(19) \end{bmatrix}^T \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (4-2)
\]

In the LP formulation in Chapter 3, it was assumed that all the LP variables (i.e., the elements of \( u \)) were positive. Except for \( u_2(k) \), this is not the case for our train; however, we can establish
definite lower bounds for each of the other elements of \( u \).
For example, since we have specified that the average speed over
the 80 second run should be 45 mph, we may be certain that the
train speed will never drop below, say, 20 mph. Hence, by
applying inequality (2-17), we know that \( u_1(k) \geq -87,000 \) for
all \( k \). Similarly, from the given effective grade specifications,
we know that \( u_3(k) \geq -0.04 \) for all \( k \). Therefore, \( u \) may be
offset appropriately so that the resultant vector contains only
positive elements. To this end, let \( u' \) be defined as:

\[
\begin{align*}
u' &= u + U \\
&= \begin{bmatrix}
87,000 & 0 & 0.04 & 87,000 & 0 & 0.04 & \ldots & 87,000 & 0 & 0.04 \\
\end{bmatrix}^T
\end{align*}
\]

where \( U \) is a constant offset vector, given in this example as:

By substituting \( u' \) for \( u \) as the LP vector, the requirement for
positive LP variables is met. The LP problem may then be
solved, and the true solution subsequently may be derived simply by
subtracting \( U \) from the optimal \( u' \) obtained.

If a \( 61 \)-dimensional cost vector \( c \) is defined as in (3-36),
then our problem is to find \( u \) to minimize \( \langle c, u' \rangle \) subject to the
following constraints:
1) **Objective Constraints:**

Define a set of 20-dimensional direction vectors \( d_{3i+1} \) for \( i = 1, 2, \) where the \((3i+1)\)th element is 1 and the remainder are zero. Then, from (3-41) and (4-3), we have the set of 42 objective constraints:

\[
\left[ \begin{array}{c}
\langle d_{3i+1}, D(k) \rangle \\
\end{array} \right] U' \leq \left[ \begin{array}{c}
\langle d_{3i+1}, D(k) \rangle \\
\end{array} \right] U - \langle d_{3i+1}, A^k x_0 \rangle
\]

\[
\text{(4-5)}
\]

for \( i = 1, 2 \) and \( k = 0, 1, \ldots, 20. \)

**Terminal Constraint:**

The only terminal constraint being set for this train \((m = 1)\) is that its position at the end of the 80 second time interval should be one mile more than its initial position. From (3-40) and (4-3), this constraint is expressed by:

\[
\left[ \begin{array}{c}
F D(20) \\
\end{array} \right] U' = \left[ \begin{array}{c}
F D(20) \\
\end{array} \right] U + f - FA^{20} x_0 
\]

\[
\text{(4-6)}
\]

where the 20-dimensional row vector \( F = [1, 0, 0, \ldots, 0] \) and \( f = 5280. \)
III) Velocity Constraints:

Define a 20-dimensional vector \( \mathbf{d}_2 \) whose second element is equal to 1 and the remainder are 0. Then from (3-44) and (4-3), we have the following 20 velocity constraints:

\[
\begin{bmatrix} \mathbf{d}_2, D(k) \end{bmatrix} \mathbf{u}^t \leq \begin{bmatrix} \mathbf{d}_2, D(k) \end{bmatrix} \mathbf{u} + v_{MAX} - \begin{bmatrix} \mathbf{d}_2, A^k \end{bmatrix} x_0 \quad \ldots \ldots \ldots (4-7)
\]

for \( k = 1, 2, \ldots, 20 \) and where \( v_{MAX} = 88 \) for this example.

IV) Tractive Effort Constraints:

With reference to Figure 3.1, reasonable values for \( v^I \) and \( v^{II} \) would be 30 and 60 mph respectively. By applying equality (2-15), the parameters of constraints (3-23b) and (3-23c) over this speed range are found to be as follows:

\( L^I = 465.91, \quad L^{II} = 61,500, \quad M^I = 659, \quad \) and \( M^{II} = 87,000. \)

Define \( \mathbf{d}_L \) as a 28-dimensional vector whose second element is equal to \( L^I \), whose eleventh element is \( +1 \), and whose remaining elements are 0. Similarly, define \( \mathbf{d}_M \) whose second element is equal to \( M^I \), whose eleventh element is \( -1 \), and whose remaining elements are 0. Then, from (3-44) and (4-3), the following 40 tractive effort constraints apply:

\[
\begin{align*}
\begin{bmatrix} \mathbf{d}_L, D(k) \end{bmatrix} \mathbf{u}^t & \leq \begin{bmatrix} \mathbf{d}_L, D(k) \end{bmatrix} \mathbf{u} + L^{II} - \begin{bmatrix} \mathbf{d}_L, A^k \end{bmatrix} x_0 \quad \ldots \ldots (4-8) \\
\begin{bmatrix} \mathbf{d}_M, D(k) \end{bmatrix} \mathbf{u}^t & \leq \begin{bmatrix} \mathbf{d}_M, D(k) \end{bmatrix} \mathbf{u} + M^{II} - \begin{bmatrix} \mathbf{d}_M, A^k \end{bmatrix} x_0 \quad \ldots \ldots (4-9)
\end{align*}
\]

for \( k = 1, 2, \ldots, 20 \).
V) Effective Grade Constraints:

From (3-47), (4-3) and (4-4), we have the following 20 grade constraints:

\[ u^i_3(k) = 0.04 + 0.01 d^i(k) \]  \hspace{1cm} (4-10)

for \( k = 0,1,\ldots,19 \) and for \( d^i(k) \) as specified in Figure 4.1 (relative to \( s_0(k) \) - see discussion in section 2.6.2).

VI) Air Brake Constraints:

Applying (3-24), we have the following 20 control variable constraints:

\[ u^i_2(k) \leq 23 \]  \hspace{1cm} (4-11)

for \( k = 0,1,2,\ldots,19 \).

The above defined problem was solved in 2.7 minutes on the Computer Science Canada UNIVAC 1108 computer by employing a system of programs called UMPIRE (Unified Mathematical Programming System Incorporating Refinements and Extensions - see reference (57) and (58)).
The complete solution procedure, including all job control statements, input information, problem definition, and output calculations, is documented in Appendix A for reference.

For convenience, the optimal control trajectory and the resultant force levels in the couplers as a function of \( k \) are presented in Figures 4.2 and 4.3 respectively.
Figure 4.2: Optimal control trajectories $u_1$ and $u_2$ (i.e., $f^*$ and $r^*$ respectively). Optimal $u_3$ is as specified by (3-27) and Figure 4.1.
Figure 4.3: Force levels in couplers resulting from the application of the optimal control depicted in Figure 4.2. Exact values are only known at discrete times $k$; dashed lines are force envelopes.
Chapter 5

Discussion of Results and Suggestions

For Future Work:

A brief discussion of the salient features of the example problem presented in Chapter 4, as well as the solution obtained, is now in order:

Firstly, as noted in Chapter 3, 2(n-1) direction vectors are ideally required to specify the "danger region" for each of the variables \( x_{3i+1} \) for \( i = 1, 2, \ldots, (n-1) \) in turn. This is because each \( x_{3i+1} \) can take on negative as well as positive values (i.e., the draft-gears can be compressed as well as extended). In essence, with reference to Figure 3.2, two hyperplanes are required for each \( x_{3i+1} \) in order to minimize its absolute value. However, in order to simplify this example, only the positive values of \( x_{3i+1} \) were chosen to be minimized, and therefore, the level of train run-ins was not included directly in the objective constraints. Nevertheless, as shown in Figure 4.3, the maximum coupler force in compression exceeded the maximum coupler force in tension by only 17.6 kips. For practical problems, additional constraints of the form (4-5) should be included in the LP formulation, with direction vectors defined such that the \( (3i+1) \)th element equals -1 and the remainder are 0. In this manner, the
absolute value of coupler forces will be minimized.

Although only two couplers were considered in this problem, the 36-car 3100-ton train was subjected to an extremely violent sequence of grade changes, as per Figure 4.1. In fact, this one mile track section is considerably more severe than that likely to be found in the main line of any Canadian railroad. Yet the maximum force levels in these two couplers was only 134 and 116 kips respectively, and both figures are well below the generally accepted critical level of about 250 kips.

However, even though the control trajectory of Figure 4.2 is optimal from the viewpoint of the mathematical problem posed, this solution could not be practically implemented. The reason for this is that there are two characteristics of the solution which conflict with known real constraints of the actual physical system. These are as follows:

1) The air brake release action depicted at time interval 10 is shown as only being a partial release. As discussed in section 2.4, this is not possible in the real system. The problem of course is that we have not completely specified the constraints (2-13) in the formulation of the optimization problem.
2) The instantaneous switching action of the locomotive throttle to dynamic braking at time intervals 7 and 9 is similarly not feasible in the actual physical system. In essence, the traction motors are being asked to instantaneously become generators, and this of course is not possible; the throttle must first be moved to the idle position for some minimal time period (usually 5 seconds) before moving to the dynamic braking region. The problem again, of course, is that this physical restraint was not incorporated as a mathematical constraint in the formulation of the optimization problem.

An obvious and direct way of circumventing these difficulties would be to simply add the appropriate constraints to the LP problem statement. Therefore, it is suggested that the design of these constraint statements is the first extension of this work which should be undertaken.

Alternatively, the most practical way of overcoming these deficiencies in the optimal trajectory might simply be to alter the obtained solution by applying common sense. In the case of our example, for instance, one might make the following modifications:
1) Make the air brake reduction \( r^*(k) \) go to zero in the tenth interval;

(ii) Rather than switching from braking to tractive effort and back to braking in the 7th, 8th and 9th intervals respectively, make \( f^*(k) \) go to zero (idle) for the 7th and 8th intervals before going into braking.

These changes would make the resultant control trajectories physically implementable. To be sure, these controls would be sub-optimal. However, a simulation of the system with the modified inputs (a relatively simple matter, since the difference equation model of the system has been obtained) would be able to quantify the degradation from optimal. Intuitively, from the trivial nature of the changes required, this deterioration should not be too severe. Additional work in this area to test this hypothesis would be well worthwhile.

What is the largest real problem that could be practically solved with the approach outlined in this dissertation? To answer this question, consider again the three separate phases of the solution procedure:
A) obtain discrete model of train

B) formulate LP problem

C) solve LP problem.

Taking these in reverse order, the largest LP problem which UMPIRE can handle is 8,000 rows and 262,000 columns (57). In consideration of this fact and (3-49), one sees that the following inequalities must hold:

\[(\bar{x} + \bar{u}) \cdot K + \bar{x} + \bar{m} \leq 8,000 \quad \ldots \ldots \ldots (5-1)\]

\[(1+2) \cdot K + 1 \leq 262,000 \quad \ldots \ldots \ldots (5-2)\]

Generally speaking, \((\bar{x} + \bar{u}) > (1+2)\) so that (5-1) will usually be the limiting constraint. (In order to take full advantage of the capabilities of UMPIRE, some work should be directed towards the formulation of the dual LP problem). Since \(K\) will usually be at least two orders of magnitude larger than \(\bar{x}, \bar{u}, \) or \(\bar{m},\) then an approximate expression of the limiting constraint for phase \(C\) is:

\[K \leq \frac{8000}{(\bar{x}+\bar{u})} \quad \ldots \ldots \ldots \ldots (5-3)\]

Let us now digress for a moment and consider the difficulties associated with Phase A, i.e., obtaining the discrete model of the
train. Although the number of state variables does not directly affect the dimension of the LP matrix, it certainly does have considerable import to the problem of calculating the state transition matrix \( \Phi(t) \). The evaluation of the Taylor series approximation of (3-10) becomes extremely costly as the dimension of the continuous system matrix \( \Delta \) grows increasingly large. The author is not aware of what the practical limit for the dimension of \( \Delta \) might be before the calculation of \( \Phi(t) \) becomes infeasible on the basis of cost and/or accuracy. However, it would seem reasonable to assume that a 61 \( \times \) 61 matrix would not present too much difficulty. Thus, a 20-member train would certainly seem capable of being treated. By considering a 100-car train to be made up of 5-car members, a discrete model for such a train may readily be obtained:

The total number of state and control variable constraints for the example of Chapter 4 was equal to 7, and this number is not likely to exceed 15 for practical trains. Therefore, from (5-3), we have that the maximum value for \( K \) should be in the order of 500. It is therefore mandatory to keep \( K \) below this approximate limit.

Consider now the effect of the choice of \( K \) on \( T \). Obviously, for any particular length of track and overall time interval,
\{t_0, t_f\}; these quantities are inversely proportional. Hence, 
for too small \( K \), \( T \) may become so large as to make the accurate 
representation of the effective grade input impossible. In 
addition, the control may be too "coarse" for too large \( T \). One 
is therefore forced to conclude that the limit on \( K \) implies a 
limit on the length of train run which can be optimized.

As an example, for average train speeds of 30 mph and 
\( T = 4 \) secs., we have \( K = 30 \) per mile. Hence, optimization 
over track lengths of about 15 miles would seem to be possible.

In summary, the problem of optimizing the operation of a 100-
car train over a 15 mile track section so that the maximum force 
levels in every fifth coupler are minimized would certainly seem 
to be feasible. By making the initial condition of 
the next track section correspond to the terminal values of the 
preceding section, effectively any track length may be considered.


44. SILLCOX, L.K., "Shoes and Stops," presented at the Massachusetts Institute of Technology, December 5, 1951.


57. Computer Sciences Canada, "Introducing UMPIRE."


63. "Factors Involved in Calculating the Stopping Distance of Freight Trains," CP Rail, Office of the Chief of Motive Power and Rolling Stock, Montreal, Quebec, April, 1967.


APPENDIX A

DOCUMENTATION OF DIGITAL COMPUTER PROCEDURE
USED TO SOLVE THE PROBLEM OUTLINED
IN CHAPTER 4
CSC Y: FO:ED
UNPHOS - U-PHOS/MATRIX GENERATOR
PROGRAM NO. 3023 VERSION 5.3

NOTATION

SUFXES
C
N = INDEX FOR OBJECTIVE CONSTRAINTS
C
I = INDEX FOR LP VECTOR
C
K = INDEX FOR TIME INTERVAL NUMBER
G
Z = DUMMY INDEX FOR LP VARIABLE PI

N  \text{ \text{NMAX}}  \text{ 2}
I  \text{ \text{INAX}}  \text{ 60}
K  \text{ \text{KMNX}}  \text{ 20}
Z  \text{ \text{ZMAX}}  \text{ 1}

VARIABLES
C
U = VECTOR SET OF LLYNKM CONTROL VARIABLES
C
PI = DUMMY LP VARIABLE WHICH IS BEING MINIMIZED

UIII  ' = II'
PIIZ  ' = 2'

EXTERNAL VALUES
FIRST CALL PREPIO

' XXXX

DECLARATIONS

PARAMETER NCARS=3  HLOCOS=1  JL=3  TAUN=3  OMEGAM=6  NSUM=10  NT=8  N=20
PARAMETER MBAR=1
PARAMETER L=2*N=NCARS+1  HLOCOS=1  NCARS=1  LL=HLOCOS+2  RN=NT=2  NT=1
PARAMETER N=NM=HLOCOS+TAUN  H=NCARS+LNTAU=TAUN+NCARS
PARAMETER IIICGA=OMEGAM/NCARS  LESS1 = NCARS-1  PLUS1 = NCARS+1
PARAMETER L=LL=MM=MM
REAL A(MM,MM,LL),D(MM,LLLYNKM),AKXO(MM,NN,NN),KX(MMM,MM,MM)
REAL X1(MMM),X2(MMM),XIAS(LL),GRADE(NK),ALPHA(LESS1,L1,L11,M1)

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<table>
<thead>
<tr>
<th>K</th>
<th>CPLR 1</th>
<th>CPLR 2</th>
</tr>
</thead>
<tbody>
<tr>
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OPTIMAL TRAJECTORY CALCULATION FOR A THREE-SECTION FREIGHT TRAIN

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184 ROWS  COST ROW IS  1
1 RIGHT HAND SIDES = 1 USED
2573 ELEMENTS

THE DENSITY IS 29.29 PER CENT

PHASE ONE OF INPUT TOOK 38.23 SECONDS
ROW 25 IS REDUNDANT
ROW 26 IS REDUNDANT

PRESOLVE FOUND:

1 NULL ROWS
40 SUB ROWS

INPUT TOOK 35.13 SECONDS
ASEN0UM = INPUT

MAXIMUM ADDRESS ASKED FOR = 363777

OPTIMAL TRAJECTORY CALCULATION FOR A THREE-SECTION FREIGHT TRAIN

ROW ID

MATRIX

FIRST B

EOF
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**COMPUTER SCIENCES CANADA, LTD.**
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Computer Science Canada, Ltd.
CONTROL MATRIX OF CONTINUOUS SYSTEM

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TITLE: OPTIMAL TRAJECTORY CALCULATION FOR A THREE-SECTION FREIGHT TRAIN

UNITS: 6 13 5 15 99999.0 0.000001

NMAX=2
IEN=60
KMAX=20
SMAX=1

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**END OF ALLOCATION**

**THIS ALLOCATION WAS DONE ON 19 JUN 73 AT 2212147**
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| NFTVS CODE | 1 16346-016370  
| NTIPS CODE | 1 16371-016695  
| 2 101601-101640  
| NHTPS CODE | 1 109790-017051  
| 2 101661-101757  
| NAOUT CODE | 1 101760-102050  
| 1 101767-021207  
| NCF1 CODE | 1 102661-102332  
| 1 821203-022454  
| NOELMT CODE | 1 102333-102402  
| 1 822455-022611  
| CHECK CODE | 1 102403-102451  
| 1 826137-023120  
| SCALFC CODE | 1 102499-102532  
| 1 825337-023145  
| NCODES CODE | 1 102499-023175  
| 2 102453-102453  
| NHTPS CODE | 1 1023176-023326  
| 2 102459-102506  
| NHTPS CODE | 1 1023327-024413  
| 2 102507-102596  
| NHTPS CODE | 1 102447-024677  
| 2 102549-102571  
| MEXPS CODE | 1 102572-102656  
| HOBCS CODE | 1 102700-024742  
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COMPUTER SCIENCE CANADA, LTD.
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FUNCTION FD7(K)
    F7 = 0.0
    KK = LL4(K-K)
    DO 10 II = 1,K
       DO 11 K = 1,II
          DII = II*II
       END
    KKK = KK + LL*(II-1) +II
    F7 = F7 + (D12(KKK) + D1II+KII)/LI + YIAS(III)
10    KK = KK +1
    F7 = F7 + LII/LI = (AKKO(2 KK) + AKKO(II+KK)/LI)
RETURN

FUNCTION FA8(I,K)
    F8 = 0.0
    KK = LL*(K-K)+I
    IF (KK,KSTALLBK) GOTO 10
    F9 = D12*KK - D1II+KK)/WI
10   RETURN

FUNCTION F09(K)
    F9 = 0.0
    KK = LL*(K-K)
    DO 10 II = 1,K
       DO 11 K = 1,II
          DII = II*II
       END
    KKK = KK + LL*(II-1) +II
    F9 = F9 + (D12(KKK) - D1II+KII)/WI + YIAS(III)
    KK = KK +1
    F9 = F9 + MII/MI = (AKKO(2 KK) - AKKO(II+KK))/MI
RETURN

FUNCTION F10(I,K)
    F10 = 0.0
    IF (II*II+KK) RETURN
    F10 = 1.0
RETURN

FUNCTION F11(K)
    F11 = 0.0
    IF (II*K+LL-1) RETURN
    F11 = 1.0
RETURN

END COMPILED "A MESSAGES"
NAMROW = 6HTELO

DO 1200 K = 1, KMAX

CALL ITMCD(NAMROW,K, ..24,12)

CALL KAP(NAMROW,PI)

ATFLO ( K ) = PI

1200 CONTINUE

C-ROW( ) FOR GPAD

190 C

NAMROW = 6HGRAD

DO 1270 K = 1, KMAX

CALL ITMCD(NAMROW,K, ..24,12)

CALL KAP(NAMROW,PI)

PGFAD ( K ) = PI

..1270 CONTINUE

C

C-ROW( ) FOR DPRAK

200 C

NAMROW = 6HBRAK

DO 1340 K = 1, KMAX

CALL ITMCD(NAMROW,K, ..24,12)

CALL KAP(NAMROW,PI)

PUPAK ( K ) = PI

1340 CONTINUE

C

C-FIND EOF MARK

209 C

RETURN

210 C

END

END-Compilation = 9 MESSAGES
KUMRL = 8
KCOM = 0
NDFLCE = 1
NAMELIST/PARAM/ KCMM,KUMRL,ISELEC
READ/PARAM,END=110
IF(KCOM) 50,30,40
30 CONTINUE
CALL MGEN(IRED)
IF(IRED.EQ.0) CALL MGINPT
I=IPROP + LT. 0) GO TO 50
WRITE (KPP,11) 32
IF(RPROP.GT. 0 IN RMAIN, JOB TERMINATED.)
CONTINUE
READ(KCOM) MGCCEV
READ(KCOM) MGCRCV
IF(KCOM) R,X0,UBIAS,ALPHA,TI
CALL MOTT(ISELEC,KUMRL)
CALL MFFAD(KUMRL).
CONTINUE
END
DO 200 I = 1, NLCONS
K = M + 1
DO 190 R(K) = 0
R(MH+1:LOCOS+1) = 1
B(KMH+1:LOCOS+2) = 1.0
WRITE (6,1007) T1
WRITE (6,1008) (ID(I,K)+K=1,LL), I=1,MIN
DO 200 I = 1, MIN
X(I) = X0(I)
DO 200 II = 1, LL
200 R(I,J) = R(I,J)-1
KK = LL*(K-1)
DO 200 S I = 1, MIN
DO 200 S II = 1, LL
K = KK+I
D(I,K) = B(I,J)
DO 200 Q I = 1, MIN
CALL MKMT (A*X2,X2,MM+1,MIN,MIN,MM)
DO 210 II = 1, MIN
X(I) = X2(I)
DO 210 III = 1, LL
210 R(I,II,III) = B2(I,II,III)
K = JN-I
KK = LL*(K-1)
DO 200 II = 1, MIN
AKX0(I,K) = X2(I)
IF (KLT.0) GO TO 220
DO 210 I = 1, LL
DO 210 II = 1, LL
KKK = KK+III
DI1+KKK = R2(I,II,III)
DO 210 CONTINUE
220 CONTINUE
WRITE (6,10141)
DO PA I = 1, 5
K1 = I
K2 = 12*K1
WRITE (6,10122) ((D(I,KK+KX2)=1,MIN)
WRITE (6,10135)
WRITE (6,10136)
WRITE (6,10137)
RETURN
END
WRITE (6+1000) (DEL2(I+K), I=1,M)
DO 70 I = 1,M
DO 70 K = 1,M
70 DELTA(I+K) = DEL1(I+K)
DO 60 I = 1,M
CALL MXM (DEL1+DELTA, DEL1+DEL+DEL+M, M, M, M)
DO 60 K = 1,M
DO 60 J = 1,M
DO 60 I = 1,M
DO 60 J = 1,M
DL1(I) = 0.
DO 60 K = 1,M
60 CONTINUE
WRITE (6+1000) (DEL2(I+K), I=1,M)
DO 100 I = 1,M
CALL MXM (DEL1+DELTA, DEL1+DEL+DEL+M, M, M, M)
DO 100 K = 1,M
DO 100 J = 1,M
DO 100 I = 1,M
DO 100 J = 1,M
DL1(I) = DEL1(I)
100 CONTINUE
RETURN
END
100 CONTINUE
COMPILED BY LEVEL 27.8  CSCX FORTRAN V ON 19 JUN 73 AT 22:21

SUBROUTINE MGINIT

C-THE S-R READS CURRENT MAXIMUMS FROM FILE NUMBER

INCLUDE MGREL

NAMELIST/MGREL/NUMAX,MAXMAX,MAXDMAX

X SPAPEA

READ(13,MGREL)RETUR

C ENTRY MGINIT

C-THE S-R READ EXTERNAL VALUES FROM CARDS AND PRINT THE DATA

CALL PREP0

C

C-XXXX -CARDS

IF (NGMCH(1),EQ,0) GOTO 110
CALL MGLINE(1)
WRITE(PPINT, 115)

230  FORMAT(2X)
   240  6HXXXX /2X,
   250     *  
   260  7H110 CONTINUE
   270  READ(10,AD, 120) KCH,IN, XXXX
   280  155 FORMAT(2A6,F5.0)
   290  CALL CDFCN(KCH,IN,6HXXXX , 1)
   300  IF (NGMCH(1),EQ,2) GOTO 120
   310  WRITE(PPINT, 115) KCH,IN, XXXX
   320  135 FORMAT(2X,2A6, F7.8)
   330  120 CONTINUE
   340  RETURN

REAL DELTA(M,H),DELTA(IN,M),DEL(IN,M),DEL(1,M),DEL(2,M),DELX(IN,H)
REAL OMEGA(M,H),OMEGA(IN,H),MASS(INCARS),ISMASS(L),LAMBDA(M,HCARS)
REAL SIGMA(MAX),SIGM(A,81),RMASS(L),LMASS(L)
INTEGER LOCORD(1,800)

DATA MASS/7.3292,0.1256627,0.25 /
DATA UETA/2.1540,0.1256627,0.25 /
DATA LMASS/2.1540,0.1256627,0.25 /
DATA ETA/2.1540,0.1256627,0.25 /

300 FORMAT (I10+3E12.5)
700 FORMAT (I10,2E12.5)
1000 FORMAT (I10,6E12.5)

1001 FORMAT (5H1///F4.5,'SYSTEM MATRIX OF CONTINUOUS SYSTEM'/////)
1002 Format (5H1///F4.5,'CONTROL MATRIX OF CONTINUOUS SYSTEM'/////)
1003 Format (5H1///F4.5,'SYSTEM TRANSITION MATRIX FOR T = 7.3'//)
     5H1 /// 'SECONDS'/////)
1004 Format (5H1///F4.5,'INTEGRAL OF SYSTEM TRANSITION MATRIX FOR T = 7.3'//)
     5H1 /// 'SECONDS'/////)
1005 Format (5H1///F4.5,'CONTROL MATRIX OF DISCRETE SYSTEM FOR T = 7.3'//)
     5H1 /// 'SECONDS'/////)
1006 Format (5H1///F4.5,'AUGMENTED CONTROL MATRIX OF DISCRETE SYSTEM FOR'//)
     5H1 /// 'T = 7.3'// 'SECONDS'/////)
1007 Format (5H1///F4.5,'AUGMENTED CONTROL MATRIX OF DISCRETE SYSTEM FOR'//)
     5H1 /// 'T = 7.3'// 'SECONDS'/////)
DO 31500 K
CALL ITORCO(HAMROW+K,12+6)
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI CON(DUMMY),0,0
31500 CONTINUE

C   ROW(S) FOR TC
C   NAPON = GHTC
C   CALL MGFLMTI C06(DUMMY),0,0
C   C   ROW(S) FOR VEL
C   NAPON = GHEL
DO 31400 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C09(DUMMY),0,0
31400 CONTINUE

C   C   ROW(S) FOR TEHI
C   NAPON = GHTEHI
DO:31510 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C12(DUMMY),0,0
31510 CONTINUE

C   C   ROW(S) FOR TELO
C   NAPON = GHTELO
DO 31580 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C15(DUMMY),0,0
31580 CONTINUE

C   C   ROW(S) FOR GRAD
C   NAPON = GHRAD
DO 31550 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C18(DUMMY),0,0
31550 CONTINUE

C   C   ROW(S) FOR BAK
C   NAPON = GBRAK
DO 31650 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C21(DUMMY),0,0
31650 CONTINUE

END

C   C   ROW(S) FOR BAK
C   NAPON = GBRAK
DO 31720 K = 1, KMAX
CALL ITORCO(HAMROW+K,24+12)
CALL MGFLMTI C24(DUMMY),0,0
31720 CONTINUE

WRITE(1,10)
10  \*
RETURN
END
MARR = 6H0BJ
IF (L2 .NE. 1160 TO 16020)
CALL MGFMT1 1=0
16020 CONTINUE
OC1000 ROW
MARR = 6H0C1000
IF (L2 .NE. 1160 TO 16040)
CALL MGFMT1 -1=0
16040 CONTINUE
OC2000 ROW
MARR = 6H0C2000
IF (L2 .NE. 1160 TO 16060)
CALL MGFMT1 -1=0
16060 CONTINUE
OC ROW
MARR = 6H0C
IF (L2 .NE. 1160 TO 16080)
DO 16070 N = 1, NMAX
16070 CONTINUE
CALL TYPED(HAARRwN :12 ; 6)
CALL TYPED(HAARRwK :24 ; 12)
CALL MGFMT1 -1=0
16076 CONTINUE
FIRST B
WRITE (3,123)
SET !1HID TO 1001 FOR FIRST B SECTION
NGID = 1001
MARR = 6H0C
MARR = 6H0C1000
CALL MGFMT1 CI0(DUMMY) : 0 , 0
MARR = 6H0C1000
CALL MGFMT1 CI0(DUMMY) : 0 , 0
MARR = 6H0C2000
CALL MGFMT1 CI0(DUMMY) : 0 , 0
MARR = 6H0C
DO 31300 II = 1, NMAX
```
00334 1196  NAMROW = $NC
00335 1200  DO 938 N = 1, NMAX
00336 1204  DO 936 K = 1, KMAX
00337 1208  CALL ITORDC(NAMROW,N)
00338 1212  CALL ITORDC(NAMROW,K) ; 24=12)
00339 1216  CALL MGFLNT(COS(DUMMY), 0, 0)
00340 1220  936 CONTINUE
00341 1224  938 CONTINUE
00350 1260  &
00351 1268  C  TC    ROW
00352 1268  C  NAMROW = $HC
00353 1272  C  CALL MGFLNT(COS(DUMMY), 0, 0)
00354 1276  C  VEL    ROW
00355 1280  C  NAMROW = $HVEL
00356 1284  DO 978 K = 1, KMAX
00357 1288  CALL ITORDC(NAMROW,K) ; 24=12)
00358 1292  CALL MGFLNT(COS(DUMMY), 0, 0)
00359 1296  978 CONTINUE
00362 1400  &
00363 1408  C  TEHI  ROW
00364 1412  C  NAMROW = $HTHI
00365 1416  DO 998 K = 1, KMAX
00366 1420  CALL ITORDC(NAMROW,K) ; 24=12)
00367 1424  CALL MGFLNT(COS(DUMMY), 0, 0)
00372 1456  &
00373 1460  C  TEO   ROW
00374 1464  C  NAMROW = $HTEO
00375 1468  DO 1018 K = 1, KMAX
00377 1472  CALL ITORDC(NAMROW,K) ; 24=12)
00379 1476  CALL MGFLNT(COS(DUMMY), 0, 0)
00402 1556  &
00403 1564  C  GRAD  ROW
00404 1568  C  NAMROW = $HGRAD
00405 1572  DO 1580 K = 1, KMAX
00410 1596  &
00411 1572  C  NAMROW = $HGRAD
00412 1576  DO 1580 K = 1, KMAX
00413 1580  CALL ITORDC(NAMROW,K) ; 24=12)
00414 1584  CALL MGFLNT(COS(DUMMY), 0, 0)
00415 1588  &
00416 1592  C  BRAK  ROW
00417 1596  C  NAMROW = $HBRK
00418 1600  DO 1580 K = 1, KMAX
00420 1604  CALL ITORDC(NAMROW,K) ; 24=12)
00421 1608  CALL MGFLNT(COS(DUMMY), 0, 0)
00424 1624  &
00425 1628  C  PI   VARIABLES
00426 1632  C  NAMCOL = $HIP
00427 1636  DO 1070 Z = 1, OMAX
00432 1648  CALL ITORDC(NAMCOL,Z) ; 18, 6)
```
SUBROUTINE MGOUT

INCLUDE MCOAH

20 FORMAT(1H1+126X)

21 FORMAT(1H10+72X)

22 FORMAT(1H10+72X)

23 FORMAT(1H10+72X)

24 FORMAT(1H10+72X)

25 FORMAT(1H10+72X)

32 FORMAT(1H10+68SETEND)

WRITE(MATRFX*20) KTITLE

WRITE(MATRFX*21)

SET MGIND Y# -1000 FOR ROW ID SECTION

MGIND:= -1000

ROW(S) FOR OBJ

NAMROW :=6HOBJ

IFLAG := 6H * Z

WRITE(MATRFX*25) IFLAG+NAMROW

WRITE(MATRFX*25) IFLAG+NAMROW

ROW(S) FOR DC1000

NAMROW :=6HOC1000

IFLAG := 6H +

WRITE(MATRFX*25) IFLAG+NAMROW

ROW(S) FOR DC2000

NAMROW :=6HOC2000

IFLAG := 6H +

WRITE(MATRFX*25) IFLAG+NAMROW

ROW(S) FOR DC

NAMROW :=6HOC

DO 470 N = 1, MMAX.

DO 460 K = 1, KNAX

CALL ITOBCDI(NAMROW+N) :=12; 61

CALL ITOBCDI(NAMROW+K) :=24; 12

WRITE(MATRFX*25) IFLAG+NAMROW

660 CONTINUE

460 CONTINUE

470 CONTINUE

ROW(S) FOR YC

NAMROW :=6HTC

IFLAG := 6H

WRITE(MATRFX*25) IFLAG+NAMROW

ROW(S) FOR VEL.
DI FORTRAN MGMAIN

COMPILED BY LEVEL 23.6 CSX FORTRAN V ON 19 JUN 73 AT 22:12

00100 1a GMAIN PROGRAM MGMAIN
00101 2a INCLUDE MGCOMM
00102 3a CALL MGHEAD(IFEND)
00103 4a IF(IFEND.EQ.0) CALL MGINPT
00104 5a IF(IFERROR.EQ.0) GO TO 20
00105 6a WRITE (KPRINT,10)
00106 7a 10 FORMAT (IM, IFERROR .GT. 0 IN MGMAIN, JOB TERMINATED.)
00107 8a STOP
00108 9a 20 CALL MGOUT
00109 10a IF(MATRISX.EQ.0) GO TO 50
00110 11a END FILE MATRIX
00111 12a REWIND MATRIX
00112 13a 50 IF(KCOMM.EQ.0) STOP
00113 14a WRITE(KCOMM,EGEV)
00114 15a WRITE(KCOMM,MATEGV)
00115 16a WRITE (KCOMM) 16K0:0:0:0:0:0:ALPHA:1
00116 17a END FILE KCOMM
00117 18a REWIND KCOMM
00118 19a STOP
00119 20a END

ENG. COMPILATION ** N MESSAGES **
DI POP: PCOMM

UNIVAC 1100 PROCEDURE DEFINITION PROCESSOR DATED MAY 5, 1969
PROC ENTERED PROCESSED ON 19 JUN 73 AT 22:12:00
PROC ORIGIN 1 ENTRY POINT 1 RWCOMM FCOPY

COMMON/RWTABS/
   1 IMBJ *IRHS *OBJVAL *NOBJF
   COMMON/RWTABS/
   2 BPI ( 1 )
   COMMON/RWTABS/
   3 SOC ( 2 10 )
   COMMON/RWTABS/
   4 VEL ( 20 )
   COMMON/RWTABS/
   5 STEH ( 20 )
   COMMON/RWTABS/
   6 STEH ( 20 )
   COMMON/RWTABS/
   7 PTEH ( 20 )
   COMMON/RWTABS/
   8 PTEO ( 20 )
   COMMON/RWTABS/
   9 PGRAD ( 20 )
   COMMON/RWTABS/
   10 SMIRK ( 20 )
   COMMON/RWTABS/
   11 PMIRK ( 20 )
   COMMON/RWTABS/
   12 SOBJ *SOC*00*POC*00+SOC200+POC200+POC
   END

PROCESSING TIME = 0 SECONDS
PROCESSING TIME = 1 SECONDS
FUNCTION F08(I,K)
  F08 = 0.0
  KK = LL*(NK-K)+1
  IF (KK.GT.LLBYNK) GO TO 10
  F08 = D12*KK - D(11,KK)/M1
10 RETURN

FUNCTION F09(K)
  F09 = 0.0
  KK = LL*(NK-K)
  DO 10 II = 1,K
    DO 10 III = 1,LL
    KK = KK + LL*(II-1) + III
    10 F09 = F09 + D0(2,KKI) - D(11,KKI)/M1*UBAS(III)
  KK = NK-K+1
  F09 = F09 + M1/M1 - KKKO(2,KKI) - KKKO(11,KKI)/M1
  RETURN

FUNCTION F10(I,K)
  F10 = 0.0
  IF (I .GT. K) RETURN
  F10 = 1.0
  RETURN

FUNCTION F11(I,K)
  F11 = 0.0
  IF (I .GT. K-1) RETURN
  F11 = 1.0
  RETURN

END
FUNCTION F05(K)
F05 = 0.0
KK = LL*(NK-K)
DO 10 II = 1+K
DO 10 III = 1+LL
KKK = KK + LL*(II-1) + III
10 F05 = F05 + D(II+KK)*UBIAS(III)
KK = NK-K+1
F05 = F05 + UMAX - AKX0(2 KK)
RETURN

FUNCTION F06(K)
F06 = 0.0
KK = LL*(NK-K)+I
IF (KK.GT.LLB+1K) GO TO 10
F06 = D(2+KK) + D(11+KK)/L1
10 RETURN

FUNCTION F07(K)
F07 = 0.0
KK = LL*(NK-K)
DO 10 II = 1+K
DO 10 III = 1+LL
KKK = KK + LL*(II-1) + III
10 F07 = F07 + D(II+KK) * D(11+KK)/L1)*UBIAS(III)
KK = NK-K+1
F07 = F07 + L12/L1 - (AKX0(2 KK) + AKX0(11 KK)/L1)
RETURN
IF (NK.GT.LLBY.NK) GO TO 10
F01 = D(3*N+1,KK)
10 RETURN

FUNCTION F02(N,K)
F02 = 0.0
KK = N*(NK-K)
DO 10 II = 1:K
DO 10 III = 1:LL
KKK = KK+LL*(II-1)*II
10 F02 = F02 + D(3*N+1+KK)*UBIAS(III)
KK = NK-K+1
F02 = F02 - AKX0(3*N+1+KK)
RETURN

FUNCTION F03(I)
F03 = 0.0
DO 10 II = 1:MK
DO 10 III = 1:LL
KK = LL*(II-1)*III
10 F03 = F03 + D(1+KK)*UBIAS(III)
F03 = F03 + FV - AKX0(I,I)
RETURN

FUNCTION F04(I,K)
F04 = 0.0
KK = LL*(NK-K)+I
IF (KK.GT.LLBY.NK) GO TO 10
F04 = D(2*KK)
10 RETURN
SUM(I) C14 = (I+1) * Eq.C16
FOR ALL K

BRK
SUM(I) C19 = U(I) * LE.C17
FOR ALL K

ELEMENTS
C01 = -XO(N)
C02 = -XO(7)
C03 = XO(I+I/K)
C04 = F02(N+K)
C05 = D1(I+1)
C06 = F03(2)
C07 = F04(I+K)
C08 = F05(I)
C11 = F06(I+K)
C12 = F07(K)
C14 = F08(I+K)
C15 = F09(K)
C16 = UBIAS(3) + 0.01*GRAGE(K)
C17 = UBIAS(3) + 25.0
C18 = F10(I+K)
C19 = F11(I+K)

FUNCTIONS

FUNCTION F03(M+I/K)
F01 = 0.0
KK = LL*(NK-K)+1
219 CONTINUE
220 CONTINUE
DO 221 k = 1, 5
221 WRITE(6,1013) (K(1:12), K(2:12), K(3:12), K(4:12), K(5:12), K(6:12), K(7:12), K(8:12), K(9:12), K(10:12), K(11:12), K(12:12))
RETURN
END
SET UP A AND B MATRICES

120 DO 120 I = 1,N
120 DO 120 K = 1,N
120 A(I,K) = DELTA(I,K)
120 DO 130 I = 1,NLOCOS
130 K = I + N
130 DO 130 II = 1,N
130 A(II,K) = OMEGAD(II,K)
130 KKK = M + NLOCOS - INCOTA + 1
140 DO 140 I = 1,NCARS
140 K = KKK + INCOTA - I
140 KK = NCARS + NLOCOS + 1 - I
140 DO 140 II = 1,N
140 A(II,K) = OMEGAD(II,KK)
140 KKK = M + NLOCOS + TAUM - INCOGA + 1
150 DO 150 I = 1,NCARS
150 K = KKK + INCOGA - I
150 KK = L + 1 - I
150 DO 150 II = 1,N
150 A(II,K) = OMEGAD(II,KK)
170 DO 170 I = 2,TAUM
170 K = N + NLOCOS + I
170 A(K-1,K) = 1.0
170 DO 170 I = 2,OMEGAN
180 K = NK + 1
180 A(K-1,K) = 1.0
WRITE (6,1900) TI
WRITE (6,1900) (A(I,K),K=1,10),I=1,1000
WRITE (6,1911)
80 DELTA(K+KK) = DELTA(K+KK).
   WRITE (6,1003) TI
   WRITE(6,1000) ((DELTA(I+K,K=1,N),I=1,N).
   CALCULATE INTEGRAL OF STM FOR TI IN DELTA

   DO 90 I = 1,N
   DO 90 II = 1,N
   DEL(I*II) = 0.0
   IF (I*II = 1) GO TO 90
   DEL(I*II) = 1.0
   DEL(I*II) = 1.0
   90 CONTINUE
   DO 100 I = 1,N
   CALL MULTRIDELX(Delta,DELTA,DEL+DELX,I=1,N)
   DO 95 II = 1,N
   DO 95 III = 1,N
   95 DELTA(II*I1) = DEL3(II,III)
   100 CALL IXADD(DEL*DELTA+DELX,DEL*DELX+DELX,I=1,N)
   CALL MULTRIDELX(DEL*DELTA+DELX,DEL+DELX+DELX,I=1,N)
   WRITE (6,1904) TI
   WRITE(6,1000) ((DELTA(I+K),K=1,N),I=1,N)
   CALCULATE DISCRETE CONTROL MATRIX
   CALL MULTRI(DELTA+OMEGA+OMEGA+OMEGA,DEL+DELX+DEL+DELX,I=1,N)
   WRITE (6,1005) TI
   WRITE(6,700) ((OMEGA(I+K),K=1,L),I=1,N)
CALCULATE DFL2 = INTEGRAL OF STM FOR T

DO 50 I = 4 + H

50 DELTAD(I:I) = 1.0

DO 60 J = 1 + NSF

II = I

TT = T / II

CALL MXADD(DELTAD, DEL1, DEL1 + M, II, M, TT)

CALL MXSCA(DELTAD + M, M, II, TT)

CALL MXADD(DELTAD + DEL2, DEL2 + M, II, M, TT)

CALL MXPLT(DELTA + DELTAD, DEL + M, M, II, M)

DO 60 K = 1 + H

60 DELTAD(K + KK) = DFL1(K + KK)

WRITE (6, 100) T

WRITE (6, 100) (DELTA(I, K), K = 1 + M, I = 1 + M)

WRITE (6, 100) T

WRITE (6, 100) (DEL1(I, K), K = 1 + M, I = 1 + M)

WRITE (6, 100) T

WRITE (6, 100) (DEL2(I, K), K = 1 + M, I = 1 + M)

C

CALCULATE STM FOR T

C

DO 70 I = 1 + M

70 DELTAD(I + K) = DEL1(I + K)

DO 40 I = 1 + HH

CALL MXPLT(DELTAD + DELTAD + DEL + M, II, M, M, T)

DO 80 K = 1 + H

80 DELTAD(K + KK) = 1.0

DO 80 KK = 1 + H

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IF (I.EQ.NCAPS) GO TO 6
9 DELTA(3*I+3*I+1) = -ALPHA(I) / MASS(I)
DELTA(3*I+3*I+2) = BETA(I) / MASS(I)
IF (I.EQ.1) GO TO 10
DELT
DELTA(3*I+3*I-1) = -(BETA(I-1)*BETA(I))/MASS(I) = 6*LAMBD
GO TO 10
2 DELTA(3*I+3*I-1) = -BETA(I) / MASS(I) = 6*LAMBD
GO TO 4
6 DELTA(3*I+3*I-1) = -BETA(I-1) / MASS(I) = 6*LAMBDA(I)
10 CONTINUE
WRITE(6,1011)
WRITE(6,1010) (DELT(I,K),K=1:NH, I=1:NH)
C
C SET UP OMEGA MATRIX
C
DO 20 K = 1:NHLCOS
   I = LOC(I,K)
20 OMEGA(3*I+I+K) = 1.0 / MASS(I)
DO 30 I = 1:NCARS
   K = I + DLHLCOS
30 OMEGA(3*I+K) = ETA / SIGMA(I)
   I = DLHLCOS + I
OME
OMEGA(K+I) = 1.0 / SIGMA(NCARS+1)
DO 40 I = 1:NCARS
   K = I + NCARS + DLHLCOS
40 OMEGA(3*I+1+K) = -6
   WRITE(6,1021)
WRITE(6,700) (OMEGA(I,K),K=1:NL,NH=1:NH)
C
C CALCULATE DEL1 = STM FOR T

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67 SECONDS

1004 FORMAT (1H1/////T3I//"INTEGRAL OF SYSTEM TRANSITION MATRIX FOR T = ",
*F7.3//SECONDS///)

1005 FORMAT (1H1/////T14//"CONTROL MATRIX OF DISCRETE SYSTEM FOR T = ",
*F7.3// SECONDS///)

1006 FORMAT (1H1/////T28//"AUGMENTED SYSTEM MATRIX OF DISCRETE SYSTEM FOR
* T = +F7.3// SECONDS///)

1007 FORMAT (1H1/////TH//"AUGMENTED CONTROL MATRIX OF DISCRETE SYSTEM FO
*R T = +F7.3// SECONDS///)

1008 FORMAT (1H1/////T16//"TRUNCATION ERROR IN SERIES APPROXIMATION OF S
*TM AND ITS INTEGRAL FOR T = +F7.3// SECONDS///)

1010 FORMAT (1H1/////)

1012 FORMAT (1PE10.3)

1013 FORMAT(1DE10.3)

1014 FORMAT(1H1, 'O MATRIX FOLLOWS')

1015 FORMAT(1H1, 'A MATRIX FOLLOWS')

T1 = T*(KT + 1)

SET UP DELTA MATRIX

DELTA(1,2) = 1.0
DELTA(I,M) = -1.0 / SIGMA(CARS + I)
DO 10 I = 1/KARS
DELTA(3*I-1,3*I) = -E(I) / MASS(I)
DELTA(3*I,3*I) = -1.0 / SIGMA(I)
IF (I.EQ.1) GO TO 2
DELTA(3*I-2,3*I-4) = 1.0
DELTA(3*I-2,3*I-1) = -1.0
DELTA(3*I-1,3*I-4) = BETA(I-1) / MASS(I)
DELTA(3*I-1,3*I-2) = ALPHA(I-1) / MASS(I)
Title: 

Optional cards in any order.

Data

<table>
<thead>
<tr>
<th>NAME</th>
<th>DESCRIPTION</th>
<th>IMPLIED IND.</th>
<th>EXTERN. IND.</th>
</tr>
</thead>
<tbody>
<tr>
<td>KCH</td>
<td>XXXX</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Card 1

Format: 1296, F, 5.3

Input:

- Data
- Subroutine: PEP10
- Delta, Delta1, Delta2, Delta3
- Omega, OMEGA, OMEGA1
- Mass, Incar, Beta, Lambda, Incar2
- Integer, Locoin, Locos1
- Data Mass, Beta, Lambda, Eta
- 300 Format: 11h=3.12.5
- 700 Format: 11h=3.12.5
- 1090 Format: 11h=1.10.12.5
- 1391 Format: 11h///73.5, System Matrix of Continuous System///
- 1392 Format: 11h///72.5, Control Matrix of Continuous System///
- 1393 Format: 11h///73.7, System Transition Matrix for T = 4, F, 7.3

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