Isomorphism with a C(Y) of the maximal ring of quotients of C(X)

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ABSTRACT

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Let C(X) denote the ring of all real-valued continuous functions on a completely regular space X . Let Q(X) denote the maximal ring of quotients of C(X) . Fine , Gillman , and Lambek stated in [1] that Q(X) is the ring of all real-valued continuous functions on the dense open sets in X (modulo the equivalence relation : $f \equiv g$ iff f and g agree on a dense open set) . Every dense set contains the set of all isolated points of X - let us denote this by Is . We conclude : if Is is dense , then $Q(X) \approx C(Is)$.

This thesis studies in detail the converse statement which was established by Hager [4] . In his paper , C(X) and Q(X) are considered as ϕ-algebras . Then elements of Q(X) are represented as real-valued continuous functions on dense open sets in the space of maximal ideals of Q(X) . A ring homomorphism of Q(X) into a C(Y) is a homomorphism of ϕ-algebras . By the properties of homomorphism of ϕ-algebras , Q(X) is isomorphic to a C(Y) iff Q(X) is isomorphic to C(R(Q(X))) , where R(Q(X)) is the subspace of all real maximal ideals of Q(X) . This is true when the cardinality of X is non-measurable , and R(Q(X)) is dense . In this case , R(Q(X)) is homeomorphic to Is , and Is is dense in X .

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Hoan Duong

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SYMBOLS

CHAPTER I. INTRODUCTION TO RINGS OF CONTINUOUS FUNCTIONS .

1.1. RINGS OF CONTINUOUS FUNCTIONS .

A is called a **partially ordered ring** if it is a ring with a partial order relation $\geq$ satisfying :

$a \geq b$ implies $a + x \geq b + x$ for all $a,b,x \in A$, and

$a \geq 0$ and $b \geq 0$ implies $ab \geq 0$.

Note that to define a partial ordering , it is enough to specify the elements ≥ 0 , subject to :

$a \geq 0$ and $-a \geq 0$ if and only if $a = 0$, and

$a \geq 0$ and $b \geq 0$ implies $a + b \geq 0$ and $ab \geq 0$,

and then $a \geq b$ if and only if $a - b \geq 0$.

A partially ordered ring A is said to be **lattice-ordered** if for every $a,b \in A$, there exists an element $a \vee b \in A$ (called sup(a,b)) such that ,

$a \leq a \vee b$, $b \leq a \vee b$, and

if $c \geq a$ and $c \geq b$, then $c \geq a \vee b$ for all $c \in A$.

We define $a \wedge b = -(-a \vee -b)$ (called inf(a,b)) . This element satisfies :

$a \geq a \wedge b$, $b \geq a \wedge b$, and

if $c \leq a$ and $c \leq b$, then $c \leq a \wedge b$ for all $c \in A$.

We also define $|a| = a \vee -a$. We can verify that $|a| \geq 0$ for every $a \in A$.

We denote the set of all continuous , real-valued functions on a topological space X by C(X) or C . Define on C pointwise operations :

Addition : $(f + g)(x) = f(x) + g(x)$,

Multiplication : $(fg)(x) = f(x).g(x)$,

Partial ordering : $f \geq g$ if $f(x) \geq g(x)$ for all $x \in X$.

With these definitions C becomes a lattice-ordered ring. The zero element and unit element are the constant functions **0** and **1** respectively . And $(f \vee g)(x) = (f(x) \vee g(x))$.

The subset of all bounded functions of C is a subring and a sublattice . It is denoted by C^* or $C^*(X)$.

Every real-valued function on a discrete space is continuous . In this case $C \neq C^*$. We call X **pseudocompact** when $C(X) = C^*(X)$. Compact spaces are pseudocompact .

Each positive element in C coincides with a square . Therefore , a ring homomorphism from a C(X) into a C(Y) is order-preserving . Moreover ,

THEOREM 1. Every (ring) homomorphism from C(Y) or $C^*(Y)$ into C(X) is a **lattice homomorphism** (preserving the lattice).

A proof is given in [2] 1.6 .

1.2. COMPLETELY REGULAR SPACES .

1.2.1. The zero-set of $f \in C(X)$ is the set :

$$Z(f) = \{ x \in X : f(x) = 0 \} .$$

A **zero-set** is a zero-set of some $f \in C$. We note that $Z(f) = Z(g)$ for $g = (f \vee -1) \wedge 1 \in C^*$. Let Z(X) denote the family of all zero-sets on X .

Since $Z(f) = f^{-}(0)$ and 0 is a G_δ (countable intersection of open sets) , every zero-set is a closed G_δ . It is known that there exists a topological space containing a closed G_δ set which is not a zero-set . However , any closed set in a

metric space is the zero-set of the distance function to it. And any subset of a discrete space is a zero-set .

$f \in C$ is a **unit** if and only if $Z(f)$ is empty . $g \in C^*$ is a unit if $|g| \geq r > 0$. We can see that the fact that $Z(g)$ is empty does not guarantee $g \in C^*$ is a unit .

Consequently , **$C(X)$ and $C^*(X)$ are semiprimitive** (i.e.,the intersection of all maximal ideals is 0) : for every $0 \neq f \in C$ (or C^*) , there exists $g \in C$ (or C^*) such that $Z(1 - fg)$ is not empty (see [8] 2.1 prop. 7) . Hence they are **semiprime** (i.e., the intersection of all prime ideals is 0) .

Remark : Every set of the form $\{ x \in X : f(x) \geq 0 \}$ is a zero-set (of $g = f \wedge 0$) , and so are the sets $\{ x \in X : f(x) \leq 0 \}$. We can replace 0 by any $r \in \mathbf{R}$.

The complement of a zero-set is called a **cozero-set** .

1.2.2. Two subsets A and B of X are said to be **completely separated** (from one another) in X if there exists $f \in C^*$ such that $0 \leq f \leq 1$, and $f(A) = 0$, $f(B) = 1$.

If there exists $g \in C$ such that $g(A) \leq 0$ and $g(B) \geq 1$, then A and B are completely separated because we can take $f = (0 \vee g) \wedge 1$. We see that 0 and 1 can be replaced by any real numbers r and s $(r < s)$.

When a zero-set Z is a neighborhood of a set A (i.e., the interior of Z contains A) , we call Z a **z-neighborhood** .

THEOREM 2. Two sets are completely separated if and only if they are contained in disjoint z-neighborhoods .

Proof : If f separates A and B , then $\{x \in X : f(x) \geq$

$2/3\}$ and $\{x \in X : f(x) \leq 1/3\}$ are two disjoint z-neighborhoods. If $A \subset Z(f)$ and $B \subset Z(g)$, then $h = |f|/(|f| + |g|)$ separates A and B ●

DEFINITION. A Hausdorff space X is said to be **completely regular** if for any closed set F and a point $x \in X\backslash F$, F and $\{x\}$ are completely separated .

Subspaces of a completely regular space are completely regular . Metric spaces , discrete spaces , compact spaces are completely regular .

A family $\underline{B}$ of closed sets is a **base** for the closed sets if every closed set is an intersection of members of $\underline{B}$ - i.e. , whenever F is closed and $x \in X\backslash F$, there exists a member of $\underline{B}$ containing F but not x .

THEOREM 3. A Hausdorff space X is completely regular if and only if the family of all zero-sets on X is a base for the closed sets .

Proof: The necessity follows from the remark above . The sufficiency follows from theorem 2 ●

Note that every neighborhood U of a point x contains a z-neighborhood of x : $X\backslash(\text{int } U)$ is completely separated from x .

Given a set X and a family C' of real-valued functions, the **weak topology** induced by C' is the weakest topology such that all functions in C' are continuous .

In practice , we take all sets of the form $f^{-}([r,\infty))$, $f \in C'$, $r \in \mathbf{R}$, as a **subbase** of closed sets (finite unions

of its members constitute a base) because the rays $[r,\infty)$ and $(-\infty,r]$, $r \in \mathbf{R}$, form a subbase of closed sets of $\mathbf{R}$.

THEOREM 4. Let X be a topological space .

The families $C(X)$ and $C^*(X)$ induce the same weak topology on X . A base for its closed sets is the family $Z(X)$.

If X is a Hausdorff space , then X is completely regular if and only if its topology coincides with the weak topology induced by $C(X)$ (or $C^*(X)$) .

Proof : By the remark in 1.2.1 , each $f^{\leftarrow}([r,\infty))$ is a zero -set . Conversely , $Z(g) = f^{\leftarrow}([0,\infty))$, with $f = -|g| \wedge 0$ .Thus the topologies induced by C and C^* have $Z(X)$ as a base for the closed sets . The second assertion is a result of theorem 3●

One important property of a completely regular space is that if $f(x) = f(y)$ for every $f \in C$, then $x = y$.

THEOREM 5. For every topological space X , there exists a completely regular space Y and a continuous mapping π from X onto Y , such that the mapping $g \to g\circ\pi$ is an isomorphism of $C(Y)$ onto $C(X)$.

Proof : Given in [2] 3.9 . Briefly :

Define an equivalence relation on X as : $x \equiv y$ if and only if $f(x) = f(y)$ for all $f \in C(X)$. Then Y is the space of all equivalence classes . π is the canonical mapping . For each $f \in C(X)$, there exists a real-valued function g defined on Y such that $f = g\circ\pi$. The weak topology induced by these functions g makes Y completely regular .

Therefore from now on , we assume that every space is

completely regular .

1.3. THE STONE - CECH COMPACTIFICATION .

1.3.1. $\mathbf{F}$ is a **z-filter** on X if it is a family of zero-sets on X and satisfies :

(i) $\mathbf{F}$ does not contain the empty set ,

(ii) if Z_1 , $Z_2 \in \mathbf{F}$, then $Z_1 \cap Z_2 \in \mathbf{F}$,

(iii) if $Z \in \mathbf{F}$, $Z' \in Z(X)$, and $Z \subset Z'$, then $Z' \in \mathbf{F}$.

Note that X belongs to every z-filter .

If I is an ideal in C(X) (not $C^*(X)$) , then the family $Z[I] = \{Z(f) : f \in I\}$ is a z-filter on X . If $\mathbf{F}$ is a z-filter on X , then the family $Z^{\leftarrow}[\mathbf{F}] = \{f : Z(f) \in \mathbf{F}\}$ is an ideal in C(X) (see [2] 2.3.) .

A **z-ultrafilter** on X is a maximal z-filter -i.e., one which is not contained in any other one . We easily see that M is a maximal ideal if and only if Z[M] is a z-ultrafilter.

DEFINITION. Let X be a subspace of T . A z-filter $\mathbf{F}$ on X **converges** to a point $p \in T$ if every neighborhood (in T) of p contains a member of $\mathbf{F}$.

Example : Let T = X . The family of all z-neighborhoods of p is a z-filter converging to p .

Since T is completely regular , no z-filter has two distinct limit points .

(a) If p is a limit point of $\mathbf{F}$, then $p \in cl_T Z$ for every $Z \in \mathbf{F}$.

An ideal I is said to be **fixed** if $\cap$ Z[I] is nonempty (otherwise it is called **free**) . If I is fixed (free) , then

$Z[I]$ is called fixed (free) .

In fact , the fixed maximal ideals in $C(X)$ are precisely the sets $M_p = \{f \in C : f(p) = 0\}$, which are distinct for distinct p . Reason : M_p is the kernel of the onto homomorphism $f \to f(p)$ from $C(X)$ onto $\mathbb{R}$; because X is completely regular , M_p's are distinct ; if M is a fixed maximal ideal , then $M = M_p$ for some $p \in \cap Z[M]$.

(b) For each $p \in X$, the family $A_p = Z[M_p] = \{Z \in Z(X) : p \in Z\}$ is a z-ultrafilter converging to p .

1.3.2. Here are some important properties of $C(X)$ when X is a compact space .

THEOREM 6. X is compact if and only if every ideal in $C(X)$ (i.e., every ideal in $C^*(X)$) is fixed .

Proof : A z-filter on X is a family of closed sets with the property that any finite subcollection has a nonempty intersection (i.e., having the finite intersection property). Hence if X is compact then each z-filter is fixed (see [9] ch.9 prop. 1) . Conversely , let $\underline{B}$ denote a family of closed subsets in X having the finite intersection property , and let $\underline{F}$ denote the family of all zero-sets each containing an intersection of finite subcollection of $\underline{B}$ We can easily verify that $\underline{F}$ is a z-filter . Since $Z(X)$ is a base for the closed sets , each member of $\underline{B}$ coincides with an intersection of a subcollection of $\underline{F}$. Thus $\cap \underline{B} = \cap \underline{F}$ is nonempty . Hence X is compact (see also [9] ch.9 prop. 1) ●

Remark: When X is compact , the **Stone topology** (see [8]

2.5 or ref. 1.4) on $M(X)$, the space of all (fixed) maximal ideals of $C(X)$, has the sets $\{ M \in M(X) : f \in M \}$, for all $f \in C(X)$, as a base for the closed sets . The relation : $p \in Z(f)$ if and only if $f \in M_p$ shows that the one-one mapping $p \to M_p$ is a homeomorphism of X onto $M(X)$. Therefore ,

THEOREM 7. Two compact spaces X and Y are homeomorphic if and only if their rings $C(X)$ and $C(Y)$ are isomorphic .

1.3.3. DEFINITION. By a **compactification** of a space X , we mean a compact space in which X is dense .

Example: X is a locally compact space if every point in X has a compact neighborhood (R and infinite discrete spaces are locally compact spaces but not compact) . Given a locally compact , non compact space X , we construct a space X^* by adding a new point to X (called the point at infinity , denoted as ∞) , then taking a set in X^* to be open if it is either an open subset of X or the complement of a compact subset in X. Then X^* is a compact Hausdorff space and is called the **one-point compactification** of X . Thus a locally compact space is completely regular .

Our goal is to have a compactification βX of X such that $C^*(X) = C(\beta X)$ by the restriction map - i.e. , every bounded continuous function on X can be extended to a continuous function on βX (we say X is **C^*-embedded** in βX) .

Remark: By theorem 6 , each z-ultrafilter on βX converges to a unique point in βX . Then the family of intersections of members of this z-ultrafilter with X is a z-ultrafilter on X

which converges to the same point . Note : each fixed z-ultrafilter on X converges to a unique point in X .

We construct βX as following :

(a) X is considered as an index set for the fixed z-ultrafilters .

(b) We enlarge this set in any convenient way to an index set for the family of all z-ultrafilters . The points of βX are defined to be the elements of this enlarged index set . The family of all z-ultrafilters on X is written

$$(A^p)_{p \in \beta X}$$

with the understanding that $A^p = A_p$ for $p \in X$.

(c) By the remark above , we have to define a topology on βX such that each A^p converges to the point $p \in \beta X$. The remark in 1.3.1(a) forces us to define

$cl_{\beta X} Z = \{p \in \beta X : Z \in A^p\}$, for all zero-sets Z in X ; we take all of these sets as a base for the closed sets in βX. X is a subspace since $cl_{\beta X} Z \cap X = Z$. X is dense because X belongs to every z-filter .

THEOREM 8. Every completely regular space X has a (**Stone-Čech**) compactification βX , with the following equivalent properties :

(i) (STONE) Every continuous mapping π from X into any compact space Y has a continuous extension π from βX into Y.

(ii) (STONE-ČECH) Every function in $C^*(X)$ has an extension to a function in $C(\beta X)$.

(iii) If a compactification T of X satisfies either (i)

or (ii) , then there exists a homeomorphism of βX onto T that leaves X pointwise fixed : βX is unique .

Proof : see [2] 6.4 , 6.5 .

Example : If X is the space of all ordinal numbers less than w_1 , the first uncountable ordinal , then βX is the space of all ordinal numbers less than $w_1 + 1$ (see [2] 5.12) .

1.4. MAXIMAL IDEAL SPACES AND EXTENSIONS OF FUNCTIONS .

Let M(X) denote the set of all maximal ideals of C(X) . We define on M(X) a (Stone) topology by taking all the sets

$$\{ M \in M(X) : f \in M \} \text{ , for each } f \in C(X) \text{ ,}$$

as a base for the closed sets .

(a) Since the sets $\{ p \in \beta X : Z \in A^p \}$ form a base for the closed sets in βX (ref. 1.3.3(c)) , the one-one mapping $p \to M^p = Z^{\leftarrow}[A^p]$ of βX onto M(X) is a homeomorphism . The image of X under this homeomorphism is the subspace of all fixed maximal ideals ($M^p = M_p$) for $p \in X$) , denoted by F(X).

We write M(f) for the residue class of f modulo M .

For all $M \in M(X)$, C/M is a **totally ordered** field with the ordering relation being defined as :

$M(f) \geq 0$ [resp. > 0] if and only if f is nonnegative [resp. positive] on some zero-set of M (see [2] 5.4) .

C/M always contains R (see [2] 5.5) . When C/M = R , M is called a **real** maximal ideal . All fixed maximal ideals are real : $M_p(f) = f(p)$ (ref. 1.3.1) . Let $\underline{R}(X)$ denote this subspace .

A totally ordered field is called **archimedean** if for

every element a , there exists $n \in N$ such that $n \geq a$. Thus a non-archimedean field is characterized (among totally ordered fields) by the presence of **infinitely large** elements , that is , elements a such that $a > n$ for every $n \in N$. Similarly , if $b > 0$ and $b < 1/n$ for every $n \in N$, then b is called **infinitely small** . An element a is infinitely large if and only if $1/a$ is infinitely small .

Every archimedean field is embeddable in **R** ([2] 0.21) . Thus C/M is archimedean if and only if M is real .

Each $f \in C$ is a continuous mapping from X into the **two-point compactification** $\mathbf{R}^* = \mathbf{R} \cup \{\pm\infty\}$ of **R** . By theorem 8 , f has an extension $f^*: \beta X \to \mathbf{R}^*$.

THEOREM 9. Let $f \in C(X)$.

(a) $f^*(p) = \pm\infty$ if and only if $|M^p(f)|$ is infinitely large.

(b) $f^*(p) = r \in \mathbf{R}$ if and only if $|M^p(f) - r|$ is either zero or infinitely small .

The proof is similar to that given in [2] 7.6 .

By 1.4 (a) , F(X) is dense in $\underline{R}(X)$. Then by the theorem above , $C(X) = C(F(X)) = C(\underline{R}(X))$ under the restriction map (ref. 2.3.2) .

CHAPTER II. INTRODUCTION TO RINGS OF QUOTIENTS OF COMMUTATIVE RINGS .

In this chapter , every ring is assumed to be commutative and to have an unit element .

2.1. RINGS OF QUOTIENTS .

2.1.1. Given a subset S in a ring A , the **annihilator** of S in A is the set

$$S^* = \{ a \in A : aS = 0 \} .$$

DEFINITION. An ideal (subring) in a ring A is said to be (rationally) **dense** if its only annihilator in A is 0 .

Example: (d) for a non-zero divisor d is a dense ideal.

An ideal is said to be **large** if it has non-zero intersection with every non-zero ideal - i.e. , with every non-zero principal ideal . Obviously , every dense ideal is large : if $a \neq 0$, then $aI \neq 0$ (I is dense) , hence $(a) \cap I \supset aI \neq 0$.

LEMMA 1. A is semiprime if and only if every large ideal in A is dense .

Proof : Assume A is semiprime , L is a large ideal , and $a \neq 0$. Since L is large , $L \cap (a) \neq 0$. Then because A is semiprime , $(L \cap (a))^2 \neq 0$. Hence L is dense since $(L \cap (a))^2 \subset La$.

Conversely , assume that every large ideal is dense , and let I be an ideal such that $I^2 = 0$. Then I^* is a large ideal : if $(a) \cap I^* = 0$, then $aI \subset (a) \cap I^* = 0$ $(aII = 0)$, then $a \in I^*$, then $a \in (a) \cap I^* = 0$. Thus I^* is dense . Since $II^* = 0$

, $I = 0$. Therefore A is semiprime ●

We observe that :

(a) A is dense .

(b) If D is dense and $D \subset D'$, then D' is dense .

(c) If D and D' are dense , then so are DD'and $D \cap D'$: Let $aDD' = 0$, then $aD = 0$ (D'is dense) , then $a = 0$ (D is dense) . And $DD' \subset D \cap D'$.

(d) If $A \neq 0$, then 0 is not dense .

2.1.2. Let B be a commutative ring containing A and having the same unit element . For $b \in B$, we define

$$b^{-1}A = \{ a \in A : ba \in A \} .$$

We can verify that $b^{-1}A$ is an ideal in A .

DEFINITION. B ($\supset A$) is a **ring of quotients** or **rational extension** of A if $b^{-1}A$ is dense in B for every $b \in B$ -that is,

(*) for $0 \neq b' \in B$, there exists $a \in A$ such that $ba \in A$ and $b'a \neq 0$.

Note that the element a can be chosen such that $b'a \in A$: pick a_1 as above ; then since $b'a_1 \neq 0$, pick a_2 such that $0 \neq b'a_1a_2 \in A$; put $a = a_1a_2$.

A ring without a proper rational extension is called **rationally complete** .

LEMMA 2. Let $A \subset B$.

(1) B is a ring of quotients of A if and only if for each non-zero $b \in B$, $b^{-1}A$ is dense in A and $b(b^{-1}A) \neq 0$.

(2) If $b(b^{-1}A) \neq 0$ for all non-zero $b \in B$, then each ideal $b^{-1}A$ is large in A .

Proof :

(1) The necessity is trivial . Conversely , for every $b \in B$ and $0 \neq b' \in B$, pick $a_1 \in A$ such that $0 \neq b'a_1 \in A$ $(b'(b'^{-1}A) \neq 0)$. Pick $a_2 \in b^{-1}A$ such that $b'a_1a_2 \neq 0$ ($b^{-1}A$ is dense in A). We have $a = a_1a_2 \in A$ satisfying $ba \in A$ and $b'a \neq 0$.

(2) Let $0 \neq a \in A$ be given . If $a \in b^{-1}A$, then $(a) \cap b^{-1}A \neq 0$. If not then $ba \neq 0$, so there exists $a' \in (ba)^{-1}A$ such that $0 \neq baa'$. We have $0 \neq aa' \in (a) \cap b^{-1}A$ ●

As a consequence , we have .

THEOREM 10. Let $A \subset B$. If A is semiprime , then B is a ring of quotients of A if and only if $b(b^{-1}A) \neq 0$ for all non-zero $b \in B$ - that is ,

(*) for $0 \neq b \in B$, there exists $a \in A$ such that $0 \neq ba \in A$.

Proof : This follows from lemma 1 and lemma 2 ●

2.2. DIRECT LIMITS .

We observe that :

(a) Let I be an ideal in A . Then $\phi : I \to A$ is called an **A-homomorphism** if ϕ is a group homomorphism , and $\phi(d.a) = \phi(d).a$, for all $d \in I$, $a \in A$. The set of all A-homomorphisms from I into A , denoted Hom (I,A) or Hom I , is an A-module .

(b) If $D' \subset D$ are ideals , then the restriction map $\phi \to \phi|D'$ ($\phi \in$ Hom D) is a homomorphism of the module Hom D into Hom D'. If D and D' are dense , then this mapping is a monomorphism : if $\phi(d) \neq 0$ for some $d \in D$, $\phi \in$ Hom D , then

there exists $d' \in D'$ such that $0 \neq \phi(d).d' = \phi(dd')$ since D' is dense . We abuse notation to write Hom $D \subset$ Hom D'.

(c) We identify each $a \in A$ with the A-homomorphism $x \to ax$, denoted a .

A family $\underline{D}$ of dense ideals in A is said to be **closed** if $A \in \underline{D}$ and the product of any two members of $\underline{D}$ is a member of $\underline{D}$. The smallest closed family is $\{A\}$, and the largest one is $\underline{D}_0(A)$, the family of all dense ideals in A .

Let $\underline{D}$ be a closed family of dense ideals . By a **standard family** of A-modules , we shall mean a family $(D^h)_{D \in \underline{D}}$ which satisfies :

(i) D^h is a submodule of Hom D ,

(ii) $1 \in A^h$,

(iii) if $D \supset D'$, then $D^h \subset D'^h$,

(iv) if ϕ_1 , $\phi_2 \in D^h$, then $\phi_1 \cdot \phi_2 \in (DD)^h$.

Certainly, $1 \in$ Hom A. If $D \supset D'$, then $D^h \subset$ Hom D'. Finally , $\phi_1 \cdot \phi_2$ is indeed defined on DD since $\phi_2(dd') = d.\phi_2(d')$.

We can see that (i) to (iii) define a functor from $\underline{D}$ into Mod(A) , the category of all A-modules . Therefore we can take the **direct limit** (see [10] ch.13) :

$$Q_h(A) = \lim_{\to D \in \underline{D}} (D^h) .$$

This may be thought of as $\cup_{D \in \underline{D}} D^h$, where we identify $\phi_1 \in D_1^h$ with $\phi_2 \in D_2^h$ whenever they agree on D_1D_2 (in fact, on any dense ideal) . The module operations in $Q_h(A)$ then reduce to the operations within each D^h . We define $\phi_1 . \phi_2 = \phi_1 \cdot \phi_2$, then $Q_h(A)$ becomes a ring . The zero and unity element in this

ring are the mappings **0** and **1** respectively .

REPRESENTATION THEOREM. Each direct limit $Q_h(A)$ is a ring of quotients of A . If B is a ring of quotients of A , then

$$B = \lim_{\to D \in Do(A)} (B \cap \mathrm{Hom}\ D) .$$

Proof:

Let $\phi \in Q_h(A)$, $0 \neq \phi' \in Q_h(A)$. Pick D such that $\phi, \phi' \in D^h$, then pick $d \in D$ such that $\phi'(d) \neq 0$. Since $(\phi \cdot d)(a) = \phi(da) = \phi(d).a$ for all $a \in A$, we have $\phi.d = \phi(d) \in A$; hence $d \in \phi^{-1}A$ and $\phi'.d \neq 0$.

If $b \in B$ and $b|D \in \mathrm{Hom}\ D$ (i.e., $bD \subset A$) , $D \in \underline{D}_o(A)$, then the correspondence $b \to b|D$ is one-one : if $b \neq 0$, then there exists $a \in A$ such that $0 \neq ba \in A$, then there exists $d \in D$ such that $b(ad) = bad \neq 0$ - that is $b|D \neq 0$. Therefore , $B \cap \mathrm{Hom}\ D$ has meaning . We easily verify that $(B \cap \mathrm{Hom}\ D)_{D \in Do(A)}$ is a standard family . Since each $b^{-1}A$ is dense and $b \in \mathrm{Hom}\ b^{-1}A$ $(b(b^{-1}A) \subset A)$, $B \subset \bigcup_{D \in Do(A)} (B \cap \mathrm{Hom}\ D)$. The reverse inclusion is trivial ●

COROLLARY. $Q(A) = \lim_{\to D \in Do(A)} \mathrm{Hom}\ D$ is the largest ring of quotients of A .

Proof : We observe that if $(D^h)_{D \in D}$ and $(E^k)_{E \in E}$ are standard families such that $\underline{D} \subset \underline{E}$, and $D^h \subset D^k$ for each $D \in \underline{D}$, then $Q_h(A) \subset Q_k(A)$. Therefore , Q(A) is the largest among $Q_h(A)$, i.e. , among rings of quotients of A ●

If C is a rational extension of B , and B is a rational extension of A , then C is a rational extension of A - Reason: given $c \in C$ and $0 \neq c' \in C$; pick $b \in B$ such that $cb \in B$ and

$0 \neq c'b \in B$; pick $a_1 \in A$ such that $cba_1 \in A$ and $c'ba_1 \neq 0$; pick $a_2 \in A$ such that $ba_2 \in A$ and $c'ba_1a_2 \neq 0$; put $a = ba_1a_2$, then $ca \in A$ ($cba_1 \in A$) and $c'a \neq 0$. Therefore , $Q(A)$ is rationally complete . $Q(A)$ is called the **the maximal ring of quotients** or **rational completion** of A .

We define the **classical** ring of quotients of a ring A as

$$Q_{cl}(A) = \lim_{\to (d) \in Do(A)} Hom(d) .$$

It is easily seen that $Q_{cl}(A) = \{ a/b : a,b \in A$ and b is a non-zero divisor $\}$.

We have two equivalent definitions of rational extensions of a commutative ring . The former formalizes elements of this ring as pairs a/b satisfying the demand that denominators never be zeroes . The latter is very useful when we study the rational extensions of - ring of continuous functions .

2.3. MAXIMAL RING OF QUOTIENTS OF A RING OF CONTINUOUS FUNCTIONS.

2.3.1. For an ideal I in C(X) , we denote

$$\text{coz } I = \bigcup_{f \in I} \text{coz } f \text{ , with coz } f = \{x \in X : f(x) \neq 0\}.$$

Then coz I is open . Conversely , if U is an open set , then $U = \text{coz } I$, for $I = \{f \in C : \text{coz } f \subset U\}$: $U \subset \text{coz } I$ because X is completely regular ; $U \supset \text{coz } I$ is obvious .

THEOREM 11. An ideal D in C (or C^*) is (rationally) dense if and only if coz D is (topologically) dense .

Proof: D is dense if and only if for all $g \in C$, $Z(g) \supset \text{coz } D$ implies $g = 0$. The latter is equivalent to coz D is dense : if coz D is dense , and if $Z(g) \supset \text{coz } D$, then $Z(g)$

is dense , hence $g = 0$; conversely , if coz D is not dense , then since X is completely regular , there exists $g \neq 0$ such that $Z(g) \supset$ coz D ●

2.3.2. We denote the family of all dense open sets in X by $V_0(X)$. If S is dense in X , then the homomorphism $f \to f|S$ from C(X) into C(S) is a monomorphism . We abuse notation and write $C(X) \subset C(S)$.

THEOREM 12. If V is a dense open set in X , then C(V) is a ring of quotients of C(X) .

Proof : Consider $0 \neq h \in C(V)$. Take $v \in$ coz $h \subset V$, there exists $f \in C^*(X)$ that vanishes on a neighborhood of $X \backslash V$ but not at v ($X \backslash V$ is closed and X is completely regular). Then $0 \neq h.f \in C(X)$. Since C(X) is semiprime , this proves that C(V) is a ring of quotients of C(X) (theorem 10) ●

In the proof , $h.f \in C(X)$ since h.f can be extended to a function $g \in C(X)$: $g = 0$ on $X \backslash V$ and $g = h.f$ on V .

Since members of $V_0(X)$ are nonempty and $V_0(X)$ is closed under finite intersection (i.e., $V_0(X)$ is a filter base) , we consider the direct limit

$$\lim_{\to V \in V_0(X)} C(V) ,$$

with respect to the restriction monomorphism $f \to f|S'$, when $f \in C(S)$ and $S \supset S'$. This direct limit may be thought of as $\bigcup_{V \in V_0(X)} C(V)$, where we identify $f \in C(S)$ with $f' \in C(S')$ if f and f' agree on $S \cap S'$.

As a result of the theorem above , $\lim_{\to V \in V_0(X)} C(V)$ is a ring of quotients of C(X) : if $h \in \lim_{\to V \in V_0(X)} C(V)$, then h

$\in C(V)$ for some $V \in V_o(X)$, then $h.(h^{-1}C(X)) \neq 0$.

2.3.3. We write $Q(X) = Q(C(X))$.

From the remark above ,

$$Q(X) = \lim_{\to D \in Do(C)} \text{Hom } D \supset \lim_{\to V \in Vo(X)} C(V) .$$

If Hom $D \subset C(V)$ for some dense open set V , then the reverse is established .

LEMMA 3. For any ideal D in $C(X)$, Hom $D \subset C(\text{coz } D)$.

Proof : Given $\phi \in$ Hom D , we have to find $g \in C(\text{coz } D)$ such that $\phi(d) = g(d) = g.d$ for all $d \in D$:

For $x \in \text{coz } D$, choose $d \in D$ such that $d(x) \neq 0$, and define

$$g(x) = \frac{\phi(d)\ (x)}{d(x)} .$$

Since $\phi(d').d = \phi(d).d'$, g is independent of d . For each $x \in \text{coz } D$, g is continuous on a neighborhood of x (namely , coz d) ; therefore g is continuous on coz D . Certainly , $\phi(d) = g.d$ for all $d \in D$ ●

REPRESENTATION THEOREM.

$$Q(X) = \lim_{\to V \in Vo(X)} C(V) .$$

Proof : The following is a result of lemma 3 and theorem 11 .

$Q(X) = \lim_{\to D \in Do(C)} \text{Hom } D \subset \lim_{\to D \in Do(C)} C(\text{coz } D) = \lim_{\to V \in Vo(X)} C(V)$ $\subset Q(X)$ ●

Similarly, $Q_{cl}(X) = Q_{cl}(C(X)) = \lim_{\to V} C(V)$,where V ranges over the dense **cozero sets** in X (see [1] 2.6).

Example : Let D be an uncountable discrete space and let

Y be the one-point compactification of D . Then the smallest dense open set in Y is D . Therefore $Q(Y) = C(D)$. Let (a_n) be a sequence of distinct points in D (then ∞ is a limit point of this sequence) . Let $f(a_n) = 1$, and let $f(y) = 0$, for $y \in D\backslash(a_n)$. Then $f \in C(D)\backslash C(Y)$, hence $Q(Y) \neq C(Y)$.

Let $G = \cap G_n$, where G_n are open sets in Y . If $G = \{\infty\}$, then $Y \setminus G_n$'s are finite sets (G_n 's are open neighborhoods of ∞) , then Y must be countable ($Y = \bigcup (Y \setminus G_n) \cup G$) . Hence $G \cap D$ is not empty . Therefore there is no $g \in C(Y)$ such that $Z(g) = \{\infty\}$. Hence no proper cozero set in X is dense . Thus $Q_{cl}(X) = C(X)$.

CHAPTER III. ISOMORPHISM WITH A C(Y) OF THE MAXIMAL RING OF QUOTIENTS OF C(X) .

From the last chapter , Q(X) is the ring of all continuous functions on dense open sets of X (modulo an equivalence relation) . Every dense set has to contain the set of all isolated points of X . If this set is dense , then obviously Q(X) is isomorphic with the ring of all continuous functions on this set (a C(Y)) . Now we try to establish the converse.

3.1. ϕ-ALGEBRA .

3.1.1. A is a **lattice-ordered algebra** if it is a lattice-ordered ring and also a (real) vector lattice - that is , $r > 0$ and $a > 0$ imply $r.a > 0$, for $a \in A$, $r \in \mathbf{R}$.

A lattice-ordered ring A is called **archimedean** if , for each $0 \neq a \in A$, the set $\{ na : n = \pm 1, \pm 2, \ldots \}$ has no upper bound in A .

DEFINITION. A **ϕ-algebra** A is an archimedean lattice-ordered algebra over **R** (real) , with an identity element **1** which is a **weak order unit** - that is :

$$\mathbf{1} > 0 , \quad \text{and} \quad b \in A \text{ and } \mathbf{1} \wedge b = 0 \text{ imply } b = 0 .$$

A **homomorphism of ϕ-algebras** is an algebra homomorphism preserving the lattice operations .

Result : **R** , C(X) , Q(X) are ϕ-algebras and a ring homomorphism of Q(X) to C(Y) which carries 1 to 1 is a homomorphism of ϕ-algebras (an analogue of theorem 1) .

3.1.2. Let K be a compact space , and let D(K) be the set of continuous functions f from K to $\mathbf{R}^* = \mathbf{R} \cup (\pm \infty)$,the two-point

compactification of the real field $\mathbb{R}$, for which $\underline{R}(f) = f^{-}(\mathbb{R})$ is (open) dense . Let $f,g,h \in D(K)$, we say $f = g + h$ if $f(x) = g(x) + h(x)$ for all $x \in \underline{R}(g) \cap \underline{R}(h)$. The product , inf and sup of two functions in D(K) are similarly (pointwise) defined. Since $\underline{R}(g) \cap \underline{R}(h)$ is dense in K , these operations are uniquely defined .

Note that D(K) is not necessary closed under addition and multiplication : Let K denote the one-point compactification of N (the discrete space of positive integers) ; let $f(x) = x + \sin x$, $f'(x) = (1/x)\sin x$, $f''(x) = -x$ if $x \in N$, and let $f(\infty) = \infty$, $f'(\infty) = 0$, $f''(\infty) = -\infty$; then $f,f',f'' \in D(K)$, but neither $f + f''$ nor $f'.f''$ is defined .

By a "ϕ-subalgebra of D(K)" we mean a subalgebra with respect to the operations in D(K) discussed above .

3.1.3. An ideal I in a lattice-ordered ring A is said to be **absolutely convex** if , whenever $|x| \leq |y|$ and $y \in I$, then $x \in I$.

Given an ideal I in a ring A , we denote the residue class of an element a modulo I by I(a) .

Remarks:

(a) If I is an absolutely convex ideal , then A/I is ordered as : $I(a) \geq 0$ if there exists $x \in A$ such that $x \geq 0$ and $a \equiv x \pmod{I}$ (see [2] 5.2) . The canonical map is a lattice homomorphism .

(b) If M is a maximal absolutely convex ideal , then M is prime , A/M is totally ordered (see [6] 1.6(ii)) , and for

$0 < a \in A$, there is $0 \leq b \in A$ such that $M(ab) \geq 1$ (see [6] 1.6(ii) and the proof of [6] 2.3 (3)) .

(c) The kernel of a homomorphism between two lattice-ordered rings is an absolutely convex ideal (see [6] 1.4) .

For a ϕ-algebra A , let M(A) denote the Stone space of all maximal absolutely convex ring ideals of A (see [8] 2.5 or ref. 1.4) .

THEOREM 13. ([6] 2.3)

(i) M(A) is a Hausdorff compact space .

(ii) A is isomorphic to a ϕ-subalgebra of D(M(A)) .

(iii) Disjoint closed subsets of M(A) are completely separated (ref. 1.2.2) by this copy of A (hence by the copy of the set of all bounded elements of A) .

The image of $a \in A$ under this isomorphism , denoted by $\hat{a}$, is given as follows :

If $a \geq 0$, then $\hat{a}(M) = \inf \{ r \in R : M(a) \leq r \}$ (where inf $\emptyset$ is understood to be $+\infty$) .

If $a \in A$ is arbitrary , then $\hat{a}(M) = \hat{a}^+(M) - \hat{a}^-(M)$ (where $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$) . Since $a^+ \wedge a^- = 0$, $M(a^+) \wedge M(a^-) = 0$, then either $M(a^+) = 0$ or $M(a^-) = 0$. Thus $\hat{a}$ is well-defined .

To illustrate this theorem , take $A = C(X)$. Every maximal ideal in C(X) is absolutely convex : if $|g| \leq |f|$, then $Z(g) \supset Z(f)$. If $f \in M$, then $Z(g) \in Z[M]$. Thus $g \in M$. Hence , $M(A) = M(X)$. And by 1.4 , $A = C(X)$ is isomorphic to $D(M(X)) = D(M(A))$.

From now on , we identify a ϕ-algebra A with its copy in D(M(A)) .

3.2. HOMOMORPHISMS OF ϕ-ALGEBRAS .

3.2.1. Note that if $M(a) = r + \delta > 0$ with an infinitely small element δ , then $M(a) = M(a^+)$, hence $\hat{a}(M) = r$. Similarly , if $M(a) = r + \delta < 0$, then $M(a) = - M(a^-)$, hence $\hat{a}(M) = r$. If $|M(a)| = M(|a|) = M(a^+) \vee M(a^-)$ is infinitely large , then $\hat{a}(M) = \infty$ or $-\infty$. Since the possibilities considered are mutually exclusive and exhaustive , we can conclude : $\hat{a}(M) = r \in \mathbf{R}$ if and only if $|M(a) - r|$ is either infinitely small or zero (an analogue of theorem 9).

$\underline{R}(A) = \cap \{\underline{R}(f) : f \in A\}$ is called the **real ideal space of A** . $M \in \underline{R}(A)$ if and only if $A/M = \mathbf{R}$: if $a > 0$ and $M(a) = \delta$, δ is infinitely small , then there exists $0 \leq b \in A$ such that $M(ab) \geq 1$ (remark 3.1.3 (b)) ; thus M(b) is infinitely large ($\hat{b}(M) = \infty$) . We can easily verify that $\hat{a}(M) = M(a)$ for $M \in \underline{R}(A)$.

3.2.2. Let μ_A (or μ) denote the homomorphism $f \to f|\underline{R}(A)$ from A into $C(\underline{R}(A))$.

LEMMA 4. μ is one-one if and only if $\underline{R}(A)$ is dense in M(A) .

Proof : The sufficiency is obvious . By part (iii) of theorem 13 , if $\underline{R}(A)$ is not dense , then there exists $0 \neq f \in A$ such that $f|\underline{R}(A) = 0$ ●

When $\underline{R}(A)$ is dense (i.e., $\cap \{ M : M \in \underline{R}(A) \} = \{0\}$) , A is called a **$\phi$-algebra of real-valued functions** .

For a continuous map $\tau: Y \to \underline{R}(A)$, we define $\tau': C(\underline{R}(A)) \to C(Y)$ as $\tau'(f) = f \circ \tau$. Then $\tau' \circ \mu$ is a homomorphism of A into C(Y) .

LEMMA 5. Let $\alpha: A \to B$ be a homomorphism of ϕ-algebras A and B . Then there exists a continuous $\tau: \underline{R}(B) \to \underline{R}(A)$ for which $\tau' \circ \mu_A = \mu_B \circ \alpha$.

Proof: For each $M \in \underline{R}(B)$, the map t_M given by $t_M(f) = \alpha(f)(M)$, for $f \in A$, is a homomorphism from A into $\mathbf{R}$. Since $t_M(r) = r$, for $r \in \mathbf{R}$, the kernel of t_M is a $\tau M \in \underline{R}(A)$ (remark 3.1.3 (c) and $A/\tau M = \mathbf{R}$) . Thus we have a map $\tau: \underline{R}(B) \to \underline{R}(A)$.

$\tau' \circ \mu_A = \mu_B \circ \alpha$:

For $f \in A$, $M \in \underline{R}(B)$, $((\tau' \circ \mu_A)(f))(M) = (\tau'(f|\underline{R}(A))(M) = (f|\underline{R}(A) \circ \tau)(M) = f(\tau M)$; and $((\mu_B \circ \alpha)(f))(M) = (\alpha(f)|\underline{R}(B))(M) = \alpha(f)(M)$; and (***) $\alpha(f)(M) = t_M(f) = \tau M(f)$ (since τM is the kernel) $= f(\tau M)$.

τ is continuous :

The Stone topology on M(A) coincides with the weak topology induced by bounded elements of A (by theorem 13 (iii) and theorem 4) . And since $f \circ \tau = \alpha(f) \in B \subset C(\underline{R}(B))$ for all $f \in A$ (from (***)) , τ is continuous (see [2] 3.8) ●

Since τM is the kernel of t_M , we have : $f \in \tau M$ if and only if $\alpha(f)(M) = 0$, i.e. , $\alpha(f) \in M$ since M is real . Hence if α is an isomorphism , then there is τ' such that $f \in \tau' M$ if and only if $\alpha^{-1}(f) \in M$. Then we have : $g \in \tau'(\tau M)$ if and only if $\alpha^{-1}(g) \in \tau M$,i.e., $g \in M$. Thus τ' is the inverse of τ . Then τ is a homeomorphism . Therefore ,

LEMMA 6. The ϕ-algebra A is isomorphic to some C(Y) if and only if A is isomorphic to $C(\underline{R}(A))$ (by μ_A) .

Proof: By the remark at the end of 1.4 and the remark above , μ_B and τ' are isomorphisms . We have $\alpha:A \longrightarrow C(Y)$, $\mu_B :C(Y) \longrightarrow C(\underline{R}(C(Y)))$, $\tau'^{-1}:C(\underline{R}(C(Y))) \longrightarrow C(\underline{R}(A))$. Hence A is isomorphic to $C(\underline{R}(A))$ by $\tau'^{-1}\cdot\mu_B\cdot\alpha = \mu A$ •

LEMMA 7. Every maximal ideal of Q(X) is absolutely convex : $M(Q(X)) = M(Q(X))$.

Proof : Given a maximal ideal M of Q(X) . Suppose $0 \leq |f| \leq |g|$, and $g \in M$. Let $V = V' \cap V''$ where V' and V" are the (dense open) domains of f and g . Define h on V as follows : $h(x) = f^2(x)/g(x)$ for $x \notin Z(g)$, and $h(x) = 0$ for $x \in Z(g)$. Since f/g is bounded on $V \backslash Z(g)$, h is the product of a continuous function and a bounded function on V . Hence h is continuous on V . Since $f^2 = h.g$ on V , $f^2 \in M$. Then because M is prime , $f \in M$ •

Result : Q(X) ($\subset D(M(Q(X)))$) is represented by functions on the space of its maximal ideals . Q(X) is isomorphic to some C(Y) if and only if the set of real maximal ideals , $\underline{R}(Q(X))$, is dense , and the restriction map from Q(X) to $C(\underline{R}(Q(X)))$ is onto .

3.3. ϕ-ALGEBRA OF EXTREMALLY DISCONNECTED SPACES .

In this section we study D(K) **when K is an extremally disconnected** space . This is done since M(Q(X)) has this property (ref. 3.4) .

3.3.1. A ϕ-algebra A is said to be **uniformly closed** if A is

complete in the metric

$$\sigma(f,g) = \sup \{|f(x) - g(x)| \wedge 1 : x \in \underline{R}(f) \cap \underline{R}(g)\} .$$

If A is not uniformly closed , the completion in σ need not be a ring (see [7] 1.8) .

Note that for $\delta < 1$, $|a - b| < \delta$ if and only if $|\hat{a}(M) - \hat{b}(M)| = |M(a) - M(b)| = M(|a - b|) < \delta$, for all $M \in \underline{R}(a) \cap \underline{R}(b)$; if and only if $\sigma(a,b) < \delta$. Thus convergences in σ and in [6] are equivalent .

LEMMA 8. Let A and B be ϕ-algebras with $\mathbf{M}(A) = \mathbf{M}(B)$. If B is the completion of A in σ , then $\underline{R}(A) = \underline{R}(B)$.

Proof : Since $A \subset B$, $\underline{R}(A) \supset \underline{R}(B)$. For each $f \in B$, there exist $f_n \in A$ converges uniformly to f on $\cap \underline{R}(f_n)$. Hence $\underline{R}(A) \subset \underline{R}(B)$ ●

3.3.2. A space X is said to be extremally disconnected if every open set has an open closure . In an extremally disconnected space , open subsets , and dense subsets are C^*-embedded . (See [2] 1H and 6M) .

LEMMA 9. D(K) is an uniformly closed ϕ-algebra and $K = \mathbf{M}(D(K))$.

Proof : For $f,g \in D(K)$, $\underline{R}(f) \cap \underline{R}(g)$ is dense , hence C^*-embedded in K . By theorem 8 (iii) , $K = \beta(\underline{R}(f) \cap \underline{R}(g))$. By theorem 8 (i) , $f.g$, $f + g \in D(K)$. Thus D(K) is a ϕ-algebra. Let $D^*(K)$ denote the set of all bounded elements of D(K) . Since $C(K) = D^*(K)$, and C(K) is complete in sup norm ([9] ch.9 lemma 31 , note that a Cauchy sequence in C(X) is an equicontinuous family) , $D^*(K)$ is uniformly closed . Hence

D(K) is uniformly closed (see [6] 3.7) . Because $D^*(K)$ contains constant functions , and elements of $D^*(K)$ separated the points of M(D(K)) (see theorem 13) ; by Stone-Weierstrass theorem (see [9] ch.9 th.28) , $D^*(K)$ is dense in C(M(D(K))). Hence $D^*(K) = C(M(D(K)))$, because $D^*(K)$ is uniformly closed. Then by theorem 7 , K = M(D(K)) ●

3.3.3. **LEMMA 10.** If the cardinal of K , or equivalently , the cardinal of D(K) , is non-measurable (see [2] ch. 12) , then $\underline{R}(D(K))$ is the set of isolated points of K .

Proof : Each isolated point of K has to lie in every $\underline{R}(f)$, hence lies in $\underline{R}(D(K))$. Conversely , if p is not isolated , then by [2] 12H 1-4 , there exists $f \in C(K)$ with f(p) = 0 and f positive on a dense subset of K . Since $1/f \in D(K)$, p does not lie in $\underline{R}(D(K))$ ●

Remark : Non-measurable cardinality and extremally disconnectedness are essential conditions for the theorem :

Let $K = \beta X$ where X is a discrete space of measurable cardinal . X is extremally disconnected , and so is K (see [2] 6M.1) . The cardinals of K and $\underline{R}(D(K))$ are larger than the cardinal of X , thus are measurable . Since D(K) = C(X) ($K = \beta X$) , by [2] 8.4 , $\underline{R}(D(K)) = \underline{R}(C(X)) = vX$ (Hewitt real-compactification of X) . Then by [2] 12.2 , $\underline{R}(D(K))$ is not discrete . Then it cannot be a set of isolated points .

Let Y be the one-point compactification of an uncountable discrete space of non-measurable cardinality . Y is not extremally disconnected (if S is an countable set from the

discrete space , then $S \cup \{\infty\}$ is the closure of S , but is not open) . $D(Y) = C(Y)$: if $f \in D(Y)$ and $f(\infty) \neq +\infty$, then there exists $g \in C(Y)$, defined as $g = 1/(f \vee 1)$ for $x \neq \infty$ and $g(\infty) = 0$, which is impossible (see example 2.3.3) . So $D(Y)$ is a ϕ-algebra . $\underline{R}(D(Y)) = Y$, and is not discrete .

3.3.4. LEMMA 11. $D(K)$ is a ϕ-algebra of real-valued functions if and only if $D(K)$ is isomorphic to some $C(Y)$.

Proof : Consider the restriction map $\mu : D(K) \to C(\underline{R}(D(K)))$. If $\underline{R}(D(K))$ is dense , it is C^*-embedded in $M(D(K)$, then $M(D(K)) = \beta(\underline{R}(D(K)))$, and this makes μ onto (theorem 8) . μ is one-one by lemma 4 . If $D(K) \approx C(Y)$, then $D(K) \approx C(\underline{R}(D(K)))$ by lemma 6 . Hence $\underline{R}(D(K))$ is dense ●

3.3.5. Let $\bar{Q}(X)$ denote the completion of $Q(X)$ in the metric defined in 3.3.1. - which is equivalent to the metric defined in [1] 4.1 (from [9] ch.7 problem 10 and mimicking the proof of [1] 5.2) ; i.e. , the convergences of a sequence are the same in two metrics ; hence $\bar{Q}(X)$ is isomorphic to the completion of $Q(X)$ in the metric [1] 4.1 . Now we have ,

LEMMA 12. $\bar{Q}(X) = D(M(Q(X)))$.

Proof : $\bar{Q}(X) = Q(C(M(Q(X))))$ by [1] 4.8 , [1] 5.5 , and [1] 5.11 . Hence $\bar{Q}(X) = \lim_{\to V} C(V)$, where V ranges over dense open sets of $M(Q(X))$. If $f \in D(M(Q(X)))$, then $f \in C(\underline{R}(f))$. Since $\underline{R}(f)$ is a dense open set of $M(Q(X))$, $f \in \bar{Q}(X)$. Note that the equivalence relations on $\bar{Q}(X)$ and $D(M(Q(X)))$ are the same . Thus $\bar{Q}(X) \supset D(M(Q(X)))$. The converse is true because $Q(X) \subset D(M(Q(X)))$ and $D(M(Q(X)))$ is complete (lemma 9) ●

Remark: From lemma 9 and the lemma above , $M(\bar{Q}(X)) = M(Q(X))$. Then from lemma 8 , $\underline{R}(\bar{Q}(X)) = \underline{R}(Q(X))$. Then from lemma 10 , if the cardinal of X , or equivalently , the cardinal of $M(Q(X))$ is non-measurable , the set of isolated points in $M(Q(X))$ is $\underline{R}(Q(X))$. Then from lemma 11 , $\bar{Q}(X)$ is isomorphic to some $C(Y)$ if and only if this set of isolated points is dense .

3.4. THE SET OF ISOLATED POINTS .

An open set is said to be **regular** if it is the interior of its closure .

The set of all regular open sets of a topological space X is a Boolean algebra (see [5] §4 theorem 1) , denoted by $B(X)$.

Theorem [1] 11.15 states that $M(Q(X)) \approx M(B(\beta X)) \approx$ an extremally disconnected compact space .

The proof of [3] 3.2 points out that there is a continuous map π from $M(B(\beta X))$ onto βX which maps proper closed subsets of $M(B(\beta X))$ onto proper subsets of βX . (In the proof : $\underline{D}(\beta X)$ is isomorphic to $B(\beta X)$ by the map $D \to \text{int } D$, $D \in \underline{D}(\beta X)$; and the Stone representation space for $B(\beta X)$ is $M(B(\beta X))$ (see [8] 2.5 (Stone) corollary)) .

As a result ,

LEMMA 13. $M(Q(X))$ is extremally disconnected , and there is a continuous map π of $M(Q(X))$ onto βX which maps proper closed subsets of $M(Q(X))$ onto proper subsets of βX (π is said to be "**irreducible**") .

LEMMA 14. Let f be an irreducible closed continuous map of K onto Z (T_1-spaces) . Then , the isolated points of K are in a one-to-one correspondence with the isolated points of Z by f , and one set is dense if and only if the other is .

Proof : If p is isolated in K , then $K\backslash\{p\}$ is closed , hence $f(K\backslash\{p\})$ is a proper closed subset . And since f is onto , $f(p) = Z\backslash f(K\backslash\{p\})$, thus f(p) is isolated . Let x be isolated in Z . Then $f^{-1}(x)$ is open . If there are $p,q \in f^{-1}(x)$ with $p \neq q$, then there exists an open set U containing p but not q such that $f(U) \subset \{x\}$ (Z is T_1-space and f is continuous) . But $f(K\backslash U) = Z$, which contradicts irreducibility . If the isolated points of K are dense , then so are the isolated points of Z since f is continuous and onto . The converse is true since f is closed and irreducible ●

LEMMA 15. The isolated points of M(Q(X)) are in a one-one correspondence with the isolated points of X . One set is dense if and only if the other is .

Proof : An isolated point of βX is isolated in X since X is dense in βX . Also by [2] 6.9(d) , an isolated point of X is isolated in βX . The rest follows from the two lemmas above ●

Note that π is closed because M(Q(X)) is compact (see [2] 0.12) .

LEMMA 16. If the cardinal of X , or equivalently , the cardinal of M(Q(X)) , is non-measurable (see [2] 12.5) , then the restriction map $\mu : Q(X) \to C(\underline{R}(Q(X)))$ is onto .

Proof : As a result of the lemma above , it suffices that each function on the isolated points of X be extendible over some dense open subset of X : assign the value 0 outside the closure of the set of isolated points ●

Remark: We cannot generalize the lemma above to all dense ϕ-subalgebras of D(K) , K extremally disconnected . Let X be an uncountable discrete space , and $K = \beta X$. Let A be the ϕ-subalgebra of functions f with f(X) countable . Since D(K) = C(X) ($K = \beta X$) , and the set of all rational numbers is dense in **R** , A is dense in D(K) . But μ_A is not onto because there exists $g \in C(\underline{R}(A))$ such that g(X) uncountable (take $g \in C^*(X)$, then extend g over $K = \beta X$) .

3.5. THE MAIN RESULT .

LEMMA. Let X have non-measurable cardinal . The following are equivalent .

(1) Q(X) is a ϕ-algebra of real-valued functions .

(2) $\overline{Q}(X)$ is a ϕ-algebra of real-valued functions .

(3) Q(X) is isomorphic to some C(Y) .

(4) $\overline{Q}(X)$ is isomorphic to some C(Y) .

(5) The isolated points of M(Q(X)) are dense in M(Q(X)).

(6) The isolated points of X are dense in X .

Proof :

(1) $\Leftrightarrow$ (2): by remark 3.3.5 . (2) $\Leftrightarrow$ (5): by remark 3.3.5 .

(1) $\Leftrightarrow$ (3): by the last lemma 16 and result 3.2 .

(2) $\Leftrightarrow$ (4): by remark 3.3.5 . (5) $\Leftrightarrow$ (6): by lemma 15 ●

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