Limit properties
of
the "Almost Lack of Memory"
distributions

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ABSTRACT

LIMIT PROPERTIES OF THE "ALMOST LACK OF MEMORY" DISTRIBUTIONS

FAIZ AHMAD

Limit behaviour of a new class of distributions having the " Almost Lack of Memory"(ALM) property is derived, when some of its parameters are close to their boundary points. The effect of combined closeness of the parameters on the distribution is also shown. Results obtained have been verified by simulation. Graphical illustrations have also been included with simulated and real distributions.

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INTRODUCTION

The exponential distribution is frequently used as a model for the distribution of times between the occurrence of successive events such as customers arriving at a service facility or calls coming on a switchboard. The reason for this is that the exponential distribution is closely related to the Poission process.

Another important application of the exponential distribution is to model the distribution of component lifetimes. A partial reason for the popularity of such application is the "memoryless" property of the exponential distribution.

While the memoryless property can be justified at least approximately in many applied problems, in other situations components deteriote with age or occasionally improve with age (at least to a certain points). More general life time models are then furnished by the gamma, weibull, or lognormal distributions. Recently a new class of models has been introduced by Chukova and Dimitrov (1992) which also models the general life time problems. They named this class of distributions as "Almost Lack of Memory" (ALM) class of distribution. It has some properties similar to the exponential distribution and hence it has some potential applications. An important question arises about the closeness of this class of distributions with that of the exponential distributions. In the present thesis the closeness between these two distributions by investigating the limit behaviour of ALM class of distributions has been studied.

The thesis consists of four chapters. The first chapter deals with the properties and important results concerning the "Almost Lack of Memory" distributions. In the second chapter the limit properties of the "Almost Lac . of Memory" distributions have been established. In chapter III, the results of chapter II have been justified by simulation. Finally chapter IV is devoted to discussions and conclusions of the work presented in this thesis.

* * *

CHAPTER I
PROBABILITY DISTRIBUTIONS HAVING THE "ALMOST LACK OF MEMORY" PROPERTY

In this chapter we have summarized some properties and important results on the "Almost Lack of Memory" probability distributions from the recent works of Chukova and Dimitrov (1991,1992) and Dimitrov et al. (1992).

§1. Properties of the "Almost Lack of Memory" Distributions

The following definitions introduce two probability distributions which play important roles in many practical problems of reliability, queuing theory, statistics and a lot of other practical situations.

Definition 1.1.1: A non negative random variable X has the *lack of memory* property iff the equation

$$P\{X \geq b + x | X \geq b\} = P\{X \geq x\} \tag{1.1.1}$$

holds for all $b > 0, x > 0$.

This definition implies that X has either exponential (if X is continuous) or geometric distribution (if X is discrete).

Definition 1.1.2: A non negative random variable X has the *almost lack of memory* (ALM) property if there exists a sequence of different constants $\{a_n\}_{n=1}^{\infty}$ such that the equation

$$P\{X \geq b + x | X \geq b\} = P\{X \geq x\} \tag{1.1.2}$$

holds for a sequence

$$b = a_n, \quad n = 1, 2, ...$$

and for all $x \geq 0$. If (1.1.2) holds for given $b = c$ and all $x > 0$, then it is said that X has the *lack of memory* (L.M.) property at the poi c.

The following Lemma has been proved by Dimitrov et al. (1992).

Lemma:1.1.1 If a random variable X has the lack of memory property at a point $c > 0$, then X has the ALM property over the sequence $\{a_n = nc\}_{n=1}^{\infty}$.

A characterization theorem established by Dimitrov et al. (1992) by using Lemma 1.1.1 is as follows.

Theorem 1.1.1. *A random variable X has the ALM distribution at a point $c > 0$, iff its c.d.f. has the form*

$$F_X(x) = 1 - \alpha^{[x/c]} + \alpha^{[x/c]}(1 - \alpha)F_Y(x - [x/c].c), \quad x \geq 0, \tag{1.1.3}$$

where $[x/c]$ denotes the integer part of x/c, $\alpha \in (0, 1)$ and $F_Y(y)$ is any c.d.f. with support $[0, c)$.

Proof: See Dimitrov et al. (1992)

Definition 1.1.3: We say that distribution $F_X(x)$ of the random variable X belongs to the $ALM(\alpha, c, F_Y(.))$ if X satisfies the Theorem 1.1.1 Here α, c and

$F_Y(.)$ are the parameters of the ALM distribution $F_X(x)$. It can be denoted as $X \in ALM(\alpha, c, F_Y(.))$.

A decomposition theorem has also been proved by Dimitrov et al. (1992), which is an equivalent representation of Theorem 1.1.1. We state it without proof:

Theorem 1.1.2. *A random variable X has the A.L.M. property at a given point $c > 0$ in the sense of Definition (1.1.2) iff X is decomposable into the form*

$$X = Y_c + cZ, \tag{1.1.4}$$

where Y_c and Z are independent r. v.'s. Y_c with probability 1 is concentrated on the interval $[0, c)$ and Z has a geometric distribution on the set $0, 1, 2, 3, \ldots$ with parameter $\alpha = P\{X \geq c\}$.

Theorem 1.1.2 is a characterizaton theorem for the distribution of random variables having the ALM property. This is very useful for simulating random variables of this type. In chapter (III), we used this representation theorem for our simulation results.

§2. ALM property for the bivariate distribution

In this section we summarize the results of bivariate distribution having ALM property. (Dimitrov and Chukova (1991))

Definition 1.2.1: The random vector $Z = (X, Y)$ with nonnegative components X and Y has the bivariate ALM property iff the following equation

$$P\{X \geq a + x, Y \geq b + y | X \geq a, Y \geq b\} = P\{X \geq x, Y \geq y\} \tag{1.2.1}$$

holds, for any $x \geq 0, y \geq 0$ and for all $a > 0, b > 0$.

The following result has been proved with the following assumptions:

Theorem 1.2.1. *Let $P\{X \geq a\} = \alpha \in (0,1)$; and $P\{Y \geq b\} = \beta \in (0,1)$.*

If (1.2.1) holds for some $a > 0, b > 0$, then the following result holds;

$$P\{X \geq na + x, Y \geq nb + y | X \geq na, Y \geq nb\} = P\{X \geq x, Y \geq y\}. \qquad (1.2.2)$$

Definition 1.2.2: The random variable X and Y are said to be almost independent iff

$$P\{X \geq na + x, Y \geq y\} = P\{X \geq na\}.P\{Y \geq y\} \qquad (1.2.3)$$

and

$$P\{X \geq x, Y \geq mb + y\} = P\{X \geq x\}.P\{Y \geq mb\}$$

hold for arbitrary $x \geq 0, y \geq 0$ and for $n = 0, 1, 2, .., m = 0, 1, 2...$

Definition 1.2.3: The distribution $F_Z(x, y)$ of the random vector $Z = (X, Y)$ belongs to the class of Bivariate distributions having ALM property iff its survival function is given by the equation

$$P\{X \geq x, Y \geq y\} = \alpha^{[x/a]}\beta^{[y/b]}[(1-\alpha)(1-\beta)G(x - [x/a]a, y - [y/b]b)$$

$$+\beta(1-\alpha)G_1(x - [x/a]a) + \alpha(1-\beta)G_2(y - [y/b]b) + \alpha\beta]. \qquad (1.2.4)$$

Here $[t]$ denotes the integer part of the argument $t \geq 0; G(x, y)$ is the survival function defined over the rectangle $\{[0, a).[0, b)\}, G_1(x)$ and $G_2(y)$ are its marginals and $\alpha \in (0,1), \beta \in (0,1)$ are arbitrary given and fixed numbers.

The following results are the consequences of the above properties.

Theorem 1.2.2. *The random vector Z is decomposable into*

$$Z = Z_1 + (X_a, Y_b), \tag{1.2.5}$$

where $Z_1 = (X_1, Y_1)$ is independent of (X_a, Y_b) and has $G(x,y)$ as a survival function and with probability 1, takes value in the rectangle $\{[0,a).[0,b)\}$. X_a and X_b are independent random variables geometrically distributed over the set $\{0, a, 2a, ...\}$ and $\{0, b, 2b, ..\}$ with parameters α and β respectively.

Theorem 1.2.3. *The p.d.f. of Z (if it exists) has the form*

$$f_Z(x,y) = \alpha^{[x/a]}\beta^{[y/b]}(1-\alpha)(1-\beta)g(x-[x/a]a, y-[y/b]b), \tag{1.2.6}$$

$$x \geq 0, y \geq 0,$$

where $g(u,v)$ is the p.d.f. of the survival function $G(x,y)$.

The proof of the above results can be found in Dimitrov and Chukova (1991).

Extension of the Lack of memory property over the real line is also discussed by Dimitrov et al. (1992) in details .

§3. Multiplicative ALM property (MALM)

Multiplicative lack of memory property is the result of replacing the *plus* sign by a *multiplication* sign in equation (1.1.1). We summarize the results of Multiplicative Almost Lack of Memory property as given by Dimitrov and Von Collani (1991) and Dimitrov et al. (1992).

Definition 1.3.1: It is said that a random variable $X \geq 0$ has the MALM property iff either

(a) for a given sequence $\{b_n\}_{n=1}^{\infty}$ of different values of $b_n \in (0,1)$ and for any $x \in (0,1)$

$$P\{X \leq xb_n|X \leq b_n\} = P\{X \leq x\} \tag{1.3.1}$$

or

(b) for infinitely many different $b_n > 1$ and for any $x \geq 1$ the relation

$$P\{X \geq xb_n|X \geq b_n\} = P\{X \geq x\} \tag{1.3.2}$$

holds.

Definition 1.3.2: If the equation (1.3.1) or (1.3.2) holds for some $b_n = c$, where $c \neq 1$, then it is said that X has the MALM property at the point c.

If equation (1.3.1) is true for all x and $b_n \in [0,1)$, then it characterizes the uniform distribution $U[0,1]$.

If (1.3.2) is true for all x and $b_n \in [1,\infty)$, then it characterizes the Pareto distribution $F_X(x) = 1 - x^{-\gamma}, \gamma > 0$.

The above result from Galambos and Kotz (1978) shows that the MALM property characterizes the uniform and Pareto distribution.

Theorem 1.3.1. *The following is true:*

Let X be a non-negative random variable with c.d.f $F_X(x)$ and X has the MALM property at a given point $c \neq 1$, then

(1) *The random variable X has the MALM property with the sequence of the form $b_n = c^n, n = 0,1,2,...;$*

(2) *The c.d.f $F_X(x)$ of X has the form defined by either the equation*

$$F_X(x) = \alpha^n(1-\alpha)[\alpha + (1-\alpha)F_Y(x.c^{-n})], \tag{1.3.3}$$

for $0 < c < 1$ and $x \in (c^{n+1}, c^n]$, $\quad n = 0, 1, 2...,$

or

$$1 - F_X(x) = \alpha^n(1-\alpha)[\alpha + (1-\alpha)(1 - F_Y(x.c^{-n}))],$$

for $c > 1$ and $x \in [c^n, c^{n+1})$, $\quad n = 0, 1, 2, ...,$

where $F_Y(y)$ is defined by the relation

$$F_Y(y) = P\{X < y | X < c\}, \quad \text{if} \quad c > 1, \quad y \in [1, c); \tag{1.3.4}$$

$$F_Y(y) = P\{X \leq y | X > c\}, \quad \text{if} \quad c < 1, \quad y \in (c, 1],$$

and

$$\alpha = P\{X \geq c\} \quad \text{for} \quad c > 1,$$

or

$$\alpha = P\{X \leq c\} \quad \text{for} \quad c < 1.$$

(3) *The random variable X is representable in the form*

$$X = Y.c^Z, \tag{1.3.5}$$

where Y is independent of Z, Y is defined by $F_Y(y)$ and Z has a geometric distribution over the $\{0, 1, 2, ...\}$ with parameter α.

§4. Characterization of the ALM distribution with a periodic failure rate function

The failure function was first introduced by Barlow and Prochan (1965).

Let $F(x)$ be the distribution function of the time to failure random variable X, with probability density function $f(x)$. Then the failure rate function is defined by

$$\lambda(x) = \frac{f(x)}{1-F(x)} = \frac{F'(x)}{1-F(x)}, \quad x \geq 0. \tag{1.4.1}$$

For the discrete case

$$\lambda(k) = P\{X = k | X \geq k\} = \frac{p(k)}{\sum_{\nu=k}^{\infty} p(\nu)}, \quad k = 1, 2, 3, \ldots . \tag{1.4.2}$$

The failure rate function may be described as the conditional probability of an equipment failing at operating age x, having survived to age x.

Definition 1.4.1: A random variable X has a periodic failure rate function (PFR) $\lambda_X(x, t)$ with period c, if there exist a positive constant $c > 0$ such that for any integer $n \geq 0$ and any $x \geq 0$ following holds

$$\lambda_X(nc + x, t) = \lambda_X(x, t) = \frac{P\{X \in [x, x+t]\}}{P\{X \geq x\}}. \tag{1.4.3}$$

Theorem 1.4.1. *(Dimitrov et al. (1992))*

(1) *A non-negative random variable X has periodic failure function of period $c > 0$ iff X has the ALM-property with $b_n = nc, \quad n = 1, 2, \ldots$;*

(2) *The PFR $\lambda_X(x, t)$ and the ALM distribution $F_X(x)$ are related by*

$$F_X(x) = exp[-\lambda(x, t)].$$

The failure function is expressed by equation

$$\lambda_X(x,t) = 1 - \alpha^{[(t+x)/c]-[x/c]} \cdot \frac{1-(1-\alpha)F_Y(t+x-[(t+x)/c])}{1-(1-\alpha)F_Y(x-[x/c]c)}, \qquad (1.4.4)$$

where

$$F_Y(y) = [1 - e^{-\lambda_X(y)}]/(1-\alpha), \quad y \in [0,c), \alpha = e^{-\lambda_X(c)}$$

with

$$\lambda_X(y) = \lambda_X(y,0) = -[y/c]ln\alpha - ln[1-(1-\alpha)F_Y(y-[y/c]c)];$$

(3) If $\lambda_X(x)$ or $\lambda_X(k)$ and α, $f_Y(x)$ or $p_Y(k)$ are the attributes determining the random variable X as PFR and having the ALM property, then the following relation holds:

(a) In the continuous case

$$\lambda_X(nc+x) = \frac{(1-\alpha)f_Y(x)}{(1-\alpha)\int_0^c f_Y(u)\,du + \alpha} = \lambda_X(x), n = 0,1,2,..., \qquad (1.4.5)$$

where

$$\alpha = exp[-\int_0^c \lambda_X(u)\,du];$$

and

$$f_Y(x) = \frac{\lambda_X(x)}{1-\alpha} exp[-\int_0^x \lambda_X(u)\,du]$$

for $x \in [0,c)$ and $f_Y(x)$ is a p.d.f on $[0,c)$ of random variable Y;

(b) In the discrete case, c is an integer and

$$\lambda_X(nc+m) = \frac{(1-\alpha)p_Y(m)}{(1-\alpha)\sum_{\nu=m}^{c-1} p_\nu + \alpha} = \lambda_X(m) \quad n = 0,1,2,...; \qquad (1.4.6)$$

where

$$\alpha = \prod_{k=0}^{c-1} [1 - \lambda_X(k)],$$

$$p_Y(k) = \frac{\lambda_X(m)}{1-\alpha} \prod_{\nu=0}^{m-1} [1 - \lambda_X(\nu)], \quad for \quad m = 0, 1, 2, \ldots c-1.$$

and

$$\{p_Y(m)\}_{m=0}^{c-1}$$

is the p.d.f of a discrete random variable Y.

For the proof of the above results see Dimitrov et al. (1992).

Theorem 1.4.2. *(Characterization theorem): A random variable X has a periodic failure rate function with a certain period iff X can be decomposed as a sum*

$$X = \eta + \xi, \tag{1.4.7}$$

of two independent components η and ξ, where ξ has a geometric distribution of the form

$$P\{\xi = nc\} = \alpha^n(1-\alpha), \quad n = 0, 1, 2, \ldots$$

with arbitrary $c > 0$, and η is distributed either on the set $\{0, 1, 2, \ldots\}$ (with c an integer in the discrete case) or the interval $[0, c)$.

Proof: The proof can be found in Chukova and Dimitrov (1993).

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CHAPTER II
LIMIT PROPERTIES IN
THE "ALMOST LACK OF MEMORY" DISTRIBUTIONS

§1.Introduction

In this chapter, we will examine the following question: What is the limit behaviour of the distribution of $X \in ALM(c, \alpha, F_Y(.))$, when some of its parameters are close to their boundaries. e.g. α close to 0 or 1; c close to 0 or ∞ with proper assumption on the behaviour of $F_Y(.)$ in the last case. We will also answer an important question about the effect of combined closeness of α, c and $F_Y(.)$, when they are close to their boundaries and how this reflects on the distribution of X.

Various kinds of normalization for X exists. We now study the following two normalization cases:

Case I : Scale Normalization

$$(\frac{X}{EX}); \tag{2.1.1}$$

Case II :Standard (scale and location) Normalization

$$(\frac{X - EX}{\sqrt{Var(X)}}). \tag{2.1.2}$$

On the basis of the results in chapter (I), we have established the limit behaviour of the above normalized random variables X.

§2. Limit behaviour of the distribution of X when α is close to its boundaries

Theorem 2.2.1. *Let the random variable X have the ALM property at a point c given by (1.1.3). If the parameter α tends to 1 then the distribution of the random variable X/EX tends to the exponential distribution.*

Proof: By the decomposition Theorem 1.1.2 of the random variable X, we have

$$X = Y_c + cZ, \tag{2.2.1}$$

where Y_c has an arbitrary distribution function on $[0, c)$, for $c > 0$; Y_c and Z are independent variables. We know that with probability 1 Y_c is concentrated on the interval $[0, c)$ and Z has a geometric distribution on the set $\{0, 1, 2, 3....\}$ with parameter $\alpha = P\{X \geq c\}$.

In our present work we use (2.2.1) to determine the limit distribution of the random variable $\frac{X}{EX}$. The representation (2.2.1) can be rewritten in the following form

$$\frac{X}{EX} = \frac{Y}{EX} + c\frac{Z}{EX}.$$

From (2.2.1) the expectation of random variable X is

$$EX = EY + cEZ; \tag{2.2.2}$$

$$= EY + \frac{\alpha c}{1-\alpha}. \tag{2.2.3}$$

Therefore,

$$\lim_{\alpha \to 1} EX = \lim_{\alpha \to 1} EY + \lim_{\alpha \to 1} \frac{\alpha c}{1-\alpha} \to \infty. \tag{2.2.4}$$

Also

$$\lim_{\alpha\to 1}(1-\alpha)EX = \lim_{\alpha\to 1}(1-\alpha)EY + \lim_{\alpha\to 1}\alpha c,$$

i.e.

$$\lim_{\alpha\to 1}(1-\alpha)EX \to c. \qquad (2.2.5)$$

Taking the Laplace-Stieltjes Transform of the random variable $\frac{X}{EX}$, we obtain:

$$\begin{aligned}\phi_{\frac{X}{EX}}(s) &= Ee^{(\frac{-sX}{EX})}\\ &= \phi_Y(\frac{s}{EX}).\frac{1-\alpha}{1-\alpha e^{\frac{-sc}{EX}}}\end{aligned}$$

Thus

$$\lim_{\alpha\to 1}\phi_{\frac{X}{EX}}(s) = \lim_{\alpha\to 1}\phi_Y(\frac{s}{EX}).\lim_{\alpha\to 1}\frac{1-\alpha}{1-\alpha e^{\frac{-sc}{EX}}},$$

and since

$$\lim_{\alpha\to 1}\phi_Y(\frac{s}{EX}) = 1,$$

we have

$$\lim_{\alpha\to 1}\phi_{\frac{X}{EX}}(s) = \lim_{\alpha\to 1}\frac{1-\alpha}{1-e^{\frac{-sc}{EX}}} \qquad (2.2.6)$$

Here

$$1-\alpha e^{\frac{-sc}{EX}} = 1-\alpha+\alpha\sum_{k=1}^{\infty}(\frac{sc}{EX})^k\frac{1}{k!}(-1)^{k+1}$$

and substituting it in (2.2.6), we obtain

$$\begin{aligned}\lim_{\alpha\to 1}\phi_{\frac{X}{EX}}(s) &= \lim_{\alpha\to 1}\frac{1-\alpha}{(1-\alpha)[1+\frac{\alpha cs}{(1-\alpha)EX}+\frac{1}{(1-\alpha)EX}o(\frac{1}{EX})]}\\ &= \lim_{\alpha\to 1}\frac{1}{[1+\frac{\alpha sc}{(1-\alpha)EX}+\frac{1}{(1-\alpha)EX}o(\frac{1}{EX})]}\\ &= \lim_{\alpha\to 1}\frac{1}{[1+\frac{\alpha sc}{(1-\alpha)EX}+\frac{1}{(1-\alpha)EX}o(\frac{1}{EX})]},\end{aligned}$$

where

$$o(\frac{1}{EX}) = \sum_{k=2}^{\infty} \frac{sc}{(EX)^{k-1}} \frac{1}{k!}(-1)^{k+1} \longrightarrow 0.$$

Hence, by (2.2.5)

$$\lim_{\alpha \to 1} \phi_{\frac{X}{EX}}(s) = \frac{1}{1+s} \quad .$$

Since $\frac{1}{1+s}$ is the Laplace Stieltjes transform of the exponential distribution with parameter one, the theorem has been proved.

Theorem 2.2.2. *Let the random variable X have the ALM property given by (1.1.3) at the point c. If the parameter α tends to 0 then the distribution of the random variable $\frac{X}{EX}$ tends to the distribution of the random variable $\frac{Y}{EY}$.*

Proof: Here, we use the decomposition theorem 1.1.2 for the random variable X .

$$X = Y_c + cZ,$$

where Y_c and Z are the same as defined earlier. Taking expectation of random variable X, we have equation (2.2.5). Therefore ,

$$\lim_{\alpha \to 0} EX = \lim_{\alpha \to 0} EY + \lim_{\alpha \to 0} \frac{\alpha c}{1-\alpha};$$

$$\lim_{\alpha \to 0} EZ = \lim_{\alpha \to 0} \frac{\alpha c}{1-\alpha} = 0, \qquad (2.2.7)$$

and

$$\lim_{\alpha \to 0} EX = EY.$$

If $\alpha \to 0$, it follows that $Z \xrightarrow{P} 0$. Indeed, for any $\epsilon > 0$,

$$P(Z > \epsilon) = 1 - P(Z = 0) = 1 - (1 - \alpha),$$

i.e

$$P(Z > \epsilon) = \alpha \to 0. \tag{2.2.8}$$

Therefore in

$$X = Y + cZ$$

from (2.2.7) and (2.2.8), we have

$$cZ \xrightarrow{P} 0.$$

By using the to Slutsky's theorem, i.e

if for

$$X_n = Y_n + Z_n.$$

$$Y_n \underset{n\to\infty}{\xrightarrow{d}} Y; \; Z_n \underset{n\to\infty}{\xrightarrow{P}} 0;$$

then

$$X_n \xrightarrow{d} Y.$$

We obtain

$$\frac{X}{EX} = \frac{Y}{EX} + \frac{cZ}{EX} \xrightarrow{d} \frac{Y}{EY}.$$

This proves the theorem.

Theorem 2.2.3. *Let the random variable X have the ALM property with the distribution function given by equation (1.1.3). If the parameter α tends to 1, then the random variable $\frac{X-EX}{\sqrt{Var(X)}}$ has the limit distribution*

$$f(t) = \begin{cases} 0, & t \leq -1; \\ e^{-(t+1)}, & t > -1. \end{cases}$$

Proof: By the decomposition theorem, the representation

$$X = Y + cZ$$

holds.

The characteristic function of $\frac{X-EX}{\sqrt{Var(X)}}$ is

$$\phi_{\frac{X-EX}{\sqrt{Var(X)}}}(t) = e^{\frac{-itEX}{\sqrt{Var(X)}}} \phi_X\left(\frac{t}{\sqrt{Var(X)}}\right), \tag{2.2.9}$$

and

$$EX = EY + \frac{\alpha c}{1-\alpha}. \tag{2.2.10}$$

Therefore, it is easy to see that

$$\lim_{\alpha \to 1} EX \longrightarrow \infty;$$

$$\lim_{\alpha \to 1} (1-\alpha) EX \longrightarrow c.$$

Moreover, we have

$$Var(X) = Var(Y) + \frac{\alpha c^2}{(1-\alpha)^2}; \tag{2.2.11}$$

$$\lim_{\alpha \to 1} \sqrt{Var(X)} = \lim_{\alpha \to 1} \sqrt{Var(Y) + \frac{\alpha c^2}{(1-\alpha)^2}} = \infty.$$

Using the normalization coefficient $(1-\alpha)$, we observe that

$$\lim_{\alpha \to 1} (1-\alpha)\sqrt{Var(X)} \longrightarrow c. \tag{2.2.12}$$

Combining (2.2.10) and (2.2.12), we obtain

$$\lim_{\alpha \to 1} \frac{EX}{\sqrt{Var(X)}} = 1, \tag{2.2.13}$$

i.e. this limit will not depend on the value of α.

Further, since $\sqrt{Var(X)} \to \infty$, and c is fixed, Y remain bounded. Thus

$$\lim_{\alpha \to 1} \phi_Y\left(\frac{t}{\sqrt{Var(X)}}\right) = \phi_Y(0) = 1. \qquad (2.2.14)$$

Moreover

$$\frac{1-\alpha}{1-\alpha \quad exp\frac{-itc}{\sqrt{Var(X)}}} = \frac{1-\alpha}{1-\alpha\left(1-\frac{itc}{\sqrt{Var(X)}}+\frac{(itc)^2}{2!Var(X)}+\ldots\right)}$$
$$= \frac{1}{\left[1+\frac{itc\alpha}{(1-\alpha)\sqrt{Var(X)}}+\frac{1}{(1-\alpha)\sqrt{Var(X)}}o\left(\frac{1}{\sqrt{Var(X)}}\right)\right]}.$$

Here $o\left(\frac{1}{\sqrt{Var(X)}}\right)$ denotes the residual sum

$$\frac{1}{2!}\left(\frac{itc}{\sqrt{Var(X)}}\right)^2 - \frac{1}{3!}\left(\frac{itc}{\sqrt{Var(X)}}\right)^3 + \ldots$$

in the expansion of $e^{\frac{-itc}{\sqrt{Var(X)}}}$. In view of (2.2.12), (2.2.13) and (2.2.14), we obtain

$$\lim_{\alpha \to 1} \frac{1-\alpha}{1-\alpha e^{\frac{-itc}{\sqrt{Var(X)}}}} = \frac{1}{1+\frac{itc}{c}} = \frac{1}{1+it}.$$

Hence, from (2.2.9) and (2.2.14), we obtain

$$\lim_{\alpha \to 1} \phi_{\frac{X-EX}{\sqrt{Var(X)}}}(t) = \frac{e^{-it}}{1+it},$$

which is the characteristic function of a shifted exponential at (-1).

Theorem 2.2.4. *Let a random variable X have the ALM property with the distribution function given by equation (1.1.3). If the parameter α tends to zero,* *then the random variable* $\frac{X-EX}{\sqrt{Var(X)}}$ *tends to the distribution of the random variable* $\frac{Y-EY}{\sqrt{Var(Y)}}$.

Proof: From the decomposition theorem, we have (2.2.1). The expectation and variance are given by (2.2.10) and (2.2.11). Therefore, the following is true

$$\lim_{\alpha \to 0} EX \to EY;$$

$$\lim_{\alpha \to 0} Var(X) \to Var(Y),$$

and

$$\lim_{\alpha \to 0} EZ \to 0.$$

Moreover, for $\alpha \to 0$, we have $Z \xrightarrow{P} 0$. Now referring to Slutsky's theorem and using the above result we obtain,

$$\frac{X}{\sqrt{Var(X)}} = \frac{Y}{\sqrt{Var(X)}} + c\frac{Z}{\sqrt{Var(X)}} \xrightarrow{P} \frac{Y}{\sqrt{Var(Y)}}.$$

The following results are well known, see e.g. C. R. Rao (1973):

If

$$X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{P} c, \tag{2.2.15}$$

then it implies that

$$X_n + Y_n \xrightarrow{d} X + c$$

as well as

$$\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}.$$

In our case, we have,

$$\frac{X}{\sqrt{Var(X)}} - \frac{EX}{\sqrt{Var(X)}} = \frac{Y}{\sqrt{Var(X)}} - \frac{EX}{\sqrt{Var(X)}} + c(\frac{Z}{\sqrt{Var(X)}} - \frac{EX}{\sqrt{Var(X)}})$$

Since

$$\frac{X}{\sqrt{Var(X)}} \xrightarrow[\alpha\to 0]{P} \frac{Y}{\sqrt{Var(Y)}};$$

$$EX \underset{\alpha\to 0}{\longrightarrow} EY;$$

$$\sqrt{Var(X)} \to \sqrt{Var(Y)},$$

it follows that

$$\frac{X - EX}{\sqrt{Var(X)}} \xrightarrow{d} \frac{Y - EY}{\sqrt{Var(Y)}}. \tag{2.2.16}$$

Therefore, if c is fixed (i.e. does not vary simultaneously with α), the limit behaviour of Z leads to the limit distribution of the random variable $X = Y + cZ$. But (2.2.16) shows that after the limit ($\alpha \to 0$) is taken, the limit behaviour of X, when c is large ($c \to \infty$) or small ($c \to 0$), will be determined by the corresponding limit behaviour of Y.

§3: Limit behaviour of X with respect to c, when it is close to its boundaries 0 or ∞

In this section we study the behaviour of an ALM$(c, \alpha, F_Y(.))$ variable X, when the parameter c is close to its boundaries and assuming the other parameter α is fixed. $F_Y(.)$ is a distribution with support over the interval $[0, c)$, varying simultaneously with c. Therefore a proper assumption on the distribution $F_Y(.)$ is also important in determining the limit behaviour of X.

Theorem 2.3.1. *Let X have the ALM property at a point c given by (1.1.3) or $X \in ALM(c, F_Y(.), \alpha)$. Assume the parameter c tends to its boundaries i.e. either to 0 or to ∞, the distribution of $F_Y(.)$ is uniform over $[0, c)$ and EX is known. Then* *the random variable* $\frac{X}{EX}$ *also has an ALM distribution, with* $Y \sim$ *Uniform over* $[0, 2\frac{1-\alpha}{1+\alpha}]$ *and* $Z \sim$ *geometric over* $k.\frac{2(1-\alpha)}{1+\alpha}, k = 0, 1, 2, \ldots$

Proof: Using the decomposition property (2.2.1), we have

$$X = Y_c + cZ,$$

where Y_c and Z are the same as defined earlier. Taking the expectation of X under the given assumptions, we obtain.

$$\begin{aligned} EX &= EY + cEZ; \\ &= \frac{c}{2} + \frac{\alpha c}{1-\alpha}; \\ &= \frac{c(1+\alpha)}{2(1-\alpha)}; \end{aligned}$$

Hence

$$\frac{c}{EX} = \frac{2(1-\alpha)}{(1+\alpha)}.$$

Thus the Laplace Stieltjes transform of $\frac{X}{EX}$, is given by

$$\begin{aligned} \phi_{\frac{X}{EX}}(s) &= Ee^{(\frac{-sx}{EX})} \\ &= \phi_Y(\frac{s}{EX})\frac{1-\alpha}{1-\alpha e^{\frac{-sc}{EX}}} \\ &= \frac{1-\alpha}{1-\alpha e^{\frac{-sc}{EX}}}\frac{e^{\frac{-cs}{EX}}-1}{\frac{2sc}{EX}}. \end{aligned}$$

Moreover

$$\begin{aligned} \lim_{c\to 0 \text{ or } c\to\infty} \phi_{\frac{X}{EX}}(s) &= \frac{1-\alpha}{1-\alpha \quad exp\frac{-2s(1-\alpha)}{1+\alpha}} \frac{1-exp\frac{-2s(1-\alpha)}{1+\alpha}}{2s\frac{1-\alpha}{1+\alpha}} \\ &= \frac{1+\alpha}{2s}.\frac{1-exp\frac{-2s(1-\alpha)}{1+\alpha}}{1-\alpha \quad exp\frac{-2s(1-\alpha)}{1+\alpha}} \qquad (2.3.1) \\ &= \frac{(1-\alpha)}{1-\alpha \quad exp\frac{-2s(1-\alpha)}{1+\alpha}} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!}(\frac{2s(1-\alpha)}{1+\alpha})^{k-1}. \end{aligned}$$

Therefore, $\frac{X}{EX}$ belongs to the class of ALM with $Y \sim$ Uniform over $[0, \frac{2(1-\alpha)}{1+\alpha}]$ and $Z \sim$ Geometric with parameter α, over the numbers $k\frac{2(1-\alpha)}{1+\alpha}, k = 0, 1, 2, \ldots$. Thus the theorem is proved.

Theorem 2.3.2. *Let X have the ALM property at c given by (1.1.3). Assume the distribution of $F_Y(.)$ is uniform over $[0, c)$. Then the random variable $\tilde{X} = \frac{X-EX}{\sqrt{Var(X)}}$ has an ALM distribution, which does not depend on c.*

Proof: The expectation of X under the given assumption is easy to get

$$EX = \frac{c}{2} + \frac{\alpha c}{1-\alpha},$$

and the variance is

$$Var(X) = \frac{c^2(1-\alpha)^2 + 12\alpha c^2}{12(1-\alpha)^2}.$$

Therefore,

$$\frac{EX}{\sqrt{Var(X)}} = \frac{\sqrt{3}(1+\alpha)}{\sqrt{(1-\alpha)^2 + 12\alpha}} = b_1, \tag{2.3.2}$$

i.e.

$$\frac{EX}{\sqrt{Var(X)}} = \text{is a constant, which does not depend on } c.$$

And

$$\frac{c}{\sqrt{Var(X)}} = \frac{\sqrt{12}(1-\alpha)}{\sqrt{(1-\alpha)^2 + 12\alpha}} = b_2 \tag{2.3.3}$$

is also a constant, which does not depend on c.

Therefore the characteristic function of $\frac{X-EX}{\sqrt{Var(X)}}$ is:

$$\phi_{\frac{X-EX}{\sqrt{Var(X)}}}(t) = e^{\frac{-itEX}{\sqrt{Var(X)}}}\phi_X(\frac{t}{\sqrt{Var(X)}})$$

$$= e^{-itb_1}\phi_Y(\frac{t}{\sqrt{Var(X)}})\frac{1-\alpha}{1-\alpha e^{itb_2}}. \tag{2.3.4}$$

From equations (2.3.2) and (2.3.3), we have

$$\phi_{\frac{X-EX}{\sqrt{Var(X)}}}(t) = e^{-itb_1}\frac{e^{itb_2}-1}{itb_2}\frac{1-\alpha}{1-\alpha e^{itb_2}}$$

Therefore,

$$\phi_{\frac{X-EX}{\sqrt{Var(X)}}}(t) = e^{-itb_1}\frac{e^{itb_2}-1}{1-\alpha e^{itb_2}}\frac{1-\alpha}{itb_2} \tag{2.3.5}$$

does not depend on c , but represents the characteristic function of a random variable $\widetilde{X}$ having an ALM distribution, shifted to the left at the point b_1, with $\widetilde{Y}$ uniformly distributed over $[0, b_2]$ and $\widetilde{Z}$- geometrically distributed with parameter α. Thus we have

$$\frac{X-EX}{\sqrt{Var(X)}} = \widetilde{Y} + b_2\widetilde{Z} - b_1, \tag{2.3.6}$$

where

$$b_1 = \frac{EX}{\sqrt{Var(X)}} = \frac{\sqrt{3}(1+\alpha)}{\sqrt{(1-\alpha)^2+12\alpha}};$$
$$b_2 = \frac{c}{\sqrt{Var(X)}} = \frac{2\sqrt{3}(1-\alpha)}{\sqrt{(1-\alpha)^2+12\alpha}}. \tag{2.3.7}$$

We conclude that when c varies, it does not affect the distribution of the normalized random variable

$$\frac{X-EX}{\sqrt{Var(X)}},$$

and this distribution remains the same, whatever the value of c is. Therefore, other normalizations must be considered.

§4:The limit behaviour of X when both α and c vary

Now, using the result (2.3.1), we can prove the following combined result.

Theorem 2.4.1. *If $X \in ALM(F_Y(.), \alpha, c)$ with Y-uniform over $[0, c)$,then*

$$\lim_{\substack{\alpha\to 1 \\ c\to\infty \text{ or } 0}} P\{\frac{X}{EX} > x\} \to exp(-x),$$

and

$$\lim_{\alpha\to 0, c\to\infty or 0} P\{\frac{X}{EX} < x\} = \begin{cases} 0, & \text{for} \quad x < 0; \\ \frac{x}{2}, & \text{for} \quad x \in [0, 2]; \\ 1, & \text{for} \quad x \geq 2. \end{cases} \qquad (2.4.1)$$

Proof: From (2.3.1) , we have,

$$\phi_{\frac{X}{EX}}(s) = \frac{1+\alpha}{2s} \frac{1 - e^{\frac{-2s(1-\alpha)}{1+\alpha}}}{1 - \alpha e^{\frac{-2s(1-\alpha)}{1+\alpha}}}.$$

Hence

$$\lim_{\alpha\to 1} \phi_{\frac{X}{EX}}(s) = \lim_{\alpha\to 1} \frac{1+\alpha}{2s} \lim_{\alpha\to 1} \frac{1 - \alpha e^{\frac{-2s(1-\alpha)}{1+\alpha}}}{1 - \alpha e^{\frac{-2s(1-\alpha)}{1+\alpha}}}.$$

Let

$$\epsilon = \frac{1-\alpha}{1+\alpha}, \quad \lim_{\alpha\to 1} \epsilon \to 0;$$

$$\alpha = \frac{1-\epsilon}{1+\epsilon}; \quad \lim_{\epsilon\to 0} \alpha \to 1;$$

Therefore

$$(\frac{1-\epsilon}{1+\epsilon})' = \frac{-(1+\epsilon)-(1-\epsilon)}{(1+\epsilon)^2} = \frac{-2}{(1+\epsilon)^2};$$

$$\begin{aligned} \lim_{\alpha\to 1} \varphi_{\frac{X}{EX}}(s) &= \frac{1}{s} \lim_{\epsilon\to 0} \frac{1 - e^{-2s\epsilon}}{1 - \frac{1-\epsilon}{1+\epsilon} e^{-2s\epsilon}} \\ &= \frac{1}{s}\frac{0}{0}, \quad \text{Now, using L'Hospital rule} \\ &= \frac{1}{s} \lim_{\epsilon\to 0} \frac{2se^{-2s\epsilon}}{[(\frac{1-\epsilon}{1+\epsilon})2se^{-2s\epsilon} + (\frac{1-\epsilon}{1+\epsilon})'e^{-2s\epsilon}]} \\ &= \lim_{\epsilon\to 0}[\frac{e^{-2s\epsilon}}{s(\frac{1-\epsilon}{1+\epsilon}) + \frac{e^{-2s\epsilon}}{(1+\epsilon)^2}}] \\ &= \frac{1}{1+s}. \end{aligned}$$

Which is the Laplace Stieltjes Transform of the exponential distribution with parameter one.

Further we have

$$\lim_{\alpha \to 0} \phi_{\frac{X}{EX}}(s) = \frac{1 - e^{-2s}}{2s},$$

which is the L.S.T. of the uniform $U[0, 2]$-distribution. Hence the theorem is proved.

Now, using the result (2.3.5) and (2.3.7), we can prove the following combined result.

Theorem 2.4.2. *If $X \in ALM$, with Y uniform over $[0, c)$, then*

$$\lim_{\alpha \to 0} P\{\frac{X - EX}{\sqrt{Var(X)}} \le x\} = \begin{cases} 0, & x < -\sqrt{3}; \\ \frac{x}{\sqrt{12}}, & x \in [-\sqrt{3}, \sqrt{3}]; \\ 1, & x \ge \sqrt{3}. \end{cases} \tag{2.4.2}$$

Proof: From (2.3.5) and (2.3.7) in the proof of Theorem 2.3.2, we have

$$b_1 \longrightarrow \sqrt{3}, \quad b_2 \longrightarrow 2\sqrt{3}$$

Further $\widetilde{Z}_1$ in (2.3.6) tends in probability to 0, since

$$\widetilde{Z}_\alpha \xrightarrow{P} 0$$

is equivalent to

$$\widetilde{Z}_\alpha \xrightarrow{d} 0.$$

Moreover

$$\widetilde{Y} \xrightarrow{d} Y_0,$$

which is uniformly distributed over $[0, 2\sqrt{3}]$.

Indeed, we have

$$\phi_{\tilde{Y}}(t) = \frac{e^{itb_2} - 1}{itb_2} \longrightarrow \frac{e^{it2\sqrt{3}} - 1}{it2\sqrt{3}}.$$

Apply the limit $\alpha \to 0$ in (2.3.6), we obtain

$$\tilde{X} = -b_1 + \tilde{Y} + \tilde{Z} \longrightarrow Y_0 - \sqrt{3}, \tag{2.4.3}$$

which is $U[-\sqrt{3}, \sqrt{3}]$.

The theorem is proved.

Theorem 2.4.3. *If $X \in ALM$, with Y uniform over $[0, c)$, then*

$$\lim_{\alpha \to 1} P\{\frac{X - EX}{\sqrt{V(X)}} > x\} = \begin{cases} e^{-\sqrt{3}x}, & x > -1; \\ 0, & x \leq -1. \end{cases} \tag{2.4.4}$$

Proof: From (2.3.5) and (2.3.7) in the proof of theorem 2.3.2, we have

$$\phi_{\frac{X-EX}{\sqrt{V(X)}}}(t) = e^{-itb_1} \frac{e^{itb_2} - 1}{itb_2} . \frac{1 - \alpha}{1 - \alpha e^{itb_2}},$$

where b_1 and b_2 are given by (2.2.23). Thus

$$\lim_{\alpha \to 1} \phi_{\frac{X-EX}{\sqrt{V(X)}}}(t) = \frac{e^{-it}}{it} \lim_{\alpha \to 1} \frac{e^{itb_2} - 1}{1 - \alpha e^{itb_2}}.$$

Let

$$\epsilon = \frac{1 - \alpha}{\sqrt{(1 - \alpha)^2 + 12\alpha}}, \quad \lim_{\alpha \to 1} \epsilon \to 0.$$

Then

$$\lim_{\alpha\to 1}\phi_{\frac{X-EX}{\sqrt{V(X)}}}(t)=\frac{e^{-it}}{it}\lim_{\epsilon\to 0}\frac{e^{2\sqrt{3}it\epsilon}-1}{1-\frac{1-\epsilon}{\sqrt{(1-\epsilon)^2+12\epsilon}}e^{2\sqrt{3}it\epsilon}}$$

$$=\frac{e^{-it}}{it}\lim_{\epsilon\to 0}\frac{2\sqrt{3}ite^{2\sqrt{3}it\epsilon}}{-\frac{(1-\epsilon)2\sqrt{3}ite^{2\sqrt{3}it\epsilon}}{\sqrt{(1-\epsilon)^2+12\epsilon}}-(\frac{(1-\epsilon)}{\sqrt{(1-\epsilon)^2+12\epsilon)}})'e^{2\sqrt{3}it\epsilon}}$$

$$=\frac{e^{-it}}{it}\frac{2\sqrt{3}it}{-2\sqrt{3}it+6}$$

$$=\frac{\frac{1}{\sqrt{3}}e^{-it}}{1-\frac{1}{\sqrt{3}}it}.$$

This is the characteristic function of the distribution (2.4.4). The theorem is proved.

§5. Normalization by the parameter c

The Normalization

$$\overset{0}{X}=\frac{X}{c} \tag{2.5.1}$$

in the case of $Y\in U[0,c]$ gives in some sense a better "standardized " form of the ALM $(\alpha, c, U[0,c])$ class of distribution.

On the basis of the representation (2.2.1), the following holds

$$X=Y+cZ;$$

$$\overset{0}{X}=\frac{Y}{c}+Z,$$

with

$$\phi_{\overset{0}{X}}=\phi_{\frac{Y}{c}}(t).\phi_Z(t)$$

$$=\frac{e^{\frac{itc}{c}}-1}{\frac{it}{c}c}\frac{1-\alpha}{1-\alpha e^{it}}$$

$$=\frac{e^{it}-1}{it}\frac{1-\alpha}{1-\alpha e^{it}}. \tag{2.5.2}$$

Therefore, whatever the constant c is , the "normalized" random variable, $\overset{0}{X}$ in the case of uniformly distributed Y over the interval $[0, c)$ is an ALM random variable from the class ALM $(\alpha, 1, U[0,1])$. Therefore, in this case the variation of c does not affect the distribution of $\overset{0}{X}$.

Remark : If Y is not uniformly distributed then the equation

$$\begin{aligned}\phi_{\overset{0}{X}}(t) &= \phi_{\frac{Y}{c}}(t).\phi_Z(t)\\ &= \phi_Y(\frac{t}{c})\frac{1-\alpha}{1-\alpha e^{it}} \quad .\end{aligned} \tag{2.5.3}$$

will control the limit distribution of $\overset{0}{X}$. Obviously, if $\frac{Y}{c} \to \overset{0}{Y}$, regardless of $c \to 0$ or $c \to \infty$, then the limit $\overset{0}{X}$ is also an independent sum $\overset{0}{Y}$ and Z, where Z has geometric distribution i.e. $\overset{0}{X} = \overset{0}{Y} + Z$.

Example 2.1:

In this representation

$$\overset{0}{X} = \frac{X}{c} + Z,$$

where Z is the geometric distribution and let us suppose Y has an exponential distribution over $[0, c)$ with parameter λ,

$$f_Y(y) = \frac{\lambda e^{-\lambda y}}{1-e^{-\lambda c}}, \quad y \in [0, c).$$

Hence the characteristic function of $\overset{0}{X}$ is

$$\begin{aligned}\phi_{\overset{0}{X}}(t) &= \phi_{\frac{Y}{c}}(t).\phi_Z(t)\\ &= \phi_Y(t/c).\frac{1-\alpha}{1-\alpha e^{it}}.\end{aligned}$$

Here

$$\begin{aligned}\phi_Y(t/c) &= \frac{\lambda}{1-e^{-\lambda c}}.\frac{e^{it-\lambda c}-1}{(it/c-\lambda)} \\ &= \frac{c\lambda(e^{it-c\lambda}-1)}{c(1-e^{-c\lambda})(it/c-\lambda)} \\ &= \frac{c\lambda(e^{it-c\lambda}-1)}{it-c\lambda-e^{-c\lambda}it+\lambda ce^{-c\lambda}}.\end{aligned}$$

Therefore

$$\lim_{c\to 0}\phi_Y(t/c) = \lim_{c\to 0}\frac{c\lambda(e^{it-c\lambda}-1)}{(c-ce^{-\lambda c})(it/c-\lambda)}$$

$$\begin{aligned}\lim_{c\to 0}\phi_Y(t/c) &= \lim_{c\to 0}\frac{\lambda e^{it}(e^{-c\lambda}-c\lambda e^{-c\lambda})-\lambda}{-\lambda+\lambda e^{-c\lambda}it+\lambda(e^{-c\lambda}-c\lambda e^{-c\lambda})}; \\ &= \frac{e^{it}-1}{it}.\end{aligned}$$

For

$$\lim_{c\to\infty}\phi_Y(t/c)$$

we have

$$\phi_Y(t/c) = \frac{\lambda}{1-e^{-\lambda c}}\frac{e^{it-\lambda c}-1}{it/c-\lambda}.$$

Let $c = \frac{1}{\epsilon}$, then

$$\lim_{c\to\infty}\frac{1}{\epsilon}\longrightarrow 0;$$

$$\begin{aligned}\lim_{c\to\infty}\phi(t/c) &= \lim_{\epsilon\to 0}\frac{\lambda}{1-e^{-\lambda/\epsilon}}.\frac{e^{it-\lambda/\epsilon}-1}{it\epsilon-\lambda}; \\ &= e^{it}.\end{aligned}$$

Hence, the representation

$$\overset{0}{X} = \overset{0}{Y} + Z$$

still holds.

* * *

CHAPTER III
"SIMULATION RESULTS"

In this chapter we have justified the results of chapter (II) by simulation. To generate the random variable $\frac{X}{EX}$, we use the following algorithm.

(1) From the decomposition property (1.1.4), the random variable $\frac{X}{EX}$ can be written as

$$\frac{X}{EX} = \frac{Y}{EX} + c\frac{Z}{EX}$$

where Y and Z are same as defined earlier. Here in doing simulation, we assumed that Y is uniformly distributed over $(0, 1)$.

(2) Generate the uniform random number $U = uniform(0, 1)$.

(3) Generate the geometric variate $Z = \left[\frac{Log(1-U)}{Log(\alpha)}\right]$.

(4) Compute EX.

(5) Divide X by EX to get the simulated random variable $\frac{X}{EX}$.

(6) Find statistical estimate $\hat{F}(x)$ of $F_x(x)$.

The generated random variable Z depends on different values of α. Since Z is geometrically distributed with parameter α, simulated random variable $\frac{X}{EX}$ also depends on α.

The difference

$$\sup_x |(1 - \hat{F}(x)) - e^{-x}| = \delta(\alpha)$$

is investigated for different values of α as the following table shows.

Table 1

α	$\delta(\alpha)$
0.10	0.1677
0.20	0.0891
0.30	0.0721
0.40	0.0461
0.50	0.0393
0.60	0.0352
0.70	0.0024
0.80	0.0097
0.85	0.0088
0.90	0.0083
0.95	0.0079
0.99	0.0067
0.995	0.0017
0.999	0.0011

Thus we find that when the value of α increases (close to 1), $\delta(\alpha)$ decreases to zero. The value of α and $\delta(\alpha)$ are plotted. From the graph between α and $\delta(\alpha)$, it can be said that the relation between α and δ is almost exponential, when α tends to 1. (see Figure 3.18)

The Empirical cumulative distribution function $P(X/EX \leq x)$ has been also estimated for the value of $\alpha = 0.90$. The values of the c.d.f is based on 1000 simulated values of the random variable X/EX. These values are plotted and the trend is found to be almost exponential. (see Figure 3.1)

A frequency histogram is also plotted for the simulated random variable X/EX for different values of α. We found that when α increases, the trend in the frequency histogram closes to exponential. (see Figure 3.14 and 3.15)

Hence from all these simulated results, we conclude that when α is close to 1, the distribution of X/EX is close to exponential.

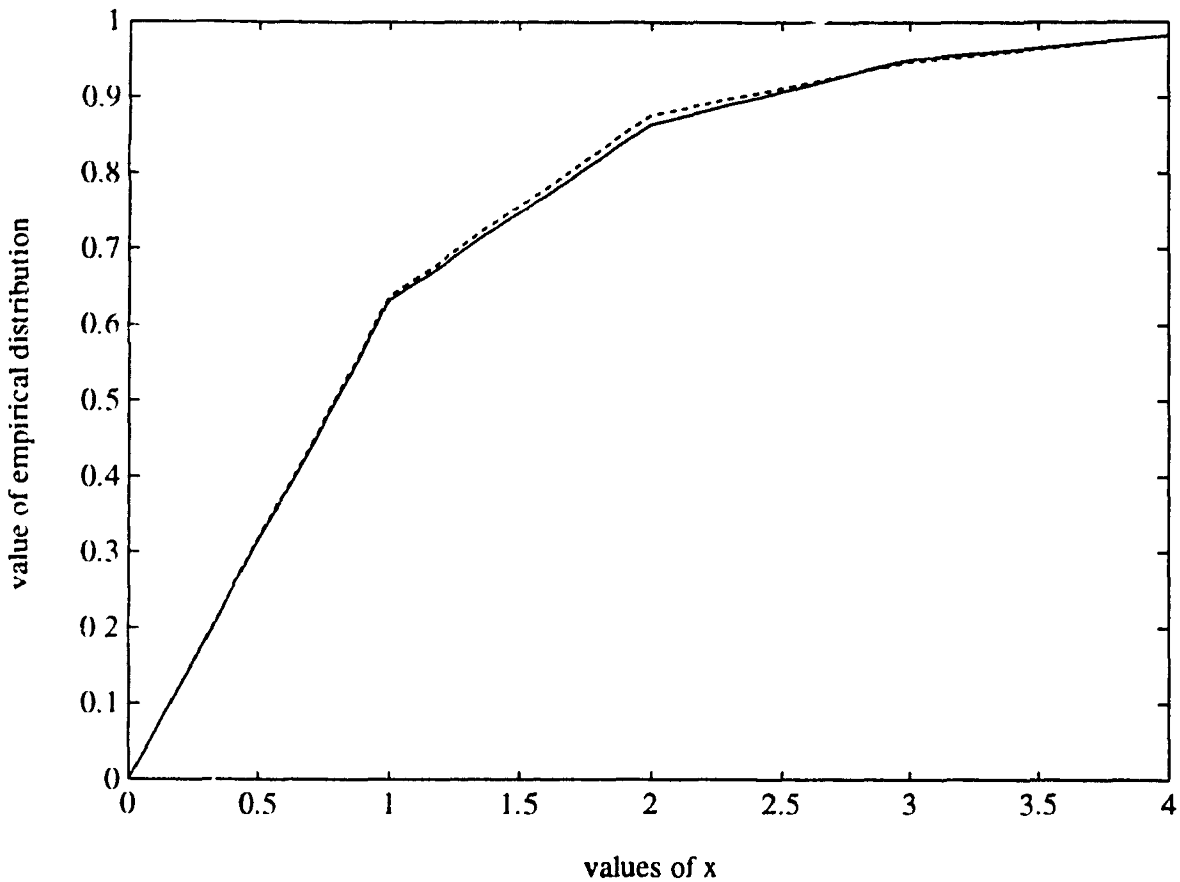

Figure 3.1: Comparison of the theoretical exponential (1) with empirical distribution for simulated random variables ($X|EX$).

Dotted Line: Empirical c.d.f

Solid Line: Theoretical c.d.f

The graphical comparison in figure shows a close match between theoretical and empirical distribution.

Dotted Line: Simulated r.v (X/EX)

Solid Line: e^{-x}

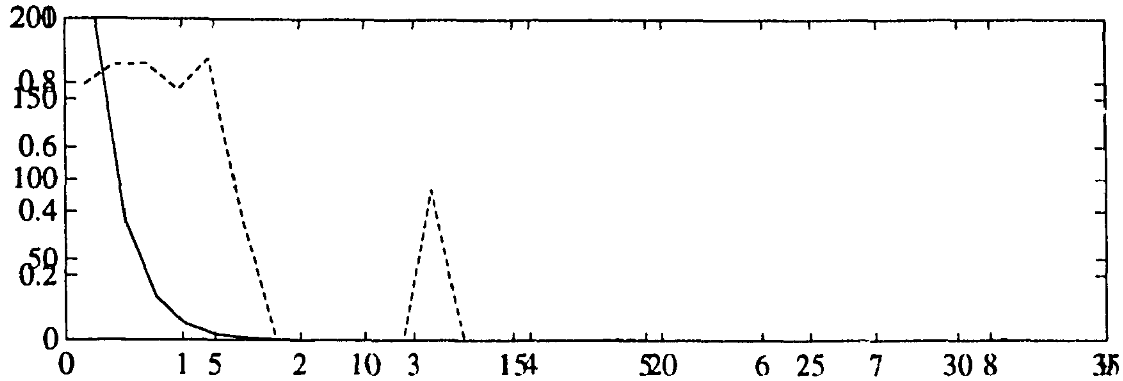

Figure 3.2: Comparison of frequency plot of simulated random variable X/EX with an exponential(1) at $\alpha = 0.01$ and $c = 1$.

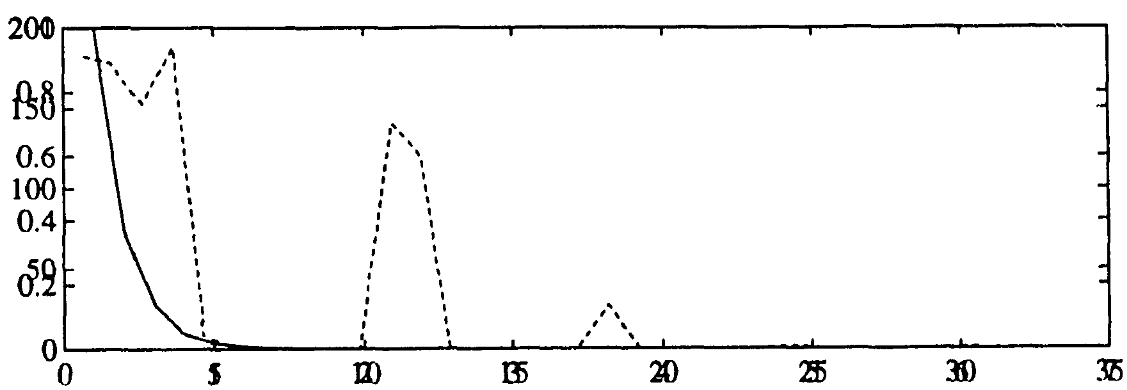

Figure 3.3: Comparison of frequency plot of simulated r.v. with an exponential(1) at $\alpha = 0.1$ and $c = 1$.

Figure 3.2 and 3.3 does not show any closeness between two plots.

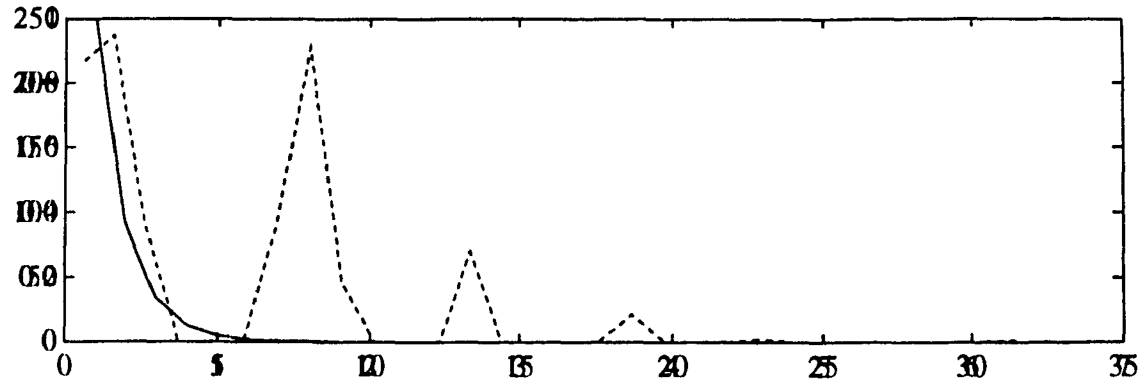

Figure 3.4: Comparison of frequency plot of simulated random variable (X/EX) with an exponential(1) at $a = 0.2$ and $c = 1$.

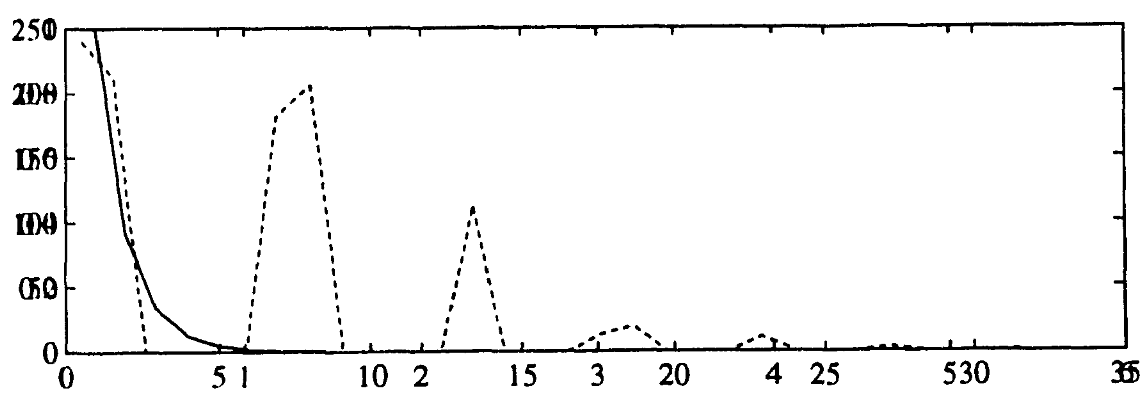

Figure 3.5: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.3$ and $c = 1$.

Figure 3.4 and 3.5 does not show any closeness between two plots.

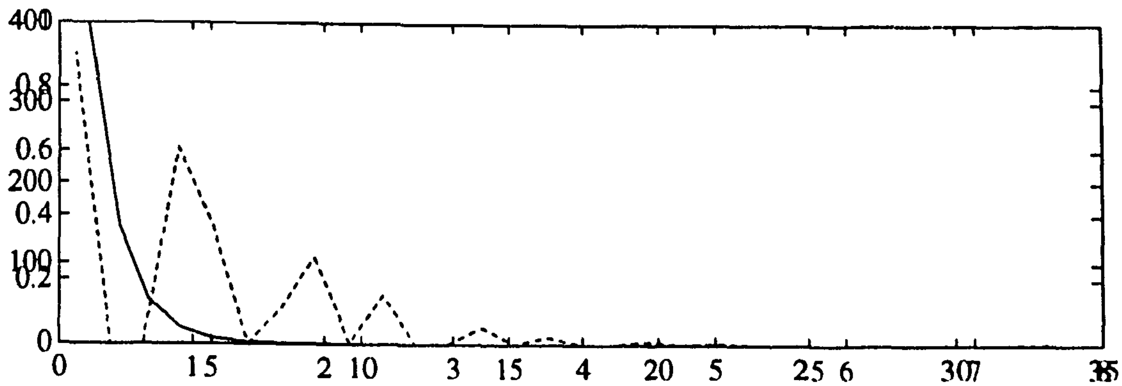

Figure 3.6: Comparison of frequency plot of simulated random variable (X/EX) with an exponential (1) at $a = 0.4$ and $c = 1$

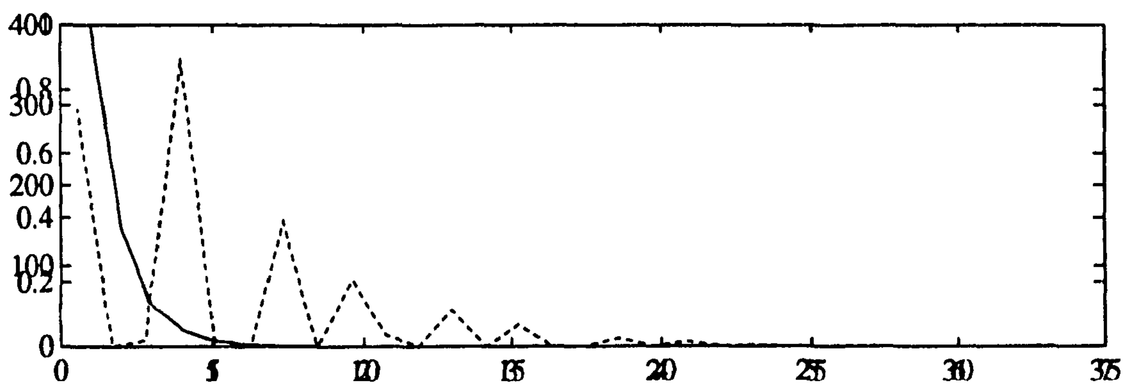

Figure 3.7: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.5$ and $c = 1$.

Figure 3.6 and 3.7 shows a slight trend towards exponential.

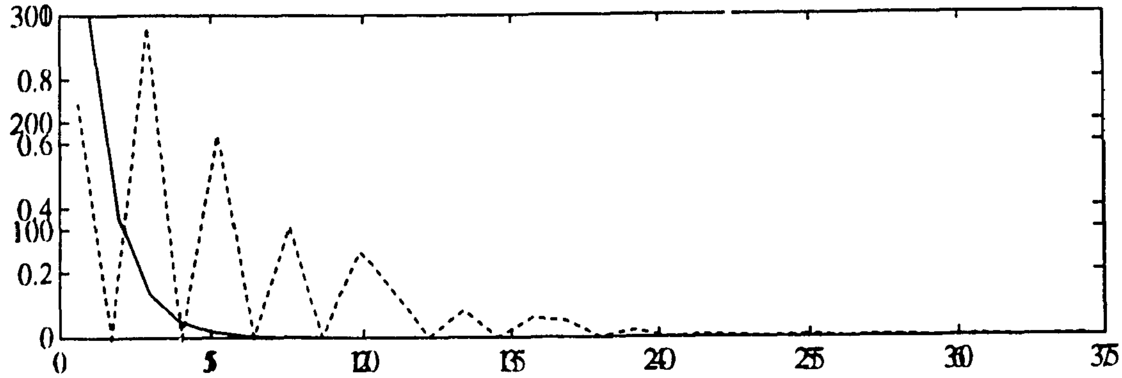

Figure 3.8: Comparison of frequency plot of simulated r.v X/EX with exponential(1) at $\alpha = 0.6$ and $c = 1$.

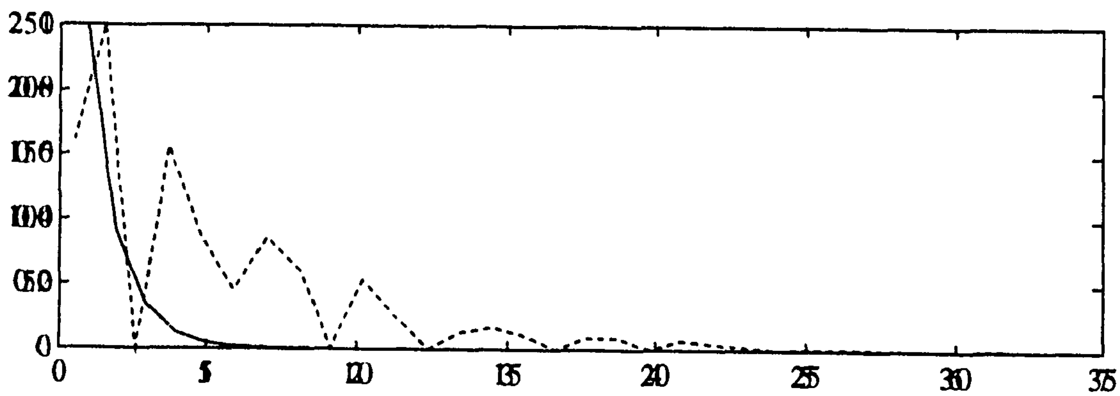

Figure 3.9: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $\alpha = 0.7$ and $c = 1$.

Figure 3.8 and 3.9 shows a trend towards exponential.

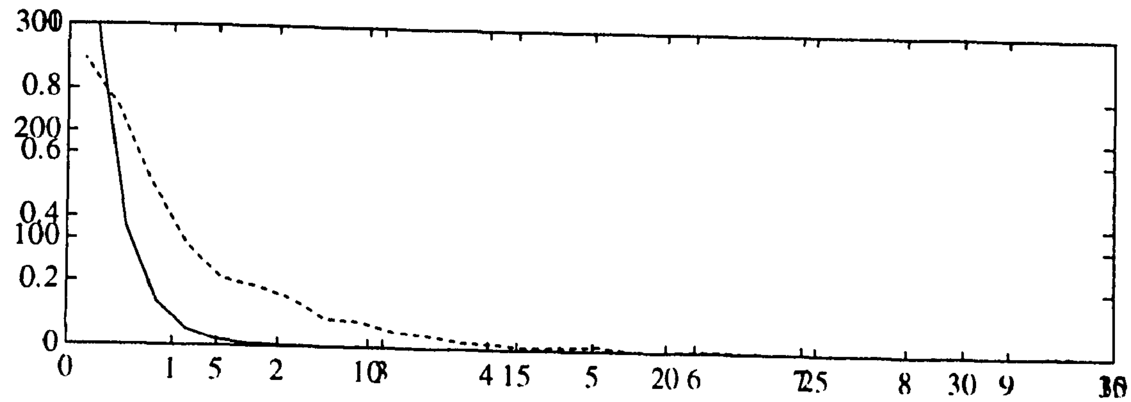

Figure 3.10: Comparison of frequency plot of simulated r.v X/EX with a exponential(1) at $\alpha = 0.8$ and $c = 1$.

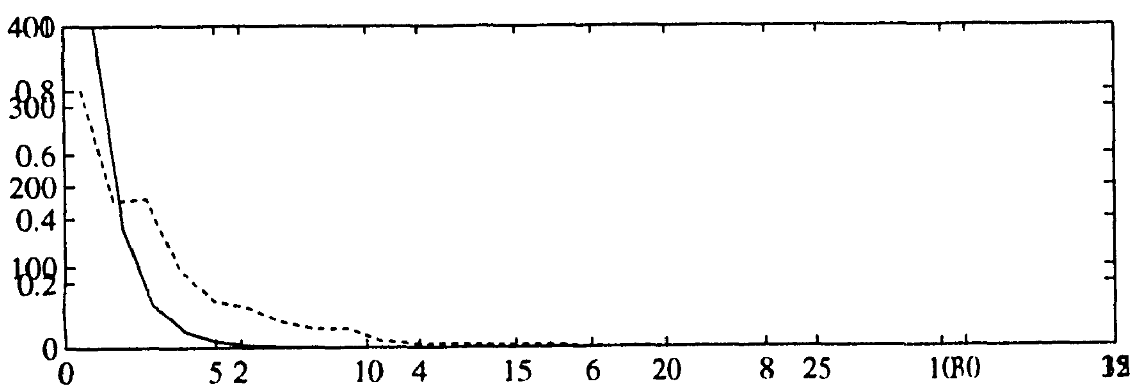

Figure 3.11: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $\alpha = 0.85$ and $c = 1$.

Figure 3.10 and 3.11 shows that a close match between two plots.

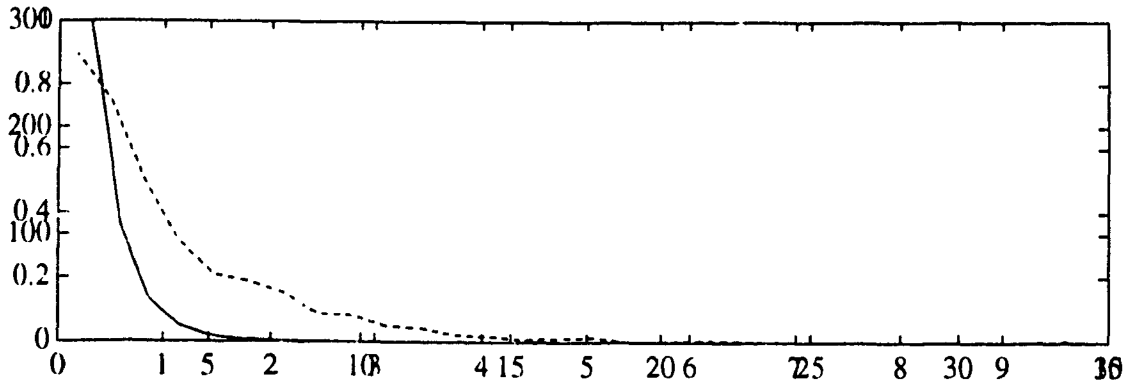

Figure 3.12: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $\alpha = 0.90$ and $c = 1$.

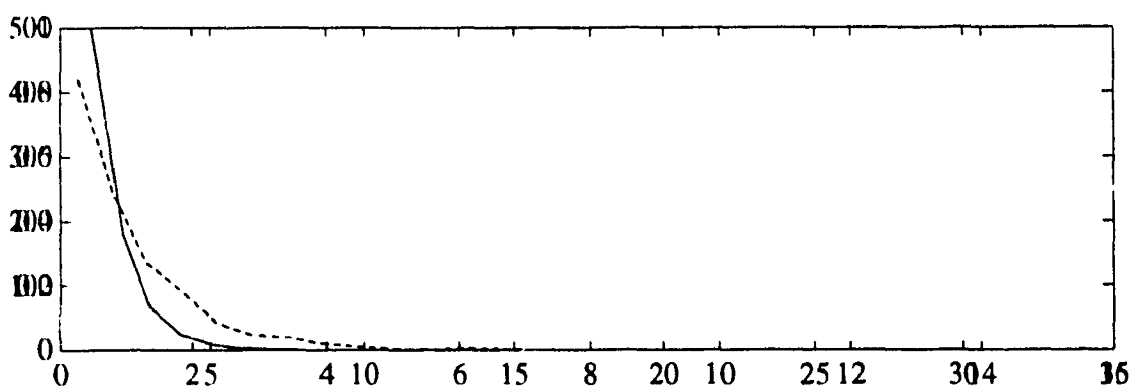

Figure 3.13: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $\alpha = 0.95$ and $c = 1$.

Figure 3.12 and 3.13 shows a very close match between plots, which is almost exponential.

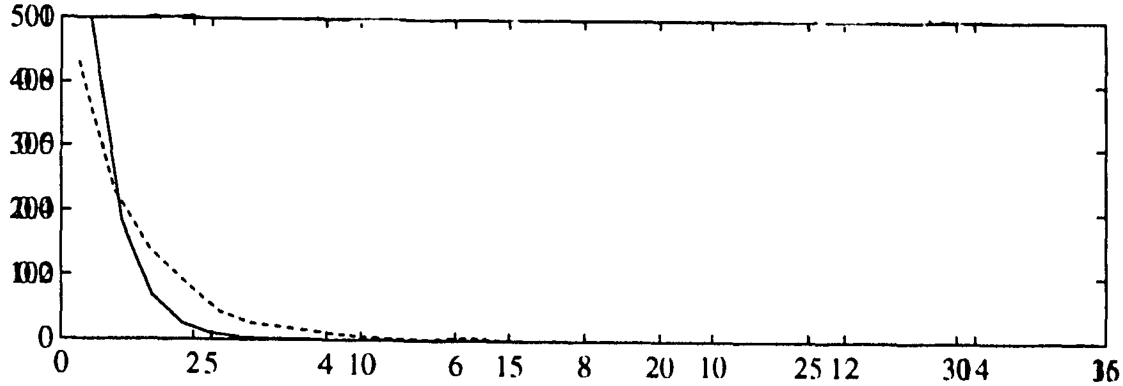

Figure 3.14: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.99$ and $c = 1$.

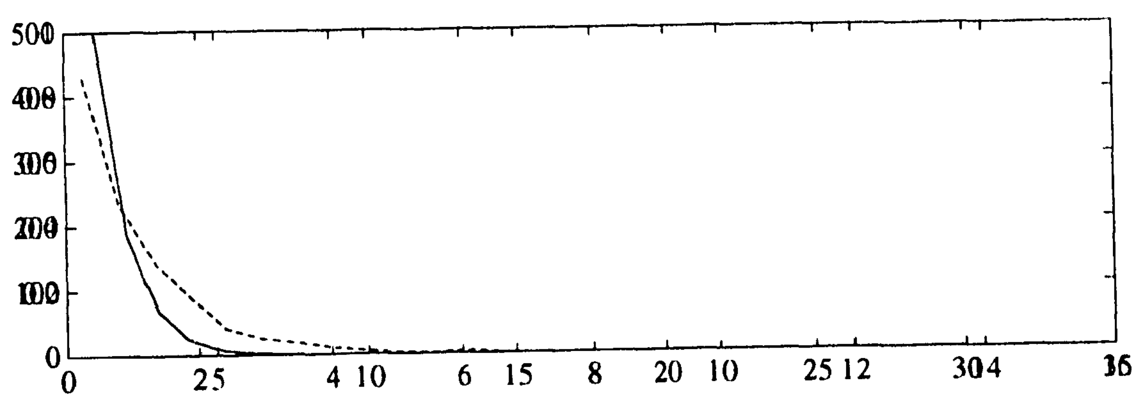

Figure 3.15: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.999$ and $c = 1$.

Figure 3.14 and 3.15 shows almost similar exponential trend.

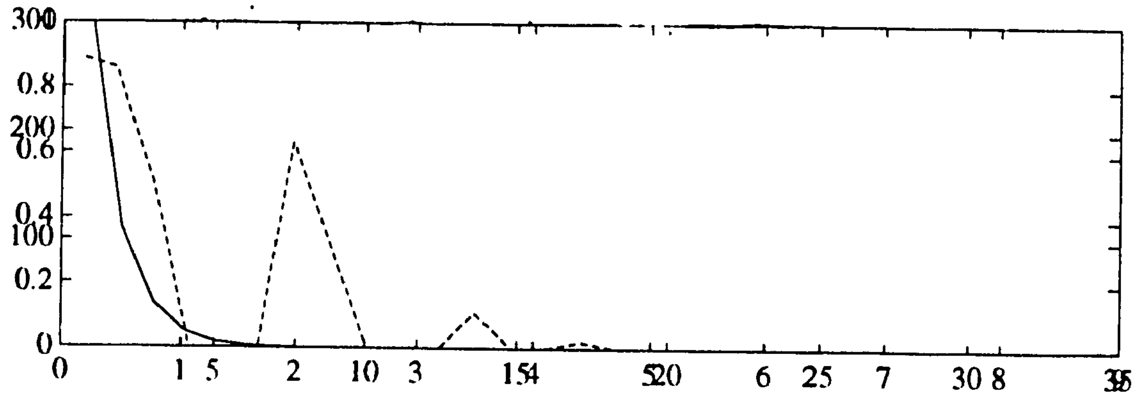

Figure 3.16: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.1$ and $c = 1000$.

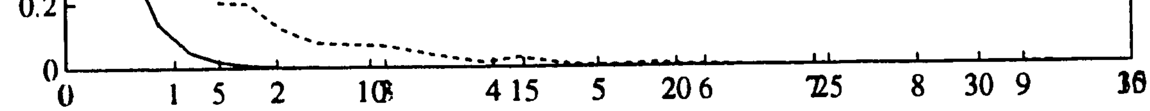

Figure 3.17: Comparison of frequency plot of simulated r.v X/EX with an exponential(1) at $a = 0.90$ and $c = 1000$.

Figure 3.16 and 3.17 shows that variation in parameter c does not affect the limit distribution.

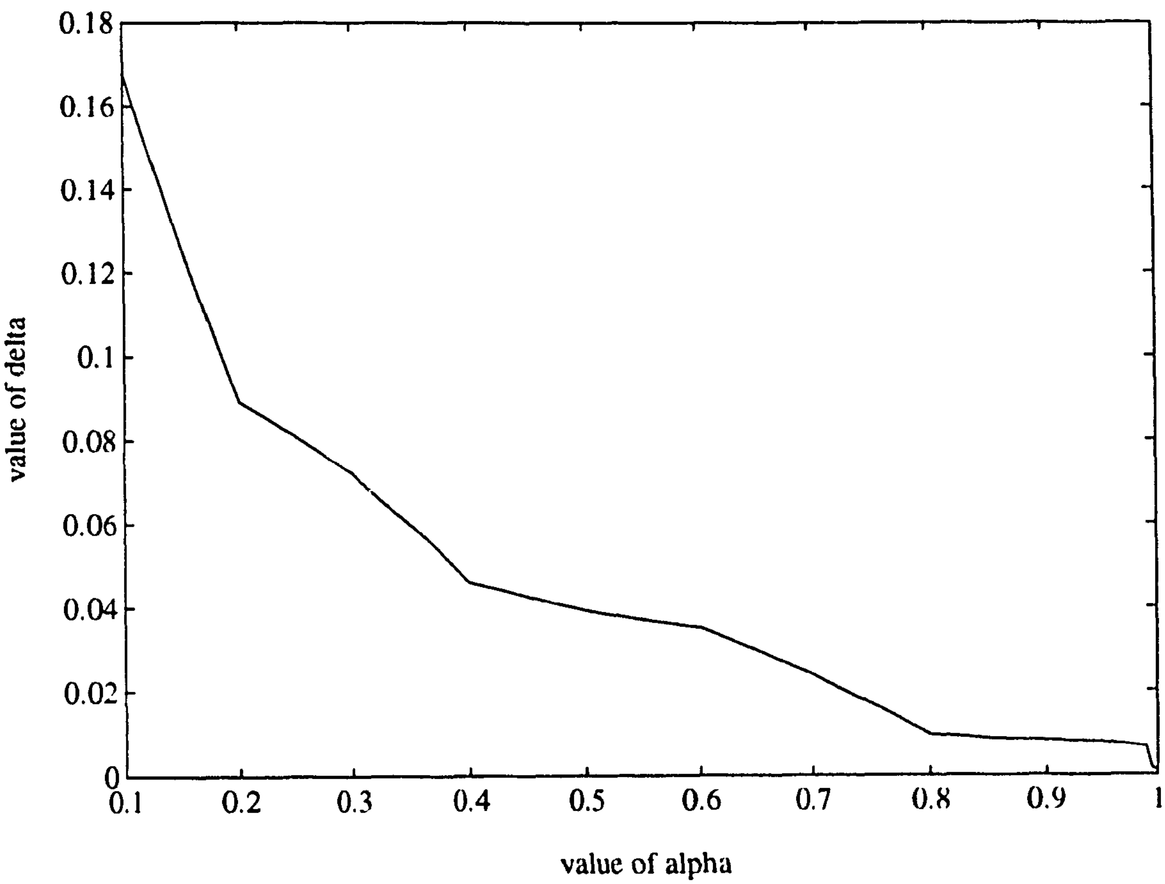

Figure 3.18: Plot between α and δ.

CHAPTER IV
Discussion and Conclusion

§4.1 Discussion

In our present work, we have investigated the limit behaviour of ALM class of distribution in two normalization cases $(i)X/EX$ and $(ii)\frac{X-EX}{\sqrt{Var(X)}}$. Theorem 2.2.1 and 2.2.2 show the limit behaviour of X when α reaches on the boundaries 0 and 1. When α is close to 1, the limit behaviour of X tends to exponential one. A graphical comparison in Figure 3.12 and Figure 3.13 shows a close match between simulated random variable (X/EX) and $y = e^{-x}$ at $\alpha = 0.9$ and $\alpha = 0.99$. But when α is close to zero the limit distribution of X will be controlled by the limiting behaviour of Y. Figure 3.1 and Figure 3.2 clearly indicates that the limit distribution belongs to the same class of ALM distribution.

Theorem 2.3.1 shows that a Variation in the parameter c does not change the limit distribution of X. Figure 3.17 and Figure 3.19 confirms the results. But when α also varies at the same time, then α controls the limiting behaviour of X. If α is close to 1 ($\alpha = 0.90$) the limit behaviour of X tends to exponential (Figure 3.17). If α is close to zero ($\alpha = 0.10$) then limiting behaviour of Y controls the limit distribution of X (Figure 3.16). Theorem 2.4.1 also states the same result in case, when α and c both are varying.

§4.2 Conclusion

The results obtained in chapter (II), can be used to generalize the limit behaviour of the distribution $X \in ALM(c, \alpha, F_Y(.))$.

The three cases, we have considered are:

(i) when c is fixed and α varies.

(ii) when α is fixed and c varies with assumptions on the distribution of Y.

(iii) when α and c are both varying.

In case (i), limit behaviour of the distiibution of Z plays a role in determining limit distribution of X. From Theorem 2.2.1 and Theorem 2.2.2, we can say that when α tends to 1 limit distribution of Z determines the limit distribution of X. But when α tends to zero then the distribution of Z converges to zero in which case, limit distribution of Y determines the distribution of X.

In case (ii), variation in c does not affect the distribution of X for both the normalization $\frac{X}{EX}$ and $\frac{X-EX}{\sqrt{Var(X)}}$ cases. In both the normalization cases the distribution of Y is assumed uniform $[0, c)$. If X is normalized by c then the limit distribution remains of the same class of ALM distribution. Also, if Y is not a uniform distribution then the distribution of X/c converges to some distribution which again belongs to the same class of ALM distribution. Hence, we can state that for any value of c , the limit distribution of X always belongs to the same class of distribution ALM and will hold the following representation

$$\overset{0}{X} = \overset{0}{Y} + c\overset{0}{Z}.$$

Where,

$$\overset{0}{X} = X/EX, \quad \text{or} \quad \frac{X - EX}{\sqrt{Var(X)}}, \quad \text{or} \quad X/c;$$

$$\overset{0}{Y} = Y/EX, \quad \text{or} \quad \frac{Y - EX}{\sqrt{Var(X)}}, \quad \text{or} \quad Y/c;$$

$$\overset{0}{Z} = Z/EX, \quad \text{or} \quad \frac{Z - EX}{\sqrt{Var(X)}}, \quad \text{or} \quad Z/c.$$

In case (iii), when both the parameters c and α are varying, then the limit behaviour of X depends on the values of α. For example, in equation 2.2.16, if we take the limit $\alpha \longrightarrow 0$, the limit behaviour of X will be determined by the corresponding limit behaviour of Y, for $c \rightarrow \infty$ or $c \rightarrow 0$. If the limit $\alpha \longrightarrow 1$ is taken then limit behaviour of X will be determined by the distribution of Z, for both, $c \rightarrow \infty$ or $c \longrightarrow 0$.

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