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LINEAR FEEDBACK, IRREDUCIBILITY, M-MATRICES AND
THE STABILITY-HOLDABILITY PROBLEM

Steve Hardy

A Thesis
in
the Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
for the Degree of Master of Science at
Concordia University
Montreal, Quebec, Canada

June 1990

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ISBN 0-315-68772-X

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ABSTRACT

LINEAR FEEDBACK, IRREDUCIBILITY, M-MATRICES AND THE STABILITY-HOLDABILITY PROBLEM

Steve Hardy

Consider a control system of n linear, time autonomous, ordinary differential equations $\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t)$. By letting $\underline{u}(t)$ be a linear feedback control, that is, $\underline{u}(t) = X \underline{x}(t)$, we study the impact on the system of making the further restrictions of holdability and stability of \mathbb{R}_+^n . The holdability restriction will be seen as equivalent to the essential non-negativity of the matrix $A+BX$. Stability, on the other hand, is equivalent to the moduli of all the eigenvalues of $A+BX$ being less than 1. The combined properties are equivalent to $A+BX$ being a non-singular M-matrix. By allowing the matrix X to vary, we study the restrictions that holdability and stability of \mathbb{R}_+^n impose on X . In either the scalar or non-scalar input case, these result in some interval conditions on x_{ij} provided by the elements of A and B , and certain polynomials. The structure of these polynomials is studied.

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LIST OF SYMBOLS

A	1	c_i	12	$Z^{n \times n}$	20
A^q	4	e^{tA}	5	α_j	45
$A \stackrel{e}{\geq} O$	6, 20	\underline{e}_j	11	β_j	44
$A \geq O$	9	$G(A)$	13	∂	15
$A > O$	9	I	44	Δ_i	24
$A \gg O$...	9	J	44	$\Delta_i(s)$	24
A_{ij}	11	κ_{n+1}	25	$\overline{\Delta_i(s)}$	25
$A \leq B$	18	κ	5, 34	$\phi(x)$	20
$A < B$	18	M^{-1}	21	γ_k	56
$A \ll B$...	18	P_i	13	Π	47
$^i A$	18, 65	\mathcal{P}_k	62	$\rho(B)$	4
$^k_i A$	41	$\mathbb{R}^{n \times n}$	1	σ_i	47
\underline{z}^A	60	\mathbb{R}_+^n	2	Σ	17
a_{ij}	9	w_i	46	τ	1, 9
$a_{ij}^{(q)}$	16	\underline{x}	1	$\ \cdot\ $	30
\overline{B}	25	x_i	1	$\nabla(t_{ij})$..	55
\underline{b}^i	38	$\underline{x}(t)$	1	\cap	57
\underline{b}_i	3	$\dot{\underline{x}}(t)$	1	$\kappa_{n,m}$	59

1 . INTRODUCTION .

A system of n homogeneous , linear , time autonomous ordinary differential equations can be written as

$$\dot{\underline{x}}(t) = A \underline{x}(t) \quad , \quad -\infty < t < \infty$$

where $t \in \mathbb{R}$ represents a scalar variable (usually time) ,

$A \in \mathbb{R}^{n \times n}$ is a constant matrix ,

$\underline{x}(t) = (x_1(t) \ x_2(t) \ \dots \ x_n(t))^T$, and

$$\dot{x}_i(t) = \frac{d}{dt} x_i(t) \quad , \quad i = 1 , 2 , \dots , n .$$

Note that since t represents a scalar time variable , it is quite natural to restrict t to $[0, \infty)$, which is assumed from here on without further comment.

An important property which is often required is that the state \underline{x} of the system be constrained to a particular set Γ for various reasons, including the possibility that the dynamics of the model break down outside of Γ . Therefore it becomes imperative to know if the trajectory $\underline{x}(t)$ is contained in Γ for all $t \geq 0$. In such cases $\underline{x}(t)$ is

said to be holdable in Γ . This problem was presented in a paper by Berman and Stern although results concerning it are almost historical. Bellman gave one of the fundamental characterizations which is still the basis for a whole area of research in stability and control theory. The holdability property is illustrated by the following example.

Example 1 . (Symbiotic species)

Consider an ecological system (possibly a bacterial culture) in which there are n species . For $i = 1, 2, \dots, n$, let $x_i(t)$ denote the biomass of species i at time t . Suppose that a linear system of differential equations $\dot{\underline{x}}(t) = A \underline{x}(t)$ serves to model the interaction of the n species . Then a_{ij} , the $(i,j)^{\text{th}}$ entry of A , is a coefficient which determines the effect of the biomass of species j on the rate of change of the biomass of species i . Note then that a_{ii} represents the difference between the birth rate and death rate of species i .

Clearly the dynamics of the above model make sense only as long as the biomass vector satisfies $\underline{x}(t) \geq \underline{0}$. In other words, for any initial (non-negative) biomass vector, no species' biomass can ever become negative, thus causing a breakdown in the dynamics of the model.

Hence it is desirable to formulate a condition on A such that the trajectory $\underline{x}(t)$ is holdable in \mathbb{R}_+^n .

Another interesting system capability is that the trajectory goes to the origin with time (i.e. $\underline{x}(t) \rightarrow \underline{0}$ as $t \rightarrow \infty$). In such cases the system is said to be stable. In terms of the symbiotic species model, this is equivalent to all the populations being doomed to extinction.

In reality, very few linear system possess either one of these two properties. A simple way to overcome this problem is to add a control function $\underline{u}(t) \in \mathbb{R}^{n \times n}$ to the original model. The new control system

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t)$$

where $B \in \mathbb{R}^{n \times m}$, provides some degree of freedom, and one can now ask under what conditions the controlled system can be made holdable and/or stable.

In the case, where $\underline{u}(t) \in \mathbb{R}^{1 \times n}$,

$$B \underline{u}(t) = \underline{b}_1 u_1(t) + \underline{b}_2 u_2(t) + \dots + \underline{b}_n u_n(t)$$

where \underline{b}_i is the i^{th} column vector of B . Again, in terms of the symbiotic species model, the control function

component $u_j(t)$ may be interpreted as the addition of a chemical j acting either as a neutral element, poison or protein on species i according to whether sign (b_{ij}) is zero, negative or positive, respectively.

The addition of a control may be restricted further by requiring that the control function satisfy a linear feedback law, that is, $u(t) = X \underline{x}(t)$ where $X \in \mathbb{R}^{m \times n}$. This latter requirement for the symbiotic species model could mean that a distributor exists which administers the chemicals automatically in a linear fashion as a function of the biomass of the species.

This thesis shall investigate the joint capability of holdability and stability through the application of a linear feedback control. A similar attempt can be found in Berman and Stern [4].

Chapter 2 is devoted to an extensive development of matrix theory background. The theory of non-negative matrices (i.e. matrices with non-negative entries) is introduced concurrently with reducible matrices (i.e. matrices permutationally similar to matrices of the form $\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$ where B and D are both square matrices). We shall demonstrate that for non-negative matrices, irreducibility

of a matrix A is equivalent to the entries of the powered matrix A^q being strictly positive for some q . A small diversion into graph theory is given in order to provide a method of verifying the reducibility property of a matrix. The spectral analysis of non-negative irreducible matrices is reviewed.

Then, attention is shifted to the review of required results from the theory of M-matrices (matrices of the form $sI-B$ where $B \geq 0$ and $s \geq \rho(B)$, the spectral radius). Finally, the chapter is closed with two basic applications of M-matrices, The Leontief economic input-output model and the Jacobi iteration procedure.

Chapter 3 is concerned with the Stabilizability - Holdability problem, denoted SH, or equivalently the problem of finding a linear feedback law $\underline{u}(t) = X \underline{x}(t)$ such that the system

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t)$$

goes to the origin as $t \rightarrow \infty$ (i.e. stabilizability) and is constrained within a certain set Γ (i.e. holdability). The set Γ considered will be assumed to be \mathbb{R}_+^n in most instances but results readily extend for general simplicial

cones K (i.e. cones of the form $K = Q \mathbb{R}_+^n$ for a non-singular matrix $Q \in \mathbb{R}^{n \times n}$). The uncontrolled system

$$\dot{\underline{x}}(t) = A \underline{x}(t)$$

is first analysed and results are demonstrated concerning the equivalence of the positive invariance of \mathbb{R}_+^n (i.e. for all $\underline{x} \in \mathbb{R}_+^n$, $e^{tA} \underline{x} \in \mathbb{R}_+^n$ for all $t \geq 0$) and the essential non-negativity of the matrix A (i.e. $a_{ij} \geq 0$ for all $i \neq j$). The equivalence of the strict positive invariance of \mathbb{R}_+^n (i.e. for all $\underline{x} \in \mathbb{R}_+^n$, $\underline{x} \neq \underline{0}$, $e^{tA} \underline{x}$ only has positive components for all $t \geq 0$) and the essential non-negativity and irreducibility of the matrix A is also demonstrated . The study of the controlled system

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t)$$

and its holdability in \mathbb{R}_+^n follows. Letting $\underline{u}(t) = X \underline{x}(t)$, \mathbb{R}_+^n is seen to be holdable if and only if a matrix X can be found such that $(A+BX) \stackrel{e}{\geq} 0$ (i.e. the matrix $(A+BX)$ is essentially non-negative) . The chapter is closed with the stabilizability-holdability problem where the equivalence of (SH) and the non-negativity of $A+BX$ with positive leading principal minors is shown in the case of scalar input (i.e. $B \in \mathbb{R}^{n \times 1}$) .

Chapter 4 begins by analysing the stabilizability - holdability problem in the case of scalar input . Certain bounds for the solution are defined and an algorithm is developed via transformation of variables such that the resulting problem becomes a vector inequality for which certain solution procedures are discussed.

The case for which one or two variables are totally unbounded is analysed and is shown to introduce a reduction in the complexity of the problem. A simple algorithm is provided to expand the existing set of solutions within the feasible convex solution set. The analysis then proceeds with the stabilizability-holdability problem in the case of non-scalar input. Some interesting generalizations of the scalar theory are developed and a procedure to reformulate the problem as a non-linear programming problem is provided. The chapter is closed by analysing the stabilizability - holdability problem for

$$\dot{\underline{x}}(t) = A \underline{x}(t)$$

where the freedom of adding a linear feedback control function $\underline{u}(t) = B X \underline{x}(t)$ is given . It will be shown that through a certain similarity transformation, the solution becomes apparent .

This problem was originally solved by Berman and Stern [4] in the case of scalar input, but their method made use of linear programming which, as we shall see, for a whole class of problems is unnecessary. So we refine the procedure within that context and extends certain results for non-scalar input.

Before proceeding any further , a remark concerning the referencing is in order . The internal referencing is provided by three numbers representing , in order : the chapter , the section and the result . In the case where the result is provided within the same chapter , the chapter number is omitted . A similar convention is adopted for sections . The square brackets following most results indicate the reference(s) in the bibliography where the result can be found. The reader will note, however, that most of the background theory can be found in Berman and Plemmons [3].

2 . MATRIX THEORY BACKGROUND

1 . Introduction .

In this chapter we consider square non-negative matrices (i.e. matrices whose elements are all non-negative) . The material developed here will be used extensively later , both implicitly and explicitly .

Section 2 is devoted to the study of non-negative and irreducible matrices and their spectral properties . Then , attention is focused on a particular set of matrices called M-matrices . Finally , in section 4 , the chapter is closed with some interesting applications .

2 . Irreducible and Non-Negative Matrices .

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$. Then we say that A is a non-negative matrix if $a_{ij} \geq 0$ for all i, j which we denote by $A \geq 0$. The special case where $A \geq 0$ but $A \neq 0$ is denoted by $A > 0$, that is , $a_{ij} \geq 0$ for all i, j with strict inequality for at least one a_{ij} . Similarly , we write $A \gg 0$ for a positive matrix A if $a_{ij} > 0$ for all i, j .

Definition 2.1 . An $n \times n$ matrix A is said to be reducible if there exists a permutation matrix P such that PAP^T has the form

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} ,$$

where B, D are both square matrices , or if $n = 1$ then $A = 0$. Otherwise , A is said to be irreducible .

As a consequence of the definition we have the following theorem .

Theorem 2.2 . [8] Let A be an $n \times n$ matrix then ,

- (a) If $a_{ij} \neq 0$ for all i, j (i.e. A has no zero elements) , then A is irreducible .
- (b) If $a_{ii} = 0$ for all i and $a_{ij} \neq 0$ for all $i \neq j$, then A is irreducible .

(c) If A is reducible then it must have at least $n-1$ elements equal to zero .

(d) If A has at least one row or column of zeroes, then A is reducible .

Corollary 2.3 . [10] Let A be an $n \times n$ positive matrix then A is irreducible .

Proof : Immediate from Theorem 2.2 (a) . ■

It is obvious that if $A \geq 0$, then $A^p \geq 0$ for any positive integer p . We might also expect that if A has a sufficiently high density of non-zero elements then , for large enough p , we would obtain $A^p \gg 0$. The following result is of this kind .

Theorem 2.4 . [19] Let the matrix $A \in \mathbb{R}^{n \times n}$ be non-negative and irreducible . Then $(I+A)^{n-1} \gg 0$.

Proof : Consider the vector $\underline{y} \in \mathbb{R}^n$ such that $\underline{y} > \underline{0}$ and write

$$\underline{z} = (I+A) \underline{y} \tag{1}$$

Since $A \geq 0$, the product $A \underline{y} \geq \underline{0}$, and so \underline{z} has at least as many non-zero (and hence positive) elements as \underline{y} . If \underline{y} is not already positive , we shall prove that \underline{z} has at least one more non-zero element than \underline{y} . Indeed , if P is a permutation matrix such that $P \underline{y} = (\underline{u}^T, \underline{0}^T)^T$

where $\underline{u} \gg \underline{0}$, then it follows from eq.(1) and the relation $P^T P = I$ that

$$P \underline{z} = \begin{pmatrix} \underline{u} \\ \underline{0} \end{pmatrix} + P A P^T \begin{pmatrix} \underline{u} \\ \underline{0} \end{pmatrix} \quad (2)$$

Hence , if we partition \underline{z} and $P A P^T$ consistently with the partition of \underline{y} ,

$$P \underline{z} = \begin{pmatrix} \underline{v} \\ \underline{w} \end{pmatrix} , \quad P A P^T = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Then eq.(2) implies

$$\underline{v} = \underline{u} + A_{11} \underline{u} \quad \text{and} \quad \underline{w} = A_{21} \underline{u} \quad (3)$$

Now observe that the matrix PAP^T is non-negative and irreducible. In particular, we have $A_{11} \geq 0$, $A_{21} > 0$, and it follows from equation (3) that $\underline{v} \gg \underline{0}$. But $\underline{u} \gg \underline{0}$ so therefore , $\underline{w} > \underline{0}$. Thus \underline{z} has at least one more positive element than \underline{y} . If $(I+A)$ is not already positive then , starting again with the element $(I+A) \underline{z} = (I+A)^2 \underline{y}$ and repeating the argument , it follows that it has at least two more positive elements than \underline{y} . Continuing this process , we find , after at most $n-1$ steps , that $(I+A)^{n-1} \underline{y} \gg \underline{0}$ for any $\underline{y} > \underline{0}$. Putting $\underline{y} = \underline{e}_j$ for $j = 1 , 2 , \dots , n$ yields the required result .

■

Observe that there is a simple result in the other direction . If $(I+A)^j \gg 0$ for some $A \in \mathbb{R}^{n \times n}$ and any positive integer j , then A must be irreducible .

Otherwise , assuming A is reducible we easily obtain a contradiction to the hypothesis .

We now proceed to examine more closely the structure of a non-negative irreducible matrix A and the elements of the powers of A . We first start with the following definition .

Definition 2.5 . Let $a_{ij}^{(q)}$ denote the $(i,j)^{th}$ element of A^q .

With the above definition at hand , we can now state the following theorem .

Theorem 2.6 . [19] A non-negative matrix A is irreducible if and only if for every (i,j) there exists a natural number q such that $a_{ij}^{(q)} > 0$.

Proof : If : Suppose $a_{ij}^{(q)} > 0$ for some q . Assume A is reducible . Then

$$A^q = F^T E^q P = P^T \begin{pmatrix} B^q & Q \\ 0 & D^q \end{pmatrix} P$$

for some matrix $Q \geq 0$, where

$$P A P^T = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

But this is a contradiction to $a_{ij}^{(q)} > 0 \quad \forall i,j$. Thus

A is irreducible .

Only if : By Theorem 2.4 , $(I+A)^{n-1} \gg 0$. So let $B = (I+A)^{n-1} A$. As a product of a positive and an irreducible matrix , B is itself positive . Let

$$B = A^n + c_{n-1} A^{n-1} + \dots + c_1 A \quad \text{where} \quad c_i = \binom{n-1}{i-1} .$$

Then

$$b_{ij} = a_{ij}^{(n)} + c_{n-1} a_{ij}^{(n-1)} + \dots + c_1 a_{ij} > 0 \quad \forall \quad i, j .$$

So for each (i,j) there exists a positive integer q such that $a_{ij}^{(q)} > 0$. ■

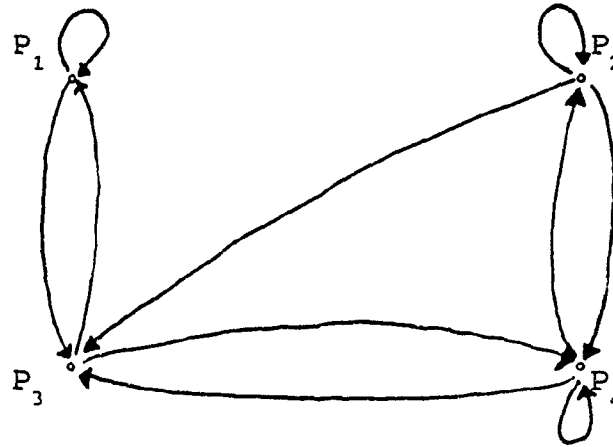
REMARK : In view of the preceding proof , an upper bound can actually be stated for q , namely , $1 \leq q \leq n$. This fact shall be used in the proof of Theorem 13 .

The characterization of irreducible matrices provided in the last theorem has a useful graph theoretic interpretation .

Definition 2.7 . [5] The associated directed graph , denoted by $G(A)$, of an $n \times n$ matrix A consists of n vertices $P_1, P_2, P_3, \dots, P_n$, where an edge leads

from P_i to P_j if and only if $a_{ij} \neq 0$.

Example 2.8 . Let $A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 6 & 5 & 4 \\ 3 & 0 & 0 & 7 \\ 0 & 8 & 9 & 1 \end{pmatrix}$ then the associated directed graph $G(A)$ is given by



Definition 2.9 . [5] A directed graph G is strongly connected if for any ordered pair (P_i, P_j) of vertices there exists a sequence of edges (i.e. a path) which leads from P_i to P_j .

One may verify that the graph in Example 2.8 is strongly connected . We now state the relation between an irreducible matrix and its associated directed graph .

Theorem 2.10 . [19] A matrix A is irreducible if and only if $G(A)$ is strongly connected .

Proof : If : We shall prove the contrapositive of the

claim (i.e. If A is reducible then $G(A)$ is not strongly connected) . Let us first argue that the graph of $P A P^T$ is obtained from A just by renumbering the nodes , and this operation does not affect the connectedness of the graph . So , without loss of generality , we may assume that the matrix A is already in reduced form . In other words ,

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \quad \text{where } B \in \mathbb{R}^{k \times k}, D \in \mathbb{R}^{(n-k) \times (n-k)}$$

Consider any directed path from a node i with $i > k$. The first segment of the path is determined by the presence of a non-zero element a_{ij} in the i^{th} row of A . This row has zeros in the first k portions , so it is possible to make a connection from node i to node j only if $j > k$. Similarly , the path can be extended from node j only to another node greater than k . Continuing in this way , it is found that the directed path from node i with $i > k$ cannot be connected to a node less than $k+1$. Hence we conclude that the directed graph is not strongly connected.

Only if : If A is irreducible , then by Theorem 2.6 this implies that $a_{ij}^{(q)} > 0 \quad \forall i, j$ which in turn implies that , for some q , there is a sequence of q edges from P_i to $P_j \quad \forall i, j$ (i.e. $G(A)$ is strongly connected) . ■

Note that it readily follows from the previous theorem that if A is irreducible, then A has at least n non-zero entries. The interested reader may find an alternate proof in Horn and Johnson ([11], p.361) which relies on the graph theoretical properties of irreducibility. A final algebraic characterization of irreducible matrices is provided here for the interest of the reader. In what follows, " ∂ " will denote the boundary of a set.

Theorem 2.11 . [14,19] The following two conditions characterize the irreducibility of a non-negative matrix A of order $n > 1$.

- (a) No eigenvector of A belongs to $\partial \mathbb{R}_+^n$.
- (b) A has exactly one (up to scalar multiplication) non-negative eigenvector, and this eigenvector is positive.

We now diverge slightly from irreducible matrices in order to state a fundamental result in the theory of non-negative matrices. It is part of the classical Perron-Frobenius theory.

Theorem 2.12 . [14,19] Let A be a non-negative square matrix. Then

- (a) $\rho(A)$, the spectral radius of A , is an

eigenvalue .

- (b) A has a non-negative eigenvector corresponding to $\rho(A)$.

Note that one can easily show that if A in the preceding theorem is assumed to be irreducible then this guarantees $\rho(A) > 0$. The proof follows directly from Theorem 2.11 . A major consequence of the previous theorem is the so called subinvariance theorem which follows .

Theorem 2.13 . [19] Let A be an irreducible non-negative matrix , s a positive number , and let $\underline{y} > \underline{0}$ be a vector satisfying $A \underline{y} \leq s \underline{y}$. Then

(a) $\underline{y} >> \underline{0}$.

(b) $s \geq \rho(A)$. Moreover , $s = \rho(A)$ if and only if $A \underline{y} = \rho(A) \underline{y}$.

Proof : (a) Suppose at least one element , say the i^{th} , of \underline{y} is zero . Then since $A^k \underline{y} \leq s^k \underline{y}$ it follows that ,

$$\sum_{j=1}^n a_{ij}^{(k)} y_j \leq s^k y_i .$$

Now since A is irreducible , for this i and any j , there exists a k such that $a_{ij}^{(k)} > 0$, and since $y_j > 0$ for some j , it follows that $y_i > 0$. But this is a contradiction . Thus $\underline{y} >> \underline{0}$.

(b) Now premultiplying the relation $A \underline{y} \leq s \underline{y}$ by $\hat{\underline{x}}^T$, a positive left eigenvector of A corresponding to $\rho(A)$, yields

$$s \hat{\underline{x}}^T \underline{y} \geq \hat{\underline{x}}^T A \underline{y} = \rho(A) \hat{\underline{x}}^T \underline{y} .$$

(i.e. $s \geq \rho(A)$). Now suppose $A \underline{y} < \rho(A) \underline{y}$. Then the preceding argument, on account of the strict positivity of $A \underline{y}$ and $\rho(A) \underline{y}$, yields $\rho(A) < \rho(A)$, which is impossible. The implication $s = \rho(A)$ follows from $A \underline{y} = s \underline{y}$ similarly. ■

We now extend the notion of inequality to matrices in general by denoting

$A \leq B$ if and only if $B-A \geq 0$,

$A < B$ if and only if $B-A > 0$, and

$A \ll B$ if and only if $B-A \gg 0$.

The following theorem relates inequalities between matrices and their spectral radii. These results are readily obtained using the Perron-Frobenius theory and Theorem 3.4 to be introduced in the next section.

Theorem 2.14 . [3] (a) If $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$.

(b) If $0 \leq A \ll B$ then $\rho(A) < \rho(B)$.

Note that the statement of part (b) of the preceding theorem could have been phrased differently as "If $0 \leq A < B$, B irreducible, then $\rho(A) < \rho(B)$ ".

Definition 2.15 . Let $A \in \mathbb{R}^{n \times n}$. Then we define the leading principal submatrices iA , $i = 1, 2, \dots, n$ to be the matrix consisting of the first i rows and columns of A .

We are now able to relate the relationship between the spectral radius of A and the spectral radii of iA .

Corollary 2.16 . [3] Let $A \geq 0$ belong to $\mathbb{R}^{n \times n}$. Then, for $i = 1, 2, \dots, n$, $\rho({}^iA) \leq \rho(A)$.

Proof : Immediate from Theorem 2.14 .

3 . M - MATRICES

Of special interest and closely related to the set of non-negative matrices is the set of Z-Matrices , denoted by $Z^{n \times n}$. It is composed of all matrices B such that the off-diagonal entries of $-B$ are non-negative . This is denoted by $-B \stackrel{e}{\geq} 0$. We then say that $-B$ is essentially non-negative (i.e. $-b_{ij} \geq 0 \quad \forall \quad i \neq j$) . Hence

$$\begin{aligned} Z^{n \times n} &= \left\{ A \in \mathbb{R}^{n \times n} : A = sI - B , B \geq 0 , s \in \mathbb{R} \right\} \\ &= \left\{ A \in \mathbb{R}^{n \times n} : -A \stackrel{e}{\geq} 0 \right\} . \end{aligned}$$

The aim of this section is to provide a characterization of a certain subset of Z-matrices called M-matrices .

Definition 3.1 . [3] A matrix $A = sI - B$, $B \geq 0$, $s \in \mathbb{R}$ is called an M-Matrix if and only if $s \geq \rho(B)$. If $s > \rho(B)$ then we obtain a Non-Singular M-Matrix .

We now delay our discussion of M-Matrices so that certain background theory may be introduced . The following elementary lemma will be used later .

Lemma 3.2 . Let $\phi(x) = \det(xI - A)$. Then $\phi(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Definition 3.3 . A matrix B is said to be convergent if $\lim_{k \rightarrow \infty} B^k$ exists and is the zero matrix .

We now state a theorem due to Varga and Oldenburger ([20] , p.13) .

Theorem 3.4 . [2,20] For a matrix $A \in \mathbb{R}^{n \times n}$, the following are equivalent :

(a) $\rho(A) < 1$.

(b) A is convergent .

(c) $(I-A)$ is non-singular and $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$,

where the notation $A^0 = I$ is adopted .

Proof : (a) \Leftrightarrow (b) The equivalence between (a) and (b) readily follows from the Jordan canonical form and is therefore omitted .

(b) \Leftrightarrow (c) We now proceed to the equivalence of (b) and (c) . First note that ,

$$(I-A) \cdot (I + A + A^2 + A^3 + \dots + A^{K-1}) = I - A^K$$

Now for k sufficiently large , A^k is uniformly close to the zero matrix , and so $I - A^k$ is close to I , and is therefore non-singular . More specifically , by the continuity of the eigenvalues of a matrix if its elements are perturbed , the eigenvalues $(I - A^k)$ must be close

to those of I for large k , the latter being all 1. Hence $(I - A^k)$ has no zero eigenvalues, and is therefore non-singular. Taking determinants yields

$$\begin{aligned} \det(I-A) \cdot \det(I + A + A^2 + A^3 + \dots + A^{K-1}) \\ = \det(I - A^K) \\ \neq 0. \end{aligned}$$

Therefore,

$$\det(I-A) \neq 0 \Rightarrow (I-A)^{-1} \text{ exists}$$

and

$$I + A + A^2 + A^3 + \dots + A^{K-1} = (I-A)^{-1} \cdot (I - A^K)$$

Letting $k \rightarrow \infty$ completes the proof of the assertion.

The converse is proved by contradiction. Suppose A is not convergent. Then $\sum_{n=0}^{\infty} A^n$ could not possibly converge to any matrix. Thus we obtain our contradiction. ■

We now provide an interesting characterization of the positive solvability of the linear system

$$(sI - A) \underline{x} = \underline{c} \quad \text{where } \underline{c} > \underline{0},$$

and A is a non-negative irreducible square matrix.

Theorem 3.5 . [19] A necessary and sufficient condition for a solution $\underline{x} \gg \underline{0}$ to the equation

$$(sI - A) \underline{x} = \underline{c} \quad (1)$$

to exist for any $\underline{c} > \underline{0}$, where $A \geq 0$ and irreducible, is that $s > \rho(A)$ (i.e. $sI - A$ is a non-singular M-matrix). In this case there is only one solution \underline{x} , which is strictly positive and given by

$$\underline{x} = (sI - A)^{-1} \underline{c}.$$

Proof : Suppose first that for some $\underline{c} > \underline{0}$, a non-negative non-zero solution \underline{x} to equation (1) exists. Then

$$\underline{c} + A \underline{x} = s \underline{x} \quad (\text{i.e. } A \underline{x} < s \underline{x})$$

This is impossible for $s \leq 0$, and if $s > 0$, the subinvariance theorem (Theorem 2.13) implies that $s > \rho(A)$. Now suppose $s > \rho(A)$. Then since $\left(\frac{A^k}{s^k} \right) \rightarrow 0$ as $k \rightarrow \infty$ it follows that

$$(sI - A)^{-1} = s^{-1} (I - s^{-1} A)^{-1} = s^{-1} \sum_{k=1}^{\infty} (s^{-1} A)^k$$

exists from Theorem 3.4 and moreover, since for any ordered pair (i, j) , $a_{ij}^{(k)} > 0$ for some $k = f(i, j)$, by irreducibility (Theorem 2.6), it follows that the right hand side is a strictly positive matrix. Hence $(sI - A)^{-1} \gg \underline{0}$ so that

$$(sI-A)^{-1} \underline{c} \gg \underline{0} \quad \text{for any } \underline{c} > \underline{0},$$

$$\text{and clearly } \underline{x} = (sI-A)^{-1} \underline{c} \gg \underline{0}.$$

It readily follows from Theorem 3.5 that $(sI-A)^{-1}$ is positive. Furthermore, if we are willing to settle for $\underline{x} > \underline{0}$, then the assumption $A \geq 0$, as we shall see later, could be weakened to $A \stackrel{e}{\geq} 0$ but not necessarily irreducible.

Before we state a fundamental characterization of M-Matrices which we will use extensively later, we must introduce the following basic definition.

Definition 3.6. Let $A \in \mathbb{R}^{n \times n}$. Then the leading principal minors of A , denoted by Δ_i , $i = 1, 2, \dots, n$, are defined to be the determinants of the leading principal submatrix iA , that is, $\Delta_i = \det({}^iA)$.

Theorem 3.7. [19] A matrix $A = sI-B$, $B \geq 0$, is a non-singular M-matrix (i.e. $s > \rho(B)$) if and only if the leading principal minors of A , denoted by $\Delta_i(s)$, are positive.

Proof: Assume first that $s > \rho(B)$. Then

$$\Delta_n(s) = \det(sI-B) = \Phi(s) > 0$$

since it is known that $\phi(x) \rightarrow \infty$ as $x \rightarrow \infty$, and s lies beyond the largest real eigenvalue $\rho(B)$ of B . Moreover, since $\rho(B) \leq \rho(B) < s$ (by Corollary 2.16) it similarly follows that

$$\Delta_i(s) > 0 \quad \text{for } i = 1, 2, \dots, n \quad \text{where } n > 1.$$

Assume now that $\Delta_i(s) > 0$ for $i = 1, 2, \dots, n$ holds for some fixed s . Since each of the $\Delta_i(s)$ is a continuous function of the entries of B , it follows that it is possible to replace all the zero entries of B by sufficiently small positive entries to produce a positive matrix \bar{B} with $\rho(\bar{B}) \geq \rho(B)$ (by Theorem 2.14) for which still $\Delta_i(s) > 0$, $i = 1, 2, \dots, n$. Thus if we can prove that $s > \rho(\bar{B})$, this will suffice. It follows, then, that it suffices to prove that $\Delta_i(s) > 0$, $i = 1, 2, \dots, n$, implies $s > \rho(B)$ for a positive matrix B .

We proceed by induction on the order of the matrix B . If $n = 1$, $\Delta_1(s) > 0$ implies

$$s > b_{11} \equiv B \equiv \rho(B).$$

Suppose the proposition is true for matrices of order n , and for a matrix B of order $n+1$ assume $\Delta_i(s) > 0$, $i = 1, 2, \dots, n+1$. We have by the induction hypothesis

that $s > \rho({}^n B)$. Let

$$K_{n+1} = \left(\Delta_n(s) / \Delta_{n+1}(s) \right) > 0$$

and consider the unique solution $\underline{x} = \underline{a}_{n+1}$ of the system

$$(sI - B) \underline{x} = \underline{e}_{n+1} \quad (2)$$

Since

$$\underline{a}_{n+1} = (sI - B)^{-1} \underline{e}_{n+1} ,$$

it follows that the $(n+1)^{th}$ element of \underline{a}_{n+1} is K_{n+1} . If we rewrite $\underline{a}_{n+1} = (\underline{a}_n^T, K_{n+1})^T$ it follows that $\underline{x} = \underline{a}_n$ must satisfy $(s^n I - {}^n B) \underline{x} = \underline{d}$, where $d_i = b_{i,n+1} K_{n+1} > 0$ for $i = 1, 2, \dots, n$. But since $s > \rho({}^n B)$, Theorem 3.5 implies that the unique solution \underline{a}_n is strictly positive . Hence $\underline{a}_{n+1} \gg \underline{0}$ and the subinvariance theorem (Theorem 2.13) applied to (2) now implies $s > \rho(B)$, as required . ■

For the sake of completeness and the reader's interest , we now state without proof a few other characterizations of M-matrices . A proof may be found in Berman and Plemmons ([3] , p.134) .

Theorem 3.8 . [3,6,9,16,17] If $A \in \mathbb{R}^{n \times n}$ then the following are equivalent :

- (a) A is a non-singular M-matrix .
- (b) All of the principal minors of A are positive
- (c) $A + \alpha I$ is a non-singular matrix for each $\alpha \geq 0$.
- (d) A is positive stable (i.e. every real part of each eigenvalue of A is positive) .
- (e) A is inverse positive ; (i.e. There exists an A^{-1} such that $A^{-1} > 0$) .
- (f) A is monotone ; (i.e. $A \underline{x} \geq \underline{0} \Rightarrow \underline{x} \geq \underline{0}$ for all $\underline{x} \in \mathbb{R}^n$) .
- (g) A has a convergent regular splitting ; (i.e. $A = M - N$ where $M^{-1} > 0$, $N \geq 0$ and $M^{-1}N$ is convergent) .
- (h) There exists $\underline{x} > \underline{0}$ such that $A \underline{x} >> \underline{0}$.

Although we are dealing only with non-singular M-matrices , there is a natural extension of the previous theory to singular M-matrices (i.e. $s = \rho(B)$). Much of the results stay the same but one interesting failure is that the non-negativity of the leading principal minors is not equivalent to non-negativity of the principal minors as demonstrated by the following counter example .

Example 3.9 . Let $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Then it is

easily verified that the leading principal minors are $\Delta_1 = 0$ and $\Delta_2 = 0$ whereas the principal minors M_{ij} are given by $M_{11} = 0$, $M_{12} = 0$, $M_{21} = 0$, and $M_{22} = -1$.

4 . APPLICATIONS OF M - MATRICES .

A : Leontief's input-output model .

Leontief's input-output analysis deals with what level of output should each of n industries in a particular economy produce , in order that it will just be sufficient to satisfy the total demand of the economy for that product .

We assume that each industry produces a single kind of commodity , and production means the transformation of several kinds of goods into a single kind of good .

To produce one unit of the j^{th} good , t_{ij} units of the i^{th} good are needed as inputs for $i = 1 , 2 , \dots , n$ in industry j , and λ units of output of the j^{th} good requires $\lambda \cdot t_{ij}$ units of the i^{th} good . The magnitudes t_{ij} are called input coefficient and are usually assumed to be constant .

Let x_i denote the output of the i^{th} good per fixed unit of time . Part of this gross output is consumed as the input needed for the production activities of the n industries . Thus $\sum_{j=1}^n t_{ij} \cdot x_j$ is consumed in the production , leaving $d_i = x_i - \sum_{j=1}^n t_{ij} \cdot x_j$ units of the i^{th} good as the net output . Hence we obtain the linear system

$$(I - T) \underline{x} = \underline{d} .$$

The solvability of the above system in the non-negative unknowns $x_i \geq 0$ means the feasibility of the Leontief model . The model just described is called the open Leontief model since the open sector lies outside the system . If this open sector is absorbed into the system as just another industry , the model is called a closed Leontief model . We now state the conditions for the feasibility of the open Leontief system .

Theorem 4.1 . [3,19] The open Leontief model with input matrix T and $A = I - T$ is feasible if and only if A is a non-singular M-matrix .

Proof : If : This direction of the proof readily follows from Theorem 3.8 (h) . (i.e. There exists $\underline{x} > \underline{0}$, with $A \underline{x} = \underline{d} > \underline{0}$) .

Only if : By Theorem 3.8 (e) , $A^{-1} > 0$. Thus $\underline{x} = A^{-1} \underline{d}$ which clearly is non-negative (i.e. the system is feasible) .

■

B : Jacobi iteration procedure .

We now provide a simple iterative procedure for which M-matrices are particularly well suited . Given a system $A \underline{x} = \underline{b}$, we write A as $D-B$ where D is a non-singular diagonal matrix . Therefore ,

$$\begin{aligned} A \underline{x} = \underline{b} &\Leftrightarrow D \underline{x} = B \underline{x} + \underline{b} \\ &\Leftrightarrow \underline{x} = D^{-1} B \underline{x} + D^{-1} \underline{b} \end{aligned}$$

Then in order to solve the system we use an initial estimate $\underline{x}^{(0)}$ and set

$$\begin{aligned} \underline{x}^{(1)} &= D^{-1} B \underline{x}^{(0)} + D^{-1} \underline{b} \\ \underline{x}^{(2)} &= D^{-1} B \underline{x}^{(1)} + \dots \end{aligned}$$

and in general ,

$$\underline{x}^{(k+1)} = D^{-1} B \underline{x}^{(k)} + D^{-1} \underline{b}$$

Let \underline{x} be a solution . If $\| \cdot \|$ denotes some vector norm on \mathbb{R}^n and the corresponding matrix norm $\| D^{-1} B \| < 1$, then $\| \underline{x}^{(k)} - \underline{x} \| \rightarrow 0$ as $k \rightarrow \infty$. Now ,

$$\begin{aligned} \underline{x}^{(1)} - \underline{x} &= (D^{-1} B \underline{x}^{(0)} + D^{-1} \underline{b}) - (D^{-1} B \underline{x} + D^{-1} \underline{b}) \\ &= D^{-1} B (\underline{x}^{(0)} - \underline{x}) . \end{aligned}$$

$$\begin{aligned}
\underline{x}^{(2)} - \underline{x} &= (D^{-1} B \underline{x}^{(1)} + D^{-1} \underline{b}) - (D^{-1} B \underline{x} + D^{-1} \underline{b}) \\
&= D^{-1} B (\underline{x}^{(1)} - \underline{x}) \\
&= (D^{-1} B)^2 (\underline{x}^{(0)} - \underline{x}) ,
\end{aligned}$$

and in general ,

$$\underline{x}^{(k)} - \underline{x} = (D^{-1} B)^k (\underline{x}^{(0)} - \underline{x}) ,$$

and

$$\begin{aligned}
\| \underline{x}^{(k)} - \underline{x} \| &= \| (D^{-1} B)^k (\underline{x}^{(0)} - \underline{x}) \| \\
&\leq \| (D^{-1} B)^k \| \cdot \| (\underline{x}^{(0)} - \underline{x}) \| \\
&\leq \| (D^{-1} B) \|^k \cdot \| (\underline{x}^{(0)} - \underline{x}) \| \\
&\longrightarrow 0 \quad \text{as } k \longrightarrow \infty \\
&\quad \text{since } \| (D^{-1} B) \| < 1 .
\end{aligned}$$

The following theorem provides an interesting convergence result concerning the iteration procedure discussed above .

Theorem 4.2 . [1,12] Let $\underline{x}^{(0)}$ be an arbitrary vector in \mathbb{R}^n and define $\underline{x}^{(i+1)} = D^{-1} B \underline{x}^{(i)} + D^{-1} \underline{b}$ for $i = 0, 1, 2, \dots, n$. If \underline{x} is the solution to $\underline{x} = D^{-1} B \underline{x} + D^{-1} \underline{b}$, then a necessary and sufficient condition for $\underline{x}^{(k)} \longrightarrow \underline{x}$ is that $\rho(D^{-1}B) < 1$.

The proof , although fairly simple , relies on the Jordan Form of $D^{-1}B$ and is rather messy , so we choose to omit it here .

Now if we assume from the start that A is a non-singular M -matrix then we may use the natural splitting $A = sI - B$ and then

$$D = sI \iff D^{-1} = (1/s) I.$$

Furthermore , if $B \in \mathbb{R}^{n \times n}$ then

$$\begin{aligned} \rho(D^{-1}B) &= (1/s) \rho(B) \\ &< 1 \quad \text{since } \rho(B) < s . \end{aligned}$$

Hence , the convergence of the procedure is guaranteed by Theorem 4.2 in the case of an M -matrix .

3. A STABILIZABILITY - HOLDABILITY PROBLEM

1. Introduction

Consider the following linear autonomous control system of differential equations :

$$\dot{\underline{x}}(t) = A \underline{x}(t) + B \underline{u}(t) \quad , \quad t \geq 0 . \quad (P)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $\underline{u}(t) \in \mathbb{R}^m$. The class of admissible controls is the set of all continuous functions $\underline{u} : \mathbb{R}_+ \rightarrow \mathbb{R}^m$, denoted by \mathcal{U} . The variation of parameters formula provides the solutions for each initial state \underline{x}_0 and each control $\underline{u} \in \mathcal{U}$. This is given by

$$\begin{aligned} \underline{x}(t) &= \underline{x}(t, \underline{x}_0, \underline{u}) \\ &= e^{tA} \underline{x}_0 + e^{tA} \int_0^t e^{-sA} B \underline{u}(s) ds \end{aligned}$$

We say that $\Gamma \subset \mathbb{R}^n$ is holdable with respect to (P) if for any initial state $\underline{x}_0 \in \Gamma$ there exists a control $\underline{u} \in \mathcal{U}$ such that the trajectory $\underline{x}(t, \underline{x}_0, \underline{u})$ stays in Γ for all $t \geq 0$. The interested reader can find an extensive development of the previous theory in Lee [13] and Macki [15] .

In Section 2 , we review the holdability problem in

the uncontrolled case . We shall see this problem has the following matrix-theoretic rephrasing.

(MT1) Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ find , if possible , a matrix $X \in \mathbb{R}^{m \times n}$ such that $A+BX \stackrel{e}{\geq} 0$ [4].

Problem (MT1) is concerned with the existence of a linear feedback law $\underline{u}(t) = X \underline{x}(t)$ such that for any $\underline{x}_0 \geq \underline{0}$, $\underline{x}(t, \underline{x}_0, \underline{u}) \geq \underline{0} \quad \forall t > 0$, or equivalently $e^{t(A+BX)} \mathbb{R}_+^n \subseteq \mathbb{R}_+^n \quad \forall t \geq 0$ [15].

Section 3 is concerned with the holdability problem for \mathbb{R}_+^n in the controlled case, while section 4 is devoted to finding , if possible , a linear feedback law $\underline{u}(t) = X \underline{x}(t)$ such that for any $\underline{x}_0 \geq \underline{0}$ we obtain $\underline{x}(t, \underline{x}_0, \underline{u}) \geq \underline{0} \quad \forall t \geq 0$, and $\underline{x}(t, \underline{x}_0, \underline{u}) \longrightarrow \underline{0}$ as $t \longrightarrow \infty$. This is referred to as the " stabilizability-holdability problem " , denoted by (SH) . It will be seen that this problem has the following matrix-theoretic rephrasal :

(MT2) Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ find , if possible , a matrix $X \in \mathbb{R}^{m \times n}$ such that $-(A+BX)$ is a non-singular M-matrix [4].

It is worth noting that all our results generalize to

a simplicial cone K (i.e. $K = Q \mathbb{R}_+^n$ for a non-singular matrix $Q \in \mathbb{R}^{n \times n}$) . This is accomplished via the transformation of coordinates $\underline{y}(t) = Q \underline{x}(t)$, for which (P) becomes

$$\dot{\underline{y}}(t) = Q A Q^{-1} \underline{y}(t) + Q B \underline{u}(t) \quad , \quad t \geq 0 \quad (\hat{P})$$

2. Uncontrolled System

We now consider the system (P) when $B = 0$. In the uncontrolled case, holdability of a set $\Gamma \subseteq \mathbb{R}_+^n$ means that $e^{tA} \Gamma \subseteq \Gamma \quad \forall t \geq 0$. We will then call Γ positively invariant. Furthermore, we say that Γ is strictly positively invariant if

$$e^{tA} [\Gamma / \{ \underline{0} \}] \subseteq \text{int}(\Gamma) \quad \forall t \geq 0.$$

Theorem 2.1. [4,18] Let $A \in \mathbb{R}^{n \times n}$. Then the non-negative orthant \mathbb{R}_+^n is positively invariant with respect to A if and only if A is essentially non-negative.

Proof: Let $A \geq^e 0$. Then $A + \alpha I \geq 0$ for some $\alpha \in \mathbb{R}$ and so $e^{t(A+\alpha I)} > 0 \quad \forall t \geq 0$. Since $e^{t(A+\alpha I)} = e^{t\alpha} e^{tA}$, it follows that $e^{tA} > 0 \quad \forall t \geq 0$, implying \mathbb{R}_+^n is positively invariant.

To complete the proof, suppose A is not essentially non-negative. Then A has a negative entry a_{ij} , $i \neq j$. We want to find $\underline{x} \geq \underline{0}$ such that $e^{tA} \underline{x} < \underline{0}$ for some positive t . Now choose $\underline{x} = \underline{e}_j$ and consider

$$\begin{aligned} e^{tA} \underline{e}_j &= \underline{e}_j + tA \underline{e}_j + \frac{t^2}{2} A^2 \underline{e}_j + \dots \\ &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + tA \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \frac{t^2}{2} A^2 \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots \end{aligned}$$

Observe that

$$(e^{tA} \underline{e}_j)_i = 0 + t a_{ij} + g(t)$$

where $\frac{g(t)}{t} \rightarrow 0$ as $t \rightarrow 0$. Then it follows that

$$\frac{(e^{tA} \underline{e}_j)_i}{t} = a_{ij} + \frac{g(t)}{t} < 0 \quad \text{for small } t, \text{ which}$$

implies that $(e^{tA})_{ij} < 0$. ■

Further assumptions improve the statement of the last theorem yielding the following result (which is not required in the sequel) .

Theorem 2.2 . [4] Let $A \in \mathbb{R}^{n \times n}$. The non-negative orthant is strictly positively invariant with respect to A if and only if A is essentially non-negative and irreducible .

Proof : Let $A \geq 0$. Then for some $\alpha > 0$, $A + \alpha I \geq 0$. Furthermore , $(A + \alpha I)^m \underline{e}_1 \gg \underline{0}$ for some $m > 0$ since A is irreducible (Theorem 2.2.4) . Therefore ,

$$\underline{e}_1 + t(A + \alpha I) \underline{e}_1 + \frac{t^2}{2} (A + \alpha I)^2 \underline{e}_1 + \dots \gg \underline{0} \quad \forall t > 0$$

$$\Rightarrow e^{t(A + \alpha I)} \underline{e}_1 \in \text{int}(\mathbb{R}_+^n)$$

$$\Rightarrow e^{tA} \underline{e}_1 \in \text{int}(\mathbb{R}_+^n)$$

$$\Rightarrow e^{tA} \underline{x} \in \text{int} (\mathbb{R}_+^n) \quad \forall \underline{x} \in \mathbb{R}_+^n / \{ \underline{0} \} .$$

Conversely , let \mathbb{R}_+^n be strictly positively invariant . Then by Theorem 2.1 , $A \succeq 0$. We need only show that A is irreducible . Suppose A is reducible . Then we will show that \mathbb{R}_+^n could not be strictly invariant . Consider the matrix $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ where $B \in \mathbb{R}^{k \times k}$, $D \in \mathbb{R}^{(n-k) \times (n-k)}$ for $1 \leq k \leq n$. Then $A^m = \begin{pmatrix} B^m & K^m \\ 0 & D^m \end{pmatrix}$ $\forall m \geq 0$. Now look at

$$\begin{aligned} e^A \underline{e}_1 &= I \underline{e}_1 + A \underline{e}_1 + \frac{A^2}{2!} \underline{e}_1 + \frac{A^3}{3!} \underline{e}_1 + \frac{A^4}{4!} \underline{e}_1 + \dots \\ &> \underline{0} \end{aligned}$$

But $e^A \underline{e}_1$ is not strictly positive since $\left[e^A \underline{e}_1 \right]_{k+1} = 0$.

Therefore this implies that

$$e^A \underline{e}_1 \notin \text{int} (\mathbb{R}_+^n)$$

contradicting the strict positive invariance of \mathbb{R}_+^n . Therefore A is irreducible . Actually one can see that all the i^{th} components of $e^A \underline{e}_1$ are zero for each $i \geq k+1$.

■

3 . Holdability of \mathbb{R}_+^n .

The focus of this section is to provide a simple characterization of the holdability of \mathbb{R}_+^n , which is the content of the following theorem .

Theorem 3.1 . [4] The non-negative orthant \mathbb{R}_+^n is holdable with respect to system (P) if and only if there exists an $X \in \mathbb{R}^{m \times n}$ which solves problem (MT1) .

Proof : It is readily noted that \mathbb{R}_+^n is holdable if and only if for each \underline{e}_i there exists a vector $\underline{x}_i \in \mathbb{R}^n$ such that

$$\begin{aligned} \langle (A+BX) \underline{e}_i , \underline{j} \rangle &= \langle A \underline{e}_i + B \underline{x}_i , \underline{j} \rangle \\ &\leq 0 \end{aligned}$$

for every outward pointing normal vector \underline{j} to \mathbb{R}_+^n at \underline{e}_i . Holdability of the non-negative orthant is then equivalent to the existence of a matrix $X = [\underline{x}_1 : \underline{x}_2 : \dots : \underline{x}_n] \in \mathbb{R}^{m \times n}$ such that $A+BX \stackrel{e}{\geq} 0$.

■

A more precise proof using the concept of subtangentiality may be found in [5], chapter 7.

Hence , checking holdability of \mathbb{R}_+^n is equivalent to solving for , if possible , a matrix X such that $A+BX$ is essentially non-negative . This involves checking the

consistency of the system of n^2-n linear inequalities

$$(*) \quad a_{ij} + \underline{b}^i \underline{x}_j \geq 0 \quad \text{for } i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \\ \text{whenever } i \neq j .$$

where \underline{b}^i is the i^{th} row of B and \underline{x}_j is the j^{th} column of X .

In a lot of problems , the artificial variable method of linear programming accomplishes this task .

4 . a Stabilizability-Holdability Problem .

We now look at the stabilizability-holdability problem (SH) , or equivalently , problem (MT2) : Find $\underline{u}(t) = X \underline{x}(t)$ such that for any $\underline{x}_0 \geq \underline{0}$, $\underline{x}(t, \underline{x}_0, \underline{u}) \geq \underline{0}$ $\forall t \geq 0$ and $\underline{x}(t, \underline{x}_0, \underline{u}) \rightarrow \underline{0}$ as $t \rightarrow \infty$. This is equivalent to finding a matrix X such that $e^{tC} \geq 0 \ \forall t \geq 0$ and $e^{tC} \rightarrow 0$ as $t \rightarrow \infty$ where $C = A+BX$. Note that the first condition of problem (MT2) simply means that $C \stackrel{e}{\geq} 0$ (by Theorem 2.1) and the second condition is characterized by

Theorem 4.1 . [4,10] $\underline{x}(t) \rightarrow \underline{0} \ \forall \underline{x}_0 \in \mathbb{R}^n$ in system (P) with $\underline{u}(t) = X \underline{x}(t)$ if and only if $\text{Re}(\lambda) < 0 \ \forall \lambda \in \text{spec}(C)$ where $C = A+BX$.

The proof of the latter is a direct consequence of the Jordan form and its exponentiation .

Recalling Theorem 2.3.8 (d) we conclude that the two criteria of the control theoretic problem (SH) posed above are identical to $-C$ being a non-singular M-matrix ; In other words , X solving (MT2) . Using Theorem 2.3.7 , implies that an alternate form for problem (SH) is given by

Theorem 4.2 . [4] Let $C = A+BX$. Then $\underline{u}(t) = X \underline{x}(t)$ solves the control problem (SH) if and only if

$$(i) \quad c_{ij} \geq 0 \quad \text{whenever} \quad 1 \leq i \neq j \leq n \quad .$$

and

$$(ii) \quad (-1)^k \Delta_k > 0 \quad \text{for} \quad k = 1, 2, \dots, n \quad .$$

Now in the case of scalar input (i.e. $B \in \mathbb{R}^{n \times 1}$) the inequalities (ii) above are linear . To show this , let ${}^k_1 A$ be the matrix obtained by taking the leading principal matrix of A of order k and replacing the i^{th} column by the first k components of B , where we adopt the convention that ${}^k_0 A$ is the leading principal matrix of order k . Hence ,

$$\begin{aligned} \Delta_k &= \det \left({}^k(A+BX) \right) \\ &= \det \left({}^k_0 A \right) + x_1 \det \left({}^k_1 A \right) + x_2 \det \left({}^k_2 A \right) + \dots \\ &\quad \dots + x_k \det \left({}^k_k A \right) \quad . \end{aligned}$$

since all other determinants would have two identical columns , namely ${}^j B$. Therefore Theorem 4.2 for $B \in \mathbb{R}^{n \times 1}$ is equivalent to the system of linear inequalities

$$(i) \quad a_{ij} \geq 0 \quad \text{whenever} \quad 1 \leq i \neq j \leq n \quad .$$

and

$$(ii) \quad (-1)^k \Delta_k > 0 \quad \text{for} \quad k = 1, 2, \dots, n \quad .$$

Example 4.3 . Solve problem (SH) when

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad X = (x_1, x_2, x_3).$$

Soln. :

The above inequalities are

$$\begin{aligned} (A+BX)_{12} &\geq 0 &\Rightarrow 1 + 0 &\geq 0 \\ (A+BX)_{13} &\geq 0 &\Rightarrow 1 + 0 &\geq 0 \\ (A+BX)_{21} &\geq 0 &\Rightarrow 1 + x_1 &\geq 0 &\Rightarrow x_1 &\geq -1 \\ (A+BX)_{23} &\geq 0 &\Rightarrow 1 + x_3 &\geq 0 &\Rightarrow x_3 &\geq -1 \\ (A+BX)_{31} &\geq 0 &\Rightarrow -x_1 &\geq 0 &\Rightarrow x_1 &\leq -1 \\ (A+BX)_{32} &\geq 0 &\Rightarrow 0 - x_2 &\geq 0 &\Rightarrow x_2 &\leq 0 \end{aligned}$$

Therefore $x_1 = -1$, $x_2 \leq 0$ and $x_3 \geq -1$. Also ,

$$-\Delta_1 = 1 > 0$$

$$\Delta_2 = -x_1 - x_2 - 1 > 0 \quad \Rightarrow \quad -x_2 > 0 \quad \Rightarrow \quad x_2 < 0$$

$$-\Delta_3 = -x_1 - x_2 - 1 > 0 \quad \Rightarrow \quad -x_2 > 0 \quad \Rightarrow \quad x_2 < 0$$

so finally $x_1 = -1$, $x_2 < 0$ and $x_3 \geq -1$ is the desired solution by Theorem 4.2 and the discussion which followed .

For instance one may verify that $\underline{x} = (-1, -1, 0)$ yields

$$-(A+BX) = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and that this is indeed a non-singular M-matrix .

4 . THE GENERAL STABILIZABILITY-HOLDABILITY PROBLEM

1 . Introduction

This chapter looks more closely at the problem of finding a matrix $X \in \mathbb{R}^{n \times 1}$ such that $-(A+BX)$ is a non-singular M-matrix, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$. Certain necessary and sufficient conditions shall be stated in Section 2. Section 3 concentrates on particular cases in which the solvability of the aforementioned problem is simplified. An algorithm to expand a given solution set is developed in Section 4. Section 5 will be concerned with the extension of the matrix X to $\mathbb{R}^{n \times n}$, where $B \in \mathbb{R}^{n \times m}$. Some of the theorems in the scalar input case shall be generalized, and an analysis of the resulting structure of the problem shall follow. Section 6 concludes by demonstrating how given $A \in \mathbb{R}^{n \times n}$, $A \stackrel{e}{\geq} 0$, we can introduce a linear feedback control function $B X \underline{x}(t)$ to the system $\dot{\underline{x}}(t) = A \underline{x}(t)$ (i.e. consider the system $\dot{\underline{x}}(t) = A \underline{x}(t) + B X \underline{x}(t)$ such that the solution of the system of differential equations becomes both stable and holdable.

2 . Scalar Input

This section is devoted to the analysis of the scalar input problem . An alternative to linear programming is also introduced for certain type of problems . Consider problem (MT2) where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $X \in \mathbb{R}^{1 \times n}$. We wish to find X such that $-(A+BX)$ is a non-singular M-matrix . The following definitions are provided in order to simplify the analysis .

Definition 2.1 . Let b_i be the i^{th} component of B .

$$\text{Then} \quad I = \{ i : \text{sign}(b_i) = 1 \}$$

$$J = \{ j : \text{sign}(b_j) = -1 \}$$

Now ,

$$(A+BX)_{ij} \geq 0 \quad \text{for } i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \\ \text{whenever } i \neq j .$$

$$\Rightarrow a_{ij} + b_i x_j \geq 0$$

$$\Rightarrow b_i x_j \geq -a_{ij}$$

and so $A+BX \stackrel{e}{\geq} 0$ only if

$$x_j \left\{ \begin{array}{ll} \geq \frac{-a_{ij}}{b_i} & \text{if } i \in I \\ \leq \frac{-a_{ij}}{b_i} & \text{if } i \in J \end{array} \right.$$

Definition 2.2 . Let $\beta_j = \min \left\{ \frac{-a_{ij}}{b_i} : i \neq j, i \in J \right\}$
and
 $\alpha_j = \max \left\{ \frac{-a_{ij}}{b_i} : i \neq j, i \in I \right\}$
for all $j = 1, 2, \dots, n$.

Note that β_j provides an upper bound for x_j .
Similarly , α_j becomes a lower bound . Armed with
Definition 2.2 , we state a necessary , although
insufficient condition for the solvability of Problem
(MT2) .

Lemma 2.3 . Problem (MT2) is solvable only if for all
 $j = 1, 2, \dots, n$ we have , $\alpha_j \leq \beta_j$.

Corollary 2.4 . If X solves (MT2) , then $x_j \in [\alpha_j, \beta_j]$
for all $j = 1, 2, \dots, n$.

Note that Corollary 2.4 can be refined by a further
assumption on the bounds .

Corollary 2.5 . If (MT2) is solvable and $\alpha_j = \beta_j$ for
all $j = 1, 2, \dots, n$, then the
solution is unique .

Proof : The statement follows trivially since the

essential non-negativity conditions are satisfied by a unique vector $X = (x_j)$ where $x_j = \alpha_j = \beta_j$.

The second criterion for the solvability of (MT2), by Theorem 3.4.2, is that $\delta_k > 0$ for $k = 1, 2, \dots, n$, where δ_k is the leading principal minor of order k of $-A+BX$. This last criterion becomes the following:

$$\begin{aligned}\delta_1 &= - \left(\det \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} A + x_1 \det \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} A \right) > 0 \\ \delta_2 &= \det \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} A + x_1 \det \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} A + x_2 \det \begin{pmatrix} 2 & \\ & 2 \end{pmatrix} A > 0 \\ \delta_3 &= - \left(\det \begin{pmatrix} 3 & \\ & 0 \end{pmatrix} A + x_1 \det \begin{pmatrix} 3 & \\ & 1 \end{pmatrix} A + \dots + x_3 \det \begin{pmatrix} 3 & \\ & 3 \end{pmatrix} A \right) > 0 \\ &\vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \qquad \qquad \qquad \vdots \\ \delta_n &= (-1)^n \left(\det \begin{pmatrix} n & \\ & 0 \end{pmatrix} A + x_1 \det \begin{pmatrix} n & \\ & 1 \end{pmatrix} A + \dots \right. \\ &\qquad \qquad \qquad \left. \dots + x_n \det \begin{pmatrix} n & \\ & n \end{pmatrix} A \right) > 0\end{aligned}$$

Defining $c_{ij} = (-1)^{i+1} \det \begin{pmatrix} 1 & \\ & j \end{pmatrix} A$ and $w_i = (-1)^i \det \begin{pmatrix} i & \\ & 0 \end{pmatrix} A$, we obtain the triangular system

$$(MT3) \quad \begin{pmatrix} c_{11} & 0 & 0 & \dots & 0 \\ c_{21} & c_{22} & 0 & \dots & 0 \\ c_{31} & c_{32} & c_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} << \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix}$$

By using the simple transformation $b_i = w_i - \epsilon_i$, where $\epsilon_i > 0$ for $i = 1, 2, \dots, n$, we obtain

$$(MT3') \quad \begin{pmatrix} c_{11} & 0 & 0 & \dots & 0 \\ c_{21} & c_{22} & 0 & \dots & 0 \\ c_{31} & c_{32} & c_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

Remark : If $\epsilon_i = \epsilon > 0$ for all i and $\prod_{i=1}^n c_{ii} \neq 0$ then it is easily verified if $\tilde{x} = C^{-1} b$ satisfies Corollary 2.4 . If so , then \tilde{x} is a solution and we can use the procedure to be described in Section 4 below in order to construct alternate solutions .

Assume that none of the x_j are totally unbounded . We will see in Section 3 how this particular case can only arise with a special type of matrix B , which actually simplifies the work . Of course x_j could still be unbounded from above or below , but not both . Applying either the transformation $x_j = \alpha_j + \epsilon_j$ or $x_j = \beta_j - \epsilon_j$, depending upon whether x_j is bounded below or above , to problem (MT3) yields , after rearrangement , the triangular system

$$(MT3'') \quad T \underline{\epsilon} \leq \underline{v}$$

where $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \dots, \epsilon_n)^T$

$$0 \leq \epsilon_1 \leq \sigma_1$$

T is then the matrix C in problem (MT3) with perhaps

some columns negated and \underline{v} is the vector obtained by the vector subtraction $\underline{v} = \underline{w} - T \underline{\xi}$, where $\underline{\xi}$ is a combination of α_i and β_i . Here $\sigma_i \in [0, \infty)$ is determined by the interval conditions; that is, $\sigma_i = \beta_i - \alpha_i$. (Note that the inequality above becomes strict if $\sigma_i \rightarrow \infty$). Hence the above criteria provides a sufficiency condition.

Remark : The possibility of $\sigma_i = 0$ has been omitted since it would imply the variables x_i , and therefore ϵ_i , are already known and this would simply reduce the complexity of the problem.

Definition 2.6 . A vector \underline{t} is said to be recessive to a vector \underline{v} , with respect to $0 \leq \epsilon \leq M$, for some $M \in \mathbb{R}^+$, provided $\epsilon t_i < v_i$ for all i and weakly recessive if $\text{sign}(t_i) \leq 0$ for those i for which the inequality fails.

Obviously a negative vector \underline{t} is always weakly recessive to any vector \underline{v} and recessive if ϵ is unbounded. In light of this new terminology, problem (MT3'') is now viewed as the vector equation,

$$(*) \quad \epsilon_1 \underline{t}_1 + \epsilon_2 \underline{t}_2 + \dots + \epsilon_n \underline{t}_n << \underline{v}$$

where \underline{t}_i is the i^{th} column vector of the matrix T .

One simple verification for the solvability of (*) is

to let $\epsilon_i = \epsilon$ for each i . Then the equation (*) reduces to

$$\epsilon \tilde{t} \ll \underline{v} \quad \text{where} \quad \tilde{t} = \sum_{i=1}^n \underline{t}_i$$

Returning to the general equation (*), an order reduction of the problem can be introduced by choosing ϵ_i as close as possible to σ_j , provided that, for some j , \underline{t}_j is weakly recessive to \underline{v} . Further reduction can be introduced if $-\underline{t}_j$ is weakly recessive to \underline{v} by letting $\epsilon_j = 0$. Note that if \underline{t}_j is weakly recessive to \underline{v} for some j and we have

$$\sigma_j > \max_j \left\{ \frac{v_j}{t_{ij}} : t_{ij} \neq 0 \right\}$$

then the problem is solved by letting

$$\epsilon_j = \max \left\{ \max_j \left\{ \frac{v_j}{t_{ij}} : t_{ij} \neq 0 \right\}, 0 \right\}$$

and $\epsilon_i = 0$ for all $i \neq j$.

Once the above assignments have been done, equation (*) is rewritten as

$$\epsilon_{i_1} \underline{t}_{i_1} + \epsilon_{i_2} \underline{t}_{i_2} + \dots + \epsilon_{i_m} \underline{t}_{i_m} \ll \underline{v} - \underline{z}$$

where $m \leq n$ and $\underline{z} = \sum_{i \in K} \epsilon_i \underline{t}_i$ for some subset K of $\{1, 2, \dots, n\}$. The same procedure is then applied to this new equation until the vector equation is no longer decomposable. Then one of two things occurs:

(1) The system has been solved .

(a) All ε_i have been chosen , or

(b) $\underline{v} - \sum_{i \in K} \varepsilon_i \underline{t}_i < \underline{0}$ for some subset

χ of $\{ 1 , 2 , \dots , n \}$. Then all

the other ε_j are set to zero .

(i.e. those ε_j for which $j \notin \chi$) .

(2) The system has a non-decomposable form $C \tilde{\underline{\varepsilon}} < \tilde{\underline{v}}$

where all the columns of C have a positive

component for a corresponding non-positive

component of $\tilde{\underline{v}}$.

Unfortunately , very little can be said about case

(2) and the alternate procedure is to rely on the theory of linear programming for the existence of a solution .

Example 2.7 . Find a matrix X such that $-(A+BX)$ is a non-singular M-matrix where

$$A = \begin{pmatrix} -3 & 0 & -1 & -1 \\ 4 & 0 & 4 & 4 \\ 5 & 2 & -6 & 5 \\ 1 & -4 & -3 & -62 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 4 \end{pmatrix} \quad X = (x_1, x_2, x_3, x_4)$$

Soln. : First note that $I = \{ 1 , 4 \}$ and $J = \{ 2 , 3 \}$ from which we obtain

$$\alpha_1 = \max \left\{ -1/4 \right\} = -1/4 \quad \beta_1 = \min \left\{ 2 , 5 \right\} = 2$$

$$\alpha_2 = \max \left\{ 0 , 1 \right\} = 1 \quad \beta_2 = \min \left\{ 2 \right\} = 2$$

$$\alpha_3 = \max \left(1, -3/4 \right) = 1 \quad \beta_3 = \min \left(2 \right) = 2$$

$$\alpha_4 = \max \left(1 \right) = 1 \quad \beta_4 = \min \left(2, 5 \right) = 2.$$

Therefore Corollary 2.4 tells us that

$$\begin{aligned} x_1 &\in [-1/4, 2] \\ x_2 &\in [1, 2] \\ x_3 &\in [1, 2] \\ x_4 &\in [1, 2] \end{aligned}$$

and the system (MT3) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & -10 & -4 & 0 \\ -148 & -282 & -132 & -16 \end{pmatrix} \underline{x} \leq \begin{pmatrix} 3 \\ 0 \\ -16 \\ -592 \end{pmatrix}$$

Letting

$$\begin{aligned} x_1 &= -1/4 + \varepsilon_1 & \varepsilon_1 &\in [0, 9/4] \\ x_2 &= 1 + \varepsilon_2 & \varepsilon_2 &\in [0, 1] \\ x_3 &= 1 + \varepsilon_3 & \varepsilon_3 &\in [0, 1] \\ x_4 &= 1 + \varepsilon_4 & \varepsilon_4 &\in [0, 1] \end{aligned} \quad \text{where}$$

yields

$$T \underline{\varepsilon} \leq \underline{v}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & -10 & -4 & 0 \\ -148 & -282 & -132 & -16 \end{pmatrix} \underline{\varepsilon} \leq \begin{pmatrix} 13/4 \\ 2 \\ -3 \\ -199 \end{pmatrix}$$

Note that all the columns are weakly recessive with respect to their corresponding ε_i . One may verify that the vector $\underline{\varepsilon} = (5/4, 0, 0, 1)^T$ is a possible solution, that is $T \underline{\varepsilon} \leq \underline{v}$ which yields $\underline{x} = (1, 1, 1, 2)^T$. Then the non-singular M-matrix $-(A+BX)$ becomes

$$-(A+BX) = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -2 & 2 & -2 & 0 \\ -4 & -1 & 7 & -3 \\ -5 & 0 & -1 & 54 \end{pmatrix}$$

3 . Totally Unbounded Variables .

In Section 2 , the assumption was made that none of variables x_i were totally unbounded . It turns out that if , say x_i , is totally unbounded , then the order of the problem is considerably reduced . (Actually , the reduction is proportionnal to the magnitude of i .)

Theorem 3.1 . Consider Problem (MT3) and suppose x_j , the j^{th} component of the row matrix X , is totally unbounded . Then b_i , the i^{th} component of the column matrix B , is equal to zero (i.e. $b_i = 0$) for every $i \neq j$.

Proof : The essential non-negativity condition states that

$$a_{ij} + b_i x_j \geq 0 \quad \forall i \neq j$$

$$\Rightarrow b_i x_j \geq -a_{ij} \quad \forall i \neq j$$

but x_j is totally unbounded . This implies that $b_i = 0$ for all $i \neq j$.

In view of the last theorem, the following remark readily follows.

Remark 3.2 . Consider Problem (MT3) . If both x_j and x_k are totally unbounded , where $j \neq k$, then $B = 0$. Hence this situation could never happen , since the problem

is non-existent in such cases.

We can deduce even more from Theorem 3.1 about the other variables .

Corollary 3.3 . If x_i is totally unbounded then for $j \neq i$,

$$x_j \in \begin{cases} \left[-a_{ij}/b_i, \infty \right) & \text{if } b_i > 0 \\ \left(-\infty, -a_{ij}/b_i \right] & \text{if } b_i < 0 \end{cases}$$

Proof : By Theorem 3.1 the matrix $A+BX$ is given by

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{j1} + b_j x_1 & a_{j2} + b_j x_2 & \dots & a_{jn} + b_j x_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

and the essential non-negativity condition yields

$$\begin{aligned} a_{ij} + b_i x_j &\geq 0 & \forall j \neq i \\ \Rightarrow b_i x_j &\geq -a_{ij} & \forall j \neq i, \end{aligned}$$

and so

$$x_j \begin{cases} \geq -a_{ij}/b_i & \text{if } b_i > 0 \\ \leq -a_{ij}/b_i & \text{if } b_i < 0 \end{cases}$$

It follows from Corollary 3 that the problem reduces to verifying that $\begin{smallmatrix} j-1 \\ 0 \end{smallmatrix}(-A)$ is an M-matrix, $a_{k1} \geq 0$ for $k \neq 1$, and solving the system (MT3''), which simplifies to

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ t_{j1} & t_{j2} & \dots & 0 \\ t_{j+1,1} & t_{j+1,2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ t_{n1} & t_{n2} & \dots & t_{nn} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \vdots \\ \epsilon_n \end{pmatrix} << \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

where $0 \leq \epsilon_j < \infty$ for all $j \neq i$. Now the system viewed as a vector equation becomes,

$$\epsilon_1 \begin{pmatrix} t_{j1} \\ t_{j+1,1} \\ \vdots \\ t_{n1} \end{pmatrix} + \epsilon_2 \begin{pmatrix} t_{j2} \\ t_{j+1,2} \\ \vdots \\ t_{n2} \end{pmatrix} + \dots + \epsilon_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ t_{nn} \end{pmatrix} << \begin{pmatrix} v_j \\ v_{j+1} \\ \vdots \\ v_{n1} \end{pmatrix}.$$

Of course if $v_i \leq 0$ for any $i = 1, 2, \dots, j-1$, then the problem is unsolvable. Also note that this is still a triangular system starting at the variable ϵ_j . Hence the algorithm of Section 2 is readily applied. Again, if two

different components of $\underline{\varepsilon}$ are totally unbounded then this is equivalent to the matrix $B = 0$. With this latter case, the problem is, therefore, nonexistent, so this possibility is omitted.

The following theorem provides a simple criteria for the non-solvability of a problem.

Theorem 3.4 . Consider problem (MT3'') and let

$$K = \left\{ k : t_{ki} \geq 0 \quad \text{for all } i \right\}$$

If $v_k < 0$ for any $k \in K$ then Problem (MT3'') is unsolvable.

The following corollary can also be used as a criteria for the non-solvability of a problem.

Corollary 3.5 Consider problem (MT3''), $T \underline{\varepsilon} < \underline{v}$. If

$$\sum_{j=1}^n \sigma_j \nabla(t_{ij}) t_{ij} \geq v_i \quad \text{for any } i \quad \text{where}$$

$$\nabla(t_{ij}) = \begin{cases} 0 & \text{if } \text{sign}(t_{ij}) \geq 0 \\ 1 & \text{if } \text{sign}(t_{ij}) < 0 \end{cases}$$

and $\sigma_j = \beta_j - \alpha_j$ then problem (MT3'') is unsolvable.

4 . Expansion of Existing Solution .

This section is devoted to the construction of a simple algorithm to expand the set of solutions of problem (MT3'') given an initial solution . The idea is to let $\underline{\varepsilon}$ be a known solution .(i.e. $T \underline{\varepsilon} << \underline{v}$) . Then we find $\gamma > 0$ such that ,

$$\begin{aligned} T (\underline{\varepsilon} + \gamma) &<< \underline{v} \\ \Rightarrow T \gamma &<< \underline{v} - T \underline{\varepsilon} = \underline{u} \quad \text{where } \underline{u} >> \underline{0} . \end{aligned}$$

$$\text{Let } \gamma_1 = \min_{1 \leq i \leq n} \left\{ \left(u_i / t_{i1} \right) : t_{i1} > 0 \right\} \quad \text{and}$$

define recursively ,

$$\gamma_j = \min_{j \leq i \leq n} \left\{ \frac{ \left(u_i - \left(\sum_{k=1}^{j-1} \gamma_k \underline{t}_k \right)_i \right) }{ t_{ij} } : t_{ij} > 0 \right\} ,$$

or let $\gamma_j \rightarrow \infty$ if $t_{ij} \leq 0$ for all i . Then any

$$\hat{\underline{\varepsilon}}_j \in [\underline{\varepsilon}_j , \underline{\varepsilon}_j + \gamma_j] \cap [0 , \beta_j - \alpha_j]$$

will also be a solution to (MT3'') .

Note that this algorithm is simply the construction of an " ε -cube" around the solution which is known to exist since the inequalities are all strict.

The following example is designed to illustrate how

to use the preceding algorithm .

Example 4.1 . Recall Example 2.7 where the system

$T \underline{\varepsilon} \leq \underline{v}$ was given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & -10 & -4 & 0 \\ -148 & -282 & -132 & -16 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix} \leq \begin{pmatrix} 13/4 \\ 2 \\ -3 \\ -199 \end{pmatrix} ,$$

had a solution $\underline{\varepsilon} = \left(5/4, 0, 0, 1 \right)^T$. Then the system $T \underline{\gamma} \leq \underline{v} - T \underline{\varepsilon} = \underline{u}$ yields ,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ -4 & -10 & -4 & 0 \\ -148 & -282 & -132 & -16 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 2 \\ 2 \\ 12 \end{pmatrix}$$

Choose

$$\begin{aligned} \gamma_1 &= \min \left(2 \right) = 2 \\ \Rightarrow \underline{w} - \gamma_1 \underline{t}_1 &= \left(0, 2, 10, 308 \right)^T \\ \Rightarrow \gamma_2 &\rightarrow \infty \quad \text{Since } t_{22}, t_{23}, t_{24} \text{ are all negative .} \\ \gamma_3 &\rightarrow \infty \quad \text{Since } t_{33}, t_{34} \text{ are all negative .} \\ \gamma_4 &\rightarrow \infty \quad \text{Since } t_{44} \text{ is negative .} \end{aligned}$$

Then ,

$$\begin{aligned} \hat{\varepsilon}_1 &\in [5/4, 13/4] \cap [0, 9/4] = [5/4, 9/4] \\ \hat{\varepsilon}_2 &\in [0, \infty) \cap [0, 1] = [0, 1] \\ \hat{\varepsilon}_3 &\in [0, \infty) \cap [0, 1] = [0, 1] \\ \hat{\varepsilon}_4 &\in [1, \infty) \cap [0, 1] = \{ 1 \} \end{aligned}$$

So for any

$$\hat{\varepsilon}_1 \in [5/4, 9/4]$$

$$\hat{\varepsilon}_2 \in [0, 1]$$

$$\hat{\varepsilon}_3 \in [0, 1]$$

$$\hat{\varepsilon}_4 \in \{1\},$$

the vector $\hat{\underline{\varepsilon}}$ is also a solution to (MT3''). That is

$$T \hat{\underline{\varepsilon}} \ll \underline{v}.$$

5 . Non-Scalar Input (i.e. $B \in \mathbb{R}^{n \times m}$, $m > 1$)

In section 3.4 , it was seen that in the case of scalar input , the determinant of the leading principal submatrix led to linear polynomials in the components of X . It turns out that a similar structure arises in the case of non-scalar input .

Theorem 5.1 . Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $X \in \mathbb{R}^{m \times n}$. Then $\det (A+BX)$ is a polynomial of degree not exceeding $\min (n , m)$.

Proof : The proof follows from the multilinearity of the determinant function and the fact that ,

- (1) $(A+BX)$ has n columns , so at most n variables can be extracted from the determinant .
- (2) B has at most m linearly independent columns which will provide a variable to be factored out .

■

The interested reader will find all the necessary background theory in Friedberg [7].

An even stronger version of Theorem 5.1 can be formulated which will actually specify the coefficients of $\det (A+BX)$, but its introduction must be delayed in order to introduce the following concepts ,

Definition 5.2 . Define $\mathbb{K}_{n,m}$ to be the set of all vectors $\underline{z} = (z_1, z_2, \dots, z_n)^T$, $z_i \in \{0, 1, \dots, m\}$ and where the integers $1, 2, \dots, m$ are not allowed to repeat .(i.e. For $i \neq j$, $z_i \neq z_j$) .

For instance , the vector $\underline{z} = (0, 0, 2, 1, 7, 0)^T \in \mathbb{K}_6$ but $\underline{x} = (0, 1, 2, 4, 2, 5)^T \notin \mathbb{K}_{6,7}$.

Definition 5.3 . Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Then define $\underline{z}A$, where $\underline{z} \in \mathbb{K}_{n,m}$, to be the matrix obtained from A by replacing the i^{th} column of A by the z_i^{th} column of B while leaving unchanged the i^{th} column of A if $z_i = 0$.

The following example is provided to illustrate definition 5.3 ,

Example 5.4 . Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

and $\underline{z} = (2, 0, 1, 0)^T \in \mathbb{K}_{4,4}$. Then ,

$$\underline{z}^A = \begin{pmatrix} b_{12} & a_{12} & b_{11} & a_{14} \\ b_{22} & a_{22} & b_{21} & a_{24} \\ b_{32} & a_{32} & b_{31} & a_{34} \\ b_{42} & a_{42} & b_{41} & a_{44} \end{pmatrix}$$

Given a product of x_{ij} from the polynomial $\det(A+BX)$, there exist a vector $\underline{z} \in \mathbb{R}_{n,m}$ associated with the given product and is constructed as follows ,

Let the product be $x_{i_1 j_1} x_{i_2 j_2} \dots x_{i_k j_k}$, $k \leq n$, where $i_k \neq i_1$ and $j_k \neq j_1$ for $k \neq 1$. Define a vector \underline{z} such that $z_{j_k} = i_k$ and all other components are zero. For instance, the product of $x_{21} x_{13}$ would result in $\underline{z} = (2, 0, 1)^T \in \mathbb{R}_{3,m}$.

Armed with this concept we can now formulate one of the main result of this section .

Theorem 5.5 . Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $X \in \mathbb{R}^{m \times n}$. The coefficient of the product $x_{i_1 j_1} x_{i_2 j_2} \dots x_{i_k j_k}$, $k \leq n$, of the polynomial $\det(A+BX)$ is given by $\det(\underline{z}^A)$ where $\underline{z} \in \mathbb{R}_{n,m}$ is as described above and the constant term is given by $\det(A)$.

Proof : The proof readily follows from the multilinearity of the determinant function and a simple analysis of the x_{ij} 's position .

In view of this new terminology , the stronger version of Theorem 5.1 can now be given .

Theorem 5.6 . Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $X \in \mathbb{R}^{m \times 1}$.
Then :

- (a) The polynomial $\det (A+BX)$ is of degree not exceeding $\min (n , \text{rank} (B))$.
- (b) There exists no variable of the form x_{11}^m for $m > 1$.
- (c) Only one variable from each row of X can appear in each term of the polynomial .
- (d) Only one variable from each column of X can appear in each term of the polynomial .

Proof : The proof is immediate from Theorem 5.1 and 5.5 . ■

Further results may be obtained from the analysis of the lower degree terms of $\det (A+BX)$.

Theorem 5.7 . Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $X \in \mathbb{R}^{m \times 1}$.
Then $\det (A+BX)$ has no terms of degree less than $\text{rank} (A)$.

If $E = -(A+BX)$ then finding an X such that E is a non-singular M-matrix must satisfy the following two

criteria . First , the essential non-negativity condition of $-E$ leads to a multivariate system of linear inequalities and second , the leading principal minor conditions leads to a series of possibly increasing degree polynomials in x_{ij} 's .

Let $\rho_k = \det ({}^k E)$. Then the problem is to find an X such that ,

$$\rho_k > 0 \quad , \quad k = 1 , 2 , \dots , n .$$

and

$$A + BX \stackrel{e}{\geq} 0$$

Unfortunately , very little can be said about this system in general even with the known structure . One alternate way of finding a solution is to try to solve the non-linear programming problem ,

$$\begin{array}{ll} \text{Maximize} & \rho_n \\ \text{Subject to} & \rho_k > 0 \quad , \quad k = 1 , 2 , \dots , n-1 . \\ & A + BX \stackrel{e}{\geq} 0 \end{array}$$

If a positive maximum exist then the X found by the preceding non-linear problem is a solution to the problem .

6 . PERMUTATIONS

Given a linear system $\dot{\underline{x}}(t) = A \underline{x}(t)$, $A \in \mathbb{R}^{n \times n}$, $A \geq 0$, the problem of how one can introduce a linear feedback control function $\underline{u}(t) = B X \underline{x}(t)$ such that the stabilizability-holdability problem is easily solved is now addressed .

First note that given $\dot{\underline{x}}(t) = A \underline{x}(t) + B X \underline{x}(t)$ and P is a permutation matrix then ,

$$\begin{aligned} P \dot{\underline{x}}(t) &= P A \underline{x}(t) + P B X \underline{x}(t) \\ &= P A I \underline{x}(t) + P B X I \underline{x}(t) \\ &= P A P^T P \underline{x}(t) + P B X P^T P \underline{x}(t) \quad \text{Since } P^T P = I \\ &= P A P^T \underline{y}(t) + P B X P^T \underline{y}(t) \\ &\quad \text{by letting } \underline{y}(t) = P \underline{x}(t) \\ &= P (A+BX) P^T \underline{y}(t) \end{aligned}$$

Note that $\dot{\underline{y}}(t) = P \dot{\underline{x}}(t)$. Hence the above equation becomes

$$\dot{\underline{y}}(t) = P (A+BX) P^T \underline{y}(t) .$$

We therefore have the following.

Theorem 6.1 . Let $\underline{x}(t)$ be the solution of $\dot{\underline{x}}(t) = A \underline{x}(t)$ and $\underline{y}(t)$ be the solution of $\dot{\underline{y}}(t) = P A P^T \underline{y}(t)$ where P is a permutation matrix . Then $\underline{y}(t) = P \underline{x}(t)$ and furthermore $\underline{x}(t) \rightarrow \underline{0}$ if and only if $\underline{y}(t) \rightarrow \underline{0}$.

Proof : The proof of the first statement follows from the

construction above and the second claim follows from the fact that P is invertible. ■

In view of this last discussion one can use the similarity transformation $P A P^T$, P a permutation matrix, so as to obtain a matrix $Q = P A P^T$ such that

$$Q = \left(\begin{array}{c|c} {}^{(n-1)}Q & D \\ \hline E & F \end{array} \right)$$

where ${}^{(n-1)}Q$ is the maximal M -submatrix and D, E, F are permuted entries of A . Add a matrix $B \in \mathbb{R}^{n \times 1}$ defined as follows

$$B = (\underline{e}_n, \underline{e}_{n-1}, \dots, \underline{e}_{n-i+1}) .$$

For the case $i = 1$ the procedure yields,

$$Q + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (x_{11}, x_{12}, \dots, x_{1n})$$

$$\begin{pmatrix} q_{11} & q_{12} & \dots & q_{1,n-1} & q_{1n} \\ q_{21} & q_{22} & \dots & q_{2,n-1} & q_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ q_{n-1,1} & q_{n-1,2} & \dots & q_{n-1,n-1} & q_{n-1,n} \\ q_{n1} + x_{11} & q_{n2} + x_{12} & \dots & q_{n,n-1} + x_{1,n-1} & q_{nn} + x_{1n} \end{pmatrix}$$

This system can easily be solved using $x_{1j} = -q_{nj}$ for $j = 1, 2, \dots, n-1$ and $x_{1n} = -q_{nn}$.

In the general case we obtain ,

$$Q + (\underline{e}_n, \underline{e}_{n-1}, \dots, \underline{e}_{n-i+1}) \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{i1} & x_{i2} & \dots & x_{in} \end{pmatrix}$$

or equivalently ,

$$\begin{pmatrix} q_{11} & q_{12} & \dots & q_{1n} \\ \vdots & \vdots & & \vdots \\ q_{n-i,1} & q_{n-i,2} & \dots & q_{n-i,n} \\ q_{n-i+1,1} + x_{11} & q_{n-i+1,2} + x_{12} & \dots & q_{n-i+1,n} + x_{1n} \\ q_{n-i+2,1} + x_{1-1,1} & q_{n-i+2,2} + x_{1-1,2} & \dots & q_{n-i+2,n} + x_{1-1,n} \\ \vdots & \vdots & & \vdots \\ q_{n-1,1} + x_{21} & q_{n-1,2} + x_{22} & \dots & q_{n-1,n} + x_{2n} \\ q_{n1} + x_{11} & q_{n2} + x_{12} & \dots & q_{nn} + x_{1n} \end{pmatrix}$$

Then letting $x_{k1} = -q_{n-k+1,1}$ for $k = 1, 2, \dots, i$
 $1 = 1, 2, \dots, n$
 $k \neq 1$

and $x_{k,n-k+1} = -q_{n-k+1,n-k+1}$

for $k = 1, 2, \dots, i$

solves the problem. In other words , for the entries below row $n-i$, make the off-diagonal entries equal to zero and the diagonal entries negative . Then the above assignment completely solves Q . Hence the linear feedback function $\underline{u}(t)$ becomes $B \times \underline{x}(t)$ and the linear system $\dot{\underline{x}}(t) = (A+BX) \underline{x}(t)$ is both stable and holdable.

One can see from the previous analysis that the assumption $A \stackrel{e}{\geq} 0$ is stronger than what is actually required and can be weakened to $a_{kj} \geq 0$ for all k and j for which

$$a_{kj} \geq 0 \quad \forall \quad k = 1, 2, \dots, n-i$$

$$j = n-i+1, n-i+2, \dots, n.$$

where $Q = P A P^T$.

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